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Behavior of Lagrange interpolants to the absolute value function in equally spaced points

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Dedicated to Aldo Ghizzetti

RIASSUNTO: *Viene trovato il limite "star" debole della successione delle misure "counting" normalizzate degli zeri delle interpolanti di Lagrange, associate a nodi equidistanti in* $[-1,1]$ *e relativi alla funzione* $f_s(x) = |x-s|$, *con* $s \in (-1,1)$. *Questo risultato viene poi utilizzato per stabilire la regione esatta, in cui le interpolanti di Lagrange convergono geometricamente.*

ABSTRACT: *We find the weak star limit of the sequence of normalized counting measures of the zeros of the Lagrange interpolants to* $f_s(x) = |x-s| (-1 < s < 1)$ *associated with equidistant nodes on* $[-1,1]$. *We use this to establish the exact region in which the Lagrange interpolants converges geometrically.*

1 – Introduction and statements of main results

For a function f defined on $[-1,1]$, let $L_n(f;\cdot)$ denote the Lagrange

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interpolating polynomial of degree at most n to f at the equidistant nodes

$$x_k^{(n)} := -1 + 2k/n,\ \ k = 0, 1, ..., n.$$

Bernstein proved that (cf. [8]) for $f(x) = |x|$, the sequence $L_n(|t|; x)$ diverges if $0 < |x| < 1$. Recently, BYRNE, MILLS and SMITH [9] considered the rate of this divergent sequence. They proved, if $0 < |x| < 1$, then

$$\limsup_{n\to\infty} |L_n(|t|; x) - |x||^{1/n} = (1 + x)^{(1+x)/2}(1 - x)^{(1-x)/2}.$$

Li and Mohapatra [6] further improved this result by showing that

$$\lim_{n\to\infty} \left| \frac{L_n(|t|; x) - |x|}{w_n(x)} \right|^{1/n} = e, \tag{1}$$

for all $x \in \mathbf{R}$ (the set of real numbers), where $w_n(x) := \prod_{k=0}^n (x - x_k^{(n)})$.

In contrast to the above results, under the assumption that f is bounded on $[-1, 1]$ and analytic at $x = 0$, the authors proved in [7] that the sequence $L_n(f; x)$ converges to f geometrically in a neighborhood (in the complex plane) of $x = 0$. This leads to the question of finding the exact region where $L_n(f; \cdot)$ converges to f. Although the answer in the general situation is still unknown, we try in this note to gain some insight by considering the special but interesting case when

$$f(x) = f_s(x) := |x - s| \quad (-1 < s < 1).$$

We determine the exact region in which $L_n(f_s; \cdot)$ converges to (an analytic continuation of) f_s geometrically. This is done by studying the zero distribution of $L_n(f_s; \cdot)$, which is equivalent to the nth root asymptotics of $L_n(f_s; z)$ in $\mathbf{C}$. Furthermore, we will show that (1) has an extension to all z in the complex plane $\mathbf{C}$.

To state our results, we first introduce some notation. The *potential* corresponding to the uniform distribution $\frac{1}{2}dt$ on $[-1, 1]$ is given by

$$U(z) := \frac{1}{2}\int_{-1}^{1} \log|z - t|\, dt.$$

The level curves of $U(z)$ are denoted by $\Gamma_s := \{z \in \mathbf{C} : U(z) = U(s)\}$, $s \in \mathbf{R}$. Let

$$\Omega_s := [-1, -s] \cup \Gamma_s \cup [s, 1] \text{ for } |s| < 1.$$

Let $\nu_n(t)$ be the normalized counting measure of the zeros of $L_n(f_s; \cdot)$, i.e.,

$$\int_B d\nu_n(t) = \frac{\text{the number of the zeros of } L_n(f_s; z) \text{ in } B}{n},$$

for every Borel set $B \subseteq \mathbf{C}$. For a compact set $S \subseteq \mathbf{C}$, we will use $\operatorname{Ext}(S)$ and $\operatorname{Int}(S)$ to denote the unbounded and the (union of) bounded components of $\overline{\mathbf{C}} \setminus S$, respectively, where $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. We need one more concept from potential theory. A measure b_s supported on Ω_s is called a *balayage* of the uniform distribution $\frac{1}{2}dt$ on $[-1, 1]$ to Ω_s if

$$\int_{\Omega_s} \log|z - t| db_s(t) = U(z) \text{ for all } z \in \operatorname{Ext}(\Omega_s).$$

On using the fact that Γ_s is regular with respect to the Dirichlet problem for $\operatorname{Int}(\Gamma_s)$, one can show that at least one such balayage b_s exists (cf. [5, §4.2]), and since $U(z)$ is continuous in $\mathbf{C}$, such a measure b_s must be unique ([5, Theorem 4.6, Corollary 2]).

We now state our results. Their proofs are given in Section 3.

THEOREM 1. *The sequence of the normalized counting measures $\{\nu_n\}$ of the zeros of $L_n(f_s; \cdot)$ converges, in the weak star topology, to the balayage b_s of $\frac{1}{2}dt$ on $[-1, 1]$ to Ω_s, as $n \to \infty$ through a subsequence Λ of positive integers.*

REMARK 1. Our proof shows that, in Theorem 1, Λ can be any sequence for which

$$\lim_{\substack{n\to\infty \\ n\in\Lambda}} |a_n|^{1/n} = e^{-U(s)}, \tag{2}$$

where a_n denotes the leading coefficient of $L_n(f_s; \cdot)$.

REMARK 2. It can be shown (by using Khinchine's theorem [10] in Lemma 3 below) that for almost all $s \in (-1, 1)$,

$$\lim_{n\to\infty} |a_n|^{1/n} = e^{-U(s)}.$$

So, by Remark 1, the whole sequence $\{\nu_n\}_{n=1}^{\infty}$ converges to b_s for almost all s.

THEOREM 2. *For $s \in (-1,1)$, we have*

$$\limsup_{n\to\infty} |L_n(f_s;z)|^{1/n} = e^{U(z)-U(s)} \tag{3}$$

quasi-everywhere in $\mathrm{Ext}(\Omega_s)$ *and*

$$\lim_{n\to\infty} L_n(f_s;z) = (s-z)\,\mathrm{sgn}(s) \tag{4}$$

geometrically for every $z \in$ $\mathrm{Int}(\Gamma_s)$.

Here we use "quasi-everywhere" to mean that the property holds except on a set having logarithmic capacity zero.

REMARK 3. Note that $U(z) > U(s)$ for $z \in \mathrm{Ext}(\Omega_s)$, so (3) implies that for quasi-every $z \in \mathrm{Ext}(\Omega_s)$ a subsequence of $L_n(f_s;z)$ tends to ∞ geometrically. It is also possible to show that (3) holds for almost all $z \in \mathrm{Ext}(\Gamma_s) \setminus \{-1,1\}$ and $s \in (-1,1)$.

REMARK 4. Using Remarks 1 and 2, it can be shown that for almost all s, $\limsup$ can be replaced by $\lim$ in (3).

REMARK 5. The relation (4) also follows from the general theorem proved in [7].

THEOREM 3. *For all $z \in \mathbf{C}$, there holds*

$$\lim_{n\to\infty} \left| \frac{L_n(|t|;z) - |z|}{w_n(z)} \right|^{1/n} = e. \tag{5}$$

Readers familiar with the subject of asymptotic zero distributions of best polynomial approximants (cf. [2], [3], [11]) will recognize that the above results and their proofs have a flavor similar to those for the best polynomial approximants. However, our proofs are a bit more involved because in the present situation, unlike the case for best polynomial approximations, the limit measure b_s is not the equilibrium measure on its support.

2 – Lemmas

Define

$$\phi_s(x) := \begin{cases} x - s, & s \le x \le 1, \\ 0, & -1 \le x \le s. \end{cases}$$

Then, if $x < s$,

$$\frac{L_n(f_s; x) - |x - s|}{2} = L_n(\phi_s; x).$$

By Newton's formula (cf. [8, p.14]),

$$L_n(\phi_s; x) = \sum_{k=0}^{n} \frac{\Delta_n^k \phi_s(-1)}{k!} \left(\frac{n}{2}\right)^k (x+1) \cdots (x + 1 - \frac{2(k-1)}{n}),$$

where

$$\Delta_n^k \phi_s(-1) = \sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r} \phi_s(-1 + \frac{2r}{n}),\ k = 0, 1, ..., n.$$

Set $k(s) := \max\{k :\ x_k^{(n)} \le s\}$. Then $k(s) = [n(s+1)/2]$ and $x_{k(s)}^{(n)}$ is the closest node to the left of s (or equal to s).

Lemma 1. (i) *If* $0 \le k \le k(s)$, *then* $\Delta_n^k \phi_s(-1) = 0$.
(ii) *If* $k(s) + 1 \le k \le n$, *then*

$$\Delta_n^k \phi_s(-1) = \frac{(-1)^{k-k(s)} (k-2)!}{(k - k(s) - 1)! k(s)!} \frac{2}{n} \left\{ \frac{n(s+1)}{2} (k-1) - \left[\frac{n(s+1)}{2}\right] k \right\}.$$

Proof. Assertion (i) is obvious. To prove (ii), we need the following two formulae [4]:

$$\sum_{k=0}^{m} (-1)^k \binom{n}{k} = (-1)^m \binom{n-1}{m} \qquad (n \ge 1) \tag{6}$$

and

$$\sum_{k=0}^{m} (-1)^k \binom{n}{k} k = (-1)^m n \binom{n-2}{m-1} \qquad (n \ge 2). \tag{7}$$

Now

$$\Delta_n^k \phi_s(-1) = \sum_{r=k(s)+1}^{k} (-1)^{k-r} \binom{k}{r} \phi_s(-1+\frac{2r}{n}) =$$

$$= \sum_{r=k(s)+1}^{k} (-1)^{k-r} \binom{k}{r} (-s-1+\frac{2r}{n}) =$$

$$= \sum_{l=0}^{k-k(s)-1} (-1)^{l} \binom{k}{l} (-s-1+\frac{2(k-l)}{n}) =$$

$$= (-s-1+\frac{2k}{n}) \sum_{l=0}^{k-k(s)-1} (-1)^{l} \binom{k}{l} - \frac{2}{n} \sum_{l=0}^{k-k(s)-1} (-1)^{l} \binom{k}{l} l =$$

$$= (-s-1+\frac{2k}{n})(-1)^{k-k(s)-1} \binom{k-1}{k-k(s)-1} +$$

$$- \frac{2}{n}(-1)^{k-k(s)-1} k \binom{k-2}{k-k(s)-2} =$$

$$= \frac{(-1)^{k-k(s)-1}(k-2)!}{(k-k(s)-1)!k(s)!} \left\{ (-s-1+\frac{2k}{n})(k-1) - \frac{2}{n} k(k-k(s)-1) \right\} =$$

$$= \frac{(-1)^{k-k(s)}(k-2)!}{(k-k(s)-1)!k(s)!} \frac{2}{n} \left\{ \frac{n(s+1)}{2}(k-1) - \left[\frac{n(s+1)}{2}\right] k \right\}.$$

This concludes the proof of Lemma 1. □

Set

$$d_k^{(n)}(s) := \frac{n(s+1)}{2}(k-1) - \left[\frac{n(s+1)}{2}\right] k$$

for $k = k(s)+1, ..., n$. Then we have the following simple lemma.

Lemma 2. *For $s \in (-1, 1)$ and $n \geq 2$, the coefficient of x^n in $L_n(f_s; x)$ is*

$$a_n := \frac{2(-1)^{n-k(s)}}{(n-1)n!}\left(\frac{n}{2}\right)^{n-1}\binom{n-1}{k(s)} d_n^{(n)}(s).$$

Proof. Note that

$$a_n = 2 \times (\text{the coefficient of } x^n \text{ in } L_n(\phi_s; x))$$

when $n \geq 2$, and

$$\text{the coefficient of } x^n \text{ in } L_n(\phi_s; x) = \frac{\Delta_n^n \phi_s(-1)}{n!}\left(\frac{n}{2}\right)^n.$$

We can now apply Lemma 1 (ii) to establish this lemma. □

Lemma 3. *For $s \in (-1, 1)$, we have*

$$\limsup_{n\to\infty} |d_n^{(n)}(s)|^{1/n} = 1.$$

Proof. Since

$$|d_n^{(n)}(s)| = \left|\frac{n(s+1)}{2}(n-1) - \left[\frac{n(s+1)}{2}\right] n\right| \leq n^2(|s|+1) \leq 2n^2,$$

we have

$$\limsup_{n\to\infty} |d_n^{(n)}(s)|^{1/n} \leq 1.$$

Write

$$d_n^{(n)}(s) = n(n-1)\left\{\frac{s+1}{2} - \frac{1}{n-1}\left[\frac{n(s+1)}{2}\right]\right\} =: n(n-1)I_n(s).$$

Then, to prove the lemma, it suffices to show

$$\limsup_{n\to\infty} |I_n(s)|^{1/n} \geq 1 \tag{8}$$

for all $s \in (-1,1)$. Assume, to the contrary, (8) is not true for some $s \in (-1,1)$. Then, there exist $r \in (0,1)$ and $N > 0$ such that

$$|I_n(s)| < r^n \tag{9}$$

for all $n \geq N$. Consequently,

$$|I_n(s) - I_{n+1}(s)| < 2r^n \tag{10}$$

for $n \geq N$. But, with $t = (s+1)/2 \in (0,1)$,

$$n(n-1)|I_n(s) - I_{n+1}(s)| = n(n-1)\left|\frac{[(n+1)t]}{n} - [nt]n - 1\right| =$$

$$= |(n-1)[(n+1)t] - n[nt]|.$$

If there are infinitely many n such that

$$(n-1)[(n+1)t] - n[nt] \neq 0, \tag{11}$$

then for those n, $n(n-1)|I_n(s) - I_{n+1}(s)| \geq 1$. So $\limsup_{n\to\infty} |I_n(s) - I_{n+1}(s)|^{1/n} \geq 1$, contradicting (10). Hence there are only finitely many n such that (11) holds. Therefore, there is a constant $M > 0$ such that $(n-1)[(n+1)t] = n[nt]$ for all $n \geq M$. Then $I_n(s) = I_{n+1}(s)$ for $n \geq M$. This, together with (9), tells us that $I_n(s) = 0$ for $n \geq M$. That is,

$$t = \frac{[nt]}{n-1}$$

for all $n \geq M$, which implies that $(n-1)t$ is an integer for every $n \geq M$. This can happen only if $t = 0$ or $t = 1$, which is impossible. This completes the proof. ∎

Lemma 4. *Let $s \in (-1,1)$. Then*

$$\limsup_{n\to\infty} |a_n|^{1/n} = e^{-U(s)}. \tag{12}$$

Proof. From Lemmas 2 and 3 and Stirling's formula we obtain

$$\limsup_{n\to\infty} |a_n|^{1/n} = \frac{e}{(1+s)^{(1+s)/2}(1-s)^{(1-s)/2}} = e^{-U(s)}.$$

□

Lemma 5. *Let $x \leq s$. Then,*

$$|L_n(\phi_s;x)| \leq \frac{2n^2}{n!}\left(\frac{n}{2}\right)^n \binom{n}{k(s)+1} |w_n(x)| =: c(n;s)|w_n(x)|,$$

and

$$\lim_{n\to\infty} c(n;s)^{1/n} = e^{-U(s)}. \tag{13}$$

Proof. Write d_k for $d_k^{(n)}(s)$, $k = k(s)+1, \ldots, n$. Then, using Lemma 1, we have

$$L_n(\phi_s;x) =$$

$$= \sum_{k=k(s)+1}^{n} \frac{(-1)^{k-k(s)}(k-2)!}{k!(k-k(s)-1)!k(s)!}\frac{2}{n}d_k\left(\frac{n}{2}\right)^k (x+1)\cdots(x+1-\frac{2(k-1)}{n}) =$$

$$= \sum_{k=k(s)+1}^{n} \frac{(-1)^{k-k(s)} w_n(x)\left(\frac{n}{2}\right)^{k-1} d_k}{k(k-1)(k-k(s)-1)!k(s)!(x+1-\frac{2k}{n})\cdots(x+1-\frac{2(n-1)}{n})(x-1)}$$

Now, note that

$$\left|x+1-\frac{2\,(k(s)+1)}{n}\right| \geq \left|s+1-\frac{2\,(k(s)+1)}{n}\right| = \frac{2\left|d_{k(s)+1}\right|}{nk(s)},$$

and

$$\left|x+1-\frac{2k}{n}\right| \geq -1+\frac{2k}{n}-\left(-1+\frac{2(k(s)+1)}{n}\right) = \frac{2}{n}(k-k(s)-1)$$

for $k = k(s)+2, \cdots, n$. Thus

$$|L_n(\phi_s;x)| =$$

$$\leq \frac{|w_n(x)|\left(\frac{n}{2}\right)^n}{n!}\binom{n}{k(s)+1}+$$

$$+\sum_{k=k(s)+2}^{n}\frac{|w_n(x)|\left(\frac{n}{2}\right)^{k-1}|d_k|}{k(k-1)(k-k(s)-1)!k(s)!\frac{2}{n}(k-k(s)-1)\cdots\frac{2}{n}(n-k(s)-1)}=$$

$$=\frac{|w_n(x)|\left(\frac{n}{2}\right)^n}{n!}\binom{n}{k(s)+1}+$$

$$+\sum_{k=k(s)+2}^{n}\frac{|w_n(x)|\left(\frac{n}{2}\right)^{k-1}|d_k|}{k(k-1)(k-k(s)-1)(n-k(s)-1)!k(s)!\left(\frac{2}{n}\right)^{n-k+1}}\leq$$

$$\leq \frac{|w_n(x)|\left(\frac{n}{2}\right)^n}{n!}\binom{n}{k(s)+1}+$$

$$+\sum_{k=k(s)+2}^{n}\frac{|w_n(x)|\left(\frac{n}{2}\right)^n 2n}{(k(s)+1)(n-k(s)-1)!k(s)!}\leq$$

$$\leq \frac{2n^2}{n!}\left(\frac{n}{2}\right)^n\binom{n}{k(s)+1}|w_n(x)|.$$

Equation (13) follows directly from an application of Stirling's formula. This completes the proof of Lemma 5. ∎

LEMMA 6. *For $z \in [-1,1]$, there holds*

$$|w_n(z)|e^{-nU(z)} \leq 3n^2.$$

PROOF. To simplify the notation, we write x_k for $x_k^{(n)}$, $k = 0, 1, ..., n$. Using the monotonicity of $\log x$ on $(0,\infty)$, we have for $k = 1, 2, ..., k(z)$,

$$\frac{1}{n}\log(z-x_k) \leq \frac{1}{2}\int_{x_{k-1}}^{x_k}\log|z-t|dt;$$

and for $k = k(z) + 1, ..., n - 1$,

$$\frac{1}{n}\log(x_k - z) \le \frac{1}{2}\int_{x_k}^{x_{k+1}} \log|z-t|dt.$$

Summing the above inequalities, we get

$$\frac{1}{n}\sum_{k=1}^{n-1}\log|z-x_k| \le \frac{1}{2}\left(\int_{-1}^{x_{k(z)}} + \int_{x_{k(z)+1}}^{1}\right)\log|z-t|dt,$$

or, equivalently,

$$\frac{1}{n}\log|w_n(z)| - \frac{1}{n}\log(1-x^2) \le U(z) - \frac{1}{2}\int_{x_{k(z)}}^{x_{k(z)+1}}\log|z-t|dt. \tag{15}$$

Now

$$\int_{x_{k(z)}}^{x_{k(z)+1}}\log|z-t|dt = g(z - x_{k(z)}) + g(x_{k(z)+1} - z),$$

where $g(u) := u\log u - u$. Note that $g'(u) = \log u < 0$ for $u \in (0,1)$, so g is decreasing and $g(u) < g(0+) = 0$ on the interval $(0,1)$. Since

$$z - x_{k(z)},\ x_{k(z)+1} - z \in (0, \frac{2}{n}),$$

it then follows that

$$\left|\int_{x_{k(z)}}^{x_{k(z)+1}}\log|z-t|dt\right| \le 2|g(\frac{2}{n})| = \frac{4}{n}(1+\log\frac{n}{2}).$$

Using this together with (15), we obtain

$$\frac{1}{n}\log|w_n(z)| \le U(z) + \frac{2}{n}(1+\log\frac{n}{2}),$$

which implies (14). □

3 – Proofs of Theorems

We are now ready to prove the theorems of Section 1.

Proof of Theorem 1. Let $s \in (-1,1)$. If $x \le s$, then, from Lemma 5, we have $|L_n(\phi_s;x)| \le c(n;s)|w_n(x)|$, and so

$$|L_n(f_s;x) - |x-s|| = 2|L_n(\phi_s;x)| \le 2c(n;s)|w_n(x)|. \tag{16}$$

If $x \ge s$, then, since

$$L_n(|t-s|;x) = L_n(|-t+s|;x) = L_n(|t-(-s)|;-x)$$

it follows from (16) and the fact that $|w_n(-x)| = |w_n(x)|$,

$$|L_n(f_s;x)-|x-s|| = |L_n(|t-(-s)|;-x)-|-x-(-s)|| \le 2c(n;-s)|w_n(x)|.$$

Hence, with $\hat{c}(n;s) := \max\{c(n;s), c(n;-s)\}$,

$$|L_n(f_s;x) - |x-s|| \le 2\hat{c}(n;s)|w_n(x)| \tag{17}$$

for all $x \in [-1,1]$. Note that, from (13),

$$\lim_{n\to\infty} \hat{c}(n;s)^{1/n} = e^{-U(s)}. \tag{18}$$

Next we need to estimate $L_n(f_s;z)$ for $z \in \mathbf{C}$. Let us first estimate $L_n(f_s;z) - |z-s|$ by extending (17) to $\mathbf{C}$. For definiteness, we assume $s > 0$; the case when $s < 0$ can be handled similarly. Define for $n \ge 2$

$$p(z) := \log|L_n(f_s;z) - (s-z)| - nU(z).$$

Then $p(z)$ is a subharmonic function in $\overline{\mathbf{C}} \setminus [-1,1]$ with $p(\infty) = \log|a_n|$. Using the maximum principle, we have

$$p(z) \le \max_{z\in[-1,1]} p(z) = \max\{\max_{z\in[-1,s]} p(z), \max_{z\in[s,1]} p(z)\},\ z \in \mathbf{C}. \tag{19}$$

By (17),

$$\max_{z\in[-1,s]} p(z) = \max_{z\in[-1,s]} \left\{\log\left|\frac{L_n(f_s;z)-|z-s|}{w_n(z)}\right| + \log|w_n(z)| - nU(z)\right\} \le$$
$$\le \max_{z\in[-1,s]} \log\left\{2\hat{c}(n;s)|w_n(z)|e^{-nU(z)}\right\} \le \log\{6\hat{c}(n;s)n^2\},$$

where in the last inequality we used (14). On the other hand, for $z \in [s,1]$,

$$\begin{aligned} e^{p(z)} &= |L_n(f_s;z) - |z-s| - 2(s-z)|\, e^{-nU(z)} \leq \\ &\leq |L_n(f_s;z) - |z-s||\, e^{-nU(z)} + 2|z-s|e^{-nU(z)} \leq \\ &\leq 2\hat{c}(n;s)|w_n(z)|e^{-nU(z)} + 4e^{-nU(z)} \leq 6\hat{c}(n;s)n^2 + 4e^{-nU(z)}, \end{aligned}$$

where in the last inequality we used (14) again. So,

$$\max_{z\in[s,1]} p(z) \leq \log\{6\hat{c}(n;s)n^2 + 4e^{-nU(s)}\}.$$

Hence, for $z \in \mathbf{C}$,

$$\begin{aligned} p(z) &\leq \max\left\{\log\left(6\hat{c}(n;s)n^2\right),\ \log\left(6\hat{c}(n;s)n^2 + 4e^{-nU(s)}\right)\right\} = \\ &= \log\left\{6\hat{c}(n;s)n^2 + 4e^{-nU(s)}\right\} =: \log K(n;s); \end{aligned} \tag{20}$$

and, by using (18), it is easy to verify that

$$\lim_{n\to\infty} K(n;s)^{1/n} = e^{-U(s)}. \tag{21}$$

An important consequence of (20) and (21) is the following: For $s > 0$,

$$\begin{aligned} \limsup_{n\to\infty} |L_n(f_s;z) - (s-z)|^{1/n} &\leq \lim_{n\to\infty} K(n;s)^{1/n} e^{U(z)} = \\ &= e^{U(z)-U(s)} < 1 \end{aligned} \tag{22}$$

for all $z \in \operatorname{Int}(\Gamma_s)$.

Now, we are ready to estimate $L_n(f_s;z)$. Let $G_s(t)$ be the Green's function for $\operatorname{Ext}(\Omega_s)$ with pole at ∞. Set $\hat{\Gamma}_\rho := \{z \in \mathbf{C} :\ G_s(z) = \rho\}$, $\rho > 0$. Since

$$\lim_{n\to\infty} \{K(n;s)e^{nU(z)}\}^{1/n} = e^{U(z)-U(s)} > 1$$

uniformly for $z \in \hat{\Gamma}_\rho$, we have, for n sufficiently large and $z \in \hat{\Gamma}_\rho$, $|z-s| \leq K(n;s)e^{nU(z)}$. Thus, using (20) we obtain, for n large and $z \in \hat{\Gamma}_\rho$,

$$|L_n(f_s;z)| \leq |z-s| + K(n;s)e^{nU(z)} \leq 2K(n;s)e^{nU(z)}. \tag{23}$$

Next, define

$$P(z) := \log|L_n(f_s; z)| - nU(z).$$

Then $P(z)$ is subharmonic in $\mathbf{C}\backslash[-1,1]$ with $P(\infty) = \log|a_n|$. From (23), for each $\rho > 0$, there is a constant $N(\rho) > 0$ such that when $n \geq N(\rho)$,

$$P(z) \leq \log\{2K(n; s)\}, \text{ for } z \in \hat{\Gamma}_\rho. \tag{24}$$

Fix $\rho^* > 0$, and let I_{ρ^*} denote the set of all the zeros of $L_n(f_s; z)$ that lie in $\text{Ext}(\hat{\Gamma}_{\rho^*})$. Choose $\rho \in (0, \rho^*)$. Let $G(z; \zeta)$ be the Green's function for $\text{Ext}(\hat{\Gamma}_\rho)$ with pole at ζ. Then $G(z; \infty) \equiv G_s(z) - \rho$. Define

$$h(z) := P(z) + \sum_{\zeta \in I_{\rho^*}} G(z; \zeta).$$

The function $h(z)$ is subharmonic in $\text{Ext}(\hat{\Gamma}_\rho)$, and by (24),

$$\limsup_{z \to \xi \in \hat{\Gamma}_\rho} h(z) = P(\xi) \leq \log\{2K(n; s)\}.$$

Hence, the maximum principle for subharmonic functions gives

$$h(z) \leq \log\{2K(n; s)\} \tag{25}$$

for all $z \in \text{Ext}(\hat{\Gamma}_\rho)$. Note that $G(\infty; \zeta) = G(\zeta; \infty) \geq \rho^* - \rho$ for $\zeta \in \text{Ext}(\hat{\Gamma}_{\rho^*})$. Thus

$$h(\infty) = \log|a_n| + \sum_{\zeta \in I_{\rho^*}} G(\infty; \zeta) \geq \log|a_n| + n\nu_n\{\text{Ext}(\hat{\Gamma}_{\rho^*})\}(\rho^* - \rho),$$

where ν_n is the normalized counting measure of the zeros of $L_n(f_s; z)$. It then follows from (25) that

$$(\rho^* - \rho) n\nu_n\{\text{Ext}(\hat{\Gamma}_{\rho^*})\} \leq \log\frac{2K(n; s)}{|a_n|}. \tag{26}$$

Now, from Lemma 4, we can find an infinite subsequence of positive integers, say Λ, such that

$$\lim_{\substack{n \to \infty \\ n \in \Lambda}} |a_n|^{1/n} = e^{-U(s)}.$$

Thus, (26) and (21) imply that

$$\limsup_{\substack{n\to\infty\\ n\in\Lambda}} \nu_n\{\,\mathrm{Ext}(\hat{\Gamma}_{\rho^*})\} \le (\rho^*-\rho)^{-1} \lim_{\substack{n\to\infty\\ n\in\Lambda}} \log\left\{\frac{2K(n;s)}{|a_n|}\right\}^{1/n} = 0,$$

and so

$$\lim_{\substack{n\to\infty\\ n\in\Lambda}} \nu_n\{\,\mathrm{Ext}(\hat{\Gamma}_{\rho^*})\} = 0 \text{ for every } \rho^*>0. \tag{27}$$

Let ν be a weak star limit of $\{\nu_n\}_{n=1}^{\infty}$. Then, from (27), $\mathrm{supp}(\nu) \subseteq \mathbf{C}\setminus\,\mathrm{Ext}(\Omega_s)$. But (22) implies that $L_n(f_s;z)\to z-s$ for $z\in\,\mathrm{Int}(\Gamma_s)$, and therefore $L_n(f_s;z)$ has only finitely many zeros in each compact subset of $\mathrm{Int}(\Gamma_s)$. Hence, we must have $\mathrm{supp}(\nu)\subseteq\Omega_s$.

We now show that the sequence $\{\nu_n\}_{n\in\Lambda}$ converges in the weak star topology to the measure b_s. Suppose that for some infinite sequence $\Lambda_0\subseteq\Lambda$, $\nu_n\to\nu$ in the weak star topology as $n\to\infty$ and $n\in\Lambda_0$. We claim that

$$\int_{\Omega_s} \log|z-t|d\nu(t) \le U(z) \tag{28}$$

for all $z\in\,\mathrm{Ext}(\Omega_s)$. Indeed, fix $z\in\,\mathrm{Ext}(\Omega_s)$. Then, by (24), we have for $n\ge n_z$

$$\int_{\mathbf{C}} \log|z-t|d\nu_n(t) - U(z) \le \log\left\{\frac{2K(n;s)}{|a_n|}\right\}^{1/n}. \tag{29}$$

Let $R>|z|+1$, so that

$$\int_{|t|\ge R} \log|z-t|d\nu_n(t) \ge 0.$$

Then (29) yields

$$\int_{|t|\le R} \log|z-t|d\nu_n(t) \le U(z) + \log\left\{\frac{2K(n;s)}{|a_n|}\right\}^{1/n},$$

and so, on letting $n \to \infty$, $n \in \Lambda_0$, we obtain

$$\limsup_{\substack{n\to\infty\\ n\in\Lambda_0}} \int_{|t|\le R} \log|z-t|d\nu_n(t) \le U(z).$$

By the lower envelope theorem (cf. [5]), we then get

$$\int_{|t|\le R} \log|z-t|d\nu(t) \le U(z), \tag{30}$$

for quasi-every $z \in \operatorname{Ext}(\Omega_s)$, with $R > |z|+1$. But since both potentials in (30) are continuous in $\operatorname{Ext}(\Omega_s)$, (recall that $\operatorname{supp}(\nu) \subseteq \Omega_s$), this inequality holds for every $z \in \operatorname{Ext}(\Omega_s)$, $R > |z|+1$. Letting $R \to \infty$ gives

$$\int_{\mathbf{C}} \log|z-t|d\nu(t) \le U(z),$$

which is equivalent to claim (28).

Since the difference of the two sides in (28) is a harmonic function in $\operatorname{Ext}(\Omega_s)$ even at ∞ with value 0, the equality in (28) must hold for all $z \in \operatorname{Ext}(\Omega_s)$. Thus ν is a balayage of $dt/2$ on $[-1,1]$ to Ω_s. By the uniqueness of b_s, $\nu = b_s$. Therefore, the sequence $\{\nu_n\}_{n\in\Lambda}$ has only one weak star limit and so it converges in the weak star topology, and the limit measure is b_s. ∎

Proof of Theorem 2. Equation (4) follows from (22). We now verify (3). Inequality (23) implies that

$$\limsup_{n\to\infty} |L_n(f_s;z)|^{1/n} \le \lim_{n\to\infty} \{2K(n;s)\}^{1/n} e^{U(z)} = e^{U(z)-U(s)} \tag{31}$$

for $z \in \hat{\Gamma}_\rho$, $\rho > 0$. By the arbitrariness of $\rho > 0$, (31) holds for all $z \in \operatorname{Ext}(\Omega_s)$. On the other hand, if Λ is chosen such that (2) holds, then $\nu_n \to b_s$ as $n \to \infty$ and $n \in \Lambda$ by Theorem 1. So, for quasi-every $z \in \operatorname{Ext}(\Omega_s)$, we have by the lower envelope theorem

$$\limsup_{\substack{n\to\infty\\ n\in\Lambda}} |L_n(f_s;z)|^{1/n} = \limsup_{\substack{n\to\infty\\ n\in\Lambda}} \left\{\exp\left(\int \log|z-t|d\nu_n(t)\right)|a_n|^{1/n}\right\} =$$

$$= e^{U(z)-U(s)}.$$

Therefore, equality holds in (31) quasi-everywhere in $\mathrm{Ext}(\Omega_s)$, and so (3) is true. ∎

Proof of Theorem 3. Equation (5) is valid for $z \in \mathbf{R}$ by [6]. So we assume $z \in \mathbf{C} \setminus \mathbf{R}$. First, note that $U(z) > U(0) = -1$ and $\lim_{n\to\infty} |w_n(z)|^{1/n} = e^{U(z)}$ for $z \in \mathbf{C} \setminus \mathbf{R}$. Next, note that (5) is a consequence of the following:

$$\text{For } z \in \mathbf{C} \setminus \mathbf{R},\ \lim_{n\to\infty} |L_n(|t|; z)|^{1/n} = e^{U(z)+1}. \tag{32}$$

Hence, we need only show (32).

Let $n' := [n/2]$. Since $L_n(|t|; x)$ is an even function, we can write $L_n(|t|; x) = P_{n'}(x^2)$ for some $P_{n'} \in \mathcal{P}_{n'}$. It is easy to verify that $P_{n'}$ is the polynomial of degree at most n' which interpolates $\sqrt{x}$ at the points $(0 \le)\ t_{n'} < t_{n'-1} < \dots < t_1 < t_0 = 1$ with $t_k = (x_k^{(n)})^2,\ k = 0, 1, \dots, n'$. Define $w_n^*(x) := \prod_{k=0}^{n'} (x - t_k)$. We now claim that

$$P_{n'}(z) - \sqrt{z} = \frac{w_n^*(z)}{\pi} \int_0^\infty \frac{\sqrt{t}\, dt}{w_n^*(-t)(t+z)},\ z \in \mathbf{C} \setminus (-\infty, 0]. \tag{33}$$

We check (33) only for the case when n is even. The proof for the case when n is odd follows the same line and is simpler. When n is even, $t_{n'} = 0$. So, the point 0 is a point of interpolation and $P_{n'}(z)/z$ is the polynomial of degree $n'-1$ that interpolates $1/\sqrt{z}$ at points $(0 <)\ t_{n'-1} < \dots < t_1 < t_0 = 1$. Then, using the Hermite formula:

$$\frac{P_{n'}(z)}{z} - \frac{1}{\sqrt{z}} = \frac{w_n^*(z)}{2z\pi i} \int_\gamma \frac{d\zeta}{\sqrt{\zeta}(w_n^*(\zeta)/\zeta)(z-\zeta)},\ z \in \mathrm{Int}(\gamma), \tag{34}$$

where γ is an arbitrary positively oriented contour in $\mathbf{C}\setminus(-\infty, 0]$ that contains $[t_{n'-1}, 1]$ in its interior. Let γ deform to the boundary of $A(\varphi, r, R) := \{z :\ |\arg(z)| \le \varphi \text{ and } r \le |z| \le R\}$ with $0 < \varphi < \pi$ and $0 < r < 1/n^2 < 1 < R$. Now, let the inner radius r tend to 0 and the outer radius R tend to ∞ and then let the angle φ tend to π. Then the integral in (34) converges to the integral in (33) multiplied by $-2/i$, from which our claim (33) follows.

In terms of $L_n(|t|; z)$, (33) yields

$$L_n(|t|; z) - z = \frac{w_n^*(z^2)}{\pi} \int_0^\infty \frac{\sqrt{t}\ dt}{w_n^*(-t)(t+z^2)},\ \mathrm{Re}(z) > 0. \tag{35}$$

Define

$$S_n(z) := \frac{(-1)^{n'+1}}{\pi} \int_0^\infty \frac{\sqrt{t}\, dt}{w_n^*(-t)(t+z)}, \ z \in \mathbf{C} \setminus (-\infty, 0].$$

Then

$$S_n(z) = \int_0^\infty \frac{\psi_n(t) dt}{t+z}, \ z \in \mathbf{C} \setminus (-\infty, 0],$$

where

$$\psi_n(t) := \frac{(-1)^{n'+1}\sqrt{t}}{\pi w_n^*(-t)} \geq 0 \text{ for } t \geq 0.$$

Thus, $S_n(z)$ is a Stieltjes function and we observe that

$$\mathrm{Im}(z) \cdot \mathrm{Im}(S_n(z)) < 0 \text{ if } \mathrm{Im}(z) \neq 0;$$

and

$$S_n(z) > 0 \text{ if } z > 0.$$

Now, we can define an analytic function $H_n(z) := \mathrm{Log}(S_n(z)/S_n(1))$ for $z \in \mathbf{C} \setminus (-\infty, 0]$. Since $|\mathrm{Im}(H_n(z))| = |\mathrm{Arg}(S_n(z)/S_n(1))| < \pi$, then we have

$$\lim_{n \to \infty} \frac{1}{n} |\mathrm{Im}(H_n(z))| = 0, \tag{36}$$

locally uniformly for $z \in \mathbf{C} \setminus (-\infty, 0]$. By Schwarz's integral formula, $H_n(z)$ can be expressed in terms of $\mathrm{Im}(H_n(z))$ and $\mathrm{Re}(H_n(z_0))$ in any disk with center z_0 contained in $\mathbf{C} \setminus (-\infty, 0]$. In particular, from (36) we have

$$\lim_{n \to \infty} \frac{1}{n} \mathrm{Re}(H_n(z)) = 0, \tag{37}$$

uniformly for $|z-1| \leq \rho$ $(\rho < 1)$. Then, by using a chain of circles we can extend (37) to all points contained in $\mathbf{C} \setminus (-\infty, 0]$. It then follows that

$$\lim_{n \to \infty} \left(\frac{1}{n} \log |S_n(z)| - \frac{1}{n} \log S_n(1) \right) = 0 \tag{38}$$

locally uniformly for $z \in \mathbf{C} \setminus (-\infty, 0]$. Using (1) with $x = 1$ in (35), we get

$$\lim_{n\to\infty} \frac{1}{n} \log S_n(1) = 1,$$

and so (38) implies that

$$\lim_{n\to\infty} \frac{1}{n} \log |S_n(z)| = 1$$

locally uniformly for $z \in \mathbf{C} \setminus (-\infty, 0]$. This, together with (35), gives us

$$\lim_{n\to\infty} \left| \frac{L_n(|t|; z) - z}{w_n(z)} \right|^{1/n} = e \tag{39}$$

locally uniformly for $\mathrm{Re}(z) > 0$. Similarly,

$$\lim_{n\to\infty} \left| \frac{L_n(|t|; z) + z}{w_n(z)} \right|^{1/n} = e \tag{40}$$

locally uniformly for $\mathrm{Re}(z) < 0$. Now, from (39) and (40), we see that (32) holds if, in addition, we assume $\mathrm{Re}(z) \neq 0$.

Finally, we verify that (32) holds when $\mathrm{Re}(z) = 0$. The proof for this case turns out to be very lengthy. We will give here only a sketch of the proof and leave the details to the reader. Assume $z = bi$ for some real number $b \neq 0$. It is easy to see that

$$w_n'(x_k^{(n)}) = \left(\frac{2}{n}\right)^n (-1)^{n-k} k!(n-k)!.$$

So, using Lagrange's formula, we have

$$L_n(|t|; bi) = \left(\frac{n}{2}\right)^n \frac{(-1)^n}{n!} w_n(bi) \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{|x_k^{(n)}|}{bi - x_k^{(n)}}. \tag{41}$$

The summation in (41) (let's call it S_n) can be written as

$$\sum_{k=0}^{n'} (-1)^k \binom{n}{k} \frac{-2bi|x_k^{(n)}|}{b^2 + (x_k^{(n)})^2} \text{ for even } n,$$

and

$$\sum_{k=0}^{n'}(-1)^k \binom{n}{k} \frac{2(x_k^{(n)})^2}{b^2+(x_k^{(n)})^2} \text{ for odd } n.$$

For odd n, we apply the residue theorem to write S_n as

$$S_n = \frac{1}{2\pi i}\int_{C_{\delta,M}} \frac{(-1)^{n'}2\Gamma(n+1)}{\Gamma(\frac{n+1}{2}+1+z)\Gamma(\frac{n+1}{2}-z)} \frac{(\frac{1+2z}{n})^2}{b^2+(\frac{1+2z}{n})^2}\frac{\pi dz}{\sin \pi z},$$

where $C_{\delta,M}$ denotes the rectangle formed by lines $\text{Re}(z) = -\delta/2$ $(0 < \delta < 1)$, $\text{Re}(z) = n/2$, and $\text{Im}(z) = \pm M$ $(M > 0)$. (This integral representation of S_n can be verified by noting that the integrand is analytic in $C_{\delta,M}$ except at $z = 0, 1, ..., (n-1)/2$ where it has simple poles and the residue at $z = (n-1)/2 - k$ is the kth term in the summation form of S_n.) Let $\Omega(z)$ denote the integrand. It can be verified that

(i) for fixed n, the integral along lines $\text{Im}(z) = \pm M$ tends to 0 as $M \to \infty$,

(ii) the integral along $\text{Re}(z) = n/2$ tends to 0 as $n \to \infty$,

(iii) the absolute value of the integral along $\text{Re}(z) = -\delta/2$ is greater than $c|\Omega(-\delta/2)|$ for some positive constant c independent of n.

Indeed, (i) follows from the (crude) estimate $\Omega(z) = O(|z|^{-2})$ (n fixed), while (ii) is proved by showing $\Omega(z) = O(n^{-1/2}(1/4+t^2)^{-1})$ $(n \to \infty)$ with $z = n/2 + it$ (t real). Assertion (iii) is verified by the saddle point method (cf. [1]). Note that the integrand $\Omega(z)$ can be written as

$$\Omega(z) = \frac{2n!(1+2z)^2}{(z+\frac{n+1}{2})(z+\frac{n+1}{2}-1)\cdots(z+\frac{n+1}{2}-n)(n^2b^2+(1+2z)^2)}.$$

Writing $z = -\delta/2 + it$, we see that $|(z+\frac{n+1}{2})(z+\frac{n+1}{2}-1)\cdots(z+\frac{n+1}{2}-n)|$ strictly increases as $|t|$ increases from 0 to ∞. So, the point $z = -\delta/2$ will play the role of a saddle point. (Actually, $z = -1/2$ is a saddle point.) Then according to [1, §5.10], we know that the absolute value of the integral along $\text{Re}(z) = -\delta/2$ can be successfully compared to $|\Omega(-\delta/2)|$ as given in (iii).

Now, by Stirling's formula, $|\Omega(-\delta/2)|^{1/n} \to 2$ as $n \to \infty$. Hence, by (i)-(iii) above, we have $\liminf_{n\to\infty,\ n \text{ odd}} |S_n|^{1/n} \geq 2$, which, together with (41), implies (32) when n is restricted to odd integers. The case when

n is even can be handled similarly, and (32) holds. This completes our proof. ∎

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