Relativistic cosmology and intrinsic spin of matter: Results and theorems in Einstein-Cartan theory

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We start by presenting the general set of structure equations for the 1+3 threading spacetime decomposition in 4 spacetime dimensions, valid for any theory of gravitation based on a metric compatible affine connection. We then apply these equations to the study of cosmological solutions of the Einstein-Cartan theory in which the matter is modeled by a perfect fluid with intrinsic spin. It is shown that the metric tensor can be described by a generic FLRW solution. However, due to the presence of torsion the Weyl tensors might not vanish. The coupling between the torsion and Weyl tensors leads to the conclusion that, in this cosmological model, the universe must either be flat or open, excluding definitely the possibility of a closed universe. In the open case, we derive a wave equation for the traceless part of the magnetic part of the Weyl tensor and show how the intrinsic spin of matter in a dynamic universe leads to the generation and emission of gravitational waves. Lastly, in this cosmological model, it is found that the torsion tensor, which has an intrinsic spin as its source, contributes to a positive accelerated expansion of the universe. Comparing the theoretical predictions of the model with the current experimental data, we conclude that torsion cannot completely replace the role of a cosmological constant.

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I. INTRODUCTION

A. The torsion tensor field and the Einstein-Cartan theory

A Riemannian spacetime geometry is uniquely described by the metric tensor field, in that the Riemann curvature tensor is solely given by the metric and its first and second derivatives. Moreover, the affine connection, which is the structure that defines parallel transport between tensors, is metric compatible and symmetric in Riemannian geometry, and called Levi-Civita connection. The geodesic equation, which determines the shortest and longest curves between two infinitesimally close points, is in Riemannian geometry also an equation for curves that transport tangent vectors in a autoparallel manner. General relativity, a spacetime theory of gravitation, has as one of its intrinsic assumptions the fact that the underlying geometry is Riemannian. In general relativity, the link between geometry and matter is provided by Einstein equation, that equates the Einstein tensor, which is a contraction of the Riemann tensor, to the matter stress-energy tensor.

A possible extension of Riemannian geometry is the Riemann-Cartan geometry in which, besides the metric tensor, there is an extra geometrical field, the torsion tensor. The Riemann curvature tensor depends now on both metric and torsion. Moreover, the metric compatible affine connection, between tensors, contains not only a symmetric part as in Riemannian geometry, but also an antisymmetric part, which is precisely the torsion tensor. In a Riemann-Cartan geometry, geodesic and autoparallel curves are different types of curves.

A natural extension of general relativity to another theory of gravitation is the Einstein-Cartan theory where the underlying geometry is the Riemann-Cartan geometry. A realization of an Einstein-Cartan gravity theory is such that the field equations are still derived from the Einstein-Hilbert action [1, 2], representing one of the simplest generalizations of general relativity. The link between geometry and matter is now provided by the Einstein equation, that equates the Einstein tensor to the canonical matter stress-energy tensor, plus an equation relating a tensor field built out of the geometrical torsion to some physical observables associated, for instance, to the density of the intrinsic angular momentum of matter, or spin. One of the interests in the Einstein-Cartan theory, within a geometric theory of gravitation, is that at extremely high densities of matter, even at densities still much less than the Planck density regime where quantum gravity rules, quantum effects on the matter may be considerable, hence the ability to include quantum corrections in a macroscopic limit, through the relation between torsion and intrinsic spin, might set the Einstein-Cartan theory to be a better classical limit of a quantum theory of gravitation than a theory without torsion like general relativity. Nonetheless, as we will show in this article, even in the low density regime, the inclusion of torsion might lead to marked contrast between the predictions of the two theories, which may be used to falsify the hypothesis.

The framework of the Einstein-Cartan theory has led to many important results, showing how the extra geometrical structure, specifically, the torsion tensor, influences the behavior of the matter fluids that permeate the spacetime and, consequently, the geometry of the manifold. Several works have worked out the properties of spacetimes with torsion and the consequences of the Einstein-Cartan theory, in particular in black hole physics and in cosmology. We mention some of them. The possibility of measuring torsion was raised in [3]. Some works showed that the inclusion of torsion could act as a repulsive force, counteracting the gravitational collapse and possibly prevent the formation of singularities both in black holes and cosmology [4, 5]. There were applications in cosmology [6, 7] as well as in rotating neutron stars [8], see also [9, 10]. The generation of solutions in Einstein-Cartan theory was provided in [11]. Some interesting spinning fluids, in particular the Weyssenhoff fluid, were introduced as generators of torsion in [12], as providers of inflation in [13], and as sources of rotating cosmological models in [14]. The possibility that the Einstein-Cartan theory is a limit of a quantum theory of the gravitational field operating at the usual microscopic and macroscopic scales has been hinted in [15]. The consequences and imprints on the curves followed by finite size test bodies was discussed in [16]. A review of Einstein-Cartan theory is in [17]. A further discussion on compact objects was given in [18], and a study of the cosmological signatures of torsion and cosmic acceleration appears in [19, 20].

B. The 1+3 spacetime decomposition

Due to the action of the torsion tensor in physical frames of reference, in gravitational theories with torsion, it is advantageous to work in a formalism that is manifestly covariant and such that the quantities that characterize the spacetime and the matter are directly associated with physical observables. A formalism with such characteristics is the covariant spacetime decomposition approach which is designed to take directly into account the symmetries and preferred directions in a manifold, and emerges as a powerful tool to analyze the geometry and dynamics of tensor fields on a spacetime. A benefit one takes from the formalism comes from the fact that it is, by construction, independent of coordinate systems. Moreover, the natural splitting of the manifold can greatly simplify the problem of finding solutions when the spacetime admits the existence of preferred directions, such as Killing vector fields.

A particular covariant spacetime decomposition is the 1+3 formalism that has been developed and used in many instances in general relativity [21–25]. In this formalism, it is assumed the existence of a congruence of smooth curves, so that any tensor quantity on the spacetime is, at each point, separated in its component along the direction of the tangent vector field to the congruence and in its components along the surfaces orthogonal to the curves of the congruence. This property of the formalism makes it especially useful from a physical point of view, since in many instances one is interested in studying the evolution of certain quantities in time. Thus, assuming the existence of a timelike congruence, the 1+3 formalism naturally decomposes the structure equations that describe the geometry of the spacetime and tensor fields in the manifold along time and spatial directions. In a geometric theory of gravity, the geometry of the spacetime is related with the matter fields that permeates it. Since the evolution and constraint equations found from the 1+3 formalism are completely covariant, they provide a clearer interpretation of the relations between the kinematics of the congruence and the properties of the matter fields.

This formalism of 1+3 decomposition of the spacetime manifold has been extensively employed to explore the properties of solutions of theories of gravitation, namely, gravitational waves, cosmological solutions, compact objects, black holes, singularities, and particle and light rays propagation. For instance, the formalism has been used in general relativity to study cosmological perturbations and their consequent gravitational waves generation [26], to analyse singularities and singularity theorems [27], to develop an effective fluid dynamics formalism [28], to find new properties of perturbed Schwarzschild black holes [29], to further investigate the Tolman-Oppenheimer-Volkoff equation [30], to discuss cosmological perfect fluid perturbations [31], and to analyze objects composed of two fluids [32]. The formalism has also been applied in $f(R)$ modified theories of gravitation to study gravitational lensing [33], to introduce black holes with emphasis in the Weyl terms [34], to describe cosmological density perturbations [35], and to search for gravitational wave solutions [36]. The formalism has further been applied in theories with torsion to explore the Raychaudhuri equation [37], to treat spacetime thermodynamics [38], to examine singularity theorems [39], and to establish focusing condition theorems [40].

C. Aim of the work

Since Hubble showed the velocity-distance relation for distant galaxies that we know the universe is expanding, and the observations of the emission spectra of type Ia supernovae have lead to the conclusion that our universe is expanding at an accelerated rate. Moreover, the high-precision data from the Hubble Space Telescope, WMAP, Planck collaboration, Sloan Digital Sky Survey, and JWST, keep confirming that, at very large scales, the present universe is very well described by the Friedman-Lemaître-Robertson-Walker (FLRW) model with the matter source having as a main component an unknown dark energy fluid. The other component, with its existence also being infered from the data from rotational curves of galaxies and from the velocities of individual members of galaxies in clusters, could be in unknown forms of dark particles or showing that the predictions of the general theory of relativity, even at a classical level, are incompatible with the observations without the inclusion of extra fundamental fields in alternative gravity theories. On the other hand, high-accuracy astrometric data of GAIA, and the regular detection of gravitational waves impose very strong constraints on these possible alternative gravity theories, notably on theories of the Einstein-Cartan type.

In this context, there has been a growing interest in studying the effects of torsion in the dynamics of the universe by considering types of torsion that have the intrinsic spin of matter as its source. Here, we are interested in studying the effects of the intrinsic spin of matter, within the canonical Einstein-Cartan theory, in the properties and evolution of the universe at very large scales. We will show that there are various aspects that have been overlooked, in particular we will show that in the Einstein-Cartan theory it is pivotal to understand the effects of the torsion field in the Weyl tensor, which to our knowledge have never been considered. As it will be shown, the coupling between the Weyl tensor and the torsion tensor field may lead to dramatic disparities between the predictions of the general theory of relativity and the Einstein-Cartan theory. Moreover, although the 1+3 formalism was initially devised to study solutions of general relativity, the paradigm of covariant spacetime decomposition is applicable to a much wider class of relativistic theories of gravitation, including theories of the Einstein-Cartan type. We will also present here the most general extension of the 1+3 formalism for spacetimes with a metric compatible affine connection valid for any metric affine gravity theory. These equations will then be used in the particular case of the Einstein-Cartan theory and will be used to study the effects of intrinsic spin in spacetimes permeated by an homogeneous and isotropic matter fluid.

D. Organization of the work

This work is organized as follows. In Sec. II, we introduce several quantities, namely, the metric, the spacetime connection with torsion, the curvature, the projection formalism with the decomposition of the torsion and Weyl tensors and the structure equations, giving as well the basic definitions and setting the adopted conventions. In Sec. III, the stress-energy tensor and the structure equations for the matter fields are given. In Sec. IV, the field equations of the Einstein-Cartan theory for a Weyssenhoff like torsion are found and compared with the results in the literature. In Sec. V, the isotropic universe and the geometry of the 3-spaces are set as a basis for a relativistic cosmology in the new set of equations for the Einstein-Cartan theory for a universe permeated by an isotropic and homogeneous matter fluid with nonvanishing intrinsic spin, and where two theorems and a proposition are proved. In Sec. VI, gravitational waves in relativitstic cosmology in Einstein-Cartan theory are studied. In Sec. VII, we analyze the tidal effects and the dynamics of the cosmic fluid in relativistic cosmology in Einstein-Cartan theory. In Sec. VIII, we further discuss the main results and conclude. In the Appendix A, we display the properties of the Laplace-Beltrami harmonics which are used in the main text. Throughout the paper we will work in the geometrized unit system where the constant of gravitation and the speed of light are set to one, and consider the metric signature $(-+++)$.

II. GEOMETRY OF LORENTZIAN MANIFOLDS WITH TORSION AND THE STRUCTURE EQUATIONS FOR THE GEOMETRIC FIELDS

A. Metric, connection with torsion, curvature, and projection formalism

1. *Metric, connection with torsion, curvature*

We start by introducing the basic definitions and setting the conventions that will be used throughout this article.

Let $(\mathcal{M}, g, S)$ be a 4-dimensional Lorentzian manifold endowed with a metric compatible affine connection. The metric tensor g is assumed to be symmetric, i.e.,

$$g_{\alpha\beta} = g_{(\alpha\beta)} \,, \tag{1}$$

with $g_{(\alpha\beta)} \equiv \frac{1}{2}\left(g_{\alpha\beta} + g_{\beta\alpha}\right)$, and the tensor S represents the torsion tensor, defined as the antisymmetric part of the connection in the lower indices, and is such that

$$S_{\alpha\beta}{}^{\gamma} = S_{[\alpha\beta]}{}^{\gamma} \,, \tag{2}$$

with $S_{[\alpha\beta]}{}^{\gamma} \equiv \frac{1}{2}\left(S_{\alpha\beta}{}^{\gamma} - S_{\beta\alpha}{}^{\gamma}\right)$. In $(\mathcal{M}, g, S)$ the covariant derivative ∇ is defined through an affine connection $C^{\gamma}_{\alpha\beta}$, such that on a $(k,m)-$tensor Y with components in a local coordinate system

$Y^{\mu_1\ldots\mu_k}{}_{\nu_1\ldots\nu_m}$, is formally given by $\nabla_\alpha Y^{\mu_1\ldots\mu_k}{}_{\nu_1\ldots\nu_m} = \partial_\alpha Y^{\mu_1\ldots\mu_k}{}_{\nu_1\ldots\nu_m} + \sum_{i=1}^{k} C^{\mu_i}_{\alpha\rho} Y^{\mu_1\ldots\rho\ldots\mu_k}{}_{\nu_1\ldots\nu_m} - \sum_{i=1}^{m} C^{\rho}_{\alpha\nu_i} Y^{\mu_1\ldots\mu_k}{}_{\nu_1\ldots\rho\ldots\nu_m}$, where ∂_α represent partial derivatives. In order that the affine connection $C^\gamma_{\alpha\beta}$ be metric g compatible, one has that $\nabla_\alpha g_{\beta\gamma} = \partial_\alpha g_{\beta\gamma} - C^\sigma_{\alpha\beta} g_{\sigma\gamma} - C^\sigma_{\alpha\gamma} g_{\beta\sigma}$ has to be identically zero, i.e.,

$$\nabla_\alpha g_{\beta\gamma} = 0\,. \tag{3}$$

A metric compatible connection $C^\gamma_{\alpha\beta}$ can always be split into two parts, namely,

$$C^\gamma_{\alpha\beta} = \Gamma^\gamma_{\alpha\beta} + K_{\alpha\beta}{}^\gamma\,, \tag{4}$$

with

$$\Gamma^\gamma_{\alpha\beta} = \frac{1}{2} g^{\gamma\sigma} \left(\partial_\alpha g_{\sigma\beta} + \partial_\beta g_{\alpha\sigma} - \partial_\sigma g_{\alpha\beta}\right)\,, \tag{5}$$

being the usual metric connection that appears in a Riemannian manifold, also referred as the Christoffel symbols, and

$$K_{\alpha\beta}{}^\gamma = S_{\alpha\beta}{}^\gamma + S^\gamma{}_{\alpha\beta} - S_\beta{}^\gamma{}_\alpha\,, \tag{6}$$

being the contorsion tensor which is a combination of torsion terms. From Eqs. (4)–(6) one finds

$$\Gamma^\gamma_{\alpha\beta} = \Gamma^\gamma_{(\alpha\beta)} = C^\gamma_{(\alpha\beta)}\,, \tag{7}$$

and

$$S_{\alpha\beta}{}^\gamma = K_{[\alpha\beta]}{}^\gamma = C^\gamma_{[\alpha\beta]}\,. \tag{8}$$

As well, from the antisymmetry of the torsion tensor in the first two indices, one can verify the following identity for the contorsion tensor,

$$K_{\alpha\beta\gamma} = K_{\alpha[\beta\gamma]}\,, \tag{9}$$

i.e., $K_{\alpha(\beta\gamma)} = 0$.

The definition of the Riemann curvature tensor associated with the connection $C^\gamma_{\alpha\beta}$ is

$$R_{\alpha\beta\gamma}{}^\rho = \partial_\beta C^\rho_{\alpha\gamma} - \partial_\alpha C^\rho_{\beta\gamma} + C^\rho_{\beta\sigma} C^\sigma_{\alpha\gamma} - C^\rho_{\alpha\sigma} C^\sigma_{\beta\gamma}\,. \tag{10}$$

This definition leads to the following relation between the commutator of two covariant derivatives of a tensor and the Riemann curvature tensor, Eq. (10),

$$\left(\nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha + 2S_{\alpha\beta}{}^\gamma\nabla_\gamma\right) Y^{\mu_1\ldots\mu_k}{}_{\nu_1\ldots\nu_m} = \sum_{i=1}^{m} R_{\alpha\beta\nu_i}{}^\rho\, Y^{\mu_1\ldots\mu_k}{}_{\nu_1\ldots\rho\ldots\nu_m} - \sum_{i=1}^{k} R_{\alpha\beta\rho}{}^{\mu_i}\, Y^{\mu_1\ldots\rho\ldots\mu_k}{}_{\nu_1\ldots\nu_m}\,, \tag{11}$$

where Y is an arbitrary (k, m)-tensor field. The Riemann curvature tensor, Eq. (10), possesses the following symmetries in its indices,

$$R_{\alpha\beta\gamma\delta} = R_{[\alpha\beta]\gamma\delta}\,, \tag{12}$$

i.e., $R_{(\alpha\beta)\gamma\delta} = 0$, and

$$R_{\alpha\beta\gamma\delta} = R_{\alpha\beta[\gamma\delta]}\ \ , \tag{13}$$

i.e., $R_{\alpha\beta(\gamma\delta)} = 0$. The symmetries of the Riemann curvature tensor given in Eqs. (12) and (13) are the same as in pure Riemannian geometry. The other index symmetry in pure Riemannian geometry, namely, ${R_{[\alpha\beta\gamma]}}^{\delta} = 0$, is, for a geometry with torsion, modified into an identity related to the covariant derivative of the torsion,

$$2\nabla_{[\alpha}\, {S_{\beta\gamma]}}^{\delta} - 4{S_{[\alpha\beta}}^{\rho}{S_{\gamma]\rho}}^{\delta} + {R_{[\alpha\beta\gamma]}}^{\delta} = 0\,, \tag{14}$$

which can be envisaged as a Bianchi identity for the torsion S, and is in this context called the first Bianchi identity. We note that the antisymetrization in the second term of Eq. (14) only refers to nondummy indices, in this case to α, β, γ, with the dummy index ρ not being affected by the process, and this is a convention that we will follow. The Riemannian Bianchi identity, namely $\nabla_{[\alpha}\, {R_{\beta\gamma]\delta}}^{\rho} = 0$, when torsion is present is modified into

$$\nabla_{[\alpha}\, {R_{\beta\gamma]\delta}}^{\rho} - 2{S_{[\alpha\beta}}^{\sigma}{R_{\gamma]\sigma\delta}}^{\rho} = 0\,, \tag{15}$$

and is in this context called the second Bianchi identity. From the index symmetry identities, Eqs. (12) and (13), and the first Bianchi identity, Eq. (14), we find that the usual symmetry of exchanging the first and second pair of indices of the Riemann tensor is modified in the presence of torsion to

$$2R_{\gamma\delta\alpha\beta} = 2R_{\alpha\beta\gamma\delta} + 3A_{\alpha\gamma\beta\delta} + 3A_{\delta\alpha\beta\gamma} + 3A_{\gamma\delta\alpha\beta} + 3A_{\beta\delta\gamma\alpha}\,, \tag{16}$$

where we have written $A_{\alpha\beta\gamma\delta} \equiv -2\nabla_{[\alpha}S_{\beta\gamma]\delta} + 4{S_{[\alpha\beta}}^{\rho}S_{\gamma]\rho\delta}$ to simplify the visualization of the equation. We remark that the results presented so far are completely general, in particular, they are valid for spacetimes of any dimension.

We will now consider the case of an orientable Lorentzian manifold $(\mathcal{M}, g, S)$ of dimension 4. In this case, a useful quantity to define is the Levi-Civita volume form, also referred as covariant Levi-Civita tensor or Levi-Civita 4-form. Introducing the Levi-Civita symbol, $\eta_{\alpha\beta\gamma\delta}$, as the totally skew tensor density whose components in any orientation preserving local coordinate system verify $\eta_{1234} = +1$, the Levi-Civita volume form is defined as

$$\varepsilon_{\alpha\beta\gamma\delta} \equiv \sqrt{|\det g|}\, \eta_{\alpha\beta\gamma\delta} \,, \tag{17}$$

where $|\det g|$ represents the absolute value of the determinant of the metric tensor. The Levi-Civita volume form verifies some useful relations, namely, (i) $\nabla_\rho \varepsilon_{\alpha\beta\gamma\delta} = 0$, (ii) $\varepsilon^{\alpha\beta\gamma\delta} = \frac{\mathrm{sign}(\det g)}{\sqrt{|\det g|}} \eta^{\alpha\beta\gamma\delta}$, (iii) $\varepsilon_{\alpha\beta\gamma\delta}\varepsilon^{\rho\sigma\mu\nu} = -24\, g^\rho{}_{[\alpha}\, g^\sigma{}_\beta g^\mu{}_\gamma g_{\delta]}{}^\nu$, (iv) $\varepsilon_{\alpha\beta\gamma\delta}\varepsilon^{\alpha\sigma\mu\nu} = -6\, g^\sigma{}_{[\beta}\, g^\mu{}_\gamma g_{\delta]}{}^\nu$, (v) $\varepsilon_{\alpha\beta\gamma\delta}\varepsilon^{\alpha\beta\mu\nu} = -4\, g^\mu{}_{[\gamma}\, g_{\delta]}{}^\nu$, and (vi) $\varepsilon_{\alpha\beta\gamma\delta}\varepsilon^{\alpha\beta\gamma\delta} = -24$. The first equality follows from the assumption that the connection is metric compatible, the second from the properties of the determinant of a matrix and in (iii) to (vi) only the lower indices are to be antisymmetrized.

The Weyl tensor $C_{\alpha\beta\gamma\delta}$ is defined as the trace-free part of the Riemann curvature tensor $R_{\alpha\beta\gamma\delta}$. In the case of a manifold of dimension 4, the components of the Weyl curvature tensor, $C_{\alpha\beta\gamma\delta}$, can be written as

$$C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - R_{\alpha[\gamma}\, g_{\delta]\beta} + R_{\beta[\gamma}\, g_{\delta]\alpha} + \frac{1}{3} R\, g_{\alpha[\gamma}\, g_{\delta]\beta} \,, \tag{18}$$

where $R_{\alpha\beta} \equiv R_{\alpha\mu\beta}{}^\mu$ is the Ricci tensor, and $R \equiv R_\mu{}^\mu$ is the Ricci scalar. The Weyl tensor inherits, from Eq. (18), the following symmetries in its indices,

$$C_{\alpha\beta\gamma\delta} = C_{[\alpha\beta]\gamma\delta} \,, \tag{19}$$

i.e., $C_{(\alpha\beta)\gamma\delta} = 0$, and

$$C_{\alpha\beta\gamma\delta} = C_{\alpha\beta[\gamma\delta]} \quad , \tag{20}$$

i.e., $C_{\alpha\beta(\gamma\delta)} = 0$. In addition, one finds

$$C_{[\alpha\beta\gamma]\delta} = R_{[\alpha\beta\gamma]\delta} + R_{[\alpha\beta}\, g_{\gamma]\delta} \,. \tag{21}$$

In the presence of torsion, the relation between the derivative of the Weyl tensor and the Riemann tensor is [18]

$$\nabla_\alpha C^{\gamma\delta\beta\alpha} = \frac{1}{2}\, \varepsilon^{\mu\nu\lambda\beta} S_{[\mu\nu}{}^\sigma R_{\lambda]\sigma\eta\rho} \varepsilon^{\eta\rho\gamma\delta} + \frac{3}{2} \left(g^{\beta\delta} S^{[\gamma\mu}{}_\sigma R^{\nu]\sigma}{}_{\mu\nu} - g^{\beta\gamma} S^{[\delta\mu}{}_\sigma R^{\nu]\sigma}{}_{\mu\nu} \right) + \nabla^{[\delta} R^{\gamma]\beta} - \frac{1}{6}\, g^{\beta[\gamma} \nabla^{\delta]} R \,, \tag{22}$$

and the dummy index σ is not involved in the antisymmetrization process.

2. Projector operator, projected covariant Levi-Civita tensor, and projected covariant derivatives

Consider a Lorentzian manifold of dimension 4, $(\mathcal{M}, g, S)$, admitting, in some open neighborhood, the existence of a congruence of timelike curves with tangent vector field u. Without loss of generality, we can foliate the manifold in 3-surfaces, $\mathcal{V}$, orthogonal, at each point, to the curves of the congruence, such that all tensor quantities are defined by their behavior along the direction of u and in $\mathcal{V}$. This procedure is usually called 1+3 spacetime decomposition. Such decomposition of the spacetime manifold relies on the existence of a projector to the hypersurface $\mathcal{V}$. Assuming each curve of the congruence to be affinely parameterized, so that $u_\alpha u^\alpha = -1$, the projector onto $\mathcal{V}$, at each point can be defined as

$$h_{\alpha\beta} \equiv g_{\alpha\beta} + u_\alpha u_\beta \,, \tag{23}$$

verifying $h_{\alpha\beta}u^\alpha = 0$, $h_{\alpha\beta} = h_{\beta\alpha}$, $h_\alpha{}^\gamma h_{\gamma\beta} = h_{\alpha\beta}$, and $h_\gamma{}^\gamma = 3$.

Another useful operator is the projected covariant Levi-Civita tensor

$$\varepsilon_{\alpha\beta\gamma} = \varepsilon_{\alpha\beta\gamma\sigma}u^\sigma \,, \tag{24}$$

derived from the Levi-Civita volume form, defined in Eq. (17), with the following properties $\varepsilon_{\alpha\beta\gamma} = \varepsilon_{[\alpha\beta\gamma]}$, $\varepsilon_{\alpha\beta\gamma}u^\gamma = 0$, $\varepsilon_{\alpha\beta\gamma}\varepsilon^{\mu\nu\sigma} = 6h^\mu{}_{[\alpha}h^\nu{}_\beta h_{\gamma]}{}^\sigma$, $\varepsilon_{\alpha\beta\gamma}\varepsilon^{\mu\nu\gamma} = 2h^\mu{}_{[\alpha}h_{\beta]}{}^\nu$, $\varepsilon_{\mu\nu\alpha}\varepsilon^{\mu\nu\beta} = 2h_\alpha{}^\beta$. Moreover, using Eq. (23) and the properties of the Levi-Civita volume form, we find the useful identities $h_\mu{}^\alpha h_\nu{}^\beta h_\rho{}^\gamma h_\lambda{}^\sigma \varepsilon_{\alpha\beta\gamma\sigma} = 0$ and $\varepsilon_{\alpha\beta\gamma\sigma} = h_\alpha{}^\mu \varepsilon_{\mu\beta\gamma\sigma} + u_\alpha \varepsilon_{\beta\gamma\sigma}$.

In order to keep the equations as compact as possible, we introduce the following notation for projected covariant derivatives. Given a tensor field $Y_{\alpha\ldots\beta}{}^{\gamma\ldots\sigma}$ we define

$$D_\mu Y_{\alpha\ldots\beta}{}^{\gamma\ldots\sigma} \equiv h_\mu{}^\nu h_\alpha{}^\rho \ldots h_\beta{}^\delta h_\lambda{}^\gamma \ldots h_\varphi{}^\sigma \nabla_\nu Y_{\rho\ldots\delta}{}^{\lambda\ldots\varphi} \,, \tag{25}$$

as the fully orthogonally projected covariant derivative on $\mathcal{V}$. On the other hand, a dot represents the covariant derivative along the integral curves of u, i.e.,

$$\dot{Y}_{\alpha\ldots\beta}{}^{\gamma\ldots\sigma} = u^\mu \nabla_\mu Y_{\alpha\ldots\beta}{}^{\gamma\ldots\sigma} \,. \tag{26}$$

B. Decomposition of the torsion tensor and Weyl tensor

We now write the 1+3 decomposition of the torsion tensor S, Eq. (8) and the Weyl tensor C, Eq. (18), in terms of their components along the direction of the tangent vector field u and on $\mathcal{V}$ with the help of the projector operator $h_{\alpha\beta}$ given in Eq. (23).

For the torsion tensor, Eq. (8), the decomposition is [18]

$$S_{\alpha\beta\gamma} = \varepsilon_{\alpha\beta}{}^{\mu}\bar{S}_{\mu\gamma} - u_{[\alpha}W_{\beta]\gamma} + S_{\alpha\beta}u_{\gamma} + u_{[\alpha}X_{\beta]}u_{\gamma} \,, \tag{27}$$

with

$$\bar{S}_{\alpha\beta} = \frac{1}{2}\varepsilon_{\alpha\mu\nu}h^{\sigma}{}_{\beta}S^{\mu\nu}{}_{\sigma} \,, \qquad W_{\alpha\beta} = 2u^{\mu}h^{\nu}{}_{\alpha}h^{\sigma}{}_{\beta}S_{\mu\nu\sigma} \,,$$
$$S_{\alpha\beta} = -h_{\alpha}{}^{\mu}h^{\nu}{}_{\beta}u^{\sigma}S_{\mu\nu\sigma} \,, \quad X_{\alpha} = 2u^{\mu}h^{\nu}{}_{\alpha}u^{\sigma}S_{\mu\nu\sigma} \,. \tag{28}$$

For the Weyl tensor, Eq. (18), the 1+3 decomposition is

$$C_{\alpha\beta\gamma\delta} = -\varepsilon_{\alpha\beta\mu}\varepsilon_{\gamma\delta\nu}E^{\nu\mu} - 2u_{\alpha}E_{\beta[\gamma}u_{\delta]} + 2u_{\beta}E_{\alpha[\gamma}u_{\delta]} - 2\varepsilon_{\alpha\beta\mu}H^{\mu}{}_{[\gamma}u_{\delta]} - 2\varepsilon_{\mu\gamma\delta}\bar{H}^{\mu}{}_{[\alpha}u_{\beta]} \,, \tag{29}$$

where

$$E_{\alpha\beta} = C_{\alpha\mu\beta\nu}u^{\mu}u^{\nu} \,, \quad H_{\alpha\beta} = \frac{1}{2}\varepsilon_{\alpha}{}^{\mu\nu}C_{\mu\nu\beta\delta}u^{\delta} \,, \quad \bar{H}_{\alpha\beta} = \frac{1}{2}\varepsilon_{\alpha}{}^{\mu\nu}C_{\beta\delta\mu\nu}u^{\delta} \,, \tag{30}$$

are defined as the electric part and the magnetic part of the Weyl tensor, respectively. In the Riemann-Cartan geometry there are two different tensor quantities associated to the magnetic part of the Weyl tensor, specifically, $H_{\alpha\beta}$ and $\bar{H}_{\alpha\beta}$, such that, in general the presence of torsion lifts a degeneracy in the magnetic part of the Weyl tensor. From the results in Eqs. (19)–(21), we see that in the presence of torsion the tensor $E_{\alpha\beta}$ has the following properties, $E_{\alpha\beta} = h_{\alpha}{}^{\mu}h_{\beta}{}^{\nu}E_{\mu\nu}$ and $E^{\alpha}{}_{\alpha} = 0$, the tensor $H_{\alpha\beta}$ has the following properties $H_{\alpha\beta} = h_{\alpha}{}^{\mu}h_{\beta}{}^{\nu}H_{\mu\nu}$ and $H_{\alpha\beta} = H_{(\alpha\beta)}$, and the tensor $\bar{H}_{\alpha\beta}$ has the following properties $\bar{H}_{\alpha\beta} = h_{\alpha}{}^{\mu}h_{\beta}{}^{\nu}\bar{H}_{\mu\nu}$ and $\bar{H}_{\alpha\beta} = \bar{H}_{(\alpha\beta)}$. Therefore, $E_{\alpha\beta}$ may not be a symmetric tensor and $H_{\alpha\beta}$ and $\bar{H}_{\alpha\beta}$ do not have to be trace-free, as in the case of Riemannian geometry. On the other hand, due to the properties of the Levi-Civita volume form and the fact that the Weyl tensor is, by definition, trace free, even in the presence of torsion, the magnetic parts, $H_{\alpha\beta}$ and $\bar{H}_{\alpha\beta}$, are symmetric under the exchange of indices.

Many of the results introduced and to be introduced are valid or easily extended for manifolds of dimension $d \geq 2$. However, quantities and identities that rely on the covariant Levi-Civita tensor, Eq. (17), notably the 1+3 decomposition of the torsion tensor, Eq. (27), and of the Weyl tensor Eq. (29), are dimension dependent, hence, the general set of structure equations that we will find will depend on the dimension of the manifold.

C. The separation vector between infinitesimally close curves of a congruence

Having introduced the definitions and properties of the basic geometric quantities and their decompositions, we will now consider the notion of separation vector between infinitesimally close curves of a congruence and relate its evolution with the kinematical quantities that characterize the congruence. For further details see [37].

Consider a congruence of curves in some open neighborhood of $(\mathcal{M}, g, S)$, with tangent vector field u. Given two points p and q in a small enough neighborhood, such that p is crossed by a curve of the congruence and q is crossed by a distinct curve of the congruence, the vector field $n \equiv q - p$ gives a meaningful notion of the separation between the curves of the congruence. Picking a curve c of the congruence as the fiducial curve, it is possible to derive an equation for the change of the separation vector n along the curve c. Indeed, one finds

$$u^\beta \nabla_\beta n^\alpha = B_\beta{}^\alpha n^\beta \,, \tag{31}$$

where

$$B_\beta{}^\alpha = \nabla_\beta u^\alpha + 2S_{\gamma\beta}{}^\alpha u^\gamma \,. \tag{32}$$

The tensor B gives the evolution of the separation vector n between two infinitesimally close curves along the fiducial curve. We note that Eq. (31) is valid for the case of u being timelike, spacelike or null, with the fiducial curve being a geodesic or not, although we will be interested in the case of a timelike curve. The first term in the right-hand side of Eq. (32) is the usual term present in pure Riemannian geometry, while the second term in the right-hand side of Eq. (32) represents an explicit contribution of the torsion tensor to the evolution of a congruence of curves.

We can now study some geometrical implications of Eqs. (31) and (32). Taking the derivative along c of the quantity $n_\alpha u^\alpha$ reads

$$u^\mu \nabla_\mu \left(n_\alpha u^\alpha \right) = n_\beta a^\beta + 2S_{\sigma\gamma\alpha} u^\sigma u^\alpha n^\gamma \,, \tag{33}$$

where the acceleration vector field a in a local coordinate system has components given by

$$a^\alpha = u^\gamma \nabla_\gamma u^\alpha \,. \tag{34}$$

The expression given in Eq. (33) can be seen as the failure of the separation vector n and the tangent vector u to stay orthogonal to each other. Indeed, if at a given point, n and u are orthogonal to each other, a nonzero acceleration a or a nonzero, general, torsion S will destroy the preservation of such orthogonality along the curve. Thus, this analysis of Eq. (33) leads to the conclusion that the tensor B, describing the behavior of the separation vector might have, even in the case of a zero acceleration a, nonzero components tangential and orthogonal to the tangent vector field associated with the fiducial curve c when torsion is present. Without loss of generality, it is then possible to write $B_{\alpha\beta}$ in terms of two components. One component, $B_{\perp\alpha\beta}$, is completely orthogonal to u, and another component, $B_{\|\alpha\beta}$, contains the remaining terms. Given a projector $h_{\alpha\beta}$ onto the surface orthogonal to the curve c at a given point, we can then write

$$B_{\alpha\beta} = B_{\perp\alpha\beta} + B_{\|\alpha\beta} \,. \tag{35}$$

Now, $B_{\perp\alpha\beta}$ is defined as $B_{\perp\alpha\beta} \equiv h_\alpha{}^\gamma h_\beta{}^\sigma B_{\gamma\sigma}$. Furthermore we can define the kinematical quantities of the congruence, namley, expansion θ, shear $\sigma_{\alpha\beta}$, and vorticity $\omega_{\alpha\beta}$, of neighboring curves of the congruence that only depend on the orthogonal part $B_\perp$ of the tensor B, so that we have the identity $B_{\perp\alpha\beta} = \frac{h_{\alpha\beta}}{h_\gamma{}^\gamma}\theta + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$. Since we are interested in 1+3 dimensions, we have $h_\gamma{}^\gamma = 3$, and so

$$B_{\perp\alpha\beta} = \frac{1}{3}\, h_{\alpha\beta}\,\theta + \sigma_{\alpha\beta} + \omega_{\alpha\beta} \,. \tag{36}$$

with

$$\theta = B_{\perp\gamma}{}^\gamma \,, \quad \sigma_{\alpha\beta} = B_{\perp(\alpha\beta)} - \frac{1}{3}\, h_{\alpha\beta}\,\theta \,, \quad \omega_{\alpha\beta} = B_{\perp[\alpha\beta]} \,, \tag{37}$$

Then, of course, given a $B_{\perp\alpha\beta}$, one uses Eq. (35) to determine $B_{\|\alpha\beta}$ as $B_{\|\alpha\beta} = B_{\alpha\beta} - B_{\perp\alpha\beta}$. The set of kinematical quantities θ, $\sigma_{\alpha\beta}$, and $\omega_{\alpha\beta}$, given in Eq. (37), characterize a congruence in a Lorentzian manifold and represent one of the building blocks of covariant spacetime decomposition approaches. Note further that the procedure that defines the projector operator strictly depends on the specific family of curves considered, i.e., depends on the tangent vector field u. Once the projector is assigned, as, e.g., in Eq. (23), one has that Eq. (32) together with Eq. (36) will give an actual expression for the derivative of the tangent vector u in terms of the kinematical quantities, the tangent vector itself, its acceleration a, and the torsion tensor S.

The results presented here are quite general and valid for curves of any kind and easily extended to spacetimes of any dimension $d \geq 2$. Nonetheless, in this work we will focus on developing the 1+3 formalism for timelike congruences in a 4-dimensional oriented Lorentzian manifold with torsion.

D. Structure equations for the geometric fields

The kinematical quantities of a congruence of curves (37), the acceleration vector field (34) and the tensors found from the decomposition of the torsion tensor, Eqs. (27) and (28), and of the Weyl tensor, Eqs. (29) and (30), and the Ricci tensor completely describe the geometry of the manifold $(\mathcal{M}, g, S)$ and the properties of a congruence of curves that permeate it. We have then to find a complete set of differential equations that describe the evolution of these quantities along u and on $\mathcal{V}$.

Now, projecting twice Eq. (32) with the projector given in Eq. (23), we find that the covariant derivative of the tangent vector field u is given by $\nabla_\alpha u_\beta = B_{\perp\alpha\beta} - W_{\alpha\beta} - u_\alpha a_\beta$, where we have used Eqs. (27), (32), and (34). Then, using Eq. (36) we find

$$\nabla_\alpha u_\beta = \frac{1}{3} h_{\alpha\beta}\theta + \sigma_{\alpha\beta} + \omega_{\alpha\beta} - W_{\alpha\beta} - u_\alpha a_\beta \,. \tag{38}$$

Applying the Ricci identity, Eq. (11), to Eq. (38), we find the propagation equations for the kinematical quantities

$$\begin{aligned} \dot{\theta} - \dot{W}_\alpha{}^\alpha =& - R_{\alpha\beta} u^\alpha u^\beta - \left(\frac{1}{3}\theta^2 + \sigma_{\alpha\beta}\sigma^{\alpha\beta} + \omega_{\alpha\beta}\omega^{\beta\alpha} \right) \\ & + D_\alpha a^\alpha + a_\alpha a^\alpha + W^{\beta\alpha} \left[\frac{1}{3} h_{\alpha\beta}\theta + \sigma_{\alpha\beta} + \omega_{\alpha\beta} \right] + X_\alpha a^\alpha \,, \end{aligned} \tag{39}$$

$$\begin{aligned} h_\mu{}^\alpha h_\nu{}^\beta \left(\dot{\omega}_{\alpha\beta} - \dot{W}_{[\alpha\beta]} \right) =& \frac{1}{2} h_\mu{}^\alpha h_\nu{}^\beta R_{[\alpha\beta]} - E_{[\mu\nu]} - \frac{2}{3}\theta\omega_{\mu\nu} + 2\sigma^\alpha{}_{[\mu}\omega_{\nu]\alpha} \\ & + D_{[\mu} a_{\nu]} + X_{[\mu} a_{\nu]} + \frac{1}{3}\theta W_{[\mu\nu]} - W^\delta{}_{[\mu} \left(\sigma_{\nu]\delta} + \omega_{\nu]\delta} \right) \,, \end{aligned} \tag{40}$$

$$\begin{aligned} h_\mu{}^\alpha h_\nu{}^\beta \left(\dot{\sigma}_{\alpha\beta} - \dot{W}_{\langle\alpha\beta\rangle} \right) =& \frac{1}{2} R_{\langle\mu\nu\rangle} - E_{(\mu\nu)} + D_{\langle\mu} a_{\nu\rangle} + a_{\langle\mu} a_{\nu\rangle} - \frac{2}{3}\sigma_{\mu\nu}\theta - \sigma^\delta{}_{\langle\mu}\sigma_{\nu\rangle\delta} \\ & - \omega_{\delta\langle\mu}\omega_{\nu\rangle}{}^\delta + X_{\langle\mu} a_{\nu\rangle} + W_{\delta\langle\mu}\sigma_{\nu\rangle}{}^\delta + W_{\delta\langle\mu}\omega_{\nu\rangle}{}^\delta + \frac{1}{3} W_{\langle\mu\nu\rangle}\theta \,, \end{aligned} \tag{41}$$

where for any 2-tensor $Y_{\alpha\beta}$ we use the angular brackets to represent the projected symmetric part without trace of it, i.e., $Y_{\langle\alpha\beta\rangle} \equiv \left[h^\mu{}_{(\alpha} h_{\beta)}{}^\nu - \frac{h_{\alpha\beta}}{3} h^{\mu\nu} \right] Y_{\mu\nu}$, and dummy indices are leftout of all the symmetrization processes. Equations (39)–(41) follow from computing the projection $h_\mu{}^\alpha u^\beta h_\nu{}^\gamma R_{\alpha\beta\gamma\delta} u^\delta$ and evaluate, respectively, its trace, antisymmetric part and symmetric part without trace. Provided the field equations of a gravity theory to relate the projection of the Ricci tensor with the stress-energy tensor, Eq. (39) represents the generalization of the Raychaudhuri equation for manifolds with nonzero torsion [37, 38], describing the evolution of the expansion of a congruence of curves. From the Ricci identity, Eq. (11), we also find the constraint equations,

$$\begin{aligned} \varepsilon^{\alpha\beta\gamma} D_\alpha \left(\omega_{\beta\gamma} - W_{\beta\gamma} \right) - \varepsilon^{\alpha\beta\gamma} a_\gamma \omega_{\alpha\beta} =& H_\gamma{}^\gamma - 2\bar{S}^{\alpha\beta} \left(\frac{1}{3} h_{\beta\alpha}\theta + \sigma_{\beta\alpha} + \omega_{\beta\alpha} - W_{\beta\alpha} \right) \\ & - \varepsilon^{\alpha\beta\gamma} a_\gamma \left(S_{\alpha\beta} + W_{\alpha\beta} \right) \,, \end{aligned} \tag{42}$$

$$\begin{aligned} \varepsilon^{\alpha\beta\langle\mu} D_\alpha \left(\sigma_\beta{}^{\nu\rangle} + \omega_\beta{}^{\nu\rangle} - W_\beta{}^{\nu\rangle} \right) + \varepsilon^{\alpha\beta\langle\mu} a^{\nu\rangle} \omega_{\beta\alpha} =& H^{\langle\mu\nu\rangle} - \varepsilon^{\alpha\beta\langle\mu} a^{\nu\rangle} \left(S_{\alpha\beta} + W_{\alpha\beta} \right) + 2 W_\delta{}^{\langle\mu} \bar{S}^{\nu\rangle\delta} \\ & - 2 \left(\frac{1}{3} h_\delta{}^{\langle\mu}\theta + \sigma_\delta{}^{\langle\mu} + \omega_\delta{}^{\langle\mu} \right) \bar{S}^{\nu\rangle\delta} \,, \end{aligned} \tag{43}$$

$$\frac{2}{3}D_\mu\theta - D_\alpha\left(\sigma_\mu{}^\alpha + \omega_\mu{}^\alpha - W_\mu{}^\alpha\right) - D_\mu W_\gamma{}^\gamma - 2a^\gamma\omega_{\mu\gamma} = - h^\alpha{}_\mu R_{\alpha\beta}u^\beta - 2\varepsilon_{\alpha\beta\mu}\bar{S}^{\alpha\gamma}W_\gamma{}^\beta + 2a^\gamma\left(S_{\gamma\mu} + W_{[\gamma\mu]}\right) + 2\varepsilon_{\alpha\beta\mu}\bar{S}^{\alpha\gamma}\left(\frac{1}{3}h_\gamma{}^\beta\theta + \sigma_\gamma{}^\beta + \omega_\gamma{}^\beta\right) . \tag{44}$$

where dummy indices do not participate in the symetrization processes. Equations (42)–(44) follow from computing the projection $\varepsilon^{\alpha\beta\lambda}h_\rho{}^\gamma R_{\alpha\beta\gamma\delta}u^\delta$ and evaluate, respectively, its trace, symmetric part without trace and antisymmetric part. These equations clearly exemplify how the presence of torsion modifies the geometry of the manifold and, consequently, the change in the evolution of a congruence of timelike curves. When comparing to the case of vanishing torsion [21, 24], we see that, in the presence of a general torsion tensor field, the magnetic part of the Weyl tensor, H, is characterized by Eqs. (42) and (43), in particular, it also depends on the divergence of the vorticity vector $\frac{1}{2}\varepsilon^{\gamma\mu\nu}\omega_{\mu\nu}$. Moreover, from Eq. (42), we conclude that the presence of torsion acts as a cause for the rotation of the congruence.

The evolution and constraint equations for the components of the Weyl tensor are found from the identity for the Weyl tensor given in Eq. (22), or, equivalently, from the second Bianchi identity, Eq. (15). For the electric part of the Weyl tensor we find the propagation equation

$$\begin{aligned}
&- h_{\alpha\mu}h_{\beta\nu}\dot{E}^{\mu\nu} + \varepsilon_{\mu\beta}{}^\nu\left(D_\nu\bar{H}^\mu{}_\alpha + a_\nu\bar{H}^\mu{}_\alpha\right) + \varepsilon^\mu{}_{\alpha\delta}a^\delta H_{\mu\beta} + \left(\sigma_{\alpha\nu} + \omega_{\alpha\nu} - W_{\alpha\nu}\right)E^\nu{}_\beta \\
&+ E_\alpha{}^\mu\left(2\sigma_{\mu\beta} + 2\omega_{\mu\beta} - W_{\mu\beta}\right) - E_{\alpha\beta}\left(\theta - W_\mu{}^\mu\right) - h_{\alpha\beta}E^{\nu\mu}\left(\sigma_{\mu\nu} + \omega_{\mu\nu} - \frac{1}{2}W_{\mu\nu}\right) = \\
&= \frac{1}{4}h_{\alpha\beta}R_{\nu\mu}\left(W^{\mu\nu} - W_\gamma{}^\gamma h^{\mu\nu} - 2\varepsilon^\mu{}_{\delta\gamma}\bar{S}^{\gamma\delta}u^\nu\right) - \varepsilon^{\mu\nu}{}_\beta X_\mu\bar{H}_{\alpha\nu} - 2\bar{S}_\beta{}^\mu\bar{H}_{\alpha\mu} + h_{\alpha\beta}\bar{S}_{\mu\nu}\bar{H}^{\mu\nu} \\
&- \frac{1}{2}W_{\alpha\nu}h_{\delta\beta}R^{\nu\delta} - \frac{1}{6}W_{\alpha\beta}R + \frac{1}{2}W_{\alpha\beta}h_{\nu\delta}R^{\nu\delta} + \frac{1}{2}h_{\alpha\nu}R^{\nu\delta}\left(W_\mu{}^\mu h_{\delta\beta} - W_{\delta\beta}\right) - \frac{1}{12}h_{\alpha\beta}\dot{R} \\
&+ \frac{1}{2}h_{\alpha\mu}h_{\beta\nu}u_\delta\nabla^\delta R^{\mu\nu} - \frac{1}{2}D_\alpha\left(u_\delta R^\delta{}_\beta\right) + u_\lambda R^{\lambda\sigma}\bar{S}_\beta{}^\mu\varepsilon_{\sigma\mu\alpha} - \frac{1}{2}R_{\mu\nu}X_\alpha u^\mu h^\nu{}_\beta \\
&+ \frac{1}{2}\left(\frac{1}{3}h_{\alpha\mu}\theta + \sigma_{\alpha\mu} + \omega_{\alpha\mu}\right)R^{\mu\nu}h_{\nu\beta} ,
\end{aligned} \tag{45}$$

and the constraint equation

$$\begin{aligned}
&D_\mu E_\alpha{}^\mu + \varepsilon^{\mu\gamma\delta}\bar{H}_{\mu\alpha}\left(\omega_{\delta\gamma} - W_{\delta\gamma} - \frac{1}{2}S_{\delta\gamma}\right) + \left(\sigma_{\delta\nu} + \omega_{\delta\nu} - \frac{1}{2}W_{\delta\nu}\right)\varepsilon^{\nu\beta\mu}h_{\alpha\beta}H_\mu{}^\delta = \frac{1}{2}R_{\beta\gamma}\bar{S}^{\mu\beta}\varepsilon^\gamma{}_{\mu\alpha} \\
&- R_{\mu\gamma}\bar{S}^{\mu\beta}\varepsilon^\gamma{}_{\beta\alpha} - \frac{1}{12}RX_\alpha + \frac{1}{2}R\varepsilon_{\mu\beta\alpha}\bar{S}^{\mu\beta} - \frac{1}{4}R^{\gamma\beta}u_\beta\left(W_{\alpha\gamma} - X_\alpha u_\gamma\right) + \frac{1}{2}h_{\nu\alpha}R^{\nu\beta}a_\beta + \frac{1}{12}D_\alpha R \\
&- \frac{1}{2}E_{\alpha\nu}X^\nu + \frac{1}{2}D_\alpha\left(R_{\mu\nu}u^\mu u^\nu\right) + R_{\nu\beta}W_\alpha{}^{(\nu}u^{\beta)} + 2\bar{S}^{\nu\beta}\varepsilon_{\nu\beta\mu}E_\alpha{}^\mu + \frac{1}{2}S_{\alpha\gamma}u^\beta R_\beta{}^\gamma - \varepsilon_{\alpha\beta\nu}\bar{S}_\mu{}^\beta E^{\mu\nu} \\
&- \frac{1}{3}\theta R_{\nu\beta}u^{(\nu}h^{\beta)}{}_\alpha - R_{\nu\beta}u^{(\nu}\left(\sigma^{\beta)}{}_\alpha - \omega^{\beta)}{}_\alpha\right) - \frac{1}{2}h_\alpha{}^\delta u^\gamma\nabla_\gamma\left(R_{\mu\nu}h^\mu{}_\delta u^\nu\right) + \frac{1}{2}R_{\mu\nu}u^\mu u^\nu a_\alpha \\
&+ \frac{1}{2}h_{\delta\alpha}R^{\delta\mu}\left(\varepsilon_{\mu\nu\beta}\bar{S}^{\beta\nu} - \frac{1}{2}W_\gamma{}^\gamma u_\mu + \frac{1}{2}X_\mu\right) ,
\end{aligned} \tag{46}$$

where only the upper indices enter in the symmetrization process. Equation (45) is found from the projection $h_{\mu\gamma}h_{\nu\beta}u_\delta\nabla_\lambda C^{\gamma\delta\beta\lambda}$ and Eq. (46) follows from $h_{\delta\alpha}u_\gamma u_\beta\nabla_\lambda C^{\gamma\delta\beta\lambda}$. For the magnetic part of the Weyl tensor we find the propagation equation

$$\begin{aligned}
&\left(2a^{\mu}E^{\nu}{}_{(\alpha} + D^{\mu}E^{\nu}{}_{(\alpha}\right)\varepsilon_{\beta)\nu\mu} - h^{\mu}{}_{(\alpha}h_{\beta)}{}^{\nu}\dot{H}_{\mu\nu} + \left(\sigma_{\mu(\alpha} + \omega_{\mu(\alpha}\right)H_{\beta)}{}^{\mu} - \left(\frac{2}{3}H_{\alpha\beta} + \frac{1}{3}\bar{H}_{\alpha\beta}\right)\theta \\
&+ \left(\frac{1}{3}h_{\alpha\beta}\theta - h_{\alpha\beta}W^{\mu}{}_{\mu} - \sigma_{\alpha\beta} + W_{(\alpha\beta)}\right)\bar{H}^{\nu}{}_{\nu} + 2\sigma_{\mu(\alpha}\bar{H}^{\mu}{}_{\beta)} - \left(\sigma_{\mu\nu} - W_{\mu\nu}\right)h_{\alpha\beta}\bar{H}^{\mu\nu} + W^{\mu}{}_{\mu}\bar{H}_{\alpha\beta} \\
&- \bar{H}^{\mu}{}_{(\alpha}W_{\beta)\mu} - W_{\mu(\alpha}\bar{H}^{\mu}{}_{\beta)} = \frac{1}{2}D^{\delta}\left(\varepsilon_{\gamma\delta(\alpha}h_{\beta)\mu}R^{\gamma\mu}\right) + E_{\mu(\alpha}\left(2\bar{S}_{\beta)}{}^{\mu} - \varepsilon_{\beta)}{}^{\mu\nu}X_{\nu}\right) \\
&- \frac{1}{2}u_{\mu}R^{\gamma\mu}\varepsilon_{\gamma\nu(\alpha}\left(\frac{1}{3}h^{\nu}{}_{\beta)}\theta + \sigma^{\nu}{}_{\beta)} + \omega^{\nu}{}_{\beta)} - W^{\nu}{}_{\beta)}\right) + \frac{1}{2}R^{\gamma\mu}u_{\gamma}h_{\mu(\alpha}\varepsilon_{\beta)}{}^{\nu\delta}\left(\omega_{\nu\delta} - W_{\nu\delta}\right) + \frac{1}{3}R\bar{S}_{(\alpha\beta)} \\
&- \bar{S}_{(\alpha}{}^{\mu}h^{\gamma}{}_{\beta)}R_{\mu\gamma} + \frac{1}{2}\varepsilon_{\mu\nu(\alpha}W^{\mu}{}_{\beta)}R^{\nu\delta}u_{\delta} + \bar{S}_{(\alpha\beta)}R_{\mu\delta}u^{\mu}u^{\delta} - \frac{1}{2}\varepsilon^{\mu\nu}{}_{(\alpha}h_{\beta)}{}^{\gamma}X_{\mu}R_{\nu\gamma}\,, \qquad (47)
\end{aligned}$$

where dummy indices are out of the symmetrization process. This equation describes the propagation of the H component of the Weyl tensor along the congruence. For the magnetic part of the Weyl tensor we find the constraint equation

$$\begin{aligned}
&-2D_{\mu}H^{\mu}{}_{\alpha} + \frac{2}{3}\varepsilon_{\alpha\beta\delta}E^{\beta\delta}\theta + 2\varepsilon_{\alpha\beta}{}^{\mu}E^{\beta\delta}\sigma_{\delta\mu} + 4E^{\beta}{}_{(\delta}\varepsilon_{\alpha)\beta}{}^{\mu}\left(\omega^{\delta}{}_{\mu} - W^{\delta}{}_{\mu}\right) = \varepsilon_{\alpha\gamma\delta}D^{\delta}\left(R^{\gamma\beta}u_{\beta}\right) \\
&+ \varepsilon_{\alpha\gamma\delta}R^{\gamma}{}_{\beta}W^{\delta\beta} - \frac{1}{3}\varepsilon_{\alpha\gamma\delta}R^{\gamma\delta}\theta - \varepsilon_{\alpha\gamma\delta}R^{\gamma\beta}\left(\sigma^{\delta}{}_{\beta} + \omega^{\delta}{}_{\beta}\right) + 2\bar{S}^{\mu}{}_{\alpha}R_{\mu\delta}u^{\delta} - 2\bar{S}_{\alpha}{}^{\beta}R_{\beta\delta}u^{\delta} - 2\varepsilon^{\mu\nu\gamma}S_{\mu\nu}E_{\gamma\alpha} \\
&+ \varepsilon^{\mu\nu\gamma}S_{\mu\nu}R_{\gamma\beta}h^{\beta}{}_{\alpha} - \varepsilon^{\mu\nu}{}_{\alpha}S_{\mu\nu}u^{\gamma}u^{\beta}R_{\gamma\beta} - \frac{1}{3}\varepsilon^{\mu\nu}{}_{\alpha}S_{\mu\nu}R - 4\bar{S}^{\gamma\beta}\varepsilon_{\gamma\beta\mu}H^{\mu}{}_{\alpha}\,. \qquad (48)
\end{aligned}$$

This equation provides the divergence of H on $\mathcal{V}$. Equation (47) is found from $\varepsilon_{\gamma\delta(\alpha}h_{\beta)\mu}\nabla_{\lambda}C^{\gamma\delta\mu\lambda}$ and Eq. (48) follows from computing the projection $\varepsilon_{\alpha\gamma\delta}u_{\beta}\nabla_{\lambda}C^{\gamma\delta\beta\lambda}$. Computing the contraction $\varepsilon_{\alpha}{}^{\gamma\delta}h^{\mu\beta}u^{\nu}\left(R_{\gamma\delta\mu\nu} - R_{\mu\nu\gamma\delta}\right)$ and using Eq. (16), we find that there is a further relation, one between the tensors H and $\bar{H}$,

$$\begin{aligned}
&H_{\alpha}{}^{\beta} - \bar{H}_{\alpha}{}^{\beta} + \varepsilon_{\alpha}{}^{\mu\beta}u^{\nu}R_{[\nu\mu]} = -\frac{1}{2}\varepsilon_{\alpha}{}^{\mu\nu}D^{\beta}\left(W_{\nu\mu} + S_{\nu\mu}\right) + \varepsilon_{\alpha\mu\nu}X^{\nu}B_{\perp}^{\beta\mu} - \varepsilon_{\alpha\mu\nu}X^{\nu}\left(W^{[\beta\mu]} + S^{\beta\mu}\right) \\
&- \frac{1}{2}\varepsilon_{\alpha}{}^{\mu\nu}a^{\beta}\left(W_{\nu\mu} + S_{\nu\mu}\right) + 2h_{\alpha}{}^{\gamma}h^{\mu\beta}u^{\nu}\nabla_{\nu}\bar{S}_{\gamma\mu} + 2\bar{S}_{\alpha}{}^{\beta}\theta - 2h_{\alpha}{}^{\beta}\bar{S}^{\mu\nu}\left(S_{\mu\nu} + W_{[\mu\nu]}\right) \\
&+ \varepsilon_{\alpha\mu\nu}a^{\nu}\left(W^{(\mu\beta)} + S^{\mu\beta}\right) - h_{\alpha}{}^{\beta}u^{\nu}\nabla_{\nu}\bar{S}_{\mu}{}^{\mu} - h_{\alpha}{}^{\beta}\bar{S}_{\mu}{}^{\mu}\theta + \varepsilon_{\alpha\mu}{}^{\nu}D_{\nu}\left(W^{(\beta\mu)} + S^{\beta\mu}\right) + \varepsilon_{\alpha}{}^{\mu\nu}X^{\beta}\omega_{\mu\nu} \\
&- 2\bar{S}^{\mu\beta}B_{\perp\mu\alpha} - \frac{1}{2}\varepsilon_{\alpha}{}^{\mu\nu}X^{\beta}\left(W_{\mu\nu} + S_{\mu\nu}\right) + 2h_{\alpha}{}^{\beta}\bar{S}^{\mu\nu}B_{\perp\mu\nu} - 2\bar{S}_{\alpha}{}^{\mu}W_{\mu}{}^{\beta}\,. \qquad (49)
\end{aligned}$$

Note that in Eq. (49) the term with the Ricci tensor on the left-hand side could be removed by taking the symmetric part in the indices α and β, however, this will add more terms and dense the notation on the right-hand side so, we opted to write the result as is. Note also that Eq. (49) shows how the presence of torsion is responsible for the degeneracy removal of the magnetic parts of the Weyl tensor. It is interesting to note that the difference between the magnetic parts of the Weyl tensor depends on the derivatives of the components of the torsion tensor, on $\mathcal{V}$ and along u, making it clear that in general both the value and the rate of change of the torsion field affect the difference between the tensors H and $\bar{H}$. Moreover, since in Eq. (49) we have an algebraic relation for the difference of the components of the tensors H and $\bar{H}$, Eqs. (47)–(49) characterize both H and $\bar{H}$, that is, we do not need to find propagation and constraint equations for $\bar{H}$ since those will not be independent of Eqs. (47)–(49). Using Eq. (14), we find the remaining equations that characterize the torsion tensor components. These equations are

$$\begin{aligned} h^{\alpha\nu}u^{\mu}R_{[\nu\mu]} = \varepsilon^{\alpha}{}_{\mu\nu}u^{\gamma}\nabla_{\gamma}\bar{S}^{\mu\nu} + \left(\varepsilon_{\gamma\mu\nu}\bar{S}^{\mu\nu} - \frac{1}{2}X_{\gamma}\right)B_{\perp}^{\alpha\gamma} + \varepsilon_{\gamma}{}^{\alpha\mu}\bar{S}_{\mu\beta}\left(B_{\perp}^{\beta\gamma} - W^{\beta\gamma}\right) + \frac{1}{2}D^{\alpha}W_{\mu}{}^{\mu} \\ - \frac{1}{2}D_{\beta}W^{\alpha\beta} + \frac{1}{2}W_{\mu}{}^{\mu}a^{\alpha} - \frac{1}{2}W^{\alpha\gamma}a_{\gamma} + \frac{1}{2}X^{\alpha}\theta + S_{\gamma}{}^{\alpha}a^{\gamma}\,, \end{aligned} \tag{50}$$

and

$$\begin{aligned} \varepsilon^{\alpha\mu\nu}R_{\mu\nu} = \varepsilon^{\alpha}{}_{\sigma\rho}\dot{S}^{\sigma\rho} + 2D_{\beta}\bar{S}^{\beta\alpha} + \varepsilon^{\alpha}{}_{\sigma\rho}D^{\sigma}X^{\rho} - \varepsilon^{\alpha}{}_{\sigma\rho}W_{\mu}{}^{\mu}\left(\omega^{\sigma\rho} - W^{\sigma\rho}\right) + 2\bar{S}^{\alpha\mu}\left(X_{\mu} + a_{\mu}\right) \\ - \varepsilon^{\alpha}{}_{\sigma\rho}W^{\rho}{}_{\beta}\left(B_{\perp}^{\beta\sigma} - W^{\beta\sigma}\right) - \varepsilon^{\alpha}{}_{\sigma\rho}\left(X^{\sigma}a^{\rho} + S^{\rho\sigma}\theta\right) - 4\varepsilon^{\beta\mu\nu}\bar{S}^{\alpha}{}_{\beta}\bar{S}_{\mu\nu}\,. \end{aligned} \tag{51}$$

Equations (50) and (51) are derived from computing the projections $h^{\sigma\alpha}u^{\gamma}R_{[\alpha\beta\gamma]}{}^{\beta}$ and $h^{\sigma\alpha}h^{\rho\gamma}R_{[\alpha\beta\gamma]}{}^{\beta}$, respectively, and using the first Bianchi identity, Eq. (14).

Equations (39)–(51) characterize the geometry of the manifold, containing exactly the same information as the Ricci and Bianchi identities.

III. THE STRESS-ENERGY-MOMENTUM TENSOR AND THE STRUCTURE EQUATIONS FOR THE MATTER FIELDS

A. The stress-energy tensor and its decomposition

For the stress-energy tensor T, that characterizes the matter fields permeating the spacetime manifold, we also want to apply the 1+3 formalism in order to study its dynamical evolution. Setting the congruence's tangent vector field u to coincide with the 4-velocity of an observer, without imposing any symmetries on T and using Eq. (23) we find the following decomposition

$$T_{\alpha\beta} = \mu\, u_{\alpha}u_{\beta} + p\, h_{\alpha\beta} + q_{1\alpha}u_{\beta} + u_{\alpha}q_{2\beta} + \pi_{\alpha\beta} + \varepsilon_{\alpha\beta}{}^{\gamma}m_{\gamma}\,, \tag{52}$$

with

$$\begin{aligned} \mu = u^{\mu}u^{\nu}T_{\mu\nu}\,, \qquad p = \frac{1}{3}h^{\mu\nu}T_{\mu\nu}\,, \quad q_{1\alpha} = -h_{\alpha}{}^{\mu}u^{\nu}T_{\mu\nu}\,, \\ q_{2\alpha} = -u^{\mu}h_{\alpha}{}^{\nu}T_{\mu\nu}\,, \quad \pi_{\alpha\beta} = T_{\langle\alpha\beta\rangle}\,, \qquad m_{\alpha} = \frac{1}{2}\varepsilon_{\alpha}{}^{\mu\nu}T_{\mu\nu}\,. \end{aligned} \tag{53}$$

where μ is the energy density measured by the chosen observer, p is the pressure, $q_{1\alpha}$ and $q_{2\alpha}$ represent energy and momentum density fluxes, $\pi_{\alpha\beta}$ is the anysotropic stress and m_{α} is a flux, in particular related with the nonconservation of intrinsic angular momentum of matter.

Note that we are free to arbitrarily choose the time-like congruence, nonetheless, in the case of a single fluid, it is useful to set the congruence's tangent vector field u to coincide with the 4-velocity of the elements of volume of the fluid, in which case the various projections of the stress-energy tensor and the kinematical quantities of the congruence directly represent the properties and evolution of the matter fluid.

B. The structure equations for the matter fields

To find the set of equations describing the dynamical evolution of the matter fields in the manifold, we will consider the general conservation law for the stress-energy tensor, given in general by

$$\nabla_\beta T^{\alpha\beta} = \Psi^\alpha \,, \tag{54}$$

where Ψ^α is some tensor to be determined by the field equations and the Bianchi identities. From Eqs. (52) and (54), the projections along u and on $\mathcal{V}$ are

$$\dot{\mu} + \left(\theta - {W_\alpha}^\alpha\right)(\mu + p) - \varepsilon^{\alpha\beta\gamma} m_\gamma \left(\omega_{\alpha\beta} - W_{\alpha\beta}\right) + \pi^{\alpha\beta}\left(\sigma_{\alpha\beta} - W_{\alpha\beta}\right) + \left(q_1^\alpha + q_2^\alpha\right) a_\alpha + D_\alpha q_2^\alpha = -u_\alpha \Psi^\alpha \,, \tag{55}$$

$$\begin{aligned}(\mu + p)\, a_\alpha + D_\alpha p + D_\mu {\pi_\alpha}^\mu + {\varepsilon_\alpha}^{\mu\nu} D_\mu m_\nu + \left(\pi_{\alpha\nu} - \varepsilon_{\alpha\mu\nu} m^\mu\right) a^\nu + {h_\alpha}^\beta \dot{q}_{1\beta} + \left(q_{1\alpha} + \frac{1}{3} q_{2\alpha}\right)\theta \\ - q_{1\alpha} {W_\beta}^\beta + q_2^\beta \left(\sigma_{\beta\alpha} + \omega_{\beta\alpha} - W_{\beta\alpha}\right) = h_{\alpha\beta}\Psi^\beta \,.\end{aligned} \tag{56}$$

At this point the imposition that the evolution equations for the matter variables are determined by (54) is given ad hoc. In practice, however, provided the field equations of a gravity theory relating the Ricci and the stress-energy tensors, the conservation equations will follow from the second Bianchi identity. Hence, these are a pivotal component to guarantee the consistency of the physical theory and system of equations.

IV. THE EINSTEIN-CARTAN THEORY FOR A WEYSSENHOFF LIKE TORSION: FIELD EQUATIONS

The general set of structure equations that arise from the 1+3 formalism can be used to study solutions of any relativistic theory of gravitation based on an affine, metric compatible connection.

In this section, we will focus on the Einstein-Cartan theory characterized by the field equations

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R + \Lambda g_{\alpha\beta} = \quad 8\pi T_{\alpha\beta}\,, \tag{57}$$

$$S^{\alpha\beta\gamma} + 2g^{\gamma[\alpha}S^{\beta]}{}_{\mu}{}^{\mu} = -8\pi\Delta^{\alpha\beta\gamma}\,, \tag{58}$$

where $T_{\alpha\beta}$ represents the canonical stress-energy tensor, $\Delta^{\alpha\beta\mu}$ is the intrinsic hypermomentum and Λ the cosmological constant The Einstein-Cartan theory defined by Eqs. (57) and Eq. (58) can be derived from the Einstein-Hilbert action $I = \frac{1}{16\pi}\int d^4x\sqrt{-g}\,(R-2\Lambda) + \int d^4x\sqrt{-g}\mathcal{L}_{\rm m}$, where the Ricci scalar contains the metric and the torsion as dynamical variables, $\mathcal{L}_{\rm m}$ is the matter Lagrangian density, and the variation of I must be performed with respect to those two fields. The conservation law is given by

$$\nabla_\beta T_\alpha{}^\beta = 2S_{\alpha\mu\nu}T^{\nu\mu} - \frac{1}{4\pi}S_{\alpha\mu}{}^{\mu}\Lambda + \frac{1}{8\pi}\left(S_{\alpha\mu}{}^{\mu}R - S^{\mu\nu\sigma}R_{\alpha\sigma\mu\nu}\right)\,. \tag{59}$$

To simplify the equations and, in agreement with what we are going to consider in the following, we will impose that the torsion tensor is characterized only by the tensor $S_{\alpha\beta}$, that is, the tensors $\bar{S}_{\alpha\beta}$, $W_{\alpha\beta}$ and X_α in Eq. (27) are considered to be identically zero, so

$$S_{\alpha\beta}{}^{\gamma} = S_{\alpha\beta}u^\gamma\,. \tag{60}$$

Given Eqs. (57)–(59) and assuming Eq. (60), the 1+3 structure equations have the following new forms.

The propagation equations for the kinematical quantities associated with u are

$$\dot\theta = -4\pi\left(\mu+3p\right) + \Lambda - \left(\frac{1}{3}\theta^2 + \sigma_{\alpha\beta}\sigma^{\alpha\beta} + \omega_{\alpha\beta}\omega^{\beta\alpha}\right) + D_\alpha a^\alpha + a_\mu a^\mu \tag{61}$$

$$h_\mu{}^\alpha h_\nu{}^\beta\dot\omega_{\alpha\beta} = -E_{[\mu\nu]} + 4\pi\varepsilon_{\mu\nu\gamma}m^\gamma - \frac{2}{3}\theta\omega_{\mu\nu} + 2\sigma^\alpha{}_{[\mu}\,\omega_{\nu]\alpha} + D_{[\mu}\,a_{\nu]}\,, \tag{62}$$

$$h_\mu{}^\alpha h_\nu{}^\beta\dot\sigma_{\alpha\beta} = -E_{(\mu\nu)} + 4\pi\left(\pi_{\mu\nu}\right) + D_{\langle\mu}\,a_{\nu\rangle} + a_{\langle\mu}\,a_{\nu\rangle} - \frac{2}{3}\sigma_{\mu\nu}\theta - \sigma^\delta{}_{\langle\mu}\,\sigma_{\nu\rangle\delta} - \omega^\delta{}_{\langle\mu}\,\omega_{\nu\rangle\delta}\,, \tag{63}$$

and the corresponding constraint equations are

$$\varepsilon^{\mu\nu\rho}D_\mu\omega_{\nu\rho} + \varepsilon^{\mu\nu\rho}a_\rho\omega_{\nu\mu} = H_\rho{}^\rho + \varepsilon^{\mu\nu\rho}a_\rho S_{\nu\mu}\,, \tag{64}$$

$$\varepsilon^{\alpha\beta\langle\mu}D_\alpha\left(\sigma_\beta{}^{\nu\rangle} + \omega_\beta{}^{\nu\rangle}\right) + \varepsilon^{\alpha\beta\langle\mu}a^{\nu\rangle}\omega_{\beta\alpha} = H^{\langle\mu\nu\rangle} - \varepsilon^{\alpha\beta\langle\mu}a^{\nu\rangle}S_{\alpha\beta}\,, \tag{65}$$

$$\frac{2}{3}D_\alpha\theta - D_\mu\left(\sigma_\alpha{}^\mu + \omega_\alpha{}^\mu\right) - 2a^\mu\omega_{\alpha\mu} = 8\pi q_{1\alpha} + 2a^\mu S_{\mu\alpha}\,. \tag{66}$$

where only upper indices enter the symetrization process.

The propagation equations for the electric and magnetic parts of the Weyl tensor are

$$\begin{aligned}
&- h_{\alpha\mu}h_{\beta\nu}\dot{E}^{\mu\nu} + \varepsilon_{\mu\beta}{}^\nu\left(D_\nu\bar{H}^\mu{}_\alpha + a_\nu\bar{H}^\mu{}_\alpha\right) + \varepsilon^\mu{}_{\alpha\delta}a^\delta H_{\mu\beta} + \left(\sigma_{\alpha\nu} + \omega_{\alpha\nu}\right)E^\nu{}_\beta + 2E_\alpha{}^\mu\left(\sigma_{\mu\beta} + \omega_{\mu\beta}\right)\\
&- E_{\alpha\beta}\theta - h_{\alpha\beta}E^{\nu\mu}\left(\sigma_{\mu\nu} + \omega_{\mu\nu}\right) = \frac{4\pi}{3}h_{\alpha\beta}\dot{\mu} + 4\pi h_{\alpha\mu}h_{\beta\nu}\dot{\pi}^{\mu\nu} + 4\pi\varepsilon_{\alpha\beta}{}^\gamma\dot{m}_\gamma + 4\pi\left(q_{1\alpha}a_\beta + a_\alpha q_{2\beta}\right)\\
&+ 4\pi D_\alpha q_{2\beta} + 4\pi\left(\frac{1}{3}h_{\alpha\delta}\theta + \sigma_{\alpha\delta} + \omega_{\alpha\delta}\right)\left[h^\delta{}_\beta\left(\mu + p\right) + \pi^\delta{}_\beta + \varepsilon^\delta{}_{\beta\gamma}m^\gamma\right]\,,
\end{aligned} \tag{67}$$

$$\begin{aligned}
&\bar{H}_\mu{}^\mu\left(\frac{1}{3}h^{\alpha\beta}\theta - \sigma^{\alpha\beta}\right) + 2\bar{H}_\mu{}^{(\alpha}\sigma^{\beta)\mu} - h^{\alpha\beta}\bar{H}^{\mu\nu}\sigma_{\mu\nu} + \left[2a_\mu E_\nu{}^{(\alpha} + D_\mu E_\nu{}^{(\alpha}\right]\varepsilon^{\beta)\nu\mu} - h^{\mu(\alpha}h^{\beta)\nu}\dot{H}_{\mu\nu}\\
&+ H^{\mu(\alpha}\left(\sigma_\mu{}^{\beta)} + \omega_\mu{}^{\beta)}\right) - \frac{1}{3}\left(2H^{\alpha\beta} + \bar{H}^{\alpha\beta}\right)\theta = 4\pi\varepsilon_\gamma{}^{\delta(\alpha}h^{\beta)\mu}q_1^\gamma\left(\sigma_{\delta\mu} + \omega_{\delta\mu}\right) + 4\pi\varepsilon^{\mu\delta(\alpha}q_2^{\beta)}\omega_{\delta\mu}\\
&+ 4\pi\varepsilon_\gamma{}^{\delta(\alpha}D_\delta\pi^{\beta)\gamma} + 4\pi D^{(\alpha}m^{\beta)} - 4\pi h^{\alpha\beta}D_\delta m^\delta\,,
\end{aligned} \tag{68}$$

and the corresponding constraint equations are,

$$\begin{aligned}
&D_\beta E_\alpha{}^\beta + \varepsilon^{\beta\gamma\delta}\bar{H}_{\beta\alpha}\left(\omega_{\delta\gamma} - \frac{1}{2}S_{\delta\gamma}\right) + \left(\sigma_{\delta\nu} + \omega_{\delta\nu}\right)\varepsilon^{\nu\beta\gamma}h_{\alpha\beta}H_\gamma{}^\delta = 4\pi D_\alpha p + \frac{8\pi}{3}D_\alpha\mu\\
&+ 4\pi\left[\pi_\alpha{}^\beta - \varepsilon^\beta{}_{\alpha\gamma}m^\gamma\right]a_\beta + 4\pi\left(q_{2\lambda} + q_{1\lambda}\right)\left(\sigma_\alpha{}^\lambda + \omega_\alpha{}^\lambda + \frac{1}{3}h_\alpha{}^\lambda\theta\right) + 4\pi h_\alpha{}^\gamma\dot{q}_{1\gamma}\\
&+ 4\pi\left(\mu + p\right)a_\alpha - 4\pi S_{\alpha\gamma}q_2^\gamma\,,
\end{aligned} \tag{69}$$

$$\begin{aligned}
&4E^\beta{}_{(\delta}\varepsilon_{\alpha)\beta}{}^\gamma\omega^\delta{}_\gamma + 2\varepsilon_{\alpha\beta}{}^\gamma E^{\beta\delta}\sigma_{\delta\gamma} - 2D_\gamma H^\gamma{}_\alpha + \frac{2}{3}\varepsilon_{\alpha\beta\delta}E^{\beta\delta}\theta = -8\pi\varepsilon_{\alpha\gamma\delta}\left[D^\delta q_1^\gamma + \omega^{\delta\gamma}\left(\mu + p\right)\right]\\
&- 8\pi\varepsilon_{\alpha\gamma\delta}\left(\pi^{\gamma\beta} + \varepsilon^{\gamma\beta}{}_\nu m^\nu\right)\left(\sigma^\delta{}_\beta + \omega^\delta{}_\beta\right) - \frac{16\pi}{3}\theta m_\alpha - \frac{8\pi}{3}\varepsilon^{\mu\nu}{}_\alpha S_{\mu\nu}\left(\mu + 3p - \frac{\Lambda}{4\pi}\right)\\
&+ 8\pi\varepsilon^{\mu\nu\gamma}S_{\mu\nu}\left(\pi_{\gamma\alpha} + \varepsilon_{\gamma\alpha\nu}m^\nu\right) - 2\varepsilon^{\sigma\nu\gamma}S_{\sigma\nu}E_{\gamma\alpha}\,,
\end{aligned} \tag{70}$$

$$H_\alpha{}^\beta - \bar{H}_\alpha{}^\beta + 4\pi\varepsilon_\alpha{}^{\mu\beta}\left(q_{1\mu} - q_{2\mu}\right) = -\frac{1}{2}\varepsilon_\alpha{}^{\mu\nu}D^\beta S_{\nu\mu} - \frac{1}{2}\varepsilon_\alpha{}^{\mu\nu}a^\beta S_{\nu\mu} + \varepsilon_{\alpha\mu\nu}a^\nu S^{\mu\beta} + \varepsilon_{\alpha\mu}{}^\nu D_\nu S^{\beta\mu}\,. \tag{71}$$

The equations that characterize the torsion tensor are,

$$4\pi\left(q_2^\alpha - q_1^\alpha\right) = S_\gamma{}^\alpha a^\gamma\,, \tag{72}$$

$$16\pi m^\alpha = \varepsilon^\alpha{}_{\rho\sigma}\left(S^{\rho\sigma}\theta + \dot{S}^{\rho\sigma}\right)\,. \tag{73}$$

The equation relating the torsion to the hypermomentum is

$$S^{\alpha\beta}u^\gamma = -8\pi\Delta^{\alpha\beta\gamma}\,. \tag{74}$$

The conservation of energy and momentum equations are

$$\dot{\mu} + \theta\,(\mu + p) + 2q_1^{\alpha} a_{\alpha} + D_{\alpha} q_2^{\alpha} + \pi^{\alpha\beta}\sigma_{\alpha\beta} + \varepsilon^{\alpha\beta\gamma} m_{\gamma}\omega_{\beta\alpha} = 0\,, \tag{75}$$

$$\begin{aligned}(\mu + p)\,a_{\alpha} + D_{\alpha} p + h_{\alpha}{}^{\beta}\dot{q}_{1\beta} + D_{\mu}\pi_{\alpha}{}^{\mu} + (\pi_{\alpha\nu} - \varepsilon_{\alpha\mu\nu} m^{\mu})\,a^{\nu} \\ + \left(q_{1\alpha} + \frac{q_{2\alpha}}{3}\right)\theta + q_2^{\beta}\,(\sigma_{\beta\alpha} + \omega_{\beta\alpha}) + \varepsilon_{\alpha}{}^{\mu\nu} D_{\mu} m_{\nu} = -\frac{1}{8\pi}\bar{H}_{\alpha}{}^{\rho} S^{\gamma\delta}\varepsilon_{\rho\gamma\delta} - S_{\alpha}{}^{\beta} q_{2\beta}\,.\end{aligned} \tag{76}$$

Once the matter model is given, Eqs. (61)–(76) completely describe the geometry of the spacetime and the evolution of the matter fluid for the Einstein-Cartan theory, for a torsion tensor of the form given in Eq. (60), i.e., $S_{\alpha\beta}{}^{\gamma} = S_{\alpha\beta} u^{\gamma}$. Note, however, that we have not yet imposed any restriction on the stress-energy tensor, and for a torsion that assumes the form of Eq. (60), the field equations, Eqs. (61)–(76), are valid for any matter model.

The form of the field equations, Eqs. (61)–(76) allow us to compare them with the results in the literature and test their validity. First, we see that our results differ from the ones in [28]. In this reference the authors seem to have not realized that in the presence of torsion, the Weyl tensor is characterized by three tensors, more specifically, the magnetic part of the Weyl tensor is described by two distinct tensors; moreover, it is quite surprising that the authors did not verify that the electric and magnetic parts of the Weyl tensor do not carry all the usual symmetries found in spacetimes with vanishing torsion. Second, setting the torsion terms in Eqs. (61)–(76) to zero, $\bar{H} = H$ and both the electric and magnetic part of the Weyl tensor are symmetric, tracefree tensors, and imposing the stress-energy tensor to be symmetric, such that $m^{\alpha} = 0$ and $q_{1\alpha} = q_{2\alpha}$, we recover the expressions for the structure equations for the theory of general relativity [21, 24].

V. RELATIVISTIC COSMOLOGY IN EINSTEIN-CARTAN THEORY: THE ISOTROPIC UNIVERSE AND THE GEOMETRY OF THE 3-SPACES

A. Field equations for the universe with homogeneous spinning fluid

The general set of structure equations for the Einstein-Cartan theory, Eqs. (61)–(76), even for a simplified torsion tensor, is extremely complicated and to find nontrivial solutions we have to impose some idealized symmetries and constraints on the matter fields. As a particular application of the previous set of equations, we will consider the effects of a neutral Weyssenhoff fluid, see, e.g., [12], in a cosmological setting.

The Weyssenhoff fluid represents a semi-classical model for a perfect fluid composed by fermions, taking into account the macroscopic effects of the intrinsic angular momentum of its constituents.

Following, Refs. [12, 14], for a comoving observer, the canonical stress-energy tensor of a Weyssenhoff fluid is such that $T = T\left(\mu, p, q_1\right)$, that is, the canonical stress-energy tensor only depends on the energy density, pressure and an heat flow term that arises from the intrinsic spin of the particles. For the Weyssenhoff fluid the intrinsic hypermomentum can be written as $\Delta^{\alpha\beta\gamma} = -\frac{1}{8\pi}\Delta^{\alpha\beta}u^{\gamma}$, where u represents the proper 4-velocity of an element of volume of the fluid and the antisymmetric spin density tensor, $\Delta^{\alpha\beta}$, verifies $\Delta^{\alpha\beta}u_{\beta} = 0$. From the field equation (74), we find that the torsion tensor is given by $S^{\alpha\beta} = \Delta^{\alpha\beta}$, and the components $\bar{S}^{\alpha\beta}$, $W^{\alpha\beta}$ and X^{α}, are identically zero for the Weyssenhoff fluid. An interesting consequence for the Weyssenhoff model is given by Eq. (72), which simplifies to $q_1^{\alpha} = -\frac{1}{4\pi}S_{\gamma}{}^{\alpha}a^{\gamma}$. This relation between the vector field q_1 and the torsion tensor was already found by Obukhov and Korotky for the Weyssenhoff fluid stress-energy tensor [14]. Of course the model found in [14] is more general, since it is independent of the considered gravitational theory, showing, nonetheless, the consistency of the results.

We are interested in studying solutions where a neutral Weyssenhoff fluid acts as a source of spin and that could be used to model the universe at very large scales, such that the cosmological principle is verified by the matter fluid. So, for the cosmological model we further assume a number of conditions. (i) The shear tensor field of the fluid is identically zero at every point and throughout the fluid's evolution, hence $\sigma_{\alpha\beta} = 0$. (ii) There are no spatial expansion gradients, such that $D_{\alpha}\theta = 0$. (iii) The matter fluid has no intrinsic preferred spatial directions, therefore we impose that there are no spatial energy density and pressure gradients, namely, $D_{\alpha}\mu = 0$ and $D_{\alpha}p = 0$. (iv) The fluid's elements of volume have zero 4-acceleration at all points and throughout the fluid's evolution, $a^{\mu} = 0$. (v) The vorticity tensor is such that $\omega_{\alpha\beta} = S_{\alpha\beta}$. This constraint is equivalent to impose that the spatial spaces, orthogonal to the curves of the congruence, are hypersurfaces [39]. (vi) The orthogonal spatial hypersurfaces are complete and simply-connected.

As we will see, these conditions guarantee that at the level of the metric there are no preferred spacial directions. On the other hand, an observer comoving with the fluid that interacts directly with the torsion tensor will in fact measure a preferred spacial direction, however this does not imply an intrinsic anisotropy of the matter fluid. We will discuss this in more detail below.

In what follows, it is useful to define the vector fields

$$\omega^{\gamma} = \frac{1}{2}\varepsilon^{\gamma\mu\nu}\omega_{\mu\nu}\,, \qquad S^{\gamma} = \frac{1}{2}\varepsilon^{\gamma\mu\nu}S_{\mu\nu}\,, \qquad \Delta^{\gamma} = \frac{1}{2}\varepsilon^{\gamma\mu\nu}\Delta_{\mu\nu}\,, \qquad E^{\gamma} = \frac{1}{2}\varepsilon^{\gamma\mu\nu}E_{\mu\nu}\,, \tag{77}$$

such that $\omega_{\mu\nu} = \varepsilon_{\mu\nu\gamma}\omega^{\gamma}$, $\Delta_{\mu\nu} = \varepsilon_{\mu\nu\gamma}\Delta^{\gamma}$, $S_{\mu\nu} = \varepsilon_{\mu\nu\gamma}S^{\gamma}$, and $E_{[\mu\nu]} = \varepsilon_{\mu\nu\gamma}E^{\gamma}$. Then, the structure equations (61)–(76), together with the previous assumptions yield the following set of equations.

We have for the kinematical quantities

$$\dot{\theta} = -4\pi\left(\mu + 3p\right) + \Lambda - \left(\frac{1}{3}\theta^2 - 2S^\sigma S_\sigma\right) \,, \tag{78}$$

$$D_\alpha \theta = 0 \,, \tag{79}$$

$$\omega^\gamma = S^\gamma \,, \tag{80}$$

$$\sigma_{\alpha\beta} = 0, \tag{81}$$

$$a^\beta = 0 \,; \tag{82}$$

for the Weyl tensor components

$$E^\gamma = \frac{1}{3}\theta S^\gamma \,, \tag{83}$$

$$E_{(\mu\nu)} = -S_{\langle\mu} S_{\nu\rangle} \,, \tag{84}$$

$$\bar{H}^{\mu\nu} = -D^{(\mu} S^{\nu)} \,, \tag{85}$$

$$H_{\alpha\beta} = \bar{H}_{\alpha\beta} - h_{\alpha\beta}\bar{H}_\sigma{}^\sigma \,, \tag{86}$$

$$u^\gamma \nabla_\gamma \bar{H}^{\alpha\beta} + \frac{4}{3}\theta \bar{H}^{\alpha\beta} = -2S_\gamma \varepsilon^{\mu\gamma(\alpha}\,\bar{H}^{\beta)}{}_\mu \,, \tag{87}$$

$$\varepsilon_\alpha{}^{\mu\nu} D_\nu \bar{H}_{\mu\beta} = \frac{1}{3}\varepsilon_{\alpha\beta\gamma} S^\gamma \left\{8\pi\mu + \Lambda - \frac{1}{3}\theta^2 - S^\sigma S_\sigma\right\} \,; \tag{88}$$

for the torsion field

$$\dot{S}^\gamma + \theta S^\gamma = 0 \,, \tag{89}$$

$$S^\gamma = \Delta^\gamma \tag{90}$$

$$\varepsilon_\alpha{}^{\mu\nu} D_\mu S_\nu = 0 \,, \tag{91}$$

$$S^\mu D_\mu S_\nu = 0 \,, \tag{92}$$

$$D_\nu \left(S^\sigma S_\sigma\right) = 0 \,; \tag{93}$$

and for the matter variables

$$\dot{\mu} + \theta\left(\mu + p\right) = 0 \,, \tag{94}$$

$$D_\alpha \mu = 0 \,, \tag{95}$$

$$D_\alpha p = 0 \,, \tag{96}$$

$$q_1^\alpha = 0 \,. \tag{97}$$

To close the system we have to either impose a function to model the pressure, $p = p(x^\alpha)$, where (x^α) is some local coordinate system on the manifold, or relate p with the energy density μ

through a barotropic equation of state, $p = p(\mu)$, i.e.,

$$p = p(x^\alpha) \quad \text{or} \quad p = p(\mu)\,. \tag{98}$$

Moreover, we see that there is no divergence equation for the torsion vector field, other than that is must be equal to minus the trace of $\bar{H}$. This is expected, since the geometry and the field equations of the theory alone cannot determine the relation between the spin density vector Δ^α and the thermodynamical variables μ and p: this is something that has to be provided by a physical model for the matter. Therefore, to completely close the system, we must either provide an ad hoc expression for the spin density vector field, such that $\Delta^\alpha = \Delta^\alpha\left(x^\alpha\right)$ or, more physically motivated, an equation that relates the spin density vector field with μ and p, $\Delta^\alpha = \Delta^\alpha(\mu, p)$, i.e.,

$$\Delta = \Delta(x^\alpha) \quad \text{or} \quad \Delta^\alpha = \Delta^\alpha(\mu, p)\,. \tag{99}$$

To further compare our results with those in the literature, notice that from Eqs. (85), (92) and (93) we find that $\bar{H}_{\alpha\rho}S^\rho = 0$. This constraint, $\bar{H}_{\alpha\rho}S^\rho = 0$, coincides with a constraint given in [3, 16], in which it is assumed that each element of volume of the fluid follows auto-parallel curves and its rest mass is constant. In our derivation of this constraint, we have not assumed that the rest mass is constant, however we have imposed that the fluid's volume elements have zero acceleration and it can be shown that this implies that their rest mass is constant, making the whole procedure consistent.

The previous set of equations can be written in a somewhat more compact form. Remembering that the magnetic parts of the Weyl tensor, H and $\bar{H}$, are symmetric tensors, Eqs. (85) and (91) can be replaced by the single equation $\bar{H}^{\mu\nu} = -D^\mu S^\nu$. In that case the relation $\bar{H}_{\alpha\rho}S^\rho = 0$, can replace the equations (92) and (93). Notwithstanding, we choose to keep all these properties explicit to avoid any confusion.

B. Geometry of the 3-spaces for the universe with homogeneous spinning fluid

We will now study some implications of the field equations, Eqs. (78)–(97). The matter equations of state, Eqs. (98) and (99), will not be used at this stage. We analyze in detail and obtain concrete results related to the geometry of the 3-spaces, i.e., 3-hypersurfaces, orthogonal to the congruence.

Without loss of generality, we will consider that the separation vector field n, introduced in Section II C, at each point is orthogonal to the tangent vector field u, such that $n^\mu u_\mu = 0$ and n^α is spacelike, i.e., we will consider the separation between points in the same orthogonal hypersurface. This is always possible since, at a given point, we may decompose a general separation vector in its components along u and orthogonal to it. Then, in the light of Eq. (33), in the considered setup, the orthogonal part will stay orthogonal to u as we move along its integral curves. From Eqs. (31) and (36) and setting $\sigma_{\alpha\beta} = 0$ in accord to Eq. (81) we have

$$\frac{D\left(n_\alpha n^\alpha\right)}{D\tau} = \frac{2}{3}\theta n^\alpha n_\alpha \,, \tag{100}$$

where we used the notation $\frac{D}{D\tau} \equiv u^\gamma \nabla_\gamma$, with τ being an affine parameter parameterizing the integral curves of u, that is τ represents up to a constant the proper time measured by an observer comoving with the fiducial curve of the congruence. Taking another derivative along u we find

$$\frac{D^2\left(n_\alpha n^\alpha\right)}{D\tau^2} = \frac{2}{3}\dot{\theta} n^\alpha n_\alpha + \frac{4}{9}\theta^2 n^\alpha n_\alpha \,, \tag{101}$$

relating the second derivative of the square of the norm of the separation vector with the expansion coefficient of the congruence and its derivative.

Since n is spacelike, we can define a length ℓ through the equation

$$\ell = \sqrt{n^\alpha n_\alpha} \,, \tag{102}$$

where in general $\ell : \mathcal{M} \to \mathbb{R}$, that is, in some local coordinate system, $\ell = \ell(x^\alpha)$, specifically of proper time τ and the spatial coordinates on the hypersurface. Nonetheless, since we have imposed $D_\alpha\theta = 0$, Eq. (79), it is always possible to define n to represent the separation vector between points at a fixed proper length at some particular hypersurface, then Eqs. (100) and (102) imply that

$$\ell = \ell\left(\tau\right) \,, \tag{103}$$

and Eq. (100) can be written as

$$\frac{1}{3}\theta = \frac{\dot{\ell}}{\ell} \,. \tag{104}$$

Now, let h_{ab}, ${}^3R_{ab}$ and ${}^3R$ to represent, respectively, the induced metric, the intrinsic Ricci tensor and the intrinsic Ricci scalar of an orthogonal 3-hypersurface. Then, in the considered setup, the Gauss embedding equation of differential geometry yields the following relations between ${}^3R_{ab}$ and ${}^3R$, and the induced metric, the kinematical and matter variables,

$${}^3R_{ab} = \frac{2}{3}h_{ab}\left(-\frac{1}{3}\theta^2 - S_\sigma S^\sigma + 8\pi\mu + \Lambda\right) \,, \tag{105}$$

$${}^3R = 2\left(-\frac{1}{3}\theta^2 - S^\sigma S_\sigma + 8\pi\mu + \Lambda\right) \,. \tag{106}$$

Equation (106) is the generalized Friedman equation for the Einstein-Cartan system we are interested. We remark that for the type of torsion that is being considered, Eq. (60), one can show that the induced connection on the orthogonal slices to u is the Levi-Civita connection, hence ${}^3R_{ab}$ and ${}^3R$ represent the Ricci tensor and Ricci scalar associated with the induced metric, h_{ab}.

From Eq. (106) we find ${}^3R\ell^2 = -6\dot{\ell}^2 - 2S^\sigma S_\sigma \ell^2 + 16\pi\mu\ell^2 + 2\Lambda\ell^2$. Taking the derivative along u of this equation and using the Raychaudhuri equation (78) and the conservation equations (89) and (94) yields $\frac{D}{d\tau}\left({}^3R\ell^2\right) = 0$, that is, the quantity ${}^3R\ell^2$ is a constant function between distinct hypersurfaces. Indeed, using Eqs. (79), (93) and (95) we conclude that

$$ {}^3R = \frac{6K}{\ell^2}\,, \tag{107} $$

where K is some constant to be dealt with and the number 6 appears for convenience. So, using Eqs. (103) and (107) we have that the orthogonal 3-hypersurfaces are manifolds of constant Ricci curvature, i.e., ${}^3R|_\tau$ = constant. This result in conjunction with Eqs. (105) and (106) leads us to conclude that the Ricci tensor of the 3-hypersurfaces is of the form ${}^3R_{ab} = \frac{2K}{\ell^2}h_{ab}$, i.e., a constant times the metric, so that in the considered setup the 3-hypersurfaces are Einstein manifolds. Now, in 3 dimensions the Riemann tensor is fully characterized by the Ricci tensor, specifically, ${}^3R_{abcd} = 2\left({}^3R_{a[c}h_{d]b} - {}^3R_{b[c}h_{d]a}\right) - {}^3R\, h_{a[c}h_{d]b}$, which in the considered setup implies ${}^3R_{abcd} = \frac{K}{\ell^2}\left(h_{ac}h_{db} - h_{ad}h_{cb}\right)$, and so the 3-hypersurfaces are surfaces of constant spatial curvature. Since we assume that the 3-hypersurfaces are complete and simply-connected, we have that the 3-hypersurfaces are isometric to the 3-hyperbolic space, to the 3-Euclidean space, or to the 3-sphere, in other words, in the considered setup, the 3-hypersurfaces are isotropic and homogeneous and the metric of the spacetime is a FLRW solution. Nonetheless, note that due to the presence of the torsion tensor, the whole spacetime is not described solely by the metric tensor. In the light of these results, we can relate the value of the integration constant K in Eq. (107) with the value of the constant curvature of each 3-hypersurface, i.e., $K = \{-1, 0, 1\}$, corresponding to the cases when the orthogonal hypersurfaces are, for the natural topology, open and hyperbolic, open and flat, or closed and spherical, respectively. Note, however, that depending on the topology, the solutions with $K = -1$ or $K = 0$ need not be necessarily open, whereas the family of solutions with $K = 1$, to which the spherical solution belongs to, is necessarily closed, see, e.g., [25].

Although the metric tensor is a FLRW solution, the presence of torsion modifies the geometry of the spacetime, in particular, we have found that the Weyl tensor does not have to vanish, see, e.g., Eq. (83). This, of course, has profound implications in the geometry of the spacetime and the type of solutions that are allowed. In the light of the field equations, we find the following results.

Theorem 1. *In the considered setup, if $S_\alpha S^\alpha \neq 0$ and $D_\alpha \Delta^\alpha = f(\mu, p)$, where f is an arbitrary differentiable function, then*

$$^3R \leq 0\,, \tag{108}$$

i.e., $K = -1$ or $K = 0$ in the FLRW metric. Moreover, each orthogonal hypersurface is flat, that is $^3R = 0$, if and only if $\bar{H}^{\alpha\beta} = 0$ for all points on the hypersurface.

Proof. From Eqs. (85) and (88) and $\bar{H}_{\alpha\rho}S^\rho = 0$, we find the following relation

$$S^\delta D_\delta \left(D_\alpha S^\alpha\right) = -\frac{2}{3} S_\delta S^\delta \left\{8\pi\mu + \Lambda - \frac{1}{3}\theta^2 - S^\sigma S_\sigma\right\} - \bar{H}^{\mu\nu}\bar{H}_{\mu\nu}\,. \tag{109}$$

Note that for the type of torsion that we are considering, Eq. (60), one has that $D_\alpha Y^\alpha$ is indeed the divergence of a vector field Y orthogonal to u, so that $D_\alpha S^\alpha$ is the divergence of the torsion vector S. Imposing that the divergence of the spin density vector is a differentiable function of the energy density μ and the pressure p, that is $D_\alpha \Delta^\alpha = f(\mu, p)$, and using Eqs. (90), (95) and (96) implies that $D_\delta \left(D_\alpha S^\alpha\right) = 0$. Using this result and the Friedman equation (106), Eq. (109) yields

$$\bar{H}^{\mu\nu}\bar{H}_{\mu\nu} = -\frac{^3R}{3} S_\delta S^\delta. \tag{110}$$

Since the hypersurfaces orthogonal to the tangent vector field u are Riemannian manifolds, the terms $\bar{H}^{\mu\nu}\bar{H}_{\mu\nu}$ and $S_\delta S^\delta$ must be non-negative, therefore a consistent solution of the field equations with $S_\delta S^\delta \neq 0$ must verify $^3R \leq 0$, i.e., $K = -1$ or $K = 0$ in the FLRW metric. Using this same argument, it follows that if $^3R = 0$, then $\bar{H}^{\mu\nu} = 0$. Of course, trivially, if $\bar{H}^{\mu\nu} = 0$ and $S_\delta S^\delta \neq 0$, then $^3R = 0$. □

The result in Theorem 1 is quite surprising. In the considered setup and for a nonvanishing torsion vector field, the orthogonal hypersurfaces that foliate the spacetime must either have negative curvature or be Ricci flat. In addition, we find the following result:

Theorem 2. *In the considered setup, if the torsion S is such that $S_\alpha S^\alpha \neq 0$ for all points on the hypersurfaces orthogonal to the congruence associated with u, then the hypersurfaces cannot be closed.*

Proof. Let us start by recalling that we have imposed the congruence u to be hypersurface orthogonal. We then chose a frame where the orthogonal slices $\mathcal{V}$ are hypersurfaces, hence there exists an embedding between each hypersurface $\mathcal{V}$ and a Riemannian manifold $(\mathcal{V}, h)$, where h represents the induced metric and, for the type of torsion that we are considering in this section, the induced torsion tensor is zero. Since an embedding exists, we can pull-back and push-forward nonvanishing orthogonal tensor fields in $(\mathcal{M}, g, S)$ to nonvanishing tensor fields in $(\mathcal{V}, h)$. In particular, the pull-back of the projected covariant derivatives of an orthogonal 1-form field Y_α, that is $Y_\alpha u^\alpha = 0$, is given by $D_a Y_b$, where Y_a represents the pull-back of Y_α and defines the induced connection in $(\mathcal{V}, h)$, which is simply the Levi-Civita connection associated with h. Of course, Y_α represents the components of $Y \in T_p^*\mathcal{M}$ in a local coordinate system and Y_a the components of $Y \in T_p^*\mathcal{V}$ in a local coordinate system, where T_p^* means the cotangent space of the corresponding manifold at the point p, however, although an abuse of language, it is much simpler and became kind of a convention to distinguish between the two tensors by using Greek and Latin letters.

From Eq. (91), if we define S_a as the pull-back of the 1-form S_α, we find that it verifies

$$\varepsilon_a{}^{bc} D_b S_c = 0 \Leftrightarrow \varepsilon_a{}^{bc} \partial_b S_c = 0\,, \tag{111}$$

where ε_{abc} represents the Levi-Civita tensor in $(\mathcal{V}, h)$. Therefore, S_a is an exact 1-form, that is, there exists a function ϕ, such that $S = d\phi$. Moreover, since h is a Riemannian metric, it is nondegenerate, hence the condition $S_a S^a \neq 0$ implies that $d\phi \neq 0$. To clarify, the induced torsion tensor on $(\mathcal{V}, h)$ is zero, meaning that the manifold is endowed with only the Levi-Civita connection. However, S_a does not have to be zero, and it should be regarded simply as a 1-form field in $T^*\mathcal{V}$ with no relation with the connection.

Now, if $\mathcal{V}$ is closed, it is, by definition, compact and it has no boundary, then, from Stokes Theorem, we have $\int_{\mathcal{V}} d\phi = 0$. However, $d\phi \neq 0$, hence $d\phi$ is a volume form and its integral over $\mathcal{V}$ cannot be zero. $\square$

This result is also surprising. In addition to the result of Theorem 1 asserting that the orthogonal hypersurfaces cannot have positive curvature, i.e., $K \leq 0$, we see that Theorem 2 establishes that they also cannot be closed, limiting the topology of these solutions. This is indeed a great disparity between the theory of Einstein-Cartan and general relativity, since in the latter there is no limitation in the sign of K nor on the topology.

The intermediate results for the proof of Theorem 1 also allow us to infer the behavior of the magnetic part of the Weyl tensor. Considering Eqs. (89), (104), (107) and (110) we have:

Proposition 1. *In the considered setup, if $S_\alpha S^\alpha \neq 0$, ${}^3R < 0$ and $D_\alpha \Delta^\alpha = f(\mu, p)$, where f is an arbitrary differentiable function, then $\bar{H}^{\mu\nu}\bar{H}_{\mu\nu} \sim \frac{1}{\ell^8}$.*

Proposition 1 establishes the behavior of the tensor $\bar{H}$, and similarly for H, in terms of the scale factor ℓ, so that if the spacetime is expanding, $\bar{H}^{\mu\nu}\bar{H}_{\mu\nu}$ tends to zero as $\frac{1}{\ell^8}$. In the next section we further study the tensor $\bar{H}$, in particular we show that it is possible to derive a wave equation for $\bar{H}$ and then study its solutions.

We use the results obtained here to clarify some confusion regarding the possibility to consider a torsion caused by an intrinsic spin of matter in a cosmological context. In [11] it was shown that, under certain conditions, the symmetries of the metric tensor, in the form of Killing vector fields, are also symmetries of the torsion tensor. Under those conditions, of course, it was then found that a torsion tensor having its origin in the intrinsic spin of matter, a torsion tensor of the form $S_{\alpha\beta}{}^\gamma = \varepsilon_{\alpha\beta\mu} S^\mu u^\gamma$, where S^μ is a spacelike vector field, is not compatible with the cosmological principle. Since the publication of [11], much of the literature considering an isotropic and homogeneous universe in the Einstein-Cartan theory has completely disregarded a torsion tensor of the previous form. However, we have to analyze the conditions under which it is valid the assertion that symmetries of the metric tensor are also symmetries of the torsion tensor. The pivotal condition is that the symmetries of the metric are also symmetries of the metric stress-energy tensor, however, in [11] it is clearly stated that this very strong condition is imposed ad hoc and, contrary to the theory of general relativity, does not follow from the field equations of the Einstein-Cartan theory. Nonetheless, it is defended that this is a reasonable assumption if the Einstein-Cartan theory is considered, in some sense, as a slight modification to general relativity. This, however, in general is not the case. As we can readily infer from the structure equations (78)–(97), the Einstein-Cartan theory, in general, is not a slight modification to general relativity. For instance, notice that the torsion tensor directly couples and acts as a source to the Weyl tensor. Of course, models with a vanishing Weyl tensor, as it is the case in general relativity, or a nonvanishing Weyl tensor, as generically presented here for the Einstein-Cartan theory we have been considering, represent very distinct physical setups. As shown above, the torsion tensor does not have to have the same symmetries of the metric tensor and the model just constructed is a consistent solution of the Einstein-Cartan theory in a cosmological context for a universe permeated by an isotropic and homogeneous matter fluid.

VI. GRAVITATIONAL WAVES IN RELATIVITSTIC COSMOLOGY IN EINSTEIN-CARTAN THEORY

A. Derivation of the gravitational wave equation for the isotropic universe

Comparing Eqs. (78)–(97) with those found in the theory of general relativity for a homogeneous and isotropic spacetime, see, e.g., [24, 25], we see that a glaring difference is that the Weyl tensor is, in general, not identically zero. This contrast between the two theories has profound implications in the evolution of the spacetime geometry and of the matter fluid. Indeed, in the previous section we have found that a nonvanishing Weyl tensor restricts the allowed geometry and topology of the orthogonal hypersurfaces, a restriction that does not exist in general relativity. In this section, we study further the Weyl tensor and its effects on the evolution of the spacetime curvature.

The Weyl tensor is known to be related with gravitational waves and tidal forces, which in fact are interconnected phenomena. In the model we are considering here, we have found that torsion and its derivatives act as a source for the Weyl tensor components, hence a natural step to understand the solutions of the structure equations is to study the presence of gravitational waves induced by the matter intrinsic spin. Due to the presence of torsion, if $K=-1$, the magnetic part of the Weyl tensor is nonvanishing. In this subsection, we will show that if the torsion tensor is caused by the matter fluid, the traceless part of $\bar{H}$, obeys a wave equation. These wave equations can be formally solved, explicitly showing that in a nonstatic universe the presence of intrinsic spin leads to the generation and emission of gravitational waves. From Eq. (86), H and $\bar{H}$ have the same traceless part, but distinct trace, namely $H_{\langle\alpha\beta\rangle}=\bar{H}_{\langle\alpha\beta\rangle}$ and $H_\alpha{}^\alpha=-2\bar{H}_\alpha{}^\alpha$. In fact, it is straightforward to show that H and $\bar{H}$ have the same eigenvectors, but associated with distinct eigenvalues. Then, in this section we will focus on studying $\bar{H}$ and all results are directly extended to H.

From the Ricci identity (11) and the field equations, we find, in the considered setup, the following expression for the projected derivative of the divergence of $\bar{H}$,

$$\begin{aligned}\left(D_\mu D_\alpha \bar{H}^\mu{}_\beta\right)-\left(D_\alpha D_\mu \bar{H}^\mu{}_\beta\right)=&\left(8\pi\mu+\Lambda-S_\sigma S^\sigma-\frac{1}{3}\theta^2\right)\bar{H}_{\langle\alpha\beta\rangle}\\&+\frac{1}{3}\theta\left(\bar{H}^\mu{}_\mu\varepsilon_{\alpha\beta}{}^\gamma-\bar{H}_{\mu\beta}\varepsilon_\alpha{}^{\mu\gamma}-\bar{H}^\mu{}_\alpha\varepsilon_{\mu\beta}{}^\gamma\right)S_\gamma\,.\end{aligned}\tag{112}$$

On the other hand, Eq. (88) implies

$$D_\mu\left(D^\alpha\bar{H}^{\mu\beta}\right)-D_\mu\left(D^\mu\bar{H}^{\alpha\beta}\right)+\frac{1}{3}\left(\bar{H}^{\alpha\beta}-h^{\alpha\beta}\bar{H}^\mu{}_\mu\right)\left(8\pi\mu+\Lambda-\frac{1}{3}\theta^2-S^\sigma S_\sigma\right)=0\,.\tag{113}$$

Taking the derivative of Eq. (87) we find

$$\frac{D^2}{d\tau^2}\bar{H}^{\alpha\beta}+\frac{4}{3}\bar{H}^{\alpha\beta}\left(\dot{\theta}-\frac{4}{3}\theta^2+3S_\gamma S^\gamma\right)-\frac{22}{3}\theta S_\gamma\varepsilon^{\mu\gamma(\alpha}\bar{H}^{\beta)}{}_\mu=2\bar{H}^\mu{}_\mu\left(h^{\alpha\beta}S^\delta S_\delta-S^\alpha S^\beta\right). \tag{114}$$

Gathering these results, yields

$$\left(\frac{D^2}{d\tau^2}-D_\mu D^\mu\right)\bar{H}_{\alpha\beta}+D_\alpha\left(D_\beta\bar{H}_\mu{}^\mu\right)+2\bar{H}_{\langle\alpha\beta\rangle}\left(8\pi\mu+\Lambda-\frac{1}{3}\theta^2+S^\sigma S_\sigma\right)+\frac{4}{3}\bar{H}_{\alpha\beta}\left(\dot{\theta}-\frac{4}{3}\theta^2\right)$$
$$+\frac{1}{3}\theta\left(\bar{H}^\mu{}_\mu\varepsilon_{\alpha\beta}{}^\nu-12\varepsilon^{\mu\nu}{}_\alpha\bar{H}_{\mu\beta}-10\varepsilon^{\mu\nu}{}_\beta\bar{H}_{\alpha\mu}\right)S_\nu+2\bar{H}^\mu{}_\mu S_{\langle\alpha}S_{\beta\rangle}=0\,. \tag{115}$$

Note that the operator $\frac{D^2}{d\tau^2}-D_\mu D^\mu$ is not the wave operator, since it is defined in terms of the total connection, nonetheless, it is equal to the wave operator plus terms in $\bar{H}$ and its first derivatives.

Now, the term $D_\alpha\left(D_\beta\bar{H}_\mu{}^\mu\right)$ in the left-hand side of the previous equation does not have to be zero. Since this term is a second order derivative of $\bar{H}$, in general the components of $\bar{H}$ are not solutions of a wave equation. Notwithstanding, it is physically reasonable to consider that the divergence of the spin density vector is a differentiable function of the energy density μ and the pressure p, that is $D_\alpha\Delta^\alpha=f(\mu,p)$: this expresses the idea that Δ^α has the matter fields as its source. In that case, Eqs. (90), (95) and (96) imply that $D_\beta\bar{H}_\mu{}^\mu=0$. Therefore, we have the following result:

Proposition 2. *In the considered setup, if $D_\alpha\Delta^\alpha=f(\mu,p)$, where f is an arbitrary differentiable function, the magnetic part of the Weyl tensor $\bar{H}$ verifies the following wave equation for the symmetric part without trace*

$$\tilde{\Box}\bar{H}_{\langle\alpha\beta\rangle}+2\bar{H}_{\langle\alpha\beta\rangle}\left(8\pi\mu+\Lambda-\frac{1}{3}\theta^2-S^\sigma S_\sigma\right)+\frac{4}{3}\bar{H}_{\langle\alpha\beta\rangle}\left(\dot{\theta}-\frac{4}{3}\theta^2\right)=0 \tag{116}$$

where $\tilde{\Box}\bar{H}_{\alpha\beta}:=\left(\frac{\tilde{D}^2}{d\tau^2}-\tilde{D}_\mu\tilde{D}^\mu\right)\bar{H}_{\alpha\beta}$, defined in terms of the Levi-Civita connection, represents the wave operator, and $\bar{H}$ verifies further the following evolution equation for the trace

$$\frac{D}{d\tau}\bar{H}_\alpha{}^\alpha+\frac{4}{3}\bar{H}_\alpha{}^\alpha\theta=0\,. \tag{117}$$

Thus, we have then found that the traceless part of $\bar{H}$ verifies a homogeneous wave equation and the trace of $\bar{H}$ verifies a first order ODE. Before we proceed to study the solutions of the previous set of two equations, we remark that the coefficient of the second term in the left-hand side of Eq. (116) is simply the Ricci scalar of the orthogonal hypersurfaces, Eq. (106).

B. The solutions

Theorem 1 establishes that the orthogonal hypersurfaces to u cannot have positive curvature and if these have zero Ricci curvature, $\bar{H}$ must be identically zero. Therefore, the only nontrivial solutions of Eqs. (116) and (117) that are of physical interest are those where $K=-1$. Notwithstanding, formally the treatment below is largely independent of the sign of K and we only have to specify the allowed values of K when we consider the initial conditions. Therefore, in an effort to be pedagogical about the covariant analysis of gravitational waves in a cosmological setting, we will keep the discussion as general as possible and only when studying the behavior of the solutions we will particularize to $K=-1$.

The second equation in Proposition 2, Eq. (117), is a first-order ODE and can be readily integrated in terms of the characteristic length ℓ. Using Eq. (104) we find $\bar{H}_\alpha{}^\alpha = \frac{C}{\ell^4}$, where $C \in \mathbb{R}$. Note that to find Eqs. (116) and (117) we have imposed that $D_\alpha \Delta^\alpha = f(\mu, p)$, which implies $D_\alpha \bar{H}_\mu{}^\mu = 0$. On the other hand, finding the solutions of the wave equation given in Eq. (116) is more involved. In that regard, we will assume that the spatial and proper-time, τ, dependence of $\bar{H}_{\langle\alpha\beta\rangle}$ are separable. Then, we will consider the eigenfunctions of the covariant Laplace-Beltrami operator $\tilde{D}_\mu \tilde{D}^\mu$ and expand $\bar{H}_{\langle\alpha\beta\rangle}$ over these eigenfunctions, such that

$$\bar{H}_{\langle\alpha\beta\rangle} = \sum_k \mathrm{h}_k^{(0)} Q_{\alpha\beta}^{(0),k} + \mathrm{h}_k^{(1)} Q_{\alpha\beta}^{(1),k} + \mathrm{h}_k^{(2)} Q_{\alpha\beta}^{(2),k}\,, \tag{118}$$

where we have used a compact notation to unify the two possibilities of k taking discrete or continuous values, such that the symbol $\sum_k$ is to be understood as either a discrete sum, if the hypersurfaces orthogonal to u have positive curvature, $K=1$, or as an integral over a continuously varying index, if these have zero or negative curvature; also the coefficients $\mathrm{h}_k^{(0)}$, $\mathrm{h}_k^{(1)}$ and $\mathrm{h}_k^{(2)}$ are in general functions of the proper time τ and $\dot{Q}_{\alpha\beta}^{(0),k} = \dot{Q}_{\alpha\beta}^{(1),k} = \dot{Q}_{\alpha\beta}^{(2),k} = 0$. Moreover, the minimum values of the eigenvalues k^2 are $k^2 = 0, 1, 3$ if, respectively, the orthogonal hypersurfaces are, for the natural topology, flat, open or closed. Nonetheless, bear in mind the since $\tilde{D}_\alpha Q^0 = 0$, even if $k^2 = 0$ is an eigenvalue of the Helmholtz equation, we have $Q_{\alpha\beta}^{(0),0} = 0$, see Appendix A. This type of decomposition is known as scalar-vector-tensor decomposition due to some properties of the harmonics $Q_{\alpha\beta}^{(0),k}$, $Q_{\alpha\beta}^{(1),k}$ and $Q_{\alpha\beta}^{(2),k}$, in particular we have that the curl of $Q_{\alpha\beta}^{(0),k}$, defined as $\varepsilon_{(\alpha|}{}^{\mu\nu} \tilde{D}_\nu Q_{\mu|\beta)}^{(0),k}$, is identically zero, $\tilde{D}^\beta \tilde{D}^\alpha Q_{\alpha\beta}^{(1),k} = 0$ and $\tilde{D}^\alpha Q_{\alpha\beta}^{(2),k} = 0$. For completeness, we list various properties of the scalar, vector and tensor harmonics in the Appendix A.

From Eq. (88), we have that $\operatorname{curl} \bar{H}_{\alpha\beta} \equiv \varepsilon_{(\alpha|}{}^{\mu\nu} D_\nu \bar{H}_{\mu|\beta)}$ vanishes. Hence, $\bar{H}$ can be described solely by the scalar harmonics $Q_{\alpha\beta}^{(0),k}$. Then, substituting the expansion $\bar{H}_{\langle\alpha\beta\rangle} = \sum_k \mathrm{h}_k^{(0)} Q_{\alpha\beta}^{(0),k}$ in the wave equation given in Eq. (116), the harmonics decouple and we find for each k the equation

$$\ddot{\mathrm{h}}_k^{(0)} + \mathrm{h}_k^{(0)} \left[\frac{k^2}{\ell^2} + \frac{4}{3} \left(\dot{\theta} - \frac{4}{3}\theta^2 \right) \right] = 0\,. \tag{119}$$

Hence, the expansion coefficients verify an equation for an harmonic oscillator with variable frequency, leading us to conclude that, in an dynamic universe, the presence of intrinsic spin may induce the emission of gravitational waves. Introducing the Hubble parameter $\mathrm{H} \equiv \frac{1}{3}\theta$, the conformal time variable t, defined such that $dt = \ell^{-1} d\tau$, and writing $\mathrm{h}_k^{(0)} = \frac{f_k(t)}{\ell^4}$, we find that these are the solutions of Eq. (119) if each $f_k\,(t)$ verifies

$$\frac{d^2 f_k}{dt^2} - 9\ell \mathrm{H} \frac{df_k}{dt} + k^2 f_k = 0\,. \tag{120}$$

Equation (120) takes a surprisingly simple form and all dependencies of the matter model are encapsulated in the quantity $\ell\mathrm{H}$, defined as the inverse comoving Hubble radius R_{H}, i.e., $R_{\mathrm{H}} \equiv (\ell\mathrm{H})^{-1}$. Now, to integrate Eq. (120) one has either to assume a model for the matter fluid or to resort to solutions valid within certain regimes. We stick to the second alternative. For this, note that in the light of Proposition 1, a consistent solution must be such that the functions f_k are bounded. Then, we can analyze the cases for which $\frac{k^2}{\ell\mathrm{H}} \gg 1$ and $\frac{k^2}{\ell\mathrm{H}} \ll 1$.

In the regime where $\frac{k^2}{\ell\mathrm{H}} \gg 1$ and such that the term in $\frac{df_k}{dt}$ is negligible, with no need for specifying the matter fields that permeate the spacetime, and further assuming f_k and its derivatives up to second order are bounded, the solutions of Eq. (119) for the higher order modes are of the form

$$\mathrm{h}_k^{(0)} = \frac{c_1 \cos(kt) + c_2 \sin(kt)}{\ell^4}\,, \tag{121}$$

where the integration constants c_1 and c_2 might change for each $h_k^{(0)}$. This result makes it clear that in the considered setup, a nonvanishing $\bar{H}$ characterizes gravitational waves induced by intrinsic spin. Moreover, we see that in an expanding universe these waves are strongly damped, as it was found in Proposition 1.

In the regime where $\frac{k^2}{\ell\mathrm{H}} \ll 1$, and also with no need for specifying the matter fields that permeate the spacetime, in the light of Theorem 1, the only nontrivial solutions for Eq. (119) that are of physical interest in the considered model are those where $K = -1$. In that case, the expansion coefficient k takes continuous values and $k \geq 1$. Now, the comoving Hubble radius verifies $\dot{R}_{\mathrm{H}} = -\ddot{\ell} R_{\mathrm{H}}^2$. Then, in an accelerating expanding universe, R_{H} is a decreasing function of the proper time. Therefore, for $K = -1$, the regime where $\frac{k^2}{\ell\mathrm{H}} \ll 1$ represents the late-time behavior of the lower order modes of the spin induced gravitational waves in an accelerating expanding universe. In this regime, assuming we can neglect the term in f_k in Eq. (120) and disregarding runaway solutions, we find that

$$\mathrm{h}_k^{(0)} = \frac{\text{constant}}{\ell^4}\,, \tag{122}$$

where the integration constant might change for each k, confirming once again that Proposition 2 is consistent with the results found in subsection V B.

VII. TIDAL EFFECTS AND DYNAMICS OF THE COSMIC FLUID IN RELATIVISTIC COSMOLOGY IN EINSTEIN-CARTAN THEORY

A. Tidal effects

In addition to the magnetic part of the Weyl tensor, the electric part of the Weyl tensor is also not identically zero in the presence of torsion. Therefore, tidal effects, i.e., the relative accelerations of nearby particles, suffer modifications when compared to the theory of general relativity.

The general formula for the tidal displacement in the presence of torsion is

$$\frac{D^2 n^\delta}{d\tau^2} = n^\mu \nabla_\mu a^\delta + R_{\alpha\beta\gamma}{}^\delta n^\alpha u^\beta u^\gamma + 2u^\sigma \nabla_\sigma \left(S_{\alpha\beta}{}^\delta u^\alpha n^\beta \right)\,, \tag{123}$$

where n represents the separation vector introduced in subsection II C. In the setup we are considering, Eq. (123) reduces to $\frac{D^2 n^\delta}{d\tau^2} = R_{\sigma\mu\nu}{}^\delta n^\sigma u^\mu u^\nu$, which is the familiar formula for geodesic deviation. Assuming without loss of generality that the separation vector is initially orthogonal to the tangent vector field u, i.e., $n^\mu u_\mu = 0$, this can further be manipulated to have the form

$$\frac{D^2 n^\delta}{d\tau^2} = n^\mu \left(\frac{1}{2} R_\mu{}^\delta + \frac{1}{2} R_{\mu\nu} u^\nu u^\delta - E_\mu{}^\delta \right) - \frac{1}{2} \left(R_{\mu\nu} u^\mu u^\nu + \frac{1}{3} R \right) n^\delta\,. \tag{124}$$

We see that Eq. (124) explicitly shows the influence of the electric part of the Weyl tensor in the tidal displacement. Using Eqs. (18), (57), (83) and (84), we find the following expression for the relative acceleration between two infinitesimally close test particles in the considered model

$$\frac{D^2 n^\delta}{d\tau^2} = n^\alpha \left(\frac{1}{3} \varepsilon_{\alpha\gamma}{}^\delta \theta \mathrm{S}^\gamma + \mathrm{S}_\alpha \mathrm{S}^\delta \right) + \frac{1}{3} \Big(\Lambda - \mathrm{S}^\sigma \mathrm{S}_\sigma - 4\pi \left(\mu + 3p \right) \Big) n^\delta\,. \tag{125}$$

From Eq. (125), we see that the presence of intrinsic spin induced torsion causes a distortion of the fluid as measured by an observer comoving with the fluid that couples directly with torsion. To interpret this result, it is clearer to consider Eq. (80), that is, the presence of torsion induces a rotation of the frame of the observer. This is a well known effect of the torsion tensor, which in fact lead to its name: a test particle, or an element of volume of the fluid, that couples directly with the torsion field, in general, will have its frame rotated. Then, the distortion of the fluid described by Eq. (125) is caused by the relative acceleration of the rotation of the fiducial observer's frame due to the presence of torsion. This rotation of the frames is due to the spacetime geometry, however it is not intrinsic to the motion of the fluid, i.e., the fluid is irrotational since elements of volume of the fluid follow metric geodesics of the spacetime, whose metric is described by a FLRW solution. Thus, observers, that do not couple directly with the torsion tensor, will not measure any relative rotation between different points in the fluid. To see this, consider an observer that does not couple directly with the torsion tensor and only perceives the effects of the intrinsic spin of matter through the metric tensor, such that its world line is a metric geodesic of the spacetime and its 4-velocity coincides with u, the tangent vector of the congruence. For this type of observer, the geodesic deviation equation, in the considered setup, reads $\frac{\tilde{D}^2 n^\delta}{d\tau^2} = \frac{1}{3}\Big(\Lambda + 2\mathrm{S}^\sigma \mathrm{S}_\sigma - 4\pi\left(\mu + 3p\right)\Big) n^\delta$, where $\frac{\tilde{D}^2 n^\delta}{d\tau^2} \equiv u^\mu \tilde{\nabla}_\mu \left(u^\nu \tilde{\nabla}_\nu n^\delta\right)$ and $\tilde{\nabla}$ represents the Levi-Civita connection. We see, then, that such observer does not measure any relative change in the rotation between nearby elements of volume of the fluid. This type of observer will only measure a relative acceleration of the distance between infinitesimally close test particles. These results exactly express the discussion in [16] where it was determined in the context of any metric-affine gravity theory, particles with no intrinsic hypermomentum will not directly experience the effects of torsion.

In addition, relative acceleration of the squared distance between infinitesimally close test particles is a physical observable, hence both type of observers, namely, those that couple directly to torsion and those that do not, will agree on its magnitude. Using the Raychaudhuri equation (78) and the generalized Friedman equation (106) in Eq. (125) or, equivalently in Eq. (101), yields

$$\frac{D^2\left(n_\delta n^\delta\right)}{d\tau^2} = \frac{\tilde{D}^2\left(n_\delta n^\delta\right)}{d\tau^2} = \frac{1}{3}\Big(8\pi\left(\mu - 3p\right) + 4\Lambda + 2\mathrm{S}^\sigma \mathrm{S}_\sigma - {}^3R\Big) n_\delta n^\delta \,, \tag{126}$$

confirming that observers that couple directly with torsion and observers that do not, will measure the same relative acceleration of the distance between nearby elements of volume of the fluid. Although it can also be inferred from the Raychaudhuri equation (78), it is explicit in Eq. (126) that the square of the norm of the torsion vector field, $\mathrm{S}^\sigma \mathrm{S}_\sigma$, has the same sign of a positive cosmological constant, therefore, the torsion field also contributes to the positive relative acceleration of the distance between infinitesimally close test particles, an effect that is measurable by both type of observers.

B. Dynamics of the cosmic fluid

In the previous subsection, we have found that the torsion vector field may contribute to a positive cosmological constant. We are then interested in understanding if it is possible to have solutions with zero cosmological constant, $\Lambda = 0$, such that the relative accelerated expansion measured in our universe is completely fueled by the torsion field.

To analyze this problem, we introduce the following dimensionless parameters

$$q \equiv -1 - \frac{\dot{\theta}}{3\mathrm{H}^2}\,, \quad \Omega_{\mathrm{K}} \equiv -\frac{{}^3R}{6\mathrm{H}^2}\,, \quad \Omega \equiv \frac{8\pi\mu}{3\mathrm{H}^2}\,, \quad \Omega_\Lambda \equiv \frac{\Lambda}{3\mathrm{H}^2}\,, \quad \Omega_{\mathrm{S}} \equiv \frac{\mathrm{S}^\sigma\mathrm{S}_\sigma}{3\mathrm{H}^2}\,, \tag{127}$$

where q is the acceleration parameter, H is the Hubble parameter defined in subsection VI, Ω, Ω_Λ and Ω_{S} represent, the matter fields, dark energy, and intrinsic spin dimensionless densities, respectively, where we note that the density parameter Ω accounts for the contribution to the energy density of all matter fields, be it baryons, photons, dark matter or neutrinos, and note also that $\Omega_{\mathrm{S}} \geq 0$. We further define the effective equation of state parameter χ as

$$\chi\,(\tau) \equiv \frac{p}{\mu}\,. \tag{128}$$

Then, we can rewrite the Raychaudhuri and the Friedman equations given in Eqs. (78) and (106) as $\frac{3}{2}\left(\chi + \frac{1}{3}\right)\Omega - \Omega_\Lambda - 2\Omega_{\mathrm{S}} = q$, and $\Omega_{\mathrm{K}} + \Omega + \Omega_\Lambda - \Omega_{\mathrm{S}} = 1$, respectively. The measured empirical results indicate that the universe is very close to being Ricci flat, hence one can put here $\Omega_{\mathrm{K}} = 0$. So, the Raychaudhuri and the Friedman equations turn into

$$\frac{3}{2}\left(\chi + \frac{1}{3}\right)\Omega - \Omega_\Lambda - 2\Omega_{\mathrm{S}} = q\,. \tag{129}$$

$$\Omega + \Omega_\Lambda - \Omega_{\mathrm{S}} = 1\,, \tag{130}$$

respectively. We see that in the Raychaudhuri equation (129), Ω_{S} has the same sign of the cosmological constant term Ω_Λ, and thus contributes to the acceleration of the expanding universe, as it could be expected, since the spin can be thought of as a source of centrifugation for the universe. On the other hand, in the Friedman equation (130), Ω_{S} has a minus sign relative to the cosmological constant term Ω_Λ, as it is also expected, since the spin can be thought of as a kinetic term and thus contributes to the balance of the kinetic energy of the universe, which is represented in the 1 of the right-hand side of the equation. Let us see more clearly the effect of Ω_{S} by turning off Λ. Setting $\Omega_\Lambda = 0$ in Eqs. (129) and (130) we find $\frac{3}{2}\left(\chi + \frac{1}{3}\right)\Omega - 2\Omega_{\mathrm{S}} = q$ and $\Omega - \Omega_{\mathrm{S}} = 1$, respectively. In this case, the Friedman equation with $\Omega_\Lambda = 0$ and $\Omega_{\mathrm{S}} \geq 0$ necessarily implies that $\Omega \geq 1$. On the other hand, the matter density in our universe is known to be less than one, $\Omega < 1$. Therefore solutions with $\Omega_\Lambda = 0$ are excluded. This fact, namely, that solutions of the considered model where $\Omega_\Lambda = 0$ are excluded, of course, does not mean that the model is excluded. This simply implies that if a torsion field of the considered type exists, $\Omega_{\rm S}$ cannot solely contribute to the expansion of the universe described by the Raychaudhuri equation, given in Eq. (129), and to the energy balance of the Friedman equation given in Eq. (130), Ω_Λ has also to contribute. How much is this $\Omega_{\rm S}$ contribution depends on the matter model. Following [12, 14] if we consider that the cosmological spinning fluid is a medium whose elements are galaxies and galaxy clusters, the torsion tensor would be caused by their macroscopic angular momenta. In that case, the spin density $\Omega_{\rm S}$ could provide a nonnegligible contribution to the accelerated expansion of the universe. On the other hand, if the only source of the torsion tensor is the intrinsic spin of elementary particles, where one finds that the value of $\Omega_{\rm S}$ is much smaller than Ω, see [8, 9], its contribution to the accelerated expansion of the universe would be negligible.

To conclude the analysis, we remark that independently of the actual value of the dimensionless spin density $\Omega_{\rm S}$, if $\Omega_{\rm S}$ is nonzero, i.e., $\Omega_{\rm S} > 0$, the results in Theorems 1 and 2 are still verified. So, even though the contribution of the torsion tensor to the accelerated expansion of the universe may be negligible, it still markedly changes the geometry and the allowed topology of the spacetime.

VIII. CONCLUSIONS

We presented the general set of structure equations for the 1+3 spacetime decomposition in 4 spacetime dimensions, valid for any theory of gravitation based on a metric compatible affine connection, showing in complete generality the relations between the kinematical quantities of the timelike congruence, the torsion tensor and the Weyl and the Ricci tensors.

The new equations were then used to study solutions of the Einstein-Cartan theory with a cosmological perfect fluid having an intrinsic spin, such that the geometry of the spacetime is described by both the metric and the torsion tensor field. The model showed that even in the presence of a torsion field originated by the intrinsic spin of matter, the metric tensor can be described by a general, spatially isotropic and homogeneous, FLRW solution. Here we would like to highlight that although we have assumed that the torsion tensor has the intrinsic spin of the fluid's constituents as its source, this does not imply that the fluid's elements of volume must have a nonzero intrinsic spin density. As was shown in [9], even if in an element of volume containing many particles with the intrinsic spins of the individual particles are randomly oriented, such that the average spin density is zero, the variance is not zero, hence the torsion tensor is not zero. This, in our view, is the correct approach to the Einstein-Cartan theory, where the fluid is described by a semi-classical model, whose elements of volume contain many particles. Of course, it could also be the case that the individual spins are aligned, and the average intrinsic spin density is not zero. In either case, in the considered model the torsion tensor is not zero. Although the metric tensor was found to be described by a FLRW model, it was shown that the Weyl tensor might not vanish, which leads to very strong constrains on the allowed geometry and topology of the spacetime. Indeed, due to the coupling between the torsion and Weyl tensors, in the considered model, the universe must either be flat or open.

In the open case, we then derived a wave equation for the traceless part of the magnetic part of the Weyl tensor, concluding that the presence of intrinsic spin of matter may induce gravitational waves, providing, to our knowledge, the first explicit result showing that the torsion field may source or influence the emission of gravitational waves in a cosmological setting. Although these waves are strongly damped in an expanding universe, this result may provide a smoking gun for the presence of spacetime torsion.

In the considered model, it was also possible to determine that a torsion tensor field originated from intrinsic spin contributes to the positive accelerated expansion of the universe, nonetheless, comparing the theoretical predictions of the model with the current experimental data, the torsion tensor cannot completely replace the role of the cosmological constant.

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Appendix A: Properties of the Laplace-Beltrami harmonics

1. Scalar harmonics

In this appendix we list some of the properties of the scalar, vector and tensor eigenfunctions of the covariantly defined Laplace-Beltrami operator on 3-hypersurfaces of constant curvature used to define the so called scalar-vector-tensor decomposition. For concreteness, we consider a spacetime endowed with a FLRW metric, such that the homogeneous spatial sections represent such 3-hypersurfaces.

Let Q^k represent the scalar eigenfunctions of the covariantly defined Laplace-Beltrami operator $\tilde{D}^2 \equiv \tilde{D}_\alpha \tilde{D}^\alpha$, where $\tilde{D}$ represents the covariant derivative operator associated with the Levi-Civita connection, such that

$$\tilde{D}^2 Q^k = -\frac{k^2}{\ell^2} Q^k \,, \tag{A1}$$

and $\dot{Q}^k = 0$, where ℓ represents the scale factor defined in Eq. (104) and the harmonic index k may take discrete or continuous values depending on whether $K = +1$, or $K \in \{-1, 0\}$, respectively, where K was introduced in Eq. (107). Then, we can define the following tensors from the scalar eigenfunctions Q^k,

$$\begin{aligned} Q^{(0),k}{}_{\alpha} &= -\frac{\ell}{k} \tilde{D}_\alpha Q^k \,, \\ Q^{(0),k}{}_{\alpha\beta} &= \frac{\ell^2}{k^2} \tilde{D}_\beta \tilde{D}_\alpha Q^k + \frac{1}{3} h_{\alpha\beta} Q^k \,, \end{aligned} \tag{A2}$$

with the following properties

$$\begin{aligned} \dot{Q}^{(0),k}{}_{\alpha} &= 0 \,, \\ \tilde{D}^\mu Q^{(0),k}{}_{\mu} &= \frac{k}{\ell} Q^k \,, \\ \tilde{D}^2 Q^{(0),k}{}_{\alpha} &= \frac{2K - k^2}{\ell^2} Q^{(0),k}{}_{\alpha} \,, \\ \tilde{D}_{[\alpha} \tilde{D}_{\beta]} Q^{(0),k}{}_{\gamma} &= \frac{K}{2\ell^2} \left(h_{\alpha\gamma} Q^{(0),k}{}_{\beta} - h_{\beta\gamma} Q^{(0),k}{}_{\alpha} \right) \,, \\ Q^{(0),k}{}_{\mu}{}^{\mu} &= 0 \,, \\ \dot{Q}^{(0),k}{}_{\alpha\beta} &= 0 \,, \\ \varepsilon_{(\alpha|}{}^{\mu\nu} D_\nu Q^{(0),k}{}_{\mu|\beta)} &= 0 \,, \\ \tilde{D}^\mu Q^{(0),k}{}_{\alpha\mu} &= \frac{2}{3\ell k} \left(k^2 - 3K \right) Q^{(0),k}{}_{\alpha} \,, \\ \tilde{D}^2 Q^{(0),k}{}_{\alpha\beta} &= \frac{6K - k^2}{\ell^2} Q^{(0),k}{}_{\alpha\beta} \,, \end{aligned} \tag{A3}$$

where in the previous expressions the k^{-1} factor is not a problem because $\tilde{D}_\alpha Q^0 = 0$.

2. Vector harmonics

Given a sufficiently smooth 1-form field Y_α in a FLRW spacetime, we can in general decompose it as

$$Y_\alpha = \sum_k \mathrm{T}^{(0)}_k Q^{(0),k}{}_{\alpha} + \mathrm{T}^{(1)}_k Q^{(1),k}_{\alpha} \,, \tag{A4}$$

where the vectors $Q^{(0),k}_{\alpha}$ are obtained from the scalar eigenfunctions Q^k, Eq. (A2), and $Q^{(1),k}{}_{\alpha}$ represent the solutions of the vector Helmholtz equation

$$\tilde{D}^2 Q^{(1),k}{}_{\alpha} = -\frac{k^2}{\ell^2} Q^{(1),k}{}_{\alpha} \,, \tag{A5}$$

with the following properties

$$\begin{aligned} \dot{Q}^{(1),k}{}_{\alpha} &= 0 \,, \\ \tilde{D}^{\mu} Q^{(1),k}{}_{\mu} &= 0 \,. \end{aligned} \tag{A6}$$

Similarly to the scalar eigeinfunctions Q^k, we can find a set of 2-tensors associated with $Q^{(1),k}{}_{\alpha}$:

$$Q^{(1),k}{}_{\alpha\beta} = -\frac{\ell}{2k}\left(\tilde{D}_{\alpha} Q^{(1),k}{}_{\beta} + \tilde{D}_{\beta} Q^{(1),k}{}_{\alpha}\right) \,, \tag{A7}$$

with the following properties

$$\begin{aligned} Q^{(1),k}{}_{\mu}{}^{\mu} &= 0 \,, \\ \dot{Q}^{(1),k}{}_{\alpha\beta} &= 0 \,, \\ \tilde{D}^{\mu} Q^{(1),k}{}_{\alpha\mu} &= \frac{k^2 - 2K}{2\ell k} Q^{(1),k}{}_{\alpha} \,, \\ \tilde{D}^2 Q^{(1),k}{}_{\alpha\beta} &= \frac{4K - k^2}{\ell^2} Q^{(1),k}{}_{\alpha\beta} \,. \end{aligned} \tag{A8}$$

3. Tensor harmonics

Given a general smooth 2-tensor field $Y_{\alpha\beta}$ in a FLRW spacetime, we can in general decompose it as

$$Y_{\alpha\beta} = \sum_k \mathrm{T}^{(0)}_k Q^{(0),k}{}_{\alpha\beta} + \mathrm{T}^{(1)}_k Q^{(1),k}{}_{\alpha\beta} + \mathrm{T}^{(2)}_k Q^{(2),k}{}_{\alpha\beta} \,, \tag{A9}$$

where the 2-tensors $Q^{(2),k}{}_{\alpha\beta}$ are defined as the solutions of the tensor Helmholtz equation

$$\tilde{D}^2 Q^{(2),k}{}_{\alpha\beta} = -\frac{k^2}{\ell^2} Q^{(2),k}{}_{\alpha\beta} \,. \tag{A10}$$

These verify

$$\begin{aligned} Q^{(2),k}{}_{\mu}{}^{\mu} &= 0 \,, \\ \dot{Q}^{(2),k}{}_{\alpha\beta} &= 0 \,, \\ \tilde{D}^{\mu} Q^{(2),k}{}_{\alpha\mu} &= 0 \,. \end{aligned} \tag{A11}$$

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