Resilience and complex dynamics — safeguarding local stability against global instability

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Abstract

We evaluate Brunnermeir's Theory of Resilience in the context of complex system dynamics where there, however, can be local and global resilience, vulnerability, loss of resilience, cycles, disruptive contractions, and persistent traps. In the paper, we refer to three-time scales. First, for shorter time scales, for the short-run market dynamics, we evaluate resilience in the context of complex market dynamics that have been studied in the history of economic theory for long. Second, with respect to a business cycle medium-term dynamics, we analytically study an endogenous cycle model, built upon Semmler and Sieveking (1993) and Semmler and Koçkesen (2017), and discuss the issue of loss of stability, corridor stability, multiple attractors, and trapping dynamics also in the light of complex dynamics. In a financial-real business cycle model, we demonstrate forces that indeed can exhibit multiple dynamic features such as local resilience, known as corridor-stability, but also other dynamic phenomena. Corridor stability pertains to small shocks with no lasting effects, but large enough shocks can lead to persistent cycles and/or contractions. We refer to the Hopf-and-Bautin-Bifurcation theorems, to establish corridor stability, and local resilience, for the interaction of real and financial variables where the trajectories can be stable or unstable in the vicinity of the equilibrium. Thus they can switch dynamic behaviour for small or large shocks.

Similar complex dynamic phenomena can be obtained from Kaleckian-Kaldorian nonlinear real business cycle models, in particular when time delays are allowed for. Third, whereas the analytical study of the dynamics is undertaken for the above second-time scale, for the longer time scale we study, in the context of multiple equilibria models, the issue of thresholds, tipping points and disruptive contractions, and persistence of traps.

Keywords: Resilience, complex dynamic models, regime change model, limit cycles, disruptive contractions
JEL: C32, E32, E44

1. Introduction

Brunnermeier's recent book "The Resilient Society" has initiated an extensive discussion of how to deal with economic, social, and political shocks and what preemptive policies should be pursued. Conventionally it is presumed that after some shocks – mostly assumed exogenous shocks – the economy rebounds quickly compared to endogenous shocks, characterized by the build-up of imbalances and delayed re-adjustment. Brunnermeier (2021) theory of resilience helps explain well some aspects of social, economic, and political dynamics. His paradigm of resilience also brings in a fresh perspective on the great meltdowns such as the financial crisis, COVID-19 pandemic, and climate disasters but also on specific macro issues, such as the dynamics of innovation, inequality, currency, inflation-deflation risks, debt dynamics, health care system, fiscal and monetary space, and the global economy. Though his new paradigm is a very refreshing one, it is challenging to evaluate his theory from the perspective of complex system dynamics.

Recent work on shocks in complex economic dynamics has already challenged traditional wisdom on economic dynamics. Generically, those shocks can be absorbed by factors increasing resilience or producing dynamic processes with multiple stability regions. Research has utilized complex system models to study the mechanisms that explain how stability and mean reversion dynamics can be achieved or not. As known by now, for example, weakening economic conditions are likely to generate adverse feedback loops in the financial sector. The application of complex system models can reveal quite adverse effects of shocks, the existence of tipping points, and multiple features in the dynamics. Economic research has, in particular, recently set out to explore the dynamic paths of variables in various fields of economics, from finance to climate change research. Time scales also play an important role in the evolution of the effects of shocks. Here, related to economic shocks, we examine dynamic economic paths for shorter and longer time scales.

First, we discuss shorter time scales in economic dynamics that have been explored in studies of how markets behave, explored since classical economics. Those studies usually refer to the issue of whether the demand and supply in markets will converge – a major theme of Adam Smith's theory of the "invisible hand". This has been explored under the topic of the dynamics of excess demand functions where excess demand is supposed to converge to zero responding disequilibrium in prices and quantities: If prices for products are too high due to a shortage of goods, there will be gains by the producers of those supplies, and this gives rise to more supply and eventually decreases prices since more products will be offered. This has been called excess demand theory (see Hahn (1982)). Such a classical mechanism has been extended by Flaschel et al. (1997) to a dual process where prices respond to imbalances of supply and demand and supply changes due to differences in profitability across sectors. Here then, since 1930, the issue has come up whether small shocks to supply and demand may indeed be self-stabilizing through a kind of local resilience or corridor stability (see Fisher (1933)). Still, large shocks may lead to resilience breakers and generate instability.

Second, in the literature on the business cycle time scale, roughly considered to be 7 to 10 years, we first propose a nonlinear model of the financial-real interaction that replicates many of the above-mentioned complex features. Local resilience has been featured here as corridor stability, on which there is already much economic literature. This pertains to small shocks with no lasting effects. However, large enough shocks can lead to different phenomena, for example, persistent cycles (limit cycles) but also persistent contractions, disruptions, and traps. In this context, one can refer to the Hopf-bifurcation (locally unstable) and Bautin-bifurcation (locally stable) theorem to characterize local and global resilience (stability) for the interaction of real and financial variables over the business cycle where the trajectories can be stable or unstable in the vicinity of the equilibrium. Corridor stability (local stability) pertains to small shocks with no lasting effects, but large enough shocks can lead to persistent cycles, contractions, or persistent traps. Thus, small shocks do not matter (keep resilience), i.e., they are mean reverting, but large shocks do matter. On the other hand, there are business cycle models considering less the financial-real interactions and more the role of nonlinearities in conjunction with time delays on the real side of the economy. We can call them real business cycles. Complex dynamic phenomena can also be obtained from the Kaleckian-Kaldorian nonlinear real business cycle model (when particular time delays are allowed for) or, for example, from Harrod's knife-edge instability principle[1].

Third, whereas the above dynamics usually assume that the relevant equilibria

[1]On business cycles an control, see Orlando and Sportelli (2021a;b), Orlando (2021), Stoop (2021).

are unique, there can be multiple equilibria. The system dynamics can move to any of those on a longer time scale after passing some tipping points and thresholds. We provide examples of loss of resilience, disruptive contractions, convergence to different attractors caused by thresholds, triggering different dynamics, and generating economic traps with considerable lock-ins. Large financial crises, such as the one in 2007/9 and the subsequent meltdowns, the spread of infectious diseases such as the COVID-19 outbreak, climate extreme events and disasters, and wars, can trigger such complex system dynamics with more or less persistent traps.

We want to note, however, that in the current paper, we do not study and evaluate empirical and econometric work that supports the dynamics on shorter, medium- and long-run time scales. We review and model continuous time approaches. There have been many empirical approaches to verify and confirm some of the complex dynamic processes discussed above, using data working with discrete time econometric methodology. This is called nonlinear econometrics. We only occasionally will refer to such work.[2]

The remainder of the paper is organized as follows. Section 2 refers to the studies of the shorter-run market dynamics, elaborating on local and global resilience (corridor or global stability). Section 3 presents an analysis and numerical results on the relation of resilience and complex dynamics in a business cycle model of business cycle medium run-time scale building on financial-real interactions. Section 4 stays with the medium run, but focuses only on the real side, yet studies the role of time delays. Section 5 moves to the theory and works on the longer time scale, referring to multiple equilibria, thresholds, tipping points, disruptions, and trapping regions. Finally, Section 6 concludes the paper. The appendix provides some technical derivations.

[2] In the empirical part, we study some features of Brunnermeier (2021) theory of resilience by using some work of Semmler and Koçkesen (2017) and undertake some nonlinear econometric study. We can show the existence of cyclical solution paths and demonstrate that the empirics of financial-real forces indeed can exhibit multiple features such as endogenous resilience (robustness against shocks) but of a globally attracting type. The essential nonlinearities and regime changes can generate a locally unstable equilibrium but global resilience (limit cycles), as reflected by the Hopf-bifurcation type dynamics. The econometrically estimated model shows local non-resilience but global resilience. For the locally unstable but globally bounded fluctuations, we can also detect asymmetric responses to shocks.

2. Shorter - run: Market dynamics and resilience

The concept of corridor stability was proposed initially by Leijonhufvud (1973) to represent the response of a market economy to an adverse income shock. Leijonhufvud suggested that the system could convert to the original equilibrium position or be taken out of the stability region and diverge, depending on the shock size. The concept that a system behaves differently given shocks of diverse intensities shows validity to theories of systemic fragility, including Minsky's theory of financial fragility.

Corridor stability measures how resilient an economy is to external shocks or disturbances, by which resilience implies local resilience (corridor stability) or global resilience – robustness against small or large shocks. Correspondingly, mean reversion refers to the return to their long-term averages or trends over time. This concept is based on the idea that extreme values or fluctuations in these variables are usually temporary and tend to balance over time as markets and economies adjust to new conditions. Overall, it refers to the natural tendency in economies and markets to maintain balance and stability over time.

However, corridor stability implies only a turn to equilibrium near the stable equilibria or stable steady states. If there are multiple equilibria, a more significant shock or disturbance can cause the economy to move from one stable state to another rather than return to its original state. Nonetheless, corridor stability may meet some tipping points and represent rapid deterioration, moving into traps and regime changes.

Brunnermeier's theory elucidates that the lack of mean reverting dynamics can destroy the ability to bounce back due to resiliency destroyers. These are often seen as externalities that break society's collective contract. In a given society, social contracts help address externalities and protect against shocks. These destabilizing loops or points of no return have potentially devastating effects on macro dynamics. However, if resiliency is present, it can help take more risks, and the ability to rebound allows for more space to grow.

Economics literature presents local and global resilience using the concept of corridor stability. The economic theory of local and global stability deals with the stability of microeconomic (market) dynamics and macroeconomic dynamics, where local stability refers to the stability of equilibrium in a small neighbourhood around the equilibrium point. In contrast, global stability refers to the stability of equilibrium for any initial condition of the economy. For an equilibrium to be locally stable, adverse economic conditions or policy changes should not cause significant economic fluctuations. Conversely, an equilibrium is globally stable if it is a basin of attraction for all other feasible economic outcomes, regardless of the initial conditions or disturbances.

Historically, classical writings on market mechanisms have already described views on market mechanisms as mentioned in Section 1. The classical market mechanism is often described as driven by market forces leading to local and global resilience (convergence) through the adjustment of excess demand functions, see Hahn (1982). This view was corrected later: Prices adjust due to quantity imbalances (excess demand), and supply adjusts through profitability differences. The classical mechanism has been extended by Flaschel et al. (1997), which is then described as a dual process where prices respond to imbalances of supply and demand and supply changes due to differences in profitability across sectors. Here then, it can already be shown that there might be convergence but also persistent cycles in prices and quantities in market economies. Since 1930, the issue has arisen whether small shocks to supply and demand may be self-stabilized through a kind of local resilience or corridor stability. Still, large shocks may lead to resilience breakers and generate instability.[3]

Those thinking on the market mechanism also impacted the research on macroeconomic adjustment mechanisms. For example, macro dynamics often present macro mechanisms leading to convergence (stable equilibrium) (see Pigou (1933; 1941)). An example is, in fact, Pigou's real balance effects that are stabilizing or globally mean reverting.[4] The study of economic dynamics of the macroeconomy was also changed when Fisher (1933), after the depression of 1929, referred to the debt-deflation process as severe enough to "rock the boat" and start its capsizing. A lasting major depression followed Fisher's analysis of the stock market crash of 1929 and during the subsequent deflation period. Later, along these lines, Keynes (1937) chapter 19 and Tobin (1975; 1980) specified the concept of corridor stability in the case when a market economy absorbs small shocks and self-adjusts enough to leave it within a corridor. Still, additional demand is however needed to correct the shocks. In disequilibrium, market economies need that additional demand when exposed not to small shocks, but to large ones.

Corridor stability and an economy's absorption of shocks on the micro and macro levels can now be found in much economic literature. For example, Dimand (2005) discussed Tobin (1975) approach, drawing an understanding from Keynes and Fisher that included corridor stability. Furthermore, Tobin (1969; 1975) notes liquidity buffer stocks as a device against funding constraints that absorbs small shocks. Nevertheless, Dimand (2005) also based his approach on Keynes (1937), helping portray

[3] See Fisher (1933) illustration where he uses the metaphor of a capsizing boot when extensively rocked.

[4] See also Keynes (1937), chapter 19.

his work on how the experience of the disruptive effect of rapid deflation on the system of financial intermediation led to the later analysis of financial instability by Minsky (1975). Minsky noted that a prolonged period of stability, as a period of tranquillity with larger risk-taking, would induce vulnerabilities and a non-sustainable process and eventually produce a fragile system, leading to disruptions, see Semmler (1987). Dimand (2005) scrutinized how Keynes's General Theory excluded extreme instabilities when small shocks occur.

In brief, the same system can exhibit different properties than those that characterized it before a shock, given how it adapts to adverse shocks and the growth dynamics leading to corridor stability (see also Semmler and Sieveking (1993)). Finally, as mentioned in Dimand (2005), Galloway et al. (1933), and Blakey (1939) believed deflation was an impeding factor to recovery. Some of those aspects of many of these ideas at the micro and macro levels will be further discussed next.

3. Medium-run: Business cycles, resilience, and complex dynamics

Other recently studied dynamics that allow the interpretation of the concept of resilience in a more complex light related to the time scale of business cycles which can be considered a medium-term time scale. We introduce and study here some typical model versions based on some financial-real interactions and present some simulations to illustrate some features of the dynamics.

3.1. A model of the real-financial interaction

The basis of our presentation here is the model by Semmler and Sieveking (1993), which is grounded in an IS-LM version for a growing real economy that links output to liquidity and credit flow. It assumes that financial flows to economic agents (to households and firms) are enhanced and credit conditions improved when the variables pass through certain thresholds.[5]

We assume that when the agents' balance sheets deteriorate (improve), creditworthiness deteriorates (improve). Using the above-cited literature, we presume that credit conditions (creditworthiness), and thus the spending of economic agents, depends on liquidity and output (or real income). As a measure of liquidity, we refer

[5]Also, liquidity and available credit may also have smoothing effects on production or consumption, at least for small shocks. Thus, actual economies may exhibit corridor stability (see Leijonhufvud (1973) and Semmler and Sieveking (1993)). In this view, small shocks do not give rise to deviation-amplifying fluctuations, but large shocks can lead to different propagation mechanisms. Thus, only large shocks are predicted to result in magnified economic activities.

to the broad definition of liquidity, including liquid assets.[6] At high levels of economic expansion, liquidity rises, default risk falls, asset prices, and creditworthiness rise.[7] The reverse may be assumed to happen during a low level of economic activity. As liquidity shrinks, default risk rises asset prices, and creditworthiness fall.[8] This is posited to occur after the variables have passed certain thresholds. Concerning spending, we may thus assume that spending accelerates (decelerates) when output and liquidity rise above (fall below) some threshold values.

The main features of a dynamic model of liquidity, credit, and output in a growing economy can give rise to regime changes through state-dependent reactions that can be represented in a deterministic form as follows.[9] We presume that economic agents respond to both financial variables (balance sheet variables) and real variables.[10] The model is written in the following generic form:

$$\begin{aligned} \dot{\lambda} &= \lambda \ \hat{\lambda}(\lambda, \rho) \\ \dot{\rho} &= \rho \ \hat{\rho}(\lambda, \rho) \end{aligned} \tag{1}$$

where $\lambda = L/K,\ \rho = Y/K$, with L denoting liquid assets, a balance sheet variable, Y, output (or real income), a real variable, and K the capital stock, while $\hat{\lambda}(\lambda, \rho)$ and $\hat{\rho}(\lambda, \rho)$ represent their growth rates, respectively. When we want to undertake the empirical estimate with data on firms, we interpret real income, Y, as firms' income and ρ as firms' income relative to capital stock. Thus, ρ denotes the rate of return on capital. A model of the type (1) can be derived from an aggregate model assuming that firms' income is linear in aggregate income. Roughly speaking,

[6] Moreover, liquidity can be a result of the central bank's monetary policy. In particular, quantitative easing provided an excess of finance for asset purchases, amplifying an asset price boom but also possibly generating its collapse when liquidity shrinks. So liquidity affects both the real and financial side of the economy.

[7] Ideally, one would like to employ net worth as collateral for borrowing, as referred to by the recent theory of the financial accelerator. Net worth should then be computed in terms of the net present value of the economic actions, where net worth is the present value of the agents' income flows reduced by the current and future debt payment commitments. Economic proxies for this variable are, however, hard to obtain. Alternatively, one could take credit lines as a proxy for creditworthiness. Time series data of sufficient length also do not exist for this variable. We are therefore left with other balance sheet variables. Given the above-mentioned role of liquidity for economic activity, one could take liquid assets as the balance sheet variable.

[8] It is thus only in this narrow sense that our model resembles the financial accelerator.

[9] For details of the model and its analytical and numerical study, the reader is referred to Semmler and Sieveking (1993).

[10] Semmler and Koçkesen (2017) shows that one can also incorporate a monetary policy reaction function.

model (1) connects the fluctuations in income and liquidity with the fluctuations in liquidity and income, respectively, since the growth rate of each variable depends on the other state variable. Moreover, it explains the fact that fluctuations in income and liquidity are more volatile the more liquidity and income are available. Note that due to this last property of model (1) the positive quadrant $\lambda > 0, \rho > 0$ is invariant, which means that if we start with positive liquidity and income they will remain positive forever.

As shown in Appendix B, the model can be thought of as being composed of two parts. First, a basic part of the model that exhibits no thresholds and regime changes, that can be expressed linearly as

$$\begin{aligned} \hat{\lambda}(\lambda,\rho) &= \alpha - \varepsilon_1\lambda - \beta\rho \\ \hat{\rho}(\lambda,\rho) &= -\gamma - \varepsilon_2\rho + \delta\lambda \end{aligned} \tag{2}$$

where the coefficients $\alpha, \beta, \varepsilon_1, \gamma, \delta, \varepsilon_2$ are positive. In model 2, α represents the natural liquidity growth for example through the central bank's growth rate of liquidity (money) supply. The term $-\beta\rho$ denotes the liquidity that is used for growth and new transaction and is thus used in the second equation with the term $+\delta\lambda$. γ is the natural loss of value of any investment, due to the ageing of capital (depreciation). The terms $\varepsilon_1\lambda$ and $\varepsilon_2\rho$ prevent the liquidity and income from growing without bounds. On the other hand, in the absence of a new injection of liquidity, the income growth rate is firmly negative, thus pushing the income toward zero if no new liquidity is provided.

Note that a non-trivial financial regime exists, i.e. that it is possible that both income and liquidity growth rate vanishes (so both income and liquidity stop their evolution) at

$$\lambda^* = \frac{\alpha\varepsilon_2 + \beta\gamma}{\beta\delta + \varepsilon_1\varepsilon_2}, \qquad \rho^* = \frac{\alpha\delta - \varepsilon_1\gamma}{\beta\delta + \varepsilon_1\varepsilon_2}. \tag{3}$$

As noted above, the sign of the coefficients of model (2) makes $\lambda^* > 0$, while $\rho^* > 0$ only if $\alpha\delta > \varepsilon_1\gamma$, i.e. a positive financial regime is possible only when the product of the natural liquidity growth α and the efficiency of the liquidity injection δ is greater than the product of the extra loss ε_1 and the investment loss γ.

Model (2) can be derived from a conventional IS-LM approach for a growing economy, see Semmler and Koçkesen (2017), although, as pointed out in Appendix B, the positivity of the model coefficients should still be subject to empirical verification. A similar system is discussed in Ozaki and Ozaki (1989), where, however, a nonlinear model in the interest rate and income is proposed.

The second part of our model explicitly allows regime changes due to state-dependent reactions. Referring to the above discussion, we may postulate regime changes to occur when the variables pass through certain thresholds. We posit that spending may accelerate (decelerate) when income and liquidity rise above (fall below) some threshold values. On the other hand, liquidity may also respond positively (negatively) when income or liquidity rises (falls) above (below) some thresholds. More formally, a model with regime changes in the cross effects between the variables can be written as follows

$$\begin{aligned} \hat{\lambda} &= \alpha - \varepsilon_1 \lambda - \beta\rho + g_1(\lambda, \rho) \\ \hat{\rho} &= -\gamma - \varepsilon_2 \rho + \delta\lambda \end{aligned} \tag{4}$$

where $g_1(\lambda, \rho)$ satisfies

$$\begin{cases} g_1(\lambda,\rho) \geq 0 & \text{for } \lambda > \mu_1 \text{ and } \rho > \varphi_1, \\ g_1(\lambda,\rho) \leq 0 & \text{for } \lambda < \mu_2 \text{ and } \rho < \varphi_2, \\ g_1(\lambda,\rho) = 0 & \text{otherwise.} \end{cases} \tag{5}$$

We further assume that $\mu_1 > \lambda^*$ and $\mu_2 < \lambda^*$, as well as $\varphi_1 > \rho^*$ and $\varphi_2 < \rho^*$, so that the regime changes can happen only 'far' from the financial regime.

Assuming the positivity of the model coefficients and the above sign structure of the perturbation terms, we can state the following propositions.[11]

Proposition 1. *If the perturbation terms $g_1(\lambda, \rho) = 0$ the system (1),(2) has a unique equilibrium globally stable.*

Proposition 2. *The trajectories of model (1),(2) remain in a positively compact invariant set for any $g_1(\lambda, \rho)$.*

Proposition 3. *If there exist $\bar{\lambda}$ such that the perturbation term $g_1(\lambda, \rho) < \varepsilon_1\lambda \ \forall \lambda > \bar{\lambda}$, then the system (1),(4) cannot indefinitely grow.*

Proposition 4. *For the perturbation terms $g_1(\lambda, \rho) \neq 0$ the trajectories of system (1),(4) with $\varepsilon_1 = \varepsilon_2 = 0$ diverges.*

We also want to note that the system ((1),(4)) may exhibit, when Proposition 3 holds, corridor stability in the sense of Leijonhufvud (1973) and Fisher (1933). For

[11]For details of the following results, see Semmler and Sieveking (1993) and the numerical extensions in Semmler and Koçkesen (2017). Note that the subsequent statements hold when the above sign structure holds. To what extent this can be empirically confirmed will be studied in another paper.

the purpose of our study, we assume the perturbation term g_1 to be concave in λ and ρ; for example,

$$g_1(\lambda, \rho) = v\left[\max(\lambda - \mu_1, 0)\max(\rho - \varphi_1, 0)\right]^{1/2}$$

where $v > 0$. In the next sections, a sampling of computer simulations illustrates the effects of perturbations of the dynamics in the different regions of the state space. As shown in Semmler and Sieveking (1993), all perturbations of the basic part of the model lead to bounded fluctuations (limit cycles, see also Appendix B). Two limit cycles can arise, a repelling and an attracting one.[12]

3.2. Numerical treatment and simulations

Further background explanations can be found in Semmler and Koçkesen (2017). The basic part of our model is consistent with a monetary growth model with an explicit LM schedule.[13] In this section, we perform a series of simulations to demonstrate the possible dynamics of the model. The solution method for our complex dynamics is with several features discussed in Appendix B.

Dynamic I

We first analyze model ((1),(2)) for $\varepsilon_1 = \varepsilon_2 = 0$. In this case, the system dynamics reduces to

$$\begin{aligned} \dot{\lambda} &= \lambda(\alpha - \beta\rho), \\ \dot{\rho} &= \rho(-\gamma + \delta\lambda). \end{aligned} \tag{6}$$

Model (6) is the well-known Lotka-Volterra model (Lotka 1910). In Appendix B it is shown that the quantity

$$V(\lambda, \rho) = \alpha \log \rho + \gamma \log \lambda - \beta\rho - \delta\lambda \quad (\lambda > 0, \rho > 0)$$

is constant along the trajectories of the system (6), thus allowing one to obtain the trajectories of this nonlinear system by simply looking at the contour plot of $V(\lambda, \rho)$.

Figure 1 depicts the trajectories of system (6) for a specific choice of the parameters. The trajectories that oscillate around a common centre point $\lambda^* = \gamma/\delta,\ \rho^* = \alpha/\beta$. The resulting phase plane plot in Figure 1 makes the cyclical relationship between the two variables very clearly showing how the swings in the income influence the swings in the liquidity and vice versa.

[12]The existence of corridor stability which gives rise to two limit cycles, a repelling and an attracting one, is studied in Semmler and Sieveking (1993).

[13]For details, see Flaschel et al. (1997, chapter 4)

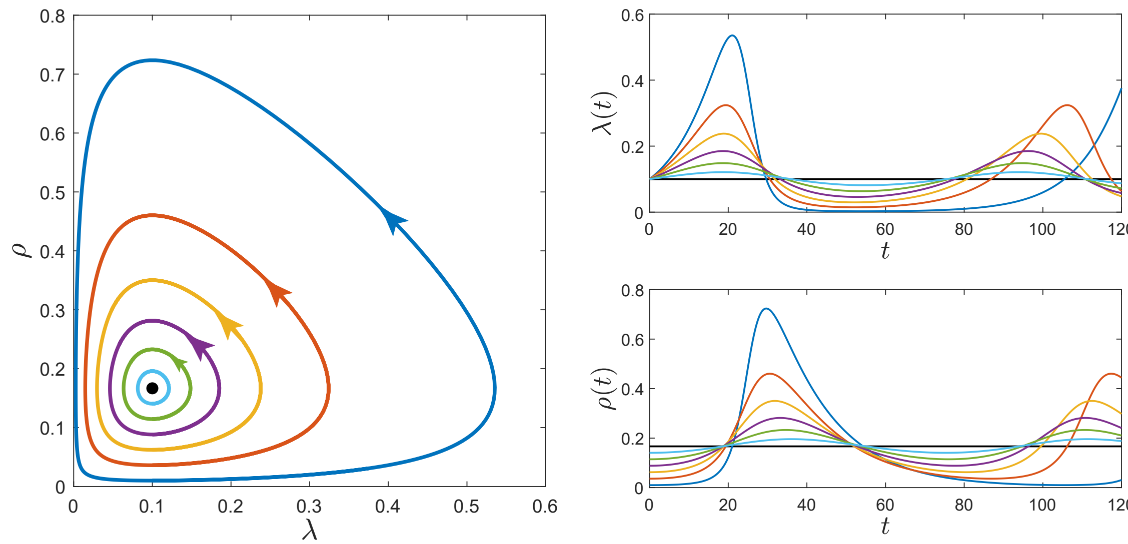


Figure 1: Simulations of model (6) with $\alpha = 0.1, \gamma = 0.07, \beta = 0.6, \delta = 0.7$ for various initial conditions. The left panel shows the phase portrait of the system, while the right panels the time evolution of each trajectory.

Dynamic II

We now analyze model ((1),(2)) for $(\varepsilon_1, \varepsilon_2) \neq 0$. The system dynamics is therefore described by

$$\begin{aligned} \dot{\lambda} &= \lambda(\alpha - \beta\rho - \varepsilon_1\lambda) \\ \dot{\rho} &= \rho(-\gamma + \delta\lambda - \varepsilon_2\rho) \end{aligned} \tag{7}$$

System (7) is a nonlinear system of differential equations of Lotka-Volterra type where with ε_1 and ε_2 as perturbations terms. However, this system has three equilibria $(\lambda_0, \rho_0) = (0, 0)$, $(\lambda_1, \rho_1) = (\frac{\alpha}{\varepsilon_1}, 0)$ and, if $\alpha\delta > \varepsilon_1\gamma$, a non-trivial equilibrium representing the positive financial regime (λ^*, ρ^*) as in system (3). The first two are saddle points and the last one is an attraction point. Except for those which start on one of the axes, all of the trajectories converge to the unique attracting point (λ^*, ρ^*).

The dynamic of system (7) is simulated by choosing economically realistic parameters in Figure 2. The economically relevant equilibrium is $\lambda^* = 0.12, r^* = 0.16$. Figure 2 depicts the trajectories of system (7) where it shows that the trajectories, though they are oscillating, asymptotically approach (λ^*, ρ^*).

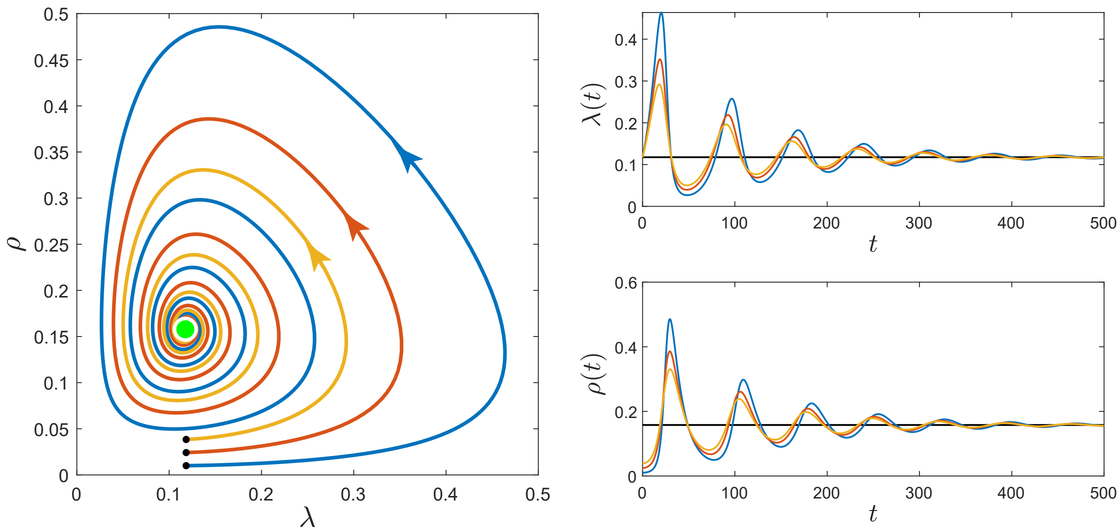


Figure 2: Simulations of model (7) with $\alpha = 0.1, \gamma = 0.07, \varepsilon_1 = 0.045, \beta = 0.6, \delta = 0.7, \varepsilon_2 = 0.078$ for various initial conditions. The left panel shows the phase portrait of the system, while the right panels the time evolution of each trajectory. Black dots represent the starting point of each trajectory, while the green point is the asymptotically stable equilibrium (λ^*, ρ^*).

Dynamic III

For the next case, we evaluate the case in which function $g_1(\lambda, \rho)$ is activated. The system dynamics is therefore described by

$$\begin{aligned} \dot{\lambda} &= \lambda(\alpha - \beta\rho - \varepsilon_1\lambda + g_1(\lambda, \rho)) \\ \dot{\rho} &= \rho(-\gamma + \delta\lambda - \varepsilon_2\rho) \end{aligned} \tag{8}$$

Function $g_1(\lambda, \rho)$ represents the notion that in business contractions, lenders' willingness to provide liquidity may depend on the state of the firms (balance sheet). In addition, agents faced with bankruptcy risk may become reluctant to use liquidity for current spending (but tend to preserve financial assets for bad times). The dissipation of liquidity, however, will entail a decline in capital outlay and investment of firms setting in motion a complicated dynamic.[14]

We assume that if the income is below a certain rate of return φ_2 (with $\varphi_2 < \rho^*$) and simultaneously liquidity drops below a certain threshold μ_2 (with $\mu_2 < \lambda^*$)

[14]One may also argue that a similar effect might occur in expansions. Since booms usually are resource constrained, we want to neglect this slight complication which is of course important as a driver of inflation rates in expansions, see Gross and Semmler (2019).

liquidity is dissipated, correspondingly affecting capital outlay and investment of firms. This mechanism can be seen as a control term in our dynamic system by the providers of liquidity representing, for example, the response of banks and firms to a decrease in liquidity and income.[15]

We shall assume that $g_1(\lambda, \rho)$ in (4) is a smooth function satisfying

(i) $g_1(\lambda, \rho) \leq 0, \ \forall (\lambda, \rho) \geq 0$,
(ii) $g_1(\lambda, \rho) \equiv 0$ in the neighborhood of (λ^*, ρ^*),
(iii) $g_1(\lambda, \rho) \not\equiv 0$.

Furthermore, if we assume that the perturbation is limited in size by the liquidity injected (subtracted) in the economy, we add $|g_1(\lambda, 0)| < \alpha$. Note that system (8) share the same equilibrium as system (7), since $g_1(\lambda^*, \mu^*) = 0$ due to assumption (ii). Effectively, due to the assumptions made on μ_2 and φ_2, in a neighborhood of (λ^*, μ^*) $g_1(\lambda, \mu) \equiv 0$: so even the property that (λ^*, μ^*) is asymptotically stable is preserved. This is because the Jacobian matrix of system (8) is the same as the one of system (7) in a neighbourhood of (λ^*, μ^*). On the contrary, the term $g_1(\lambda, \mu)$ pushes the trajectories toward the axes as soon as λ and ρ decline below μ_2 and φ_2, respectively. Note that the terms ε_1, and ε_2 guarantee that Proposition 3 holds, i.e. that system cannot diverge, since they make the growth rate negative as λ and ρ becomes too big, keeping the trajectories in a compact set. The exact analytical study of the impact of the perturbation terms on the Lotka-Volterra dynamics is given in Appendix B. These results serve as a hint to understand the possible effect of the perturbation to system (7). In the following, we illustrate the possible scenarios through a simulation study. To create our simulation, we choose:

$$g_1(\lambda, \rho) = -v[\max(\mu_2 - \lambda, 0) \max(\varphi_2 - \rho, 0)]^{1/2} \tag{9}$$

with $\mu_2 = \varphi_2 = 0.15$, and $v = 0.5$. The state portrait of the system is shown in Figure 3.

We can distinguish three scenarios. For small shocks trajectories still converge toward the equilibrium (violet trajectory): in fact, since $g_1(\lambda, \rho) \equiv 0$ in a neighbourhood of (λ^*, ρ^*), the state portrait of the system in that neighbourhood is the same as the one of system (7), shown in Figure 2. Since $g_1(\lambda, \rho)$ is continued, the state portrait then deforms with continuity as we leave (λ^*, ρ^*). At a certain distance, trajectories

[15]On the other hand, one can think that central banks can inject liquidity, impacting or modifying this dynamics. A magnifying effect can also come from the financial market; shrinking liquidity and falling income are usually highly correlated with asset prices shrinking.

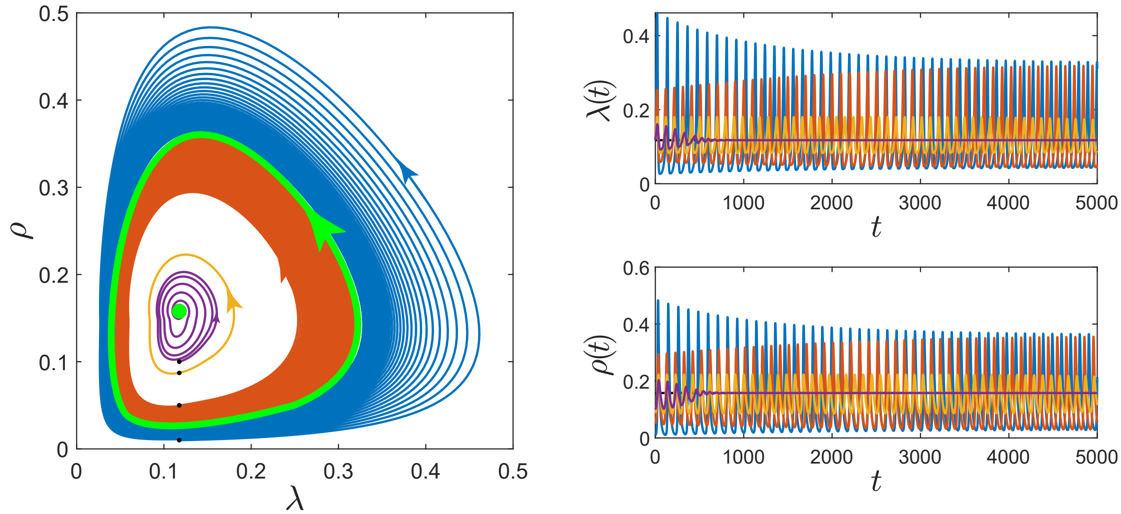


Figure 3: Simulations of model (8),(9) with $\alpha = 0.1, \gamma = 0.07, \varepsilon_1 = 0.045, \beta = 0.6, \delta = 0.7, \varepsilon_2 = 0.078$ for various initial conditions. The left panel shows the phase portrait of the system, while the right panels the time evolution of each trajectory. In green, are the two possible attractors of the system.

stop converging (λ^*, ρ^*), and a repelling limit cycle is present (yellow trajectory). If we start outside that limit cycle, the system will converge toward another attractor, a stable (attracting) limit cycle (highlighted in green), that can be generally reached starting from an initial condition 'sufficiently' far from (λ^*, μ^*). Note that this means that there exist shocks that can bring our system from the non-trivial (stationary) financial regime to a new oscillatory regime, with oscillations that persist in time. Technically speaking, the system is bi-stable and presents two alternative asymptotic regimes, one stationary and one characterized by persistent oscillations, and a shock can bring the economy from one to the other behaviour.

On the other hand, as can be shown if the reaction coefficient v becomes larger than 2 a disruptive behaviour can emerge where both liquidity and income go to zero. If we think that the stationary regime is more desirable, so, for example, if monetary policy prevents g_1 to be 'activated' for a sufficiently long period, then the system will spontaneously enter the basin of attraction of (λ^*, μ^*) - resulting in local corridor stability (local resilience).

4. Medium-run: Delays, complex dynamics, and policy impacts

Another well-known medium-run framework to be discussed in the context of the resilience concept is the real business cycle modelling, with their assumption of time-to-build, see Kydland and Prescott (1982). Real business cycle models with time-to-build, are models of time delay. Such models with a time delay between investment decisions and actual investment were already developed in 1930. Thus, real business cycle models due to delays were already developed earlier, using nonlinearities, possibly resulting in complex dynamics, such as the one devised by Kaldor (1940), and with reference to delay effects by Kalecki (1935; 1937). Kalecki hereby used a differential delay system in which the idea of lag related to the implementation of investment decisions was introduced. For delay effects in monetary policy causing possibly destabilizing dynamics see Chen et al. (2022)

For Kalecki, investment is key to determining aggregate demand and production and the course of the business cycle because investment expenditure determines the level of savings by changing the level of national income and the economic recovery and declines follow the movement of investments (Lopez and Mott 1999). The level of economic activity, among other things, depends on the pricing policy of enterprises and the distribution of income. Moreover, the ratio of investment to total production and employment under conditions of 'imperfect' competition, the rate of change of production equals the rate of change of investment, provided that the 'degree of monopoly' is unchanged. However, if the degree of monopoly were to decline while investment was falling, then the output would drop proportionately less than investment (at the limit, it might not drop at all) as decreases in prices relative to money wages would increase real wages and consumer spending. Thus, Kalecki anticipated, but with altogether different arguments, the 'New Keynesian' idea that unemployment may be partly due to rigidities in prices and profit margins" (Lopez and Mott 1999).

It should be stressed, however, that Kalecki's conclusion contrasts with that of the New Keynesians in that he demonstrates that real wages must rise for increases in consumption to compensate for decreases in investment. In stark contrast to the 'neoclassical synthesis' or hybrid Keynesian precept, wage stickiness is something to be desired, rather than avoided. This is because it prevents profit margins from expanding when capital expenditure falls. Keynes himself argued in Chapter 19 of The General Theory that downward flexibility of wages might decrease aggregate demand.[16]

[16] On the points in common and the differences between Kalecki and Keynes, a reader may refer

Back to the business cycles modelling, still in the 1930s, Kalecki (1937) and Kaldor (1940) suggested ordinary differential equations and nonlinear investment and savings functions to model business cycles. The latter is key in determining endogenous oscillations between economic growth and recession. Periodic solutions and the existence of limit cycles were extensively investigated in the literature through the years Chang and Smyth (1971), Varian (1979), Semmler (1987), Grasman and Wentzel (1994), De Cesare and Sportelli (2022). Among the others, Szydłowski and Krawiec (2005) proposed a modified Kaldor model with Kalecki time delay in investment via a second-order nonlinear delay differential equation with negative feedback

$$\begin{aligned} \dot{Y} &= \alpha[I(Y(t), K(t)) - S(Y(t), K(t))] \\ \dot{K} &= I(Y(t-T), K(t)) - \delta K(t) \end{aligned} \tag{10}$$

where investment $I(Y, K)$ and saving $S(Y, K)$ depend on income Y and capital K and the parameters α, δ, and T denote the speed of adjustment in the market of goods, the capital depreciation rate, and a time delay in the investment function, respectively. The authors show that the system (10) is equivalent to an autonomous dynamical system of infinite dimension and, by means of the central manifold method, they provide the condition for the stability of a limit cycle solution. The latter is important for deciding on the economic relevance of the solutions. As for the central manifold method, it is a useful tool for finding supercritical and subcritical Hopf bifurcations that occur when linear stability is lost.

Among the extensions, we mention Sasaki (2013) who presented a disequilibrium macrodynamic model incorporating employment and income distribution from Goodwin, investment function independent of savings and mark-up pricing in oligopolistic goods markets from Kalecki, and the reserve army from Marx. The model describes the dynamics of profit share, rate of utilization, and rate of employment and displays limit cycles depending on the size of the unemployment rate. The author shows that "if the stable long-run equilibrium corresponds to the profit-led growth regime, an

to Kregel (1989). Here we just recall Kalecki's aphorism that capitalists 'get what they spend' while workers 'spend what they get'. This means that capitalists cannot decide to earn more but they can decide to spend more. Workers can neither decide to earn more nor to spend more. So, for Kalecki, the distribution of income determines aggregate spending and money plays no role. For Keynes, instead, through the interest rate, money affects the propensity to consume and the marginal efficiency of capital. The distribution of income plays a secondary role. On the empirical side, Fazzari and Mott (1986) undertook a study of the investment theories of Kalecki and Keynes and confirmed that effective demand and firms' financial conditions are primary determinants of investment. This in turn implies that macroeconomic dynamics depend on the availability of finance, as also suggested by Minsky and Davidson.

increase in the bargaining power of workers increases the rate of unemployment; conversely, if the equilibrium corresponds to the wage-led growth-regime, an increase in the bargaining power of workers decreases the rate of unemployment" (Sasaki 2013).

4.1. Chaotic businesses cycles within a Kaldor-Kalecki framework

A version of the Kaldor model with lagged investment 'à la Kalecki', has been proposed by Orlando (2016; 2018).[17]

$$\begin{aligned} Y_{t+1} - Y_t &= \alpha(I_t - S_t) = \alpha[I_t - (Y_t - C_t)] \\ K_{t+1} - K_t &= I_t - \delta K_t \end{aligned} \tag{11}$$

where Y, I, S, K denote income, investment, saving and capital, respectively. As before, the parameter α represents the rate of adjustment of production to excess investment and δ is the depreciation rate of capital. Moreover, $I = I(Y, K)$ and $S = S(Y, K)$ are nonlinear functions of income and capital where the first depends on the difference between desired capital K^d and owned capital K, i.e.

$$I_t = K_{t-1} \cdot f_1(K^d_{t-1} - K_{t-1}) \tag{12}$$

Instead of modelling savings, we model its complement, i.e. the consumption which we assume depends on the difference between desired Y^d and current income Y, i.e.

$$C_t = Y_t \cdot f_2(Y^d_t - Y_t) \tag{13}$$

Regarding the mappings f_1 and f_2, they are increasing and s-shaped. In addition, $f_1(K^d - K)$ and $f_2(Y^d - Y)$ (representing the consumed fraction of Y) is bounded from below i.e. exists a constant $c > 0$ such that $c < f_2 < 1$ everywhere. Contrary to most of the literature, instead of taking a trigonometric function as arctangent, we consider an expanding form as the following two variants of the hyperbolic tangent:

$$f_1(x) = \rho \frac{\exp(2x/\tau_1)}{\exp(2x/\tau_1) + 1} \qquad \text{and} \qquad f_2(y) = \frac{\exp(2y/\tau_2) + c}{\exp(2y/\tau_2) + 1} \tag{14}$$

so that $f_1(x)$ goes to 0 as $x \to -\infty$ and tends to ρ as $x \to \infty$ whereas $f_2(y)$ goes to c as $y \to -\infty$ and tends to 1 as $x \to \infty$. The parameters τ_1 and τ_2 are the 'knees' of

[17] For an alternative model addressing both growth and business cycles in an open economy see Orlando and Della Rossa (2019). In that work, Harrod's speculation that opening the economy to foreign trade could lead to a reduction in cyclical instability is tested on real-world data.

the function determining the reactivity of the dependent variable to the independent variable, ρ is the upper bounds to investment. At this point, we assume that,

$$K_t^d - K_t = g_1 \left(\frac{Y_t - Y_{t-1}}{Y_{t-1}} - \frac{K_t - K_{t-1}}{K_{t-1}} \right) \tag{15}$$

$$Y_t^d - Y_t = g_2 \left(\frac{Y_t - Y_{t-1}}{Y_{t-1}} - \frac{C_t - C_{t-1}}{C_{t-1}} \right) \tag{16}$$

where $g_1 = g_2 = g$ takes the form

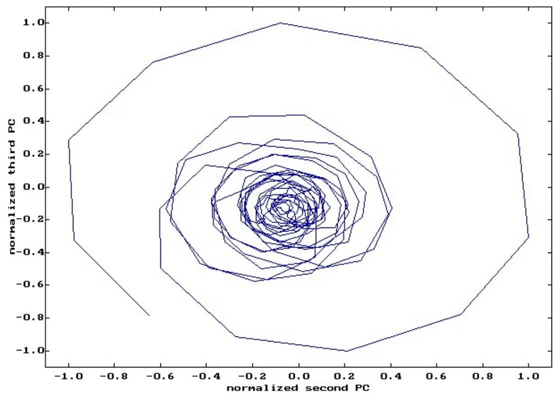
$$g(x) = \begin{cases} h & \text{for all} \quad x \leq 0 \\ -\log((k/x)^s - 1) & \text{for all} \quad x \in (0, k] \end{cases}$$

with $h < 0$ such that f_1 is close to zero and k a parameter depending on the state of the economy. In other words, a higher k correlates with a more volatile economic period. Notice that the shape of the g function is an approximation of the value function by Kahneman and Tversky (1979) where on the left the function is assumed to be straight. This lower bound accounts for a minimum level of subsistence in case of consumption or a minimum level of capital upkeep in case of investment.

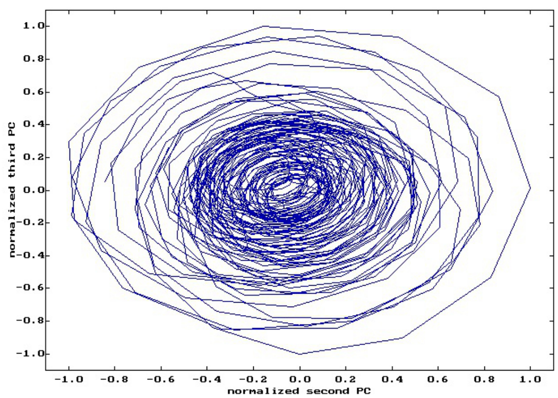
To sum up, the proposed modified Kaldor-Kalecki model is

$$\begin{aligned} Y_{t+1} - Y_t &= \alpha \left[f_1 \left(g \left(\tfrac{Y_{t-1} - Y_{t-2}}{Y_{t-2}} - \tfrac{K_{t-1} - K_{t-2}}{K_{t-2}} \right) \right) + f_2 \left(g \left(\tfrac{Y_t - Y_{t-1}}{Y_{t-1}} - \tfrac{C_t - C_{t-1}}{C_{t-1}} \right) \right) - Y_t \right] \\ K_{t+1} - K_t &= f_1 \left(g \left(\tfrac{Y_{t-1} - Y_{t-2}}{Y_{t-2}} - \tfrac{K_{t-1} - K_{t-2}}{K_{t-2}} \right) \right) - \delta K_t \end{aligned} \tag{17}$$

The model described by (17), for a rather wide range of parameters, shows chaotic dynamics and is capable of endogenously generating black swan events (Orlando and Zimatore 2020a). Figure 4 shows a strange attractor for the USA GDP time series BEA, U.S. Bureau of Economic Analysisìs (2020) and a simulated time series from system (17) as obtained with Van den Bleek (1994). The attractor of the dynamical system is reconstructed using Takens embedding rule and singular value decomposition Takens (1981). Notice that "the resilience of complex systems is a critical ability to regain desirable behavior after perturbations" (Zou et al. 2023), the presence of an attractor ensures the said resilience at least within a given domain.



(a) Phase space along second and third dimension for USA GDP BEA, U.S. Bureau of Economic Analysisìs (2020)



(b) Phase space along second and third dimension for the system in (17)

Figure 4: Strange attractor for the real versus simulated data. The plot shows the third principal component of the variable on the Y-axis as a function of the second principal component of the variable on the X-axis. Notice that real data consist of 290 (quarterly) points while simulated data are 10,000. Source (Orlando and Zimatore 2020a)

Moreover, Figure 5 depicts a simulation of the four macroeconomic variables Y, I, S and K as generated by the system (17) which displays a black swan event.

In terms of empirical tests, business cycles versus the model proposed in system (17) have been extensively investigated through recurrence quantification analysis (RQA) Marwan et al. (2007), Orlando et al. (2021b), principal component analysis (PCA) and Poincaré Plot with related quantifiers. Chaos analysis brought evidence on the fractal dimension and entropy measures for both real data and model simulations. Therefore, it has been shown that the real and simulated business cycle

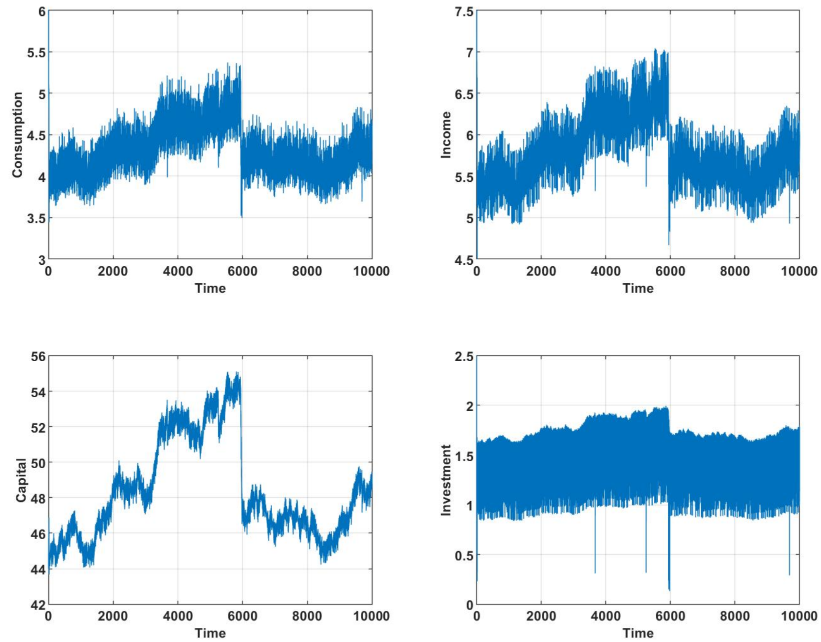


Figure 5: A simulation of the system Eq. (17) for Y, K, I and C with a black swan event. Source (Orlando and Zimatore 2020a)

dynamics share characteristics that make the proposed model a suitable tool to simulate economic reality Orlando and Zimatore (2020b). In particular, in the case of RQA, it has been demonstrated that it can be used for the early detection of recessions and that it can distinguish between stock and flow macroeconomic variables as well as between real and nominal data Orlando and Zimatore (2017; 2018b;a). The above methodology is useful for discovering the underlying dynamics of economic time series, especially where other methods may fail due to the randomness, nonlinearity and non-stationarity of the data. Given that the economy may be deterministic and chaotic, a number of empirical studies have been performed on real-world data, from the market share of operating systems such as Android Stoop et al. (2022), to financial markets Orlando et al. (2022), from credit risk Orlando and Bufalo (2022c) to corporate dynamics Orlando (2022). In the studies mentioned above, it has been shown that real data can be explained by a deterministic model that mimics the bursts and chaos regularization of a neuronal brain cell. Furthermore, the aforementioned model is at least on par in data explanation and prediction (in-sample and out-of-sample performance) with the classic ARIMA-GARCH (and related variants) econometric model explicitly designed to model autoregression, cointegration, moving average, and heteroscedastic volatility.

5. Longer-run: Multiple equilibria, thresholds, disruptive contractions, and traps

Concerning longer time scales, we want to study the issue of thresholds, tipping points, disruptive contractions, and persistent traps in the context of multiple equilibria models. Recent economic history, in particular of advanced market economies, has shown that there might be some longer period of growth and tranquillity, whereby imbalances and vulnerability have been built up. Then there might be some sudden disruptive contractions and, at times, longer trapping periods. Those are not much captured in the concept of resilience since, in those cases, there does not seem to be robustness to shocks. Brunnermeier mentions those features as a result of what he calls points of resilience destroyers.

Some recent examples are the financial market meltdown of the years 2007-9, and the European sovereign debt crisis, triggering a great and long recession. Other examples are known as growth and development traps, climate extreme events and disasters, and the sudden disruptive contractions caused by the spread of epidemiological diseases – all those might generate persistent trapping periods. Many of those disruptive meltdowns have been studied and characterized by features such as rising vulnerability, imbalances and risks built up, that lead to those disruptions and partly longer periods of stagnation.

Most of these have the feature that there are multiple equilibria involved, with sudden shifts from one to another equilibrium triggered by the above-mentioned thresholds and tipping points. We will restrict ourselves here to three examples, where we will not explicitly present the analytics and numerics. We employ here a recently developed solution methodology, the NMPC method, for rather simple models and depict graphs to illustrate those complex dynamics.[18]

Financial and sovereign debt crises

The financial market meltdown of the years 2007-9, and the European sovereign debt crisis, can be described by a nonlinear macro model where some longer-term forces are at work. Before the large meltdown of those years, the housing prices went up steeply, mortgages were easily granted at flexible rates, and the banking sector

[18]See Grüne et al. (2015) and for a sketch of the method see Appendix A.

outsourced the mortgage risks, also for low-income households with a high risk of insolvency using financial engineering methods such as mortgage and asset-backed securities.

But once the mortgages could not be repaid, insolvency started in the housing sector when the interest rate rose, spread to the banking system, and became a worldwide meltdown. In Europe, this accelerated through the sovereign debt crisis to the point that, in the years of 2011-12, sovereign debt was at the centre of the storm, with the threat of sovereign debt defaults and fast jumping up of risk premia for sovereign debt, particularly for Greece, Italy, Spain, Portugal, and Ireland.

The features of sovereign debt dynamics and crisis can be stylized as follows. We use a slightly modified approach suggested by Blanchard (1983). The below sovereign debt dynamics equations represent an extension of the Blanchard (1983) study of sovereign debt, whereby we follow Blanchard (2019) who introduces a "good" and "bad" debt equilibria that can be solved by NMPC as in Grüne et al. (2015). We write a finite horizon decision problem for sovereign debt as follows.

$$V(k,b) = \max_{c_t, g_t} E_t \int_0^T e^{-rt} U((c_t) - \chi(\mu_t - \mu^*)^2) dt \tag{18}$$

$$\dot{k}_t = (g_t - \delta) k_t \tag{19}$$

$$\dot{b}_t = (r b_t - (y_t - c_t - i_t)) \tag{20}$$

Alternatively, equation (20) can be formulated as

$$\dot{b}_t = (r(s_t|\gamma, c^*) b_t - (y_t - c_t - i_t)) \tag{21}$$

with a state-dependent risk premium in equation (21) such as:

$$r(s_t|\gamma, c^*) = [1 + \exp(-\gamma(s_t - c^*))]^{-1}, \gamma > 0, c^* > 0 \tag{22}$$

with sovereign debt b, capital stock k, the growth rate of capital stock g, depreciation rate of capital stock δ, and a pay-off from welfare from consumption $U((c_t) - \chi(\mu_t - \mu^*)^2)$ reduced by adverse effects from high sovereign debt, with $(\mu_t - \mu^*)$ the excess of debt to capital stock ratio above a threshold. The variables y, c and $i = gk$ are output, consumption, and investment, respectively. The policy decision variables are c and g. Hereby then, sovereign debt is allowed to increase through borrowing from abroad.

The "good" debt equilibrium operates with low and stable interest rate r on the sovereign debt dynamics of equation (21). The "bad" debt equilibrium exhibits

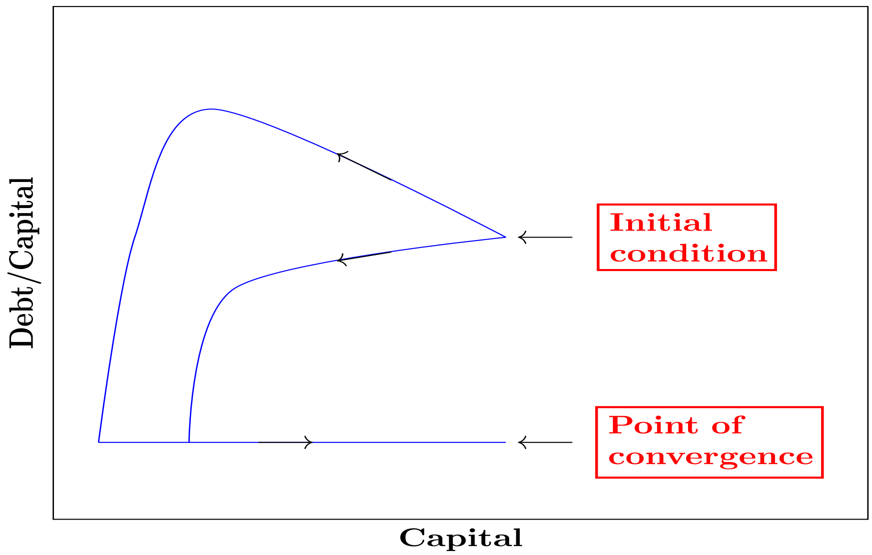


Figure 6: Debt dynamics convergence to sustainable debt or good equilibrium with a low-risk premium. The two initial conditions $k(0) = 0.9, b(0) = 0.9$ (large), the left trajectory with $A = 0.1$, the right with $A = 0.2$, and one other initial condition $k(0) = 0.2, b(0) = 0.08$ (small); all trajectories converging to steady state $\mu* = 0.3$, with $r = 0.04$. See Mittnik and Semmler (2018)

a state-dependent risk premium defined in equation (22), depending on a logistic function with a threshold given by the "bad" state of the dynamics and the value threshold value c^*. The left-hand sides of equations (19) - (21) are time derivatives. Solving the debt dynamic model, indicating a regime-switching between two equilibria, a good one being stable and the bad one unstable, can emerge.

Figure 6 shows the dynamic paths of assets and leveraging for low and constant interest rate, for three initial conditions Using equations (18) - (21), where the interest rate r is assumed to be small and fixed, Figure 6 represents several solution paths with different initial conditions, for a good equilibrium.

As can be observed, there is global stability – global resilience of debt dynamics. All shocks, represented by initial conditions in Figure 6, approach due to the dynamics, the Point of convergence.

Next, we explore the vicinity of the bad equilibrium. The behaviour of the trajectories, capital stock, and the debt-to-capital ratio in the neighbourhood of the bad equilibrium is observable in Figure 7.

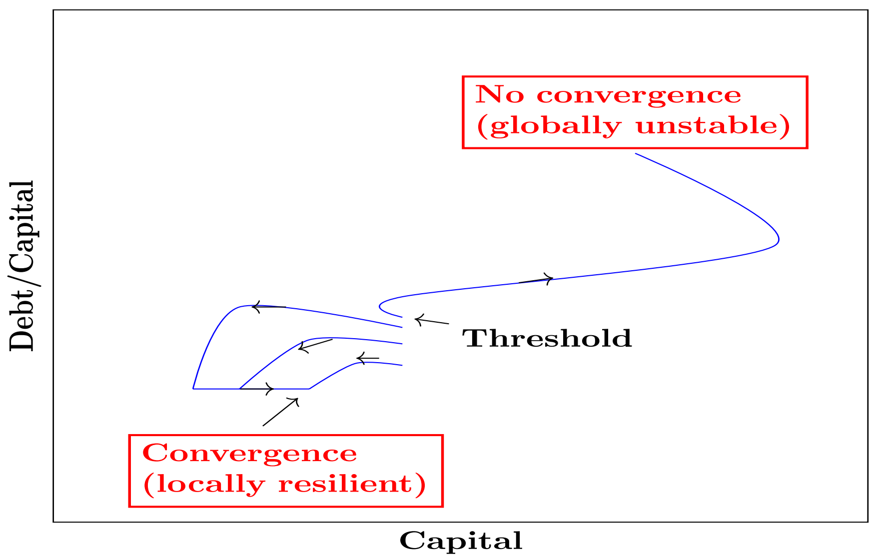


Figure 7: Debt dynamics for nonlinear financial stress or bad equilibrium; lower three trajectories for the low-stress case with borrowing cost below a threshold, for three initial conditions, convergence to some steady state, even in the high-stress regime but for low-credit costs; yet triggering of unstable dynamics, upper trajectory; for initial conditions $k(0) = 1, b(0) = 0.9$ with high credit cost. See Mittnik and Semmler (2018)

For the bad debt equilibrium case, we observe only a local convergence and local resilience (see the Convergence box). Only trajectories starting near that equilibrium are locally resilient. In fact, when the bad debt equilibrium in the years 2011-12 was experienced in Greece, Italy, Portugal, and Ireland the monetary policy, the ECB changed its monetary policy and massively purchased treasury bonds in order to bring the sovereign debt spread down and to rescue the Euro.[19]

Development and climate disaster traps

Next, we explore nonlinear models of development and growth that have been studied for long, and where it has been shown that there may be, in the long run, some lock-ins and difficulties in moving out from low growth traps; in earlier times, studied by Scitovsky (1959) and Gunnar et al. (2017). This situation could also lead to long-term poverty traps Banerjee and Duflo (2007), Barrett et al. (2018). On the other hand, the experience of natural and climate disasters and weather extremes (such as flooding, heat waves and droughts, forest fires, storms, typhoons, and hurricanes) can leave the region in a state of long-term disaster effects without recovery.[20] These are stable trapping regions, and it is often hard to get out of them. There are certain mechanisms for why it is difficult to move out of those traps.

In recent times, growth models with multiple steady states have been developed to address those issues, see Azariadis and Stachurski (2005), Semmler and Ofori (2007), Kovacevic and Semmler (2021), the latter is a model with stochastic shocks trapping probabilities. In such models, large disasters may change the steady state and persistently produce a lock-in with lower growth rates. Usually, in those models, there are three equilibria observable, whereby the middle one is unstable, which is called a Skiba point.[21]

Below we describe the possible self-enforcing trapping regions

1. **Real side**: increasing/decreasing returns

$$y(k(t) = ak(t)^{\alpha_k(t)}$$

[19] ECB president Draghi made the famous statement that referenced this bad equilibrium by saying in a news conference (September 6th, 2012): “The assessment of the Governing Council is that we are in a situation now where you have large parts of the Euro Area in what we call a bad equilibrium, namely an equilibrium where you have self-fulfilling expectations. You may have self-fulfilling expectations that feed upon themselves and generate adverse, very adverse scenarios. So there is a case for intervening to, in a sense, break these expectations [...]”

[20] For a stochastic model on the expected value and the volatility of natural disasters occurrences see Orlando and Bufalo (2022a;b).

[21] Skiba (1978) was the first theorist who discovered such tipping points in optimally controlled dynamic systems.

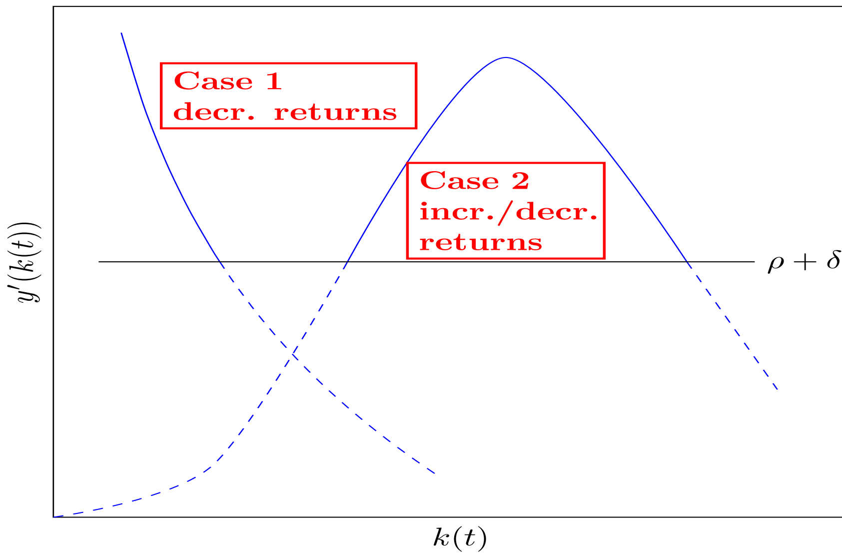


Figure 8: Development and climate disaster traps seen on increasing and decreasing returns with default risk and a finance premium; Case 1 is obtained if $\alpha_k(t) > 1$, holds forever, the marginal product of capital, $y(k)$ would approach the line given by the discount rate ρ plus capital depreciation, δ, from above if depreciation is allowed; Case 2 illustrates the marginal product of capital $y(k)$ first approaching $\rho + \delta$ from below, then move above this line, $\rho + \delta$, and eventually decrease again. See Semmler and Ofori (2007)

$$\alpha_k(t) = \begin{cases} \bar{\alpha}_k & \text{if} \quad k(t) > \bar{k}(t) \\ \underline{\alpha}_k & \text{otherwise} \end{cases}$$

2. **Financial side**: risk premia depending on net worth, with net worth low the risk premia are high, leading to a failure to obtain financial resources and generating poverty lock-ins
3. **Human side**: human capital loss and migration after climate disasters

The above three mechanisms are usually operating here to create poverty traps which are sketched on the left-hand side of Figure 8. Countries can move into trapping regions, which are self-enforcing, due to the form of a production function, its shape depending on the level of capital input $k(t)$. In the Case 1, the production function is S-shaped, given by $y(k(t)) = ak(t)^{\alpha_k(t)}$, with a a coefficient, and the exponent $\alpha_k(t)$ switching sign when the capital stock rises; first it is $\alpha_k(t) > 1$ and after the threshold it becomes $\alpha_k(t) < 1$. So the production function is first convex, then concave, a widely used assumption in the early development literature.

Large climate-related disasters can also lead to a higher trapping probability (for small and middle-income countries or certain regions). The derivative of the production function with respect to k is in Figure 8 denoted by y' which is also called the marginal product of capital (returns to capital). In standard economic growth models when the marginal product of capital, k, is greater than the discount rate and capital depreciation $\rho + \delta$, capital grows, and it declines if y' is smaller than $\rho + \delta$. With our convex-concave production function, however, capital returns are shrinking if below $\rho + \delta$ and rising when above. So there are three equilibria, and the middle one is unstable, also called Skiba point.

In Case 2, is illustrated that other mechanisms than originating in the production function are also possible. With lower per capita income, there are often restrictions in the financial market, such as credit constraints and/or high-risk premia, explored in many studies on the role of the financial markets in developing economies. In Case 3, we show that there can be a third effect when low growth and poverty traps emerge, for example resulting from climate extreme events. In this case, but also in other examples, there will usually be an exit of human capital, skills, and entrepreneurs, from that region, also generating long-term lock-ins and growth traps.

We finally want to note empirical estimates of poverty traps are made for 90 countries in Semmler and Ofori (2007), using Markov matrices, where one can show that the middle range of the Markov matrices gets empty over time.

Climate change, tipping surfaces, and climate disasters

As recently shown, complex systems of higher dimensions, such as studying the climate-economy interaction models, can exhibit tipping surfaces instead of tipping points. They are commonly presented in the context of DICE-type models (Nordhaus 2013) (and related numerous variants and extensions). If the Hansen et al. (2008) prediction holds the earth's temperature will face a tipping point –with possible subsequent tipping points in extreme weather events and long-run indirect effects, such as lower growth and less food production in agriculture Nordhaus (2019). The Hansen et al. (2008) prediction concerns the energy balance on the earth, known as the albedo effect: with rising CO_2 emissions and rising temperature, the fraction of energy coming from the sun and absorbed by the earth will increase and there will be less energy reflected back to space. So, with the rising temperature, the temperature will rise faster since more energy is absorbed by the earth. Thus, the higher the temperature, the more energy the earth absorbs, and the warmer it gets, generating more climate disasters.

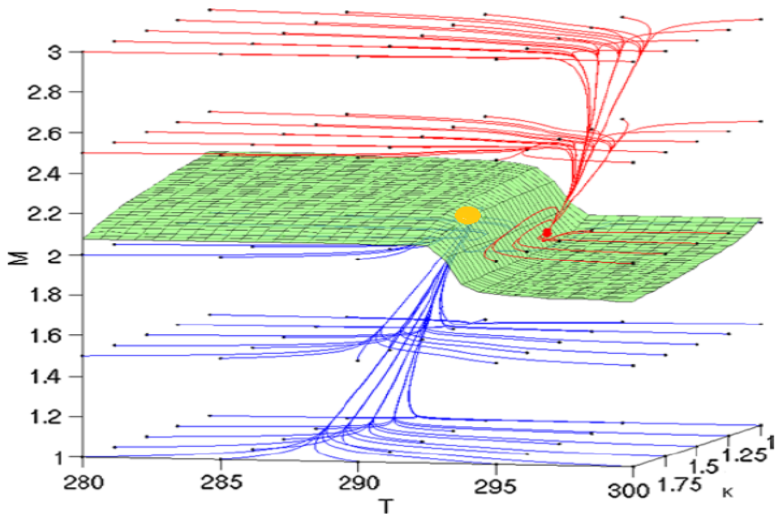


Figure 9: Tipping surface shown on a Skiba plane in the $(T - K - M)$ space. A Skiba plane in the $(T - K - M)$ space where on the lower axes (T) is the average global surface temperature measured in Kelvin, and the vertical axis is (M) the CO_2 concentration in the atmosphere or greenhouse gases $(GHGs)$ measured in parts per million (ppm). Source Greiner et al. (2010)

The non-linearities of these models, such as in Greiner et al. (2010), and the versatility of these models help shed light on the economy and society's climate vulnerability to loss of resiliency and the importance of stabilizing policies.[22] Those effects in a higher dimensional multiple equilibria model suggest the outcome as illustrated in Figure 9. As shown, there is one equilibrium with lower temperatures (i.e. the blue lines coming from below), but as the CO_2 concentration rises above the green surface, there is likely to be another equilibrium, one with a higher temperature, see the red lines coming from above. Thus, Figure 9 displays Skiba surfaces instead Skiba points: Above the green surface, there are possibly irreversible dynamics to a higher temperature with subsequent greater severity and frequency and disasters, and possibly less increase in global output in the long run. So, the model shows local resilience but global non-resilience.[23]

[22] See also Brock et al. (2008).

[23] A similar conclusion was reached by Caleiro et al. (2019) who used an evolutionary game theory approach, where the size of the risk-reward penalty might lead to either getting out of a Skiba surface or reaching a global stability point.

6. Conclusions

Given the nowadays popular concept of resilience, we tried to evaluate this new idea in terms of what has been achieved by studies of economic and financial complex system dynamics. We considered the relation of resilience to complex system dynamics for a short-run time scale, the well-known market dynamics studied since classical economics, a medium-run time scale usually represented by business cycle dynamics, and by a long-run time scale frequently used in growth theory, development economics and in studies of the climate-economy linkages.

Economic, social and/or climate-related shocks occurring on these three time scales may show different features of the subsequent dynamics. Whereas on the time scale of market dynamics, driven by supply and/or demand shocks, there is often local endogenous resilience to be found, in the literature called corridor stability where small shocks are mean reverting but large shocks less so – but they may be globally bounded. For business cycle models there is also resilience stated in the sense of mean reverting to some endogenous corridor stability path, but there are also often destabilization and amplifying mechanisms at work that may generate large fluctuations or even produce loss of resilience, persistent (limit) cycles, and complex dynamics, see also Orlando et al. (2021a). On longer time scales there are frequently experienced a loss of local resilience, long cycles, multiple attractors, and disruptive contractions, lasting for a longer period Orlando et al. (2022), Stoop et al. (2022).

These challenges of nonlinear and complex dynamics on different time scales have created challenges for statistical and econometric efforts which have generated considerable work on nonlinear econometrics. Such empirical work has shown that state-dependent reactions and regime changes in cross-effects between the variables at different time scales can occur. Whereas theoretical models often use continuous time models, sometimes introducing differential delay systems, econometric studies usually face time-discrete data and often work with time discrete models. For this purpose nonlinear continuous time models need to be discretized. This is done through some direct, continuous time methods (Euler scheme and Ozaki's local linearization method), as well as a discrete-time version for example through some discrete-time regime change models. Although the first is easier to apply, it may generate instability and cannot always be mapped into a coherent discrete-time method. These are further research challenges in particular for a higher dimensional state space, where the number of observations to be used is limited, and the computation is very time-consuming. Although the direct method revealed some expected dynamic behaviour, such as regime change behaviour, when estimated, lag and delay effects do play here an important role. For the latter, see Orlando (2018) and Chen et al. (2022).

As shown, a linear model often fits the data and exhibits a convergent behaviour, indicating a stable steady state. In other words, if one were to assume that the data had a linear representation, the system would be regarded as being stable around the steady state, and the cause of fluctuations would have to be attributed to exogenous shocks. As standard models often exhibit, one would always observe a mean reversion behaviour of the variables. However, a regime change model, which we claim is often a better representation of the data, reveals that the actual dynamics of the system are characterized by a locally unstable steady state contained by stable outer regions. Regime change models are capable of asking and answering more exciting questions, which have been prevalent in the theoretical literature but are searching for a way to be examined by empirical analysis. Threshold principle time series models seem to be particularly useful in this endeavour. We could show that one can study multiple steady states and global resilience even empirically, though local resilience might or might not exist.[24]

On the other hand, we have pointed out in the theoretical Sections 3 and 4 that the opposite can also hold, generating local resilience, called corridor stability, and global resilience, but in between some destabilizing forces might be working and complex behaviour might emerge. Higher dimensional models can also be studied where such thresholds exist, not as threshold points but as threshold surfaces, see Section 5. Transitions of a system from order to chaos may be induced by additive and parametric random noise even in a two-dimensional system as per the Kaldor model (Bashkirtseva et al. 2018). The alternative behaviour in which the economic system is randomly periodic or converges towards normality can be made dependent on some

[24]For instance, SIR (Susceptible, Infectious, or Recovered) models such the one developed by Aliano et al. (2023) may explain how financial crises may spread out and if there exists a link between financial systems and ecosystems. Time delay and incubation period are critical in the sense that a long incubation makes risk-free equilibrium can be globally stable whilst "a sufficiently short incubation, together with a short immunity period lets the endemic steady state be locally stable so that risk remains in the economy in the long run". In contrast to deterministic models, stochastic models such as dynamic stochastic general equilibrium models (DSGE) are employed for policy analysis and interpretation of policy effects and market shocks. Chen and Semmler (2021), by identifying in rates cuts a regime identifier monetary response, show "that the financial stress shocks have a large and persistent negative impact on the real side of the economy, and their impact is stronger in the non-rate-cut regime than in the rate-cut regime". In terms of energy, a DSGE model proposed by Aminu (2019) found that shocks in prices are important drivers of economic activity, they are temporary because of stationarity. This implies that monetary authorities may intervene to offset falls in output by lowering the interest rate and quantitative easing by asset purchase.

threshold value (Li et al. 2017). Random transitions between stable attractors in the context of the Goodwin-type economy have been shown as a feature of weak noise levels regime switching and low saving rates. On the contrary "increased uncertainty can induce an essentially unpredictable income process out of an apparently stable high-income level situation" (Jungeilges and Ryazanova 2017).

On the same line, existing work shows that the complex dynamics can generate steady states as a trapping region, as bad attractors, where the dynamics nearby can be trapped in a low-level equilibrium with features of a persistent trap, see Mittnik et al. (2020), Kovacevic and Semmler (2021), Orlando (2022). They provide examples of this type arising for instances from severe climate disasters with disruptive contractions. The latter models can also help study mechanisms that explain how counter-policy could be used to avoid such disruptions and traps.

7. Acknowledgements

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Appendix A. Solution Method NMPC

As a solution method to solve complex dynamic systems, without and with control decisions, we use the method of Nonlinear Model Predictive Control (NMPC) which solves one trajectory at a time, see Grüne et al. (2015). This solution method is used to obtain the dynamic solution paths of the models in Section 5, where the complex dynamics are described and then solved via the Euler or Runge-Kutta procedure for solving differential equations with one or more policy control variables. If the control variable or control variables are non-zero, the following solution algorithm from NMPC can be used.

$$maximize \sum_{k=0}^{N} \beta^k (x_{k,i}, u_{k,i}) \tag{A.1}$$

$$N \in \mathbb{N} \text{ with } x_{k+1,i} = \varphi(h, x_{k,i} u_{k,i})$$

Figure A.10 shows the predictions (open loop) illustrated by the black paths and the NMPC results (closed loop) illustrated by the red path, which gives us the solution paths of the dynamic system with controls for finite decision and time

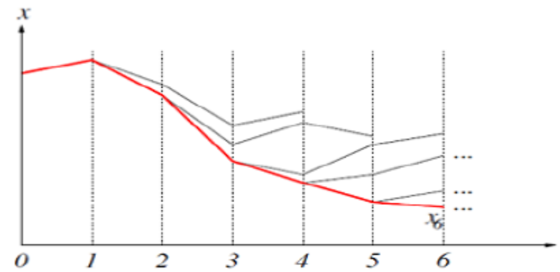

Figure A.10: Predictions and decisions using NMPC; black lines representing a 4-period decision horizon, and the red line, as envelop of the black lines, depict the optimal path of the variables x for the time horizon of 6 periods, for details Grüne et al. (2015)

horizons. Hereby the index i indicates the number of iterations and β^k is the discount factor.

If there are no control variables, the complex system can be solved directly through NMPC.

Appendix B. The Lotka-Volterra system and perturbations

For a detailed explanation see Semmler and Sieveking (1993). The simple Lotka-Volterra system, described in system (6), is most simply analyzed with the aid of the Lyapunov function

$$V(\lambda, \rho) = \alpha \log \rho + \gamma \log \lambda - \beta\rho - \delta\lambda \quad (\lambda > 0, \rho > 0).$$

In fact, along a generic trajectory $(\lambda(t), \rho(t))$ of system (6), $V(\lambda(t), \rho(t))$ does not change its value. To show that, we can simply compute

$$\frac{dV(\lambda(t), \rho(t))}{dt} = \frac{dV(\lambda, \rho)}{d\lambda}\frac{d\lambda(t)}{dt} + \frac{dV(\lambda, \rho)}{d\rho}\frac{d\rho(t)}{dt} = \left(\gamma\frac{1}{\lambda} - \delta\right)\dot{\lambda} + \left(\alpha\frac{1}{\rho} - \beta\right)\dot{\rho}$$
$$= \gamma(\alpha - \beta\rho) - \delta\lambda(\alpha - \beta\rho) + \alpha(-\gamma + \delta\lambda) - \beta\rho(-\gamma + \delta\lambda) = 0.$$

Thus, trajectories in the positive quadrant, coincide with the level curves of $V(\lambda, \rho)$, i.e. the set

$$\mathcal{O} = \{(\lambda, \rho) \in (0, \infty) \times (0, \infty) : \ H(\lambda, \rho) = c, \ \forall c \in \mathbb{R}\}$$

Being $V(\lambda, \rho)$ a convex function, the orbits of (6) are closed orbits around $\gamma/\delta, \alpha/\beta)$. Note that system (6) is defined also for $\lambda = 0$ and $\rho = 0$: to obtain the entire set of orbits we have to complete $\mathcal{O}$ with $\{(0, 0), \ \{0\} \times (0, \infty), \ (0, \infty) \times \{0\}\}$.

A first perturbation

We now consider system (7), to prove Propositions 1 and 2. Let's start with Proposition 2. The fact that trajectories cannot become negative has already been highlighted as a general property of model (1), since variables fluctuations are proportional to their value. We need to show that variables cannot indefinitely grow. This can be easily shown by noting that the growth rate $\hat{\lambda}(\lambda, \rho)$ as defined in (2) is negative if $\lambda > \bar{\lambda} = \alpha/\varepsilon_1$. Therefore, any trajectory eventually enters the region $\lambda < \bar{\lambda}$ and stays there forever. In this set, also the growth rate $\hat{\rho}(\lambda, \rho)$ is negative if $\rho < \bar{\rho} = (\delta\bar{\lambda} + \gamma)/\varepsilon_2 = (\delta\alpha + \gamma\varepsilon_1)/\varepsilon_1\varepsilon_2$. Therefore, every trajectory eventually enters the box $[0, \bar{\lambda}] \times [0, \bar{\rho}]$ and stays there forever.

To show 1 we start noting that system (7) has the same phase portrait in the positive quadrant of the system

$$\begin{aligned} \dot{\lambda} &= \frac{1}{\lambda\rho}\lambda(\alpha - \beta\rho - \varepsilon_1\lambda) &= \frac{\alpha}{\rho} - \beta - \varepsilon_1\frac{\lambda}{\rho} \\ \dot{\rho} &= \frac{1}{\lambda\rho}(-\gamma + \delta\lambda - \varepsilon_2\rho) &= -\frac{\gamma}{\lambda} + \delta - \varepsilon_2\frac{\rho}{\gamma} \end{aligned} \tag{B.1}$$

since being $\lambda > 0$ and $\rho > 0$, dividing for the positive quantity $1/(\lambda\rho)$ does not change the vector field orientation but changes only the velocity at which the trajectories are travelled. The divergence or system (B.1) is

$$\frac{\partial\dot{\lambda}}{\partial\lambda} + \frac{\partial\dot{\rho}}{\partial\rho} = -\varepsilon_1\frac{1}{\rho} - \varepsilon_2\frac{1}{\lambda}$$

is negative in all the first quadrants, and therefore, due to the Dulac-Bendixon theorem (Burton 2005), system (7) cannot admit limit cycles. Being the box $[0, \bar{\lambda}] \times [0, \bar{\rho}]$ forward invariant, all the trajectories must converge in the unique forward attractor present in the box, that is the equilibrium (λ^*, ρ^*), thus proving Proposition 1. Note that, even if all the positive quadrants converge to (λ^*, ρ^*), the trajectories that depart on the vertical axis, i.e. starting from a point $(0, \rho(0))$, remain on the vertical axis (since $\dot{\lambda} = 0$) and converge to $(0, 0)$ since $\dot{\rho} < 0$. Similarly, the trajectories that depart on the horizontal axis, i.e. starting from a point $(\lambda(0), 0)$, cannot leave the horizontal axis (since $\dot{\rho} = 0$) and converge to the point $(\alpha/\varepsilon_1, 0)$. These two trajectories, which are the stable manifolds of the saddles $(0, 0)$ and $(\alpha/\varepsilon_1, 0)$, do not affect the global stability of (λ^*, ρ^*), since they represent a null set in the phase plane.

A second perturbation

We now consider system (8), to prove Propositions 3 and 4. The proofs of these Propositions are possible by looking at how the dynamics change with respect to the one of the Lotka-Volterra model (6).

In fact, under the assumptions of Proposition 3 the growth rate $\hat{\lambda}(\lambda, \rho)$ defined in (4) can be bounded as

$$\hat{\lambda} = \alpha - \varepsilon_1 \lambda - \beta\rho + g_1(\lambda, \rho) < \alpha - \beta\rho \quad \forall \lambda \geq \bar{\lambda}.$$

The obtained growth rate is the one of model (6): this means that starting on a point of the trajectories depicted in Figure 1 with $\lambda \geq \bar{\lambda}$, model (8) has a growth rate of the first variable that is smaller, i.e. the evolution of $\lambda(t)$ is bounded by outside the trajectory of model (6), as shown in the left panel of Figure B.11.

Similarly, under the assumptions of Proposition 4 ($\varepsilon_1 = \varepsilon_2 = 0$), the growth rate $\hat{\lambda}(\lambda, \rho)$ defined in (4) is

$$\hat{\lambda} = \alpha - \beta\rho + g_1(\lambda, \rho).$$

Remembering the assumptions we made on $g_1(\lambda, \rho)$ and the bounds for $\mu_1,\ \mu_2,\ \varphi_1$, and φ_2, it is easy to show that

$$\begin{array}{lllllllll} \hat{\lambda} & = & \alpha - \beta\rho + g_1(\lambda, \rho) & \leq & \alpha - \beta\rho & \text{if} & \alpha - \beta\rho \geq 0 & \text{and} & -\gamma + \delta\lambda \leq 0, \\ \hat{\lambda} & = & \alpha - \beta\rho + g_1(\lambda, \rho) & \geq & \alpha - \beta\rho & \text{if} & \alpha - \beta\rho \leq 0 & \text{and} & -\gamma + \delta\lambda \geq 0. \end{array}$$

This means that $\hat{\lambda}(\lambda, \rho)$ of model (8) may be smaller than the one of model (6) when it is positive, giving to the other variable more time to decrease (since there $\hat{\rho}$ is negative). On the other hand, it may be bigger than the one of model (6) when it is negative, giving to the other variable more time to increase (since there $\hat{\rho}$ is positive). This means that starting on a point of the trajectories depicted in Figure 1, model (8) has trajectories that grow up, thus causing the divergence of the system. This can be easily seen looking at the vector field, depicted in Fig. B.11, right panel, where, for example, is shown that the vector field at the point $\lambda = \mu_1,\ \rho = \varphi_1$ is bent rightward to the one of model (6), causing the trajectory moving toward outside.

Note that Propositions 3 and 4 allow us to conclude that the system, for a suitable choice of the function $g_1(\lambda, \rho)$ and sufficiently small $(\varepsilon_1, \varepsilon_2)$, must have a stable limit cycle, since trajectories must diverge to a level curve of model (6) and cannot indefinitely growth, thus being confined in a closed bounded region in which no equilibria are present, due to the Poincaré-Bendixon theorem (Burton 2005).

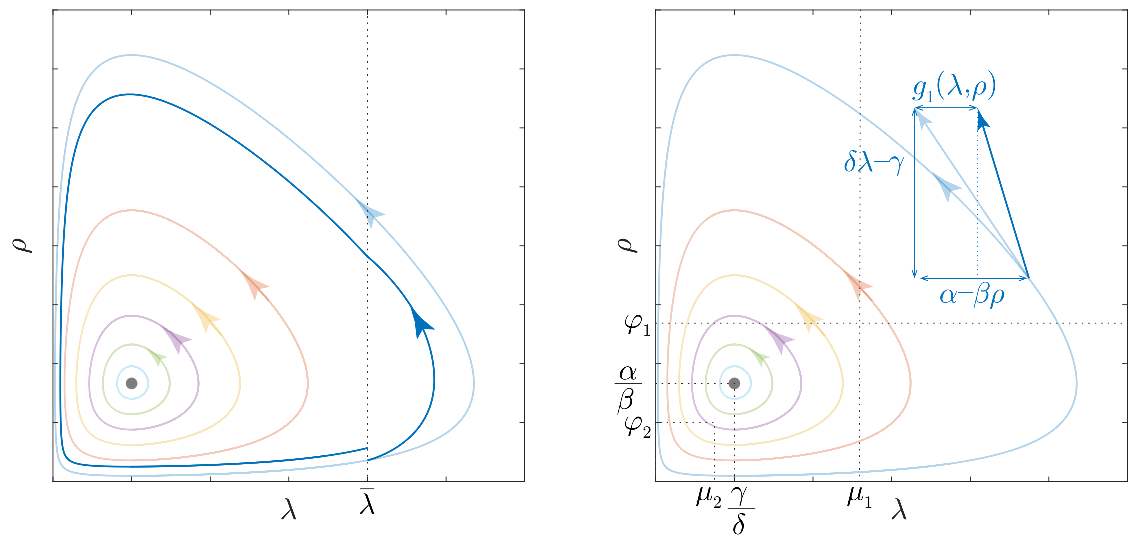

Figure B.11: The panel shows how the state portrait of model (6) is modified by the introduction of the perturbation described in this subsection. In the left panel, it is shown that the fact that $g_1(\lambda,\rho) < \varepsilon_1$ for $\lambda > \bar{\lambda}$ makes the system not diverge (the blue trajectory comes back at the same point in an inner position). In the right panel, it is shown that the perturbation, when $\varepsilon_1 = \varepsilon_2 = 0$, makes instead the system to diverge.

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