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FRACTIONAL-VALUED MODAL LOGIC AND SOFT BILATERALISM

Abstract

In a recent paper, under the auspices of an unorthodox variety of bilateralism, we introduced a new kind of proof-theoretic semantics for the base modal logic **K**, whose values lie in the closed interval $[0, 1]$ of rational numbers [13]. In this paper, after clarifying our conception of bilateralism – dubbed "soft bilateralism" – we generalize the fractional method to encompass extensions and weakenings of **K**. Specifically, we introduce well-behaved hypersequent calculi for the deontic logic **D** and the non-normal modal logics **E** and **M** and thoroughly investigate their structural properties.

Keywords. Modal logic 03B45; proof theory, general (including proof-theoretic semantics) 03F03; many-valued logics 03B50.

1. Introduction

From a general perspective, the distinctive aspect of bilateralism is that it recognizes and isolates two different dimensions of logic which are placed on a par: *assertion* and *denial*. Although often neglected in the history of logic, denial can be seen as a perfectly sensible logical notion which follows its own specific inferential trajectories [6, 17]. Since the notion of logical denial admits several consistent meanings, the proper logical realm of bilateralism is still a matter of philosophical controversy. Therefore, over the last few decades, various proposals concerning the possibility of a bilateral reading of logic have flourished [19, 4, 22, 17].

On the one hand, Rumfitt has argued that the natural theoretical backdrop against which bilateralism takes place is classical logic; and in effect, bilateralism has traditionally been adopted to give a coherent proof-theoretic account of classical logic. On the other hand, more recently, this view has been challenged by Kürbis, who claims that a bilateral account of intuitionistic logic is also possible [8, 9]. This stance seems perfectly sensible, as the acts of assertion and denial can also be rephrased in proper intuitionistic terms.

In what follows, we propose a particular conception of bilateralism, which can accommodate non-classical logics or extensions of classical logic, such as substructural logics and modal logic. As it is well known, the notion of denial in bilateralism is primitive and cannot be reduced to the assertion of a negation. Our proposal is based on interpreting the act of denial by means of the logically "soft" notion of *rejection*. A formula A can be considered as rejected just in case it does not admit a proof within the reference system. For example, in classical propositional logic contradictions and truth-functional contingencies all qualify as rejectable formulas [18]. This is why we label this type of bilateralism as "soft" to distinguish it from other narrower interpretations, whereby denial is logically analyzed as refutation, i.e. in terms of a derivation of grounds for the denial of the proposition.

In this paper, we introduce calculi for a family of modal logics that operate within a soft bilateral framework by combining rules for handling derivable as well as underivable sequents.[1] This hybrid approach to inference rules is both technically useful, as it allows for a more comprehensive understanding of the logic without reducing it to the set of its theorems, and conceptually profound, as it is closely linked to the venerable notion of analyticity, which is essential for manipulating information about underivability in a well-behaved proof-theoretic setting.

Mainstream proof-theoretic semantics embraces the meaning-as-use paradigm, which entails shifting the focus from analyzing truth-conditions to understanding the inference patterns that govern the recursive construction of proofs [21, 15, 5]. In proof-theoretic semantics, the meaning of connectives is primarily conveyed through the top-down reading of their respective introduction rules.

[1] Proof-systems combining together rules for dealing with valid and invalid syntactic expressions are sometimes called 'hybrid' in the literature on rejection systems [20, 6].

As standard bilateralism is conceptually linked to proof-theoretic semantics, our account of bilateralism also yields its peculiar semantics in terms of proofs, which we call *fractional semantics*. While proof-theoretic semantics is mainly concerned with intuitionistic logic, we have recently shown how a fractional semantics can be provided for a wide class of logics, including classical logic [12], the minimal normal modal logic K [13], and the multiplicative-additive fragment of linear logic MALL [14].

The term "fractional" is used to describe semantics in which formulas are interpreted as values in the closed interval $[0, 1]$ of rational numbers. In the fractional setting, a reference proof system is used as an algorithm to decompose a formula A into a set of clauses $\mathcal{C}(A)$, which are ordinary sequents in the case of classical logic, and hypersequents when K and MALL are being analyzed. The interpretation of A, denoted by $[\![A]\!]$, is obtained by calculating the ratio of true clauses in $\mathcal{C}(A)$ to the total number of clauses produced by the decomposition. This interpretation function measures the degree to which A is satisfied, or the "quantity of truth" in A[2]. Needless to say, we must be able to carry out such a decomposition for *any* formula A in the language, including the case in which A is neither provable nor refutable. Therefore, a "soft" variety of bilateralism is necessary to ensure that this is possible.

Methodologically, the proof-theoretic platform on which the fractional evaluation is built needs to meet the following requirements:

- *Invertibility:* for each logical rule in the calculus, the derivability of the conclusion always implies the derivability of (each of) the premise(s).
- *Stability*: any complete decomposition of the endsequent (end-hypersequent) always returns the *same* set of top-sequents (top-hypersequents).
- *Termination of the proof search*: any decomposition of a given endsequent (end-hypersequent) always terminates yielding either a proof

[2]In interpreting the formulas of classical logic, we use Kleene's system $\mathsf{G4}$ enriched with a 'complementary' axiom introducing whatever clause $\Gamma \vdash \Delta$ such that $\Gamma \cap \Delta = \varnothing$ [12]. Consider for instance the formula $A \equiv p \to (p \wedge q)$. The enriched system decomposes it into the set of clauses $\{p \vdash p; p \vdash q\}$, so that $[\![A]\!] = {}^1/_2 = 0.5$. Actually, this formula can be rewritten as $(p \to p) \wedge (p \to q)$ and this form clearly displays that fact that A is formed by two components of which only one displays an identity.

or a rejection.

On one hand, invertibility and termination guarantee the possibility of turning any set of clauses $\mathcal{C}(A)$ into some sort of canonical form for A (its conjunctive normal form, in classical logic). On the other hand, stability is what allows us to call the described fractional evaluation a 'semantics', making the value $[\![A]\!]$ a derivation-invariant.

The technical aim of this paper is to extend the fractional approach proposed for modal logic to other systems beyond **K**. After reviewing the main proof-theoretic ingredients, the paper shows how to apply the fractional approach to basic deontic logic **D** as well as non-normal modal logics **E** and **M**. **E** is the minimal non-normal modal logic characterized by neighborhood semantics. **M** extends **E** by introducing the axiom of distributivity of $\Box$ over conjunction. The paper investigates the structural properties of these systems and establishes the admissibility of the rules of weakening, contraction, and cut using purely finitary and constructive methods.

2. The systems

2.1. Separating modality and classicality

As we have remarked above, in order to apply the fractional method to modal logic, we need to design a calculus which meets some proof-theoretic *desiderata*. In particular, stability, finiteness of the proof-search space and invertibility..

Achieving finiteness of the proof-search space is perhaps the most delicate item when dealing with non-classical logics or extensions of classical logic. In fact, if we stick to a standard sequent calculus setting, we often lose invertibility. On the other hand, if we supplement the structure of sequents, we can obtain invertible rules, but often at the cost of losing finiteness of the proof-search space.

To meet all of these requirements, we find it natural to switch to a hypersequent formulation of the modal logics we are considering. The use of hypersequents proves to be well-suited as it maintains a strong version of the formula interpretation, meaning that any syntactic object can be interpreted as a formula in the language. Furthermore, hypersequents provide a way to disentangle the classical content of a sequent from its modal residual elements, which is a key step in obtaining finiteness of the proof-search space.

2.2. The calculus $\overline{\overline{\overline{\mathsf{HK}}}}$

We shall be mainly working with *hypersequents*, introduced under a different name by Mints in the early seventies of the last century [11, 10] and independently by Pottinger [16], then further elaborated (and so named) by Avron [1, 2, 3]. Hypersequents come as a generalization of the standard notion of sequent in the style of Gentzen. A *sequent* is a syntactic expression of the form $\Gamma \Rightarrow \Delta$, where Γ, Δ are finite multisets of modal formulas from the set $\mathscr{F}$ recursively defined by the grammar:

$$\mathscr{F} ::= AT \,|\, \neg\mathscr{F} \,|\, \mathscr{F} \rightarrow \mathscr{F} \,|\, \mathscr{F} \wedge \mathscr{F} \,|\, \mathscr{F} \vee \mathscr{F} \,|\, \Box\mathscr{F}$$

with AT collecting the atomic sentences. As usual, $\Diamond A$ is taken to abridge the formula $\neg\Box\neg A$. If $\Gamma = [A_1, A_2, \ldots, A_n]$, then $\bigwedge\Gamma$ and $\bigvee\Gamma$ are the two formulas $A_1 \wedge A_2 \wedge \cdots \wedge A_n$ and $A_1 \vee A_2 \vee \cdots \vee A_n$, respectively. If $\Gamma = \varnothing$, then we set $\bigwedge\Gamma = \top$ and $\bigvee\Gamma = \bot$, where $\top$ and $\bot$ stand for an arbitrarily selected tautology and contradiction, respectively. With $\Box\Gamma$ we mean the multiset $[\Box A_1, \Box A_2, \ldots, \Box A_n]$. For any formula A we denote with A^n the multiset containing exactly n occurrences of A.

In general, if M and N are two multisets, we indicate with $M \uplus N$ and $\#M$ their multiset union and M's cardinality, respectively. A *hypersequent*, denoted by $\mathcal{G}, \mathcal{H}, \ldots$, is defined as a finite (possibly empty) multiset of sequents written as follows:

$$\Gamma_1 \Rightarrow \Delta_1 \,|\, \Gamma_2 \Rightarrow \Delta_2 \,|\, \cdots \,|\, \Gamma_n \Rightarrow \Delta_n.$$

We shall keep calling 'sequents' those hypersequents listing exactly one sequent. The set collecting hypersequents is here indicated with $\mathscr{H}$. Practically speaking, a hypersequent $\mathcal{G}$ turns out to be *valid* whenever at least one of the sequents listed in $\mathcal{G}$ is valid. Here the meaning of the term 'valid' has to be specified in progress, depending on the logical context.

The following definition introduces the notion of *hyperclause* which extends that of clause for standard sequents of classical logic.

Definition 2.1 (Hyperclauses). A hyperclause is a hypersequent

$$\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$$

such that no rule of the calculus can be upwardly applied to it. An *identity* hyperclause is such that, for some i, $\Gamma_i \uplus \Delta_i \neq \varnothing$; otherwise, it is *complementary*.

Example 2.2. An identity hyperclause and a complementary hyperclause, respectively:

$$p \Rightarrow p \mid \Box(\Box p \to p) \Rightarrow \qquad\qquad \Rightarrow p \mid \Rightarrow p \mid \Box(\Box p \to p) \Rightarrow$$

Figure 1 presents the 'softly' bilateral hypersequent calculus $\overline{\overline{\mathsf{HK}}}$. The rules of $\overline{\overline{\mathsf{HK}}}$ operate on hypersequents prefixed by the symbols '$\vdash$' and '$\dashv$': we write $\vdash \mathcal{G}$ and $\dashv \mathcal{G}$ to assert the validity and *in*validity of $\mathcal{G}$, respectively. For the sake of a more compact notation, in Figure 1 the $\overline{\overline{\mathsf{HK}}}$ rules are expressed by writing $\overset{1}{\vdash}$ and $\overset{0}{\vdash}$ to indicate the two signs '$\vdash$' and '$\dashv$', respectively. The calculus is equipped with two axiom rules: the ordinary ax-rule introduces any identity hyperclause, whilst the $\overline{ax}$-rule specifically introduces complementary hyperclauses.

From now on, we will indicate derivations with small Greek letters $\pi, \rho, \ldots$. We recall that the height $h(\pi)$ of a derivation π is given by the number of hypersequents figuring in one of its longest branches. Moreover, we indicate with $\mathsf{top}(\pi)$ the multiset of π's top-hypersequents.

Example 2.3. Figure 2 displays a $\overline{\overline{\mathsf{HK}}}$-derivation ending in $\dashv \Rightarrow \Box(\Box p \to p) \to \Box p$, that is a formal *rejection* for the sequent $\Rightarrow \Box(\Box p \to p) \to \Box p$.

Remark 2.4. The $\Box$-rule is the only inference schema in which the hypersequent structure comes effectively into play. Intuitively speaking, a $\Box$-application in its bottom-up reading allows us to decompose a sequent-component in a hypersequent by splitting its classical part from modal residues. In fact, each time the rule is applied, a new hypersequent component is added, thus starting a parallel derivation.

Furthermore, notice that the side condition on the $\Box$-rule about contexts Γ' and Δ' is crucial to avoid pathological situations like the one indicated below, in which $\overline{\overline{\mathsf{HK}}}$ proves both $\vdash \mathcal{G}$ and $\dashv \mathcal{G}$.

AXIOMS

$$\frac{}{\overset{1}{\vdash} \Box\Pi_1, \Gamma_1, p \Rightarrow \Delta_1, p \mid \cdots \mid \Box\Pi_n, \Gamma_n \Rightarrow \Delta_n}\; ax$$

$$\frac{}{\overset{0}{\vdash} \Box\Pi_1, \Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Box\Pi_n, \Gamma_n \Rightarrow \Delta_n}\; \overline{ax} \quad \text{where } \Gamma_i \cap \Delta_i = \varnothing \text{ for each } 1 \leqslant i \leqslant n$$

LOGICAL RULES

$$\frac{\overset{i}{\vdash} \mathcal{G} \mid \Gamma \Rightarrow \Delta, A}{\overset{i}{\vdash} \mathcal{G} \mid \Gamma, \neg A \Rightarrow \Delta}\; \neg\Rightarrow \qquad \frac{\overset{i}{\vdash} \mathcal{G} \mid A, \Gamma \Rightarrow \Delta}{\overset{i}{\vdash} \mathcal{G} \mid \Gamma \Rightarrow \Delta, \neg A}\; \Rightarrow\neg$$

$$\frac{\overset{i}{\vdash} \mathcal{G} \mid \Gamma, A, B \Rightarrow \Delta}{\overset{i}{\vdash} \mathcal{G} \mid \Gamma, A \wedge B \Rightarrow \Delta}\; \wedge\Rightarrow \qquad \frac{\overset{i}{\vdash} \mathcal{G} \mid \Gamma \Rightarrow \Delta, A \qquad \overset{j}{\vdash} \mathcal{G} \mid \Gamma \Rightarrow \Delta, B}{\overset{i \cdot j}{\vdash} \mathcal{G} \mid \Gamma \Rightarrow \Delta, A \wedge B}\; \Rightarrow\wedge$$

$$\frac{\overset{i}{\vdash} \mathcal{G} \mid \Gamma, A \Rightarrow \Delta \qquad \overset{j}{\vdash} \mathcal{G} \mid \Gamma, B \Rightarrow \Delta}{\overset{i \cdot j}{\vdash} \mathcal{G} \mid \Gamma, A \vee B \Rightarrow \Delta}\; \vee\Rightarrow \qquad \frac{\overset{i}{\vdash} \mathcal{G} \mid \Gamma \Rightarrow \Delta, A, B}{\overset{i}{\vdash} \mathcal{G} \mid \Gamma \Rightarrow \Delta, A \vee B}\; \Rightarrow\vee$$

$$\frac{\overset{i}{\vdash} \mathcal{G} \mid \Gamma \Rightarrow \Delta, A \qquad \overset{j}{\vdash} \mathcal{G} \mid \Gamma, B \Rightarrow \Delta}{\overset{i \cdot j}{\vdash} \mathcal{G} \mid \Gamma, A \to B \Rightarrow \Delta}\; \to\Rightarrow \qquad \frac{\overset{i}{\vdash} \mathcal{G} \mid \Gamma, A \Rightarrow \Delta, B}{\overset{i}{\vdash} \mathcal{G} \mid \Gamma \Rightarrow \Delta, A \to B}\; \Rightarrow\to$$

MODAL OPERATOR RULE

$$\frac{\overset{i}{\vdash} \mathcal{G} \mid \Gamma \Rightarrow A \mid \Box\Gamma, \Gamma' \Rightarrow \Box\Delta, \Delta'}{\overset{i}{\vdash} \mathcal{G} \mid \Box\Gamma, \Gamma' \Rightarrow \Box A, \Box\Delta, \Delta'}\; \Box, \quad \text{where } \Gamma' \uplus \Delta' \subseteq AT$$

Figure 1. The $\overline{\overline{\mathsf{HK}}}$ sequent calculus (read $\overset{1}{\vdash}$ as $\vdash$ and $\overset{0}{\vdash}$ as $\dashv$).

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{\dashv\, \Rightarrow p \mid\, \Rightarrow p \mid \Box(\Box p \to p) \Rightarrow}\ \overline{ax}}{\dashv\, \Rightarrow \Box p, p \mid \Box(\Box p \to p) \Rightarrow}\ \Box \qquad \overline{\vdash p \Rightarrow p \mid \Box(\Box p \to p) \Rightarrow}\ ax.}{\dashv \Box p \to p \Rightarrow p \mid \Box(\Box p \to p) \Rightarrow}\ \to\Rightarrow}{\dashv \Box(\Box p \to p) \Rightarrow \Box p}\ \Box}{\dashv\, \Rightarrow \Box(\Box p \to p) \to \Box p}\ \Rightarrow\to$$

Figure 2. An example of $\overline{\overline{\mathsf{HK}}}$ proof

$$\dfrac{\dfrac{\overline{\vdash t \mid p \Rightarrow p}\ ax}{\vdash p \Rightarrow p, \Box t}\ \Box \qquad \dfrac{\overline{\vdash t \Rightarrow t \mid \Box t \Rightarrow p}\ ax}{\vdash p, \Box t \Rightarrow \Box t}\ \Box}{\vdash p, p \to \Box t \Rightarrow \Box t}\ \to\Rightarrow \qquad \dfrac{\dfrac{\overline{\vdash\, \Rightarrow t \mid p \Rightarrow p}\ ax \qquad \overline{\dashv\, \Rightarrow t \mid p, \Box t \Rightarrow}\ \overline{ax}}{\dashv\, \Rightarrow t \mid p, p \to \Box t \Rightarrow}\ \to\Rightarrow}{\dashv p, p \to \Box t \Rightarrow \Box t}\ \Box$$

The other modal systems are obtained by adjusting the system $\overline{\overline{\mathsf{HK}}}$ as indicated below.

- $\overline{\overline{\mathsf{HD}}}$ is obtained by adding to $\overline{\overline{\mathsf{HK}}}$ the rule:

  $$\dfrac{\stackrel{i}{\vdash} \mathcal{G} \mid \Pi \Rightarrow \Sigma \mid \Gamma \Rightarrow}{\stackrel{i}{\vdash} \mathcal{G} \mid \Box\Gamma, \Pi \Rightarrow \Sigma}\ \mathsf{d} \quad \text{where } \Pi, \Sigma \subset \mathsf{AT}$$

  and by revising the $\overline{ax}$-rule as follows:

  $$\overline{\dashv \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n}\ \text{where } \Gamma_i, \Delta_i \subset AT$$

- $\overline{\overline{\mathsf{HM}}}$ is obtained by substituting the $\Box$-rule in $\overline{\overline{\mathsf{HK}}}$ with the following inference pattern:

  $$\dfrac{\stackrel{i}{\vdash} \mathcal{G} \mid A_1 \Rightarrow B \mid \ldots \mid A_n \Rightarrow B \mid \Box A_1, \ldots, \Box A_n, \Pi \Rightarrow \Box\Delta, \Sigma}{\stackrel{i}{\vdash} \mathcal{G} \mid \Box A_1, \ldots, \Box A_n, \Pi \Rightarrow \Box\Delta, \Box B, \Sigma}\ \mathsf{m}$$

  where Π, Σ are multisets of atomic formulas, $i \in \{1, ..., m\}$, and $j \in \{1, ..., n\}$. We also need to replace the $\overline{ax}$-rule with the following version:

  $$\overline{\dashv \Box\Pi_1, \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n, \Box\Sigma_n}\ \text{where } \Gamma_i, \Delta_i \subset AT$$

- The system $\overline{\overline{\overline{\mathsf{HE}}}}$ is obtained from $\overline{\overline{\overline{\mathsf{HK}}}}$ by replacing the $\Box$-rule with the following inference schema:

$$\frac{\stackrel{i}{\vdash} \mathcal{G} \,|\, [\Rightarrow A_i \leftrightarrow B_j] \,|\, \Gamma \Rightarrow \Delta}{\stackrel{i}{\vdash} \mathcal{G} \,|\, \Box A_1, ..., \Box A_m, \Gamma \Rightarrow \Delta, \Box B_1, ..., \Box B_n} \, \mathsf{e}$$

where Γ, Δ are multisets of atomic formulas and $i \in \{1, ..., m\}$ and $j \in \{1, ..., n\}$. We also need to replace the $\overline{ax}$-rule with the following version:

$$\frac{}{\dashv \Box \Pi_1, \Gamma_1 \Rightarrow \Delta_1 \,|\, \ldots \,|\, \Gamma_n \Rightarrow \Delta_n, \Box \Sigma_n} \text{ where } \Gamma_i, \Delta_i \subset AT$$

3. Structural analysis

In this section we spell out the details of a purely syntactical cut-elimination procedure for these systems. In a previous work [14], cut-elimination was established in the form of closure under cut due to soundness and completeness of the system. We shall now give a purely syntactic proof thereof.

We recall the standard proof-theoretic definitions and measures. In particular, the *degree* of a formula is defined as the number of occurrences of connectives in it.

We also recall that a rule is *height-preserving admissible* when (i) the derivability of the premises entails the derivability of the conclusion and (ii) the height of the conclusion's derivation does not exceed that of the derivations of the premises. Additionally, we need the following notation: given a calculus $\overline{\overline{\overline{\mathbf{HX}}}}$, we denote by **HX** the calculus obtained by removing its complementary axiom."

Lemma 3.1. *The rules of the calculus* **HK** *are height-preserving invertible.*

Proof: The proof is by induction on the height of the derivation of the conclusion of the rule. We consider only the case of the modal operator, the other ones are routine. Given a hypersequent shaped as

$$\vdash \mathcal{G} \,|\, \Box\Gamma, \Gamma' \Rightarrow \Box A, \Box\Delta, \Delta',$$

by inspection of the rules of the system, it can only come as a conclusion of the $\Box$-rule. On the other hand, if $\Box A$ is the principal formula, then the

$$\frac{\vdash \mathcal{G} \,|\, \Gamma \Rightarrow \Delta}{\vdash \mathcal{G} \,|\, A, \Gamma \Rightarrow \Delta}\, LW \qquad \frac{\vdash \mathcal{G}}{\vdash \mathcal{G} \,|\, \mathcal{H}}\, EW \qquad \frac{\vdash \mathcal{G} \,|\, A, A, \Gamma \Rightarrow \Delta}{\vdash \mathcal{G} \,|\, A, \Gamma \Rightarrow \Delta}\, LC$$

$$\frac{\vdash \mathcal{G} \,|\, \Gamma \Rightarrow \Delta}{\vdash \mathcal{G} \,|\, \Gamma \Rightarrow \Delta, A}\, RW \qquad \frac{\vdash \mathcal{G} \,|\, \Gamma \Rightarrow \Delta \,|\, \Gamma \Rightarrow \Delta}{\vdash \mathcal{G} \,|\, \Gamma \Rightarrow \Delta}\, EC \qquad \frac{\vdash \mathcal{G} \,|\, \Gamma \Rightarrow \Delta, A, A}{\vdash \mathcal{G} \,|\, \Gamma \Rightarrow \Delta, A}\, RC$$

$$\frac{\vdash \mathcal{G} \,|\, \Gamma \Rightarrow \Delta, A \qquad \vdash \mathcal{H} \,|\, A, \Pi \Rightarrow \Sigma}{\vdash \mathcal{G} \,|\, \mathcal{H} \,|\, \Gamma, \Pi \Rightarrow \Delta, \Sigma}\, Cut$$

Figure 3. Admissible structural rules

premise is the desired conclusion. If the principal formula is a formula in $\Box\Delta$, say $\Box B$, then we have:

$$\frac{\vdash \mathcal{G} \,|\, \Gamma \Rightarrow B \,|\, \Box\Gamma, \Gamma' \Rightarrow \Box A, \Box\Delta'', \Delta'}{\vdash \mathcal{G} \,|\, \Box\Gamma, \Gamma' \Rightarrow \Box A, \Box\Delta'', \Box B, \Delta'}\, \Box$$

Since the height gets decreased, we can apply the induction hypothesis which yields a derivation ending in $\vdash \mathcal{G} \,|\, \Gamma \Rightarrow B \,|\, \Box\Gamma, \Gamma' \Rightarrow \Box A, \Box\Delta'', \Delta'$. The desired conclusion then follows by a final application of the $\Box$-rule. $\Box$

Lemma 3.2. *The weakening rules* (EW), (LW) *and* (RW) *are both admissible.*

Proof: Admissibility of the rule of external weakening (EW) follows from a straightforward induction on the height of derivations. On the contrary, to establish the admissibility of the weakening rules (LW) and (RW) we need to argue by double induction, with the main induction hypothesis on the degree of the formula to be added and the secondary induction hypothesis on the height of the derivation under consideration. In particular:

If $n = 0$, then if the hypersequent $\vdash \mathcal{G} \,|\, \Box\Gamma, \Gamma' \Rightarrow \Delta$ is derivable, so are both $\vdash \mathcal{G} \,|\, A, \Box\Gamma, \Gamma' \Rightarrow \Delta$ and $\vdash \mathcal{G} \,|\, \Box\Gamma, \Gamma' \Rightarrow \Delta, A$.

If $n > 0$ and the last rule is not a $\Box$-application, then we apply the secondary induction hypothesis to the premise(s) and then the rule again. Otherwise, if the last rule applied is a $\Box$-application, we distiguish three subcases.

- If A is an atomic formula, then we apply the secondary induction hypothesis and then the rule again.

- If A is a modal formula $\Box B$ we have:

$$\dfrac{\vdash \mathcal{G} \,|\, \Gamma \Rightarrow C \,|\, \Box\Gamma, \Gamma' \Rightarrow \Box\Delta, \Delta'}{\vdash \mathcal{G} \,|\, \Box\Gamma, \Gamma' \Rightarrow \Box C, \Box\Delta, \Delta'}\ \Box$$

  If we want to add $\Box B$ to the succedent we can simply apply the secondary induction hypothesis and then the rule again. Otherwise, we get the following configuration:

$$\dfrac{\dfrac{\dfrac{\vdash \mathcal{G} \,|\, \Gamma \Rightarrow C \,|\, \Box\Gamma, \Gamma' \Rightarrow \Box\Delta, \Delta'}{\vdash \mathcal{G} \,|\, \Gamma \Rightarrow C \,|\, \Box\Gamma, \Box B, \Gamma' \Rightarrow \Box\Delta, \Delta'}\ LW}{\vdash \mathcal{G} \,|\, \Gamma, B \Rightarrow C \,|\, \Box\Gamma, \Box B, \Gamma' \Rightarrow \Box\Delta, \Delta'}\ LW}{\vdash \mathcal{G} \,|\, \Box\Gamma, \Gamma', \Box B, \Rightarrow \Box C, \Box\Delta, \Delta'}\ \Box$$

  The first application of LW is removed by secondary induction hypothesis, while the second by the primary induction hypothesis.

- It remains to consider the case in which A is a formula whose principal connective is one among $\wedge$, $\vee$, and $\rightarrow$. In these case, we decompose the formula A by applying invertibility of the rules for the classical connectives, then we add the formulas as described in the preceding subcases. $\Box$

Lemma 3.3. *The rules of contraction* (LC) *and* (RC) *and external contraction* (EC) *are all height-preserving admissible.*

Proof: By simultaneous induction on the height of derivations. External contraction follows by a straightforward induction on the height of the derivation under analysis by applying height-preserving invertibility of the logical rules.

Internal contraction is slightly more delicate to handle. The critical situation is the one in which we have a hypersequent $\vdash \mathcal{G} \,|\, \Box\Gamma, \Gamma' \Rightarrow \Box A, \Box A, \Box\Delta, \Delta'$ and the formula $\Box A$ is principal in the last rule applied. In this case, we consider the premise

$$\vdash \mathcal{G} \,|\, \Gamma \Rightarrow A \,|\, \Box\Gamma, \Gamma' \Rightarrow \Box A, \Box\Delta, \Delta'$$

and we proceed in the following way

$$\dfrac{\dfrac{\dfrac{\vdash \mathcal{G} \,|\, \Gamma \Rightarrow A \,|\, \Box\Gamma, \Gamma' \Rightarrow \Box A, \Box\Delta, \Delta'}{\vdash \mathcal{G} \,|\, \Gamma \Rightarrow A \,|\, \Gamma \Rightarrow A \,|\, \Box\Gamma, \Gamma' \Rightarrow \Box\Delta, \Delta'}\, \mathit{Inv}\text{-}\Box}{\vdash \mathcal{G} \,|\, \Gamma \Rightarrow A \,|\, \Box\Gamma, \Gamma' \Rightarrow \Box\Delta, \Delta'}\, \text{EC}}{\vdash \mathcal{G} \,|\, \Box\Gamma, \Gamma' \Rightarrow \Box A, \Box\Delta, \Delta'}\, \Box$$

□

Theorem 3.4. *The cut-rule is admissible.*

Proof: The proof is by double induction with main induction hypothesis on the degree of the cut-formula and the secondary induction hypothesis on the sum of the height of the derivation of the premises of the cut.

We distinguish the following cases. If the right premise of the cut is an initial sequent, then, when the cut formula is not active, we remove it. Otherwise, the conclusion follows by weakening.

If the right premise of the cut is the conclusion of a logical rule different from $\Box$, we distinguish two subcases according to whether the cut-formula is principal or not. In the former case, we apply the invertibility of the corresponding rule and we replace the cut-application under consideration with cuts on formulas of smaller degree. In the latter case we permute the cut upwards.

If the last inference step is a $\Box$-application, then the cut-formula is either atomic or a modal formula. In both cases, we argue by induction on the left premise of the cut. The relevant case is the one in which the last rule applied is $\Box$. We have:

$$\dfrac{\dfrac{\vdash \mathcal{G} \,|\, \Gamma \Rightarrow A \,|\, \Box\Gamma, \Gamma' \Rightarrow \Box\Delta, \Delta'}{\vdash \mathcal{G} \,|\, \Box\Gamma, \Gamma' \Rightarrow \Box\Delta, \Box A, \Delta'}\, \Box \qquad \dfrac{\vdash \mathcal{H} \,|\, A, \Pi \Rightarrow B \,|\, \Box A, \Box\Pi, \Pi' \Rightarrow \Box\Sigma, \Sigma'}{\vdash \mathcal{H} \,|\, \Box A, \Box\Pi, \Pi' \Rightarrow \Box\Sigma, \Box B, \Sigma'}\, \Box}{\vdash \mathcal{G} \,|\, \mathcal{H} \,|\, \Box\Gamma, \Box\Pi, \Gamma', \Pi' \Rightarrow \Box\Delta, \Box\Sigma, \Box B, \Delta', \Sigma'}\, \mathit{Cut}$$

The cut is removed as follows (we avoid writing the contexts for better readability). First, we apply a cross-cut:

$$\dfrac{\vdash \Box\Gamma, \Gamma' \Rightarrow \Box\Delta, \Box A, \Delta' \qquad \vdash A, \Pi \Rightarrow B \,|\, \Box A, \Box\Pi, \Pi' \Rightarrow \Box\Sigma, \Sigma'}{\vdash A, \Pi \Rightarrow B \,|\, \Box\Gamma, \Box\Pi, \Gamma', \Pi' \Rightarrow \Box\Delta, \Box\Sigma, \Delta', \Sigma'}\, \mathit{Cut}$$

The cut is removed by applying the secondary induction hypothesis. The reduction is then completed as follows:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\vdash \Gamma \Rightarrow A \,|\, \Box\Gamma, \Gamma' \Rightarrow \Box\Delta, \Delta' \qquad \vdash A, \Pi \Rightarrow B \,|\, \Box\Gamma, \Box\Pi, \Gamma', \Pi' \Rightarrow \Box\Delta, \Box\Sigma, \Delta', \Sigma'}{\vdash \Gamma, \Pi \Rightarrow B \,|\, \Box\Gamma, \Gamma' \Rightarrow \Box\Delta, \Delta' \,|\, \Box\Gamma, \Box\Pi, \Gamma', \Pi' \Rightarrow \Box\Delta, \Box\Sigma, \Delta', \Sigma'}\,\text{Cut}}{\vdash \Gamma, \Pi \Rightarrow B \,|\, (\Box\Gamma, \Box\Pi, \Gamma', \Pi' \Rightarrow \Box\Delta, \Box\Sigma, \Delta', \Sigma')^2}\,LW, RW}{\vdash \Gamma, \Pi \Rightarrow B \,|\, \Box\Gamma, \Box\Pi, \Gamma', \Pi' \Rightarrow \Box\Delta, \Box\Sigma, \Delta', \Sigma'}\,\text{EC}}{\vdash \Box\Gamma, \Box\Pi, \Gamma', \Pi' \Rightarrow \Box\Delta, \Box B, \Box\Sigma, \Delta', \Sigma'}\,\Box
$$

where the cut-rule is removed by primary induction hypothesis on the degree of the cut-formula. □

We consider now the system **HD**. In this case the analysis proceeds analogously. Of course, the admissibility of the structural rules needs to be established once again.

LEMMA 3.5. *Every rule is height-preserving invertible in* **HD**.

PROOF: The only new case to be detailed is the one involving the rule d. In this case the proof is immediate, as the only applicable rule is d which acts on all the formulas in the antecedents. □

LEMMA 3.6. *The weakening rules* (EW)*,* (LW) *and* (RW) *are admissible.*

PROOF: External weakening is established by a straightforward induction on the height of the derivation. Proving the admissibility of W requires a double induction, with main induction hypothesis on the degree of the formula and secondary induction hypothesis on the height of derivations.

The only new case to detail is the one involving rule d. As usual, we need to proceed by cases. If the formula to be added is an atomic formula, then we simply apply the secondary induction hypothesis and then the rule again. If it is a boxed formula to be added in the antecedent, then we apply the primary induction hypothesis on the degree of the formula and then the rule again.

In the remaining cases we first decompose the formulaand we then obtain some hypersequents which contain only boxed formulas in the antecedents of the components and atomic formulas. Hence we apply the primary induction hypothesis and then we apply the rules in the reverse order. □

LEMMA 3.7. *The rules of contraction are height-preserving admissible.*

PROOF: The proof is by induction on the height of the derivation. The only new case to discuss is the one involving the rule d. We have:

$$\dfrac{\vdash \mathcal{G} \,|\, A, A, \Gamma \Rightarrow \ \,|\, \Pi \Rightarrow \Sigma}{\vdash \mathcal{G} \,|\, \Box A, \Box A, \Box\Gamma, \Pi \Rightarrow \Sigma}\ \mathsf{d}$$

We proceed as follows:

$$\dfrac{\dfrac{\vdash \mathcal{G} \,|\, A, A, \Gamma \Rightarrow \ \,|\, \Pi \Rightarrow \Sigma}{\vdash \mathcal{G} \,|\, A, \Gamma \Rightarrow \ \,|\, \Pi \Rightarrow \Sigma}\ LC}{\vdash \mathcal{G} \,|\, \Box A, \Box\Gamma, \Pi \Rightarrow \Sigma}\ \mathsf{d}$$

The application of LC is removed by the induction hypothesis on the height of the derivation. □

Theorem 3.8. *The cut rule is admissible in* **HD**.

Proof: By double induction. We discuss only the new interesting case.

$$\dfrac{\dfrac{\vdash \mathcal{G} \,|\, \Gamma \Rightarrow A \,|\, \Box\Gamma, \Gamma' \Rightarrow \Box\Delta, \Delta'}{\vdash \mathcal{G} \,|\, \Box\Gamma, \Gamma' \Rightarrow \Box\Delta, \Box A, \Delta'}\ \Box \qquad \dfrac{\vdash \mathcal{H} \,|\, A, \Pi \Rightarrow \ \,|\, \Theta \Rightarrow \Sigma}{\vdash \mathcal{H} \,|\, \Box A, \Box\Pi, \Theta \Rightarrow \Sigma}\ \mathsf{d}}{\vdash \mathcal{G} \,|\, \mathcal{H} \,|\, \Box\Gamma, \Box\Pi, \Gamma', \Theta \Rightarrow \Box\Delta, \Sigma, \Delta'}\ Cut$$

We proceed as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash \mathcal{G} \,|\, \Gamma \Rightarrow A \,|\, \Box\Gamma, \Gamma' \Rightarrow \Box\Delta, \Delta' \qquad \vdash \mathcal{H} \,|\, A, \Pi \Rightarrow \ \,|\, \Theta \Rightarrow \Sigma}{\vdash \mathcal{G} \,|\, \mathcal{H} \,|\, \Gamma, \Pi \Rightarrow \ \,|\, \Theta \Rightarrow \Sigma \,|\, \Box\Gamma, \Gamma' \Rightarrow \Box\Delta, \Delta'}\ Cut}{\vdash \mathcal{G} \,|\, \mathcal{H} \,|\, \Box\Gamma, \Box\Pi, \Theta \Rightarrow \Sigma \,|\, \Box\Gamma, \Gamma' \Rightarrow \Box\Delta, \Delta'}\ \mathsf{d}}{\vdash \mathcal{G} \,|\, \mathcal{H} \,|\, (\Box\Gamma, \Box\Pi, \Gamma', \Theta \Rightarrow \Box\Delta, \Sigma, \Delta')^2}\ LW, RW}{\vdash \mathcal{G} \,|\, \mathcal{H} \,|\, \Box\Gamma, \Box\Pi, \Gamma', \Theta \Rightarrow \Box\Delta, \Sigma, \Delta'}\ EC$$

The cut is replaced by a cut on a formula of smaller degree and the conclusion is obtained applying the rule d followed by weakening and contraction. □

We now consider the case of **HM**. Since by now the reader should be acquainted with the strategies employed to establish the structural properties of this kind of calculi we shall not get into the details.

Lemma 3.9. *Every rule is height-preserving invertible.*

Proof: We deal with the rule m. If $\vdash \mathcal{G} \,|\, \Box A_1, \dots, \Box A_n, \Pi \Rightarrow \Box\Delta, \Box B, \Box C, \Sigma$ is an initial sequent, so is $\vdash \mathcal{G} \,|\, A_1 \Rightarrow C \,|\, \dots \,|\, A_n \Rightarrow C \,|\, \Box A_1, \dots, \Box A_n, \Pi \Rightarrow$

$\Box\Delta, \Box B, \Sigma$. If it is the conclusion of a rule, we apply the induction hypothesis to each of the premises and then the rule again. For example, we have:

$$\dfrac{\vdash \mathcal{G} \,|\, A_1 \Rightarrow B \,|\, \ldots \,|\, A_n \Rightarrow B \,|\, \Box A_1, \ldots, \Box A_n, \Pi \Rightarrow \Box\Delta, \Box C, \Sigma}{\vdash \mathcal{G} \,|\, \Box A_1, \ldots, \Box A_n, \Pi \Rightarrow \Box\Delta, \Box B, \Box C, \Sigma}\ \mathsf{m}$$

We proceed as follows:

$$\dfrac{\dfrac{\vdash \mathcal{G} \,|\, A_1 \Rightarrow B \,|\, \ldots \,|\, A_n \Rightarrow B \,|\, \Box A_1, \ldots, \Box A_n, \Pi \Rightarrow \Box\Delta, \Box C, \Sigma}{\vdash \mathcal{G} \,|\, A_1 \Rightarrow B \,|\, \ldots \,|\, A_n \Rightarrow B \,|\, A_1 \Rightarrow C \,|\, \ldots \,|\, A_n \Rightarrow C \,|\, \Box A_1, \ldots, \Box A_n, \Pi \Rightarrow \Box\Delta, \Sigma}\ \mathrm{IH}}{\vdash \mathcal{G} \,|\, A_1 \Rightarrow C \,|\, \ldots \,|\, A_n \Rightarrow C \,|\, \Box A_1, \ldots, \Box A_n, \Pi \Rightarrow \Box\Delta, \Box B, \Sigma}\ \mathsf{m}$$

$\Box$

Lemma 3.10. *The rules* $(EW), (LW)$ *and* (RW) *are admissible.*

Proof: EW. Straightforward by induction on the height of the derivation. With respect to W we argue by double induction as above with minor changes. $\Box$

Lemma 3.11. *The rules* $(EC), (LC)$ *and* (RC) *are height-preserving admissible.*

Proof: By induction on the height of the derivation. We deal with the only relevant cases.

$$\dfrac{\vdash \mathcal{G} \,|\, A_1 \Rightarrow B \,|\, \ldots \,|\, A_n \Rightarrow B \,|\, \Box A_1, \ldots, \Box A_n, \Pi \Rightarrow \Box\Delta, \Box B, \Sigma}{\vdash \mathcal{G} \,|\, \Box A_1, \ldots, \Box A_n, \Pi \Rightarrow \Box\Delta, \Box B, \Box B, \Sigma}\ \mathsf{m}$$

We proceed as follows:

$$\dfrac{\dfrac{\dfrac{\vdash \mathcal{G} \,|\, A_1 \Rightarrow B \,|\, \ldots \,|\, A_n \Rightarrow B \,|\, \Box A_1, \ldots, \Box A_n, \Pi \Rightarrow \Box\Delta, \Box B, \Sigma}{\vdash \mathcal{G} \,|\, (A_1 \Rightarrow B)^2 \,|\, \ldots \,|\, (A_n \Rightarrow B)^2 \,|\, \Box A_1, \ldots, \Box A_n, \Pi \Rightarrow \Box\Delta, \Sigma}\ Inv\text{-}\mathsf{m}}{\vdash \mathcal{G} \,|\, A_1 \Rightarrow B \,|\, \ldots \,|\, A_n \Rightarrow B \,|\, \Box A_1, \ldots, \Box A_n, \Pi \Rightarrow \Box\Delta, \Sigma}\ EC}{\vdash \mathcal{G} \,|\, \Box A_1, \ldots, \Box A_n, \Pi \Rightarrow \Box\Delta, \Box B, \Sigma}\ \mathsf{m}$$

If the formula to contract is in the antecedent, we proceed analogously, possibly exploiting external contraction and the induction hypothesis on the height of the derivation. □

The last step is the cut-elimination theorem.

THEOREM 3.12. *The cut rule is admissible in* **HM**.

PROOF: By double induction on the degree of the cut formula and the sum of the height of the derivations of the premises of the cut. We discuss the case in which the cut formula is principal in both the premises in an application of the rule m.

$$\dfrac{\dfrac{\mathcal{G} \,|\, A_1 \Rightarrow C_1 \,|\, \dots \,|\, A_n \Rightarrow C_1 \,|\, \Box A_1, \dots, \Box A_n, \Gamma \Rightarrow \Box\Delta, \Delta'}{\mathcal{G} \,|\, \Box A_1, \dots, \Box A_n, \Gamma \Rightarrow \Box\Delta, \Box C_1, \Delta'}\,\mathsf{m} \qquad \dfrac{\mathcal{H} \,|\, C_1 \Rightarrow D \,|\, \dots \,|\, C_n \Rightarrow D \,|\, \Box C_1, \dots, \Box C_n, \Pi \Rightarrow \Box\Sigma, \Sigma'}{\mathcal{H} \,|\, \Box C_1, \dots, \Box C_n, \Pi \Rightarrow \Box\Sigma, \Box D, \Sigma'}\,\mathsf{m}}{\mathcal{G} \,|\, \mathcal{H} \,|\, \Box A_1, \dots, \Box A_n, \Gamma, \Box C_2, \dots, \Box C_n, \Pi \Rightarrow \Box\Sigma, \Box D, \Sigma', \Box\Delta, \Delta'}\,\text{Cut}$$

We construct the following derivation (we omit the contexts for better readability):

$$\dfrac{\vdash \Box A_1, \dots, \Box A_m, \Gamma \Rightarrow \Box\Delta, \Box C_1, \Delta' \qquad \vdash C_1 \Rightarrow D \,|\, \dots \,|\, C_n \Rightarrow D \,|\, \Box C_1, \dots, \Box C_n, \Pi \Rightarrow \Box\Sigma, \Sigma'}{\vdash C_1 \Rightarrow D \,|\, \dots \,|\, C_n \Rightarrow D \,|\, \Box A_1, \dots, \Box A_m, \Gamma, \Box C_2, \dots, \Box C_n, \Pi \Rightarrow \Box\Sigma, \Sigma', \Box\Delta, \Delta'}\,\text{Cut}$$

The cut is removed by secondary induction hypothesis. Next, we cut on C_1. We write $\mathcal{S}$ as an abbreviation for $\vdash \Box A_1, \dots, \Box A_m, \Gamma, \Box C_2, \dots, \Box C_n, \Pi \Rightarrow \Box\Sigma, \Sigma', \Box\Delta, \Delta'$. We have:

$$\dfrac{\dfrac{\vdash A_1 \Rightarrow C_1 \,|\, \dots \,|\, A_m \Rightarrow C_1 \,|\, \Box A_1, \dots, \Box A_m, \Gamma \Rightarrow \Box\Delta, \Delta' \qquad \vdash C_1 \Rightarrow D \,|\, \dots \,|\, C_n \Rightarrow D \,|\, \mathcal{S}}{\vdash A_1 \Rightarrow D \,|\, \dots \,|\, A_m \Rightarrow C_1 \,|\, \dots \,|\, C_n \Rightarrow D \,|\, \Box A_1, \dots, \Box A_m, \Gamma \Rightarrow \Box\Delta, \Delta' \,|\, \mathcal{S}}\,Cut}{\vdash A_1 \Rightarrow D \,|\, \dots \,|\, A_m \Rightarrow C_1 \,|\, \dots \,|\, C_n \Rightarrow D \,|\, \mathcal{S}}\,LW, RW, EC$$

We now apply again a cut on C_1 between $\vdash A_1 \Rightarrow D \,|\, \dots \,|\, A_m \Rightarrow C_1 \,|\, \dots \,|\, C_n \Rightarrow D \,|\, \mathcal{S}$ and $\vdash C_1 \Rightarrow D \,|\, \dots \,|\, C_n \Rightarrow D \,|\, \mathcal{S}$ which yields (modulo contraction)

$$\vdash A_1 \Rightarrow D \,|\, A_2 \Rightarrow D \,|\, \dots \,|\, A_m \Rightarrow C_1 \,|\, \dots \,|\, C_n \Rightarrow D \,|\, \mathcal{S}$$

By repeating this procedure (formalizable by induction on m), we get:

$$\vdash A_1 \Rightarrow D \,|\, A_2 \Rightarrow D \,|\, \dots \,|\, A_m \Rightarrow D \,|\, \dots \,|\, C_n \Rightarrow D \,|\, \mathcal{S}$$

An application of the rule m gives the desired conclusion. □

The last system that we analyze is **HE**. We state the preliminary structural properties omitting the proofs which can be obtained along the same lines as the previously discussed systems.

PROPOSITION 3.13. The rule of weakening is admissible. Every rule of the calculus is height-preserving invertible. The rule of contraction is height-preserving admissible.

To conclude the section we discuss cut-elimination for the case of **HE**. Instead of lingering on abstract technicalities, we give a concrete example of reduction and we leave to the reader the generalization of the argument.

$$\dfrac{\dfrac{\vdash \mathcal{G} \mid \Rightarrow A \leftrightarrow C \mid \Rightarrow B \leftrightarrow C \mid \Gamma \Rightarrow \Delta}{\vdash \mathcal{G} \mid \Box A, \Box B, \Gamma \Rightarrow \Delta, \Box C}\,\mathsf{e} \qquad \dfrac{\vdash \mathcal{G}' \mid \Rightarrow C \leftrightarrow D \mid \Rightarrow C \leftrightarrow E \mid \Pi \Rightarrow \Sigma}{\vdash \mathcal{G}' \mid \Box C, \Pi \Rightarrow \Sigma, \Box D, \Box E}\,\mathsf{e}}{\vdash \mathcal{G} \mid \mathcal{G}' \mid \Box A, \Box B, \Gamma, \Pi \Rightarrow \Delta, \Sigma, \Box D, \Box E}\,Cut$$

We first observe that the rule:

$$\dfrac{\vdash \mathcal{G} \mid \Rightarrow A \leftrightarrow B \qquad \vdash \mathcal{G}' \mid \Rightarrow B \leftrightarrow C}{\vdash \mathcal{G} \mid \mathcal{G}' \mid \Rightarrow A \leftrightarrow C}\,Eq$$

is admissible via cuts on formulas of lower size. Hence we propose the following reduction containing applications of Eq (we omit the contexts and the turnstiles and the applications of the rule EC for reasons of space):

$$\dfrac{\dfrac{\Rightarrow A \leftrightarrow C \mid \Rightarrow B \leftrightarrow C \qquad \dfrac{\dfrac{\Rightarrow A \leftrightarrow C \mid \Rightarrow B \leftrightarrow C \qquad \Rightarrow C \leftrightarrow D \mid \Rightarrow C \leftrightarrow E}{\Rightarrow A \leftrightarrow D \mid \Rightarrow B \leftrightarrow C \mid \Rightarrow C \leftrightarrow E} \qquad \Rightarrow C \leftrightarrow D \mid \Rightarrow C \leftrightarrow E}{\Rightarrow A \leftrightarrow D \mid \Rightarrow B \leftrightarrow D \mid \Rightarrow C \leftrightarrow E}}{\Rightarrow A \leftrightarrow D \mid \Rightarrow B \leftrightarrow D \mid \Rightarrow A \leftrightarrow E \mid \Rightarrow B \leftrightarrow C} \qquad \Rightarrow A \leftrightarrow D \mid \Rightarrow B \leftrightarrow D \mid \Rightarrow C \leftrightarrow E}{\Rightarrow A \leftrightarrow D \mid \Rightarrow B \leftrightarrow D \mid \Rightarrow A \leftrightarrow E \mid \Rightarrow B \leftrightarrow E}$$

All the cuts are removed by primary induction hypothesis on the degree of the cut formula.

THEOREM 3.14. *The cut rule is admissible in* **HE**.

As a matter of fact, proofs in the hypersequent calculi here proposed amount to the decomposition of the endsequent into non further analyzable top-hypersequents. The calculi enjoy invertibility of every rule with preservation of the height. In addition, as it will be shown in the next section, the decomposition is unique or, which is equivalent, the calculus enjoys the stability property.

4. Development of fractional semantics

4.1. Conservativity over the base logic

Conservativity stems from the soundness and the completeness of the calculus. Soundness is established with respect to structures which interpret modal logics.

Definition 4.1. An **E**-*neighborhood model* is a triple $\langle \mathcal{W}, \mathcal{I}, \mathcal{V} \rangle$, where $\mathcal{W}$ is a non-empty set, $\mathcal{I} : \mathcal{W} \to \mathcal{P}(\mathcal{P}(\mathcal{W}))$ and $\mathcal{V} : AT \to \mathcal{P}(\mathcal{W})$. Truth conditions for a formula A in a world x in a model are inductively defined as follows:

- $x \Vdash p$ if and only if $x \in \mathcal{V}(P)$.
- $x \Vdash B \wedge C$ if and only if $x \Vdash B$ and $x \Vdash C$.
- $x \Vdash B \vee C$ if and only if $x \Vdash B$ or $x \Vdash C$.
- $x \Vdash \neg B$ if and only if $x \nVdash B$.
- $x \Vdash \Box B$ if and only if $\{y \,|\, y \Vdash B\} \in \mathcal{I}(x)$.

An **M**-*neighborhood model* is an **E**-*neighborhood model* with the additional condition: if $a \in \mathcal{I}(x)$ and $a \subseteq b$ then $b \in \mathcal{I}(x)$. A **K**-*neighborhood model* is an **M**-*neighborhood model* in which, if $a \in \mathcal{I}(x)$ and $b \in \mathcal{I}(x)$ then we get both $a \cap b \in \mathcal{I}(x)$ and $\mathcal{I}(x) \neq \emptyset$, for every x. A **D**-*neighborhood model* is a **K**-*neighborhood model* satisfying the following additional condition: $a \in \mathcal{I}(x) \Rightarrow a^c \notin \mathcal{I}(x)$.

The definition of validity for a hypersequent in this setting is as follows: $\mathcal{G}$ is valid if one of its components it valid.

Proposition 4.2. If **HX** proves $\vdash \Rightarrow A$, then A is valid.

Proof: The proof is by induction on the height of the derivation in the corresponding hypersequent calculus. We discuss the case of **HE** as an example. Suppose the hypersequent $\vdash \mathcal{G} \,|\, [\Rightarrow A_i \leftrightarrow B_j] \,|\, \Gamma \Rightarrow \Delta$ is valid, hence one of the components is valid. If any component in $\mathcal{G}$ or $\Gamma \Rightarrow \Delta$ is valid, then so is the conclusion, trivially. If for some i, j $A_i \leftrightarrow B_j$ is valid, then this implies that $\Box A_i \leftrightarrow \Box B_j$ is valid and therefore the validity of the conclusion follows. □

As regards completeness, it suffices to establish that whenever we have a derivation of the Hilbert style calculus for a given modal logic, the corresponding sequent is derivable in our calculus too.

We recall here the modular presentation of the Hilbert style systems for the logics considered here.

- The system **E** is axiomatized by adding to a Hilbert-style calculus for classical propositional logic the rule:

$$\dfrac{\vdash A \leftrightarrow B}{\vdash \Box A \leftrightarrow \Box B}\ \mathrm{E}$$

- The system **M** is axiomatized by adding to **E** the rule:

$$\dfrac{\vdash A \to B}{\vdash \Box A \to \Box B}\ \mathrm{M}$$

- The system **K** is axiomatized by adding to a Hilbert-style calculus for classical propositional logic the axiom $\Box(A \to B) \to (\Box A \to \Box B)$ and the rule:

$$\dfrac{\vdash A}{\vdash \Box A}\ \mathrm{RN}$$

- The system **D** is axiomatized by adding to **K** the axiom $\Box A \to \Diamond A$.

THEOREM 4.3. *If* **X** *proves* $\vdash A$*, then* $\overline{\overline{\overline{\mathbf{HX}}}} \vdash\Rightarrow A$ *for* $\mathbf{X} \in \{\mathbf{K}, \mathbf{M}, \mathbf{D}\}$.

PROOF: The proof is by induction on the height of the derivation in the system **X**. We give an example of the derivation of the axiom **D** in **HD**:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash A \Rightarrow A}{\vdash A, \neg A \Rightarrow}\ \mathrm{L}\neg}{\vdash \Box A, \Box \neg A \Rightarrow}\ \mathsf{d}}{\vdash \Box A \Rightarrow \neg \Box \neg B}\ \mathrm{R}\!\to}{\vdash\Rightarrow \Box A \to \neg \Box \neg A}\ \mathrm{R}\!\to$$

With respect to the rules of the calculus, we show the admissibility of the rule M in the calculus **HM**:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash\Rightarrow A \to B}{\vdash A \Rightarrow B}\ \mathrm{Inv}\!\to}{\vdash A \Rightarrow B \,|\, \Box A \Rightarrow}\ \mathrm{EW}}{\vdash \Box A \Rightarrow \Box B}\ \mathsf{m}}{\vdash\Rightarrow \Box A \to \Box B}\ \mathrm{R}\!\to$$

of modus ponens:

$$\dfrac{\vdash\Rightarrow A \qquad \dfrac{\vdash\Rightarrow A \to B}{\vdash A \Rightarrow B}\,\text{Inv}\!\to}{\vdash\Rightarrow B}\,\text{Cut}$$

and of the E rule in **HE**:

$$\dfrac{\dfrac{\dfrac{\vdash\Rightarrow A \leftrightarrow B}{\vdash \Box B \Rightarrow \Box A}\,\mathsf{e}}{\Rightarrow \Box B \to \Box A}\,\text{R}\!\to \qquad \dfrac{\dfrac{\vdash\Rightarrow B \leftrightarrow A}{\vdash \Box A \Rightarrow \Box B}\,\mathsf{e}}{\vdash\Rightarrow \Box A \to \Box B}\,\text{R}\!\to}{\vdash\Rightarrow \Box A \leftrightarrow \Box B}\,\text{R}\wedge$$

□

As a corollary of the embedding we get the completeness of the resulting system. Soundness is obtained as usual through a straightforward induction on the height of the derivation of the system and thus we omit the details.

Corollary 4.4. The systems $\overline{\overline{\overline{\mathbf{HX}}}}$ are sound and complete with respect to the logics **X**.

Proof: If A is valid, then it is derivable in the corresponding axiomatic calculus and so in $\overline{\overline{\overline{\mathbf{HX}}}}$. □

4.2. Fractional valued non-normal modal logics

In order to develop a fractional interpretation of non-normal modal logics, we need to show that the assignment of values to formula does not depend on the specific shape of the derivations.

Theorem 4.5 (Stability). *If π and ρ are two $\overline{\overline{\overline{\mathbf{HX}}}}$-derivations ending with the same hypersequent, then $\mathsf{top}(\pi) = \mathsf{top}(\rho)$.*

Proof: The proof is standardly led by induction on the height n of the derivation of π. If $n = 0$, then the claim comes straightforwardly. Otherwise we distinguish cases according to the last rule applied. We consider the case in which the last inference is an application of a unary rule, that is:

$$\dfrac{\begin{matrix}\pi' \\ \vdots \\ \vdash^{i} \mathcal{G}'\end{matrix}}{\vdash^{i} \mathcal{G}}\ \mathsf{r}$$

We apply the invertibility of the rule r to get a derivation ρ' of $\mathcal{G}'$. Since the height of π' is strictly lower than that of π, we can apply the induction hypothesis to get $\mathsf{top}(\pi') = \mathsf{top}(\rho')$, which immediately yields the desired conclusion. □

Due to the stability property, we can now consider the multiset of top-hypersequents associated with a given formula as a derivation-invariant notion. That is, the multiset decomposition remains stable through different derivations of the same hypersequent.

Definition 4.6. Given a formula A, $\mathsf{top}_{\mathbf{X}}(A)$ is the multiset of the top-hyperclauses in any of the $\overline{\overline{\mathbf{HX}}}$-derivation ending in $(\vdash \text{ or } \dashv) \Rightarrow A$. The multiset $\mathsf{top}_{\mathbf{X}}(A)$ is partitioned into the two multisets $\mathsf{top}^{1}_{\mathbf{X}}(A)$ and $\mathsf{top}^{0}_{\mathbf{X}}(A)$ collecting all the hyperclauses signed by '$\vdash$' and the hyperclauses signed by '$\dashv$', respectively.

Definition 4.7 (Fractional evaluation function). Let $\mathbb{Q}^* = [0,1] \cap \mathbb{Q}$, i.e., $\mathbb{Q}^*$ is the set of the rational numbers in the closed interval $[0,1]$. For each system $\mathbf{X}$, the evaluation function $[\![\cdot]\!]_{\mathbf{X}} : \mathscr{F} \mapsto \mathbb{Q}^*$ is defined as follows: for any logical formula A,

$$[\![A]\!]_{\mathbf{X}} = \frac{\#\mathsf{top}^{1}_{\mathbf{X}}(A)}{\#\mathsf{top}_{\mathbf{X}}(A)}$$

Let us emphasize some basic features about the evaluation function defined above. First, as already noticed, the Stability property makes the fractional evaluation of formulas a derivation-invariant, therefore the fractional method can be regarded as a semantics to all intents and purposes. Second, invertibility of of the rules of the calculus ensures that the relevant information stored in the conclusion is entirely preserved through the decomposition procedure. Third, the assignment is conservative over the base logic, as valid formulas are mapped to the maximum fractional value The next theorem establishes the latter point.

THEOREM 4.8 (Conservativity). *The formula A is* **X***-valid just in case* $\llbracket A \rrbracket_{\mathbf{X}} = 1$.

PROOF: ($\Leftarrow$) If $\llbracket A \rrbracket_{\mathbf{X}} = 1$, then there is a **HX** ending in $\vdash\Rightarrow A$. By applying the soundness theorem we can infer the **X**-validity of A.

($\Rightarrow$) If A is **X**-valid, then by completeness there is a **HX** derivation ending in $\vdash\Rightarrow A$, so every initial top-hypersequent expresses an identity and therefore we get

$$\llbracket A \rrbracket_{\mathbf{X}} = \frac{\#\mathsf{top}^1_{\mathbf{X}}(A)}{\#\mathsf{top}_{\mathbf{X}}(A)} = \frac{\#\mathsf{top}^1_{\mathbf{X}}(A)}{\#\mathsf{top}^1_{\mathbf{X}}(A)} = 1$$

□

Let $\mathscr{F}^c$ be the language of classical propositional logic. The next theorem establishes the surjectivity of the interpretation function $\llbracket \cdot \rrbracket$. In particular, we have:

THEOREM 4.9. *For any $q \in \mathbb{Q}^*$: (i) there is a formula $A \in \mathscr{F}^c$ s.t. $\llbracket A \rrbracket_{\mathbf{X}} = q$, and (ii) there is a formula $B \in \mathscr{F} - \mathscr{F}^c$ s.t. $\llbracket B \rrbracket_{\mathbf{X}} = q$.*

PROOF: Let $q = {}^{m}/_{n}$, where $m, n \in \mathbb{N}^+$ and $m \leqslant n$. (i) Consider the formula $\bigwedge(p \vee \neg p)^m \wedge \bigwedge p^{n-m}$. It is immediate to see that $\llbracket \bigwedge(p \vee \neg p)^m \wedge \bigwedge p^{n-m} \rrbracket_{\mathbf{X}} = {}^{m}/_{n} = q$.

(ii) We provide details for the modal logic **E**, other systems can be handled analogously. We consider now the modal formula $\bigwedge(\Box p \to \Box p)^m \wedge \bigwedge(\Box p)^{2n-m}$ in $\mathscr{F} - \mathscr{F}^c$. It turns out, similarly, that $\llbracket \bigwedge(\Box p \to \Box p)^m \wedge \bigwedge(\Box p)^{2n-m} \rrbracket_{\mathbf{X}} = {}^{2m}/_{2n} = {}^{m}/_{n} = q$. □

Remark 4.10. By combining Theorem 4.9 and the density of $\mathbb{Q}^*$, it is easy to verify that for, any modal system **X** and any pair of modal formulas A, B with $\llbracket A \rrbracket_{\mathbf{X}} < \llbracket B \rrbracket_{\mathbf{X}}$, we can always find a third formula $C \in \mathscr{F}^c$ such that $\llbracket A \rrbracket_{\mathbf{X}} < \llbracket C \rrbracket_{\mathbf{X}} < \llbracket B \rrbracket_{\mathbf{X}}$.

The previous theorem extends the result that has already been established for the modal logic **K** and serves as a bridge between classical and modal propositional logic. Specifically, for any modal formula, it is possible to provide a classical formula that has the same identity content as the modal one, as determined by the fractional interpretation. To illustrate this qualitative analysis, consider the modal formula $\Box(\Box p \to p) \to \Box p$ such that $[\![\Box(\Box p \to p) \to \Box p]\!]_{\mathbf{M}} = 0.5$. The decomposition algorithm ejects the modal component and returns the classical formula $(p \vee \neg p) \wedge p$ whose fractional interpretation is $[\![(p \vee \neg p) \wedge p]\!]_{\mathbf{M}} = 0.5$. In fact, the decomposition of the formula leads to two initial sequents: a tautological one and a complementary one.

5. Concluding remarks

We have developed new logical calculi for modal logic **D**, as well as the non-normal modal logics **M** and **E**. These systems are able to combine some of the most important proof-theoretic features: the subformula property (as a consequence of the cut-elimination theorem), finiteness of the proof-search space, and invertibility of the logical rules. By fine-tuning a variety of bilateralism based on the notion of rejection as underivability, we showed how to articulate a proof-based interpretation of the modal logics under focus.

We acknowledge that there are differences between canonical proof-theoretic semantics and fractional semantics, to the extent that a semantics in terms of proofs does not necessarily qualify as proof-theoretic. In particular, the fractional technique results in a multi-valued interpretation of the formulas in the language, whereas proof-theoretic semantics is completely disengaged from any "quantitative" form of evaluation. This fact deserves special consideration as it suggests that, when decidable systems are under consideration, the syntax/semantics dichotomy can be overcome by means of a proof-based interpretation, which nonetheless entails a quantitative evaluation of the formulas in the language.

To conclude, we would like to say something about the problem of devising a proof-theoretic semantics for the modal operator of necessity. According to Kürbis, a proof-theoretic semantics should be seriously regarded as defective without a proper account of the $\Box$-modality [7]. The technical achievements in this paper show that modal formulas can be maximally analyzed by means of a set of logical rules which have the effect of progressively detecting the modal components as residual elements. That is, the "quantity of identity" present in a modal formula can be measured in essentially the same way as in classical logic, provided that the classical content has been properly isolated. The lesson to be learned is that, if we consider the fractional method as a legitimate variant of proof-theoretic semantics, the issue raised by Kürbis can be circumvented inasmuch as modal formulas can be evaluated without taking the meaning of the $\Box$-modality directly into account. In this sense, we believe that our work is a step towards a proof-theoretic semantics for modal logics Nonetheless, the problem of providing a fully satisfactory proof-theoretic account of the $\Box$-modality remains an open and challenging task, which requires further investigation and research.

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