New Perspectives on Integrable Hamiltonian Systems via the Algebraic Geometry of Twisted Hitchin Moduli Spaces: A Case Study on the Calogero-Françoise Integrable System

A thesis submitted to the
College of Graduate and Postdoctoral Studies
in partial pulfillment of the requirements
for the degree of Master of Science
in the Department of Mathematics and Statistics
University of Saskatchewan
Saskatoon

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Abstract

Integrable systems are dynamical systems that exhibit very special properties. These systems are exactly solvable, have deep connections with algebraic geometry, and give rise to a maximal set of conserved quantities. Integrable systems have proven to be essential as they arise naturally in various branches of mathematics and physics such as differential and algebraic geometry, partial differential equations, statistical mechanics, quantum field theories, string theory and even more. One of the unique features of integrable systems is that all known integrable systems seem to be inherently related in some sense. This special feature has inspired many mathematicians to attempt and find a single origin of all known integrable systems. One of the most prominent approaches towards this unification process is through realizing different integrable systems as symmetry reductions of the self-dual-Yang-Mills (SDYM) equations. In fact, most known integrable systems (at least in lower dimensions) fit in this paradigm and this thesis is a further step towards this unification.

Four integrable systems are in the central attention of this thesis. Namely, these are Euler's equations for the motion of a rigid body, Nahm's equations, the Hitchin system, and the the Calogero-Françoise integrable system. Euler's equations are the most classical example of an integrable system. Moreover, Nahm's equations are obtained as a dimensional reduction of the SDYM equations to one dimension. Furthermore, the Hitchin system is an algebraically completely integrable system that arises as the space of solutions to Hitchin's equations. Hitchin's equations are a coupled system of non-linear partial differential equations that arise as a dimensional reduction of the SDYM equations to two dimensions. Finally, the Calogero-Françoise (CF) integrable system is a finite-dimensional Hamiltonian system that arises as a generalization of the Camassa-Holm (CH) dynamics.

In this thesis, we show that the dynamics of Euler's equations and the CF system can be perceived by realizing both systems as twisted Hitchin systems. More specifically, we obtain an explicit solution to Euler's equations by transforming them to Nahm's equations and then studying the evolution of the Higgs field of Nahm's equations. The solution method is different from the classical ones since we used a different formulation than the one usually presented in the literature. More specifically, we formulated the problem on the Lie algebra $\mathfrak{su}(2)$ rather than formulating it on $\mathfrak{so}(3)$ as usually done. Furthermore, we study the dynamics of the Calogero-Françoise (CF) integrable system while focusing on the special case of peakon anti-peakon interactions ($d = 2$). We show explicitly by embedding the CF system into a (twisted) Hitchin system that the CF dynamics is completely governed by the evolution of the corresponding Higgs field. In particular, we show that different singularities in the CF system correspond to very special Higgs fields in the underlying Hitchin system. Furthermore, we show that a periodization (compactification) of the CF dynamics corresponds to a compactification of the underlying Hitchin system. This result is then a direct manifestation of the correspondence between the CF dynamics and the dynamics of the associated Higgs field in the underlying Hitchin system.

Acknowledgements

First of all, I would like to thank my supervisors Dr. Steven Rayan and Dr. Jacek Szmigielski for their guidance and support during these two years. More specifically, I would like to express my deepest appreciation and gratitude for the immense support Steve has given me since I arrived here in Canada. I can not overestimate the role of his support and kindness in helping me reach this point despite all the obstacles I have come through. For that, I am heavily indebted to him and I am sincerely grateful for everything he has done for me. Furthermore, I would like to thank Jacek for the exceptional energy and enthusiasm he brought to every maths discussion we had. I am also grateful to him for taking the time and effort to read every write-up I sent with care and diligence and for guiding me to constantly enhance my writing.

I would also like to thank my family for their unparalleled love and their unwavering support and encouragement throughout my life. I would like to send my sincere love and affection to my angel mother not only for being my life-long best friend but also for being my peerless source of support and unconditional love. Moreover, I would like to thank my father for being nothing but supportive and reassuring specially when it comes to my financial security. Furthermore, I would like to thank my brother whose presence in my life is irreplaceable and whose positive impact on my life can not be overstated. Last but not least, I would like to thank my lovely sister and her admirable children for being the main source of joy and delight in my life. I would like to thank Anas (also known as Super Mario), Leen and Basmala for being so cute, exceedingly loving and for enlightening our lives over the past few years.

I would also like to thank all the amazing people I have met during my stay in Saskatoon. More specifically, I would like to thank my colleagues: Jason, Sheldon, Kazuki, Kuntal, Christopher, Madeline, Mandana, Isaac, Yaindrilla, Tara and Osama. Although we only met a few times, these people are amazing and I am glad to get to know them all. I would also like to thank Manuela for being so kind and understanding and for considering my teaching preferences every semester. I would also like to thank Kyla for being so lively and for her constant effort to bring joy and happiness to the mathematics department.

Last but not least, I would like to thank all my friends here in Saskatoon and back in Egypt. Above all, I would like to express my gratitude to Abdelrahman for being a great friend although we only knew each other for a very short time. I would also be happy to thank my dear friends: Lemo, Mohab, Yehia, Kareem, Basem, Ashraf, Hazem, Ahmed Emad, Yousef, Amr and others who never ceased to ask about me since I left Egypt.

Finally, I would like to thank Allah for all the blessings he bestowed upon me, for giving me a great family and friends, for guiding me throughout my life, and for giving me the patience to get through the hardest of times. Undoubtedly, he has given me more than what I actually deserve and for that I am utterly grateful.

"And He has given you from all you have asked Him, and if you count the blessings of Allah you shall never be able to enumerate them. Indeed, mankind are (generally) highly unjust, exceedingly ungrateful."

To my dear aunt Yasmeen who left us during the pandemic, you will always be remembered.

Contents

1 Introduction

Generally speaking, there is no universal agreement on a formal definition of an integrable system. However, integrable systems are known to exhibit certain features that make them special and worthy of studying, For example, these systems give rise to many conserved quantities and are analytically solvable in some sense. Furthermore, there is an inherent geometric picture that is usually hidden in the formulation of these systems and therefore they can be studied using techniques from geometry. On the other hand, integrability can be defined more concretely in classical mechanics. Such systems are known as completely integrable Hamiltonian systems and constitute a large body of the integrable systems studied in the literature. Another fascinating feature of integrable systems is that many of them can be obtained as special cases of the self-dual Yang Mills (SDYM) equations by a process known as "dimensional reduction". These systems include the KdV hierarchies, the Calogero-Moser system, the Sine-Gordon equation, the non-linear Schrödinger equation, Nahm's equations and most importantly Hitchin's self-duality equations. Hitchin's self-duality equations arise as a dimensional reduction of the SDYM equations to two dimensions. These are two coupled non-linear partial differential equations whose solutions give rise to the widely celebrated Hitchin integrable system discovered in [26]. In low dimensions, many integrable systems can be realized as a (maybe twisted) Hitchin system of some sort. Furthermore, there is a common belief that all classical integrable systems fit in this paradigm. The goal of this thesis is to give two explicit examples of the realization of a certain integrable system as a Hitchin system and therefore as a dimensional reduction of the SDYM equations. More specifically, we will show that the dynamics of Euler's equations and the Calogero-Françoise integrable system can be realized by the behavior of the associated Higgs field in the underlying Hitchin system.

One of the special features of many integrable systems is the existence of a pair $A(\lambda)$, $B(\lambda)$ that satisfies the Lax pairs equation

$$\frac{dA}{dt} = [A, B].$$

where $A(\lambda)$ and $B(\lambda)$ are polynomials with matrix coefficients and $\lambda \in \mathbb{P}^1$. In fact, one can study the Lax pairs equation by utilizing the theory of line and vector bundles over Riemann surfaces. The utility of this approach manifests in three interrelated aspects. First of all, the existence of integrals of motion becomes obvious in this case as the trace of any positive power of $A(\lambda)$ is an invariant of motion. Secondly, this formulation enables us to easily define the spectral curve Y associated to the problem. This is an algebraic curve that is defined by the following equation

$$\det(\eta \mathbb{I} - A(\lambda)) = 0,$$

and which plays a critical role in studying integrable systems using the formulation of vector bundles. The spectral curve Y is then the set of points (λ, η) in the total space of the line bundle $\mathcal{O}(n)$ over $\mathbb{P}^1$ that satisfy the characteristic equation. Here n is the highest degree of λ in $A(\lambda)$ and therefore

$$A(\lambda) = A_0 + A_1\lambda + ... + A_n\lambda^n.$$

Generally speaking, the spectral curve Y is an $r:1$ branched cover of $\mathbb{P}^1$ where r is the rank of matrix $A(\lambda)$. Finally, the non-linear flow of $A(\lambda)$ given by the Lax pairs equation can be realized as a linear flow of the eigenspace line bundle on the Jacobian of the spectral curve $\mathcal{J}(Y)$. The geometric approach to the Lax pairs equation will be reviewed in some detail in chapter 2. While the two main results in that chapter are proved in detail, the proofs of the classical theorems are all omitted and the relevant references will be mentioned accordingly.

Euler's equations for the motion of a rigid body is probably the most classical example of an integrable system. These equations were studied heavily in the literature and their solutions are usually written in terms of elliptic functions. On the other hand, contrary to classical integrable systems, Nahm's equations did not arise from a classical mechanical system. Instead, they arose in the construction of monopoles and they can be obtained as a dimensional reduction of the SDYM equations to one dimension [25], [39]. These are three coupled differential equations for the three Higgs fields T_1, T_2 and T_3 given as follows

$$\frac{dT_1}{dt} = [T_2, T_3], \quad \frac{dT_2}{dt} = [T_3, T_1], \quad \frac{dT_3}{dt} = [T_1, T_2].$$

It is well-known that Nahm's equations are related to Euler's equations and our job is to use this relation to attain a new approach of integrating Euler's equations and thus obtaining an explicit solution in terms of the Weierstrass elliptic function. This part will occupy chapter 3, in which we will give an explicit realization of Euler's equations as a twisted Hitchin system.

Furthermore, Hitchin's self-duality equations are two coupled non-linear partial differential equations for a pair $(\mathcal{D}, A)$, where $\mathcal{D}$ is a connection on a holomorphic vector bundle on a compact Riemann surface $E \to \Sigma$ while A is a holomorphic 1-form with values in the bundle of endomorphisms of the bundle $E \to \Sigma$. These equations are given as follows

$$\mathcal{R}_{\mathcal{D}} + [A \wedge A^*] = 0,$$
$$\bar{\partial}_E A = 0,$$

where $\mathcal{R}_{\mathcal{D}}$ is the curvature of the given connection $\mathcal{D}$. The 1-form A is usually called the Higgs field and a solution $(\mathcal{D}, A)$ to Hitchin's equations is equivalent to a Higgs bundle (E, A), that is a holomorphic vector bundle E along with a Higgs field A. Over the years, the moduli space of Higgs bundles has proved to play a pivotal role in the development of many areas in mathematics and physics. Maybe the most notable example of their crucial importance is that they were used by Ngô in his proof of the fundamental lemma which led him to win Fields medal in 2010 [42], [43]. The total space of the moduli space of Higgs bundle is an algebraically completely integrable system known as the Hitchin system. The fibre of each point in the

Hitchin base is a torus that is the Jacobi variety of the underlying the spectral curve associated to the Higgs field A. This is known as the Hitchin fibration and the Hitchin map sends every pair (E, A) to the coefficients of the associated spectral curve

$$H : \mathcal{M}(r, d) \to H^0(\Sigma, K^r)$$

where K is the cotangent bundle on the Riemann surface Σ. The asymptotic behavior of a Higgs bundle (E, A) as it approaches the ends of the moduli space is very special. Loosely speaking, the Higgs field becomes singular at this limit and a degeneration phenomena occurs to the zeros of A. However, a desingularization process exists such that one glues the diverging sequence of solutions to a limiting configuration, a pair $(\mathcal{D}_\infty, A_\infty)$ that satisfies a decoupled version of Hitchin's equations

$$\begin{aligned} \mathcal{R}_{\mathcal{D}_\infty} &= 0, \\ [A_\infty \wedge A_\infty^*] &= 0, \\ \bar{\partial}_E A_\infty &= 0. \end{aligned} \tag{1.1}$$

The dimension of the moduli space of limiting configurations is exactly half the dimension of the moduli space of Higgs bundles. Therefore, by this gluing method, any diverging sequence of solutions to Hitchin's equations converges to a limiting configuration $(\mathcal{D}_\infty, A_\infty)$ [35] [36]. This construction amounts to a compactification of the moduli space of Higgs bundles and this fact will be the precursor of the main results of this thesis. We will review the moduli space of Higgs bundles along with its asymptotic behavior in chapter 4.

Finally, the Calogero-Françoise integrable system is a finite-dimensional Hamiltonian system that arose as a generalization to the Camassa-Holm dynamics. Similar to the CH system, the CF system gives rise to peaked solitons (peakons). These peakons collide when their positions coincide at one point in space and the collisions between the CH peakons were studied in detail in [7]. Even though the CF system has a priori a non-periodic Hamiltonian, its dynamics are related to the dynamics of the periodic CH system over a window its domain [6], [5]. At the boundaries of this domain, which we will refer to as the confinement box, the CF dynamics becomes singular and therefore a question arises, whether there is a natural analytic continuation past these singularities? We will prove in chapter 5 that in the case $d = 2$ (peakon anti-peakon pair), there is a natural analytic continuation in which the momenta of the peakons exchange signs and the slope of distance between them flips its sign. In other words, if the peakons were initially approaching, then they should move apart after the collision. On the other hand, if they were initially moving apart, then they should approach each other after colliding with the walls of the confinement box. These facts suggest that one may analytically continue the CF dynamics to be periodic and therefore coincide with periodic CH dynamics. However, one should naturally ask why is this choice relevant and whether there is a geometrical interpretation to this periodization? To answer this question, we relate the CF dynamics with the asymptotics of the moduli space of Higgs bundles mentioned earlier. We show that the periodization (compactification) of the CF dynamics corresponds to a compactification of the underlying moduli space. Therefore, the CF dynamics is captured by the behavior of associated the Higgs field in the underlying Hitchin system. This is then another example of the realization of a classical integrable system as a (twisted) Hitchin system. These facts will be discussed in detail in chapter 5 for the case $d = 2$. Finally, we will proceed to study some results regarding the dynamics of the case $d = 3$. We will prove that the underlying Higgs field prohibits certain dynamics from occurring. More specifically, triple collisions are forbidden and collisions between two non-neighbouring peakons are also forbidden. Although we will not discuss in detail the geometry of the compactification of the underlying moduli space in this case, the geometric picture shall be similar to the one obtained in the case $d = 2$.

2 Integrable Systems Through Vector Bundles

2.1 The Jacobi Variety of a Riemann Surface.

Let Σ be a non-singular algebraic curve of genus g, then the Jacobi variety of Σ, also known as the Jacobian, is an Abelian variety $\mathcal{J}(\Sigma)$ of complex dimension g. When Σ is a Riemann surface, then $\mathcal{J}(\Sigma)$ is a complex torus that parametrizes the line bundles of degree zero on Σ. In fact, the Jacobian plays an essential role in the theory of integrable systems and vector bundles over Riemann surfaces. Having said that, the main purpose of this section is to define the Jacobian of a Riemann surface formally and explore its relation to the Riemann-theta function.

2.1.1 Theta functions

The Riemann theta-function is a special multi-variable complex function that arises in many parts of mathematics and physics such as Abelian varieties, moduli spaces and quantum field theories. It is a quasi-periodic function of g complex variables that is usually defined in terms of its Fourier series as will shown shortly. For full treatment of the Riemann theta function and its role in solving the Jacobi inversion problem, one may consult [16],[17].

Definition 2.1.1. *A Riemann matrix is a symmetric $(g \times g)$ matrix $R = (R_{jk})$ with negative definite real part. That is, if $R_r = \Re(R)$, then the quantity*

$$z^T R_r z < 0,$$

for any non-zero vector $z \in \mathbb{C}^g$.

Definition 2.1.2. *A Riemann theta function is a quasi-periodic function defined by its Fourier series as follows*

$$\theta(z \mid R) = \sum_N \exp\left(\frac{1}{2}\langle RN, N\rangle + \langle N, z\rangle\right), \tag{2.1}$$

where $z = (z_1, ..., z_g) \in \mathbb{C}^g$ is a complex vector, the inner product is the standard Euclidean inner product, and $N = (N_1, ..., N_g) \in \mathbb{Z}^g$ is a lattice vector.

Obviously, the function $\theta(z \mid R)$ is analytic over the whole space $\mathbb{C}^g$. From now on, we fix a Riemann matrix R and simply write $\theta(z)$ instead of $\theta(z \mid R)$. Let $(e_1, ..., e_g)$ be the standard basis of vectors in $\mathbb{C}^g$, that is

$$(e_j)_k = \delta_{jk}, \quad (j, k = 1, ..., g).$$

Furthermore, let the vectors $(r_1, ..., r_g) \in \mathbb{C}^g$ be defined as follows

$$r_j = Re_j, \quad (j = 1, ..., g),$$

and therefore, $(r_j)_k = R_{jk}$. Then, the Riemann theta function has the following transformation laws

$$\begin{aligned} \theta(z + 2\pi i e_j) &= \theta(z), \\ \theta(z + r_j) &= \exp(-\frac{1}{2}R_{jj} - z_j)\theta(z), \end{aligned} \tag{2.2}$$

where $j = 1, ..., g$. Therefore, the basis $2\pi e_1, ..., 2\pi e_g$ and $r_1, ..., r_g$ are called the periods and the quasi-periods of the Riemann theta function respectively. Combining both transformations, one gets

$$\theta(z + 2\pi i N + RM) = \exp\left(-\frac{1}{2} < RM, M > - < M, z >\right)\theta(z), \tag{2.3}$$

where $N, M \in \mathbb{Z}^g$.

Proposition 2.1.3. *The vectors $2\pi i e_1, ..., 2\pi i e_g$ and $r_1, ..., r_g$ defined on $\mathbb{C}^g$ are linearly independent over the field of real numbers $\mathbb{R}$.*

We call the lattice generated by these vectors the period lattice of the Riemann theta function $\theta(z)$ and denote it by Γ. A vector in Γ has the following form

$$(2\pi i N + RM),$$

where $N, M \in \mathbb{Z}^g$. Furthermore, the quotient $\mathbb{C}^g/\Gamma$ is a complex g-dimensional torus. This torus is defined in terms of the periods and the quasi-periods of the Riemann theta function $\theta(z \mid R)$ and therefore is denoted by $T^g(R)$. In fact, this torus is uniquely determined by the Riemann matrix R up to a symplectic transformation as follows. Let $(2\pi i e'_j, r'_j)$ be another basis for the lattice Γ such that $r'_j = R'e'_j$ and R' is another Riemann matrix. Then the transformation from the old to the new basis is given as follows

$$\begin{aligned} 2\pi i e'_j &= \sum_k d_{jk} 2\pi i e_k + c_{jk} r_k, \\ r'_j &= \sum_k b_{jk} 2\pi i e_k + a_{jk} r_k, \end{aligned}$$

where $a, b, c, d \in GL(g, \mathbb{Z})$. If the transformation matrix

$$T = \begin{bmatrix} a & b \\ c & c \end{bmatrix}$$

is symplectic, that is if $T \in Sp(2g, \mathbb{Z})$, then the two Riemann matrices R and R' determine the same torus, that is

$$T^g(R) = T^g(R').$$

One can also define a θ-function with characteristics as follows. Let $\alpha, \beta \in \mathbb{R}^g$, then the θ function with characteristics $[\alpha, \beta]$, denoted by $\theta[\alpha, \beta](z)$, is defined as follows

$$\theta[\alpha, \beta](z) = \exp\left(\frac{1}{2}\langle R\alpha, \alpha\rangle + \langle z + 2\pi i\beta, \alpha\rangle\right)\theta(z + 2\pi i\beta + R\alpha). \tag{2.4}$$

Obviously, $\theta[0,0](z)=\theta(z)$ and $\theta[N,M](z)=\theta(z)$ where $N,M\in\mathbb{Z}^g$. Therefore, without loss of generality, one can only consider $0<\alpha_j,\beta_j<1$ where $j=1,...,g$. Furthermore, Fourier series of these functions are given as follows

$$\theta[\alpha,\beta](z)=\sum_N \exp\left(\frac{1}{2}\langle R(N+\alpha),N+\alpha\rangle+\langle z+2\pi i\beta,N+\alpha\rangle\right). \tag{2.5}$$

Finally, their transformation law is given as follows

$$\theta[\alpha,\beta](z+2\pi iN+RM)=\exp\left(2\pi i(\langle\alpha,N\rangle-\langle\beta,M\rangle)-\frac{1}{2}\langle RM,M\rangle-\langle z,M\rangle\right)\theta[\alpha,\beta](z).$$

If all the elements α_j,β_j of the characteristics $[\alpha,\beta]$ are either 0 or 1/2, then $[\alpha,\beta]$ are called half periods. Furthermore, a half period is called even if $4\langle\alpha,\beta\rangle\equiv 0(mod\ 2)$ and odd otherwise.

Proposition 2.1.4. *The function $\theta[\alpha,\beta](z)$ is even or odd if $[\alpha,\beta]$ is even or odd respectively.*

All over this work, we will be dealing with algebraic curves of genus $g=1$ (elliptic curves). Therefore, we will be specifically dealing with θ functions of one variable; such functions are usually called elliptic functions. In this case, the Riemann matrix is a single number $r\in\mathbb{C}$ such that $\Re(r)<0$. Moreover, there are only four half periods that correspond to the following θ functions

$$\begin{aligned}
i\theta_1(z)&\equiv\theta\left[\frac{1}{2},\frac{1}{2}\right](z)=\sum_j \exp\left(\frac{1}{2}r\left(j+\frac{1}{2}\right)^2+\left(\frac{2j+1}{2}\right)(z+\pi i)\right),\\
\theta_2(z)&\equiv\theta\left[\frac{1}{2},0\right](z)=\sum_j \exp\left(\frac{1}{2}r\left(\frac{2j+1}{2}\right)^2+\frac{2j+1}{2}z\right),\\
\theta_3(z)&\equiv\theta[0,0](z)=\sum_j \exp\left(\frac{1}{2}j^2r+jz\right),\\
\theta_4(z)&\equiv\theta\left[0,\frac{1}{2}\right](z)=\sum_j \exp\left(\frac{1}{2}j^2r+j(z+\pi i)\right),
\end{aligned}$$

where $j\in\mathbb{Z}$. It is obvious that all the half-periods are even except the first one which is odd. Therefore, $\theta_1(z)$ is odd while $\theta_2(z)$, $\theta_3(z)$ and $\theta_4(z)$ are even. For example, θ_1 has zero at the origin $z=0$ as well as (by periodicity) at all the vertices of the period lattice $z=2\pi im+rn$. Consider now the Weierstrass $\wp$ function defined as follows

$$\wp(z)=\frac{1}{z^2}+\sum_{n^2+m^2\neq 0}\left[\frac{1}{(z-2n\omega_1-2m\omega_2)^2}-\frac{1}{(2n\omega_1+2m\omega_2)^2}\right], \tag{2.6}$$

where $n,m\in\mathbb{Z}$ and ω_1,ω_2 are the periods of a 1-dimensional complex torus $T(R)$. It is quite obvious that this function is doubly-periodic,

$$\wp(z+2n\omega_1+2m\omega_2)=\wp(z).$$

Furthermore, one can show that the series (2.6) converges uniformly on every compact set in $\mathbb{C}/\{2n\omega_1+2m\omega_2\}$. Therefore, $\wp$ is a meromorphic function of z that has a double pole at $z=0$ as well as at each vertex of the period lattice. Since these points constitute the zero locus of $\theta_1(z)$, one may expect that the two functions are related. This is in fact true and

$$\wp(z) = -\frac{d^2}{dz^2}\ log(\theta_1) + c, \tag{2.7}$$

where c is a constant.

2.1.2 Differential Forms

Definition 2.1.5. *A differential form of order n, or simply an n-form, is a skew-symmetric tensor of type $(0, n)$.*

Let the space of n-forms on a differentiable manifold Σ be denoted $\Lambda^n(\Sigma)$. Obviously, $\Lambda^0(\Sigma) = C^\infty(\Sigma)$ and $\Lambda^n(\Sigma)$ is the empty set for $n > d$, where d is the dimension of Σ. Furthermore, the space of differential forms on Σ, which we denote by $\Lambda(\Sigma)$, is given as follows

$$\Lambda(\Sigma) = \bigoplus_{n=0}^{d} \Lambda^n(\Sigma).$$

The space $\Lambda(\Sigma)$ is equipped with two basic operations, the wedge (exterior) product and the exterior derivative. These are defined as follows

Definition 2.1.6. *The wedge (exterior) product $\wedge$ is defined as follows*

$$\begin{aligned} \wedge : \Lambda^m(\Sigma) \times \Lambda^n(\Sigma) &\to \Lambda^{m+n}(\Sigma), \\ \Omega_1 \wedge \Omega_2 &:= \frac{(m+n)!}{m!n!}\ \hat{A}(\Omega_1 \otimes \Omega_2), \end{aligned} \tag{2.8}$$

where $\hat{A}$ is the alternating operator that picks up the skew-symmetric part of the $(0, m+n)$-tensor $\Omega_1 \otimes \Omega_2$.

Proposition 2.1.7. *The wedge product satisfies the following properties*

1. $\Omega_1 \wedge (\Omega_2 \wedge \Omega_3) = (\Omega_1 \wedge \Omega_2) \wedge \Omega_3$.
2. *If $\Omega_1 \in \Lambda^m(\Sigma)$ and $\Omega_2 \in \Lambda^n(\Sigma)$, then $\Omega_1 \wedge \Omega_2 = (-1)^{mn}\ \Omega_2 \wedge \Omega_1$.*

Furthermore, in a local coordinate system $(z^1, ..., z^d)$, an n-form has the following representation

$$\Omega = \frac{1}{n!} \sum_{j_1,\ldots,j_n} \Omega_{j_1 \ldots j_n}\ dz^{j_1} \wedge ... \wedge dz^{j_n}. \tag{2.9}$$

Finally, the exterior derivative is defined as follows

Definition 2.1.8. *Let $\Omega \in \Lambda^n(\Sigma)$, then the exterior derivative is given as follows*

$$d : \Lambda^n(\Sigma) \to \Lambda^{n+1}(\Sigma),$$

and given a local coordinate system $(z_1, ..., z_n)$ on Σ, it has the following representation

$$d\Omega = \frac{1}{n!} \sum_{k,j_1,\ldots,j_n} \frac{\partial \Omega_{j_1 \ldots j_n}}{\partial z^k}\ dz^k \wedge dz^{j_1} ... \wedge dz^{j_n},$$

where $k = n+1, ..., d$.

For simplicity, consider now differential forms on a Riemann surface Σ. Let Ω be a differential 1-form on Σ, which we simply call a differential. In a local coordinate system, it has the following representation

$$\Omega = g_1(z)\ dz + g_2(z)\ d\bar{z}.$$

where g_1 and g_2 are differentiable functions on Σ. Therefore, if f is a differentiable function on Σ. Then in local coordinates, the exterior derivative of f is given by

$$df = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}.$$

Definition 2.1.9. *A differential 1-form Ω on a Riemann surface Σ is called holomorphic if there exists a neighbourhood of any point $p \in \Sigma$ such that*

$$\Omega = f(z)dz,$$

where $f(z)$ is a holomorphic function and z is a local coordinate on Σ. Similarly, if

$$\Omega = g(z)d\bar{z}$$

for some differentiable function $g(z)$, then Ω is called anti-holomorphic. Finally a differential form is closed if

$$d\Omega = 0.$$

We denote the vector spaces of holomorphic and anti-holomorphic 1-forms on Σ by $\Lambda^{1,0}(\Sigma)$ and $\Lambda^{0,1}(\Sigma)$ respectively. Then, by construction

$$\Lambda^1(\Sigma) = \Lambda^{1,0}(\Sigma) \oplus \Lambda^{0,1}(\Sigma).$$

Furthermore, the exterior derivative also splits in a natural way

$$d = \partial + \bar{\partial},$$

where

$$\partial : \Lambda^0(\Sigma) \to \Lambda^{1,0}(\Sigma),$$
$$\bar{\partial} : \Lambda^0(\Sigma) \to \Lambda^{0,1}(\Sigma).$$

This construction extends naturally for any n-form as follows

$$\Lambda^n(\Sigma) = \bigoplus_{p+q=n} \Lambda^{p,q}(\Sigma),$$

where the exterior derivative acts as follows

$$\partial : \Lambda^{p,q}(\Sigma) \to \Lambda^{p+1,q}(\Sigma),$$
$$\bar{\partial} : \Lambda^{p,q}(\Sigma) \to \Lambda^{p,q+1}(\Sigma).$$

We summarize the properties of the exterior derivative for $f \in \Lambda^0(\Sigma), \Omega \in \Lambda^1(\Sigma)$ as follows

$$d^2(f) = \partial^2 f = \bar{\partial}^2 f = 0,$$
$$d\Omega = \partial\Omega + \bar{\partial}\Omega,$$
$$d(f\Omega) = df \wedge \Omega + f d\Omega.$$

It follows from these properties that every holomorphic 1-form on a Riemann surface is closed [18].

2.1.3 The Jacobian

We are now ready to define the Jacobian of a Riemann surface. More specifically, we will focus on the case of hyperelliptic curves defined by the equations

$$\eta^2 = q^{2g+1}(z), \quad \eta^2 = q^{2g+2}(z), \tag{2.10}$$

where $q^{2g+1}(z)$ and $q^{2g+2}(z)$ are polynomials without multiple roots of degrees $2g+1$ and $2g+2$ respectively. Both equations (2.10) define a hyperelliptic curve Σ of genus g. Consider now the 1-dimensional homology group $H_1(\Sigma) = \bigoplus_{j=1}^{2g} \mathbb{Z}$, one can choose a basis of cycles in $H_1(\Sigma)$ such that

$$a_j \circ a_k = b_j \circ b_k = 0, \quad a_j \circ b_k = \delta_{jk}, \qquad (j, k = 1, ..., g). \tag{2.11}$$

We will now utilize this basis of cycles to construct the Jacobian of Σ. For the hyperelliptic curve Σ defined in (2.10), one can construct g holomorphic differentials as follows

$$\Omega_j = \frac{z^{j-1}}{\sqrt{q^{2g+1}}} dz, \quad (j = 1, ..., g). \tag{2.12}$$

These differentials are in fact linearly independent and one can prove the following.

Proposition 2.1.10. *The space of holomorphic differentials on a Riemann surface of genus g is g-dimensional.*

For the proof see [16].

Consider now a closed differential Ω on a Riemann surface Σ; its periods over the cycles $(a_1, ..., a_g, b_1, ..., b_g)$ are defined as follows

$$\oint_{a_j} \Omega = A_j, \quad \oint_{b_j} \Omega = R_j, \quad (j = 1, .., g). \tag{2.13}$$

Furthermore, if all the a-periods of Ω vanish, then Ω is identically zero. Therefore, if $\Omega_1, ..., \Omega_g$ is the set of linearly independent differentials on Σ, then the matrix of their a-periods

$$A_{jk} = \oint_{a_k} \Omega_j, \quad (j = 1, ..., g)$$

is non singular. Choose now another basis for the space of holomorphic differentials on Σ

$$\omega_j = \sum_{k=1}^{g} c_{jk} \Omega_k, \quad (j = 1, ..., g),$$

where $c_{jk} = 2\pi i (A_{jk})^{-1}$. Then the new basis satisfies the following condition

$$\oint_{a_k} \omega_j = 2\pi i \delta_{jk}, \quad (j, k = 1, .., g). \tag{2.14}$$

This new basis is called canonically dual to the given basis of cycles.

Theorem 2.1.11. *Let ω_j be the basis of holomorphic differentials defined in (2.14), then the matrix of b-periods*

$$R_{jk} = \oint_{b_k} \omega_j \quad (j, k = 1, ..., g) \tag{2.15}$$

is a Riemann matrix.

Therefore, given any Riemann surface Σ of genus g and a basis of cycles $(a_1, ..., a_g, b_1, .., b_g)$, one can construct a Riemann matrix $R(\Sigma)$.

Definition 2.1.12. *The Abelian torus $T^g(R)$ constructed from the Riemann matrix $R(\Sigma)$ is called the Jacobian of the Riemann surface Σ and is denoted by $\mathcal{J}(\Sigma)$ where*

$$\mathcal{J}(\Sigma) = T^g(R) = \frac{\mathbb{C}^g}{\{2\pi i N + RM\}} \tag{2.16}$$

and $\{2\pi i N + RM\}$ is the period lattice of the Riemann theta function $\theta(z \mid R)$.

In fact, one can show that the definition of the Jacobian does not depend on the choice of the basis of cycles. That is, given another basis of cycles with the same intersection indices (2.11), then the new Riemann matrix is equivalent to the old one. Therefore, the Jacobian $\mathcal{J}(\Sigma)$ does not depend on the choice of the basis of cycles. Furthermore, the functions $\theta(z|R)$ constructed from the matrix of periods are called the θ functions of the Riemann surface Σ. The zero locus of a theta function on $\mathcal{J}(\Sigma)$ is known as the theta divisor Θ. The theta divisor forms a subvariety of complex dimension $g-1$ in the complex g-dimensional torus $\mathcal{J}(\Sigma)$. Therefore, for $g = 1$, the theta divisor will be a single point (it can not be a bunch of scattered points because Θ has to be connected).

The tori $\mathcal{J}(\Sigma) = T^g(R)$ are usually called Abelian tori and they admit a Kähler metric [16], [21]. Furthermore, meromorphic functions on these tori are called Abelian functions. These are meromorphic doubly periodic functions of g complex variables. We have then considered one example of Abelian functions in section (2.1.1), that is the Weierstrass $\wp$ function defined by (2.7).

2.2 Completely Integrable Hamiltonian Systems.

It is well known that the classical Hamiltonian dynamics can be formulated in the language of symplectic geometry. This beautiful geometric picture is well-studied in the literature and various classical Hamiltonian systems were investigated by utilizing this scheme. In this section, we will introduce a very economical version of this picture in order to give a formal definition of a classical integrable Hamiltonian system. For full treatment, one can consult the following references [2], [8], [13], [14].

A dynamical system consists of two ingredients: a space of physical states which is usually called the phase space $\mathcal{P}$ and a vector field F on $\mathcal{P}$ that defines the dynamics. The evolution of a state $t \to f(t)$ satisfies the following equation on $\mathcal{P}$

$$\dot{f}(t) = F(f(t)), \quad f(0) = f_0. \tag{2.17}$$

Hamiltonian systems are a particular class of dynamical systems whose flow is governed by Hamilton's equations

$$\dot{q}^j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q^j}, \quad (j = 1, ..., n), \tag{2.18}$$

where $(q^1, ..., q^n)$ and $(p_1, ..., p_n)$ are the generalized coordinates and generalized momenta respectively. Our goal now is to show that these two equations arise from a concrete geometric picture in symplectic geometry.

Definition 2.2.1. *A symplectic manifold is a pair* $(\mathcal{P}, \Omega)$ *where* $\mathcal{P}$ *is an (even-dimensional) manifold and* Ω *is a closed, non-degenerate 2-form on* $\mathcal{P}$*. That is,* Ω *satisfies the following conditions*

- $d\Omega = 0$*,*
- *If for any* $f \in \mathcal{P}$*,* $\Omega_f(u, v) = 0 \ \ \forall u \in T_f\mathcal{P}$*, then* $v = 0$*.*

Ω is usually called the symplectic form and one can prove that the non-degeneracy of Ω forces the phase space $\mathcal{P}$ to be even-dimensional. Let the dimension of $\mathcal{P}$ be $2n$, and let $(z^1, ..., z^{2n})$ be local coordinates on $\mathcal{P}$, then Ω can be written in component form as follows

$$\Omega = \frac{1}{2}\sum_{k,l=1}^{2n} \Omega_{kl}\ dz^k \wedge dz^l,$$

where $\wedge$ is the exterior product. Therefore, the condition $d\Omega = 0$ can be written in component form as follows

$$\partial_k\Omega_{lm} + \partial_m\Omega_{kl} + \partial_l\Omega_{mk} = 0. \tag{2.19}$$

for all $k, l, m = 1, ..., 2n$. The simplest example of a symplectic manifold is an even-dimensional Euclidean space $\mathbb{R}^{2n}$. In this case, the symplectic form is given as follows

$$\Omega = \begin{pmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{pmatrix}. \tag{2.20}$$

Evidently, the symplectic form is constant and therefore is closed as required. In fact Darboux theorem implies that any symplectic form is locally constant and therefore given by (2.20). Then the corresponding coordinates (q^j, p_j) are called canonical coordinates. Furthermore, the most prominent examples of a symplectic manifold are the co-tangent bundle on a manifold and the co-adjoint orbit of a Lie group G. In fact, the phase space of most classical Hamiltonian systems is the cotangent bundle of the corresponding configuration space.

Let $g \in C^\infty(\mathcal{P})$ be a smooth function on $\mathcal{P}$, such a function is usually called a classical observable. One then defines the Hamiltonian vector field as follows

Definition 2.2.2. *The vector field* F_g *on* $\mathcal{P}$ *defined by*

$$\Omega(F_g,\ .\ ) = dg \tag{2.21}$$

is called the Hamiltonian vector field relative to g*.*

Defining a local coordinates (z^j) on $\mathcal{P}$, the components of the vector field F_g are given by

$$(F_g)^k = \sum_{j=1}^{2n} \Omega^{jk} \frac{\partial g}{\partial z_j}, \tag{2.22}$$

where Ω^{jk} is the inverse of Ω_{jk} defined as follows

$$\sum_l \Omega_{kl}\Omega^{lm} = \delta_k^m.$$

One can now define the Poisson bracket as follows

$$\{g,h\} := \Omega(F_g,F_h) = \sum_{j,k=1}^{2n} \Omega^{jk} \frac{\partial g}{\partial z_j}\frac{\partial h}{\partial z_k}, \tag{2.23}$$

where $g,h \in C^\infty$ and $F_g,\ F_h$ are the Hamiltonian vector fields relative to g and h respectively. Note that in local canonical coordinates (q^j, p_j), one recovers the standard formula for the Poisson bracket

$$\{g,h\} = \sum_{j=1}^{n} (\frac{\partial g}{\partial q^j}\frac{\partial h}{\partial p_j} - \frac{\partial h}{\partial q^j}\frac{\partial g}{\partial p_j}). \tag{2.24}$$

Now we are fully equipped to define a Hamiltonian system.

Definition 2.2.3. *A Hamiltonian system is a triple* $(\mathcal{P},\Omega,H)$*, where* $(\mathcal{P},\Omega)$ *is a symplectic manifold and a function* $H \in C^\infty(\mathcal{P})$ *called the Hamiltonian of the system.*

Let F_H be the Hamiltonian vector field relative to H, then the Hamiltonian dynamics is defined by

$$\dot{f}(t) = F_H(f(t)). \tag{2.25}$$

Note that in canonical local coordinates, this formula simplifies to the standard Hamiltonian equations

$$\dot{q}^j = \frac{\partial H}{\partial p_j} = \{q^j, H\}, \quad \dot{p}_j = -\frac{\partial H}{\partial q^j} = \{p_j, H\}, \quad (j = 1, ..., n). \tag{2.26}$$

Furthermore, given two symplectic manifolds $(\mathcal{P}_1,\Omega_1)$ and $(\mathcal{P}_2,\Omega_2)$, then a map that preserves the symplectic form is called a symplectomorphism. That is, a map $\varsigma : \mathcal{P}_1 \to \mathcal{P}_2$ is a symplectomorphism if

$$\varsigma^*\Omega_2 = \Omega_1. \tag{2.27}$$

In fact, if $\mathcal{P}_1 = \mathcal{P}_2 = \mathcal{P}$, then symplectic transformations of $\mathcal{P}$ are the transformations that preserve the symplectic form and these transformations form a group that is a subgroup of of the group of diffeomorphisms of $\mathcal{P}$. If $\mathcal{P} = \mathbb{R}^{2n}$, then this subgroup is known as the symplectic group

$$Sp(n) := \{M \in GL(2n,\mathbb{R}) | M\Omega M^T = \Omega\}. \tag{2.28}$$

where Ω is given by (2.20).

Consider now the Hamiltonian system $(\mathcal{P},\Omega,H)$. A function $G : \mathcal{P} \to \mathbb{R}$ is a constant of motion, also called a first integral, if

$$\{G,H\} = 0. \tag{2.29}$$

In fact, first integrals play a crucial rule in solving the Hamiltonian dynamics as will be shown shortly. Moreover, note that any Hamiltonian system has at least one integral of motion, that is the Hamiltonian itself. On the other hand, a system with n degrees of freedom can have up to $2n$ independent constants of motion. However, it turns out that in order to completely solve the system, one only needs to determine n integrals of motion that are in involution (Poisson commuting). In this case, the Hamiltonian system is called integrable.

Definition 2.2.4. *A Hamiltonian system is said to be integrable if there exists* $n = \dim \mathcal{P}/2$ *integrals of motion* G_j *such that*

- $\{G_j, G_k\} = 0 \quad (j, k = 1, ..., n)$,
- *the functions* G_j *are independent on a level set of* $G = (G_1, ..., G_n)$ *at any point* $x = (x_1, ..., x_n) \in \mathbb{R}^n$. *In other words, the functions* G_j *satisfy the following equation*

$$dG_1 \wedge dG_2 \wedge ... \wedge dG_n \neq 0, \tag{2.30}$$

on a level set

$$G^{-1}(x) := \{f \in \mathcal{P} \mid G_j(f) = x_j \ ; \ j = 1, ..., n\}.$$

Integrable systems are special not only because they are solvable but also because they exhibit certain periodicity conditions. More precisely, the Hamiltonian flow is quasi-periodic when defined in terms of certain variables known as the action-angle variables. To see this explicitly, consider the two classical theorems of Liouville and Arnold.

Theorem 2.2.5. *(Liouville) For an integrable system, there exists* n *functions* $\Phi_j : \mathcal{P} \to \mathbb{R}$ *that satisfy the following conditions*

$$\{\Phi_j, \Phi_k\} = 0, \quad \{\Phi_j, G_k\} = \delta_{jk}, \tag{2.31}$$

for all $j, k = 1, ..., n$. *Furthermore, the functions* Φ_j *are unique up to the following transformation*

$$\Phi_j \to \Phi_j + \frac{\partial V}{\partial G_j}$$

where $V : \mathbb{R}^n \to \mathbb{R}$ is an arbitrary function.

Theorem 2.2.6. *(Arnold) If a level set* $G^{-1}(x)$ *is compact and connected, then it is diffeomorphic to an* n*-dimensional torus defined as follows*

$$G^{-1}(x) \cong T^n = \{(\Phi_1, ..., \Phi_n)/2\pi\}.$$

Furthermore, the Hamiltonian flow on $G^{-1}(x)$ *is quasi-periodic; that is*

$$\frac{d\Phi_j}{dt} = \omega_j(x), \quad (j = 1, ..., n). \tag{2.32}$$

where $(\omega_1, ..., \omega_n)$ *are the frequencies of the quasi-periodic motion.*

One can also show that if the frequencies are linearly dependent, then the Hamiltonian flow is strictly periodic on the torus. In other words, the trajectory of the flow given by (2.32) is closed on T^n. On the other hand, if the frequencies are linearly independent, then the trajectory is only quasi-periodic.

Let $a_1, ..., a_n$ be a basis of cycles dual to the basis of differentials $(d\Phi_1, ..., d\Phi_n)$ defined as follows

$$\oint_{a_j} d\Phi_k = 2\pi\delta_{jk} \quad (j, k = 1, , , ., n).$$

Then, define the standard action variables as follows

$$I_j(x) := \frac{1}{2\pi}\oint_{a_j}\sum_{k=1}^{n} p_k \wedge dq^k, \quad (j = 1, .., n). \tag{2.33}$$

Furthermore, if

$$\det\left\{\frac{\partial I_j}{\partial x_k}\right\} \neq 0,$$

then there exists a canonical transformation

$$(q^1, ..., q^n, p_1, ..., p_n) \to (I_1, ..., I_n, \Phi_1, ..., \Phi_n)$$

where the canonical set $(I_1, ..., I_n, \Phi_1, ..., \Phi_n)$ are known as the action-angle variables. One can then show that the Hamilton equations simplify remarkably when expressed in terms of these new variables, namely

$$\frac{dI_j}{dt} = 0, \quad \frac{d\Phi_j}{dt} = \omega_j, \quad (j = 1, ..., n). \tag{2.34}$$

These equations are solved simply to give

$$I(t) = I(0), \quad \Phi(t) = \Phi(0) + \omega t. \tag{2.35}$$

This concludes our brief survey of the classical integrable Hamiltonian systems. In the upcoming section, we will consider integrable systems from a different point of view. More precisely, we will study integrable systems that satisfy the Lax pairs equation by utilizing the theory of vector bundles over Riemann surfaces.

2.3 Vector bundles and Lax Pairs.

In the last section, we defined integrable Hamiltonian systems and discussed the unique features of these systems. In this section, we will be concerned with a specific type of integrable systems, those satisfying the Lax pairs equation

$$\frac{dA}{dt} = [A, B], \tag{2.36}$$

where $A(\lambda) = A_0 + A_1\lambda + ... + A_m\lambda^m$ and $B(\lambda) = B_0 + B_1\lambda + ... + B_n\lambda^n$ are polynomials with matrix coefficients. Evidently, the Lax pairs equation when expanded gives a system of differential equations for different powers of λ in the coefficients of $A(\lambda)$ and $B(\lambda)$. In fact, many classical and non-classical integrable systems fit in this paradigm. For example, Euler's equation for the motion of a rigid body around its center of mass fits in this paradigm. Furthermore, Nahm's equations that arose in the study of monopoles also fits in this paradigm. As will be shown, integrable systems that satisfy the Lax pairs equation can be studied geometrically by utilizing the theory of vector bundles. An excellent treatment of this geometric approach to the Lax pairs equation which we will follow throughout this section is [28].

Definition 2.3.1. *A holomorphic line bundle over a Riemann surface Σ is a two-dimensional complex manifold L endowed with a holomorphic projection $\pi : L \to \Sigma$ such that the fibre of each point $\lambda \in \Sigma$ is a 1-dimensional vector space. Furthermore, each $\lambda \in \Sigma$ has a neighbourhood U and a homeomorphism Φ_U such that the following diagram commutes.*

$$
\begin{tikzcd}
\pi^{-1}(U) \arrow[r, dashed, "\Phi_U", "\cong"'] \arrow[d, "\pi"] & U \times \mathbb{C} \arrow[dl] \\
U &
\end{tikzcd}
$$

Finally, $\Phi_{U_k} \circ \Phi_{U_j}^{-1} : U_j \times \mathbb{C} \to U_k \times \mathbb{C}$ *has the following form*

$$(\lambda, \eta) \to (\lambda, T_{jk}(\lambda)\eta),$$

where the function $T_{jk}(\lambda) : U_j \cap U_k \to \mathbb{C}$ *is holomorphic and non-vanishing.*

The functions T_{jk} are known as the transition functions of the line bundle L, and Φ_{U_j} is the local trivialization over U_j.

Definition 2.3.2. *A holomorphic section of a line bundle* L *over* Σ *is a holomorphic map* $s : \Sigma \to L$ *such that* $\pi \circ s = \mathbb{I}_\Sigma$

where $\mathbb{I}_\Sigma$ is the identity map on Σ. In a local trivialization, the sections are defined by holomorphic functions. For example, let U_1 and U_2 be two patches on Σ, then a section s is given as a local holomorphic function on each patch: s_1 on U_1 and s_2 on U_2. On the overlap, these are related as follows

$$s_1 = T_{12}s_2.$$

In fact, the set of global sections of a line bundle $L \to \Sigma$ forms a vector space which we denote $H^0(\Sigma, L)$. Moreover, there are few examples of line bundles which will be central in the following discussions; these are the following

- The trivial line bundle L over Σ is simply $\Sigma \times \mathbb{C}$. If a line bundle L admits a non-vanishing global holomorphic section s, then this section gives an isomorphism between L and the trivial bundle

$$\Sigma \times \mathbb{C} \to L,$$
$$(\lambda, \eta) \to \eta s(\lambda).$$

- The canonical bundle K, also called the cotangent bundle, is the bundle whose fibres at each point $\lambda \in \Sigma$ is the cotangent space $T^*_\lambda(\Sigma)$. Sections of the canonical bundle K are holomorphic 1-forms. In fact, every Riemann surface has a canonical line bundle and the dimension of the vector space of global sections of this bundle $H^0(\Sigma, K)$ is an important invariant that is equal to the genus of the Riemann surface. Formally,

  Definition 2.3.3. *Let* Σ *be a compact Riemann surface, then the genus of* Σ *is defined to be the dimension of the vector space* $H^0(\Sigma, K)$*.*

- For each point $p \in \Sigma$, take a neighbourhood U_1 of p and a local coordinate z such that $z(p) = 0$ and let $U_2 = \Sigma \setminus \{p\}$. Then z is a holomorphic, non-vanishing function on $U_1 \cap U_2$. Therefore, we can consider $T_{12} = z$ to be a transition function that defines a line bundle L_p on Σ. This line bundle has a canonical section s_λ that has only one simple zero at λ.

- Finally, let $\Sigma = \mathbb{P}^1$ with the usual coordinate patches U_0 and U_1. Then, the transition function $T_{01} = \lambda^n$ on $U_0 \cap U_1 \cong \mathbb{C}^*$ defines a line bundle that is denoted by $\mathcal{O}(n)$. A section of this bundle is given by two functions s_0 on U_0 and s_1 on U_1 related as follows

$$s_0(\lambda) = \lambda^n s_1(\tilde{\lambda}).$$

One can then show that every section of this line bundle is given by a polynomial of the following form

$$\sum_{k=0}^{n} a_k \lambda^k,$$

and therefore, the dimension of $H^0(\mathbb{P}^1, \mathcal{O}(n))$ is $n+1$.

Recall now that we are considering matrix-valued polynomials $A(\lambda) = A_0 + ... + A_m\lambda^m$. One can then interpret this matrix in a different way. Rather than thinking about polynomials, one can regard $A(\lambda)$ as a matrix whose elements belong to the space of sections $H^0(\mathbb{P}^1, \mathcal{O}(m))$. Finally, we state some properties of line bundles.

- Given a line bundle L one can construct its dual bundle L^* whose transition functions are defined by $T_{jk}(L^*) = T_{jk}^{-1}(L)$.

- Given two line bundles L and $\tilde{L}$, one can form their tensor product such that it has the following transition function $T_{jk}(L \otimes \tilde{L}) = T_{jk}(L)T_{jk}(\tilde{L})$. Furthermore, let s be a section of L and let $\tilde{s}$ be section of $\tilde{L}$, then $s\tilde{s}$ is a section of $L \otimes \tilde{L}$.

- One can also form the homomorphism bundle $\hom(L, \tilde{L}) \cong L^* \otimes \tilde{L}$. The fibre of this line bundle at each point $\lambda \in \Sigma$ is a homomorphism between the vector spaces L_λ and $\tilde{L}_\lambda$.

Furthermore, the degree of a line bundle is defined to be the degree of the divisor of any non-zero section. That is, the degree of L is the number of times a generic section of L vanishes on the Riemann surface Σ. This degree is also equal to the first Chern class of the line bundle which we denote by $\deg(L)$. One can then prove that if $\deg(L) < 0$, then L has no non-trivial holomorphic sections. Moreover, the degree is additive with respect to direct products, that is $\deg(L \otimes \tilde{L}) = \deg(L) + \deg(\tilde{L})$.

Up until now, we have been using local descriptions in our treatment of line bundles. In order to obtain a global description of these objects, one has to utilize the sheaf theory and its cohomology groups.

Definition 2.3.4. *Let Σ be a topological space; a sheaf $\mathcal{S}$ on Σ associates to each open set $U \subset \Sigma$ an abelian group $\mathcal{S}(U)$ and a restriction map $r_V^U : \mathcal{S}(U) \to \mathcal{S}(V)$ where $V \subset U$. Furthermore, the restriction map has to satisfy the sheaf axioms given as follows*

1. *Given $W \subset V \subset U$, then $r_W^U = r_W^V \circ r_V^U$.*

2. *Given $\tau_1 \in \mathcal{S}(U)$ and $\tau_2 \in \mathcal{S}(V)$ such that $r_{U\cap V}^U(\tau_1) = r_{U\cap V}^V(\tau_2)$, then there exists $\tau_3 \in \mathcal{S}(U \cup V)$ such that $r_U^{U\cup V}(\tau_3) = \tau_1$ and $r_V^{U\cup V}(\tau_3) = \tau_2$.*

3. Given $\tau \in \mathcal{S}(U \cup V)$ such that $r_U^{U\cup V}(\tau) = 0$ and $r_V^{U\cup V}(\tau) = 0$, then $\tau = 0$.

Note that we defined a sheaf of abelian groups but one can similarly define a sheaf of vector sets, rings, vector spaces, etc. Here are some examples of sheaves that will be used later on. Let $U \subset \Sigma$, then

- $\mathcal{O}(U)$: the sheaf of holomorphic functions on U.
- $\mathcal{O}^*(U)$: the sheaf of non-vanishing holomorphic functions on U.
- $\mathcal{O}(L)(U)$: the sheaf of sections of a holomorphic line bundle L on U.

One can now form the cohomology groups of a sheaf $\mathcal{S}$ as follows. Take a locally finite covering $\{U_j\}_{j\in I}$ of Σ and let

$$\begin{aligned}
\mathcal{S}^0 &= \bigoplus_j \mathcal{S}(U_j), \\
\mathcal{S}^1 &= \bigoplus_{j_0 \neq j_1} \mathcal{S}(U_{j_0} \cap U_{j_1}), \\
\mathcal{S}^k &= \bigoplus_{j_0 \neq \ldots \neq j_k} \mathcal{S}(U_{j_0} \cap \ldots \cap U_{j_k}).
\end{aligned}$$

Finally, let $\mathcal{C}^k$ be the alternating elements in $\mathcal{S}^k$. In other words, every element in $\mathcal{S}^k$ is an element $\mathcal{C}^k$ given that for a permutation of the indices $j_1, ..., j_k$ that element is multiplied by a sign of the permutation. Furthermore, let $\delta : \mathcal{C}^k \to \mathcal{C}^{k+1}$ be the boundary operator defined by

$$(\delta f)_{j_0,\ldots,j_{k+1}} = \sum_n (-1)^n f_{j_0\ldots\hat{j}_n\ldots j_{k+1}}|_{U_{j_0}\cap\ldots\cap U_{j+1}},$$

and satisfying $\delta^2 = 0$. Then, one can define the cohomology groups as follows

Definition 2.3.5. *The $k-th$ cohomology group of the sheaf S, relative to the open covering $\{U_j\}_{j\in I}$ is defined as follows*

$$H^k(\Sigma, \mathcal{S}) = \frac{\text{ker } \ \delta : \mathcal{C}^p \to \mathcal{C}^{p+1}}{im\ \delta : \mathcal{C}^{p-1} \to \mathcal{C}^p}.$$

We consider now two crucial examples

- **Example 1**: Let L be a holomorphic line bundle over Σ and $\mathcal{S}$ be its sheaf of holomorphic sections. If $h \in \mathcal{C}^0$, then $(\delta h)_{j_1 j_2} = h_{j_1} - h_{j_2}$. Therefore, (δh) vanishes if and only if it admits a global section constructed by gluing the local sections together. That is,

  $$H^0(\Sigma, L) = ker\ \delta$$

  is the space of global holomorphic sections of L. This example justifies the notation used earlier for the vector space $H^0(\Sigma, L)$.

  The next example is actually the starting point in connecting Line bundles with the Lax pairs equation.

- **Example 2**: Let L be a line bundle with transition functions T_{jk} defined by the local trivializations Φ_j and Φ_k. Then, $T_{jk} = T_{kj}^{-1}$ and therefore, $T_{jk} \in \mathcal{C}^1$ for the sheaf $\mathcal{O}^*$. Furthermore, one can easily show that these transition functions satisfy the cocycle condition $(\delta T)_{jkl} = \mathbb{I}$ where $\mathbb{I}$ is the identity element on $U_j \cap U_k \cap U_l$. Recall now that the local trivialization is not unique and one could change Φ_j to $h_j\Phi_j$ and then T becomes $T(\delta h)$. We conclude that the transition function of two isomorphic line bundles differ by δh for some $h \in \mathcal{C}^1$. Therefore, the isomorphism classes of line bundles on a Riemann surface are given by the elements of the group $H^1(\Sigma, \mathcal{O}^*)$. In fact, one can show that this group fits into the following short exact sequence

$$0 \to \frac{H^1(\Sigma, \mathcal{O})}{H^1(\Sigma, \mathbb{Z})} \cong \frac{\mathbb{C}^g}{\mathbb{Z}^{2g}} \to H^1(\Sigma, \mathcal{O}^*) \to \mathbb{Z} \to 0. \tag{2.37}$$

The group $H^1(\Sigma, \mathcal{O}^*)$ is known as the Picard group of Σ. Note also that the first term in the sequence is topologically a complex g-dimensional torus as it is the quotient of $\mathbb{C}^g$ by a lattice. Therefore, each line bundle is characterized by an integer invariant, that is its degree. Furthermore, the equivalence classes of line bundles of some degree d is a g-dimensional torus which we denote by $\mathcal{J}^d$. In fact, all these tori are isomorphic to the Jacobian of the Riemann surface Σ. Our main goal in the next section is to show that the time evolution of the operator $A(\lambda)$ defined in (2.36) corresponds to a linear flow of some line bundle on the torus $\mathcal{J}^d$.

Finally, the following two theorems are crucial in the upcoming analysis; for the proofs see [24] [18]

Theorem 2.3.6. *Given a short exact sequence of sheaves on a Riemann surface* Σ

$$0 \to \mathcal{S}_1 \to \mathcal{S}_2 \to \mathcal{S}_3 \to 0,$$

then there exists a long exact sequence of cohomology groups on Σ *given as follows*

$$0 \to H^0(\Sigma, \mathcal{S}_1) \to H^0(\Sigma, \mathcal{S}_2) \to H^0(\Sigma, \mathcal{S}_3) \xrightarrow{\delta_0} H^1(\Sigma, \mathcal{S}_1) \to ...$$
$$... \to H^p(\Sigma, \mathcal{S}_1) \to H^p(\Sigma, \mathcal{S}_2) \to H^p(\Sigma, \mathcal{S}_3) \xrightarrow{\delta_p} H^{p+1}(\Sigma, \mathcal{S}_1) \to ...$$

Theorem 2.3.7. *(Serre Duality) Given a line bundle* L *on a compact Riemann surface* Σ*, then*

$$H^1(\Sigma, L) \cong H^0(\Sigma, L^* \otimes K)^*. \tag{2.38}$$

Furthermore, the pullback of a line bundle is defined as follows

Definition 2.3.8. *Let* $f : \tilde{\Sigma} \to \Sigma$ *be a holomorphic map between Riemann surfaces, then the pull back of a line bundle* L *on* Σ *is defined as follows*

$$f^*L = \{(z, \tilde{\lambda}) \in L \times \tilde{\Sigma} : \pi(z) = f(\tilde{\lambda})\},$$

where the transition functions of f^*L *are* $T_{jk} \circ f$ *where* T_{jk} *are the transition functions for the line bundle* L*. Moreover, the sections of* f^*L *are the holomorphic maps* $s : \tilde{\Sigma} \to L$ *such that* $\pi \circ s = f$*.*

We now have a complete description of a line bundle over a Riemann surface. The next step is then to define and classify vector bundles over a Riemann surface Σ.

Definition 2.3.9. *A rank r vector bundle over a Riemann surface Σ is a complex manifold E endowed with a holomorphic projection $\pi : E \to \Sigma$ such that the fibre of each point $\lambda \in \Sigma$ is an r-dimensional vector space. Furthermore, each $\lambda \in \Sigma$ has a neighbourhood U and a homeomorphism Φ_U such that the following diagram commutes.*

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, dashed, "\Phi_U", "\cong"'] \arrow[d, "\pi"] & U \times \mathbb{C}^r \arrow[dl] \\ U & \end{tikzcd}$$

Finally, $\Phi_{U_k} \circ \Phi_{U_j}^{-1} : U_j \times \mathbb{C}^r \to U_k \times \mathbb{C}^r$ has the following form

$$(\lambda, \eta) \to (\lambda, T_{jk}(\lambda)\eta),$$

where $T_{jk}(\lambda) : U_j \cap U_k \to GL(r, \mathbb{C})$ is a holomorphic map to the space of invertible $r \times r$ matrices.

As before, the matrices T_{jk} are called transition functions and they satisfy the following relation $T_{jk}T_{kl} = T_{jl}$, and Φ_{U_j} is a local trivialization of the vector bundle over U_j. Obviously, line bundles are rank-1 vector bundles and therefore all the properties discussed earlier of line bundles extend simply to vector bundles as follows

- Given two vector bundles E and $\tilde{E}$, one can form their direct sum $E \oplus \tilde{E}$ and their direct product $E \otimes \tilde{E}$ where $\mathrm{rk}(E \otimes \tilde{E}) = \mathrm{rk}(E)\deg(\tilde{E}) + \mathrm{rk}(\tilde{E})\deg(E)$ and $\mathrm{rk}(E \oplus \tilde{E}) = \mathrm{rk}(E) + \mathrm{rk}(\tilde{E})$.
- Given a vector bundle E, one can form its dual E^* whose transition functions are the inverses of the matrices T_{jk}.
- The highest exterior power of a vector bundle E forms a line bundle defined by $\det(E) = \wedge^r E$. This line bundle has transition functions $\det(T_{jk})$ and the degree of the vector bundle E is defined to be the degree of this line bundle.
- Finally, the Serre duality applies to vector bundles as well.

Furthermore, let $H^0(\Sigma, E)$ be the vector space of global holomorphic sections of the vector bundle E, then

Theorem 2.3.10. *(Riemann-Roch) Given a vector bundle E on a compact Riemann surface Σ of genus g, then*

$$\dim(H^0(\Sigma, E)) - \dim(H^1(\Sigma, E)) = \deg(E) + (1-g)rk(E). \tag{2.39}$$

The theorem is usually proven by induction on the the rank of E as in [28] .

In what follows, our Riemann surface will always be the projective line $\mathbb{P}^1$. Fortunately, the classification of line and vector bundles is quite simple in this case. Recall that we showed that the equivalence classes of line bundles are classified by the Picard group $H^1(\Sigma, \mathcal{O}^*)$ that fits in the short exact sequence (2.37). As mentioned before, the first term is topologically a g-dimensional torus which is trivial in this case since the genus $\mathbb{P}^1$ is zero. Therefore, the short exact sequence simplifies to

$$0 \to H^1(\Sigma, \mathcal{O}^*) \to \mathbb{Z} \to 0. \tag{2.40}$$

Hence, the Picard group is isomorphic to $\mathbb{Z}$ and line bundles are classified (up to an isomorphism) by the integers. That is, equivalence classes of line bundles on $\mathbb{P}^1$ are classified by an integer that is equal to their degree. Fortunately, the classification of vector bundles is also simple in this case [22].

Theorem 2.3.11. *(Birkhoff-Grothendieck) Let E be a rank-r holomorphic vector bundle over $\mathbb{P}^1$, then*

$$E \cong \bigoplus_{j=1}^{r} \mathcal{O}(a_j)$$

for some $a_j \in \mathbb{Z}$.

Recall that the line bundle L_p has degree one and so is isomorphic to $\mathcal{O}(1)$ and similarly $L_{p_1} \otimes ... \otimes L_{p_m} \cong \mathcal{O}(m)$. An interesting corollary is the following

Corollary 2.3.12. *A holomorphic vector bundle E on $\mathbb{P}^1$ is trivial if and only if $\deg(E) = 0$ and $H^0(\mathbb{P}^1, E(-1)) = 0$ where $E(-1) = E \otimes \mathcal{O}(-1)$.*

We can now use this information to construct vector bundles on a Riemann surface Σ from line bundles on a covering space $\tilde{\Sigma}$ as follows. Let $f : \tilde{\Sigma} \to \Sigma$ be a holomorphic map, then the degree of f is defined as follows

$$\deg(f) := \deg(f^* L_p)$$

for some $p \in \Sigma$. Therefore, a section of the line bundle $f^* L_p$ vanishes with multiplicity $\deg(f)$; that is simply the number of points in $f^{-1}(\lambda)$ if λ is a regular point in Σ. Furthermore, given a sheaf $\mathcal{S}$ on $\tilde{\Sigma}$, one can define in a canonical way the direct image of this sheaf on Σ as follows

$$(f_* \mathcal{S})(U) := \mathcal{S}(f^{-1}(U)).$$

Furthermore, one can compute the degree of E using the following proposition

Proposition 2.3.13. *Let L be a holomorphic line bundle on $\tilde{\Sigma}$ and let $\mathcal{S} = \mathcal{O}(L)$ be its sheaf of sections, then*

- *$H^0(\Sigma, f_* \mathcal{O}(L)) \cong H^0(\tilde{\Sigma}, \mathcal{O}(L))$.*
- *$f_* \mathcal{O}(L) = \mathcal{O}(E)$ where E is a rank-r vector bundle on Σ and $r = \deg(f)$.*
- *If $\tilde{E}$ is a holomorphic vector bundle on Σ, then*

$$f_* \mathcal{O}(L \otimes f^* \tilde{E}) \cong \mathcal{O}(E \otimes \tilde{E}).$$

For the proof see [23]. Finally, using the above construction, one gets a relation between the degrees of the vector bundle E and the line bundle L.

Theorem 2.3.14. *(Grothendieck-Riemann-Roch) Given the line bundle L and the vector bundle E defined above, then*

$$\deg(E) = \deg(L) + (1 - \tilde{g}) - (1 - g)\ \deg(f), \tag{2.41}$$

where $\tilde{g}$ is the genus of $\tilde{\Sigma}$ and g is the genus of Σ.

Furthermore, an immediate consequence of corollary (2.3.11) and proposition (2.3.12) is the following

Proposition 2.3.15. *Let L and E be defined as above where $\deg(E) = 0$, then E is trivial if and only if*

$$L(-1) = L \otimes f^*\mathcal{O}(-1)$$

has no nontrivial holomorphic sections.

Now consider the degree-r map $f : \Sigma \to \mathbb{P}^1$, if $\deg(L) = r + (\tilde{g} - 1)$, then $\deg(L(-1)) = (\tilde{g} - 1)$. Therefore, if $L(-1)$ has no non-zero holomorphic sections, then $E = f_*L(-1)$ is a trivial vector bundle on $\mathbb{P}^1$. Furthermore, recall that the theta divisor $\Theta \subset \mathcal{J}^{g-1}$ is the set of isomorphism classes of holomorphic line bundles with non-zero global holomorphic section. That is, the isomorphism classes of the trivial line bundle on $\mathcal{J}^{g-1}$. Then

$$H^0(\Sigma, L) = H^1(\Sigma, L) = 0$$

for the line bundle $L \in \mathcal{J}^{g-1} \setminus \Theta$. Now, let $U \subset \mathbb{P}^1$, $L(n) = L \otimes f^*\mathcal{O}(n)$, $E(n) = E \otimes \mathcal{O}(n)$ and let $\eta \in H^0(\Sigma, f^*\mathcal{O}(n))$, then multiplication by η defines a linear map

$$\eta : H^0(f^{-1}(U), L) \to H^0(f^{-1}(U), L(n)).$$

This map descends to a homomorphism of E defined by

$$A : H^0(U, E) \to H^0(U, E(n)).$$

Since E is trivial, A is globally defined and therefore

$$A : H^0(\mathbb{P}^1, E) \to H^0(\mathbb{P}^1, E(n)),$$

where $H^0(\mathbb{P}^1, E) \cong \mathbb{C}^r$ and $H^0(\mathbb{P}^1, E(r)) \cong \mathbb{C}^r \otimes H^0(\mathbb{P}^1, \mathcal{O}(n))$. Thus, A is an $r \times r$ matrix-valued holomorphic section of $\mathcal{O}(n)$. Namely

$$A(\lambda) = A_0 + A_1\lambda + ... + A_n\lambda^n \in H^0(\mathbb{P}^1, \mathcal{O}(n)). \tag{2.42}$$

Finally, we have constructed a matrix-valued polynomial from a line bundle L over the Riemann surface Σ. Our aim now is to show that to every $A(\lambda)$ there is attached an algebraic curve, called the spectral curve defined by the following equation

$$\det(\eta - A(\lambda)) = 0. \tag{2.43}$$

In fact, the eigenvalues of $A(\lambda)$ are not single-valued functions of λ since for each λ there are r solutions to the equation (2.43). Therefore, there is an r-fold cover of $\mathbb{P}^1$ over which the eigenvalues η are single-valued. This covering is the spectral curve of $A(\lambda)$ on which the eigenvalues η are single-valued functions of λ.

Let $\lambda \in \mathbb{P}^1$, and let U be a neighborhood of λ, then the preimage of U under the map f consists of r neighborhoods U_j and by proposition (2.3.12)

$$H^0(U, E) = \bigoplus_{j=1}^{r} H^0(U_j, L).$$

Then, one can choose a basis of sections $(s_1, ..., s_r)$ of E over U that correspond to a basis of sections $(\sigma_1, ..., \sigma_r)$ of L over $f^{-1}(U)$ that satisfy

$$\sigma_j|U_k = 0, \quad j \neq k.$$

Note that E is trivial and therefore s_j are vector-valued functions. Recall now that η acts on $H^0(f^{-1}(U), L)$ by multiplication and therefore, the basis of sections $(\sigma_1, ..., \sigma_r)$ satisfy the following equation

$$A\sigma_j = \eta|_{U_j}\sigma_j.$$

By continuity at every point of Σ, $A(\lambda)$ satisfies the following equation

$$\det(\eta - A(\lambda)) = 0. \tag{2.44}$$

This equation defines the spectral curve of $A(\lambda)$ mentioned earlier. Furthermore, the section η defines the following commutative diagram

$$\begin{tikzcd} \Sigma \arrow[r, "\eta"] \arrow[d, "f"] & \mathcal{O}(n) \arrow[dl, "\pi"] \\ \mathbb{P}^1 & \end{tikzcd}$$

Let S be the total space of the line $\mathcal{O}(n)$, then η embeds Σ into S and the image is given by the spectral curve equation (2.44). In conclusion, we have established the following correspondence

- The spectrum of $A(\lambda)$ is a Riemann surface Σ.
- The eigenspace of $A(\lambda)$ is a line bundle on Σ.

This correspondence is given formally by the following theorem.

Theorem 2.3.16. *Let Y be a smooth curve, and let X be the space of all $A \in H^0(\mathbb{P}^1, \mathcal{O}(r))$ with a spectral curve Y. Then $PGL(r, \mathbb{C})$ acts freely on X by conjugation and the quotient is identified with $\mathcal{J}^{g-1}(Y) \setminus \Theta$.*

Proof. This proof follows [27].

Let L be a degree $\tilde{g} - 1$ line bundle on a Riemann surface Σ that has genus $\tilde{g}$, and let f be degree$-r$ covering map $f : \Sigma \to \mathbb{P}^1$. Then the line bundle L pushes forward to a vector bundle $E = f_*L$ on $\mathbb{P}^1$. Furthermore, multiplication by $\eta : H^0(\Sigma, f^*\mathcal{O}(n))$ defines a linear map

$$\eta : H^0(f^{-1}(U), L) \to H^0(f^{-1}(U), L(n)),$$

and this map pushes forward to a homomorphism on $\mathbb{P}^1$

$$A : H^0(\mathbb{P}^1, E) \to H^0(\mathbb{P}^1, E(n)).$$

If L does not belong in the theta divisor, that is if $L \in \mathcal{J}^{g-1} \setminus \Theta$, then

$$H^1(\Sigma, L) = H^0(\Sigma, L) = 0.$$

Therefore, by the functorial properties of the direct image this is equivalent to

$$H^1(\mathbb{P}^1, E) = H^0(\mathbb{P}^1, E) = 0.$$

In other words, the bundle E is trivial and therefore by Grothendieck-Riemann-Roch

$$E = \bigoplus_{j=1}^{r} \mathcal{O}(-1),$$

and so $Hom(E, E) = Hom(\mathcal{O}^r, \mathcal{O}^r)$. Therefore, one can interpret $A \in H^0(\mathbb{P}^1, \mathcal{O}(n))$. □

2.4 The Linear Flow

One can now use the above construction to discuss the geometric nature of the Lax pairs equation given by

$$\frac{dA}{dt} = [A, B], \tag{2.45}$$

where $A(\lambda) = A_0 + A_1\lambda + ... + A_m\lambda^m$ and $B(\lambda) = B_0 + B_1\lambda + ... + B_n\lambda^n$. The Lax pairs equation implies that the spectrum of A is conserved while the corresponding eigenvectors vary with time. That is, the eigenspace bundle varies on a complex torus as time flows. Our purpose in this section is to show that if the eigenspace bundle follows a straight line on the complex torus $L_t(-1) \in \mathcal{J}^{g-1} \setminus \Theta$, then there is a basis for the trivial bundle on $\mathbb{P}^1$ such that $A(\lambda)$ and $B(\lambda)$ satisfy the Lax pairs equation. As will be shown shortly, this construction will force $B(\lambda)$ to have a specific form. The problem of realizing non-linear flows given by the Lax pairs equation as a linear flow of the eigenspace line bundle was studied using cohomological techniques in [20]. However, our approach in this section closely follows the algebraic geometric approach presented in [28].

First of all, we need a good description of S the total space of the line bundle $\mathcal{O}(n)$ over $\mathbb{P}^1$. Since S is two-dimensional complex manifold, one can use the standard patches $S = U \cup \tilde{U}$ where U and $\tilde{U}$ are both isomorphic to $\mathbb{C}^2$ and $U \cap \tilde{U} \cong \mathbb{C}^* \times \mathbb{C}$. Furthermore, let (λ, η) and $(\tilde{\lambda}, \tilde{\eta})$ be local coordinates on U and $\tilde{U}$ respectively. On the overlap, these coordinates are related as follows

$$\begin{aligned} \tilde{\lambda} &= \lambda^{-1}, \\ \tilde{\eta} &= \eta\lambda^{-n}. \end{aligned} \tag{2.46}$$

Recall now that Σ is embedded in S as follows

$$\eta(\Sigma) \subset S \quad \text{such that} \quad \det(\eta - A(\lambda)) = 0.$$

But since

$$\det(\eta - A(\lambda)) = \eta^m + a_1(\lambda)\eta^{m-1} + ... + a_m(\lambda),$$

then it is the zero set of a holomorphic section $\pi^*\mathcal{O}(mn)$ on S, where $\pi : S \to \mathbb{P}^1$ is the canonical projection of the line bundle. Therefore, there is a short exact sequence

$$0 \to \mathcal{O}(mn) \xrightarrow{\det(\eta - A(\lambda))} \mathcal{O}_S \to \mathcal{O}_\Sigma \to 0$$

that gives rise to the long exact sequence

$$H^1(S, \mathcal{O}(mn)) \xrightarrow{\det(\eta - A(\lambda))} H^1(S, \mathcal{O}) \to H^1(\Sigma, \mathcal{O}) \to 0.$$

Note that $H^2(S, L)$ vanishes for any $L \in S$ since S is covered using only two patches. Moreover, from the properties of the exact sequence, we see that $H^1(\Sigma, \mathcal{O})$ can be described as $H^1(S, \mathcal{O})$ modulo the image of $\det(\eta - A(\lambda))$. Note that $H^1(S, \mathcal{O})$ is described by holomorphic functions on $U \cap \tilde{U}$ modulo functions that extend to U and $\tilde{U}$. In other words, $H^1(S, \mathcal{O})$ is described as functions

$$\sum_{j=-\infty}^{\infty} \sum_{k=0}^{\infty} a_{jk}\eta^j\lambda^k$$

on $U \cap \tilde{U} \cong \mathbb{C}^* \times \mathbb{C}$ modulo functions

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{jk}\eta^j\lambda^k$$

on $U \cong \mathbb{C}^2$, and functions

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \tilde{a}_{jk}\tilde{\eta}^j\tilde{\lambda}^k = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \tilde{a}_{jk}\eta^j\lambda^{-nk-j}$$

on $\tilde{U} \cong \mathbb{C}^2$. Finally, $H^1(\Sigma, \mathcal{O})$ is described as $H^1(S, \mathcal{O})$ modulo the image of the equation

$$\det(\eta - A(\lambda)) = 0.$$

Hence, any class in $H^1(\Sigma, \mathcal{O})$ can be described by

$$\Upsilon(\eta, \lambda) = \sum_{j=1}^{r-1} \frac{b_j(\lambda)\eta^j}{\lambda^N}, \tag{2.47}$$

where $b_j(\lambda)$ are polynomials and $N \in \mathbb{Z}^+$. It follows that

$$\exp\{\Upsilon(\eta, \lambda)\}$$

gives a class in $H^1(\Sigma, \mathcal{O}^*)$ that corresponds to a degree zero line bundle. Moreover, one can now use the patches $\Sigma \cap U$ and $\Sigma \cap \tilde{U}$ to cover Σ. In this case, one can express any line bundle of degree d as a line bundle with transition function

$$T \exp\{\Upsilon(\eta, \lambda)\},$$

where T is the transition function of some fixed line bundle of degree d. Then making a linear variation in $H^1(\Sigma, \mathcal{O})$, we get a family of line bundles L_t that correspond to the transition functions

$$T \exp\{t\Upsilon(\eta, \lambda)\},$$

where T is independent of time.

Suppose now that we have a family of sections varying with time, each section in

$$H^0(\Sigma, L_t) = H^0(\mathbb{P}^1, E_t) = \mathbb{C}^r,$$

where E_t is the direct image of L_t. These sections will be given by functions $s(t)$ and $\tilde{s}(t)$ on U and $\tilde{U}$ respectively, and they are related as follows on the overlap $U \cap \tilde{U}$

$$s(t) = T \exp\{\Upsilon(\eta, \lambda)\}\tilde{s}(t).$$

Therefore, taking a varying family of global sections of L_t

$$\sigma_1(t), ..., \sigma_r(t),$$

then these are represented by functions s_j and $\tilde{s}_j$ that are related by

$$s_j(t) = T \exp\{t\Upsilon(\eta, \lambda)\}\tilde{s}_j(t),$$

where $j = 1, ..., r..$ Differentiating with respect to t, we get

$$\frac{\partial s_j}{\partial t} = \Upsilon s_j + T \exp\{\Upsilon(\eta, \lambda)\}\frac{\partial \tilde{s}_j}{\partial t}. \tag{2.48}$$

Now recall that η acts on these sections by multiplication as follows

$$\eta s_j = \sum_k A_{kj} s_k, \tag{2.49}$$

where A_{jk} are the components of the matrix $A(\lambda)$ given in this bases of sections. Recalling that $\Upsilon(\eta, \lambda)$ is a polynomial in η, we can write (2.48) as follows

$$\frac{\partial s_j}{\partial t} = \sum_k \Upsilon(A(\lambda), \lambda)_{kj} s_k + T \exp\{\Upsilon(\eta, \lambda)\}\frac{\partial \tilde{s}_j}{\partial t}. \tag{2.50}$$

Moreover, since $\Upsilon(A(\lambda), \lambda)$ is a finite matrix-valued Laurent series in λ, one can split it into polynomials Υ^+ in λ and Υ^- in λ^{-1}.

$$\Upsilon(A(\lambda), \lambda) = \Upsilon^+ + \Upsilon^-.$$

Note that the constant term can be put in either polynomial Υ^+ or Υ^- and it can also split between the two. Either way, this is not going to affect the following argument as will be shown shortly. Hence, (2.50) splits as follows

$$\frac{\partial s_j}{\partial t} - \sum_k \Upsilon^+_{kj} s_k = \sum_k \Upsilon^-_{kj} s_k + T \exp\{\Upsilon(\eta, \lambda)\}\frac{\partial \tilde{s}_j}{\partial t}.$$

But since

$$s_k(t) = T\exp\{\Upsilon(\eta,\lambda)\}\tilde{s}_k(t),$$

then

$$\frac{\partial s_j}{\partial t} - \sum_k \Upsilon^+_{kj} s_k = T\exp\{\Upsilon(\eta,\lambda)\}\left(\sum_k \Upsilon^-_{kj}\tilde{s}_k + \frac{\partial \tilde{s}_j}{\partial t}\right).$$

Since the L.H.S is holomorphic in λ and the R.H.S is holomorphic in λ^{-1}, the two sides defines a global section $u_j(t)$ of L_t. Recalling that we chose $\sigma_1, ..., \sigma_m$ to be a basis of global holomorphic sections of L_t, then

$$\frac{\partial s_j}{\partial t} - \sum_k \Upsilon^+_{kj} s_k = \sum_k C_{kj} s_k,$$

where C_{kj} are the components of a matrix $C(t)$ that depends holomorphically on t. Consider now (2.49), differentiating both sides with respect to time, one gets

$$\begin{aligned} \eta\frac{\partial s_j}{\partial t} &= \sum_k \left(\frac{\partial A}{\partial t}\right)_{kj} s_k + \sum_k A_{kj}\frac{\partial s_k}{\partial t}, \\ &= \sum_l \left(\frac{\partial A}{\partial t}\right)_{lj} s_l + \sum_{k,l} A_{kj}(\Upsilon^+_{lk} + C_{lk}) s_l. \end{aligned} \tag{2.51}$$

However,

$$\eta\frac{\partial s_j}{\partial t} = \eta\sum_k(\Upsilon^+_{kj} + C_{kj})s_k = \sum_{k,l}(\Upsilon^+_{kj} + C_{kj})A_{lk}s_l. \tag{2.52}$$

Combining (2.51) and (2.52), we get the following equation

$$\frac{\partial A}{\partial t} = [A, \Upsilon^+ + C(t)] \tag{2.53}$$

Now recall that we chose an arbitrary moving basis $\sigma_1(t), ..., \sigma_m(t)$ for the space global sections of L_t. Choosing another basis amounts to a change of frame and in this case $A(\lambda)$ transforms by a gauge transformation as follows

$$\hat{A} = G^{-1}AG. \tag{2.54}$$

Then

$$\frac{\partial A}{\partial t} = \frac{dG}{dt}\hat{A}G^{-1} + G\frac{\partial \hat{A}}{\partial t}G^{-1} + G\hat{A}\frac{dG^{-1}}{dt},$$

and using the fact that

$$\frac{dG^{-1}}{dt} = -G^{-1}\frac{dG}{dt}G^{-1},$$

then

$$\frac{\partial A}{\partial t} = \frac{dG}{dt}\hat{A}G^{-1} + G\frac{\partial \hat{A}}{\partial t}G^{-1} - G\hat{A}G^{-1}\frac{dG}{dt}G^{-1}.$$

At the same time, the R.H.S of the Lax pairs equation becomes

$$[G\hat{A}G^{-1}, \Upsilon^+ + C(t)].$$

One can then show that if

$$\frac{dG}{dt} = -C(t)\ G, \tag{2.55}$$

then, $\hat{A}$ satisfies the following Lax pairs equation

$$\frac{d\hat{A}}{dt} = [\hat{A}, \Upsilon^{+}(\hat{A}, \lambda)]. \tag{2.56}$$

Therefore, solving the linear equation (2.55) gives a moving frame in which $\hat{A}(\lambda)$ and $B(\lambda) = \Upsilon^{+}(\hat{A}, \lambda)$ satisfy the Lax pairs equation.

In conclusion, if we vary the line bundle L_t linearly in time, then there exists a basis for the trivial bundle $E_t \to \mathbb{P}^1$ such that $A(\lambda)$ satisfies the Lax pairs equation

$$\frac{dA}{dt} = [A, B].$$

In this case, $B(\lambda) = \Upsilon^{+}(A, \lambda)$ is the projection over the polynomial part of the quantity

$$\Upsilon(A, \lambda) = \sum_{j=1}^{r-1} \frac{b_j(\lambda) A^j}{\lambda^N}, \tag{2.57}$$

where $N \in \mathbb{Z}^{+}$ and $b_j(\lambda)$ are polynomials. Finally, note that $B(\lambda)$ is not unique. In particular, any transformation of the form $B(\lambda) \to B(\lambda) + CA^j(\lambda)$ does not alter the Lax pairs equation where C is a constant and j is an integer.

3 Euler's Equations for The Motion of a Free Rigid Body

3.1 Lax Pairs and the spectral curve

Euler's equations describes the motion of a free rigid body around its center of mass. It is one of the most extensively studied classical Hamiltonian systems that exhibit the notion of integrability [4], [32]. In three dimensional space $\mathbb{R}^3$, Euler's equation can be written in the following vector form

$$\frac{d}{dt}\boldsymbol{M} - \boldsymbol{M} \times \boldsymbol{\omega} = 0, \tag{3.1}$$

where $\mathbf{M}$ is the angular momentum vector defined by $\boldsymbol{M} = I\boldsymbol{\omega}$ such that I is the moment of inertia matrix (tensor) given by $I = diag(I_1, I_2, I_3)$ and $\boldsymbol{\omega}$ is the angular velocity vector. Note that the right-hand side is zero because the system is torque-free.

The configuration space is the space of positions the system can attain and this space is spanned by the generalized coordinates of the system. Therefore, the configuration space of the motion of a rigid body is the group of rotations $SO(3)$ because each element of $SO(3)$ describes a certain orientation of the body in space. Furthermore, the motion is described by a trajectory $t \to f(t) \in SO(3)$ and the velocity is described by an element in the tangent space $\dot{f} \in T_f SO(3)$. Recall now that $\mathfrak{so}(3)$ is the Lie algebra of $SO(3)$ whose elements are represented by skew-symmetric 3×3 matrices. The elements of this Lie algebra represent infinitesimal rotations as they are obtained as elements in the tangent space at the identity of the rotation Lie group $SO(3)$. However, the tangent space at any element $f \in SO(3)$ may be mapped to the Lie algebra $\mathfrak{so}(3)$ by left or right translation. Therefore, one may consider the angular velocities to be elements of the lie algebra $\mathfrak{so}(3)$

$$\omega \in \mathfrak{so}(3).$$

Now recall that the Lie algebra $\mathfrak{so}(3)$ can be also identified with its dual $\mathfrak{so}^*(3)$ using the Killing form. Similarly, since the inertia tensor is represented by a symmetric positive definite matrix, it defines an inner product for elements in the Lie algebra $\mathfrak{so}(3)$. Therefore, it defines a map that identifies the Lie algebra $\mathfrak{so}(3)$ with its dual

$$I : \mathfrak{so}(3) \to \mathfrak{so}(3)^*.$$

In this case, one can show that Euler's equation can be written as an Euler-Arnold equation, also known as

Euler-Poincaré equation, as follows

$$\frac{d}{dt}M = ad^*_\omega M, \tag{3.2}$$

where ad^*_ω is the co-adjoint action of the Lie algebra defined as follows

$$\langle ad^*_\omega M, \iota\rangle := \langle M, [\omega, \iota]\rangle,$$

where $\iota \in \mathfrak{so}(3)$. Finally, equation (3.2) is also equivalent to the Lax pairs equation

$$\frac{d}{dt}M = [IM, M]. \tag{3.3}$$

where I is the inertia tensor. With this background in mind, our approach to Euler's equation is different from this classical approach. In fact, we start by noting that the Lie algebra $\mathfrak{so}(3)$ is isomorphic to the Lie algebra $\mathfrak{su}(2)$ of traceless, skew-Hermitian matrices. We will then formulate the problem on $\mathfrak{su}(2)$ and define new Lax pairs that are compatible with this Lie algebra. We will then be able to relate the Lax pairs of Nahm's equations with that of Euler's equations. Therefore, rather than solving Euler's equations directly, one may transform Euler's Lax pairs to Nahm's Lax pairs, solve Nahm's equations and then pull the solution back to obtain an explicit solution to Euler's equations. This will be our technique in appraoching the problem.

In component form, Euler's equations are given as follows

$$\begin{aligned} \dot{M}_1 &= \frac{(I_2 - I_3)}{I_2 I_3} M_2 M_3, \\ \dot{M}_2 &= \frac{(I_3 - I_1)}{I_3 I_1} M_3 M_1, \\ \dot{M}_3 &= \frac{(I_1 - I_2)}{I_1 I_2} M_1 M_2. \end{aligned} \tag{3.4}$$

In order to simplify calculations, we introduce the following constants

$$\begin{aligned} a_1 &= \sqrt{\frac{I_2 - I_3}{I_2 I_3}}, \\ a_2 &= \sqrt{\frac{I_3 - I_1}{I_3 I_1}}, \\ a_3 &= \sqrt{\frac{I_1 - I_2}{I_1 I_2}}. \end{aligned} \tag{3.5}$$

Then, equations (3.4) will have the following form

$$\begin{aligned} \dot{M}_1 &= a_1^2 M_2 M_3, \\ \dot{M}_2 &= a_2^2 M_3 M_1, \\ \dot{M}_3 &= a_3^2 M_1 M_2. \end{aligned} \tag{3.6}$$

Now, to formulate the problem on the Lie algebra $\mathfrak{su}(2)$, one can choose the following matrices to be the generators of the Lie algebra $\mathfrak{su}(2)$

$$S_1 = \frac{1}{2i}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$
$$S_2 = \frac{1}{2i}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \tag{3.7}$$
$$S_3 = \frac{1}{2i}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

In fact, these generators are defined in terms of the Pauli matrices σ_j as follows

$$S_j = \frac{1}{2i}\sigma_j, \quad (j = 1, ..., 3). \tag{3.8}$$

One can then show that these matrices have the following commutation relations

$$[S_j, S_k] = \epsilon_{jkl} S_l \quad (j, k, l = 1, 2, 3). \tag{3.9}$$

where ϵ_{jkl} is the anti-symmetric Levi-Civita symbol. Furthermore, since $\sigma^2 = \mathbb{I}_2$ where $\mathbb{I}_2$ is the 2×2 identity matrix, then

$$S_j^2 = \frac{-1}{4}\mathbb{I}_2, \quad (j = 1, 2, 3).$$

Now, let $m_j = M_j S_j$, where $j = 1, ..., 3$. Then Euler's equations (3.6) can be written as follows

$$\begin{aligned} \dot{m}_1 &= a_1^2[m_2, m_3], \\ \dot{m}_2 &= a_2^2[m_3, m_1], \\ \dot{m}_3 &= a_3^2[m_1, m_2]. \end{aligned} \tag{3.10}$$

We would like now to find the specific form of the Lax pairs for Euler's equations when formulated on the Lie algebra $\mathfrak{su}(2)$. For that purpose, let

$$\begin{aligned} A(\lambda) &= \alpha m_1 + \beta m_2 + \lambda \gamma m_3 + \lambda^2(\alpha' m_1 + \beta' m_2), \\ B(\lambda) &= \delta m_3 + \lambda(\alpha' m_1 + \beta' m_2). \end{aligned} \tag{3.11}$$

and we require that this pair satisfies the Lax equation

$$\frac{d}{dt}A(\lambda) = [A(\lambda), B(\lambda)]. \tag{3.12}$$

Substituting $A(\lambda)$ and $B(\lambda)$ in (3.12) and matching the coefficients for different powers of λ we get the following system of equations

$$\begin{aligned} &\lambda^0 : \alpha \dot{m}_1 + \beta \dot{m}_2 = -\alpha\delta[m_3, m_1] + \beta\delta[m_2, m_3], \\ &\lambda^1 : \gamma \dot{m}_3 = (\alpha\beta' - \beta\alpha')[m_1, m_2], \\ &\lambda^2 : \alpha' \dot{m}_1 + \beta' \dot{m}_2 = \alpha'(\gamma - \delta)[m_3, m_1] + \beta'(\delta - \gamma)[m_2, m_3]. \end{aligned} \tag{3.13}$$

These equations can now be solved for the coefficients $\alpha, \beta, \gamma, \delta, \alpha'$, and β'. Evidently, the choice of these coefficients is not unique. However, one has to make sure that any choice made is consistent with Euler's equations (3.10). Solving this system of equations, we get the following expressions for constants

$$\begin{aligned}\alpha &= -\alpha' = ia_2a_3, \\ \beta &= \beta' = a_1a_3, \\ \gamma &= 2\delta = 2ia_1a_2.\end{aligned} \tag{3.14}$$

Hence, $A(\lambda)$ has the following form

$$\begin{aligned}A(\lambda) &= \alpha(1-\lambda^2)m_1 + \beta(1+\lambda^2)m_2 + \lambda\gamma m_3, \\ &= \frac{1}{2i}\begin{bmatrix} \gamma M_3\lambda & \alpha(1-\lambda^2)M_1 - i\beta(1+\lambda^2)M_2 \\ \alpha(1-\lambda^2)M_1 + i\beta(1+\lambda^2)M_2 & -\gamma M_3\lambda \end{bmatrix}.\end{aligned} \tag{3.15}$$

Note that since we formulated the problem on the Lie algebra $\mathfrak{su}(2)$, $A(\lambda)$ is traceless and skew-hermitian as expected. Furthermore, since $A(\lambda)$ is quadratic in λ, then $A(\lambda) \in H^0(\mathbb{P}^1, \mathcal{O}(2)) \otimes \mathfrak{su}(2)$. Since we have the explicit form of the matrix $A(\lambda)$, we can now study the spectral curve associated to Euler's equations, namely,

$$det(\eta - A(\lambda)) = 0. \tag{3.16}$$

Since $A(\lambda)$ is traceless, this equation simplifies to the following form $\eta^2 = -det(A)$, therefore

$$\eta^2 = b_0(1+\lambda^4) + b_1\lambda^2 \tag{3.17}$$

where

$$\begin{aligned}b_0 &= \frac{-1}{4}\left[\alpha^2M_1^2 + \beta^2M_2^2\right], \\ &= \frac{1}{4}\left[a_2^2a_3^2M_1^2 - a_1^2a_3^2M_2^2\right], \\ b_1 &= \frac{1}{2}\left[\alpha^2M_1^2 - \beta^2M_2^2 - 2\delta^2M_3^2\right], \\ &= \frac{1}{2}\left[2a_1^2a_2^2M_3^2 - a_2^2a_3^2M_1^2 - a_1^2a_3^2M_2^2\right].\end{aligned} \tag{3.18}$$

By Riemann-Hurwitz formula, the spectral curve has genus one. Consequently, this spectral curve Y is an elliptic curve and therefore is a double cover of the Riemann sphere $\mathbb{P}^1$. As mentioned before, the eigenvalues η of $A(\lambda)$ are single-valued functions of λ on this cover. Furthermore, the spectral curve Y is isomorphic to the Jacobian $\mathcal{J}(Y)$ and the theta divisor is just a single point in $\mathcal{J}(Y)$ which we may take to be the origin. Finally, the coefficients b_0 and b_1 of the spectral curve are constants of motion

$$\frac{d}{dt}\,b_0 = \frac{d}{dt}\,b_1 = 0.$$

In fact, by a simple computation one can use Euler's equations verify this explicitly in this case. Now, one can ask an interesting question. Since these coefficients are time-invariant, they should be related in some way to the constants of motion of the system. Since the system is torque-free, the kinetic energy and the angular momentum are conserved quantities defined as follows

$$\begin{aligned} E &= \frac{1}{2}\left(\frac{M_1^2}{I_1} + \frac{M_2^2}{I_2} + \frac{M_3^2}{I_3}\right), \\ M^2 &= M_1^2 + M_2^2 + M_3^2. \end{aligned} \tag{3.19}$$

As one would expect, b_0 and b_1 can be expressed as linear combinations of these two constants of motion. Namely, one can show the following

$$\begin{aligned} b_0 &= \frac{a_3^2}{4}\left[E - \frac{M^2}{I_3}\right], \\ b_1 &= \left[\frac{a_1^2}{I_1} - \frac{a_2^2}{I_2}\right]\frac{M^2}{2} - \left[a_1^2 + a_3^2\right]\frac{E}{2}. \end{aligned} \tag{3.20}$$

3.2 The linear flow

Before attempting to solve Euler's equations explicitly, we might ask whether the problem can be linearized using the scheme of the last chapter or not. Recall that we showed in the last chapter that the flow of the eigenspace line bundle L_t for some $A(\lambda)$ linearizes on the Jacobian $J(Y)$ if $B(\lambda)$ has the specific form

$$\Upsilon^+(A, \lambda),$$

where $\Upsilon(\eta, \lambda)$ is given by (2.47). In the present case, $r = 2$ so that

$$\Upsilon(A, \lambda) = \frac{b(\lambda)A(\lambda)}{\lambda^N}.$$

Let $N = 1$, and $b(\lambda) = 1$, then

$$\Upsilon(A, \lambda) = \frac{A(\lambda)}{\lambda} = \frac{\alpha m_1 + \beta m_2}{\lambda} + \gamma m_3 + \lambda(\alpha' m_1 + \beta' m_2),$$

and therefore

$$\Upsilon(A, \lambda)^+ = B(\lambda) = \delta m_3 + \lambda(\alpha' m_1 + \beta' m_2).$$

Hence, the flow of the line bundle $t \mapsto L_t$ for Euler's equations linearizes on the jacobian $J(Y)$. The next goal is now to integrate the problem explicitly. For that purpose, we take a detour to solve Nahm's equations and then we will utilize this solution to solve Euler's equations.

3.3 Nahm's equations

Nahm's equations originated in the study of monopoles and since then, they have played a crucial role in several parts in geometry and physics [39] [25]. These equations can be obtained as a dimensional reduction of the self-dual Yang-Mills equations over a 4-dimensional manifold by imposing translational symmetry in three spatial directions. More specifically, consider the space $\mathbb{R}^4$ with a positive Euclidean metric $\sum_{j=1}^{4} dx_j^2$ and then apply a 3-dimensional group of translations

$$(x_0, x_1, x_2, x_3) \to (x_0, x_1 + c_1, x_2 + c_2, x_3 + c_3).$$

Then, a translation-invariant connection gives three Higgs fields T_1, T_2, T_3 that satisfy the Nahm's equations

$$\begin{aligned} \frac{dT_1}{dt} &= [T_2, T_3], \\ \frac{dT_2}{dt} &= [T_3, T_1], \\ \frac{dT_3}{dt} &= [T_1, T_2]. \end{aligned} \tag{3.21}$$

For more details about this reduction process, one may consult [33], . Our strategy here will be to transform the Lax pairs of Euler's equations to that of Nahm's equations, then explicitly solve Nahm's equations in terms of theta functions. Finally, we will be able then to pull the solution back to obtain an explicit solution for Euler's equations. Furthermore, the solution method of Nahm's equations presented here closely follows Hitchin's work [27] with some minor adaptations. For our purpose, we consider Nahm's equations for which $T_j \in \mathfrak{sl}(2, \mathbb{C})$. One can now introduce $A(\lambda) \in H^0(\mathbb{P}^1, \mathcal{O}(2))$ given by

$$A(\lambda) = (T_1 + iT_2) + 2T_3\lambda - (T_1 - iT_2)\lambda^2. \tag{3.22}$$

By theorem (2.3.16), $A(\lambda)$ corresponds to a the spectral curve $Y \subset \mathcal{O}(2)$ defined by $det(\eta - A(\lambda)) = 0$ and a line bundle $L_t(-1) \in \mathcal{J}(Y)/\Theta$. Then, Letting $B(\lambda) = A(\lambda)/\lambda$ and projecting over the polynomial part one gets

$$B(\lambda) = \left(\frac{A(\lambda)}{\lambda}\right)^+ = T_3 - (T_1 - iT_2)\lambda.$$

Then, one can explicitly show that $A(\lambda)$, and $B(\lambda)$ satisfy the Lax pairs equation. Furthermore, since $B(\lambda)$ has the form $\Upsilon^+(A, \lambda)$, then the flow of $A(\lambda)$ corresponds to a linear flow of the line bundle L_t on the Jacobian $\mathcal{J}(Y)$ where Y is defined by the following equation

$$\eta^2 = -det(A) = \frac{1}{2}Tr(A^2) = \sum_{j=0}^{4} a_j \lambda^j, \tag{3.23}$$

where the coefficients of the spectral curve a_j are invariants of motion given explicitly as follows

$$\begin{aligned} a_0 &= \frac{1}{2}\,[Tr(T_1^2) - Tr(T_2^2) + 2iTr(T_1T_2)], \\ a_1 &= 2\left[Tr(T_1T_3) + iTr(T_2T_3)\right], \\ a_2 &= \ [2Tr(T_3)^2 - Tr(T_1^2) - Tr(T_2)^2], \\ a_3 &= 2\left[iTr(T_2T_3) - Tr(T_1T_3)\right], \\ a_4 &= \frac{1}{2}\,[Tr(T_1^2) - Tr(T_2^2) - 2iTr(T_1T_2)]. \end{aligned} \tag{3.24}$$

These can also be written as follows

$$
\begin{aligned}
a_1 + a_3 &= 4i\ Tr(T_2 T_3), \\
a_1 - a_3 &= 4\ Tr(T_1 T_3), \\
a_0 + a_4 &= Tr(T_1^2) - Tr(T_2^2), \\
a_0 - a_4 &= 2i\ Tr(T_1 T_2), \\
a_2 &= \ [2Tr(T_3)^2 - Tr(T_1^2) - Tr(T_2)^2].
\end{aligned}
\tag{3.25}
$$

Since the L.H.S of each equation is an invariant of motion, therefore the right hand side of each of these equations is also an invariant of motion. These are the holomorphic functions on the Jacobian $\mathcal{J}(Y)$. Obviously, the functions $Tr(T_j^2)$ where $j = 1, ..., 3$ are not in the space spanned by these functions. Therefore, these are not constant functions on the Jacobian. Recalling that the only holomorphic functions on a compact, connected manifold are the constants, we conclude that these functions are meromorphic with poles along the theta-divisor. In fact, this conclusion is also a direct consequence of theorem (2.3.16) since every $A(\lambda)$ acquires a pole on Θ. Our goal now is to examine the nature of these poles. In fact, it is shown in [25] that at $t = 0$ as well as at any point of Θ, each $T_j(t)$ acquires a simple pole. In the present case, Θ is a single point which we take to be the origin $t = 0$. Therefore, in a neighborhood of $t = 0$, the triple $T_j(t)$ are given as follows

$$
T_j(t) = \frac{\chi_j}{t} + \zeta_j + \tau_j t + ... \quad (j = 1, ..., 3). \tag{3.26}
$$

where $\chi_j, \zeta_j, \tau_j \in \mathfrak{sl}(2, \mathbb{C})$. Then, imposing Nahm's equations (3.21) on this representation, we get the following relations

$$
\begin{aligned}
[\chi_2, \chi_3] &= -\chi_1, \\
[\chi_2, \zeta_3] + [\zeta_2, \chi_3] &= 0, \\
[\tau_2, \chi_3] + [\zeta_2, \zeta_3] + [\chi_2, \tau_3] &= \tau_1,
\end{aligned}
\tag{3.27}
$$

and the same expressions hold for any cyclic permutation of the indices. The first equation suggests that the matrices χ_j can be taken to be the generators of the Lie algebra $\mathfrak{su}(2)$. This is in fact true up to conjugacy as was shown in [25]. Therefore, up to conjugacy, these matrices can taken to be the matrices S_j defined in (3.7). But since

$$
S_j^2 = -\frac{1}{4}\mathbb{I}_{2\times 2}, \quad (j = 1, 2, 3),
$$

then

$$
Tr(T_j^2)(t) = \frac{-1}{2t^2} + \frac{2}{t}\ Tr(\chi_j \zeta_j) + 2\ Tr(\chi_j \tau_j) + ... \quad (j = 1, 2, 3). \tag{3.28}
$$

Our purpose is to show that $\zeta_j = 0$, and therefore the residue in the previous formula vanishes. Consider the second relation in (3.27), it implies that

$$
[\chi_1, [\chi_2, \zeta_3]] = [\chi_1, [\chi_3, \zeta_2]]. \tag{3.29}
$$

Factoring the L.H.S through Jacobi identity

$$\begin{aligned}[\chi_1,[\chi_2,\zeta_3]] &= -[\zeta_3,[\chi_1,\chi_2]] - [\chi_2,[\zeta_3,\chi_1]], \\ &= [\zeta_3,\chi_3] + [\chi_2,[\chi_1,\zeta_3]].\end{aligned} \tag{3.30}$$

Similarly

$$[\chi_1,[\chi_3,\zeta_2]] = -[\zeta_2,\chi_2] - [\chi_3,[\zeta_2,\chi_1]]. \tag{3.31}$$

Thus, from (3.29)

$$\begin{aligned}[\zeta_3,\chi_3] + [\zeta_2,\chi_2] &= -[\chi_2,[\chi_1,\zeta_3]] - [\chi_3,[\zeta_2,\chi_1]], \\ &= [\chi_2,[\zeta_1,\chi_3]] + [\chi_3,[\chi_2,\zeta_1]], \\ &= -[\zeta_1,\chi_1].\end{aligned}$$

Therefore,

$$\sum_{j=1}^{3}[\zeta_j,\chi_j] = 0. \tag{3.32}$$

As a consequence,

$$[\chi_1,[\chi_1,\zeta_1]] + [\chi_1,[\chi_2,\zeta_2]] + [\chi_1,[\chi_3,\zeta_3]] = 0. \tag{3.33}$$

Again, factoring the last two terms through Jacobi identity and cancelling terms through the relation (3.27) we get

$$[\chi_1,[\chi_1,\zeta_1]] + [\chi_2,[\chi_2,\zeta_1]] + [\chi_3,[\chi_3,\zeta_1]] = 0.$$

In other words,

$$((ad\chi_1)^2 + (ad\chi_2)^2 + (ad\chi_3)^2)\zeta_1 = 0, \tag{3.34}$$

and same equation holds for ζ_2 and ζ_3. Now, recall that these matrices are defined up to conjugacy. Furthermore, recall that for the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ one can always choose a basis for the Lie algebra such that the adjoint representation of χ_1 is diagonal while the adjoint representations for χ_2 and χ_3 are nilpotent. Since $\mathfrak{sl}(2,\mathbb{C})$ is the complexification of the Lie algebra $\mathfrak{su}(2)$, the same is true for the Lie algebra $\mathfrak{su}(2)$ and one can always choose a basis χ_1, χ_2 and χ_3 such that $ad(\chi_1)$ is diagonal while

$$(ad\chi_2)^2 = (ad\chi_3)^2 = 0.$$

Therefore, equation (3.34) simplifies to

$$(ad\chi_1)^2\zeta_1 = 0. \tag{3.35}$$

Therefore, ζ_1 commutes with χ_1. However, since equation (3.34) is also true for ζ_2 and ζ_3, we conclude that

$$[\chi_1,\zeta_j] = 0, \quad (j = 1,2,3).$$

Consider now permutations of the second equation in (3.27)

$$\begin{aligned}[\chi_3,\zeta_1] + [\zeta_3,\chi_1] &= 0, \\ [\chi_1,\zeta_2] + [\zeta_1,\chi_2] &= 0.\end{aligned}$$

Since $[\zeta_3, \chi_1] = 0$ and $[\chi_1, \zeta_2] = 0$, then $[\chi_3, \zeta_1] = 0$ and $[\zeta_1, \chi_2] = 0$. Therefore, we conclude that

$$[\zeta_1, \chi_j] = 0, \quad (j = 1, 2, 3).$$

One can also do the same calculation for ζ_2 and ζ_3. Therefore

$$[\chi_j, \zeta_k] = 0, \quad (j, k = 1, ..., 3).$$

However, one can show explicitly that this equation implies that $\zeta_j = 0$ for each j. Therefore, we conclude that

$$\zeta_j = 0, \quad (j = 1, 2, 3).$$

Hence,

$$Tr(T_j^2) = \frac{-1}{2t^2} + 2\ Tr(\chi_j \tau_j) + ... \quad (j = 1, 2, 3). \tag{3.36}$$

Therefore, $Tr(T_j^2)$ are meromorphic functions on the Jacobian $\mathcal{J}(Y)$ with vanishing residues and double poles at Θ.

3.4 Solving Euler's Equations

Now we utilize the obtained solution to Nahm's equations to obtatin an explicit solution to Euler's equations. Recall now from chapter 2 that the Weierstrass $\wp$ function is a doubly periodic meromorphic function defined on the Jacobian of a Riemann surface and has the following expansion in a neighborhood of $z = 0$

$$\wp(z) = \frac{1}{z^2} + \frac{g_2 z^2}{20} + \frac{g_3 z^4}{28} + ...,$$

where g_2 and g_3 are constants defined as follows

$$g_2 = 60 \sum_{m^2+n^2 \neq 0} \frac{1}{(2m\omega + 2n\omega')^4},$$
$$g_3 = 140 \sum_{m^2+n^2 \neq 0} \frac{1}{(2m\omega + 2n\omega')^6},$$

and ω and ω' are the periods of $\mathcal{J}(Y)$. Let $z = \kappa t$ for some constant κ, then

$$\wp(\kappa t) = \frac{1}{\kappa^2 t^2} + \frac{g_2 \kappa^2 t^2}{20} + \frac{g_3 \kappa^4 t^4}{28} + ...$$

Note that the principal parts of $-\frac{\kappa^2}{2}\wp(\kappa t)$ and $Tr(T_j^2)(t)$ coincide. Therefore,

$$Tr(T_j^2) + \frac{\kappa^2}{2}\wp(\kappa t) \quad (j = 1, 2, 3)$$

is a holomorphic function on $\mathcal{J}(Y)$ and therefore is a constant. We conclude that

$$Tr(T_j^2) = -\frac{\kappa^2}{2}\wp(\kappa t) + C_j, \quad (j = 1, 2, 3), \tag{3.37}$$

for some constants C_j.

Now consider the Lax pairs of Euler's equation (3.11) and that of Nahm's equation (3.22). These two are related by the following transformation

$$T_1 = \alpha m_1, \quad T_2 = -i\beta m_2, \quad T_3 = \delta m_3. \tag{3.38}$$

Therefore,

$$\begin{aligned} Tr(T_1^2) &= \alpha^2 M_1^2 Tr(S_1^2) = \frac{1}{2} a_2^2 a_3^2 M_1^2, \\ Tr(T_2^2) &= -\beta^2 M_2^2 Tr(S_2^2) = \frac{1}{2} a_1^2 a_3^2 M_2^2, \\ Tr(T_3^2) &= \delta^2 M_3^2 Tr(S_3^2) = \frac{1}{2} a_1^2 a_2^2 M_3^2. \end{aligned} \tag{3.39}$$

In conclusion, the solution to Euler's in a neighbourhood of the point $t = 0$ is given as follows

$$\begin{aligned} a_2^2 a_3^2 M_1^2 &= -\kappa^2 \wp(\kappa t) + C_1, \\ a_1^2 a_3^2 M_2^2 &= -\kappa^2 \wp(\kappa t) + C_2, \\ a_1^2 a_2^2 M_3^2 &= -\kappa^2 \wp(\kappa t) + C_3. \end{aligned} \tag{3.40}$$

Few crucial points should be noted here. First of all, in the transition from Euler's equations to Nahm's equations we switched from the Lie algebra $\mathfrak{su}(2)$ to the Lie algebra $sl(2, \mathbb{C})$. However, $\mathfrak{sl}(2, \mathbb{C})$ can be obtained from $\mathfrak{su}(2)$ by complexification, that is

$$\mathfrak{su}(2) \oplus i\mathfrak{su}(2) = \mathfrak{sl}(2, \mathbb{C}),$$

so there is no harm in this transition. Furthermore, since we have set $z = \kappa t$, therefore κ is in general a complex parameter in this solution. Finally, the solution is local and singular at $t = 0$ since we chose the theta-divisor to occur at $z = 0$. However, we could choose another local coordinate in order to translate the theta-divisor to some point other than $z = 0$. In this case, the singularity will also be translated to another moment in time $t \neq 0$. However, we can not get rid of this singularity since the solution has to become singular once the eigenspace bundle hits the theta divisor. However, there is no harm in this singularity since, as mentioned in theorem (2.3.16), the solution is only defined on $\mathcal{J}(Y) \setminus \Theta$.

3.5 Alternative Solution

In the last section, we showed one can obtain an explicit solution to Euler's equations by transforming them into Nahm's equation. In this section, we provide another solution by integrating the equations explicitly. Namely, We will exploit the constants of motion M^2 and E defined earlier in order to express the angular velocities ω_j in terms of the Jacobi elliptic function $sn(x, k)$. This solution is then a manifestation of the fact that the integrals of motion reduce the degrees of freedom of the system and therefore enable us to find the trajectory of motion. In the present case, the configuration space is three dimensional and we have three constants of motion. Therefore, utilizing these two constants reduce the configuration space of the rigid body to one dimension. Hence, one can explicitly find the trajectory for the element $t \to f(t) \in SO(3)$ representing the motion of the rigid body.

The Jacobi elliptic sine function is defined implicitly as follows

$$x = \int_0^{sn(x,k)} \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}} \quad (k^2 \leq 1) \tag{3.41}$$

Therefore, the function $sn(x, k)$ is the inverse of the elliptic integral of the first kind. Now, consider the two constants of motion

$$\begin{aligned} M^2 &= M_1^2 + M_2^2 + M_3^2, \\ E &= \frac{1}{2}\left(\frac{M_1^2}{I_1} + \frac{M_2^2}{I_2} + \frac{M_3^2}{I_3}\right). \end{aligned} \tag{3.42}$$

One can then use (3.42) to express M_2^2 and M_3^2 in terms of M_1^2. After some calculation, one gets the following formulas

$$\begin{aligned} M_2^2 &= \frac{1}{a_1^2}\left[a_3^2 M_1^2 + 2E - \frac{M^2}{I_2}\right] \\ M_3^2 &= \frac{1}{a_1^2}\left[\frac{M^2}{I_3} - 2E + a_2^2 M_1^2\right] \end{aligned} \tag{3.43}$$

where a_1, a_2 and a_3 are the constants given by (3.5). Consider now the first equation in (3.6); squaring both sides one gets

$$\dot{M}_1^2 = a_1^4 M_2^2 M_3^2,$$

then substituting (3.43) into this equation we get

$$\dot{M}_1^2 = b_0 + b_1 M_1^2 + b_2 M_1^4, \tag{3.44}$$

where the real constants b_0, b_1, and b_2 are given as follows

$$\begin{aligned} b_0 &= 2EM^2\left(\frac{1}{I_3} + \frac{1}{I_2}\right) - 4E^2 - \frac{M^4}{I_2 I_3}, \\ b_1 &= M^2\left(\frac{a_3^2}{I_3} - \frac{a_2^2}{I_2}\right) + 2E(a_2^2 - a_3^2), \\ b_2 &= a_2^2 a_3^2. \end{aligned} \tag{3.45}$$

Now, define the two roots of the polynomial in the R.H.S of (3.44) as follows

$$r_\pm = \frac{-b_1 \pm \sqrt{b_1^2 - 4b_0 b_2}}{2b_2}. \tag{3.46}$$

Then, one can write (3.44) as follows

$$\begin{aligned} \int_0^t dt &= \int_0^{M_1(t)} \frac{dy}{\sqrt{b_0 + b_1 y^2 + b_2 y^4}}, \\ t &= \frac{1}{\sqrt{r_+}} \int_0^{M_1(t)} \frac{d(y/\sqrt{r_-})}{\sqrt{(y^2/r_+ - 1)(y^2/r_- - 1)}}. \end{aligned} \tag{3.47}$$

Now let $u = y/\sqrt{r_-}$ and rearrange, then

$$t\sqrt{r_+} = \int_0^{M_1(t)} \frac{du}{\sqrt{(1-u^2)(1-(r_-/r_+)u^2)}}. \tag{3.48}$$

Comparing this with the Jacobi elliptic function defined in equation (3.41), we see that

$$M_1(t) = sn(t\sqrt{r_+}, \sqrt{r_-/r_+}). \tag{3.49}$$

One can also repeat the same calculation for $M_2(t)$ and $M_3(t)$ to obtain similar formulas. Alternatively, substituting the obtained solution in (3.43), we get explicit solutions for $M_2(t)$ and $M_3(t)$.

Note that $sn(x,k)$ is an odd function in x. Therefore, $M_1(t)$ is an odd function in t as we may expect. In other words, applying a time-reversal transformation $t \to -t$ will transform $M_1(t) \to -M_1(t)$. On the other hand, $\wp(\kappa t)$ is an even function in t, and therefore $M_1^2(t)$ is an even function in t and $M_1^2(t)$ is invariant under time-reversal transformations. Therefore, these results are consistent with our physical intuition regarding how these classical observables transform when acted on by time-reversal transformations.

4 The Moduli Space of Higgs Bundles

4.1 Introduction

In classification problems, one usually has a collection of objects and an equivalence relation between them and the goal is to describe equivalence classes of such objects. Usually, there exists discrete invariants that partition these equivalence classes into a collection of dissociated subsets. Classification problems in geometry are usually studied by introducing the notion of a moduli space; this is a space whose points represent equivalence classes of certain objects. In this chapter, our goal is to present a quick review of the moduli space of Higgs bundles over a compact Riemann surface Σ. A Higgs bundle over a Riemann surface Σ is a pair (E, A) consisting of a holomorphic vector bundle E and a twisted $End(E)$-valued holomorphic 1-form called the Higgs field A. Higgs bundles first appeared in Hitchin's study of the self-duality equations over a Riemann surface [26]. They also appeared in the PhD thesis of Carlos Simpson and his subsequent work on non-Abelian Hodge theory [50]. Since then, Higgs bundles have held a privileged position as they arise at the crossroads of many areas in mathematics and physics. For example, they play a crucial role in gauge theory, Kähler geometry, integrable systems, mirror symmetry, Laglands duality, and even supersymmetric quantum field theories. For a review of the moduli space of Higgs bundles and some of its old and recent applications see [1], [9], [47], [49], [51]. In order to classify Higgs bundles and study their moduli space, it would be more convenient to start by classifying Hermitian vector bundles over a Riemann surface Σ. Recall that holomorphic vector bundles over $\mathbb{P}^1$ are classified by the Birkhoff-Grothendieck theorem. Furthermore, holomorphic vector bundles are classified in the genus one case by the work of Atiyah [3]. However, classifying isomorphism classes of holomorphic vector bundles over a Riemann surface with higher genus is a much more subtle endeavor. Here we assume that Σ is a Riemann surface of genus $g \geq 2$. Furthermore, we will study holomorphic vector bundles from the prespective of the $\bar{\partial}_E$ operators. The advantage of adopting this approach is that the transition from the moduli space of vector bundles to the moduli space of Higgs bundles will be smoother and more comprehensible. As usual, we will omit the proofs of most theorems presented here and refer to the relevant references accordingly. For full treatment, one may consult [29], [52].

4.2 Hermitian Vector Bundles

We start here by introducing the basic notions and theorems of Hermitian vector bundles using the same notation as in chapter 1. However, for the purpose of this chapter, a holomorphic vector bundle is a pair

$(\mathbb{E}, \bar{\partial}_E)$ where $\mathbb{E}$ is a smooth vector bundle and the operator $\bar{\partial}_E$ defines the holomorphic structure on $\mathbb{E}$ as will be shown shortly. A holomorphic vector bundle is then $E = (\mathbb{E}, \bar{\partial}_E)$ and we use both notations interchangeably throughout the chapter and it should be clear that both E and $(\mathbb{E}, \bar{\partial}_E)$ refer to the same thing.

Definition 4.2.1. *Let $E \to \Sigma$ be a rank-r holomorphic vector bundle over a Riemann surface Σ, and let U be an open set of Σ. Then a frame of E over U is a set of r sections $\mathcal{F} = (s_1, ..., s_r)$ such that $(s_1(\lambda), ..., s_r(\lambda))$ are linearly independent and therefore form a basis of the fibre $\pi^{-1}(\lambda)$ for every $\lambda \in U$.*

Note that we mentioned a change of frame informally when we changed the basis of global sections to transform equation (2.53) to be of autonomous Lax pairs form. Now recall that any holomorphic vector bundle E is locally trivial and let Φ_U be a local trivialization over U such that

$$\Phi_U : E|_U \xrightarrow{\cong} U \times \mathbb{C}^r.$$

Then, this map induces a map on the sections of E given by

$$\Phi_{U*} : H^0(U, E) \xrightarrow{\cong} H^0(U, U \times \mathbb{C}^r).$$

Now let $(e_j)_k = \delta_{jk}$ where $j, k = 1, ..., r$ be the standard basis of the vector space $\mathbb{C}^r$. This basis forms a constant frame on the trivial bundle $U \times \mathbb{C}^r$. Therefore, $\Phi_{U*}^{-1}(e_j)$ forms a frame of $E|_U$. Thus, having a frame is equivalent to having a local trivialization over E and therefore having a global frame over all of Σ is equivalent to the vector bundle E being trivial. Furthermore, a change of frame amounts to a holomorphic mapping

$$G : U \to GL(r, \mathbb{C}),$$

acting on a frame $\mathcal{F}$ to give a new frame $\mathcal{F}'$. Usually, a change of frame G is known as a gauge transformation. Moreover, given any two frames $\mathcal{F}, \mathcal{F}'$ over a subset U, one can find a gauge transformation defined on U such that $\mathcal{F}' = G\mathcal{F}$. The advantage of studying vector bundles in terms of frames is that this approach enables us to give a local matrix representation of every geometric object in hand. For example, a section $s \in H^0(U, E)$ has the following representation in terms of a local frame $\mathcal{F} = (e_1, ..., e_r)$

$$s = \sum_{j=1}^{r} s^j(\mathcal{F})\ e_j,$$

where $s^j(\mathcal{F})$ are holomorphic functions on U that are uniquely determined by $\mathcal{F}$. Therefore, a representation of a section of E with respect to a frame $\mathcal{F}$ is simply a vector-valued function. Furthermore, changing a frame by a gauge transformation G transforms a section as follows

$$s(\mathcal{F}') = G^{-1} s(\mathcal{F}), \tag{4.1}$$

where $s(\mathcal{F})$ and $s(\mathcal{F}')$ are the vector representations of the given section in the old and new frame respectively.

Definition 4.2.2. *Let $E \to \Sigma$ be a holomorphic vector bundle. A hermitian metric h on E is a holomorphic assignment of a Hermitian inner product $\langle\ ,\ \rangle$ to each fibre E_λ of E. Given an open set $U \in \Sigma$ and $s_1, s_2 \in H^0(U, E)$, then*

$$\begin{aligned} &\langle s_1, s_2 \rangle : U \to \mathbb{C}, \\ &\langle s_1, s_2 \rangle(\lambda) = \langle s_1(\lambda), s_2(\lambda) \rangle. \end{aligned} \tag{4.2}$$

If a holomorphic vector bundle E is endowed with a Hermitian metric h, we call it a Hermitian vector bundle. Moreover, given a frame on E, then a Hermitian metric can be represented locally by an $r \times r$ positive-definite, Hermitian matrix over each open set $U \subset \Sigma$ as follows

$$h(\mathcal{F})_{jk} = \langle e_j, e_k \rangle, \quad (j, k = 1, ..., r).$$

In terms of this local representation, the inner product is defined as follows

$$\langle s_1, s_2 \rangle = s_1^\dagger h(\mathcal{F}) s_2, \tag{4.3}$$

where $s_1, s_2 \in H^0(U, E)$. Finally, applying a gauge transformation G from a frame $\mathcal{F}$ to a frame $\mathcal{F}'$ transforms the metric h as follows

$$h(\mathcal{F}') = G^\dagger h(\mathcal{F}) G. \tag{4.4}$$

Finally, one can prove the following [52]

Theorem 4.2.3. *Every holomorphic vector bundle $E \to \Sigma$ admits a Hermitian metric.*

Consider now E-valued holomorphic n-forms; these are holomorphic sections of the vector bundle $\wedge^n K \otimes E$. That is, let Ω_E be an E-valued holomorphic n-form, then

$$\Omega_E \in H^0(\Sigma, (\wedge^n K) \otimes E),$$

where $\wedge^n K$ is the anti-symmetric nth tensor product of the cotangent bundle. Using this definition, we may consider the holomorphic n-forms to be sections of same bundle when E is trivial. In other words, differential n-forms defined in chapter two are actually sections of the bundle $\wedge^n K$. One can use these E-valued forms to define a connection over the vector bundle E as follows

Definition 4.2.4. *Let $E \to \Sigma$ be a vector bundle. Then a connection $\mathcal{D}$ on E is a $\mathbb{C}$-linear mapping*

$$\mathcal{D} : H^0(\Sigma, E) \to H^0(\Sigma, E \otimes K) \tag{4.5}$$

such that

$$\mathcal{D}(\Omega\varphi) = (d\Omega)\ \varphi + \Omega\ \mathcal{D}\varphi, \tag{4.6}$$

where $\Omega \in H^0(\Sigma, E)$ and $\varphi \in H^0(\Sigma, E \otimes K)$.

For simplicity, we let $H^0(\Sigma, E) = \Lambda^0(\Sigma, E)$ and $H^0(\Sigma, \wedge^n K \otimes E) = \Lambda^n(\Sigma, E)$. Obviously, when the vector bundle E is trivial, then a connection simplifies to the exterior derivative

$$d : \Lambda^0(\Sigma) \to \Lambda^1(\Sigma).$$

Therefore, a connection is an extension of the exterior derivative to vector-valued differential forms. Furthermore, given a frame $\mathcal{F}$ on $U \subset \Sigma$, a connection can be represented locally by a 1-form-valued $r \times r$ matrix, which we denote by $D(\mathcal{F})$, defined as follows

$$\mathcal{D}(\mathcal{F}) = (d\, \mathbb{I}_r + D(\mathcal{F}))s(\mathcal{F}), \tag{4.7}$$

where $s \in H^0(U, E)$, d is the regular exterior derivative, and $\mathbb{I}_r$ is the $r \times r$ identity matrix. The matrix $D(\mathcal{F})$ transforms by a gauge transformation as follows

$$D(\mathcal{F}') = G^{-1} D(\mathcal{F}) G.$$

Recall now that the exterior derivative decomposes into a holomorphic and an anti-holomorphic parts as follows

$$d = \partial + \bar{\partial},$$

where

$$\partial : \Lambda^0(\Sigma) \to \Lambda^{1,0}(\Sigma), \quad \bar{\partial} : \Lambda^0(\Sigma) \to \Lambda^{0,1}(\Sigma).$$

Similarly, a connection decomposes as follows

$$D : \partial_E + \bar{\partial}_E,$$

where

$$\partial_E : \Lambda^0(\Sigma, E) \to \Lambda^{1,0}(\Sigma, E), \quad \bar{\partial}_E : \Lambda^0(\Sigma, E) \to \Lambda^{0,1}(\Sigma, E).$$

In other words, ∂_E and $\bar{\partial}_E$ project into the holomorphic and the anti-holomorphic parts of $\Lambda^0(\Sigma, E)$ respectively. We call the $\mathbb{C}$-linear map $\bar{\partial}_E$ a pseudo-connection over the Riemann surface Σ.

Definition 4.2.5. *A connection $\mathcal{D}$ is called compatible with a Hermitian metric h on the vector bundle $E \to \Sigma$ if*

$$\mathcal{D}\langle s_1, s_2 \rangle = \langle \mathcal{D}s_1, s_2 \rangle + \langle s_1, \mathcal{D}s_2 \rangle,$$

where $s_1, s_2 \in H^0(U, E)$.

Formally speaking, the compatibility condition should be written as follows

$$\mathcal{D}_v\langle s_1, s_2 \rangle = \langle (\mathcal{D}_v)s_1, s_2 \rangle + \langle s_1, (\mathcal{D}_v)s_2 \rangle,$$

for any vector field v; this condition must hold for all vector fields $v \in \Gamma(T_\Sigma)$. In this case, we interpret the connection $\mathcal{D}_v$ as a map $\mathcal{D}_v : T_\Sigma \otimes E \to E$ rather than $\mathcal{D}_v : E \to E \otimes T^*_\Sigma$. Then, $(\mathcal{D}_v)s_1$ is a section and the R.H.S is a Hermitian inner product of two sections. On the other hand, $\mathcal{D}_v\langle s_1, s_2\rangle$ means $\mathcal{D}_v$ applied to $\langle s_1, s_2\rangle \otimes 1$ where 1 is a constant section of E. Therefore, one can use the local form of the connection $\mathcal{D}_v$ given in (4.7) to act on the inner product function $\langle s_1, s_2\rangle$. But since the connection matrix acts on the constant section, then the local form of $\mathcal{D}_v$ reduces to the exterior derivative d acting on the function $\langle s_1, s_2\rangle$. Having said that, however, we will keep using the short-hand notation $\mathcal{D}\langle s_1, s_2\rangle$ for convenience.

Now, given a connection $\mathcal{D}$ that is compatible with a Hermitian metric h, one can prove the following [52].

Theorem 4.2.6. *Let h be a Hermitian metric on a holomorphic vector bundle $E \to \Sigma$, then h induces a canonical connection $\mathcal{D}$ on E that satisfies the following conditions*

- *$\mathcal{D}$ is compatible with the metric h.*
- *Given a holomorphic section s on E, then $\bar{\partial}_E^2(s) = 0$.*

Given a connection $\mathcal{D} : \Lambda^0(\Sigma, E) \to \Lambda^1(\Sigma, E)$, it extends naturally to n-forms such that

$$\mathcal{D} : \Lambda^n(\Sigma, E) \to \Lambda^{n+1}(\Sigma, E).$$

In this case, a connection is usually called a covariant derivative. Then the operator $\mathcal{R}_{\mathcal{D}} = \mathcal{D} \cdot \mathcal{D}$ is known as the curvature of the connection $\mathcal{D}$. Formally,

Definition 4.2.7. *Let $\mathcal{D}$ be a connection on the Hermitian vector bundle $E \to \Sigma$, then the curvature of $\mathcal{D}$ is a $\mathbb{C}$-linear mapping*

$$\mathcal{R}_{\mathcal{D}} : \Lambda^0(\Sigma, E) \to \Lambda^2(\Sigma, E)$$

such that $\mathcal{R}_{\mathcal{D}} = \mathcal{D} \cdot \mathcal{D}$.

Given a frame $\mathcal{F}$ on E, then the curvature is represented locally by the following matrix

$$R_D(\mathcal{F}) = dD(\mathcal{F}) + D(\mathcal{F}) \wedge D(\mathcal{F}),$$

where $D(\mathcal{F})$ is the matrix representation of the connection $\mathcal{D}$ with respect to the frame $\mathcal{F}$. Finally, the curvature matrix transforms under the action of a gauge transformation as follows

$$R_D(\mathcal{F}') = G^{-1} R_D(\mathcal{F}) G.$$

4.3 The Moduli Space of Vector Bundles

As seen in the last section, a pseudo-connection is a $\mathbb{C}$-linear mapping

$$\bar{\partial}_E : \Lambda^0(\Sigma, E) \to \Lambda^{0,1}(\Sigma, E) \tag{4.8}$$

that satisfies the following equation

$$\bar{\partial}_E(s\varphi) = \bar{\partial}s\ \varphi + s\ \bar{\partial}_E\varphi, \tag{4.9}$$

where $s \in \Lambda^0(\Sigma)$ and $\varphi \in \Lambda^0(\Sigma, E)$. In fact, this map can be extended to a $\mathbb{C}$-linear mapping as follows

$$\bar{\partial}_E : \Lambda^{0,1}(\Sigma, E) \to \Lambda^{0,2}(\Sigma, E) \tag{4.10}$$

such that $\bar{\partial}_E^2 = 0$. But since our base space is a Riemann surface Σ, and $\Lambda^{0,2}(\Sigma) = 0$ over any Riemann surface, then the condition $\bar{\partial}_E^2 = 0$ is automatic in this case. Conversely, given a smooth vector bundle $\mathbb{E}$ and an operator $\bar{\partial}_E$ that satisfies the condition $\bar{\partial}_E^2 = 0$, then one can find holomorphic transition functions of $\mathbb{E}$ by locally solving the equation

$$\bar{\partial}_E T_{jk} = 0,$$

and thus endowing $\mathbb{E}$ with a holomorphic structure [30]. In other words, there is a bijection between the set of all operators $\bar{\partial}_E$ defined on $\mathbb{E}$, which we denote by $\mathcal{C}$, and the set of holomorphic structures on $\mathbb{E}$. Therefore, the pair $(\mathbb{E}, \bar{\partial}_E)$ defines a holomorphic vector bundle over Σ. Let $\mathcal{G}$ be the group of automorphisms (gauge transformations) of the vector bundle $\mathbb{E}$, then an element $G \in \mathcal{G}$ acts on $\mathcal{C}$ as follows

$$G \cdot \bar{\partial}_E = G^{-1} \bar{\partial}_E G. \tag{4.11}$$

The following theorem formalizes the previous discussion.

Theorem 4.3.1. *The group $\mathcal{G}$ acts on $\mathcal{C}$ by conjugation and the quotient $\mathcal{C}/\mathcal{G}$ can be identified with the isomorphism classes of holomorphic vector bundles of rank r and degree d on Σ.*

The issue with this construction is that the space $\mathcal{C}/\mathcal{G}$ is not a "nice" space in a sense that it is not Hausdorff [41]. It turns out that one has to impose an extra condition on the vector bundles to obtain a "nice" space. This condition arises naturally from Mumford's geometric invariant theory and is usually called the slope stability [38].

Definition 4.3.2. *Given a holomorphic vector bundle E over a compact Riemann surface Σ, then the slope of E is defined to be*

$$\mu(E) = deg(E)/rank(E). \tag{4.12}$$

Then the vector bundle E is said to be stable if

$$\mu(F) < \mu(E), \tag{4.13}$$

for every proper sub-bundle $F \subset E$.

Using the slope stability condition, one can get a "nice" moduli space of holomorphic vector bundles as follows. Let $\mathcal{C}^s$ be the set of all holomorphic structures on a stable vector bundle E, that is

$$\mathcal{C}^s = \{\bar{\partial}_E \in \mathcal{C} : (\mathbb{E}, \bar{\partial}_E) \text{ is stable}\}. \tag{4.14}$$

Then, the moduli space of stable holomorphic vector bundles over a Riemann surface Σ is defined as follows

$$M^s(r, d) = \mathcal{C}^s/\mathcal{G}. \tag{4.15}$$

Then, one can prove the following [29]

Theorem 4.3.3. *The moduli space of stable holomorphic vector bundles $M^s(r,d)$ of rank r and degree d over a Riemann surface Σ has the structure of a manifold of complex dimension $1 + r^2(g-1)$ where g is the genus of Σ.*

Note that since every line bundle is stable, then $M^s(1,d)$ is simply $Pic^d(\Sigma)$. In particular, $M^1(1,0)$ is the Jacobian of the Riemann surface $\mathcal{J}(\Sigma)$.

In general, the manifold $M^s(r,d)$ is not compact. However, if one considers semi-stable vector bundles, then this manifold can be compactified. A holomorphic vector bundle E is semi-stable if

$$\mu(F) \leq \mu(E)$$

for every subbundle $F \subset E$.

Furthermore, a vector bundle E is poly-stable if it satisfies the following condition

$$E = \bigoplus_j E_j, \quad \mu(E_j) = \mu(E), \tag{4.16}$$

where E_j are stable. Obviously, a poly-stable bundle is, in particular, semi-stable. Moreover, if E is a semi-stable bundle that is not stable, then there is a subbundle $F \subset E$ that has the same slope as E; that is $\mu(F) = \mu(E)$. Furthermore, the quotient E/F is semi-stable and $\mu(E) = \mu(E/F)$. Iterating this process, one gets what is known as the Jordan–Hölder filtration of E by semi-stable bundles; that is

$$0 \subset E_1 \subset E_2 \subset ... \subset E_k = E$$

for some index k. In this case, E_j/E^{j-1} is stable and has the same slope as E. Then the graded vector bundle of E is defined as follows

$$gr(E) = \bigoplus_j (E_j/E_{j-1}).$$

Even though the graded vector bundle is not uniquely determined, its equivalence class is uniquely determined and one calls two vector bundles E_1 and E_2 to be S-equivalent if $gr(E_1) \cong gr(E_2)$. Note that if E is strictly stable, then the bundle has a trivial Jordan-Hölder filtration consisting of itself and the zero bundle. Therefore, the graded vector bundle of E is trivial and an S-equivalence class of E is simply the isomorphism class of E. One can now define the moduli space of semi-stable vector bundles $M(r,d)$ to be the set of S-equivalence classes of semi-stable bundles of rank r and degree d. However, note that since the graded bundle of a semi-stable vector bundle is polystable, then $M(r,d)$ is also the set of isomorphism classes of polystable vector bundles of rank r and degree d.

Theorem 4.3.4. *The moduli space of $M(r,d)$ of semi-stable vector bundles has the structure of an irreducible algebraic variety that contains $M^s(r,d)$ as an open smooth subvariety.*

For the construction of this moduli space and its basic properties one may consult [37], [40]. Note that if $gcd(r,d) = 1$, then there are no strictly semi-stable vector bundles and therefore $M(r,d)$ coincides with $M^s(r,d)$.

Corollary 4.3.5. *If $gcd(r,d) = 1$, then $M^s(r,d) = M(r,d)$ and therefore $M(r,d)$ is a connected projective manifold of dimension $1 + r^2(g-1)$, where g is the genus of the underlying Riemann surface Σ.*

Consider now a holomorphic vector bundle $E \to \Sigma$, and a Hermitian metric h on E. Let $\mathcal{A}$ be the set of all connections of E that are compatible with the metric h. Furthermore, let $\mathcal{G}_h$ be the subgroup of the gauge transformations $\mathcal{G}$ of E that preserve h. Then, the action of $\mathcal{G}_h$ on $\mathcal{A}$ is given as follows

$$G \cdot \mathcal{D} = G^{-1} \mathcal{D} G, \tag{4.17}$$

where $G \in \mathcal{G}_h$ and $\mathcal{D} \in \mathcal{A}$. Moreover, the connection $\mathcal{D} \in \mathcal{A}$ is said to have constant central curvature if

$$\mathcal{R}_{\mathcal{D}} = \lambda, \tag{4.18}$$

where

$$\lambda = -i\mu(E) I_E \Omega,$$

I_E is the identity endomorphism of E and Ω is the Kähler form of the Hermitian metric h. Then, one can prove the following [52]

Proposition 4.3.6. *Let E be a Hermitian vector bundle on Σ with a hermitian metric h. Furthermore, let $\mathcal{R}_{\mathcal{D}}$ be the curvature of the canonical connection $\mathcal{D}$. If $\mathcal{R}_{\mathcal{D}}$ satisfies equation (4.18), then E is polystable.*

Furthermore, noting that the action of $\mathcal{G}_h$ on both sides of equation (4.18) is the same. Then the equation is invariant under the action of $\mathcal{G}_h$. Furthermore, let

$$\mathcal{A}_0 = \{\mathcal{D} \in \mathcal{A} : \mathcal{R}_{\mathcal{D}} = \lambda\} \tag{4.19}$$

be the set of all connections on a Hermitian vector bundle E with constant central curvature. Then, the quotient $\mathcal{A}_0/\mathcal{G}_h$ is the moduli space of connections on E with constant central curvature. Moreover, a connection $\mathcal{D} \in \mathcal{A}$ of a vector bundle E with hermitian metric h is called reducible if

$$E = E_1 \oplus E_2, \quad h = h_1 \oplus h_2, \quad \mathcal{D} = \mathcal{D}_1 \oplus \mathcal{D}_2,$$

where $\mathcal{D}_1$ is a connection on E_1 compatible with the metric h_1 while $\mathcal{D}_2$ is a connection on E_2 compatible with the metric h_2. We call a connection $\mathcal{D}$ irreducible if it is not reducible. Finally, we let $\mathcal{A}^{irr}$ and $\mathcal{A}_0^{irr}$ be the subsets of $\mathcal{A}$ and $\mathcal{A}_0$ respectively of irreducible connections. Then

Theorem 4.3.7. *The moduli space of irreducible connections with constant central curvature $\mathcal{A}_0^{irr}/\mathcal{G}_h$ on a Hermitian vector bundle $E \to \Sigma$ is a smooth manifold with dimension $1 + r^2(g-1)$ where g is the genus of Σ. Furthermore, one has the following isomorphisms*

$$\begin{aligned} \mathcal{A}_0/\mathcal{G}_h &\cong \mathcal{C}^{ps}/\mathcal{G}, \\ \mathcal{A}_0^{irr}/\mathcal{G}_h &\cong \mathcal{C}^{s}/\mathcal{G}, \end{aligned} \tag{4.20}$$

where $\mathcal{C}^{ps}$ is the set of holomorphic structures on the polystable vector bundle E.

This is the well-known theorem of Narasimhan and Seshadri, see [41],[15].

4.4 The Moduli Space of Higgs Bundles

As mentioned before, the main goal of this chapter is to introduce the moduli space of Higgs bundles. A Higgs bundle over a Riemann surface Σ is a pair (E, A) where $E \to \Sigma$ is a holomorphic vector bundle and A is an $End(E)$-valued holomorphic 1-form known as the Higgs field, namely

$$A \in H^0(\Sigma, End(E) \otimes K). \tag{4.21}$$

A Higgs bundle (E, A) is called stable if every A-invariant subbundle of E is stable. That is, if $\mu(F) < \mu(E)$ for every proper subbundle F such that $A(F) \subset F \otimes K$. Furthermore, semi-stability poly-stability, Jordan–Hölder filtration and S-equivalence classes are defined as in the case of vector bundles. Let $\mathcal{M}^s(r, d)$, be the moduli space of stable Higgs bundles with vector bundle $E \to \Sigma$ of rank r and degree d. Moreover, let $\mathcal{M}(r, d)$ be the moduli space of S-equivalence classes of semi-stable Higgs bundles of rank r and degree d. In [44], Nitsure constructed the moduli space of semi-stable Higgs bundles using geometric invariant theory and he showed the following

Theorem 4.4.1. *The moduli space of semi-stable Higgs bundles $\mathcal{M}(r, d)$ is a complex-quasi-projective variety which contains $\mathcal{M}^s(r, d)$ as an open smooth sub-variety of complex dimension $2 + 2r^2(g-1)$, where g is the genus of Σ.*

As in the case of vector bundles, if $gcd(r, d) = 1$, then $\mathcal{M}(r, d) \cong \mathcal{M}^s(r, d)$. We have also shown that the stability of a vector bundle E is connected to the differential geometric picture of having a connection $\mathcal{D}$ with a constant central curvature; see proposition (4.3.6). Interestingly, the same is true in the case of Higgs bundles. Given a Higgs bundle (E, A) where $E \to \Sigma$ is a Hermitian vector bundle with metric h, canonical connection $\mathcal{D}$, and curvature $\mathcal{R}_{\mathcal{D}}$. One may now ask whether the stability of the Higgs bundle (E, A) equivalent to the fact that the metric h satisfies an equation similar to (4.18). This is in fact true and the desired equation is in fact the celebrated Hitchin's self-duality equations

$$\begin{gathered} \mathcal{R}_{\mathcal{D}} + [A \wedge A^*] = -i\mu(E) I_E \Omega, \\ \bar{\partial}_E A = 0. \end{gathered} \tag{4.22}$$

Then, one can prove the following [26], [50]

Theorem 4.4.2. *Let (E, A) be a Higgs bundle where $E \to \Sigma$ is a Hermitian vector bundle with a metric h and a canonical connection $\mathcal{D}$ that satisfies (4.22). Then, the Higgs bundle (E, A) is polystable. Conversely, if (E, A) is stable, then there exists a Hermitian metric h on E and a canonical connection $\mathcal{D}$ that satisfies (4.22).*

Consider now the set of all solutions to Hitchin's equations

$$\aleph = \{(\mathcal{D}, A) \in \mathcal{A} \times H^0(\Sigma, End(E) \otimes K) \text{ that satisfies (4.22)}\}, \tag{4.23}$$

then again the set $\aleph$ is invariant under the action of $\mathcal{G}_h$. Then we define the moduli space of solutions to Hitchin's self-duality equations to be $\aleph/\mathcal{G}_h$. Reduciblity of a pair $(\mathcal{D}, A)$ is defined exactly as in the case of vector bundles but with the additional condition $A = A_1 \oplus A_2$. As before, we call a pair $(\mathcal{D}, A)$ irreducible if it is not reducible and we let

$$\aleph^{irr} \subset \aleph \tag{4.24}$$

be the subset of irreducible solutions to Hitchin's equations. Then, one can prove the following [26]

Theorem 4.4.3. *The moduli space of irreducible solutions to Hitchin's equations $\aleph^{irr}/\mathcal{G}_h$ has the structure of a smooth manifold of complex dimension $2 + 2r^2(g-1)$.*

Finally, to see the analog of Narasimhan and Seshadri correspondence, fix a smooth vector bundle $\mathbb{E} \to \Sigma$ and let

$$\mathcal{H} = \{(\bar{\partial}_E, A) \in \mathcal{C} \times H^0(\Sigma, End(E) \otimes K) : \bar{\partial}_E A = 0\}. \tag{4.25}$$

Then consider the action of the gauge group $\mathcal{G}$ on $\mathcal{H}$. This action is defined exactly as in the case of vector bundles and $\mathcal{G}$ acts on A as follows

$$G \cdot A = G^{-1} A G,$$

where $G \in \mathcal{G}$. Obviously, a pair $(\bar{\partial}_E, A) \in \mathcal{H}$ is equivalent to a Higgs bundle (E, A) where $E = (\bar{\partial}_E, \mathbb{E})$. Now let

$$\mathcal{H}^s = \{(\bar{\partial}_E, A) \in \mathcal{H} : (E, A) \text{ stable}\} \tag{4.26}$$

be the set of stable Higgs bundles on the Riemann surface Σ. Then, $\mathcal{H}^s/\mathcal{G}$ is the space of isomorphism classes of stable Higgs bundles and

$$\mathcal{M}^s(r, d) \cong \mathcal{H}^s/\mathcal{G}.$$

Similarly, let

$$\mathcal{H}^{ps} = \{(\bar{\partial}_E, A) \in \mathcal{H} : (E, A) \quad \text{polystable}\} \tag{4.27}$$

be the set of polystable Higgs bundles on Σ, then the correspondence is given as follows [26] [52]

Theorem 4.4.4. *Let $\mathcal{H}^s$, $\mathcal{H}^{ps}$, $\aleph$, $\aleph^{irr}$, $\mathcal{G}$, and $\mathcal{G}_h$ be defined as before, then*

$$\begin{aligned} \aleph/\mathcal{G}_h &\cong \mathcal{H}^{ps}/\mathcal{G}, \\ \aleph^{irr}/\mathcal{G}_h &\cong \mathcal{H}^s/\mathcal{G}. \end{aligned} \tag{4.28}$$

From now on, we assume that any given Higgs field is irreducible. Furthermore, we will always be concerned with the moduli space of stable Higgs bundles so that we will drop the superscripts refering to stability from now on.

4.4.1 Fixed Determinant Case

Recall that given a vector bundle $E \to \Sigma$, one can form the determinant line bundle $L = \det(E)$ given by the highest exterior power $\wedge^r E$. The transition function of this line bundles are simply $\det(T_{jk})$ where T_{jk} are the transition functions of the vector bundle E. From now on, we fix the determinant line bundle on E. Furthermore, let $\mathcal{D}$ be a unitary connection on E, then the curvature of $\mathcal{D}$ decomposes as follows

$$\mathcal{R}_{\mathcal{D}} = \mathcal{R}^0_{\mathcal{D}} + \frac{1}{r}Tr(\mathcal{R}_{\mathcal{D}}) \otimes I_E,$$

where $\mathcal{R}^0_{\mathcal{D}}$ and $\frac{1}{r}Tr(\mathcal{R}_{\mathcal{D}}) \otimes I_E$ are the trace-free part and the central part of the curvature respectively. In fact, the central part of the curvature is also the curvature of the induced connection on the line bundle $L = \det(E)$. But since we have fixed this line bundle, then this central part is also fixed. Hence, Hitchin's equations reads

$$\begin{aligned} \mathcal{R}^0_{\mathcal{D}} + \frac{1}{r}Tr(\mathcal{R}_{\mathcal{D}}) \otimes I_E + [A \wedge A^*] &= -i\mu(E) I_E \Omega, \\ \bar{\partial}_E A &= 0. \end{aligned} \tag{4.29}$$

However, there exists a unitary connection $\mathcal{D}$ on E such that $Tr(\mathcal{R}_{\mathcal{D}}) = -i\ \Omega\ deg(E)$. Hence, fixing this as a background connection and considering only connections which $\mathcal{D}$ that induce the same connection on $L = \det(E)$, then Hitchin's equations read

$$\begin{aligned} \mathcal{R}^0_{\mathcal{D}} + [A \wedge A^*] &= 0, \\ \bar{\partial}_E A &= 0. \end{aligned} \tag{4.30}$$

Evidently, the previous choices correspond to fixing a holomorphic structure on $L = \det(E)$. Furthermore, since the trace of the Higgs field is constant, we restrict to trace-free Higgs fields. In this case, the emerging moduli space of Higgs bundles is defined as follows

$$\mathcal{M}_{SL}(r,d) = \{(\mathcal{D}, A) :\ \text{solution to (4.30)}\}/\mathcal{G}_{SL}, \tag{4.31}$$

where $\mathcal{G}_{SL}$ is the set of gauge transformations of E that has unit determinant. From now on, when we mention the moduli space of Higgs bundles we mean this fixed determinant moduli space $\mathcal{M}_{SL}(r,d)$. In fact, all the previous constructions and theorems presented earlier carry on directly to this fixed determinant case. However, the moduli space $\mathcal{M}_{SL}(r,d)$ is of complex dimension

$$\mathcal{M}(r,d) = (r^2-1)(2g-2). \tag{4.32}$$

Furthermore, we also set $r = 2$ from now on and therefore, the Higgs field will be represented simply by a 2×2 traceless matrix acting on a two-dimensional vector spaces parametrized by $\lambda \in \Sigma$.

Assume now that the Higgs field is simple, meaning that $\det(A)$ has simple zeros, then the Higgs bundle (E, A) is stable. To see this explicitly, assume that there is a lime bundle $L \subset E$ that is preserved by the action of A. Then there is a frame $\mathcal{F}$ on E and a local coordinate on Σ such that

$$A(\lambda) = \begin{bmatrix} a(\lambda) & b(\lambda) \\ 0 & -a(\lambda) \end{bmatrix}.$$

Obviously, $\det(A)$ has double zeros at every root of $a(\lambda)$. We conclude that when $r = 2$, the simplicity of $A(\lambda)$ is equivalent to the stability of the Higgs bundle (E, A).

4.4.2 Co-Higgs bundles

Since the beginning of this chapter, we assumed that the base space of the vector bundle $E \to \Sigma$ is a Riemann surface of genus $g \geq 2$. However, as will be seen in the following chapter, the base space of our vector bundle is going to be $\mathbb{P}^1$. This case was studied in [45] [46] where the underlying Higgs bundles were called co-Higgs bundles. In this case, the definition of the Higgs bundles is the same but with the cotangent bundle K replaced with the tangent bundle K^*. Otherwise, all the definitions that we introduced previously are basically the same. It was shown in the same reference that co-Higgs bundles are only stable on $\mathbb{P}^1$, that is in the genus zero case. But since Higgs bundles are never stable on $\mathbb{P}^1$ [26], then one can consider co-Higgs bundles to be an extension of Higgs bundles to the genus zero case.

Recall now that the Birkhoff-Grothendieck theorem classifies vector bundles over $\mathbb{P}^1$ up to an isomorphism by the splitting

$$E \cong \bigoplus_{j=1}^{r} \mathcal{O}(a_j)$$

for some integers a_j. Then the following result was proved in [45]

Theorem 4.4.5. *Let $E = \bigoplus_{j=1}^{r} \mathcal{O}(a_j)$ be a holomorphic vector bundle of rank r on $\mathbb{P}^1$ such that $a_1 \geq a_2 \geq ... \geq a_r$. Then E admits a semi-stable Higgs field $A \in H^0(\mathbb{P}^1; End(E) \otimes \mathcal{O}(m))$ if and only if $a_j \leq a_{j+1} + m$ for all $1 \geq j \geq r-1$ and $m \geq 2$.*

Note that in the original paper this theorem was proved for $m = 2$; however, it can be shown that the result is still valid for any $m \geq 2$ as stated here. Consider now the case $r = 2$, $d = 0$ and $m = 2$. In this case, the only stable bundles on $\mathbb{P}^1$ are $E \cong \mathcal{O} \oplus \mathcal{O}$ and $E \cong \mathcal{O}(-1) \oplus \mathcal{O}(1)$. Furthermore, in the case $r = 2$, $d = -1$ and $m = 2$, then every stable vector bundle on $\mathbb{P}^1$ is isomorphic to $E \cong \mathcal{O}(-1) \oplus \mathcal{O}$.

Now, consider $\mathcal{M}_{SL}(2, -1)$ where every Higgs field in this space has underlying bundle $E = \mathcal{O}(-1) \oplus \mathcal{O}$ and can be represented by the following matrix

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \tag{4.33}$$

where a, b and c are sections of $\mathcal{O}(2)$, $\mathcal{O}(3)$, and $\mathcal{O}(1)$ respectively. Stability ensures that c is not identically zero and therefore has a unique zero $\lambda_0 \in \mathbb{P}^1$. It was shown in [45] that it is possible to provide a global description of the moduli space $\mathcal{M}_{SL}(2, -1)$ as a universal elliptic curve as follows. Let $\pi : S \to \mathbb{P}^1$ be the projection from the total space of $\mathcal{O}(2)$ to the projective line. Then, one can assign to each stable Higgs bundle $(E,A) \in \mathcal{M}_{SL}(2,-1)$ a point in the 6-dimensional space M defined as follows

$$M = \{(y,D) \in S \times H^0(\mathbb{P}^1; \mathcal{O}(4)) : \eta^2(y) = D(\pi(y))\}.$$

The isomorphism from $\mathcal{M}_{SL}(2,-1)$ into M is defined by sending A to $(\lambda_0, a(\lambda_0), -\det(A))$. Note that, $(\lambda_0, a(\lambda_0))$ is a point in the total space of $\mathcal{O}(2)$ since $\lambda_0 \in \mathbb{P}^1$ and a is a section of $\mathcal{O}(2)$ and therefore $a(\lambda_0) \in \mathcal{O}(2)$. In fact, one can show explicitly that this point in M is defined uniquely by the Higgs field A up to a gauge transformation. Therefore, the moduli space $\mathcal{M}_{SL}(2,-1)$ has a global description as a double cover of the Riemann sphere and the preimage of any point $\lambda \in \mathbb{P}^1$ corresponds to two points in $\mathcal{M}_{SL}(2,-1)$ with two distinct Higgs fields A_+ and A_-. More precisely, let $\lambda_0 \in \mathbb{P}^1$ be the zero of the $\mathcal{O}(1)$ section in A and let $\pi : \mathcal{M}_{SL}(2,-1) \to \mathbb{P}^1$ be the restriction of the map $\pi : S \to \mathbb{P}^1$ to M, then $\pi^{-1}(\lambda)$ gives two Higgs fields which we denote by $A_+(\lambda_0)$ and $A_-(\lambda_0)$. To see this explicitly, we fix the spectrum of A by evaluating its determinant at the point λ_0 as follows

$$D_0 := D(\lambda_0) = -a^2(\lambda_0).$$

Then, $(\lambda_0, \pm\sqrt{D_0}, -D_0)$ are the corresponding two points in M and $A(\lambda)$ has two normal forms corresponding to each point

$$A_+(\lambda) = \begin{bmatrix} \sqrt{D_0} & \alpha(\lambda) \\ \lambda - \lambda_0 & -\sqrt{D_0} \end{bmatrix}, \quad A_-(\lambda) = \begin{bmatrix} -\sqrt{D_0} & \alpha(\lambda) \\ \lambda - \lambda_0 & \sqrt{D_0} \end{bmatrix}, \tag{4.34}$$

where $\alpha(\lambda)$ is a unique degree-3 polynomial. One could explicitly check that there is no gauge transformation that takes A_+ to A_-. Therefore, if λ_0 is not a ramification point, then each preimage corresponds to two distinct Higgs fields A_+ or A_-. Furthermore, the two points in M corresponding to these two Higgs fields are projected by $\pi : M \to \mathbb{P}^1$ onto λ_0. Yet, if λ_0 is a Weierstrass point of the curve Y, that is if $a(\lambda_0) = 0$, then $A_+(\lambda_0) = A_-(\lambda_0)$ and this point corresponds to a unique Higgs field.

This discussion can be reformulated by considering the spectral viewpoint. Consider the spectral curve M, the Grothendieck-Riemann-Roch theorem tells us that the preimage of $E = \mathcal{O}(-1) \oplus \mathcal{O}$ on $\mathbb{P}^1$ is a degree one line bundle on the spectral curve M. Then, the Riemann-Roch theorem tells us that this line bundle has a one-dimensional space of sections. That is, all its sections vanish at one point that can be either $(\lambda_0, \sqrt{D_0}, D_0)$ or $(\lambda_0, -\sqrt{D_0}, D_0)$. Whether one gets the Higgs field A_+ or A_- depends on which sheet contains the point at which these sections vanish and the covering map projects the two points onto λ_0. Again, if λ_0 is a Weierstrass point, then the two points $(\lambda_0, \sqrt{D_0}, D_0)$ and $(\lambda_0, -\sqrt{D_0}, D_0)$ coincide and the two line bundles corresponding to the two different sections coincide (collide), and we get a single stable Higgs field $A_+ = A_-$.

Consider now the moduli space $\mathcal{M}_{SL}(2,0)$. As mentioned before, every stable Higgs bundle in this moduli space has an underlying vector bundle $E = \mathcal{O} \oplus \mathcal{O}$ or $E = \mathcal{O}(-1) \oplus \mathcal{O}(1)$. Unfortunately, this moduli space does not have such an explicit description as in the case of $\mathcal{M}_{SL}(2,-1)$. However, it was shown in the same reference that up to isomorphism, there is a unique Higgs field in $\mathcal{M}_{SL}(2,0)$ that has an underlying vector bundle $\mathcal{O}(-1)\oplus\mathcal{O}(1)$. Therefore, every other Higgs field in this moduli space has an underlying vector bundle $E = \mathcal{O}\oplus\mathcal{O}$.

4.5 Ends of the Moduli Space of Higgs bundles

In [36], Mazzeo et al studied the ends of the moduli space of Higgs bundles. Their initial motivation was to understand the asymptotic nature of the Hyperkähler metric and therefore understand the asymptotic geometry of that moduli space. This work was also reviewed in [35] from a different and simpler prespective. As mentioned before, the moduli space of stable Higgs bundles is non-compact. A simple way to make sense of this non-compactness is to note that the Higgs field is in general unbounded and its elements can take any large values one may like. In fact, this is exactly what happens when one reaches the ends of the moduli space. The main purpose of this section is to review this work and to state some results of the aforementioned papers that will be crucial in the upcoming chapter.

As before, we will fix the rank of the vector bundle E to be $r = 2$ and we restrict our discussion to the fixed-determinant moduli space. We also set Σ^* to be $\Sigma \setminus q^{-1}(0)$ where $q(\lambda)$ is the determinant of the Higgs field A. We want now to consider what is called the fiducial solutions to Hitchin's equations. These are elements of a one-parameter radial family of global solutions $(\mathcal{D}_t^{fid}, tA_t^{fid})$ on $\mathbb{C}$ that satisfy the rescaled Hitchin's equations

$$\begin{aligned} \mathcal{R}^0_{\mathcal{D}_t^{fid}} + t^2[A_t^{fid}\wedge(A_t^{fid})^\dagger] &= 0, \\ \bar{\partial}_E A &= 0. \end{aligned} \tag{4.35}$$

This family of solutions approaches the ends of the moduli space as $t\to\infty$. In other words, what we actually mean by the ends of the moduli space of Higgs bundles is that we follow a trajectory in the Hitchin base such that the determinant of the Higgs field is growing is growing without a bound. Put differently, the determinant of the Higgs field and therefore the spectral curve itself blow up as the trajectory approaches the ends of the moduli space. We provide a schematic diagram of a trajectory in the moduli space of Higgs bundles in figure (4.1) at the end of the section.

Moreover, the fiducial solutions for $0 < t < \infty$ give rise to solutions of the self-dual Yang-Mills equation which are rotationally symmetric and translation invariant in two direction. The connection between this type of symmetric solutions and integrable systems was studied by Mason and Woodhouse in [34]. Furthermore, since the function

$$\begin{aligned} &\mathcal{M}_{SL}\to\mathbb{R}, \\ &[\mathcal{D}, A]\to ||A^2||_{L^2} \end{aligned}$$

is a proper Morse-Bott function, then if the sequence of solutions to Hitchin's equations $(\mathcal{D}_t^{fid}, tA_t^{fid})$ is bounded, then it lies in a compact subset of $\mathcal{M}_{SL}$. Evidently, the Higgs field in the family of fiducial solutions $(\mathcal{D}_t^{fid}, tA_t^{fid})$ becomes singular as $t \to \infty$. Furthermore, assume that $(\mathcal{D}_t^{fid}, A_t^{fid}) \to (\mathcal{D}_\infty^{fid}, A_\infty^{fid})$ as $t \to \infty$, then this limiting element of the diverging family of solutions satisfies the decoupled version of Hitchin's equations

$$\begin{aligned} \mathcal{R}^0_{\mathcal{D}_\infty^{fid}} &= 0, \\ [A_\infty^{fid} \wedge (A_\infty^{fid})^\dagger] &= 0, \\ \bar{\partial}_E A_\infty^{fid} &= 0. \end{aligned} \tag{4.36}$$

In other words, the connection $\mathcal{D}_\infty^{fid}$ is flat and the Higgs field A_∞^{fid} is normal. Since the limiting Higgs field is normal, then by the spectral theorem there exists a unitary gauge transformation that diagonalizes A_∞. In other words, one can find a gauge transformation such that A_∞ has the following form

$$\begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}. \tag{4.37}$$

Therefore, the spectrum of A_∞^{fid} degenerates at this limit. In other words, if $\det(A_\infty^{fid}) \in H^0(\Sigma, K^2) = 0$ at a point $\lambda \in \mathbb{P}^1$, then the order of vanishing must be at least two unless A_∞^{fid} is nilpotent. Surprisingly, starting with a simple Higgs field, then the spectrum of its associated family of solutions degenerates as it approaches the ends of the moduli space.

On the other hand, given a pair $(\mathcal{D}_\infty, A_\infty)$ such that A_∞ is a simple and normal Higgs field and therefore satisfies (4.36), then this solution is known as a "limiting configuration" and it is defined formally as follows.

Definition 4.5.1. *Let (E, A) be a Hermitian Higgs bundle where the Higgs field A is simple. A limiting configuration is a pair $(\mathcal{D}_\infty, A_\infty)$ defined on Σ^* which satisfies the decoupled version of Hitchin's equations (3.36) and which agrees with $(\mathcal{D}_\infty^{fid}, A_\infty^{fid})$ near each point in $q^{-1}(0)$ with respect to some holomorphic coordinate system and some unitary frame on E.*

Then, one can obtain a global non-singular solution to Hitchin's equations by gluing a family of diverging fiducial solutions $(\mathcal{D}_t^{fid}, A_t^{fid})$ for $0 < t < \infty$ to a limiting configuration $(\mathcal{D}_\infty, A_\infty)$. However, does every diverging family of solutions to Hitchin's equations converges to a limiting configuration? This is in fact true. To see this, consider the following theorem.

Theorem 4.5.2. *The moduli space of limiting configurations with a fixed simple determinant q which we denote by $\mathcal{T}_q$ is a torus of complex dimension $3g - 3$.*

However, recall that by (4.32), that the moduli space of stable Higgs bundles in the fixed determinant case is of complex dimension $(r^2 - 1)(2g - 2)$. Therefore, when $r = 2$,

$$dim(\mathcal{M}_{SL}(2, d)) = 6g - 6.$$

In other words, the dimension of $\mathcal{T}_q$ is half the dimension of the corresponding moduli space $\mathcal{M}_{SL}(2, d)$. Therefore, gluing $\mathcal{T}_q$ to the boundaries of $\mathcal{M}_{SL}(2, d)$ one achieves a compactification of the moduli space of Higgs bundles in the fixed determinant case. Then, one can prove the following

Theorem 4.5.3. *Let $(\mathcal{D}_t^{fid}, tA_t^{fid})$ be a diverging sequence of solutions to Hitchin's equations (3.35) such that the determinant of A_t is simple and fixed. Then there is a limiting configuration $(\mathcal{D}_\infty, A_\infty)$ such that $(\mathcal{D}_t^{fid}, A_t^{fid})$ converges to $(\mathcal{D}_\infty, A_\infty)$ uniformally at an exponential rate in t on Σ^*.*

Simply speaking, this theorem states given a pair $(\mathcal{D}_\infty, A_\infty)$ where A_∞ is a simple and normal Higgs field, then there exists a diverging sequence of solutions that converges to $(\mathcal{D}_\infty, A_\infty)$. By this gluing process, we then obtain a compactification of the moduli space of Higgs bundles $\mathcal{M}_{SL}(2, d)$. Schematically, the compactified moduli space looks as follows

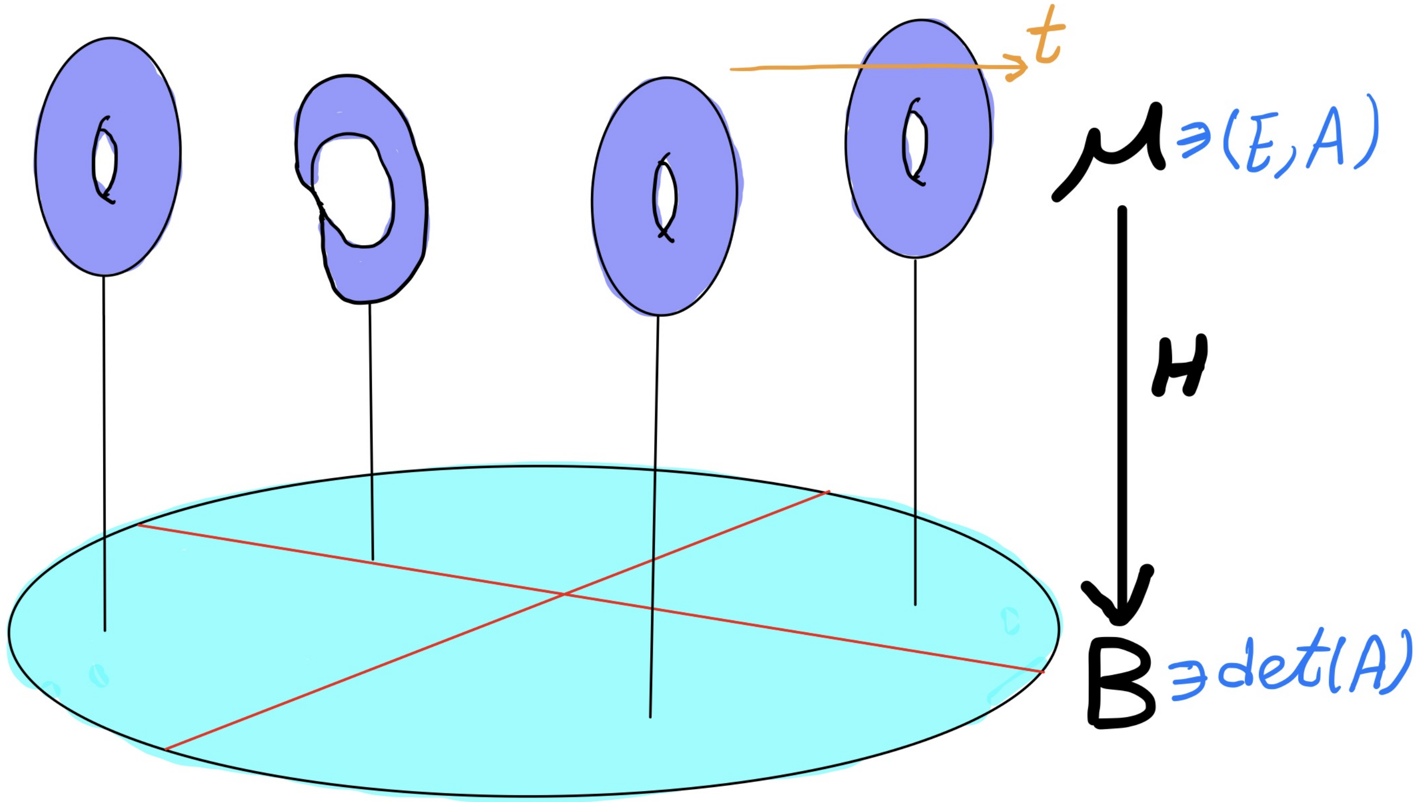

Figure 4.1: The moduli space of Higgs bundles.

5 The Calogero-Françoise Integrable System

5.1 Introduction

The Camassa-Holm integrable system is a non-linear partial differential equation that was first discovered by Fokas and Fuchssteiner in [19]. This equation was then derived from physical considerations by Camassa and Holm as a model for a unidirectional propagation of shallow water waves [12]. The discovery of the Camassa-Holm equation was followed by thorough investigations of this equation and it turned out to be one of most extensively studied integrable Hamiltonian systems in the past few decades. The special behavior of the Camassa-Holm system is that it gives rise solitary waves (solitons) that are similar to the solitons arising as solutions to the KdV equation [31]. However, unlike the KdV solitons , the solitons arising as solutions to the CH equation have peaks (sharp edges) and this is why they are called peakons.

After that, Calogero and Françoise introduced a finite dimensional integrable Hamiltonian system that generalizes the Camassa-Holm dynamics [10], [11]. The Hamiltonian is a priori non-periodic and it also gives rise to peaked solitons, which we will simply call peakons, as in the Camassa-Holm case. The goal of this chapter is to study the dynamics of the Calogero-Françoise integrable system from an algebraic-geometric point of view. More specifically, we are going to introduce an embedding of the CF system into the moduli space of Higgs bundles and then we will study the evolution of the Higgs field associated to the system. We will focus here on the singular dynamics that occur when two peakons interact together through one of two processes called collisions and pseudo-collisions. We will show that the emerging singularity obtained at the time of pseudo-collision amounts to the fact that the Higgs field is becoming asymptotically far away in its moduli space. On the other hand, collisions correspond to a very special form of Higgs bundles known as limiting configurations. These are Higgs bundles with normal Higgs fields that arise as limits of solutions to Hitchin's equations. Furthermore, we will show that the peakons are confined in a box whose boundaries are defined by the constants associated to the problem. When the peakons hit these boundaries, the solution becomes singular and we call this phenomenon a pseudo-collision. We will also show that there is a natural analytic continuation past these singularities in which the system is periodized. In other words, we will glue the boundaries of the confinement box so that the peakons are moving on a circle and therefore collisions and pseudo-collisions will be basically the same dynamics. Finally, we will show that the introduced periodization (compactification) of the CF dynamics corresponds to a compactification of the underlying moduli space. We conclude that the CF dynamics can be realized completely by the behavior of the associated Higgs field in the underlying Hitchin system. These results are then a direct manifestation of fact that the CF system can be realized as a twisted Hitchin system. Hence, these results are consistent with the common belief that all classical integrable systems can be realized as a (maybe twisted) Hitchin system.

5.2 Brief review of the CF system

The spectral problem associated with the CF system is given as follows. Let [6]

$$L(\lambda)\phi = (\frac{d^2}{dx^2} - \nu^2 - 2\nu\lambda m)\phi = 0, \tag{5.1}$$

where λ is the spectral parameter and ν is a constant. Then the operator $L(\lambda)$ is compatible with the generalized Lax evolution

$$\frac{d}{dt}L(\lambda) = [L(\lambda), B(\lambda)] + 2u_x L(\lambda), \tag{5.2}$$

where

$$B(\lambda) = \left\{\frac{1}{2\nu\lambda} - u(x)\right\}\frac{d}{dx} + \frac{1}{2}u_x(x). \tag{5.3}$$

One can then show that equation (4.2) is equivalent to the following system of equations

$$m_t = 2u_x m + u m_x, \quad 2m_x = 4\nu^2 u_x - u_{xxx}. \tag{5.4}$$

We assume that $m(x,t)$ is a discrete sum of Dirac measures given as follows

$$m(x,t) = \sum_{j=1}^{d} m_j(t)\delta(x - x_j(t)). \tag{5.5}$$

We also assume that at $t = 0$, $x_1 < x_2 < ... < x_d$ and that $x_0 = -\infty$ and $x_{d+1} = \infty$. Then, the CF system represents a finite dimensional Hamiltonian system given by

$$H(x_1, ..., x_d, m_1, ..., m_d) = \frac{1}{2}\sum_{j,k=1}^{d} m_j m_k \; G_{\nu,\beta}(x_j - x_k), \tag{5.6}$$

where x_j and m_j are the positions and masses of the peakons respectively and Green's function for the system is given as follows

$$G_{\nu,\beta}(x) = \frac{\beta_+}{2\nu}e^{2\nu|x|} + \frac{\beta_-}{2\nu}e^{-2\nu|x|}. \tag{5.7}$$

Then, the function $u(x,t)$ is given as follows

$$u(x,t) = \sum_{j=1}^{d} m_j(t) \; G_{\nu,\beta}(x - x_j(t)). \tag{5.8}$$

In this case, (5.4) is equivalent to the following system of equations

$$\dot{x}_j = u(x_j), \quad \dot{m}_j = -m_j < u_x > (x_j), \tag{5.9}$$

where $< u_x > (x_j)$ is the arithmetic average of the left and right limits of u_x at the point x_j. Furthermore, let

$$M_\pm = \sum_{j=1}^{d} e^{\pm 2\nu x_j} m_j, \tag{5.10}$$

then $u(x) = \sum_{j=1} m_j \; G_{\nu,\beta}(x - x_j)$ is given asymptotically as follows

$$u(x) = \frac{\beta_-}{2\nu} M_- e^{2\nu x} + \frac{\beta_+}{2\nu} M_+ e^{-2\nu x}, \quad x < x_1,$$
$$u(x) = \frac{\beta_-}{2\nu} M_+ e^{-2\nu x} + \frac{\beta_+}{2\nu} M_- e^{2\nu x}, \quad x > x_d.$$

Moreover, in each region, (x_{j-1}, x_j) where $m(x,t) = 0$, a solution of (5.1) is given as follows

$$\phi_j(x,t) = a_j e^{\nu x} + b_j e^{-\nu x}.$$

Then, equation (5.1) translates to a continuity equation and a jump discontinuity as follows

$$\begin{aligned} a_{k+1}\mathrm{e}^{\nu x_k} + b_{k+1}\mathrm{e}^{-\nu x_k} &= a_k \mathrm{e}^{\nu x_k} + b_k \mathrm{e}^{-\nu x_k}, \\ a_{k+1}\mathrm{e}^{\nu x_k} - b_{k+1}\mathrm{e}^{-\nu x_k} &= a_k \mathrm{e}^{\nu x_k} - b_k \mathrm{e}^{-\nu x_k} + \lambda m_k \left(a_k \mathrm{e}^{\nu x_k} + b_k \mathrm{e}^{-\nu x_k} \right). \end{aligned} \tag{5.11}$$

Writing these equations in a matrix form, we get

$$\begin{bmatrix} a_{k+1} \\ b_{k+1} \end{bmatrix} = \begin{bmatrix} 1 + \lambda m_k & \lambda m_k \mathrm{e}^{-2\nu x_k} \\ -\lambda m_k \mathrm{e}^{2\nu x_k} & 1 - \lambda m_k \end{bmatrix} \begin{bmatrix} a_k \\ b_k \end{bmatrix}. \tag{5.12}$$

Then, the transfer matrix $T(\lambda)$ that gives the transition from the representation defined on the interval (x_0, x_1) to the one defined on (x_d, x_{d+1}) is given as follows

$$T(\lambda) = T_d(\lambda) T_{d-1}(\lambda) ... T_1(\lambda), \tag{5.13}$$

and since $Det(T_k) = 1$, then $Det(T) = 1$. Furthermore, the asymptotics of $T(\lambda)$ are given as follows

$$\begin{aligned} T(\lambda) &= \mathbb{I} + \lambda \begin{bmatrix} M & M_- \\ -M_+ & -M \end{bmatrix} + \mathcal{O}(\lambda^2), \qquad \lambda \to 0, \\ T(\lambda) &= \lambda^d \prod_{k=1}^{d} m_k \prod_{k=2}^{d} \left(1 - \mathrm{e}^{-2\nu(x_k - x_{k-1})} \right) \begin{bmatrix} 1 & \mathrm{e}^{-2\nu x_1}, \\ -\mathrm{e}^{2\nu x_d} & -\mathrm{e}^{2\nu(x_d - x_1)} \end{bmatrix} + O\left(\lambda^{d-1}\right), \quad \lambda \to \infty. \end{aligned} \tag{5.14}$$

where $M = \sum_{k=1}^{d} m_k$ is the total momentum and $M_\pm$ are given by (5.10). It was then shown in [6] that the matrix $T(\lambda)$ satisfies the following equation

$$\frac{d}{dt} T(\lambda) = T(\lambda) B_-(\lambda) - B_+(\lambda) T(\lambda), \tag{5.15}$$

where

$$\begin{aligned} B_-(\lambda) &= \begin{bmatrix} \frac{1}{2\lambda} & \beta_- M_- \\ -\beta_+ M_+ & -\frac{1}{2\lambda} \end{bmatrix}, \\ B_+(\lambda) &= \begin{bmatrix} \frac{1}{2\lambda} & \beta_+ M_- \\ -\beta_- M_+ & -\frac{1}{2\lambda} \end{bmatrix} \end{aligned} \tag{5.16}$$

are the vector representations of the operator $B(\lambda)$ on the intervals $(x < x_1)$ and $x > x_d$ respectively. Furthermore, setting

$$A(\lambda) = T(\lambda) \beta, \tag{5.17}$$

where

$$\beta = \begin{bmatrix} \gamma & 0 \\ 0 & \gamma^{-1} \end{bmatrix},$$

and $\gamma = \sqrt{\beta_-/\beta_+}$. Then, $A(\lambda)$ satisfies the following Lax pairs equation

$$\frac{d}{dt}A(\lambda) = [A(\lambda), B_+(\lambda)]. \tag{5.18}$$

But since the Lax pairs equation (5.18) implies that the trace of $A(\lambda)$ is a conserved quantity, then one can always choose $A(\lambda)$ to be traceless as follows. Let

$$A(\lambda) \to A'(\lambda) = A(\lambda) - \frac{1}{2}Tr(A)\mathbb{I}. \tag{5.19}$$

Then,

$$\frac{d}{dt}A'(\lambda) = \frac{d}{dt}A(\lambda), \qquad [A'(\lambda), B(\lambda)] = [A(\lambda), B(\lambda)].$$

Therefore, without loss of generality, we can always consider the matrix $A(\lambda)$ to be traceless. We will always work with this traceless matrix and for convenience we will denote it by $A(\lambda)$ instead of $A'(\lambda)$.

5.3 Two types of Singularities.

For d peakons, the CF Hamiltonian is given as follows

$$H(x_1, ..., x_d, m_1, ..., m_d) = \frac{1}{2}\sum_{j,k=1}^{d} m_j m_k \ G_{\nu,\beta}(x_j - x_k), \tag{5.20}$$

where the Green's function $G_{\nu,\beta}$ is given by (5.7). Furthermore, we set $\beta_- - \beta_+ = 1$; this conditions is equivalent to the fact that the Green's function satisfies the following equation

$$\left(\frac{d^2}{dx^2} - 4\nu^2\right) G_{\nu,\beta} = -2\delta(x).$$

For $d = 2$, the Hamiltonian is given as follows

$$\begin{aligned} H(x_1, x_2, m_1, m_2) &= \frac{1}{2}\sum_{j,k=1}^{2} m_j m_k \ G_{\nu,\beta}(x_j - x_k), \\ &= \frac{1}{2}(m_1^2 + m_2^2) \ G_{\nu,\beta}(0) + m_1 m_2 \ G_{\nu,\beta}(x_1 - x_2), \\ &= \frac{1}{2}M^2 \ G_{\nu,\beta}(0) + m_1 m_2 \ [G_{\nu,\beta}(x_1 - x_2) - G_{\nu,\beta}(0)]. \end{aligned} \tag{5.21}$$

Now let

$$\begin{aligned} H^0 &= \frac{1}{2}M^2 \ G_{\nu,\beta}(0), \\ H^i &= m_1 m_2 \ [G_{\nu,\beta}(x_1 - x_2) - G_{\nu,\beta}(0)] \end{aligned}$$

be the stationary Hamiltonian and the interaction Hamiltonian respectively, then $H = H^0 + H^i$. Since the Hamiltonian H as well as the total momentum $M = m_1 + m_2$ are conserved, then H^0 and H^i are also conserved. In other words, $m_1 m_2$ blows up if and only if $[G_{\nu,\beta}(x_1 - x_2) - G_{\nu,\beta}(0)] \to 0$ such that their product remains constant.

Now, consider the following quantity

$$[G_{\nu,\beta}(x_1 - x_2) - G_{\nu,\beta}(0)] = \frac{\beta_+}{2\nu}(e^{2\nu|x_1-x_2|} - 1) + \frac{\beta_-}{2\nu}(e^{-2\nu|x_1-x_2|} - 1). \tag{5.22}$$

Let $2\nu|x_1 - x_2| = \ln(\beta_-/\beta_+)$, then (5.22) becomes

$$\begin{aligned}[G_{\nu,\beta}(x_1 - x_2) - G_{\nu,\beta}(0)] &= \frac{\beta_+}{2\nu}(\beta_-/\beta_+) - 1) + \frac{\beta_-}{2\nu}(\beta_+/\beta_- - 1),\\ &= \frac{1}{2\nu}(\beta_- - \beta_+ + \beta_+ - \beta_-) = 0.\end{aligned} \tag{5.23}$$

Therefore, the product $m_1 m_2$ blows up whenever $2\nu|x_1 - x_2| = \ln(\beta_-/\beta_+)$. Note that for this type of collision, which we call pseudo-collision from now on, the separation $|x_1 - x_2|$ depends explicitly on the values of the constants ν and β_+. On the other hand, at the time of collision, which we denote by t_c, $[G_{\nu,\beta}(x_1 - x_2) - G_{\nu,\beta}(0)] \to 0$ as $|x_1 - x_2| \to 0$. Note that this is valid for any value of the constants ν and β_+. In other words, this regular collision does not depend explicitly on these two constants. However, these constants still implicitly determine whether this type of collision happens or not as will be shown in what follows. Now, our task is to study these two types of collisions separately and to determine the geometric interpretation of each case. For this purpose, we have to embed the CF system into a Hitchin system and then study the evolution of the associated Higgs field.

5.4 The Higgs Field for $d = 2$

Now, recall that $A(\lambda)$ is a matrix-valued polynomial of degree d in λ. This matrix acts by multiplication on the vector spaces $\mathbb{C}^2$ parametrized by $\lambda \in \mathbb{P}^1$. Therefore, we may consider $A(\lambda) = \sum_{k=0}^{d} A_k \lambda^k$ to be acting on the holomorphic vector bundle $\mathbb{P}^1 \times \mathbb{C}^2$; this is a vector bundle of rank 2 and degree 0 with trivial holomorphic structure. Furthermore, by the Birkhoff-Grothendieck splitting theorem, the vector bundle E decomposes uniquely as $E = \mathcal{O} \oplus \mathcal{O}$ where $\mathcal{O} = \mathbb{P}^1 \times \mathbb{C}$ is the trivial line bundle on $\mathbb{P}^1$. Therefore, one can view $A(\lambda)$ as a Higgs field that acts on this trivial vector bundle. In other words,

$$A \in H^0(\mathbb{P}^1; End(E) \otimes \mathcal{O}(d)), \tag{5.24}$$

and so the pair (E, A) is a (twisted) Higgs bundle. Therefore, we have embedded the CF system into a twisted Hitchin system. This new prespective allows us to study the CF dynamics by considering the evolution of the Higgs field $A(\lambda)$. For this purpose, our goal now is to compute the Higgs field $A(\lambda)$ explicitly. For simplicity, we will limit the present discussion to the case $d = 2$. In this case the Higgs field $A(\lambda)$ is given as follows $A(\lambda) = A_0 + A_1\lambda + A_2\lambda^2$. Recall now that we limited our discussion to traceless Higgs fields and this Higgs field is traceful. Therefore, we have to apply the transformation

$$A(\lambda) \to A(\lambda) - \frac{1}{2}Tr(A)\mathbb{I}_2,$$

Then we obtain a traceless Higgs field given at any point $\lambda \in \mathbb{P}^1$ as follows

$$A(\lambda) = \begin{bmatrix} \frac{1}{2}(\gamma T_{11} - \gamma^{-1} T_{22}) & \gamma^{-1} T_{12} \\ \gamma T_{21} & -\frac{1}{2}(\gamma T_{11} - \gamma^{-1} T_{22}) \end{bmatrix}, \tag{5.25}$$

where T_{jk} for $j = 1, 2$ are elements of the transfer matrix $T = T_2(\lambda)T_1(\lambda)$ given as follows

$$\begin{aligned} T = T_2 T_1 &= \left(\mathbb{I} + \lambda m_2 \begin{bmatrix} 1 & e^{-2\nu x_2} \\ -e^{2\nu x_2} & -1 \end{bmatrix} \right) \left(\mathbb{I} + \lambda m_1 \begin{bmatrix} 1 & e^{-2\nu x_1} \\ -e^{2\nu x_1} & -1 \end{bmatrix} \right), \\ &= \mathbb{I} + \lambda \begin{bmatrix} M & M_- \\ -M_+ & -M \end{bmatrix} + \lambda^2 m_1 m_2 \begin{bmatrix} 1 - e^{2\nu(x_1 - x_2)} & e^{-2\nu x_1}(1 - e^{-2\nu(x_2 - x_1)}) \\ e^{2\nu x_1}(1 - e^{2\nu(x_2 - x_1)}) & 1 - e^{2\nu(x_2 - x_1)} \end{bmatrix}. \end{aligned}$$

Therefore, the elements of the matrix $A(\lambda)$ defined in (5.25) are given as follows

$$\begin{aligned} A_{11} = -A_{22} &= \frac{1}{2} \left[(\gamma - \gamma^{-1}) + (\gamma + \gamma^{-1}) M \lambda + m_1 m_2 \lambda^2 \left(\gamma (1 - e^{-2\nu(x_2 - x_1)}) - \gamma^{-1}(1 - e^{2\nu(x_2 - x_1)}) \right) \right], \\ A_{21} &= -\gamma e^{2\nu x_2} \lambda \left[(m_1 e^{-2\nu(x_2 - x_1)} + m_2) - \lambda m_1 m_2 (e^{-2\nu(x_2 - x_1)} - 1) \right], \\ A_{12} &= \gamma^{-1} e^{-2\nu x_2} \lambda \left[(m_1 e^{2\nu(x_2 - x_1)} + m_2) + \lambda m_1 m_2 (e^{2\nu(x_2 - x_1)} - 1) \right]. \end{aligned} \tag{5.26}$$

Note that the trace of the original (traceful) Higgs field is an invariant of motion. In this case, it is given as follows

$$Tr(A) = \gamma T_{11} + \gamma^{-1} T_{22} = (\gamma + \gamma^{-1}) + \lambda M (\gamma - \gamma^{-1}) + \lambda^2 m_1 m_2 [(1 - e^{-2\nu(x_2 - x_1)})(\gamma - \gamma^{-1} e^{2\nu(x_2 - x_1)})]. \tag{5.27}$$

Therefore, the total momentum M as well as the following quantity

$$C_1 = m_1 m_2 [(1 - e^{-2\nu(x_2 - x_1)})(\gamma - \gamma^{-1} e^{2\nu(x_2 - x_1)})] \tag{5.28}$$

are invariants of motion.

5.4.1 The Spectral Curve and Möbius Transformations

As mentioned before, the spectral curve associated to this Higgs field defined by the following equation

$$Det(\eta \mathbb{I} - A(\lambda)) = 0. \tag{5.29}$$

But since the matrix $A(\lambda)$ is traceless, the equation simplifies to the following form

$$\eta^2 = -Det(A(\lambda)) = \frac{1}{2} Tr(A^2(\lambda)). \tag{5.30}$$

But since the Lax pairs equation implies that

$$\frac{d}{dt} Tr(A^n) = 0$$

for any $n \in \mathbb{Z}^+$, then we see again that this spectral curve is invariant under the flow. This spectral curve, which we denote by Y, is elliptic ($g = 1$) by the Riemann-Hurwitz formula. Therefore, it represents a double cover of the Riemann sphere $\mathbb{P}^1$ on which η is a single-valued function of λ. The branch points of Y are those points for which $Det(A) = 0$. Writing these roots explicitly, one can easily see that the Higgs field is simple. As mentioned before, since $r = 2$ and the Higgs field $A(\lambda)$ is simple, then the Higgs bundle (E, A) is stable. Furthermore, since the spectral curve is elliptic, one can always find a Möbius transformation that sends three of the four roots of $Det(A)$ to $0, 1, \infty$. Formally, let the roots of the spectral curve be $\lambda_1, \lambda_2, \lambda_3$, and λ_4, then

$$M\ddot{o}b(\lambda) = \left(\frac{\lambda - \lambda_1}{\lambda - \lambda_3}\right)\left(\frac{\lambda_2 - \lambda_3}{\lambda_2 - \lambda_1}\right) \tag{5.31}$$

is the desired Möbius transformation. Obviously, this transformation sends $\lambda_1 \to 0$, $\lambda_2 \to 1$, and $\lambda_3 \to \infty$. In fact, this transformation can be applied to any elliptic curve to send three of its branch points points to $0, 1, \infty$. This reflects the fact that the moduli space of elliptic curves over a Riemann surface is just $\mathbb{P}^1$.

5.4.2 Collision vs Pseudo-collision Higgs Fields.

Now consider the point of collision at which $x_1 = x_2$. At this point,

$$m_1 m_2 (1 - e^{-2\nu(x_2 - x_1)}) \to \frac{C_1}{\gamma - \gamma^{-1}}, \tag{5.32}$$

while

$$m_1 m_2 (1 - e^{2\nu(x_2 - x_1)}) \to \frac{C_1}{\gamma^{-1} - \gamma}. \tag{5.33}$$

Therefore, the collision Higgs field is given as follows

$$A_c(\lambda) = \begin{bmatrix} \frac{1}{2}\left[(\gamma - \gamma^{-1}) + (\gamma + \gamma^{-1})M\lambda + (\beta_- + \beta_+)C_1\lambda^2\right] & \gamma^{-1}e^{-2\nu x_2}\lambda\left(M + \frac{C_1}{\gamma - \gamma^{-1}}\lambda\right) \\ -\gamma e^{2\nu x_2}\lambda\left(M + \frac{C_1}{\gamma - \gamma^{-1}}\lambda\right) & \frac{-1}{2}\left[(\gamma - \gamma^{-1}) + (\gamma + \gamma^{-1})M\lambda + (\beta_- + \beta_+)C_1\lambda^2\right] \end{bmatrix}. \tag{5.34}$$

Interestingly, the roots of the two elements A_{12} and A_{21} collide at the point of collision. This fact will have tremendous impact in our analysis regarding the geometric interpretation of the collision point. Now consider the following gauge transformation

$$G = \begin{bmatrix} g & 0 \\ 0 & g^{-1} \end{bmatrix}. \tag{5.35}$$

Obviously this is an $SL(2, \mathbb{C})$ gauge. Since this is an automorphism of the trivial bundle $E = \mathcal{O} \oplus \mathcal{O}$, then $g \in \mathbb{C}^*$. Let

$$g^2 = \gamma e^{2\nu x_2},$$

then the gauged Higgs field is given as follows

$$\begin{aligned} A_c'(\lambda) &= G^{-1}A_c(\lambda)G, \\ &= \begin{bmatrix} \frac{1}{2}\left[(\gamma - \gamma^{-1}) + (\gamma + \gamma^{-1})M\lambda + (\beta_- + \beta_+)C_1\lambda^2\right] & \lambda\left(M + \frac{C_1}{\gamma - \gamma^{-1}}\lambda\right) \\ -\lambda\left(M + \frac{C_1}{\gamma - \gamma^{-1}}\lambda\right) & \frac{-1}{2}\left[(\gamma - \gamma^{-1}) + (\gamma + \gamma^{-1})M\lambda + (\beta_- + \beta_+)C_1\lambda^2\right] \end{bmatrix}. \end{aligned} \tag{5.36}$$

Obviously, the collision Higgs field is special since the off-diagonal elements of A'_c are identical up to a minus sign. This fact will have a huge impact on the geometrical interpretation of the collision point and we will elaborate on this fact in section (5.6).

On the other hand, pseudo-collision occurs when $\frac{1}{2\nu}(x_2 - x_1) = \ln(\beta_-/\beta_+) = \ln(\gamma^2)$ and therefore

$$e^{2\nu(x_2-x_1)} = \gamma^2, \quad e^{-2\nu(x_2-x_1)} = \gamma^{-2}. \tag{5.37}$$

Hence, the Higgs field is given as follows at the time of pseudo-collision

$$A_p(\lambda) = \begin{bmatrix} \frac{1}{2}\left[(\gamma-\gamma^{-1}) + (\gamma+\gamma^{-1})M\lambda + 2m_1m_2\lambda^2(\gamma-\gamma^{-1})\right] & e^{-2\nu x_2}\lambda\left[(m_1\gamma + m_2\gamma^{-1}) + \lambda m_1m_2(\gamma-\gamma^{-1})\right] \\ -e^{2\nu x_2}\lambda\left[(m_1\gamma^{-1} + m_2\gamma) + \lambda m_1m_2(\gamma^{-1}-\gamma)\right] & -\frac{1}{2}\left[(\gamma-\gamma^{-1}) + (\gamma+\gamma^{-1})M\lambda + 2m_1m_2\lambda^2(\gamma-\gamma^{-1})\right] \end{bmatrix}. \tag{5.38}$$

However, recall that the momenta m_1 and m_2 blow up at the time of pseudo-collision t_p as well as at the time of collision t_c. Therefore, it is quite evident that the Higgs field blows up at t_p. Another way to say this is that the holomorphic structure on the Higgs bundle (E, A) breaks down at t_p. We will discuss this fact further in section (5.6) when we consider the periodization of the dynamics. But before that, we need to prove some theorems concerning the dynamics and this is the topic of the upcoming section.

5.5 Peakons Confinement

We have shown in section (5.3) that the momenta m_1 and m_2 blow up at t_c as well as at t_p. A natural question arises, is there is a natural analytic continuation past these singularities that extends smooth dynamics? As one would expect, this is in fact true and the purpose of this section is to show that at t_c as well as at t_p, there is a natural analytic continuation of the solution such that m_1 and m_2 exchange signs and the slope of distance between the two peakons reverse direction. In other words, we prove that the two quantities

$$m_1 - m_2, \quad \frac{d}{dt}(x_2 - x_1)$$

change their signs at t_c and at t_p. However, we begin by proving some theorems that are the essential tools for proving the main result.

Theorem 5.5.1. *Given M_- and M_+ given by (5.10), then*

$$\begin{aligned} \frac{\dot{M}_-}{M_-} &= (\beta_- + \beta_+)M + \sum_{j=1}^{d-1}\frac{1}{\lambda_{1j}}, \\ \frac{\dot{M}_+}{M_+} &= -(\beta_- + \beta_+)M - \sum_{j=1}^{d-1}\frac{1}{\lambda_{2j}}, \end{aligned} \tag{5.39}$$

where λ_{1j} and λ_{2j} are the non-zero roots of A_{12} and A_{21} respectively.

Proof. The Lax pairs equation (5.18) implies that the off-diagonal elements of the matrix $A(\lambda)$ evolve as follows

$$\dot{A}_{12} = \frac{-1}{\lambda}A_{12} + \beta_+ M_-(A_{11} - A_{22}),$$
$$\dot{A}_{21} = \frac{1}{\lambda}A_{21} + \beta_- M_+(A_{11} - A_{22}), \tag{5.40}$$

where $(A_{11} - A_{22}) = (\gamma - \gamma^{-1}) + (\gamma + \gamma^{-1})M\lambda + O(\lambda^2)$. Furthermore,

$$\frac{\dot{A}_{12}}{A_{12}} = \frac{-1}{\lambda} + \beta_+ M_- \frac{A_{11} - A_{22}}{A_{12}},$$
$$\frac{\dot{A}_{21}}{A_{21}} = \frac{1}{\lambda} + \beta_- M_+ \frac{A_{11} - A_{22}}{A_{21}}. \tag{5.41}$$

Now, let the non-zero roots of A_{12} and A_{21} be denoted by λ_{1j} and λ_{2j} respectively where $j = 1, ..., d-1$. Then

$$A_{12} = \gamma^{-1} M_- \lambda \prod_{j=1}^{d-1}(1 - \frac{\lambda}{\lambda_{1j}}),$$
$$A_{21} = -\gamma M_+ \lambda \prod_{j=1}^{d-1}(1 - \frac{\lambda}{\lambda_{2j}}). \tag{5.42}$$

It follows that

$$\frac{\dot{A}_{12}}{A_{12}} = \frac{\dot{M}_-}{M_-} + \frac{d}{dt}log(\lambda) + \frac{d}{dt}\sum_{j=1}^{g} log(1 - \frac{\lambda}{\lambda_{1j}}),$$
$$\frac{\dot{A}_{21}}{A_{21}} = \frac{\dot{M}_+}{M_+} + \frac{d}{dt}log(\lambda) + \frac{d}{dt}\sum_{j=1}^{g} log(1 - \frac{\lambda}{\lambda_{2j}}). \tag{5.43}$$

Thus,

$$\lim_{\lambda\to 0}\frac{\dot{A}_{12}}{A_{12}} = \frac{\dot{M}_-}{M_-},$$
$$\lim_{\lambda\to 0}\frac{\dot{A}_{21}}{A_{21}} = \frac{\dot{M}_+}{M_+}. \tag{5.44}$$

From (5.42),

$$\frac{\gamma^{-1}M_-}{A_{12}} = \frac{1}{\lambda\prod_{j=1}^{g}(1 - \frac{\lambda}{\lambda_{1j}})},$$
$$\frac{\gamma M_+}{A_{21}} = \frac{-1}{\lambda\prod_{j=1}^{g}(1 - \frac{\lambda}{\lambda_{2j}})}. \tag{5.45}$$

Substituting both of these equations into (5.41) and simplifying, one finds that

$$\lim_{\lambda\to 0}\frac{\dot{A}_{12}}{A_{12}} = (\beta_- + \beta_+)M + \sum_{j=1}^{d-1}\frac{1}{\lambda_{1j}},$$
$$\lim_{\lambda\to 0}\frac{\dot{A}_{21}}{A_{21}} = -(\beta_- + \beta_+)M - \sum_{j=1}^{d-1}\frac{1}{\lambda_{2j}}. \tag{5.46}$$

This proves the theorem.

$\square$

Theorem 5.5.2. *The non-zero roots λ_{1j} and λ_{2j} of A_{12} and A_{21} respectively evolve as follows*

$$\begin{aligned} \dot{\lambda}_{1j} &= -\gamma\beta_+ \frac{A_{11}(\lambda_{1j}) - A_{22}(\lambda_{1j})}{\prod_{k\neq j}(1-\lambda_{1j}/\lambda_{1k})}, \\ \dot{\lambda}_{2j} &= \gamma^{-1}\beta_- \frac{A_{11}(\lambda_{2j}) - A_{22}(\lambda_{2j})}{\prod_{k\neq j}(1-\lambda_{2j}/\lambda_{2k})}, \end{aligned} \tag{5.47}$$

where $j = 1, ..., d-1$.

Proof. Differentiating (5.42) and evaluating at the roots, one get

$$\begin{aligned} \dot{A}_{12}(\lambda_{1j}) &= \gamma^{-1} M_- \lambda_{1j} \left[\frac{d}{dt} \prod_{k=1}^{g} (1 - \frac{\lambda}{\lambda_{1k}}) \right]_{\lambda=\lambda_{1j}}, \\ &= -\gamma^{-1} M_- \dot{\lambda}_{1j} \prod_{k\neq j} (1 - \lambda_{1j}/\lambda_{1k}). \\ \dot{A}_{21}(\lambda_{2j}) &= -\gamma M_+ \lambda_{2j} \left[\frac{d}{dt} \prod_{k=1}^{g} (1 - \frac{\lambda}{\lambda_{2k}}) \right]_{\lambda=\lambda_{2j}}, \\ &= \gamma M_+ \dot{\lambda}_{2j} \prod_{k\neq j} (1 - \lambda_{2j}/\lambda_{2k}). \end{aligned} \tag{5.48}$$

On the other hand, evaluating (5.40) at the roots, we get

$$\begin{aligned} \dot{A}_{12}(\lambda_{1j}) &= \beta_+ M_- (A_{11}(\lambda_{1j}) - A_{22}(\lambda_{1j})), \\ \dot{A}_{21}(\lambda_{2j}) &= \beta_- M_+ (A_{11}(\lambda_{2j}) - A_{22}(\lambda_{2j})). \end{aligned} \tag{5.49}$$

Equating (5.48) with (5.49) proves the theorem □

The previous two theorems where valid for any $d \geq 2$. However, the following two theorems are only valid in the case $d = 2$. We also set

$$X = \frac{1}{2\nu} \ln(\beta_-/\beta_+).$$

Lemma 5.5.3. *Assume that at $t = 0$, $|x_2 - x_1| < X$. If $m_1 m_2 > 0$, then*

$$0 < [G(0) - G(x_1 - x_2)] \leq \frac{1}{2\nu}(\sqrt{\beta_-} - \sqrt{\beta_+})^2$$

for all $t > 0$.

On the other hand, if $m_1 m_2 < 0$, then

$$0 \leq [G(0) - G(x_1 - x_2)] \leq \frac{1}{2\nu}(\sqrt{\beta_-} - \sqrt{\beta_+})^2$$

for all $t > 0$ and $[G(0) - G(x_1 - x_2)] = 0$ only when $t = t_c$ or $t = t_p$.

Proof. By the second equation in (5.9), the momenta m_j do not flip their signs and therefore the quantity $m_1 m_2$ preserves its sign as time flows. Furthermore, we have already shown in section (5.3) that $[G(0) - G(x_1 - x_2)] = 0$ only at t_c and at t_p. Therefore to prove this lemma, it suffices to show that

$$0 < [G_{\nu,\beta}(0) - G_{\nu,\beta}(x_1 - x_2)] \leq \frac{1}{2\nu}(\sqrt{\beta_-} - \sqrt{\beta_+})^2$$

for all $t > 0$ during the smooth dynamics. Recall that

$$G(0) - G(x_1 - x_2) = \frac{1}{2\nu}[(\beta_- + \beta_+) - (\beta_- e^{-2\nu|x_1-x_2|} + \beta_+ e^{2\nu|x_1-x_2|})]. \tag{5.50}$$

Let $z = |x_1 - x_2|$ and $f(z) = G(0) - G(x_1 - x_2)$, then

$$f'(z) = 0 \to \beta_-/\beta_+ = e^{4\nu z}. \tag{5.51}$$

Therefore, the function $f(z)$ has only one critical point $z_0 = \frac{1}{4\nu}\ln(\beta_-/\beta_+)$. Since $f''(z_0) = -4\nu\sqrt{\beta_-\beta_+} < 0$, then $f(z)$ has a local maximum at z_0 where

$$f(z_0) = \frac{1}{2\nu}(\sqrt{\beta_-} - \sqrt{\beta_+})^2.$$

Furthermore, during the smooth dynamics $0 < (x_2 - x_1) < X$. At both boundaries $f(z) = 0$ and therefore zero is the global minimum of the function $f(z)$ in that region. This proves the theorem. □

We will now prove the main theorem of this section, that is the two peakons are confined in a moving box whose width is defined by the value of the constants ν and β_+.

Theorem 5.5.4. *For $d = 2$, at the time of collision t_c as well as at the time of pseudo-collision t_p, the quantity, $(m_1 - m_2)$ flips sign and so does $\frac{d}{dt}(x_2 - x_1)$. Hence, if at $t = 0$, $(x_2 - x_1) < X$, then at every moment $t > 0$, $(x_2 - x_1) \leq X$. Therefore, the two peakons are then confined in a moving 1-dimensional box whose width is given by the equation*

$$(x_2 - x_1) = X.$$

Furthermore, the two peakons collide with the boundaries of the box if and only if $m_1 m_2 < 0$.

Proof. We begin by showing that the quantity $m_1 - m_2$ flips sign if and only if $\lambda_{11} - \lambda_{21}$ also does. Note that by (5.26) the roots λ_{11} and λ_{21} are given as follows for any time t

$$\begin{aligned} \lambda_{21} &= \frac{m_1 e^{-2\nu(x_2-x_1)} + m_2}{m_1 m_2 (e^{-2\nu(x_2-x_1)} - 1)}, \\ \lambda_{11} &= -\frac{m_1 e^{2\nu(x_2-x_1)} + m_2}{m_1 m_2 (e^{2\nu(x_2-x_1)} - 1)}. \end{aligned} \tag{5.52}$$

One can show by direct calculation, that

$$\lambda_{21} - \lambda_{11} = \frac{m_1 - m_2}{m_1 m_2}. \tag{5.53}$$

As mentioned before, the quantity $m_1 m_2$ preserves its sign at all times and therefore, the quantity $m_1 - m_2$ flips sign if and only if $\lambda_{11} - \lambda_{21}$ also does. Furthermore, the first equation in (5.9) implies that

$$\dot{x_2} - \dot{x_1} = u(x_2) - u(x_1) = (m_1 - m_2)\left[G(x_1 - x_2) - G(0)\right].$$

However, we proved in the previous lemma that the quantity $[G(x_1 - x_2) - G(0)]$ is bounded from above by zero and therefore preserves its sign at every time . Hence, whenever the quantity $(m_1 - m_2)$ flips its sign,

$\frac{d}{dt}(x_2 - x_1)$ also does. To complete the proof, it suffices then to show that at t_c as well as at t_p, the quantity $(\lambda_{11} - \lambda_{21})$ flips signs. Now, consider the collision moment t_c at which $x_1 = x_2$, then $M_- = Me^{-2\nu x_1}$ and $M_+ = Me^{2\nu x_1}$. Thus,

$$\frac{\dot{M}_-}{M_-} + \frac{\dot{M}_+}{M_+} = 0.$$

Then theorem (5.5.1) implies that $\lambda_{11} = \lambda_{21}$ (In fact, we have shown that explicitly in (5.33). Furthermore, theorem(5.5.2) implies that

$$\dot{\lambda}_{11} = -\dot{\lambda}_{21}. \tag{5.54}$$

Thus, if at $t \to t_c^-$, λ_{11} and λ_{22} approach $\lambda_c = -M/C_2$ from different directions, then λ_{11} and λ_{22} changes sign at t_c. Consider now (5.52), as $t \to t_c^-$, $e^{2\nu(x_2-x_1)} \to 1 - 2\nu(x_2 - x_1) + O((x_2 - x_1)^2)$. Hence

$$\lambda_{11} \to \frac{-M - 2\nu(x_2 - x_1)m_1}{C_2},$$
$$\lambda_{21} \to \frac{-M + 2\nu(x_2 - x_1)m_1}{C_2}.$$

and thus λ_{11} and λ_{21} approach λ_c from different directions depending on the sign of m_1. We conclude then that at t_c, the quantity $\lambda_{11} - \lambda_{21}$ flips sign and so do $(m_1 - m_2)$ and $\frac{d}{dt}(x_2 - x_1)$.

On the other hand, as $t \to t_p^-$, the roots are given up to a constant as follows

$$\lambda_{11} = \frac{M}{-m_1 m_2} - \frac{1}{m_1},$$
$$\lambda_{21} = \frac{M}{-m_1 m_2} - \frac{1}{m_2}.$$

Therefore, λ_{11} and λ_{21} approach $\lambda = 0$ from different directions depending on the signs of m_1 and m_2. Furthermore, since at t_p $\lambda_{11} = \lambda_{21} = 0$, then theorem (5.5.2) again implies that $\dot{\lambda}_{11} = -\dot{\lambda}_{21}$. Therefore, we conclude that at t_p, the quantity $\lambda_{11} - \lambda_{21}$ flips sign and so do $(m_1 - m_2)$ and $\frac{d}{dt}(x_2 - x_1)$.

However, since before t_c, the two peakons are approaching and therefore $(x_2 - x_1)$ is decreasing. Then just after t_c, $(x_2 - x_1)$ is increasing and therefore the two peakons are moving apart. Similarly, before t_p $(x_2 - x_1)$ is increasing, then just after t_p the two peakons will be approaching. We conclude if at $t = 0$ $|x_2 - x_1| < X$, then $|x_2 - x_1| \leq X$ for all $t > 0$. Now assume that at $t = 0$, $(x_2 - x_1) < X$, the previous conclusions imply that the 2 peakons will never cross the boundaries of the confinement box whose boundaries are defined by $(x_2 - x_1) = X$ and we showed in section (5.3) that pseudo-collision only occur when $m_1 m_2 < 0$. This proves the theorem. $\square$

5.6 The Geometry of Collisions and Analytic Continuation

We will now provide a concrete geometric interpretation of the collision and the pseudo-collision points as well as the natural analytic continuation that we discussed in the last section. Recall that since $(E, A) \in \mathcal{M}_{SL}(2,0)$, we have to restrict to $SL(2,\mathbb{C})$ gauge transformations. Therefore, we can now use the following gauge transformation

$$G_c = \begin{bmatrix} 1/\left(M + \frac{C_1}{\gamma-\gamma^{-1}}\lambda\right) & 0 \\ 0 & \left(M + \frac{C_1}{\gamma-\gamma^{-1}}\lambda\right) \\ & \end{bmatrix}$$

to transform the trivial bundle $E = \mathcal{O} \oplus \mathcal{O}$ to the bundle $E = \mathcal{O}(-1) \oplus \mathcal{O}(1)$ as follows

$$A_c''(\lambda) = G_c^{-1} A_c'(\lambda) G_c = \begin{bmatrix} A_{11} & \lambda\left(M + \frac{C_1}{\gamma-\gamma^{-1}}\lambda\right)^3 \\ -\lambda/\left(M + \frac{C_1}{\gamma-\gamma^{-1}}\lambda\right) & -A_{11} \end{bmatrix}. \tag{5.55}$$

Therefore, the collision Higgs field is singular. However, as mentioned in section (4.4), the vector bundle $E = \mathcal{O}(-1) + \mathcal{O}(1)$ has a unique Higgs field that, as a section of the Hitchin system, intersects each torus in one point. In fact, this point is the theta divisor and therefore the theta divisor is the collision locus that highlights the singular dynamics. However, recall that by theorem (2.3.16) up to conjugation, the space of all Higgs fields $A(\lambda)$ with the same spectral curve Y can be identified with $\mathcal{J}(Y)/\Theta$. Furthermore, the Higgs field becomes singular when the eigenspace line bundles hits the theta-divisor. Therefore, both results confirm the fact that the theta-divisor is the locus of the singular dynamics. This analysis first appeared in [48] but we included it here for completion.

As shown in (5.36), the collision point admits a unique behavior in a sense that, up to a constant, the non-diagonal elements of the Higgs field coincide at this moment. We will show now that this unique behavior of the Higgs field at t_c enables us to normalize it using gauge transformations. All we need to do is to turn-off the diagonal elements. For that purpose, we write (5.36) as follows

$$A_c' = \begin{bmatrix} A_{11} & A_{12} \\ -A_{12} & -A_{11} \end{bmatrix},$$

and we transform A_c' by the following gauge

$$G_1 = \frac{1}{\sqrt{A_{12}}} \begin{bmatrix} 1 & 0 \\ 0 & A_{12} \end{bmatrix}, \quad G_1^{-1} = \sqrt{A_{12}} \begin{bmatrix} 1 & 0 \\ 0 & 1/A_{12} \end{bmatrix}.$$

Then, A_c' becomes (for simplicity we will keep calling the transformed matrix A_c')

$$A'(c) \to G_1^{-1} A_c' G_1 = \begin{bmatrix} A_{11} & 1 \\ -A_{12}^2 & -A_{11} \end{bmatrix}.$$

Then, we apply the following gauge transformation

$$G_2 = \begin{bmatrix} 1 & 0 \\ -A_{11} & 1 \end{bmatrix}, \quad G_2^{-1} = \begin{bmatrix} 1 & 0 \\ A_{11} & 1 \end{bmatrix}. \tag{5.56}$$

Then, A'_c transforms as follows

$$A'_c \to G_2^{-1} A'_c G_2 = \begin{bmatrix} 0 & 1 \\ A_{11}^2 - A_{12}^2 & 0 \end{bmatrix}. \tag{5.57}$$

However, from (5.36) $det(A'_c) = q = A_{12}^2 - A_{11}^2$. Therefore, the gauged A'_c has the following form

$$A'_c = \begin{bmatrix} 0 & 1 \\ -q(\lambda) & 0 \end{bmatrix}. \tag{5.58}$$

Finally, we transform A'_c by the following gauge transformation

$$G_3 = \frac{1}{\sqrt[4]{q}} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{q} \end{bmatrix} \to G_3^{-1} = \sqrt[4]{q} \begin{bmatrix} 1 & 0 \\ 0 & 1/\sqrt{q} \end{bmatrix}, \tag{5.59}$$

then

$$A'_c \to A'_c = G_3^{-1} A'_c G_3 = \begin{bmatrix} 0 & \sqrt{q} \\ -\sqrt{q} & 0 \end{bmatrix}. \tag{5.60}$$

This is the normal form of A'_c that we aspire.

Recall now from section (4.5) that limiting configurations are pairs $(\mathcal{D}_\infty, A_\infty)$ that arise as limits of solutions to Hitchin's equations and satisfy the decoupled version of Hitchin's equations (4.36). However, since the Higgs field A'_c is normal, then it satisfies the second equation in (4.36), namely

$$[A, A^\dagger] = 0.$$

In other words, the collision Higgs field $A'_c(\lambda)$ is naturally equipped with a flat connection $\mathcal{D}_c$ and the pair $(\mathcal{D}_c, A'_c)$ represents a limiting configuration. However, as pointed our before, every limiting configuration arises as a limit of a diverging sequence of solutions to Hitchin's equations. Our job now is to determine which sequence of solution converges to A'_c and how does this connect with the analytic continuation for the problem.

Recall that we proved in theorem (5.5.4) that after t_p the two peakons shall be approaching each other while after t_c the two peakons shall be moving apart. We concluded that the two peakons are confined in a one-dimensional box whose boundaries are determined by the constants ν and β_+. These results may hint that the two peakons collide elastically with the walls of the box, reverse their directions and bounce back just as billiard balls. However, this is not the case as we will show now. Consider the velocities of the two peakons which are given as follows

$$v_1 = u(x_1) = m_1 G(0) + m_2 G(x_1 - x_2),$$
$$v_2 = u(x_2) = m_1 G(x_2 - x_1) + m_2 G(0).$$

Then, at t_c as well as at t_p,

$$v_1 = v_2 = MG(0) = \frac{M(\beta_- + \beta_+)}{2\nu}.$$

Therefore, if $M > 0$ then the two peakons are moving to the right with the same speed at t_c and at t_p. If $M < 0$, then they are moving to the left with the same speed. Finally, if $M = 0$, then they stop momentarily, and then bounce back in different directions. Obviously, our previous interpretation concerning elastic collisions may make sense when $M = 0$, that is when $m_1 = -m_2$. However, in general, it is indeed not true. Another hint towards a proper analytic continuation is to note that the Green's function of the CF system and the Green's function of the periodic Camassa-Holm are identical on the interval $-X < x < X$ given that

$$X = \frac{1}{2\nu} \ln(\beta_-/\beta_+)$$

is the period of the CH dynamics. Therefore, if we assume that peakons in the CF system satisfy the following periodization condition

$$x_{j+d}(0) = x_j(0) + X, \quad m_{j+d}(0) + X = m_j(0),$$

then the periodic CH and the CF dynamics coincide. This amounts to periodizing the CF dynamics such that the CF peakons are moving on a circle. However, recall that the CF peakons where confined in a box whose boundaries are defined by $|x_2 - x_1| < X$. Hence, this periodization process is simply gluing the boundaries of the confinement box. In this new setting, a pseudo-collision becomes simply a collision on the circle. In other words, the pseudo-collision Higgs field A_p is glued to the collision Higgs field A_c. However, starting with the Higgs field (5.25) with an appropriate initial conditions such that the two peakons are moving apart. Then the Lax pairs evolution gives us a (continuous) family of Higgs fields that diverge as $t = t_p$. Furthermore, note that the degeneration phenomena of the limiting element in the sequence manifests in A_p since all the roots of its elements become $\lambda = 0$. Hence, gluing this diverging sequence of solutions to the limiting configuration A'_c amounts to a desingularization of the diverging family solutions ending with A_p. However, this process also amounts to a desingularization of the dynamics. Furthermore, as discussed in the last chapter, this gluing process in $\mathcal{M}_{SL}(2,0)$ amounts to a compactification of the moduli space of Higgs bundles. In this case, we get a family of Higgs fields evolving with time such that $A(\lambda, t)$ is bounded for all t. Hence, this family of Higgs bundles is moving in a compact subset of $\mathcal{M}_{SL}(2,0)$. We conclude that a compactification of the CF dynamics to occur on a circle is equivalent to the compactification of the underlying moduli space of Higgs bundles.

Hence, we have shown that the CF dynamics is completely governed (reflected) by the evolution of the associated Higgs field $A(\lambda)$ in the underlying moduli space. These results are then a manifestation of the fact that the dynamics of the CF system can be realized by the dynamics of the underlying Hitchin system. In other words, the CF system can be realized as a twisted Hitchin system. This is then a concrete example of the fact that many classical integrable systems can be realized as Hitchin systems.

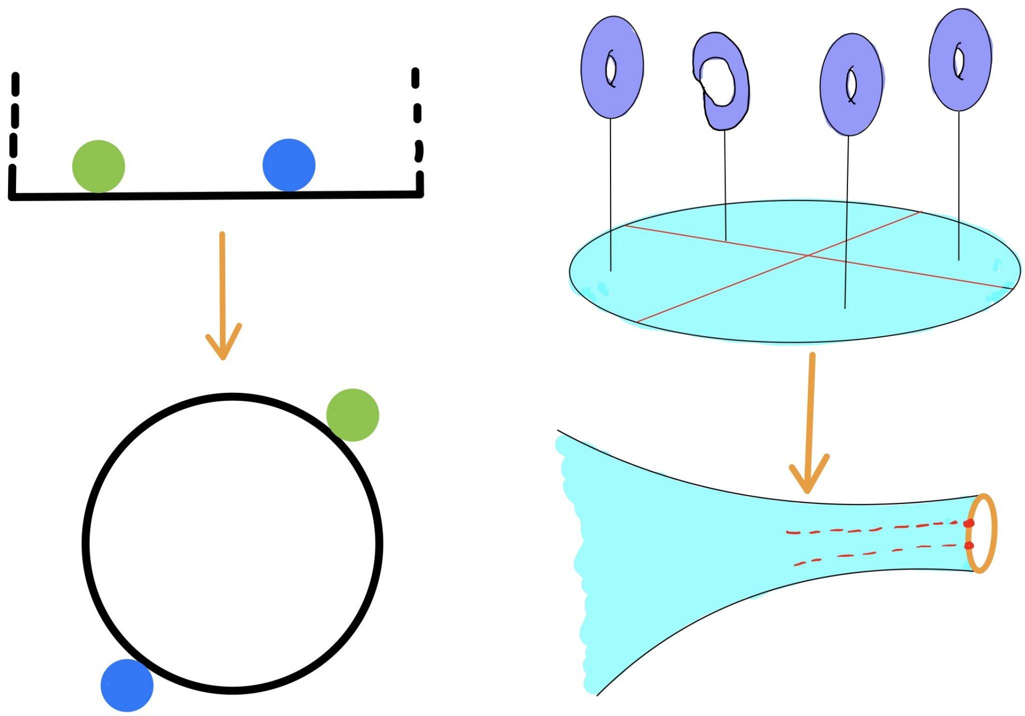

Figure 5.1: Compactification of the domain of the CF system is equivalent to compactification of the underlying moduli space of Higgs bundles.

5.7 Three Peakons Dynamics ($d = 3$).

In this section, we will study the dynamics of three peakons interactions. We start by analyzing the Hamiltonian $H(x_1, x_2, x_3, m_1, m_2, m_3)$ given as follows

$$
\begin{aligned}
H &= \frac{1}{2}\sum_{j,k=1}^{3} m_j m_k \; G_{\nu,\beta}(x_j - x_k), \\
&= \frac{1}{2}(m_1^2 + m_2^2 + m_3^2) \; G_{\nu,\beta}(0) + m_1 m_2 \; G_{\nu,\beta}(x_1 - x_2) + m_1 m_3 \; G_{\nu,\beta}(x_1 - x_3) + m_2 m_3 \; G_{\nu,\beta}(x_2 - x_3), \\
&= \frac{1}{2}M^2 \; G_{\nu,\beta}(0) + m_1 m_2 \; [G_{\nu,\beta}(x_1 - x_2) - G_{\nu,\beta}(0)] \\
&+ m_2 m_3 \; [G_{\nu,\beta}(x_2 - x_3) - G_{\nu,\beta}(0)] + m_1 m_3 \; [G_{\nu,\beta}(x_1 - x_3) - G_{\nu,\beta}(0)], \\
&= H^0 + H^i,
\end{aligned}
$$

where, $M = m_1 + m_2 + m_3$ is the total momentum and H^0 and H^i are the stationary and the interaction Hamiltonians respectively. They are defined as follows

$$
\begin{aligned}
H^0 &= \frac{1}{2}M^2 \; G_{\nu,\beta}(0), \\
H^i &= \sum_{j<k} m_j m_k [G(x_j - x_k) - G(0)],
\end{aligned}
\tag{5.61}
$$

where $j, k = 1, .., 3$. Obviously, the interaction Hamiltonian is nothing but the sum of the three interaction Hamiltonians representing the interactions between pairs of peakons. Namely,

$$H^i = \sum_{j<k} H^i_{jk} = H^i_{12} + H^i_{23} + H^i_{13}, \tag{5.62}$$

where H^i_{jk} is the interaction Hamiltonian between the peakons in the j-th and k-th positions respectively

$$H^i_{jk} = m_j m_k \; [G_{\nu,\beta}(x_j - x_k) - G_{\nu,\beta}(0)]. \tag{5.63}$$

Since the interaction Hamiltonian factors in this way, one might expect that the dynamics of the three peakons mimics the dynamics of $d = 2$ between each pair separately. In other words, each pair of peakons can collide or pseudo-collide at certain times. However, as will be shown shortly, there are other constraints that will prevent certain dynamics from occurring. Furthermore, similar to the interaction Hamiltonian, the total momentum M also factors out as follows

$$\begin{aligned} M &= m_1 + m_2 + m_3, \\ &= \frac{1}{2}[(m_1 + m_2) + (m_2 + m_3) + (m_1 + m_3)], \\ &= \frac{1}{2}\sum_{j<k} M_{jk}, \end{aligned} \tag{5.64}$$

where $j, k = 1, ..., 3$ and $M_{jk} = m_j + m_k$ is the total momentum for the peakon pair in the positions j and k respectively. This factorization also suggests that interactions only occur in pairs; this fact will be made concrete while studying the Higgs field for the three peakons.

Consider now the Higgs field $A(\lambda)$ for the three peakons. In this case, it is given as follows

$$A(\lambda) \in H^0(\mathbb{P}^1, End(E) \otimes \mathcal{O}(3)),$$

where

$$A(\lambda) = T_3 T_2 T_1 \beta = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}. \tag{5.65}$$

In this case,

$$A_{11} = \gamma \left[1 + \lambda M + \lambda^2 \sum_{j<k} m_j m_k (1 - e^{2\nu(x_j - x_k)}) + \lambda^3 m_1 m_2 m_3 (1 + \sum_{j<k} (-1)^{k-j} e^{2\nu(x_j - x_k)}) \right],$$

$$A_{12} = \gamma^{-1}\lambda \left[\sum_k m_k e^{-2\nu x_k} + \lambda \sum_{j<k} m_j m_k e^{-2\nu x_j} (1 - e^{2\nu(x_j - x_k)}) + \lambda^2 m_1 m_2 m_3 e^{-2\nu x_3} (1 + \sum_{j<k} (-1)^{k-j} e^{-2\nu(x_j - x_k)}) \right],$$

$$A_{21} = \gamma\lambda \left[- \sum_k m_k e^{2\nu x_k} + \lambda \sum_{j<k} m_j m_k e^{2\nu x_j} (1 - e^{-2\nu(x_j - x_k)}) - \lambda^2 m_1 m_2 m_3 e^{2\nu x_3} (1 + \sum_{j<k} (-1)^{k-j} e^{2\nu(x_j - x_k)}) \right],$$

$$A_{22} = \gamma^{-1} \left[1 - \lambda M + \lambda^2 \sum_{j<k} m_j m_k (1 - e^{-2\nu(x_j - x_k)}) - \lambda^3 m_1 m_2 m_3 (1 + \sum_{j<k} (-1)^{k-j} e^{-2\nu(x_j - x_k)}) \right].$$

First of all note that this is the traceful Higgs field. Meaning that we have not yet applied the transformation

$$A \to A - \frac{1}{2}Tr(A).$$

As we already know, the dynamics of the system is completely governed by the behavior of the off-diagonal terms of the Higgs field. Therefore, we will keep this traceful form of $A(\lambda)$ since it is more convenient for the purposes of this section. The trace of the Higgs field is given as follows

$$\begin{aligned} Tr(A(\lambda)) &= A_{11} + A_{22}, \\ &= (\gamma + \gamma_{-1}) + (\gamma - \gamma^{-1})\lambda M + \lambda^2 \sum_{i<j} m_i m_j \left[(\gamma + \gamma^{-1}) - (\gamma e^{2\nu(x_i - x_j)} + \gamma^{-1} e^{-2\nu(x_i - x_j)}) \right] \\ &+ \lambda^3 m_1 m_2 m_3 \left[(\gamma - \gamma^{-1}) + \sum_{i<j} (-1)^{j-i} (\gamma e^{2\nu(x_i - x_j)} - \gamma^{-1} e^{-2\nu(x_i - x_j)}) \right]. \end{aligned} \tag{5.66}$$

But since the $Tr(A)$ is a constant of motion, then the total mass M along with the following two formulas are time invariant

$$\begin{aligned} C_2 &= \sum_{i<j} m_i m_j \left[(\gamma + \gamma^{-1}) - (\gamma e^{2\nu(x_i - x_j)} + \gamma^{-1} e^{-2\nu(x_i - x_j)}) \right], \\ C_3 &= m_1 m_2 m_3 (1 - e^{-2\nu(x_2 - x_1)})(1 - e^{-2\nu(x_3 - x_2)})(\gamma - \gamma^{-1} e^{2\nu(x_3 - x_1)})). \end{aligned} \tag{5.67}$$

Note that C_2 is again nothing more than the sum of the constant C_1 given by (5.28) of the the Higgs fields for the three different pairs. Namely

$$C_2 = \sum_{j<k} C_1^{jk}, \quad (j, k = 1, .., 3), \tag{5.68}$$

where C_1^{jk} is the constant C_1 given by the trace of Higgs field of the two peakons in the j-th and the k-th positions respectively given by (5.28). In fact, it is easy to show that the $d = 3$ Higgs field given by (5.65) factors as follows

$$A(\lambda) = \frac{1}{3} \sum_{j<k} A^{jk} + A^{aux}, \quad (j, k- = 1, ..., 3), \tag{5.69}$$

where A^{jk} are the Higgs fields of peakon pairs given by

$$A^{jk}(\lambda) = T^k T^j \beta,$$

and

$$A^{aux} = \lambda^3 m_1 m_2 m_3 = \begin{bmatrix} \gamma(1 + \sum_{j<k} (-1)^{k-j} e^{2\nu(x_j - x_k)}) & \gamma^{-1} e^{-2\nu x_3} (1 + \sum_{j<k} (-1)^{k-j} e^{-2\nu(x_j - x_k)}) \\ -\gamma e^{2\nu x_3} (1 + \sum_{j<l} (-1)^{k-j} e^{2\nu(x_j - x_k)}) & -\gamma^{-1} (1 + \sum_{j<k} (-1)^{k-j} e^{-2\nu(x_j - x_k)}) \end{bmatrix}. \tag{5.70}$$

Note: this factorization is in fact true up to a factor of 3 multiplying the elements $m_j m_k$ in the three Higgs fields A^{jk}.

Recall now that the interaction Hamiltonian as well as the total momentum factored out as sums of their corresponding quantities for pairs of peakons. Surprisingly, the Higgs field does not factor out as a sum of three Higgs fields for the three peakon pairs since there is an additional auxiliary Higgs field. In fact, it turns out that this auxiliary Higgs field is necessary to prevent certain interactions that are forbidden by the dynamics. To see this explicitly, note that A^{aux} can be rewritten as follows

$$A^{aux} = \lambda^3 m_1 m_2 m_3 (1 - e^{-2\nu(x_2 - x_1)})(1 - e^{-2\nu(x_3 - x_2)}) \begin{bmatrix} \gamma & \gamma^{-1} e^{-2\nu x_1} \\ -\gamma e^{2\nu x_3} & -\gamma^{-1} e^{2\nu(x_3 - x_1)} \end{bmatrix}. \tag{5.71}$$

Consider now the collision between the peakons in the positions x_1 and x_3. In this case, H^i_{13}, M_{13}, are conserved as shown previously in the case $d = 2$. At the same time, the three Higgs fields A^{jk} for $j < k$ are finite. So without considering the Higgs field A^{aux}, there is no restriction for such collision to occur. Consider now A^{aux}, obviously the Higgs field blows up when $x_1 = x_3$. Hence, the total Higgs field $A(\lambda)$ diverges and therefore it becomes asymptotically far in $\mathcal{M}_{SL}(2,0)$ when $x_1 = x_3$. Similarly, the Higgs field A^{13} blows up when $(x_3 - x_1) = X$. Therefore, following the interpretation of the previous sections, the total Higgs field $A(\lambda)$ becomes asymptotically far in the $\mathcal{M}_{SL}(2,0)$ at this moment. As before, a periodization of the problem will get rid of these two singularities by gluing each sequence of diverging solutions to a limiting configuration. Hence, gluing the walls of the confinement box will enable x_1 and x_3 to collide on the domain circle. As mentioned before, this compactification of the CF dynamics will then correspond to a compactification of the underlying moduli space of Higgs bundles.

5.7.1 Triple collisions

The natural question to ask now is whether triple collision occur or not. Can the three peakons collide at one point in space such that $x_1 = x_2 = x_3$? The answer is indeed no, such collision is forbidden.

Theorem 5.7.1. *Consider the CF dynamics in the case $d = 3$, peakons collide only in pairs and triple collisions do not occur.*

Proof. Consider the interaction Hamiltonian

$$H^i = \sum_{j<k} H^i_{jk} = H^i_{12} + H^i_{23} + H^i_{13},$$

where

$$H^i_{jk} = m_j m_k \; [G_{\nu,\beta}(x_j - x_k) - G_{\nu,\beta}(0)].$$

As mentioned before, since $H = H^0 + H^i$ and H as well as H^0 are conserved by the evolution of t, then H^i is also conserved. However, by lemma (5.5.3), the quantities

$$[G_{\nu,\beta}(x_j - x_k) - G_{\nu,\beta}(0)]$$

preserve their signs for all $t > 0$ and are equal to zero only at a moment of singularity. But since the quantities $m_j m_k$ also preserve their signs for all $t > 0$, then each H^i_{jk} preserves its sign for all $t > 0$. Assume now that triple collisions do occur such that at some moment $t > 0$, $x_1 = x_2 = x_3$. Since the three quantities

H^i_{12}, H^i_{23}, and H^i_{13} are identical, they have the same order of poles and zeros; therefore, they behave similarly at the time of triple collision. However, since $m_1m_2m_3 < 0$, two of these quantities have the same sign and the third one has an opposite sign. Therefore, assuming that each H^i_{jk} blows up at the moment of triple collision, then two of them will cancel (since they are identical with opposite signs) and the remaining one will blow up. In other words, the interaction Hamiltonian blows up at the moment of triple collision. This is a contradiction and we conclude that each H^i_{jk} remains finite at the time of triple collision. In other words, the poles of each m_jm_k cancel the zeros of $[G_{\nu,\beta}(x_j - x_k) - G_{\nu,\beta}(0)]$ such that each H^i_{jk} remains finite for all $t > 0$. Note that this is consistent with our previous results in the case $d = 2$.

Consider now the following two constants of motion

$$\begin{aligned} C_2 &= \sum_{j<k} m_j m_k [(1 - e^{-2\nu(x_k - x_j)})(\gamma - \gamma^{-1} e^{2\nu(x_k - x_j)})], \\ C_3 &= m_1 m_2 m_3 (1 - e^{-2\nu(x_2 - x_1)})(1 - e^{-2\nu(x_3 - x_2)})(\gamma - \gamma^{-1} e^{2\nu(x_3 - x_1)})). \end{aligned} \tag{5.72}$$

where we have rewritten C_1^{jk} to be of the form (5.28). Assuming that $x_1 = x_2 = x_3$, then

$$m_1 m_2 (1 - e^{-2\nu(x_2 - x_1)}) \to C_1^{12}/(\gamma - \gamma^{-1}).$$

We now know that this limit exists because of the argument in the previous paragraph. However, at this moment C_3 is given as follows

$$C_3 = C_1^{12} m_3 (1 - e^{-2\nu(x_3 - x_2)}).$$

But since $m_3(1 - e^{-2\nu(x_3 - x_2)}) \to 0$, therefore, $C_3 = 0$ when $x_1 = x_2 = x_3$. But since C_3 is conserved by the Lax pairs equations, this is a contradiction and we conclude that triple collisions do not occur in the CF system and peakons only collide in pairs. $\square$

Simply speaking, there is no way the order of zeros can be matched such that C_2 and C_3 are both conserved at the time of triple collisions.

Appendix A

Graphs of Peakons

Finally, we conclude this work with some graphs of the smooth and singular dynamics of the CF system in the two cases $d = 2$ and $d = 3$. The first two graphs signify the smooth dynamics when all $m_j > 0$. The confinement of peakons is showed in both graphs but surely no singularities emerge in these two cases. The following two graphs signify the singular dynamics for $d = 2$. As proved in theorem (5.5.4), with the proper analytic continuation, these singular dynamics can be continued such that the motion takes place on a circle.

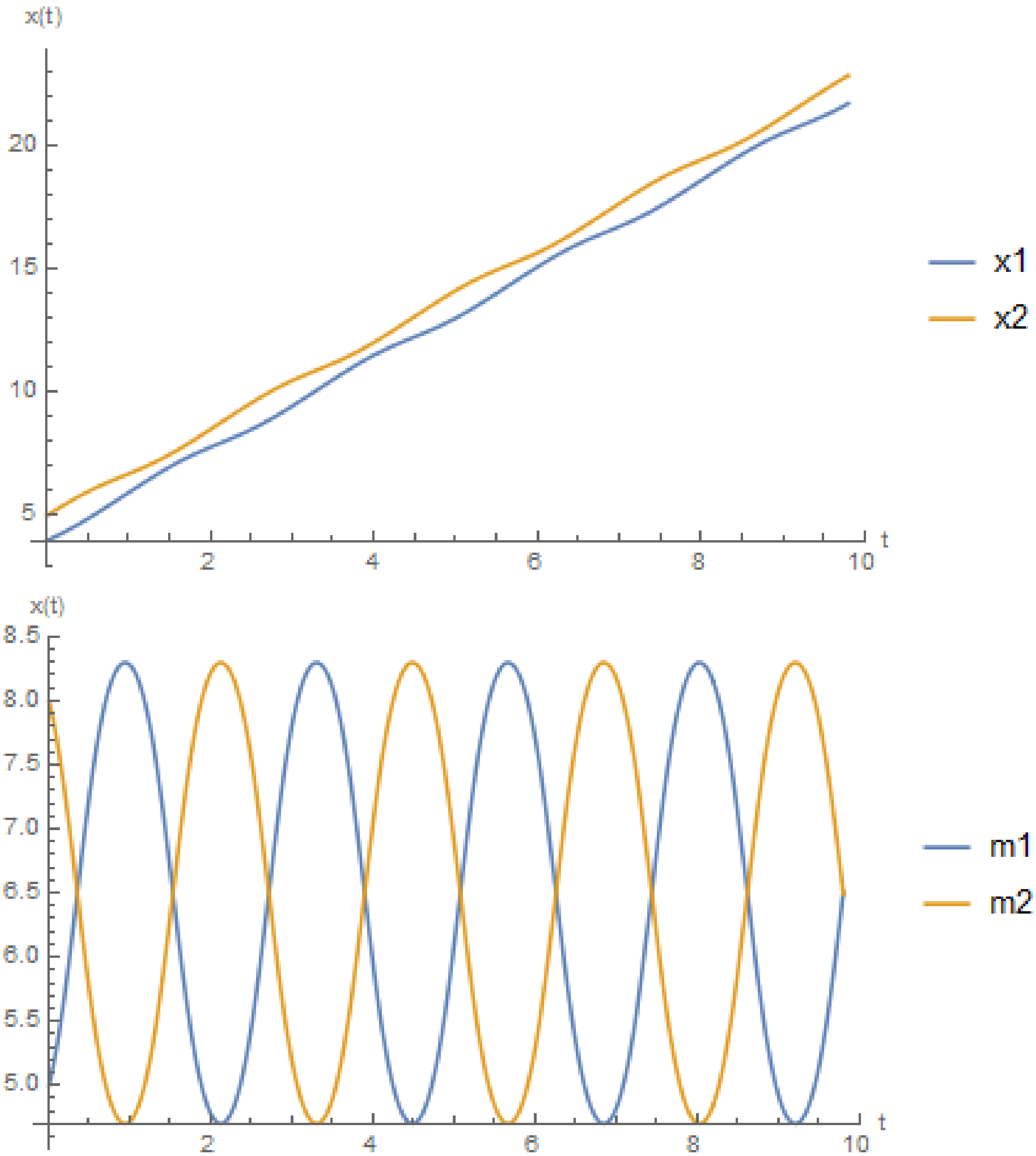

Figure A.1: Peakons confinement for $\beta_+ = 0.015$, $\nu = 2$, $x_1(0) = 4$, $x_2(0) = 5$, $m_1(0) = 5$, $m_2(0) = 8$.

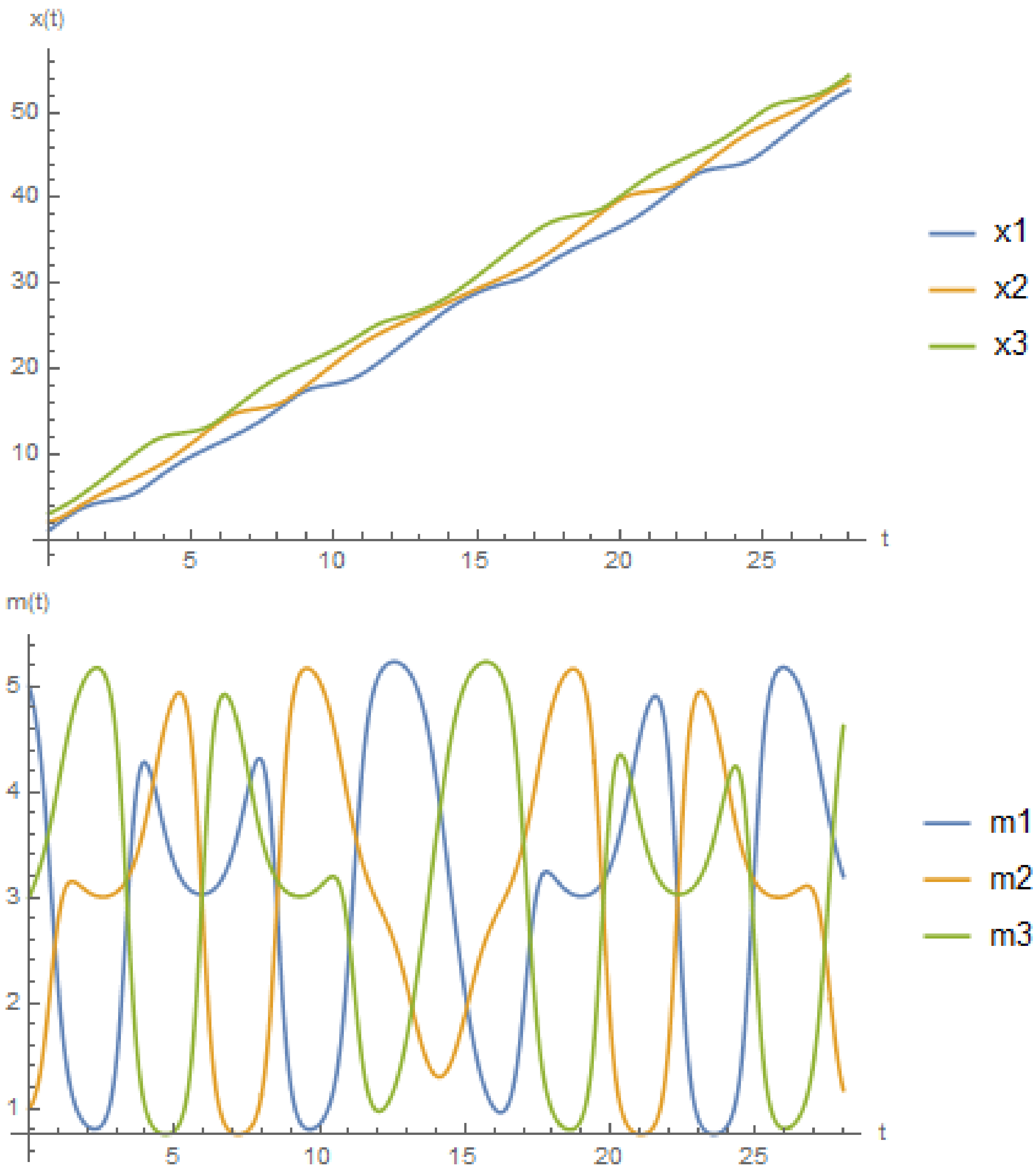

Figure A.2: Three Peakons confinement for $\beta_+ = 0.00004$, $\nu = 1$, $x_1(0) = 1$, $x_2(0) = 2$, $x_3(0) = 3$, $m_1(0) = 5$, $m_2(0) = 1$, $m_3(0) = 3$.

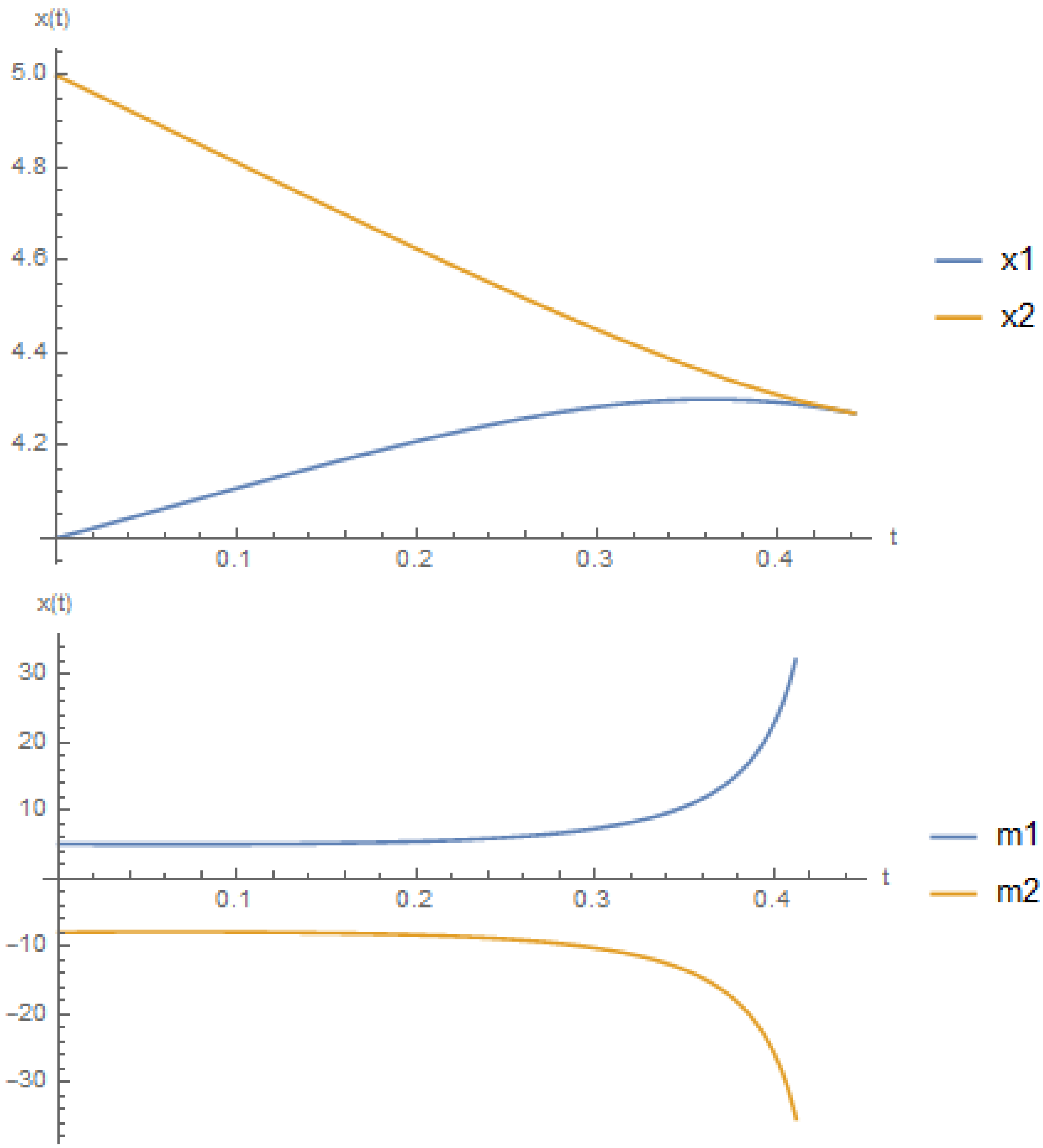

Figure A.3: Peakons collide for $\beta_{+} = 0.0015$, $\nu = 2$, $x_1(0) = 4$, $x_2(0) = 5$, $m_1(0) = 5$, $m_2(0) = -8$.

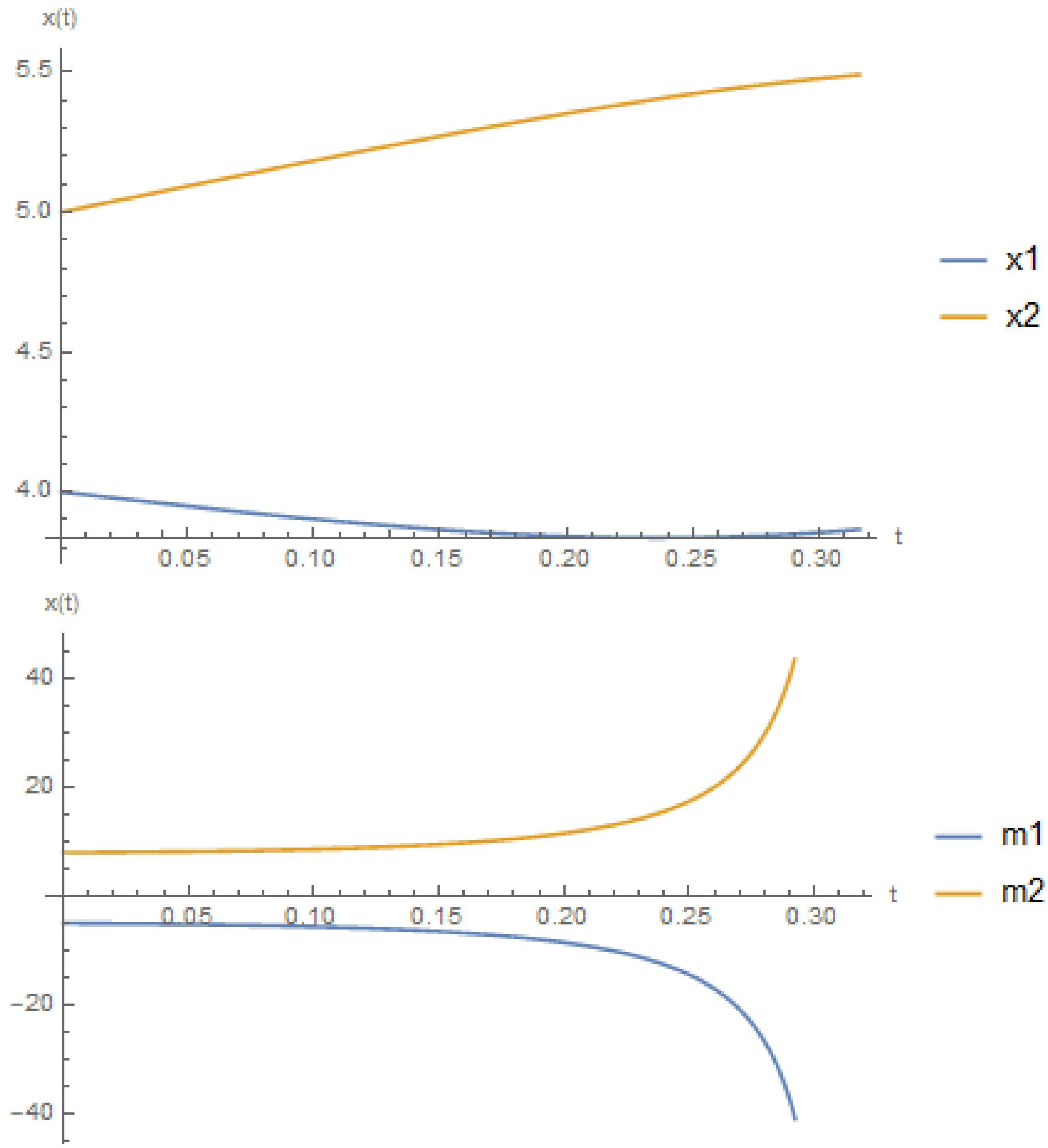

Figure A.4: Peakons pseudo-collide for $\beta_+ = 0.0015$, $\nu = 2$, $m_1(0) = -5$, $m_2(0) = 8$.

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