Z. Angew. Math. Phys. (2022) 73:218

0044-2275/22/050001-29
published online September 28, 2022
https://doi.org/10.1007/s00033-022-01855-0

Zeitschrift für angewandte Mathematik und Physik ZAMP

Deformation-induced coupling of the generalized external actions in third-gradient materials

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Abstract. In this study, diverse typologies of external actions are outlined, which turn out to be admissible for the third-gradient modeling of elastic materials. It is shown how such loading, when prescribed over the boundary surface, along the border edges and at the wedges of a deformable body in the Eulerian configuration, can be transformed into the Lagrangian description generating multiple interactions, with a surprising deformation-induced coupling. Such a phenomenon becomes more and more important at increasing the order of the β-forces, specified by duality as covectors expending work on the βth normal derivative of the virtual displacements, being herein at most $\beta = 2$. Insights are provided into the true nature of such generalized forces, resting on the differential geometric features of the deformation process.

Mathematics Subject Classification. 74A99, 74B20, 74G99.

Keywords. Continuum mechanics, Third-gradient materials, Principle of virtual work, Nonstandard boundary conditions, Generalized actions, Lagrangian formulation, Eulerian formulation.

1. Introduction

The expression *third-gradient materials* denotes a class of elastic models for continuous bodies, whose deformation energy density depends on the derivatives of the placement map up to the third order see e.g. [1,2]. To govern the inner virtual work for such materials, the double-rank stress tensor is no more sufficient, and hyperstress tensors of order three and four are required, referred to as double and triple stresses, see e.g. [3]. By a variational approach, through iterated applications of the integration by parts and of the divergence theorem extended to submanifolds with boundary [4,5], the inner virtual work, namely the first variation of the deformation energy functional, can be represented by the sum of volume, surface, edge and wedge integrals [6]. In this way, the Cauchy representation theorem for external loads in terms of stress tensor is extended, so that generalized external forces are equated to hyperstress tensors applied to the shape of Cauchy cuts, see [7,8]. Such generalized forces are defined by duality as covectors that expend work on the virtual displacements and on their directional derivatives: up to the second order along the direction of the normal, over the boundary faces, and along the direction of the normal vectors, for the curved edges having codimension two. From the mathematical standpoint, such a problem can be interpreted as the representation of a finite-order distribution through the sum of a finite number of addends, each including function derivatives with increasing order, see e.g. [9,10]. In this way, nonstandard boundary conditions are deduced: a complex expression is met also for the generalized contact pressures over the boundary surface, depending not only on the normal but also on the local curvature and on its square. In this respect, the principle of virtual work turns out to be more powerful and versatile than the postulation of mechanics resting on the balance of forces and of moments of forces.

By the third-gradient modeling, several phenomena of solid and fluid mechanics have been effectively addressed, which are not fully consistent with Cauchy's first gradient theory [11]. Among the others, we can mention size effects depending on characteristic length scales [12,13], dislocation theory [14], corner and surface effects, boundary layers [15,16], reflection and transmission of waves in the presence of discontinuities [17–19], surface tension in fluids [20,21], fabric sheets [22,23], joints and interfaces. Moreover, the present approach is especially suitable to describe metamaterials constituted of a beam lattice [24–27], which in a sense mimic the crystal structure: their properties, depending on the microscopic architecture, can be tailored for the specific applications in civil, mechanical and aerospace engineering. To investigate several of these phenomena, during the last years novel experimental techniques are being utilized, providing high-quality information at the microscale never achieved before, see e.g. [28,29]. For instance, in wettability experiments at high temperatures and under inert atmosphere, droplets of a molten alloy at rest on a ceramic substrate can be monitored "in situ" by X-ray computed microtomography, see [30]: such droplets exhibit a three-dimensional profile and a contact angle which depend on both surface tension and adhesion. During mechanical tests, full-field kinematic measurements can be provided by multiscale digital image correlation procedures, see e.g. [31,32], over the free surface (2D DIC) or within the bulk (3D Volume DIC), as carried out in [33,34] on pantographic metamaterials.

The main feature of the present approach resides in the very high number of constitutive parameters governing the deformation energy density [35]: besides the enriched experimental information mentioned above, it is therefore necessary to develop robust identification strategies to calibrate such parameters at the macroscale, see e.g. [36,37], or through homogenization techniques on the basis of simple assumptions at the lower scales, as proposed in [38–41]. As a further difficulty, the *synthesis* of third-gradient metamaterials still represents a prohibitive issue. In fact, in the literature strategies are not yet available to design a microstructure giving rise, through homogenization, to a mechanical response at the macroscale consistent with assigned third-gradient features [42], solving a sort of nonparametric inverse problem. An attempt in this direction has been carried out with reference to beams constituted of microscopic trusses [24,43]. Moreover, fundamental results concerning existence, uniqueness and stability of third-gradient models have not been yet addressed (see e.g. [44]), differently from second-gradient materials for which a few contributions exist concerning weak solutions [45], well-posedness [46] and strong ellipticity conditions [47,48]. Numerical implementations of third-gradient models based on finite elements and NURBS have been developed, see e.g. [49–51].

In some of his works [52], Gabrio Piola started the study of generalized actions that expend work on the βth normal derivative of the virtual displacements. Notwithstanding this, a debate characterized the modern (re-)discovery of higher-order gradient models involving Mindlin, Sedov, Toupin and Germain among the others on the one side, and Truesdell and his followers on the other, concerning the nature of such "exotic" β-forces and their role in mathematical physics, see e.g. [53–55]. It is true that with extreme difficulty the generalized loading can enter the postulation of mechanics based on the balance of forces and of moments of forces: Some attempts have been made, leading to an extension named "quasi-balanced power" [56]. But it is still more clear that the β-forces appear naturally by duality in the postulation of mechanics based on the principle of virtual work, see e.g. the first attempt by [57] and the more recent contribution [8].

In previous studies, the governing equations for second-gradient continua have been deduced by a truly variational approach, see [58,59], and the external actions were specified consistently. Through some novel results of differential geometry serving as intermediate steps [4], a strategy was developed to transform such second-gradient governing equations from the Eulerian to the Lagrangian description [5,9]. More recently, in [6] the third-gradient equilibrium equations were deduced, endowed by suitable relationships between Lagrangian and Eulerian hyperstress tensors of order lower or equal to four. In the present study, the transport of the external actions from the Eulerian to the Lagrangian form is carried out for the first time with reference to such third-gradient continua. Surprisingly, Eulerian loading of high-order generate multiple Lagrangian actions, with a coupling induced by the deformation process. On the basis of the present results, a few recursive mathematical structures have been identified, useful for further generalizations.

The paper is organized as follows. Section 2 recalls the main assumptions of third-gradient modeling and outlines the admissible external actions. In Sect. 3, the transformation of the external loading is carried out from the Eulerian to the Lagrangian description, finding a surprising coupling of actions, especially for the higher-order terms. An attempt to identify possible recursive structures in the transformation formulae is carried out in Sect. 4. Section 5 is devoted to the practical evaluation of the diverse integral contributions, taking into account the differential geometric structure of the oriented boundary constituted of multiple regular faces with piecewise regular border edges, including in turn a finite number of wedges. Finally, basic properties of surface and edge projectors, the formulation of the divergence theorem extended to submanifolds with boundary, transformation formulae for the edge vectors enriched by some useful corollaries, are made available in three short Appendices, labeled as A, B and C, respectively.

1.1. Notation

In what follows recourse is made to tensor notation, in the classical syntax by Levi Civita and Ricci, and to the Einstein convention on the implicit sum of repeated indices, distinguishing between contravariant and covariant components. Whenever possible, Eulerian quantities are denoted by lowercase letters, and their Lagrangian counterparts by uppercase ones. Consistently, to distinguish valences acting on Lagrangian vectors from those specifying Eulerian components, e.g., as in F^a_A, the former are indicated by uppercase letters, i.e., $A, B, \cdots$, the latter by lowercase ones, $a, b, \cdots$. The Lagrangian gradient is denoted by the symbols $\nabla \equiv \frac{\partial}{\partial X^A}$, with the obvious extension to kth-order gradients as $\nabla^{(k)} = \nabla\, \nabla^{(k-1)}$. The derivative along the direction of the unit vector N^R is denoted as $\frac{\partial}{\partial N} \equiv N^R \frac{\partial}{\partial X^R}$; the Lagrangian divergence operator is seldom indicated by symbol DIV, while symbol $\text{DIV}_{\|}$ is adopted for the surface divergence. With reference to a domain D with dimension i $(i = 3, 2, 1)$, symbol ∂D denotes its differential border, having dimension $(i - 1)$, with the obvious extension to the pth-order border as $\partial^{(p)} D \equiv \partial\, \partial^{(p-1)} D$. The word *action*, widely utilized in the jargon of structural mechanics, is adopted herein as a synonym of loading and is not to be confused with the Hamilton functional.

2. Admissible external actions

Let us consider a three-dimensional body which occupies the volume $\Omega_\star \subset \mathcal{R}^3$ (marked by a star) at a reference instant $t_\star$, and the volume $\Omega \subset \mathcal{R}^3$ at the generic instant $t > t_\star$. The former is usually referred to as the Lagrangian, material or reference configuration, the latter as the Eulerian, spatial or current one. In the Lagrangian configuration we denote the piecewise regular boundary surface, the regular border edges, discontinuity loci for the face normals, and the discrete set of wedges, in turn discontinuity loci for the edge tangent, as differential borders of increasing order starting from the volume occupied by the deformable body, namely $\Sigma_\star \equiv \partial\Omega_\star$, $L_\star \equiv \partial\Sigma_\star \equiv \partial\partial\Omega_\star$, and $P_\star \equiv \partial L_\star \equiv \partial\partial\Sigma_\star \equiv \partial\partial\partial\Omega_\star$, respectively. For the relevant domains in the Eulerian configuration, we adopt the same symbols without the subscript $\star$. All the above domains with different codimensions are oriented consistently with the boundary surface $\Sigma_\star$, starting from the orientation of the normal field N^R (as usual, the normal pointing outwards is assumed as positive). Both the material and the spatial configurations are equipped with an orthonormal vector basis: symbol $\langle \mathbf{a}, \mathbf{b} \rangle_g = g_{rs} a^r b^s$ will denote the inner (scalar) product according to the metric tensor $\mathbf{g}$. If the usual Euclidean norm is utilized, one has $\langle \mathbf{a}, \mathbf{a} \rangle = \|\mathbf{a}\|^2$. Along each border edge $L_\star$ of a boundary face, we can define the Darboux orthonormal moving frame (repère mobile) oriented counterclockwise, constituted of the vectors $\mathbf{T}, \mathbf{N}$ and $\mathbf{B} \equiv \mathbf{T} \wedge \mathbf{N}$, where symbol $\wedge$ denotes the wedge (cross) product. Both the edge tangent $\mathbf{T}$ and the edge normal $\mathbf{B}$ belong to the plane tangent to the face: at each point of the border edge, a rigid rotation around the tangent axis distinguishes such a frame from the Frenet–Serret moving basis.

The deformation process of the body is herein described as a diffeomorphism between the above volumes $\Omega_\star$ and Ω, regarded as submanifolds with boundary [60–62] embedded in the ambient space $\mathcal{R}^3$. Hence, a one-to-one map is specified, continuous and differentiable with a continuous and differentiable inverse, so that each point $\mathbf{x} \in \Omega$ can be uniquely obtained as the image of a point in the Lagrangian configuration, labeled as $\mathbf{X} \in \Omega_\star$, namely $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$. As an alternative, additional kinematic descriptors can be introduced, e.g. see [63]. We will denote as usual the tangent operator as $\mathbf{F} = \partial\boldsymbol{\chi}/\partial\mathbf{X}$, with the assumption $J = \det(\mathbf{F}) > 0$ to prevent matter interpenetration. In third-gradient continua, the deformation energy density is assumed of the form $\mathrm{W}\left(\mathbf{F}, \nabla\mathbf{F}, \nabla^{(2)}\mathbf{F}\right)$, see e.g. [6]. In the postulation of mechanics based on the virtual work principle, the contributions to the external work can be specified only after that the inner virtual work, first variation of the stored energy functional, has been represented in a unique and no more reducible way as the sum of volume, surface, edge and wedge terms. For the third-gradient continua, the contributions to the external virtual work in the Lagrangian description are as follows:

$$
\begin{aligned}
\delta\,\mathcal{E}^{\mathrm{EXT}} = & \int\limits_{\Omega_\star} \mathcal{F}^{\mathrm{ext}}_{\Omega_\star\, i}\, \delta\chi^i \,\mathrm{d}\Omega_\star \ {\scriptstyle (V0)} \\
& + \int\limits_{\Sigma_\star} \mathcal{F}^{\mathrm{ext}}_{\Sigma_\star\, i}\, \delta\chi^i \,\mathrm{d}\Sigma_\star \ {\scriptstyle (S0)} + \int\limits_{\Sigma_\star} \mathcal{F}^{\mathrm{ext}}_{\Sigma_\star N\, i}\, \frac{\partial\,\delta\chi^i}{\partial\, N} \,\mathrm{d}\Sigma_\star \ {\scriptstyle (S1)} + \int\limits_{\Sigma_\star} \mathcal{F}^{\mathrm{ext}}_{\Sigma_\star NN\, i}\, \frac{\partial^2\,\delta\chi^i}{\partial\, N^2} \,\mathrm{d}\Sigma_\star \ {\scriptstyle (S2)} \\
& + \int\limits_{L_\star} \mathcal{F}^{\mathrm{ext}}_{L_\star\, i}\, \delta\chi^i \,\mathrm{d}L_\star \ {\scriptstyle (L0)} + \int\limits_{L_\star} \mathcal{F}^{\mathrm{ext}}_{L_\star N\, i}\, \frac{\partial\,\delta\chi^i}{\partial\, N} \,\mathrm{d}L_\star \ {\scriptstyle (L1)} + \int\limits_{L_\star} \mathcal{F}^{\mathrm{ext}}_{L_\star B\, i}\, \frac{\partial\,\delta\chi^i}{\partial\, B} \,\mathrm{d}L_\star \ {\scriptstyle (L2)} \\
& + \sum_{\mathrm{w}=1}^{\mathrm{ntotwedge}} \mathcal{F}^{\mathrm{ext}}_{P_{\star\,\mathrm{w}}\, i}\, \delta\chi^i(\mathbf{P}_{\star\,\mathrm{w}}) \ {\scriptstyle (W0)}\,;
\end{aligned} \tag{1}
$$

For the sake of clarity, each contribution in Eq. (1) was labeled by a letter, as a volume (V), surface (S), edge (L) and wedge (W) term, followed by a progressive number from 0 to 2. The above symbols possess the following meaning: $\mathcal{F}^{\mathrm{ext}}_{\Omega_\star\, i}(\mathbf{X})$, $\mathcal{F}^{\mathrm{ext}}_{\Sigma_\star\, i}(\mathbf{X})$, $\mathcal{F}^{\mathrm{ext}}_{L_\star\, i}(\mathbf{X})$ and $\mathcal{F}^{\mathrm{ext}}_{P_{\star\,\mathrm{w}}\, i}$ denote Eulerian vectors defined in the Lagrangian domain ($\Omega_\star$), over its boundary face ($\Sigma_\star$), along the its edges ($L_\star$) and at relevant wedges ($\{P_{\star\,\mathrm{w}}\}$ with coordinates $\mathbf{P}_{\star\,\mathrm{w}}$, $\mathrm{w} = 1, \cdots, \mathrm{ntotwedge}$), dimensionally equal to force densities per unit volume, per unit surface, per unit length and to a point force, respectively; $\mathcal{F}^{\mathrm{ext}}_{\Sigma_\star N\, i}(\mathbf{X})$ and $\mathcal{F}^{\mathrm{ext}}_{\Sigma_\star NN\, i}(\mathbf{X})$ indicate Eulerian vector fields defined over the Lagrangian boundary surfaces, referred to as external (surface) double and triple force densities, respectively, dimensionally equal to a force per unit surface multiplied by a length (or a work per unit surface), and to a force per unit surface multiplied by a length squared; symbols $\mathcal{F}^{\mathrm{ext}}_{L_\star N\, i}(\mathbf{X})$ and $\mathcal{F}^{\mathrm{ext}}_{L_\star B\, i}(\mathbf{X})$ indicate Eulerian vector fields defined over the Lagrangian boundary edge, referred to as external (edge) double force densities, dimensionally equal to a force. It can be proven that, selecting suitable pairs of work conjugate variables, the governing equations exhibit the same mathematical form in the Lagrangian and in the Eulerian configuration, see [5,6]. Hence, the expression of the external virtual work results to be the same, on condition of substituting the Lagrangian variables and the integration domains with their Eulerian counterparts (without the subscript $\star$). As detailed in the next sections, the Eulerian loading will be denoted by lowercase symbols, maintaining the same convention of Lagrangian actions, for instance $f^{\mathrm{ext}}_{\Omega\, i}$ will indicate the Eulerian counterpart of the volume loading density $\mathcal{F}^{\mathrm{ext}}_{\Omega_\star\, i}$, $f^{\mathrm{ext}}_{\Sigma\, i}$ will be used in the spatial configuration for the surface loading density $\mathcal{F}^{\mathrm{ext}}_{\Sigma_\star\, i}$, etc .

Equation (1), deduced by a variational approach in [6], must be regarded as a particularization of the general representation formula for the external virtual work, provided in [7] for the Hth gradient material occupying the volume $\Omega_\star$, namely

$$\delta\,\mathcal{E}^{\mathrm{EXT}} = \int\limits_{\Omega_\star} \mathcal{F}_{\Omega_\star\,i}\;\delta\chi^i\,\mathrm{d}\mathcal{H}^3 + \sum_{\beta=0}^{H-1}\int\limits_{\partial\Omega_\star} \mathcal{F}^{(\beta)}_{\partial\Omega_\star\,i}\,\frac{\partial^{(\beta)}\delta\chi^i}{\partial N^{(\beta)}}\,\mathrm{d}\mathcal{H}^2 + \sum_{\beta=0}^{H-2}\int\limits_{\partial\partial\Omega_\star} \mathcal{F}^{(\beta)}_{\partial\partial\Omega_\star\,i}\,\frac{\partial^{(\beta)}\delta\chi^i}{\partial N^{(\beta)}}\,\mathrm{d}\mathcal{H}^1 + \sum_{\beta=0}^{H-3}\int\limits_{\partial\partial\partial\Omega_\star} \mathcal{F}^{(\beta)}_{\partial\partial\partial\Omega_\star\,i}\,\frac{\partial^{(\beta)}\delta\chi^i}{\partial N^{(\beta)}}\,\mathrm{d}\mathcal{H}^0\,; \tag{2}$$

where symbols have the following meaning: $\mathrm{d}\mathcal{H}^{(3-p)}$ indicates a $(3-p)$-dimensional Hausdorff measure over the pth differential border of $\Omega_\star$, i.e., $\partial^{(p)}\Omega_\star$, being $\partial^{(0)}\Omega_\star \equiv \Omega_\star$ and $\mathrm{d}\mathcal{H}^0$ a measure on the discrete set of wedges; $\frac{\partial^{(\beta)}\delta\chi^i}{\partial N^{(\beta)}}$ denotes the βth normal derivative of the virtual displacements, which for $\beta = 0$ retrieves the virtual displacements themselves; vector $\mathcal{F}_{\Omega_\star\,i}$ denotes a long-range force density, while vectors $\mathcal{F}^{(\beta)}_{\partial\cdots\partial\Omega_\star\,i}$ indicate contact interaction densities of (β)th order prescribed over domains with different codimensions. For the present case with $H = 3$, we can notice that over the surface contributions including up to the second normal derivative are expected, while along the border edge we attain at most the first normal derivative but along two different "normal" directions, since in this case the submanifold exhibits codimension two. Finally, at the wedges, only terms expending work on the virtual displacements are found. For the sake of clarity, in what follows symbols utilized in Eq. (1) will be preferred.

3. Loading transformation

3.1. Volume forces

The Eulerian volume loading labeled as $(V0)$, expending work on the virtual displacement vector, can be transformed into its Lagrangian counterpart as follows:

$$\int\limits_{\Omega} f^{\mathrm{ext}}_{\Omega\,i}\,\delta\chi^i\,\mathrm{d}\Omega = \int\limits_{\Omega_\star} \mathrm{d}\Omega_\star\,J\,f^{\mathrm{ext}}_{\Omega\,i}\circ\boldsymbol{\chi}\;\delta\chi^i\,; \tag{3}$$

where $J = \det(\mathbf{F}) = \mathrm{d}\Omega/\mathrm{d}\Omega_\star$ denotes the Jacobian determinant of the placement map regarded as a diffeomorphism between the material and the spatial configuration. Symbol $\circ$ indicates the composition of functions, so that $f^{\mathrm{ext}}_{\Omega\,i}\circ\boldsymbol{\chi}(\mathbf{X})$ is defined in the Lagrangian configuration: This change of variables for the Eulerian loading will be assumed implicitly in what follows. The present Eulerian term gives rise exclusively to the Lagrangian action $\mathcal{F}^{\mathrm{ext}}_{\Omega_\star\,i} = J\,f^{\mathrm{ext}}_{\Omega\,i}$: This transformation coincides with that occurring for Cauchy's materials.

3.2. Contact pressures

The Eulerian surface loading labeled as $(S0)$, expending work on the virtual displacements (retrieved in Eq. 2 by setting $\beta = 0$ over $\partial\Omega_\star$), can be transformed into its Lagrangian counterpart through the transformation rule of the area elements [4,58,60], namely $\mathrm{d}\Sigma = \|J\,\mathbf{F}^{-T}\mathbf{N}\|\,\mathrm{d}\Sigma_\star$, obtaining

$$\int\limits_{\Sigma} f^{\mathrm{ext}}_{\Sigma\,i}\,\delta\chi^i\,\mathrm{d}\Sigma = \int\limits_{\Sigma_\star} \mathrm{d}\Sigma_\star\|J\,\mathbf{F}^{-T}\mathbf{N}\|\,f^{\mathrm{ext}}_{\Sigma\,i}\,\delta\chi^i\,; \tag{4}$$

This term gives rise uniquely to the Lagrangian action $\mathcal{F}^{\mathrm{ext}}_{\Sigma_\star\,i} = \|J\,\mathbf{F}^{-T}\mathbf{N}\|\,f^{\mathrm{ext}}_{\Sigma\,i}$.

3.3. Double force

Now let us focus on the Eulerian double force ($S1$), expending work on the normal derivative of the virtual displacement vector. This action can be retrieved in Eq. 2 by setting $\beta = 1$ over $\partial\Omega$. By the usual transformation of the area element and of the covariant normal n_s when passing from the Eulerian to the Lagrangian configuration [4,58] (see "Appendix C"), we obtain

$$\begin{aligned}
\int\limits_{\Sigma} f^{\text{ext}}_{\Sigma n\, i} \frac{\partial\, \delta\chi^i}{\partial\, n}\, \mathrm{d}\Sigma &= \int\limits_{\Sigma_\star} \|J\mathbf{F}^{-T}\mathbf{N}\|\; f^{\text{ext}}_{\Sigma n\, i}\; g^{rs} n_s \frac{\partial\, \delta\chi^i}{\partial\, x^r}\, \mathrm{d}\Sigma_\star \\
&= \int\limits_{\Sigma_\star} \|J\mathbf{F}^{-T}\mathbf{N}\|\; f^{\text{ext}}_{\Sigma n\, i}\; g^{rs} \frac{\left(\mathbf{F}^{-1}\right)^{Q}_{s} N_Q}{\|\mathbf{F}^{-T}\mathbf{N}\|} \left(\mathbf{F}^{-1}\right)^{R}_{r} \frac{\partial\, \delta\chi^i}{\partial\, X^R}\, \mathrm{d}\Sigma_\star \\
&= \int\limits_{\Sigma_\star} J f^{\text{ext}}_{\Sigma n\, i}\, g^{\star\, QR} N_Q \frac{\partial\, \delta\chi^i}{\partial\, X^R}\, \mathrm{d}\Sigma_\star
\end{aligned} \tag{5}$$

Symbol $g^{\star\, RS} \equiv g^{rs} \left(\mathbf{F}^{-1}\right)^{R}_{r} \left(\mathbf{F}^{-1}\right)^{S}_{s}$ denotes the *pullback* metric tensor in doubly contravariant form (also indicated by the musical symbol $\mathbf{g}^{\star\,\sharp}$), having the property $g^{\star\, RQ} N_R N_Q = \langle \mathbf{N}, \mathbf{N}\rangle_{\mathbf{g}^{\star\,\sharp}} = \|\mathbf{F}^{-T}\mathbf{N}\|^2$, see [4,58,60]. For the sake of simplicity, in what follows we will denote by $N^{\star R} \equiv g^{\star\, RS} N_S$ the contravariant form of the normal vector according to the pullback metrics, for which in general one has $N^{\star R} \neq N^R$. Making recourse to the additive decomposition of the gradient by the complementary surface projectors $[M_\perp]^S_R = N^S N_R$ and $[M_\parallel]^S_R = \delta^S_R - [M_\perp]^S_R$ (see "Appendix A" and [59]), exploiting idempotence and integration by parts one can write

$$\begin{aligned}
&= + \int\limits_{\Sigma_\star} J f^{\text{ext}}_{\Sigma n\, i}\, N^{\star\, R} \underbrace{\left([M_\perp]^S_R + [M_\parallel]^S_R\right)}_{=\delta^S_R} \frac{\partial\, \delta\chi^i}{\partial\, X^S}\, \mathrm{d}\Sigma_\star \\
&= + \int\limits_{\Sigma_\star} J f^{\text{ext}}_{\Sigma n\, i}\; \underbrace{N^{\star R} N_R}_{=\|\mathbf{F}^{-T}\mathbf{N}\|^2}\; N^S \frac{\partial\, \delta\chi^i}{\partial\, X^S}\, \mathrm{d}\Sigma_\star \\
&\quad + \int\limits_{\Sigma_\star} \mathrm{d}\Sigma_\star\, [M_\parallel]^S_{R'} \frac{\partial}{\partial\, X^S} \left\{ J f^{\text{ext}}_{\Sigma n\, i}\, N^{\star R}\, \delta\chi^i\, [M_\parallel]^{R'}_R \right\} \\
&\quad - \int\limits_{\Sigma_\star} \mathrm{d}\Sigma_\star\, [M_\parallel]^S_{R'} \frac{\partial}{\partial\, X^S} \left\{ J f^{\text{ext}}_{\Sigma n\, i}\, N^{\star R} [M_\parallel]^{R'}_R \right\} \delta\chi^i \\
&= \boxed{+ \int\limits_{\Sigma_\star} \|\mathbf{F}^{-T}\mathbf{N}\|^2\, J f^{\text{ext}}_{\Sigma n\, i} \frac{\partial\, \delta\chi^i}{\partial\, N}\, \mathrm{d}\Sigma_\star} \\
&\quad \boxed{+ \int\limits_{L_\star} \mathrm{d}L_\star\, J f^{\text{ext}}_{\Sigma n\, i} \left(B_R N^{\star R}\right) \delta\chi^i} \\
&\quad \boxed{- \int\limits_{\Sigma_\star} \mathrm{d}\Sigma_\star\, [M_\parallel]^S_{R'} \frac{\partial}{\partial\, X^S} \left\{ J f^{\text{ext}}_{\Sigma n\, i}\, N^{\star R}\, [M_\parallel]^{R'}_R \right\} \delta\chi^i};
\end{aligned} \tag{6}$$

where the boxes graphically emphasize the irreducible terms. The divergence theorem for submanifolds with boundary allowed us to transport the second addend from the boundary surface $\Sigma_\star$ to the border edge $L_\star$, whose outward unit normal is denoted by B^R (see "Appendix B" and [5,58,64] for more details).

It is worth emphasizing that the Eulerian double force $f^{\text{ext}}_{\Sigma n\, i}$ gives rise in the Lagrangian configuration to a double force (after multiplication by a local scaling factor), to surface contact pressures, via the surface divergence operator, and to an edge force.

Lagrangian actions generated by the Eulerian force $f^{\text{ext}}_{\Sigma n\, i}$ (S1)

$$\begin{aligned}\mathcal{F}^{\text{ext}}_{\Sigma_\star N\, i} &= \|\mathbf{F}^{-T}\mathbf{N}\|^2\, J f^{\text{ext}}_{\Sigma n\, i}\,; \\ \mathcal{F}^{\text{ext}}_{\Sigma_\star\, i} &= -\,[M_\parallel]^S_{R'}\,\frac{\partial}{\partial X^S}\left\{J f^{\text{ext}}_{\Sigma n\, i}\, N^{\star R}\,[M_\parallel]^{R'}_R\right\}\,; \\ \mathcal{F}^{\text{ext}}_{L_\star\, i} &= J f^{\text{ext}}_{\Sigma n\, i}\,\left(B_R N^{\star R}\right)\,;\end{aligned} \tag{7}$$

It is worth noting that in Eq. (7) the coupling of the Lagrangian actions $\mathcal{F}^{\text{ext}}_{\Sigma_\star\, i}$ and $\mathcal{F}^{\text{ext}}_{L_\star\, i}$ with $\mathcal{F}^{\text{ext}}_{\Sigma_\star N\, i}$ is induced by the deformation process: in fact, the contact pressures over the surface and the edge loading include in turn the products $N^{\star R}\,[M_\parallel]^{R'}_R$ and $B_R N^{\star R}$, which vanish when the deformation gradient tends to the unit tensor, namely when $F^a_A \to \delta^a_A$. In that scenario, the pulled-back normal $N^{\star R}$ is still normal to the tangent plane (and hence to the border edge normal), and $\mathcal{F}^{\text{ext}}_{\Sigma_\star N\, i}$ remains the only Lagrangian counterpart of the Eulerian double force $f^{\text{ext}}_{\Sigma n\, i}$. The surprising fact (and in a sense counterintuitive) is that the double force, generating work on the normal derivative of the virtual displacements over the boundary surface, in the deformation process gives rise also to contact pressures over the same face through the surface divergence operator, generating work on the virtual displacements, and can be transported to the border edge like any other surface force with a tangential component.

3.4. Triple force

Let us analyze the Eulerian triple force $f^{\text{ext}}_{\Sigma nn\, i}$ labeled as $(S2)$, generating work on the second normal derivative of the virtual displacements. This action can be retrieved in Eq. 2 by setting $\beta = 2$ over $\partial\Omega$. Considering the transformation formula for the covariant normal n_r, by permuting the partial derivatives according to the Schwarz' theorem one finds

$$\begin{aligned}\int_\Sigma f^{\text{ext}}_{\Sigma NN\, i}\,\frac{\partial^2\,\delta\chi^i}{\partial\, n^2}\,\mathrm{d}\Sigma &= \int_{\Sigma_\star}\mathrm{d}\Sigma_\star\|J\,\mathbf{F}^{-T}\mathbf{N}\|\; f^{\text{ext}}_{\Sigma nni}\; n^r n^s\,\frac{\partial^2\,\delta\chi^i}{\partial\, x^r\partial\, x^s} \\ &= \int_{\Sigma_\star}\mathrm{d}\Sigma_\star\|J\,\mathbf{F}^{-T}\mathbf{N}\|\; f^{\text{ext}}_{\Sigma nn\, i}\, g^{rt}\,\frac{\left(\mathbf{F}^{-1}\right)^Q_t N_Q}{\|\mathbf{F}^{-T}\mathbf{N}\|}\,g^{sv}\,\frac{\left(\mathbf{F}^{-1}\right)^V_v N_V}{\|\mathbf{F}^{-T}\mathbf{N}\|}\,\left(\mathbf{F}^{-1}\right)^R_r\left(\mathbf{F}^{-1}\right)^S_s\,\frac{\partial^2\,\delta\chi^i}{\partial\, X^R\partial\, X^S} \\ &= \int_{\Sigma_\star}\mathrm{d}\Sigma_\star\,\frac{J\, f^{\text{ext}}_{\Sigma nn\, i}}{\|\mathbf{F}^{-T}\mathbf{N}\|}\; N^{\star\, R}\, N^{\star\, S}\,\frac{\partial^2\,\delta\chi^i}{\partial\, X^R\partial\, X^S}\end{aligned} \tag{7}$$

where we have set $N^{\star\, R} = g^{\star\, RS} N_S$, being $g^{\star\, RS} = g^{rs}\left(\mathbf{F}^{-1}\right)^R_r\left(\mathbf{F}^{-1}\right)^S_s$ the pullback metric tensor. By the additive decomposition of the gradient through the complementary surface projectors $\mathbf{M}_\perp = \mathbf{N}\otimes\mathbf{N}$ and $\mathbf{M}_\parallel = \mathbf{1} - \mathbf{M}_\perp$ (see "Appendix A" and [4,59]), one finds

$$\begin{aligned}&= \int_{\Sigma_\star}\mathrm{d}\Sigma_\star\,\frac{J\, f^{\text{ext}}_{\Sigma nn\, i}}{\|\mathbf{F}^{-T}\mathbf{N}\|}\; N^{\star\, R}\, N^{\star\, S}\left([M_\perp]^A_R + [M_\parallel]^A_R\right)\frac{\partial^2\,\delta\chi^i}{\partial\, X^A\partial\, X^S} \\ &= \int_{\Sigma_\star}\mathrm{d}\Sigma_\star\,\frac{J\, f^{\text{ext}}_{\Sigma nn\, i}}{\|\mathbf{F}^{-T}\mathbf{N}\|}\;\underbrace{\left(N_R N^{\star\, R}\right)}_{=\|\mathbf{F}^{-T}\mathbf{N}\|^2}\, N^{\star\, S} N^A\,\frac{\partial^2\,\delta\chi^i}{\partial\, X^A\partial\, X^S}\end{aligned}$$

$$+\int_{\Sigma_\star} \mathrm{d}\Sigma_\star\, [M_\parallel]^A_{R'} \frac{\partial}{\partial X^A} \left\{ \frac{J\, f^{\mathrm{ext}}_{\Sigma nn\, i}}{\|\mathbf{F}^{-T}\mathbf{N}\|}\; N^{\star\, R}\, N^{\star\, S}\, \frac{\partial\, \delta\chi^i}{\partial\, X^S}\, [M_\parallel]^{R'}_R \right\}$$

$$-\int_{\Sigma_\star} \mathrm{d}\Sigma_\star\, [M_\parallel]^A_{R'} \frac{\partial}{\partial X^A} \left\{ \frac{J\, f^{\mathrm{ext}}_{\Sigma nn\, i}}{\|\mathbf{F}^{-T}\mathbf{N}\|}\; N^{\star\, R}\, N^{\star\, S}\, [M_\parallel]^{R'}_R \right\} \frac{\partial\, \delta\chi^i}{\partial\, X^S} \tag{8}$$

In the last equality, we exploited the idempotence of the tangential projector, namely $[M_\parallel]^A_R = [M_\parallel]^A_{R'}\, [M_\parallel]^{R'}_R$ (see "Appendix A"), and the integration by parts before applying the divergence theorem. One has

$$= \int_{\Sigma_\star} \mathrm{d}\Sigma_\star\, \|\mathbf{F}^{-T}\mathbf{N}\| J\, f^{\mathrm{ext}}_{\Sigma nn\, i}\, N^{\star\, S} N^A \frac{\partial^2\, \delta\chi^i}{\partial\, X^S \partial\, X^A}$$

$$+\int_{L_\star} \mathrm{d}L_\star\, \frac{J\, f^{\mathrm{ext}}_{\Sigma nn\, i}}{\|\mathbf{F}^{-T}\mathbf{N}\|} \left(B_R N^{\star\, R}\right) N^{\star\, S} \frac{\partial\, \delta\chi^i}{\partial\, X^S}$$

$$-\int_{\Sigma_\star} \mathrm{d}\Sigma_\star\, [M_\parallel]^A_{R'} \frac{\partial}{\partial X^A} \left\{ \frac{J\, f^{\mathrm{ext}}_{\Sigma nn\, i}}{\|\mathbf{F}^{-T}\mathbf{N}\|}\; N^{\star\, R}\, N^{\star\, S}\, [M_\parallel]^{R'}_R \right\} \frac{\partial\, \delta\chi^i}{\partial\, X^S}\,; \tag{9}$$

The procedure carried out to reduce the first partial derivative of the virtual displacements can be iterated for the second partial derivative by virtue of the Schwarz' theorem. Let us analyze separately the three addends in Eq. (9) at next points (i)–(iii).

(i) For the first contribution, one finds

$$\int_{\Sigma_\star} \mathrm{d}\Sigma_\star\, \|\mathbf{F}^{-T}\mathbf{N}\| J\, f^{\mathrm{ext}}_{\Sigma nn\, i}\, N^{\star\, S} N^A \left([M_\perp]^L_S + [M_\parallel]^L_S\right) \frac{\partial^2\, \delta\chi^i}{\partial\, X^L \partial\, X^A}$$

$$= \int_{\Sigma_\star} \mathrm{d}\Sigma_\star\, \|\mathbf{F}^{-T}\mathbf{N}\| J\, f^{\mathrm{ext}}_{\Sigma nn\, i}\, \underbrace{\left(N^{\star\, S} N_S\right)}_{=\|\mathbf{F}^{-T}\mathbf{N}\|^2} N^A N^L \frac{\partial^2\, \delta\chi^i}{\partial\, X^L \partial\, X^A}$$

$$+\int_{\Sigma_\star} \mathrm{d}\Sigma_\star\, \|\mathbf{F}^{-T}\mathbf{N}\| J\, f^{\mathrm{ext}}_{\Sigma nn\, i}\, N^{\star\, S} N^A\, [M_\parallel]^L_{S'}\, [M_\parallel]^{S'}_S \frac{\partial^2\, \delta\chi^i}{\partial\, X^L \partial\, X^A}$$

$$= \int_{\Sigma_\star} \mathrm{d}\Sigma_\star\, \|\mathbf{F}^{-T}\mathbf{N}\|^3 J\, f^{\mathrm{ext}}_{\Sigma nn\, i} \frac{\partial^2\, \delta\chi^i}{\partial\, N^2}$$

$$+\int_{\Sigma_\star} \mathrm{d}\Sigma_\star\, [M_\parallel]^L_{S'} \frac{\partial}{\partial\, X^L} \left\{ \|\mathbf{F}^{-T}\mathbf{N}\| J\, f^{\mathrm{ext}}_{\Sigma nn\, i}\, N^{\star\, S} N^A\, [M_\parallel]^{S'}_S \frac{\partial\, \delta\chi^i}{\partial\, X^A} \right\}$$

$$-\int_{\Sigma_\star} \mathrm{d}\Sigma_\star\, [M_\parallel]^L_{S'} \frac{\partial}{\partial\, X^L} \left\{ \|\mathbf{F}^{-T}\mathbf{N}\| J\, f^{\mathrm{ext}}_{\Sigma nn\, i}\, N^{\star\, S} N^A\, [M_\parallel]^{S'}_S \right\} \frac{\partial\, \delta\chi^i}{\partial\, X^A} \tag{10}$$

In the first addend of the last equality, we can recognize the Lagrangian second derivative along the direction of the surface normal. Reiterating the application of the divergence theorem, one can write

$$= \boxed{\int_{\Sigma_\star} \mathrm{d}\Sigma_\star\, \|\mathbf{F}^{-T}\mathbf{N}\|^3 J\, f^{\mathrm{ext}}_{\Sigma nn\, i} \frac{\partial^2\, \delta\chi^i}{\partial\, N^2}}$$

$$\boxed{+\int_{L_\star} \mathrm{d}L_\star\, \|\mathbf{F}^{-T}\mathbf{N}\| J\, f^{\mathrm{ext}}_{\Sigma nn\, i} \left(B_S N^{\star\, S}\right) \frac{\partial\, \delta\chi^i}{\partial\, N}}$$

$$\underbrace{-\int\limits_{\Sigma_\star} \mathrm{d}\Sigma_\star\, [M_\|]^L_{S'} \frac{\partial}{\partial X^L} \left\{ \|\mathbf{F}^{-T}\mathbf{N}\| J\, f^{\mathrm{ext}}_{\Sigma nn\, i}\, N^{\star\, S} N^A\, [M_\|]^{S'}_S \right\} \frac{\partial\, \delta\chi^i}{\partial\, X^A}}_{=(\diamond)} \tag{11}$$

The first two addends above, concerning the boundary surface $\Sigma_\star$ and the border edge $L_\star$, are now complete: a box surrounding the relevant contributions graphically emphasizes this circumstance. The last addend can be further reduced through the additive decomposition of the gradient resting on surface projectors, namely

$$\begin{aligned}
(\diamond) = &-\int\limits_{\Sigma_\star} \mathrm{d}\Sigma_\star\, [M_\|]^L_{S'} \frac{\partial}{\partial X^L} \left\{ \|\mathbf{F}^{-T}\mathbf{N}\| J\, f^{\mathrm{ext}}_{\Sigma nn\, i}\, N^{\star\, S} N^A\, [M_\|]^{S'}_S \right\} \underbrace{N^Q N_A}_{=[M_\perp]^Q_A} \frac{\partial\, \delta\chi^i}{\partial\, X^Q} \\
&-\int\limits_{\Sigma_\star} \mathrm{d}\Sigma_\star\, [M_\|]^L_{S'} \frac{\partial}{\partial X^L} \left\{ \|\mathbf{F}^{-T}\mathbf{N}\| J\, f^{\mathrm{ext}}_{\Sigma nn\, i}\, N^{\star\, S} N^A\, [M_\|]^{S'}_S \right\} \underbrace{[M_\|]^Q_{A'} [M_\|]^{A'}_A}_{=[M_\|]^Q_A} \frac{\partial\, \delta\chi^i}{\partial\, X^Q} \\
= &-\int\limits_{\Sigma_\star} \mathrm{d}\Sigma_\star\, [M_\|]^L_{S'} \frac{\partial}{\partial X^L} \left\{ \|\mathbf{F}^{-T}\mathbf{N}\| J\, f^{\mathrm{ext}}_{\Sigma nn\, i}\, N^{\star\, S} N^A\, [M_\|]^{S'}_S \right\} N_A \frac{\partial\, \delta\chi^i}{\partial\, N} \\
&-\int\limits_{\Sigma_\star} \mathrm{d}\Sigma_\star\, [M_\|]^Q_{A'} \frac{\partial}{\partial X^Q} \left\{ [M_\|]^L_{S'} \frac{\partial}{\partial X^L} \left(\|\mathbf{F}^{-T}\mathbf{N}\| J\, f^{\mathrm{ext}}_{\Sigma nn\, i}\, N^{\star\, S} N^A\, [M_\|]^{S'}_S \right) [M_\|]^{A'}_A\, \delta\chi^i \right\} \\
&+\int\limits_{\Sigma_\star} \mathrm{d}\Sigma_\star\, [M_\|]^Q_{A'} \frac{\partial}{\partial X^Q} \left\{ [M_\|]^L_{S'} \frac{\partial}{\partial X^L} \left(\|\mathbf{F}^{-T}\mathbf{N}\| J\, f^{\mathrm{ext}}_{\Sigma nn\, i}\, N^{\star\, S} N^A\, [M_\|]^{S'}_S \right) [M_\|]^{A'}_A \right\} \delta\chi^i
\end{aligned} \tag{12}$$

where the addend with the tangential projector was integrated by parts. Hence, through the divergence theorem (see "Appendix B") we attain the complete expressions

$$\begin{aligned}
= &\boxed{-\int\limits_{\Sigma_\star} \mathrm{d}\Sigma_\star\, [M_\|]^L_{S'} \frac{\partial}{\partial X^L} \left(\|\mathbf{F}^{-T}\mathbf{N}\| J\, f^{\mathrm{ext}}_{\Sigma nn\, i}\, N^{\star\, S} N^A\, [M_\|]^{S'}_S \right) N_A \frac{\partial\, \delta\chi^i}{\partial\, N}} \\
&\boxed{-\int\limits_{L_\star} \mathrm{d}L_\star\, [M_\|]^L_{S'} \frac{\partial}{\partial X^L} \left(\|\mathbf{F}^{-T}\mathbf{N}\| J\, f^{\mathrm{ext}}_{\Sigma nn\, i}\, N^{\star\, S} N^A\, [M_\|]^{S'}_S \right) B_A\, \delta\chi^i} \\
&\boxed{+\int\limits_{\Sigma_\star} \mathrm{d}\Sigma_\star\, [M_\|]^Q_{A'} \frac{\partial}{\partial X^Q} \left\{ [M_\|]^L_{S'} \frac{\partial}{\partial X^L} \left(\|\mathbf{F}^{-T}\mathbf{N}\| J\, f^{\mathrm{ext}}_{\Sigma nn\, i}\, N^{\star\, S} N^A\, [M_\|]^{S'}_S \right) [M_\|]^{A'}_A \right\} \delta\chi^i};
\end{aligned} \tag{13}$$

In the first two addends, a surface and an edge integral, we recognize the surface divergence operator, namely $\mathrm{DIV}_\|(\mathbf{v}_\|)$, with one additional valence contracted with the face normal N_A for the former, and with the edge normal B_A for the latter. The covector in the third addend represents indeed a double surface divergence operator, namely $\mathrm{DIV}_\| \left\{ \left[\mathrm{DIV}_\|(\mathbf{v}_\|)\right]_\| \right\}$, requiring the argument $\mathbf{v}$ to possess two free valences (at least).

(ii) For reducing the second addend in Eq. (9), we utilize complementary edge projectors (marked by subscript L), tangential and normal (wrt to the border edge), namely $\mathbf{M}_{L\,\|} = \mathbf{T} \otimes \mathbf{T}$ and $\mathbf{M}_{L\,\perp} = \mathbf{B} \otimes \mathbf{B} + \mathbf{N} \otimes \mathbf{N}$, respectively, resulting hence $\mathbf{M}_{L\,\perp} + \mathbf{M}_{L\,\|} = \mathbf{1}$. As expected, the border edge has codimension two and the orthogonal complement of the tangent space is spanned by two normals, $\mathbf{B}$ and $\mathbf{N}$. Hence, one finds

$$
\begin{aligned}
&+\int_{L_\star} \mathrm{d}L_\star \frac{J\, f^{\mathrm{ext}}_{\Sigma nn\, i}}{\|\mathbf{F}^{-T}\mathbf{N}\|} \left(B_R N^{\star\, R}\right) N^{\star\, S} \underbrace{\left([M_{L\,\perp}]^W_S + [M_{L\,\|}]^W_S\right)}_{=\delta^W_S} \frac{\partial\, \delta\chi^i}{\partial\, X^W} \\
&= + \int_{L_\star} \mathrm{d}L_\star \frac{J\, f^{\mathrm{ext}}_{\Sigma nn\, i}}{\|\mathbf{F}^{-T}\mathbf{N}\|} \left(B_R N^{\star\, R}\right) \underbrace{N^{\star\, S} N_S}_{=\|\mathbf{F}^{-T}\mathbf{N}\|^2} N^W \frac{\partial\, \delta\chi^i}{\partial\, X^W} + \\
&+ \int_{L_\star} \mathrm{d}L_\star \frac{J\, f^{\mathrm{ext}}_{\Sigma nn\, i}}{\|\mathbf{F}^{-T}\mathbf{N}\|} \left(B_R N^{\star\, R}\right) N^{\star\, S} B^W B_S \frac{\partial\, \delta\chi^i}{\partial\, X^W} \\
&+ \int_{L_\star} \mathrm{d}L_\star [M_{L\,\|}]^W_{S'} \frac{\partial}{\partial\, X^W} \left\{ \frac{J\, f^{\mathrm{ext}}_{\Sigma nn\, i}}{\|\mathbf{F}^{-T}\mathbf{N}\|} \left(B_R N^{\star\, R}\right) N^{\star\, S} [M_{L\,\|}]^{S'}_S\, \delta\chi^i \right\} \\
&- \int_{L_\star} \mathrm{d}L_\star [M_{L\,\|}]^W_{S'} \frac{\partial}{\partial\, X^W} \left\{ \frac{J\, f^{\mathrm{ext}}_{\Sigma nn\, i}}{\|\mathbf{F}^{-T}\mathbf{N}\|} \left(B_R N^{\star\, R}\right) N^{\star\, S} [M_{L\,\|}]^{S'}_S \right\} \delta\chi^i
\end{aligned}
\tag{14}
$$

Rearranging terms and applying the divergence theorem to the third addend, we obtain

$$
\begin{aligned}
= & \boxed{+ \int_{L_\star} \mathrm{d}L_\star \|\mathbf{F}^{-T}\mathbf{N}\| J\, f^{\mathrm{ext}}_{\Sigma nn\, i} \left(B_R N^{\star\, R}\right) \frac{\partial\, \delta\chi^i}{\partial\, N}} \\
& \boxed{+ \int_{L_\star} \mathrm{d}L_\star \frac{J\, f^{\mathrm{ext}}_{\Sigma nn\, i}}{\|\mathbf{F}^{-T}\mathbf{N}\|} \left(B_R N^{\star\, R}\right) \left(B_S N^{\star\, S}\right) \frac{\partial\, \delta\chi^i}{\partial\, B}} \\
& \boxed{+ \sum_{\mathrm{e}=1}^{\mathrm{ntotedge}} \left[\frac{J\, f^{\mathrm{ext}}_{\Sigma nn\, i}}{\|\mathbf{F}^{-T}\mathbf{N}\|} \left(B_R N^{\star\, R}\right) \left(T_S N^{\star\, S}\right) \delta\chi^i \right]^{P_{\star\, w_{\mathrm{e}}(2)}}_{P_{\star\, w_{\mathrm{e}}(1)}}} + \\
& \boxed{- \int_{L_\star} \mathrm{d}L_\star T^W T_{S'} \frac{\partial}{\partial\, X^W} \left\{ \frac{J\, f^{\mathrm{ext}}_{\Sigma nn\, i}}{\|\mathbf{F}^{-T}\mathbf{N}\|} \left(B_R N^{\star\, R}\right) \left(T_S N^{\star\, S}\right) T^{S'} \right\} \delta\chi^i};
\end{aligned}
\tag{15}
$$

Symbol $[t]^b_a \equiv t(b) - t(a)$ possesses the usual meaning derived from the fundamental theorem of calculus: herein, along each curved branch of the oriented face border, the function within the square parentheses must be evaluated at the ends, coinciding with two wedges, before computing the above difference. A suitable connectivity matrix can be used to specify in the global array the indices of the wedge pair belonging to the eth oriented edge, say $P_{\star\, w_{\mathrm{e}}(1)}$ and $P_{\star\, w_{\mathrm{e}}(2)}$. Index e spans all the ntotedge border edges, taking into account the local orientation of the contiguous faces sharing the support of each edge: To avoid a cumbersome notation, further highlights for the correct evaluation of this term will be provided in Sect. 5.

(iii) Finally, let us reduce the third addend in Eq. (9). Through an additive decomposition by complementary surface projectors, one has

$$
\begin{aligned}
&-\int\limits_{\Sigma_\star} \mathrm{d}\Sigma_\star\, [M_\|]^A_{R'} \frac{\partial}{\partial X^A} \left\{ \frac{J\, f^{\mathrm{ext}}_{\Sigma nn\, i}}{\|\mathbf{F}^{-T}\mathbf{N}\|}\; N^{\star\, R}\, N^{\star\, S}\, [M_\|]^{R'}_R \right\} \left([M_\perp]^Y_S + [M_\|]^Y_S \right) \frac{\partial\, \delta\chi^i}{\partial\, X^Y} \\
&\quad = -\int\limits_{\Sigma_\star} \mathrm{d}\Sigma_\star\, [M_\|]^A_{R'} \frac{\partial}{\partial X^A} \left\{ \frac{J\, f^{\mathrm{ext}}_{\Sigma nn\, i}}{\|\mathbf{F}^{-T}\mathbf{N}\|}\; N^{\star\, R}\, N^{\star\, S}\, [M_\|]^{R'}_R \right\} \underbrace{[M_\perp]^Y_S}_{=N_S N^Y} \frac{\partial\, \delta\chi^i}{\partial\, X^Y} \\
&\quad - \int\limits_{\Sigma_\star} \mathrm{d}\Sigma_\star\, [M_\|]^A_{R'} \frac{\partial}{\partial X^A} \left\{ \frac{J\, f^{\mathrm{ext}}_{\Sigma nn\, i}}{\|\mathbf{F}^{-T}\mathbf{N}\|}\; N^{\star\, R}\, N^{\star\, S}\, [M_\|]^{R'}_R \right\} [M_\|]^Y_S \frac{\partial\, \delta\chi^i}{\partial\, X^Y} \\
&\quad = -\int\limits_{\Sigma_\star} \mathrm{d}\Sigma_\star\, [M_\|]^A_{R'} \frac{\partial}{\partial X^A} \left\{ \frac{J\, f^{\mathrm{ext}}_{\Sigma nn\, i}}{\|\mathbf{F}^{-T}\mathbf{N}\|}\; N^{\star\, R}\, N^{\star\, S}\, [M_\|]^{R'}_R \right\} N_S \frac{\partial\, \delta\chi^i}{\partial\, N} \\
&\quad - \int\limits_{\Sigma_\star} \mathrm{d}\Sigma_\star\, [M_\|]^Y_{S'} \frac{\partial}{\partial\, X^Y} \left\{ [M_\|]^A_{R'} \frac{\partial}{\partial X^A} \left(\frac{J\, f^{\mathrm{ext}}_{\Sigma nn\, i}}{\|\mathbf{F}^{-T}\mathbf{N}\|}\; N^{\star\, R}\, N^{\star\, S}\, [M_\|]^{R'}_R \right) \delta\chi^i\; [M_\|]^{S'}_S \right\} + \\
&\quad + \int\limits_{\Sigma_\star} \mathrm{d}\Sigma_\star\, [M_\|]^Y_{S'} \frac{\partial}{\partial\, X^Y} \left\{ [M_\|]^A_{R'} \frac{\partial}{\partial X^A} \left(\frac{J\, f^{\mathrm{ext}}_{\Sigma nn\, i}}{\|\mathbf{F}^{-T}\mathbf{N}\|}\; N^{\star\, R}\, N^{\star\, S}\, [M_\|]^{R'}_R \right) [M_\|]^{S'}_S \right\} \delta\chi^i
\end{aligned}
\tag{16}
$$

Hence, we can write

$$
\begin{aligned}
&= \boxed{-\int\limits_{\Sigma_\star} \mathrm{d}\Sigma_\star\, [M_\|]^A_{R'} \frac{\partial}{\partial X^A} \left\{ \frac{J\, f^{\mathrm{ext}}_{\Sigma nn\, i}}{\|\mathbf{F}^{-T}\mathbf{N}\|}\; N^{\star\, R}\, N^{\star\, S}\, [M_\|]^{R'}_R \right\} N_S \frac{\partial\, \delta\chi^i}{\partial\, N}} \\
&\quad \boxed{-\int\limits_{L_\star} \mathrm{d}L_\star\, [M_\|]^A_{R'} \frac{\partial}{\partial X^A} \left(\frac{J\, f^{\mathrm{ext}}_{\Sigma nn\, i}}{\|\mathbf{F}^{-T}\mathbf{N}\|}\; N^{\star\, R}\, N^{\star\, S}\, [M_\|]^{R'}_R \right) B_S\, \delta\chi^i} \\
&\quad \boxed{+\int\limits_{\Sigma_\star} \mathrm{d}\Sigma_\star\, [M_\|]^Y_{S'} \frac{\partial}{\partial\, X^Y} \left\{ [M_\|]^A_{R'} \frac{\partial}{\partial X^A} \left(\frac{J\, f^{\mathrm{ext}}_{\Sigma nn\, i}}{\|\mathbf{F}^{-T}\mathbf{N}\|}\; N^{\star\, R}\, N^{\star\, S}\, [M_\|]^{R'}_R \right) [M_\|]^{S'}_S \right\} \delta\chi^i};
\end{aligned}
\tag{17}
$$

Before closing this section, it is worth emphasizing that some of the above addends admit alternative expressions. For instance, in Eq. (13) by product differentiation and through well-known properties of the normal vector gradient (see “Appendix C”) one can write

$$
\begin{aligned}
&-\int\limits_{\Sigma_\star} \mathrm{d}\Sigma_\star\, [M_\|]^B_{S'} \frac{\partial}{\partial\, X^B} \left\{ \|\mathbf{F}^{-T}\mathbf{N}\|\, J\, f^{\mathrm{ext}}_{\Sigma nn\, i}\, N^{\star\, S}\, N^A\, [M_\|]^{S'}_S \right\} N_A \frac{\partial\, \delta\chi^i}{\partial\, N} \\
&\quad = -\int\limits_{\Sigma_\star} \mathrm{d}\Sigma_\star\, [M_\|]^B_{S'} \frac{\partial}{\partial\, X^B} \left\{ \|\mathbf{F}^{-T}\mathbf{N}\|\, J\, f^{\mathrm{ext}}_{\Sigma nn\, i}\, N^{\star\, S}\, [M_\|]^{S'}_S \right\} \frac{\partial\, \delta\chi^i}{\partial\, N};
\end{aligned}
\tag{18}
$$

Other formulae including the tangential edge projector $\mathbf{M}_{L\,\|} = \mathbf{T}\otimes\mathbf{T}$ can be simplified through the relationships at point (vi) of "Appendix C" and Eqs. (48)-(49) in "Appendix A."

By rearranging the addends originated from the Eulerian loading ($S2$), one has

$$
\begin{aligned}
&= \int\limits_{\Sigma_\star} \mathrm{d}\Sigma_\star \,\|\mathbf{F}^{-T}\mathbf{N}\|^3\, J\, f^{\mathrm{ext}}_{\Sigma nn\, i}\, \frac{\partial^2\,\delta\chi^i}{\partial\, N^2} \\
&\quad - \int\limits_{\Sigma_\star} \mathrm{d}\Sigma_\star\, [M_{\|}]^B_{S'}\, \frac{\partial}{\partial\, X^B} \left\{ \|\mathbf{F}^{-T}\mathbf{N}\|\, J\, f^{\mathrm{ext}}_{\Sigma nn\, i}\, N^{\star\, S} N^A\, [M_{\|}]^{S'}_S \right\} N_A\, \frac{\partial\,\delta\chi^i}{\partial\, N} \\
&\quad - \int\limits_{\Sigma_\star} \mathrm{d}\Sigma_\star\, [M_{\|}]^B_{S'}\, \frac{\partial}{\partial X^B} \left\{ \frac{J\, f^{\mathrm{ext}}_{\Sigma nn\, i}}{\|\mathbf{F}^{-T}\mathbf{N}\|}\, N^{\star\, S}\, N^{\star\, A}\, [M_{\|}]^{S'}_S \right\} N_A\, \frac{\partial\,\delta\chi^i}{\partial\, N} \\
&\quad + \int\limits_{\Sigma_\star} \mathrm{d}\Sigma_\star\, [M_{\|}]^L_{A'}\, \frac{\partial}{\partial\, X^L} \left\{ [M_{\|}]^B_{S'}\, \frac{\partial}{\partial\, X^B} \left(\|\mathbf{F}^{-T}\mathbf{N}\|\, J\, f^{\mathrm{ext}}_{\Sigma nn\, i}\, N^{\star\, S} N^A [M_{\|}]^{S'}_S \right) [M_{\|}]^{A'}_A \right\} \delta\chi^i \\
&\quad + \int\limits_{\Sigma_\star} \mathrm{d}\Sigma_\star\, [M_{\|}]^L_{A'}\, \frac{\partial}{\partial\, X^L} \left\{ [M_{\|}]^B_{S'}\, \frac{\partial}{\partial X^B} \left(\frac{J\, f^{\mathrm{ext}}_{\Sigma nn\, i}}{\|\mathbf{F}^{-T}\mathbf{N}\|}\, N^{\star\, S}\, N^{\star\, A}\, [M_{\|}]^{S'}_S \right) [M_{\|}]^{A'}_A \right\} \delta\chi^i \\
&\quad + 2 \int\limits_{L_\star} \mathrm{d}L_\star\, \|\mathbf{F}^{-T}\mathbf{N}\|\, J\, f^{\mathrm{ext}}_{\Sigma nn\, i}\, \left(B_S N^{\star S}\right) \frac{\partial\,\delta\chi^i}{\partial\, N} + \int\limits_{L_\star} \mathrm{d}L_\star\, \frac{J\, f^{\mathrm{ext}}_{\Sigma nn\, i}}{\|\mathbf{F}^{-T}\mathbf{N}\|}\, \left(N^{\star R} B_R\right)^2 \frac{\partial\,\delta\chi^i}{\partial\, B} \\
&\quad - \int\limits_{L_\star} \mathrm{d}L_\star\, T^L T_{S'}\, \frac{\partial}{\partial\, X^L} \left\{ \frac{J\, f^{\mathrm{ext}}_{\Sigma nn\, i}}{\|\mathbf{F}^{-T}\mathbf{N}\|}\, \left(N^{\star\, R} B_R\right) \left(T_S N^{\star\, S}\right) T^{S'} \right\} \delta\chi^i \\
&\quad - \int\limits_{L_\star} \mathrm{d}L_\star\, [M_{\|}]^B_{S'}\, \frac{\partial}{\partial\, X^B} \left\{ \|\mathbf{F}^{-T}\mathbf{N}\|\, J\, f^{\mathrm{ext}}_{\Sigma nn\, i}\, N^{\star\, S} N^A [M_{\|}]^{S'}_S \right\} B_A\, \delta\chi^i \\
&\quad - \int\limits_{L_\star} \mathrm{d}L_\star\, [M_{\|}]^B_{S'}\, \frac{\partial}{\partial X^B} \left\{ \frac{J\, f^{\mathrm{ext}}_{\Sigma nn\, i}}{\|\mathbf{F}^{-T}\mathbf{N}\|}\, N^{\star\, S}\, N^{\star\, A}\, [M_{\|}]^{S'}_S \right\} B_A\, \delta\chi^i \\
&\quad + \sum_{\mathrm{e}=1}^{\mathrm{ntotedge}} \left[\frac{J\, f^{\mathrm{ext}}_{\Sigma nn\, i}}{\|\mathbf{F}^{-T}\mathbf{N}\|}\, \left(B_R N^{\star\, R}\right) \left(T_S N^{\star\, S}\right)\, \delta\chi^i \right]^{P_{\star\, w_{\mathrm{e}}(2)}}_{P_{\star\, w_{\mathrm{e}}(1)}}
\end{aligned}
\tag{19}
$$

It is worth emphasizing that the Eulerian triple force $f^{\mathrm{ext}}_{\Sigma nn\, i}$ when transported to the Lagrangian configuration gives rise to all the typologies of actions: a triple force, after multiplication by a local scale factor; a double force, through the surface divergence operator; contact pressures over the face, through a double surface divergence operator; edge double forces; edge forces via the edge divergence operator; finally, wedge forces. Concerning the edge and wedge contributions, we can recognize the presence of several *pull-back* products between edge vectors, i.e., $N^{\star\, R} B_R = g^{\star\, QR} B_R N_Q$ (even squared), $N^{\star\, R} T_R = g^{\star\, QR} T_R N_Q$ and $N^{\star\, S}[M_{\|}]^{S'}_S$, all vanishing when $F^a_A \to \delta^a_A$.

For the sake of clarity, we orderly gather the Lagrangian loading corresponding to the Eulerian action $f^{\mathrm{ext}}_{\Sigma nn\, i}$ in a synopsis.

Lagrangian actions generated by the Eulerian force $f^{\text{ext}}_{\Sigma nn\,i}$ (S2)

$$\mathcal{F}^{\text{ext}}_{\Sigma_\star NN\,i} = \|\mathbf{F}^{-T}\mathbf{N}\|^3\, J\, f^{\text{ext}}_{\Sigma nn\,i}\,; \tag{21}$$

$$\mathcal{F}^{\text{ext}}_{\Sigma_\star N\,i} = -[M_\|]^B_{S'}\,\frac{\partial}{\partial X^B}\left\{\|\mathbf{F}^{-T}\mathbf{N}\|\, J\, f^{\text{ext}}_{\Sigma nn\,i}\, N^{\star S}[M_\|]^{S'}_S\left(N^A + \frac{N^{\star A}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2}\right)\right\} N_A\,;$$

$$\mathcal{F}^{\text{ext}}_{\Sigma_\star\, i} = +[M_\|]^L_{A'}\,\frac{\partial}{\partial X^L}\left\{[M_\|]^B_{S'}\,\frac{\partial}{\partial X^B}\left[\|\mathbf{F}^{-T}\mathbf{N}\|\, J\, f^{\text{ext}}_{\Sigma nn\,i}\, N^{\star S}\,[M_\|]^{S'}_S\right.\right.$$
$$\left.\left.\times\left(N^A + \frac{N^{\star A}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2}\right)\right][M_\|]^{A'}_A\right\};$$

$$\mathcal{F}^{\text{ext}}_{L_\star N\,i} = 2\,\|\mathbf{F}^{-T}\mathbf{N}\|\, J\, f^{\text{ext}}_{\Sigma nn\,i}\,\left(B_S N^{\star S}\right)\,;$$

$$\mathcal{F}^{\text{ext}}_{L_\star B\,i} = +\frac{J\, f^{\text{ext}}_{\Sigma nn\,i}}{\|\mathbf{F}^{-T}\mathbf{N}\|}\,\left(B_R N^{\star R}\right)^2\,;$$

$$\mathcal{F}^{\text{ext}}_{L_\star\, i} = -T^L T_{S'}\,\frac{\partial}{\partial X^L}\left\{\frac{J\, f^{\text{ext}}_{\Sigma nn\,i}}{\|\mathbf{F}^{-T}\mathbf{N}\|}\,\left(B_R N^{\star R}\right)\,\left(T_S N^{\star S}\right)\,T^{S'}\right\}$$
$$-[M_\|]^B_{S'}\,\frac{\partial}{\partial X^B}\left\{\|\mathbf{F}^{-T}\mathbf{N}\|\, J\, f^{\text{ext}}_{\Sigma nn\,i}\, N^{\star S}\,[M_\|]^{S'}_S\left(N^A + \frac{N^{\star A}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2}\right)\right\} B_A\,;$$

$$\sum_{\mathrm{w}=1}^{\text{ntotwedge}} \mathcal{F}^{\text{ext}}_{P_{\star\mathrm{w}}\,i}\,\delta\chi^i(\mathbf{P}_{\star\mathrm{w}}) = \sum_{\mathrm{e}=1}^{\text{ntotedge}}\left[\frac{J\, f^{\text{ext}}_{\Sigma nn\,i}}{\|\mathbf{F}^{-T}\mathbf{N}\|}\,\left(B_R N^{\star R}\right)\,\left(N^{\star S} T_S\right)\,\delta\chi^i\right]^{P_{\star\, w_{\mathrm{e}}(2)}}_{P_{\star\, w_{\mathrm{e}}(1)}}\,;$$

On the basis of the relationships outlined in the "Appendices A and C," concerning the directional derivatives of the edge vectors and the properties of the edge projectors, the Lagrangian edge action in Eq. (21) can be written equivalently as follows:

$$\mathcal{F}^{\text{ext}}_{L_\star\, i} = -T^L\,\frac{\partial}{\partial X^L}\left\{\frac{J\, f^{\text{ext}}_{\Sigma nn\,i}}{\|\mathbf{F}^{-T}\mathbf{N}\|}\,\left(B_R N^{\star R}\right)\,\left(N^{\star S} T_S\right)\right\}$$
$$-T^B\frac{\partial}{\partial X^B}\left\{\|\mathbf{F}^{-T}\mathbf{N}\|\, J\, f^{\text{ext}}_{\Sigma nn\,i}\,\left(N^{\star S} T_S\right)\left(N^A + \frac{N^{\star A}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2}\right)\right\} B_A$$
$$-B^B\frac{\partial}{\partial X^B}\left\{\|\mathbf{F}^{-T}\mathbf{N}\|\, J\, f^{\text{ext}}_{\Sigma nn\,i}\,\left(N^{\star S} B_S\right)\left(N^A + \frac{N^{\star A}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2}\right)\right\} B_A +$$
$$-T^B T_{S'}\frac{\partial B^{S'}}{\partial X^B}\,\left(N^{\star S} B_S\right)^2\,\frac{J\, f^{\text{ext}}_{\Sigma nn\,i}}{\|\mathbf{F}^{-T}\mathbf{N}\|}\,; \tag{22}$$

3.5. Edge contributions

The Eulerian edge contribution including $f^{\text{ext}}_{L\,i}$ and labeled as $(L0)$ can be easily transformed into the Lagrangian description. In fact, one finds

$$\int_L f^{\text{ext}}_{L\,i}\,\delta\chi^i\,\mathrm{d}L = \int_{L_\star} \|\mathbf{FT}\|\, f^{\text{ext}}_{L\,i}\,\delta\chi^i\,\mathrm{d}L_\star\,; \tag{23}$$

resulting for the length element $dL = \|\mathbf{FT}\|\,\mathrm{d}L_\star$. One obtains $\mathcal{F}^{\text{ext}}_{L_\star\, i} = \|\mathbf{FT}\|\, f^{\text{ext}}_{L\,i}$, see e.g. [9,58]: such a contribution does not generate any other loading action in the Lagrangian configuration. This action can be retrieved in Eq. 2 by setting $\beta = 0$ along $\partial\partial\Omega$.

Let us consider the Eulerian edge double force $f^{\text{ext}}_{Ln\,i}$, expending work on the (face) normal derivative of the virtual displacements and above labeled as $(L1)$. This action can be retrieved in Eq. 2 by setting

$\beta = 1$ over $\partial\partial\Omega$. One can write

$$\int_L f^{\text{ext}}_{Ln\,i}\, g^{rs} n_s \frac{\partial\,\delta\chi^i}{\partial\, x^r}\,\mathrm{d}L = \int_{L_\star} \|\mathbf{FT}\|\, f^{\text{ext}}_{Ln\,i}\, g^{rs} \frac{\left(\mathbf{F}^{-1}\right)^Q_s N_Q}{\|\mathbf{F}^{-T}\mathbf{N}\|} \left(\mathbf{F}^{-1}\right)^S_r \frac{\partial\,\delta\chi^i}{\partial\, X^S}\,\mathrm{d}L_\star \tag{24}$$

In this case, we decompose the Lagrangian gradient by means of orthogonal projectors defined along the border edge (marked by subscript L, see "Appendix A"), namely through the tangential edge projector $\mathbf{M}_{L\,\|} = \mathbf{T}\otimes\mathbf{T}$ and the normal edge projector $\mathbf{M}_{L\,\perp} = \mathbf{B}\otimes\mathbf{B} + \mathbf{N}\otimes\mathbf{N}$, for which the complementarity relationship holds, i.e., $\mathbf{M}_{L\,\perp} + \mathbf{M}_{L\,\|} = \mathbf{1}$. Introducing as above the pullback metric tensor $g^{\star QS} = g^{rs}\left(\mathbf{F}^{-1}\right)^Q_s \left(\mathbf{F}^{-1}\right)^S_r$ with the simplified notation $N^{\star\,Q} = g^{\star QS} N_S$, one has

$$\begin{aligned}
&= + \int_{L_\star} \frac{\|\mathbf{FT}\|}{\|\mathbf{F}^{-T}\mathbf{N}\|}\, f^{\text{ext}}_{Ln\,i}\, N^{\star\,S}\, \underbrace{\left([M_{L\,\perp}]^C_S + [M_{L\,\|}]^C_S\right)}_{=\delta^C_S} \frac{\partial\,\delta\chi^i}{\partial\, X^C}\,\mathrm{d}L_\star \\
&= + \int_{L_\star} \frac{\|\mathbf{FT}\|}{\|\mathbf{F}^{-T}\mathbf{N}\|}\, f^{\text{ext}}_{Ln\,i}\, \underbrace{N^{\star\,S} N_S}_{=\|\mathbf{F}^{-T}\mathbf{N}\|^2}\, N^C \frac{\partial\,\delta\chi^i}{\partial\, X^C}\,\mathrm{d}L_\star \\
&\quad + \int_{L_\star} \frac{\|\mathbf{FT}\|}{\|\mathbf{F}^{-T}\mathbf{N}\|}\, f^{\text{ext}}_{Ln\,i} \left(N^{\star\,S} B_S\right) B^C \frac{\partial\,\delta\chi^i}{\partial\, X^C}\,\mathrm{d}L_\star \\
&\quad + \int_{L_\star} \frac{\|\mathbf{FT}\|}{\|\mathbf{F}^{-T}\mathbf{N}\|}\, f^{\text{ext}}_{Ln\,i}\, N^{\star\,S}\, [M_{L\,\|}]^C_S \frac{\partial\,\delta\chi^i}{\partial\, X^C}\,\mathrm{d}L_\star
\end{aligned} \tag{25}$$

Through the integration by parts and applying the divergence theorem along the one-dimensional curved edge (see "Appendix B"), one obtains

$$\begin{aligned}
&= + \int_{L_\star} \|\mathbf{FT}\| \|\mathbf{F}^{-T}\mathbf{N}\|\, f^{\text{ext}}_{Ln\,i} \frac{\partial\,\delta\chi^i}{\partial\, N}\,\mathrm{d}L_\star \\
&\quad + \int_{L_\star} \frac{\|\mathbf{FT}\|}{\|\mathbf{F}^{-T}\mathbf{N}\|}\, f^{\text{ext}}_{Ln\,i} \left(N^{\star\,S} B_S\right) \frac{\partial\,\delta\chi^i}{\partial\, B}\,\mathrm{d}L_\star \\
&\quad + \int_{L_\star} \mathrm{d}L_\star\, [M_{L\,\|}]^C_{S'} \frac{\partial}{\partial\, X^C} \left\{ \frac{\|\mathbf{FT}\|}{\|\mathbf{F}^{-T}\mathbf{N}\|}\, f^{\text{ext}}_{Ln\,i}\, N^{\star\,S}\, \delta\chi^i\, [M_{L\,\|}]^{S'}_S \right\} \\
&\quad - \int_{L_\star} \mathrm{d}L_\star\, [M_{L\,\|}]^C_{S'} \frac{\partial}{\partial\, X^C} \left\{ \frac{\|\mathbf{FT}\|}{\|\mathbf{F}^{-T}\mathbf{N}\|}\, f^{\text{ext}}_{Ln\,i}\, N^{\star\,S}\, [M_{L\,\|}]^{S'}_S \right\} \delta\chi^i \\
&= \boxed{+ \int_{L_\star} \|\mathbf{FT}\| \|\mathbf{F}^{-T}\mathbf{N}\|\, f^{\text{ext}}_{Ln\,i} \frac{\partial\,\delta\chi^i}{\partial\, N}\,\mathrm{d}L_\star} \\
&\boxed{+ \int_{L_\star} \frac{\|\mathbf{FT}\|}{\|\mathbf{F}^{-T}\mathbf{N}\|}\, f^{\text{ext}}_{Ln\,i} \left(N^{\star\,S} B_S\right) \frac{\partial\,\delta\chi^i}{\partial\, B}\,\mathrm{d}L_\star} \\
&\boxed{+ \sum_{\mathrm{e}=1}^{\text{ntotedge}} \left[\frac{\|\mathbf{FT}\|}{\|\mathbf{F}^{-T}\mathbf{N}\|}\, f^{\text{ext}}_{Ln\,i} \left(T_S N^{\star\,S}\right) \delta\chi^i \right]^{P_{\star\,\mathrm{w}_{\mathrm{e}(2)}}}_{P_{\star\,\mathrm{w}_{\mathrm{e}(1)}}}} +
\end{aligned}$$

$$\boxed{-\int\limits_{L_\star} \mathrm{d}L_\star \, [M_{L\,\|}]^C_{S'} \frac{\partial}{\partial X^C} \left\{ \frac{\|\mathbf{FT}\|}{\|\mathbf{F}^{-T}\mathbf{N}\|} \, f^{\mathrm{ext}}_{Ln\,i} \, N^{\star\,S} \, [M_{L\,\|}]^{S'}_S \right\} \delta\chi^i} ; \tag{26}$$

Hence, the Eulerian edge double force $f^{\mathrm{ext}}_{Ln\,i}$, working on the derivative of the virtual displacements along the direction of the face normal n^r, has generated in the Lagrangian configuration, besides its Lagrangian counterpart of equal order and typology, an edge double force working on the directional derivative of the virtual displacements along the edge normal B^R, an edge force through the surface divergence operator, and concentrated forces at the wedges. All the above contributions include the same functional group, i.e., $f^{\mathrm{ext}}_{Ln\,i} N^{\star\,S}\|\mathbf{FT}\|/\|\mathbf{F}^{-T}\mathbf{N}\|$, but each time the pulled-back normal $N^{\star\,S}$ is multiplied by a different vector of the Darboux moving frame, namely by N_S, B_S and T_S (see Eq. 25).

Lagrangian actions generated by the Eulerian edge double force $f^{\mathrm{ext}}_{Ln\,i}$ $(L1)$

$$\mathcal{F}^{\mathrm{ext}}_{L_\star N\,i} = \|\mathbf{FT}\|\|\mathbf{F}^{-T}\mathbf{N}\| \, f^{\mathrm{ext}}_{Ln\,i} \, ; \tag{27}$$

$$\mathcal{F}^{\mathrm{ext}}_{L_\star B\,i} = \frac{\|\mathbf{FT}\|}{\|\mathbf{F}^{-T}\mathbf{N}\|} \, f^{\mathrm{ext}}_{Ln\,i} \left(B_S N^{\star\,S}\right) \; ;$$

$$\mathcal{F}^{\mathrm{ext}}_{L_\star\,i} = -\,T^C T_{S'} \frac{\partial}{\partial X^C} \left\{ \frac{\|\mathbf{FT}\|}{\|\mathbf{F}^{-T}\mathbf{N}\|} \, f^{\mathrm{ext}}_{Ln\,i} \left(N^{\star\,S} T_S\right) T^{S'} \right\} ;$$

$$\sum_{\mathrm{w}=1}^{\mathrm{ntotwedge}} \mathcal{F}^{\mathrm{ext}}_{P_\star{}_\mathrm{w}\,i} \, \delta\chi^i \left(\mathbf{P}_{\star\,\mathrm{w}}\right) = \sum_{\mathrm{e}=1}^{\mathrm{ntotedge}} \left[\frac{\|\mathbf{FT}\|}{\|\mathbf{F}^{-T}\mathbf{N}\|} \, f^{\mathrm{ext}}_{Ln\,i} \left(N^{\star\,S} T_S\right) \delta\chi^i \right]^{P_{\star\,\mathrm{w}_{\mathrm{e}(2)}}}_{P_{\star\,\mathrm{w}_{\mathrm{e}(1)}}} ;$$

At this stage, we can consider the edge contribution to the Eulerian virtual work expending work on the derivative of the virtual displacements along the edge normal, namely $f^{\mathrm{ext}}_{L\,b\,i}$ labeled as $(L2)$. Through the transformation formula for the contravariant edge normal b^r (see "Appendix C" and [4,5]), one can write

$$\begin{aligned}
&\int\limits_L f^{\mathrm{ext}}_{Lb\,i} \, b^r \frac{\partial\,\delta\chi^i}{\partial\,x^r} \, \mathrm{d}L \\
&\quad = \int\limits_{L_\star} \mathrm{d}L_\star \, \|\mathbf{FT}\| \, f^{\mathrm{ext}}_{Lb\,i} \left\{ F^r_R B^R - \frac{\langle\mathbf{FB},\mathbf{FT}\rangle}{\langle\mathbf{FT},\mathbf{FT}\rangle} F^r_R T^R \right\} \frac{\|\mathbf{FT}\|}{\|J\mathbf{F}^{-T}\mathbf{N}\|} \left(\mathbf{F}^{-1}\right)^S_r \frac{\partial\,\delta\chi^i}{\partial\,X^S} \\
&\quad = \int\limits_{L_\star} \mathrm{d}L_\star \, \frac{\|\mathbf{FT}\|^2}{\|J\mathbf{F}^{-T}\mathbf{N}\|} \, f^{\mathrm{ext}}_{Lb\,i} \left\{ B^S - \frac{\langle\mathbf{FB},\mathbf{FT}\rangle}{\langle\mathbf{FT},\mathbf{FT}\rangle} T^S \right\} \frac{\partial\,\delta\chi^i}{\partial\,X^S}
\end{aligned} \tag{28}$$

resulting $F^r_R \left(\mathbf{F}^{-1}\right)^S_r = \delta^S_R$. Through complementary orthogonal projectors defined along the edge (marked by subscript L, see "Appendix A"), exploiting the integration by parts for the term including the tangential edge projector one can write

$$\begin{aligned}
&= +\int\limits_{L_\star} \mathrm{d}L_\star \, \frac{\|\mathbf{FT}\|^2}{\|J\mathbf{F}^{-T}\mathbf{N}\|} \, f^{\mathrm{ext}}_{Lb\,i} \left\{ B^S - \frac{\langle\mathbf{FB},\mathbf{FT}\rangle}{\langle\mathbf{FT},\mathbf{FT}\rangle} T^S \right\} \underbrace{\left([M_{L\,\perp}]^Q_S + [M_{L\,\|}]^Q_S\right)}_{=\delta^Q_S} \frac{\partial\,\delta\chi^i}{\partial\,X^Q} \\
&= +\int\limits_{L_\star} \mathrm{d}L_\star \, \frac{\|\mathbf{FT}\|^2}{\|J\mathbf{F}^{-T}\mathbf{N}\|} \, f^{\mathrm{ext}}_{Lb\,i} \underbrace{\left\{ B^S - \frac{\langle\mathbf{FB},\mathbf{FT}\rangle}{\langle\mathbf{FT},\mathbf{FT}\rangle} T^S \right\} B_S}_{=1} \, B^Q \frac{\partial\,\delta\chi^i}{\partial\,X^Q}
\end{aligned}$$

$$+\int\limits_{L_\star} \mathrm{d}L_\star\, \frac{\|\mathbf{FT}\|^2}{\|J\mathbf{F}^{-T}\mathbf{N}\|}\; f^{\mathrm{ext}}_{Lb\,i}\, \underbrace{\left\{B^S - \frac{\langle \mathbf{FB},\mathbf{FT}\rangle}{\langle \mathbf{FT},\mathbf{FT}\rangle} T^S\right\} N_S}_{=0}\; N^Q\, \frac{\partial\,\delta\chi^i}{\partial\, X^Q}$$

$$+\int\limits_{L_\star} \mathrm{d}L_\star\, [M_{L\,\|}]^Q_{S'}\, \frac{\partial}{\partial\, X^Q}\left(\frac{\|\mathbf{FT}\|^2}{\|J\mathbf{F}^{-T}\mathbf{N}\|}\; f^{\mathrm{ext}}_{Lb\,i}\left\{B^S - \frac{\langle \mathbf{FB},\mathbf{FT}\rangle}{\langle \mathbf{FT},\mathbf{FT}\rangle} T^S\right\}\; \delta\chi^i\, [M_{L\,\|}]^{S'}_S\right)$$

$$-\int\limits_{L_\star} \mathrm{d}L_\star\, [M_{L\,\|}]^Q_{S'}\, \frac{\partial}{\partial\, X^Q}\left(\frac{\|\mathbf{FT}\|^2}{\|J\mathbf{F}^{-T}\mathbf{N}\|}\; f^{\mathrm{ext}}_{Lb\,i}\left\{B^S - \frac{\langle \mathbf{FB},\mathbf{FT}\rangle}{\langle \mathbf{FT},\mathbf{FT}\rangle} T^S\right\}\; \underbrace{[M_{L\,\|}]^{S'}_S}_{=T^{S'}T_S}\right)\delta\chi^i \tag{29}$$

taking into account that the normalized vectors of Darboux moving frame, B^S, T^S and N^S, are mutually orthogonal. Recalling that $[M_{L\,\|}]^Q_{S'} = T^Q T_{S'}$ and $\|\mathbf{FT}\|^2 = \langle \mathbf{FT},\mathbf{FT}\rangle$, one obtains

$$= \boxed{+\int\limits_{L_\star} \mathrm{d}L_\star\, \frac{\|\mathbf{FT}\|^2}{\|J\mathbf{F}^{-T}\mathbf{N}\|}\; f^{\mathrm{ext}}_{Lb\,i}\, \frac{\partial\,\delta\chi^i}{\partial\, B}}$$

$$\boxed{-\sum_{\mathrm{e}=1}^{\mathrm{ntotedge}}\left[\frac{\langle \mathbf{FB},\mathbf{FT}\rangle}{\|J\mathbf{F}^{-T}\mathbf{N}\|}\; f^{\mathrm{ext}}_{Lb\,i}\;\delta\chi^i\right]_{P_{\star\,\mathrm{w}_{e(1)}}}^{P_{\star\,\mathrm{w}_{e(2)}}}}$$

$$\boxed{+\int\limits_{L_\star} \mathrm{d}L_\star\, T^L T_{S'}\, \frac{\partial}{\partial\, X^L}\left(\frac{\langle \mathbf{FB},\mathbf{FT}\rangle}{\|J\mathbf{F}^{-T}\mathbf{N}\|}\; f^{\mathrm{ext}}_{Lb\,i}\; T^{S'}\right)\delta\chi^i}\,; \tag{30}$$

Then, the Eulerian edge double force $f^{\mathrm{ext}}_{Lb\,i}$, working on the derivative of the virtual displacements along the edge normal b^r, has generated in the Lagrangian configuration, besides its Lagrangian counterpart of equal order and typology, an edge force through the surface divergence operator, and forces concentrated at the wedges. It is worth noting that, differently from the edge double force working on the displacement derivative along the face normal, i.e., $f^{\mathrm{ext}}_{Ln\,i}$ see Eq. (27), the present Eulerian action does not generate any Lagrangian edge double force $\mathcal{F}^{\mathrm{ext}}_{L_\star N\,i}$.

Lagrangian actions generated by the Eulerian edge double force $f^{\mathrm{ext}}_{Lb\,i}$ $(L2)$

$$\mathcal{F}^{\mathrm{ext}}_{L_\star B\,i} = +\frac{\|\mathbf{FT}\|^2}{\|J\mathbf{F}^{-T}\mathbf{N}\|}\; f^{\mathrm{ext}}_{Lb\,i}\,; \tag{31}$$

$$\mathcal{F}^{\mathrm{ext}}_{L_\star\,i} = +T^L T_{S'}\, \frac{\partial}{\partial\, X^L}\left\{\frac{\langle \mathbf{FB},\mathbf{FT}\rangle}{\|J\mathbf{F}^{-T}\mathbf{N}\|}\; f^{\mathrm{ext}}_{Lb\,i}\, T^{S'}\right\}\,;$$

$$\sum_{\mathrm{w}=1}^{\mathrm{ntotwedge}} \mathcal{F}^{\mathrm{ext}}_{P_{\star\,w}\,i}\,\delta\chi^i(\mathbf{P}_{\star\,w}) = -\sum_{\mathrm{e}=1}^{\mathrm{ntotedge}}\left[\frac{\langle \mathbf{FB},\mathbf{FT}\rangle}{\|J\mathbf{F}^{-T}\mathbf{N}\|}\; f^{\mathrm{ext}}_{Lb\,i}\,\delta\chi^i\right]_{P_{\star\,\mathrm{w}_{e(1)}}}^{P_{\star\,\mathrm{w}_{e(2)}}}\,;$$

The same results of Eqs. (30) and (31) can be obtained by using in Eq. (28) the metric tensor g^{rs} with the transformation formula for the covariant edge normal b_s, and then considering the relationships reported in "Appendix C" at point (iv). Considering Eq. (29) before the final developments, it appears clear that the above integral contributions include the same functional group, each time multiplied by a different edge vector.

The Eulerian loading concentrated at the discrete set of wedges are easily referred to the Lagrangian configuration by composition with the placement map $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$. Since the same composition is considered for the virtual displacements at the corresponding points, the forces concentrated at the wedges do not require any transformation formula. Assuming $\mathbf{p}_w = \boldsymbol{\chi}(\mathbf{P}_{\star w})$, one has $\delta\chi^i(\mathbf{P}_{\star w}) = \delta\chi^i(\mathbf{p}_w)$.

Lagrangian actions generated by the Eulerian wedge forces $f^{\text{ext}}_{p_w i}$ labeled as $(W0)$

$$\mathcal{F}^{\text{ext}}_{P_{\star w} i} = f^{\text{ext}}_{p_w i} \quad \forall \text{w} = 1, ..., \text{ntotwedge}; \tag{32}$$

4. Recursive formal structures

At this stage, we can consider synoptically the transformed contributions to the external virtual work, to seek for the possible presence of recursive formal structures, suitable for further generalizations. Considering only the transformation of the Eulerian contact pressures, the surface double and triple forces, their Lagrangian surface counterparts can be outlined as follows:

$$\begin{aligned}
&\mathcal{F}^{\text{ext}}_{\Sigma_\star NN i} = \|\mathbf{F}^{-T}\mathbf{N}\|^3\, J\, f^{\text{ext}}_{\Sigma nn i}\,;\\
&\quad\mathcal{F}^{\text{ext}}_{\Sigma_\star N i} = \|\mathbf{F}^{-T}\mathbf{N}\|^2\, J f^{\text{ext}}_{\Sigma n i}\\
&\quad - [M_\|]^B_{S'}\frac{\partial}{\partial X^B}\left\{\|\mathbf{F}^{-T}\mathbf{N}\|\, J\, f^{\text{ext}}_{\Sigma nn i}\, N^{\star S}[M_\|]^{S'}_S\left(N^A + \frac{N^{\star A}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2}\right)\right\}N_A\,;\\
&\quad\mathcal{F}^{\text{ext}}_{\Sigma_\star i} = \|J\,\mathbf{F}^{-T}\mathbf{N}\|\, f^{\text{ext}}_{\Sigma i}\\
&\quad - [M_\|]^S_{R'}\frac{\partial}{\partial X^S}\left\{J f^{\text{ext}}_{\Sigma n i}\, N^{\star R}\,[M_\|]^{R'}_R\right\}\\
&\quad + [M_\|]^L_{A'}\frac{\partial}{\partial X^L}\left\{[M_\|]^B_{S'}\frac{\partial}{\partial X^B}\left[\|\mathbf{F}^{-T}\mathbf{N}\|\, J\, f^{\text{ext}}_{\Sigma nn i}\, N^{\star S}\,[M_\|]^{S'}_S\right.\right.\\
&\quad \left.\left.\times\left(N^A + \frac{N^{\star A}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2}\right)\right]\,[M_\|]^{A'}_A\right\}\,;
\end{aligned}\tag{33}$$

In the above addends, we recognize a top-down structure which is typical of the higher-order gradient materials: in [6], we already noticed the same architecture with reference to the relationships between Eulerian and Lagrangian hyperstresses. Firstly, the Eulerian loading of highest order generates Lagrangian actions of order lower or equal. Such a transformation implies the application of the surface divergence operator for each lower level: thus, with reference for instance to the Eulerian triple force $f^{\text{ext}}_{\Sigma nn i}$, a surface divergence operator appears in the Lagrangian surface double force $\mathcal{F}^{\text{ext}}_{\Sigma_\star N i}$, while a double surface divergence operator contributes to the Lagrangian surface force $\mathcal{F}^{\text{ext}}_{\Sigma_\star i}$. Moreover, one can notice that the sign of the pth surface divergence is alternating; hence, it must be multiplied by $(-1)^p$. The contributions of the Eulerian triple force to the Lagrangian double force and to the Lagrangian contact pressures in terms of surface divergence and double surface divergence, respectively, must be split into two addends, since they include the sum of N^A and $N^{\star A}/\|\mathbf{F}^{-T}\mathbf{N}\|^2$. Furthermore, the Eulerian action of order β over the boundary face ($\beta = 0, 1, 2$) gives rise to the Lagrangian loading of equal order and typology after multiplication by the factor $\|\mathbf{F}^{-T}\mathbf{N}\|^{\beta+1}\, J$: In fact, for the triple force, the double force and the generalized contact pressures, one has $\mathcal{F}^{\text{ext}}_{\Sigma_\star NN i} = \|\mathbf{F}^{-T}\mathbf{N}\|^3\, J\, f^{\text{ext}}_{\Sigma nn i}$, $\mathcal{F}^{\text{ext}}_{\Sigma_\star N i} = \|\mathbf{F}^{-T}\mathbf{N}\|^2\, J f^{\text{ext}}_{\Sigma n i}$, and $\mathcal{F}^{\text{ext}}_{\Sigma_\star i} = \|J\,\mathbf{F}^{-T}\mathbf{N}\|\, f^{\text{ext}}_{\Sigma i}$, respectively.

As for the Lagrangian edge force, considering the original expressions one has

$$\begin{aligned}
&\mathcal{F}^{\text{ext}}_{L_\star i} = \|\mathbf{F}\mathbf{T}\|\, f^{\text{ext}}_{L i}\\
&\quad - [M_{L\,\|}]^Q_{S'}\frac{\partial}{\partial X^Q}\left(\frac{\|\mathbf{F}\mathbf{T}\|^2}{\|J\mathbf{F}^{-T}\mathbf{N}\|}\; f^{\text{ext}}_{Lb i}\left\{B^S - \frac{\langle \mathbf{F}\mathbf{B}, \mathbf{F}\mathbf{T}\rangle}{\langle \mathbf{F}\mathbf{T}, \mathbf{F}\mathbf{T}\rangle}T^S\right\}[M_{L\,\|}]^{S'}_S\right)
\end{aligned}$$

$$
\begin{aligned}
&- [M_{L\,\|}]^Q_{S'} \frac{\partial}{\partial X^Q} \left\{ \frac{\|\mathbf{FT}\|}{\|\mathbf{F}^{-T}\mathbf{N}\|} \, f^{\mathrm{ext}}_{Ln\,i} \, N^{\star\,S} \, [M_{L\,\|}]^{S'}_S \right\} \\
&+ J f^{\mathrm{ext}}_{\Sigma n\,i} \left(B_R N^{\star R} \right) \\
&- [M_{L\,\|}]^Q_{S'} \frac{\partial}{\partial X^Q} \left\{ \frac{J\, f^{\mathrm{ext}}_{\Sigma nn\,i} \, N^{\star\,S}}{\|\mathbf{F}^{-T}\mathbf{N}\|} \left(B_R N^{\star\,R} \right) \; [M_{L\,\|}]^{S'}_S \right\} + \\
&- [M_{\|}]^B_{S'} \frac{\partial}{\partial X^B} \left\{ \|\mathbf{F}^{-T}\mathbf{N}\| \, J\, f^{\mathrm{ext}}_{\Sigma nn\,i} \, N^{\star\,S} [M_{\|}]^{S'}_S \left(N^A + \frac{N^{\star\,A}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} \right) \right\} B_A \, ; \qquad (34)
\end{aligned}
$$

As expected, the Eulerian edge force $f^{\mathrm{ext}}_{L\,i}$ gives rise only to its Lagrangian counterpart $\mathcal{F}^{\mathrm{ext}}_{L_\star\,i}$, after multiplication by a local scale factor. On the contrary, the Eulerian edge double forces $f^{\mathrm{ext}}_{Lb\,i}$ and $f^{\mathrm{ext}}_{Ln\,i}$, expending work on the derivatives of the virtual displacements along the face normal and the edge normal, respectively, are transformed through an edge divergence into the Lagrangian edge action. The Eulerian surface triple force $f^{\mathrm{ext}}_{\Sigma nn\,i}$ gives rise to the Lagrangian edge action through the edge and the surface divergence operators, while the Eulerian surface double force $f^{\mathrm{ext}}_{\Sigma n\,i}$ is transformed via a scale factor as a result of the divergence theorem.

As for the edge double forces, expending work on the derivatives of the virtual displacement vector along the two normal directions along the border edge, one finds

$$
\begin{aligned}
\mathcal{F}^{\mathrm{ext}}_{L_\star N\,i} &= \|\mathbf{FT}\|\|\mathbf{F}^{-T}\mathbf{N}\| \; f^{\mathrm{ext}}_{Ln\,i} + 2\,\|\mathbf{F}^{-T}\mathbf{N}\| \, J\, f^{\mathrm{ext}}_{\Sigma nn\,i} \left(B_S N^{\star\,S} \right) \\
\mathcal{F}^{\mathrm{ext}}_{L_\star B\,i} &= + \frac{\|\mathbf{FT}\|^2}{\|J\mathbf{F}^{-T}\mathbf{N}\|} \; f^{\mathrm{ext}}_{Lb\,i} \; + \frac{\|\mathbf{FT}\|}{\|\mathbf{F}^{-T}\mathbf{N}\|} \; f^{\mathrm{ext}}_{Ln\,i} \left(B_S N^{\star\,S} \right) + \frac{J\, f^{\mathrm{ext}}_{\Sigma nn\,i}}{\|\mathbf{F}^{-T}\mathbf{N}\|} \; \left(B_R N^{\star\,R} \right)^2 \; ; \qquad (35)
\end{aligned}
$$

With reference to Eq. (35), it can be noticed that the Eulerian edge double force $f^{\mathrm{ext}}_{Ln\,i}$ when transformed into the Lagrangian description gives rise to both the Lagrangian actions of equal order relevant to the normals N^R and B^R, namely $\mathcal{F}^{\mathrm{ext}}_{L_\star N\,i}$ and $\mathcal{F}^{\mathrm{ext}}_{L_\star B\,i}$, while the Eulerian edge double force $f^{\mathrm{ext}}_{Lb\,i}$ gives rise exclusively to the Lagrangian force $\mathcal{F}^{\mathrm{ext}}_{L_\star B\,i}$ relevant to the edge normal B^R. Moreover, we observe that the Eulerian double force over the boundary surface $f^{\mathrm{ext}}_{\Sigma n\,i}$ does not affect the above Lagrangian 1-forces along the edge. For the readers' convenience, in "Appendix A" at points (i)-(ii) we have reported some relationships which might be useful to attain simpler expressions of the above equations, in view of further generalizations.

5. Boundary geometry and virtual work integrals

In [6], we deduced the equilibrium equations for third-gradient materials: in the adopted variational approach, the boundary surface, the border edge and the discrete set of wedges were dealt with as differential borders of increasing order of the reference volume, namely as $\partial^{(p)}\Omega_\star$ where p varies from 0 to 3. To avoid a cumbersome notation for the mathematical expressions, details concerning the actual geometry of the boundary were not made explicit so far. Herein we recall that the boundary surface of the body volume is constituted of a finite number of disjoint oriented faces (with the positive normal pointing outwards), say nface, having in common two by two the support of their border edges, namely $\Sigma_\star \equiv \bigcup_{h=1}^{\mathrm{nface}} \Sigma_{\star\,h}$. Hence, the surface integrals for finite additivity must be split into the sum of contributions each one extended to a singular (regular) boundary face. For instance, the surface integral in Eq. (11) originated from $(S2)$ can be evaluated as follows:

$$
\int_{\Sigma_\star} \mathrm{d}\Sigma_\star \, \|\mathbf{F}^{-T}\mathbf{N}\|^3 \, J\, f^{\mathrm{ext}}_{\Sigma nn\,i} \frac{\partial^2 \delta\chi^i}{\partial N^2} = \sum_{h=1}^{\mathrm{nface}} \int_{\Sigma_{\star\,h}} \mathrm{d}\Sigma_{\star\,h} \, \|\mathbf{F}^{-T}\mathbf{N}\|^3 \, J\, f^{\mathrm{ext}}_{\Sigma nn\,i} \, N^R N^S \frac{\partial^2 \delta\chi^i}{\partial X^R X^S} \, ; \qquad (36)
$$

Quantities entering the above integral, such as the normal vector, are defined over the entire face $\Sigma_{\star\,h}$ without ambiguity.

Instead, when considering edge and wedge contributions to the virtual work, some clarifications are needed. The curved border of each face represents a discontinuity locus for the normal vector field: such a border in turn is constituted of a finite number of regular branches, say ntotedge, namely $L_\star \equiv \bigcup_{\mathrm{e}=1}^{\mathrm{ntotedge}} L_{\star\,\mathrm{e}}$, where the wedges represent discontinuity points for the edge tangents. Two faces may share the support of an edge: Hence, there are two oriented edges with the same support, each ideally belonging to one of the contiguous faces and oriented consistently with it ($[T^+]^R = -[T^-]^R$). The positive direction of the tangent $\mathbf{T}$ along the face border is suggested by a counterclockwise curling of the fingers when the thumb points the positive normal of that face. For the readers' convenience, let us consider an edge integral in Eq. (26) originated from the Eulerian edge loading $f_{Ln\,i}^{\mathrm{ext}}$ labeled as $(L1)$. This edge contribution must be evaluated as the sum of ntotedge integrals each one extended to a singular regular edge, say $L_{\star\,e}$, whose support is shared between two contiguous faces (inducing on it two opposite orientations). Hence, one can write

$$\int\limits_{L_\star} \|\mathbf{F}\mathbf{T}\|\|\mathbf{F}^{-T}\mathbf{N}\|\; f_{Ln\,i}^{\mathrm{ext}}\,\frac{\partial\,\delta\chi^i}{\partial\,N}\;\mathrm{d}L_\star = \sum_{\mathrm{e}=1}^{\mathrm{ntotedge}}\;\int\limits_{L_{\star\,\mathrm{e}}} f_{Ln\,i}^{\mathrm{ext}}\left\{\|\mathbf{F}[\mathbf{T}^+]\|\|\mathbf{F}^{-T}[\mathbf{N}^+]\|\;[N^+]^R + \right.$$
$$\left.\|\mathbf{F}[\mathbf{T}^-]\|\|\mathbf{F}^{-T}[\mathbf{N}^-]\|\;[N^-]^R\right\}\frac{\partial\,\delta\chi^i}{\partial\,X^R}\;\mathrm{d}L_{\star\,\mathrm{e}}\,; \tag{37}$$

where symbol $[N^+]^R$ (resp. $[N^-]^R$) denotes the normal to the boundary face at the left (resp. at the right) of the edge in point, and $[T^+]^R = -[T^-]^R$ indicates the edge tangent consistent with the orientation of the face at the left. In the above formula, the orientation assumed for the edge tangent can be noticed to be irrelevant due to the norm.

With the intention to clarify as much as possible the present procedure, we are going to illustrate other three edge contributions. The first is derived from Eq. (17), originated from the Eulerian surface loading $f_{\Sigma nn\,i}^{\mathrm{ext}}$ labeled as $(S2)$. Also in this case, if a unique index e spans all the ntotedge border edges, each counted once, we must include contributions from the contiguous faces sharing the support of any singular edge, namely

$$+\int\limits_{L_\star}\mathrm{d}L_\star\,\frac{J\,f_{\Sigma nn\,i}^{\mathrm{ext}}}{\|\mathbf{F}^{-T}\mathbf{N}\|}\;\left(B_R N^{\star Q}\right)^2\;\frac{\partial\,\delta\chi^i}{\partial\,B}$$
$$=\sum_{\mathrm{e}=1}^{\mathrm{ntotedge}}\;\int\limits_{L_{\star\,\mathrm{e}}}\mathrm{d}L_{\star\,\mathrm{e}}\,J\left\{f_{\Sigma nn\,i}^{\mathrm{ext}\,+}\frac{\left([B^+]_R[N^+]^{\star R}\right)^2}{\|\mathbf{F}^{-T}[\mathbf{N}^+]\|}[B^+]^R\right.$$
$$\left.+f_{\Sigma nn\,i}^{\mathrm{ext}\,-}\frac{\left([B^-]_R[N^-]^{\star R}\right)^2}{\|\mathbf{F}^{-T}[\mathbf{N}^-]\|}[B^-]^R\right\}\frac{\partial\,\delta\chi^i}{\partial\,X^R}\,; \tag{38}$$

Also for the surface loading $f_{\Sigma nn\,i}^{\mathrm{ext}\,+}\circ\boldsymbol{\chi}(\mathbf{X})$ we specified the face (at the right or at the left), since it may be prescribed over one face only. No specifications are needed for the deformation gradient $\mathbf{F}$ and the Jacobian J, since they are continuous along the edge.

The second edge contribution comes from Eq. (17). In the presence of multiple faces with their border edges consistently oriented, we find

$$-\int\limits_{L_\star}\mathrm{d}L_\star\,[M_\parallel]^A_{R'}\,\frac{\partial}{\partial X^A}\left\{\frac{J\,f_{\Sigma nn\,i}^{\mathrm{ext}}}{\|\mathbf{F}^{-T}\mathbf{N}\|}\;N^{\star\,R}\,N^{\star\,S}\,[M_\parallel]^{R'}_R\right\}B_S\,\delta\chi^i$$
$$=-\sum_{\mathrm{e}=1}^{\mathrm{ntotedge}}\;\int\limits_{L_{\star\,\mathrm{e}}}\mathrm{d}L_{\star\,\mathrm{e}}\,\delta\chi^i\left\{[M_\parallel^+]^A_{R'}\,\frac{\partial}{\partial X^A}\left\{\frac{J\,f_{\Sigma nn\,i}^{\mathrm{ext}\,+}}{\|\mathbf{F}^{-T}[\mathbf{N}^+]\|}\;[N^+]^{\star R}\,[N^+]^{\star S}\,[M_\parallel^+]^{R'}_R\right\}[B^+]_S\right.$$
$$\left.+[M_\parallel^-]^A_{R'}\,\frac{\partial}{\partial X^A}\left\{\frac{J\,f_{\Sigma nn\,i}^{\mathrm{ext}\,-}}{\|\mathbf{F}^{-T}[\mathbf{N}^-]\|}\;[N^-]^{\star R}\,[N^-]^{\star S}\,[M_\parallel^-]^{R'}_R\right\}[B^-]_S\right\}\,; \tag{39}$$

The third and last example comes from Eq. (30), originated from the Eulerian edge loading $f^{\text{ext}}_{Lb\,i}$ labeled as $(L2)$. One can write

$$+\int\limits_{L_\star} \mathrm{d}L_\star\, T^L T_{S'} \frac{\partial}{\partial X^L}\left(\frac{\langle \mathbf{FB},\mathbf{FT}\rangle}{\|\mathbf{F}^{-T}\mathbf{N}\|}\, J^{-1}\, f^{\text{ext}}_{Lb\,i}\, T^{S'}\right)\delta\chi^i$$

$$=\sum_{\text{e}=1}^{\text{ntotedge}}\ \int\limits_{L_{\star\,\text{e}}} \mathrm{d}L_{\star\,\text{e}}\left\{[T^+]^L[T^+]_{S'}\frac{\partial}{\partial X^L}\left(\frac{\langle \mathbf{F}[\mathbf{B}^+],\mathbf{F}[\mathbf{T}^+]\rangle}{\|\mathbf{F}^{-T}[\mathbf{N}^+]\|}\, J^{-1}\, f^{\text{ext}}_{Lb\,i}\,[T^+]^{S'}\right)\right.$$

$$\left.+[T^-]^L[T^-]_{S'}\frac{\partial}{\partial X^L}\left(\frac{\langle \mathbf{F}[\mathbf{B}^-],\mathbf{F}[\mathbf{T}^-]\rangle}{\|\mathbf{F}^{-T}[\mathbf{N}^-]\|}\, J^{-1}\, f^{\text{ext}}_{Lb\,i}\,[T^-]^{S'}\right)\right\}\delta\chi^i \tag{40}$$

It is worth noting that for a given boundary face the number of regular branches constituting its border equals the global number of *wedges* belonging to it. Hence, over the same boundary face a pair of oriented border edges share the same *wedge*, which necessarily coincides with the first end of one edge and with the last end of the other edge. For a while let us exploit such a wedge-centered perspective. To balance the external force concentrated at one wedge, in fact, we must simply gather the contributions from all the edges converging to it, and hence from all the pairs of contiguous faces sharing the support of those edges. Hence, if index w runs over all the ntotwedge wedges, a nested loop must be included by index e spanning all the oriented edges converging to the wth wedge, the number of which is indicated by nedg(w). Also in this case, distinct contributions are provided by the contiguous oriented faces sharing the support of the eth edge. Indicating by $\mathbf{P}_{\star\,\text{w}}$ the coordinates of the wth wedge, from Eq. (19) one has

$$\sum_{\text{e}=1}^{\text{ntotedge}}\left[\left(\frac{J\, f^{\text{ext}}_{\Sigma nn\,i}}{\|\mathbf{F}^{-T}\mathbf{N}\|}\, N^{\star\,R}B_R\, N^{\star\,S}T_S\right)\delta\chi^i\right]_{P_{\star\,\text{w}_{e(1)}}}^{P_{\star\,\text{w}_{e(2)}}}$$

$$=\sum_{\text{w}=1}^{\text{ntotwedge}}\delta\chi^i\,(\mathbf{P}_{\star\,\text{w}})\,J(\mathbf{P}_{\star\,\text{w}})\sum_{\text{e}=1}^{\text{nedg(w)}}\left\{f^{\text{ext}\,+}_{\Sigma nn\,i}\,\alpha^+\,\frac{[B^+]_R[N^+]^{\star R}\ [T^+]_S[N^+]^{\star S}}{\|\mathbf{F}^{-T}[\mathbf{N}^+]\|}\right.$$

$$\left.+f^{\text{ext}\,-}_{\Sigma nn\,i}\,\alpha^-\,\frac{[B^-]_R[N^-]^{\star R}\ [T^-]_S[N^-]^{\star S}}{\|\mathbf{F}^{-T}[\mathbf{N}^-]\|}\right\};\tag{41}$$

where $\alpha(\text{e},\text{w})=\pm1$ depending on whether the wth wedge coincides with the last or the first end of the eth curved edge, whose support is shared between contiguous faces for which one has $\alpha^+=-\alpha^-$. Surface loading $f^{\text{ext}\,\pm}_{\Sigma nn\,i}\circ\boldsymbol{\chi}(\mathbf{X})$ may be prescribed over one face only.

6. Closing remarks

In this study, the external actions consistent with third-gradient modeling of elastic bodies are outlined and their transformation from the Eulerian to the Lagrangian description is carried out. The results presented above, resting on the geometric differential features of the deformation process, enlighten the role of the generalized loading that such materials can sustain. In particular, the Eulerian *double* and *triple forces* prescribed over the boundary face, expending work in the order on the first and second normal derivatives of the virtual displacements, give rise to multiple Lagrangian actions over domains with a dimension lower or equal, namely surfaces, edges and even wedges (for the latter). Analogous results are met for the edge *double forces*, working on the directional derivative of the virtual displacements along the two normals to the border curves. Such a surprising coupling is a consequence of the fact that the diffeomorphisms between the material and the spatial configurations do not preserve neither the metrics nor the structure of the external interactions when regarded as distributions. The Eulerian derivative along the direction of the normal to the Eulerian boundary face corresponds, in the Lagrangian configuration, to a derivative along a direction with both normal and tangential components with respect to the relevant material face: hence, in the transformation procedure the term including the tangential derivative can be integrated by parts, obtaining one contribution along the boundary due to the divergence theorem, and another one over the same domain expressed through the divergence operator. Accordingly, the surface divergence operator and the pullback metric tensor play a crucial role in such a transformation.

To correctly evaluate face, edge and wedge contributions to the external virtual work, detailed formulae are provided, taking into account the actual geometry of the volume boundary, constituted of multiple faces with piecewise regular border edges consistently oriented, including in turn a finite number of wedges. Although in the resulting equations some algebraic structures have been recognized, the specification of a global recursive formula for third- and higher-gradient models, apt to draw the Lagrangian external actions once their Eulerian counterparts have been prescribed, remains an open problem, which needs further investigation.

The present results represent a significant, although intermediate step to complete the transformation of the governing equations for the third-gradient materials from the Eulerian to the Lagrangian description. This transformation is expected to play an important role in several nonlinear theories for materials mechanics, such as fracture and damage, allowing one to describe effectively the elastic deformation including also wedge and edge loading, besides the higher-order actions, and to specify more accurately the homogenized response of novel metamaterials.

The deformation-induced coupling among Lagrangian interactions observed in the presence of high-order Eulerian loading could represent a starting point to develop a geometrically nonlinear theory for third- and higher-order gradient materials. We expect a strong interest for possible results in the study of micro and nano-systems. Moreover, the present theoretical study has to be regarded as a further contribution to improve the numerical tools based on the third-gradient equations, with the aim to provide reliable predictions for a variety of engineering scenarios.

Funding Information Open access funding provided by Politecnico di Milano within the CRUI-CARE Agreement.

Appendix A. Properties of surface and edge projectors

In this appendix, some basic properties of surface and edge orthogonal projectors are briefly recalled, see [4,58,60,61]). As well known, at each point of the same regular face $\Sigma_\star \equiv \partial\Omega_\star$ a pair of complementary linear operators can be defined, apt to project orthogonally any vector of the ambient space onto the tangential and normal spaces at that point, referred to as the (Lagrangian) tangential and normal projectors and denoted by symbols $[M_\parallel]^A_B$ and $[M_\perp]^A_B$, respectively. Such projectors possess the following noteworthy properties (in both index and matrix notation):

$$[M_\parallel]^A_B + [M_\perp]^A_B = \delta^A_B\,; \quad \mathbf{M}_\parallel + \mathbf{M}_\perp = \mathbf{1}\,;$$
$$[M_\perp]^C_A = N^C\,N_A\,; \quad [\mathbf{M}_\perp] = \mathbf{N}\otimes\mathbf{N}\,;$$
$$[M_\parallel]^C_A = \delta^C_A - N^C\,N_A\,; \quad [\mathbf{M}_\parallel] = \mathbf{1} - \mathbf{N}\otimes\mathbf{N}\,;$$

$$
\begin{aligned}
&[M_\parallel]^A_B [M_\parallel]^B_C = [M_\parallel]^A_C\,; \quad \mathbf{M}^2_\parallel = \mathbf{M}_\parallel\,;\\
&[M_\perp]^A_B [M_\perp]^B_C = [M_\perp]^A_C\,; \quad \mathbf{M}^2_\perp = \mathbf{M}_\perp\,;
\end{aligned} \tag{42}
$$

where the Kronecker symbol $\delta^A_B = \mathbf{1} = g^A_B$ represents the unit operator, coincident with the mixed form of the metric tensor. The property expressed in the first row is usually referred to as complementarity of the projector pair, while the idempotence of both the projectors is defined by the relationships in the last two rows. The following equality can be easily proven [6]

$$
[M_\parallel]^R_S \frac{\partial}{\partial X^R} [M_\parallel]^S_D = -N_D \frac{\partial N^S}{\partial X^S} = \frac{2}{R_m} N_D \tag{43}
$$

where symbol R_m denotes the local mean curvature over the boundary face.

Analogously, at each point of a border edge $L_\star \equiv \partial\Sigma_\star \equiv \partial\partial\Omega_\star$, which is a unidimensional manifold with codimension two, a pair of complementary linear operators can be defined, apt to project orthogonally any vector of the ambient space onto the tangent space, spanned by the tangent vector $\mathbf{T}$ at that point, and onto its orthogonal complement (i.e., the normal space), spanned by any linear combination of the face normal $\mathbf{N}$ and of the edge normal $\mathbf{B}$. Such projectors will be denoted by symbols $[M_{L\,\parallel}]^A_B$ and $[M_{L\,\perp}]^A_B$, respectively, marked by subscript L. One has

$$
\begin{aligned}
&[M_{L\parallel}]^E_A = T^E\,T_A\,; \quad \mathbf{M}_{L\,\parallel} = \mathbf{T}\otimes\mathbf{T}\,;\\
&[M_{L\perp}]^E_A = B^E\,B_A + N^E\,N_A\,; \quad \mathbf{M}_{L\,\perp} = \mathbf{B}\otimes\mathbf{B} + \mathbf{N}\otimes\mathbf{N}\,;\\
&\delta^E_A = [M_{L\parallel}]^E_A + [M_{L\perp}]^E_A = T^E\,T_A + B^E\,B_A + N^E\,N_A\,;\\
&\mathbf{1} = \mathbf{M}_{L\parallel} + \mathbf{M}_{L\perp} = \mathbf{T}\otimes\mathbf{T} + \mathbf{B}\otimes\mathbf{B} + \mathbf{N}\otimes\mathbf{N}\,;
\end{aligned} \tag{44}
$$

It is worth noting that, along a border edge, the edge tangent vector and the edge normal belong to the plane tangent to the boundary face at that point. Hence, the surface tangential projector acts on them as the identity, namely

$$
[M_\parallel]^E_A T^A = T^E\,; \quad [M_\parallel]^E_A B^A = B^E\,; \tag{45}
$$

When evaluated along an edge, the above face projectors can be represented in terms of edge vectors (constituting the Darboux frame), namely

$$
[M_\parallel]^E_{A'} = T^E\,T_{A'} + B^E\,B_{A'}\,; \quad [M_\perp]^E_{A'} = N^E\,N_{A'}\,;
$$

Two expressions frequently met in the study of third-gradient models are considered at the next point (i) and (ii).

(i) Let us consider the following expression included in some edge integrals (the involved variables are restricted to such an edge)

$$
[M_\parallel]^E_{A'} \frac{\partial}{\partial X^E} \left\{ g^{\star AQ} N_Q [M_\parallel]^{A'}_A \right\} \tag{46}
$$

Exploiting the relationships specified in "Appendix C" at points (v)–(vi), one can write

$$
\begin{aligned}
&= \left(T^E\,T_{A'} + B^E\,B_{A'}\right) \frac{\partial}{\partial X^E} \left\{ g^{\star AQ} N_Q \left(T^{A'}\,T_A + B^{A'}\,B_A \right) \right\}\\
&= T^E\,T_{A'} \frac{\partial}{\partial X^E} \left\{ g^{\star AQ} N_Q T_A T^{A'} \right\} + T^E\,T_{A'} \frac{\partial}{\partial X^E} \left\{ g^{\star AQ} N_Q B_A\,B^{A'} \right\}\\
&\quad + \underbrace{B^E\,B_{A'} \frac{\partial}{\partial X^E} \left\{ g^{\star AQ} N_Q T_A T^{A'} \right\}}_{=0} + B^E\,B_{A'} \frac{\partial}{\partial X^E} \left\{ g^{\star AQ} N_Q B_A\,B^{A'} \right\}\\
&= T^E\,T_{A'} \frac{\partial}{\partial X^E} \left\{ g^{\star AQ} N_Q T_A T^{A'} \right\} + T^E\,T_{A'} \frac{\partial}{\partial X^E} \left\{ g^{\star AQ} N_Q B_A\,B^{A'} \right\}
\end{aligned}
$$

$$+ B^E \underbrace{B_{A'} B^{A'}}_{=1} \frac{\partial}{\partial X^E} \left\{ g^{\star AQ} N_Q B_A \right\} \tag{47}$$

since the derivatives of both the tangent and the edge normal along the direction of the edge normal vanish. The above equality by the product rule can be written as follows:

$$= T^E \underbrace{T_{A'} T^{A'}}_{=1} \frac{\partial}{\partial X^E} \left\{ g^{\star AQ} N_Q T_A \right\} + g^{\star AQ} N_Q T_A \underbrace{T^E T_{A'} \frac{\partial T^{A'}}{\partial X^E}}_{=0}$$

$$+ T^E \underbrace{T_{A'} B^{A'}}_{=0} \frac{\partial}{\partial X^E} \left\{ g^{\star AQ} N_Q B_A \right\} + g^{\star AQ} N_Q B_A \, T^E T_{A'} \frac{\partial B^{A'}}{\partial X^E} + B^E \frac{\partial}{\partial X^E} \left\{ g^{\star AQ} N_Q B_A \right\}$$

$$= T^E \frac{\partial}{\partial X^E} \left\{ g^{\star AQ} N_Q T_A \right\} + g^{\star AQ} N_Q B_A \, T^E T_{A'} \frac{\partial B^{A'}}{\partial X^E} + B^E \frac{\partial}{\partial X^E} \left\{ g^{\star AQ} N_Q B_A \right\} \tag{48}$$

(ii) Another expression frequently entering the virtual work contributions over the boundary surface is the following:

$$[M_\|]^B_S \frac{\partial}{\partial X^B} \left\{ \frac{g^{\star AQ} N_Q}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} \right\} N_A \tag{49}$$

Recalling that $g^{\star AQ} \equiv g^{rs} \left(\mathbf{F}^{-1}\right)^A_r \left(\mathbf{F}^{-1}\right)^Q_s$ and $g^{\star AQ} N_Q N_A = \|\mathbf{F}^{-T}\mathbf{N}\|^2$, by the product rule one can write

$$[M_\|]^B_S \frac{\partial}{\partial X^B} \left\{ \frac{g^{rs} \left(\mathbf{F}^{-1}\right)^A_r \left(\mathbf{F}^{-1}\right)^Q_s N_Q}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} \right\} N_A =$$

$$\begin{aligned}
&= [M_\|]^B_S \frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} g^{rs} \left(\mathbf{F}^{-1}\right)^Q_s N_Q N_A \frac{\partial}{\partial X^B} \left(\mathbf{F}^{-1}\right)^A_r \\
&+ [M_\|]^B_S \frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} g^{rs} \left(\mathbf{F}^{-1}\right)^A_r N_A N_Q \frac{\partial}{\partial X^B} \left(\mathbf{F}^{-1}\right)^Q_s \\
&+ [M_\|]^B_S \frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} g^{rs} \left(\mathbf{F}^{-1}\right)^A_r \left(\mathbf{F}^{-1}\right)^Q_s N_A \frac{\partial N_Q}{\partial X^B} \\
&+ g^{rs} \left(\mathbf{F}^{-1}\right)^A_r \left(\mathbf{F}^{-1}\right)^Q_s N_A N_Q [M_\|]^B_S \frac{\partial}{\partial X^B} \left(\frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} \right)
\end{aligned} \tag{50}$$

Hence, exploiting the symmetry of the metric tensor one obtains

$$\begin{aligned}
&= 2 [M_\|]^B_S \frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} g^{rs} \left(\mathbf{F}^{-1}\right)^Q_s N_Q N_A \frac{\partial}{\partial X^B} \left(\mathbf{F}^{-1}\right)^A_r \\
&+ [M_\|]^B_S \frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} g^{rs} \left(\mathbf{F}^{-1}\right)^A_r \left(\mathbf{F}^{-1}\right)^Q_s N_A \frac{\partial N_Q}{\partial X^B} \\
&+ \underbrace{g^{rs} \left(\mathbf{F}^{-1}\right)^A_r \left(\mathbf{F}^{-1}\right)^Q_s N_A N_Q [M_\|]^B_S \frac{\partial}{\partial X^B} \left(\frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} \right)}_{=(\square)}
\end{aligned} \tag{51}$$

The last addend can be developed as follows:

$$(\square) = g^{rs} \left(\mathbf{F}^{-1}\right)^A_r \left(\mathbf{F}^{-1}\right)^Q_s N_A N_Q \left(- \frac{1}{\left[g^{lm} \left(\mathbf{F}^{-1}\right)^L_l \left(\mathbf{F}^{-1}\right)^M_m N_L N_M \right]^2} \right.$$

$$
\begin{aligned}
&\times 2\, g^{fq}\, \left(\mathbf{F}^{-1}\right)_f^F N_F\, [M_{\|}]_S^B \frac{\partial}{\partial X^B} \left(\left(\mathbf{F}^{-1}\right)_q^G N_G\right) \\
&= -\frac{1}{g^{lm}\, \left(\mathbf{F}^{-1}\right)_l^L \left(\mathbf{F}^{-1}\right)_m^M N_L N_M} \left\{ 2\, g^{fq}\, \left(\mathbf{F}^{-1}\right)_f^F N_F\, [M_{\|}]_S^B \frac{\partial}{\partial X^B} \left(\left(\mathbf{F}^{-1}\right)_q^G N_G\right) \right\} \\
&= -\frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} \left\{ 2\, g^{fq} \left(\mathbf{F}^{-1}\right)_f^F N_F N_G\, [M_{\|}]_S^B \frac{\partial}{\partial X^B} \left(\mathbf{F}^{-1}\right)_q^G \right\} \\
&- \frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} \left\{ 2\, g^{fq}\, \left(\mathbf{F}^{-1}\right)_f^F \left(\mathbf{F}^{-1}\right)_q^G N_F\, [M_{\|}]_S^B \frac{\partial N_G}{\partial X^B} \right\}; \qquad (52)
\end{aligned}
$$

Hence, the expression ($\square$) cancels out the first two addends in the last equality of Eq. 50, which finally becomes

$$
= -[M_{\|}]_S^B \frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|^2}\, g^{rs} \left(\mathbf{F}^{-1}\right)_r^A \left(\mathbf{F}^{-1}\right)_s^Q N_A \frac{\partial N_Q}{\partial X^B} = -\frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|^2}\, g^{\star AQ}\, N_A \frac{\partial N_Q}{\partial X^S}; \qquad (53)
$$

Appendix B. The divergence theorem for submanifolds with boundary

An important achievement in differential geometry concerns the extension of the divergence theorem to vector fields defined over Riemannian submanifolds with boundary, in particular those defined over bidimensional surfaces or along unidimensional curves embedded in a three-dimensional ambient space, see e.g. [61,62]. The formulation outlined herein makes reference to the Lagrangian configuration, but the result equally holds for the Eulerian configuration. Firstly, let us consider the boundary surface $\Sigma_\star \subset \mathcal{R}^3$, and let $L_\star = \partial\Sigma_\star$ denote its border edge, with the unit normal vector indicated by symbol $\mathbf{B}$. At each point of this surface, a tangential projector $[\mathbf{M}_{\|}]_A^C$ can be defined, whose basic properties were outlined in the "Appendix A": such a linear operator projects orthogonally onto the local tangent space any vector of the space environment. Moreover, the tangential surface projector allows one to express the surface divergence operator with respect to the coordinates of the space environment, avoiding any intrinsic representation of the surface. Let W^B be a vector field defined over the curved face $\Sigma_\star$. Under such assumptions, the following equality holds:

$$
\int\limits_{\Sigma_\star} [\mathbf{M}_{\|}]_A^C \frac{\partial}{\partial X^C} \left([\mathbf{M}_{\|}]_B^A W^B\right) \mathrm{d}\Sigma_\star = \int\limits_{L_\star} [\mathbf{M}_{\|}]_A^B W^A B_B\, \mathrm{d}L_\star \qquad (54)
$$

It is worth noting that at the LHS the outer tangential projector has one valence contracted with the partial derivative, and the other valence contracted with the inner projector within the parentheses. The integrand at LHS has the meaning of surface divergence of a tangential vector field, namely $\mathrm{DIV}_{\|}\left(\mathbf{W}_{\|}\right)$.

In the case of a simple, compact curve $L_\star \subset \mathcal{R}^3$, the differential border $\partial L_\star$ of such a unidimensional submanifold is a discrete set constituted of its two ends, say $P_{w\star R}$ and $P_{w\star S}$ (in the order consistent with the curve orientation), and the normal to its border, denoted above by symbol $\mathbf{B}$, is provided by the tangent vector evaluated at the ends, oriented outwards with respect to the interior domain. Accordingly, the tangential projector along the curve can be expressed as $[\mathbf{M}_{L\,\|}]_A^C = T^C T_A$, while the normal space at each point possesses dimension two, see "Appendix A". By formulae, one has

$$
\int\limits_{L_\star} [\mathbf{M}_{L_\star\,\|}]_A^C \frac{\partial}{\partial X^C} \left([\mathbf{M}_{L\,\|}]_B^A W^B\right) \mathrm{d}L_\star = \int\limits_{\partial L_\star} [\mathbf{M}_{L\,\|}]_A^B W^A B_B\, d\partial L_\star = [W^A T_A]_{P_{w\star R}}^{P_{w\star S}}; \qquad (55)
$$

Well-known symbol $[W^A T_A]_{P_{w\star R}}^{P_{w\star S}} \equiv \left(W^A T_A\right)|_{P_{w\star S}} - \left(W^A T_A\right)|_{P_{w\star R}}$ emanates from the fundamental theorem of integral calculus.

Appendix C. Transformation formulae for the edge vectors

Herein, some basic formulae are recalled, apt to transform Lagrangian edge vectors into their Eulerian counterparts. It is worth emphasizing that such relationships not always are available in a closed form, turn out to be not necessarily unique, and are different for the covariant and contravariant components of the same tensor. In any case, the metric tensor can be used for raising or lowering the indices.

(i) The following Eulerian–Lagrangian transformation formula holds for the contravariant tangent vector:

$$t^r = \frac{F^r_R T^R}{\|\mathbf{F}\mathbf{T}\|} = F^r_R T^R \|\mathbf{F}^{-1}\mathbf{t}\| \,;$$
$$\left(t_r = g_{rs} t^s = g_{rs} \frac{F^s_R T^R}{\|\mathbf{F}\mathbf{T}\|} = g_{rs} \frac{F^s_R g^{RS} T_S}{\|\mathbf{F}\mathbf{T}\|} \right) \tag{56}$$

The above formula exploits the relationship between the tangent vector to a curve drawn over a surface in the material configuration, and the tangent to its image through the placement map, at corresponding points, in the Eulerian configuration.

(ii) Another widely adopted transport formula concerns the covariant normal vector to the face (see e.g. [4,58]), namely

$$n_r = \frac{\left(\mathbf{F}^{-1}\right)^R_r N_R}{\|\mathbf{F}^{-T}\mathbf{N}\|} = \left(\mathbf{F}^{-1}\right)^R_r N_R \|\mathbf{F}^T \mathbf{n}\| \,;$$
$$\left(n^r = g^{rs} n_s = g^{rs} \left(\mathbf{F}^{-1}\right)^R_r g_{RS} N^S \frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|}; \right) \tag{57}$$

This equation can be derived by expressing the normal to a surface in the Eulerian configuration as the wedge (cross) product of two independent (Eulerian) tangent vectors, which in turn are images of two independent Lagrangian tangent vectors, and exploiting the well-known relationship between matrix cofactors and determinant. By formulae

$$\mathbf{n} = \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{\|\mathbf{x}_u \wedge \mathbf{x}_v\|} = \frac{\mathbf{F}\mathbf{X}_U \wedge \mathbf{F}\mathbf{X}_V}{\|\mathbf{F}\mathbf{X}_U \wedge \mathbf{F}\mathbf{X}_V\|} = \frac{J\,\mathbf{F}^{-T}\left(\mathbf{X}_U \wedge \mathbf{X}_V\right)}{\|J\,\mathbf{F}^{-T}\left(\mathbf{X}_U \wedge \mathbf{X}_V\right)\|} = \frac{\mathbf{F}^{-T}\mathbf{N}}{\|\mathbf{F}^{-T}\mathbf{N}\|} \,; \tag{58}$$

It is worth noting that u and v are related to the local parametric representation of the surface in the Eulerian configuration, i.e., $\mathbf{x} = \mathbf{x}(u, v)$, so that $\mathbf{x}_u$ and $\mathbf{x}_v$ constitute the local tangent basis vectors. Uppercase symbols U and V play the same role for the material configuration.

(iii) The following transport formulae, proposed in [4,9], hold for the contravariant and covariant edge normal components

$$b_r = \left\{ \left(\mathbf{F}^{-1}\right)^R_r B_R - \frac{\langle \mathbf{F}^{-T}\mathbf{B}, \mathbf{F}^{-T}\mathbf{N}\rangle_g}{\langle \mathbf{F}^{-T}\mathbf{N}, \mathbf{F}^{-T}\mathbf{N}\rangle_g} \left(\mathbf{F}^{-1}\right)^R_r N_R \right\} \frac{\|\mathbf{F}^{-T}\mathbf{N}\|}{\|J^{-1}\mathbf{F}\mathbf{T}\|} \,;$$
$$b^r = \left\{ F^r_R B^R - \frac{\langle \mathbf{F}\mathbf{B}, \mathbf{F}\mathbf{T}\rangle_g}{\langle \mathbf{F}\mathbf{T}, \mathbf{F}\mathbf{T}\rangle_g} F^r_R T^R \right\} \frac{\|\mathbf{F}\mathbf{T}\|}{\|J\mathbf{F}^{-T}\mathbf{N}\|} \,; \tag{59}$$

being as usual $b_r b^r = 1$. These formulae are derived by assuming affine relationships as ansatz, and prescribing the orthogonality conditions $b_r t^r = 1$ and $b_r n^r = 1$. Moreover, they can be regarded as the results of Gram–Schmidt orthogonalization procedures: in fact, although the tangent operator $\mathbf{F}$ transforms material tangent vectors into spatial tangent vectors, the orthogonality between $\mathbf{FB}$ and $\mathbf{FT}$ is not guaranteed.

(iv) Substituting in the relationship $g^{rs} b_r = b^s$ the above transformation formulae for the edge normals and multiplying both the sides by $\left(\mathbf{F}^{-1}\right)^S_s$ one obtains

$$g^{rs} \left\{ \left(\mathbf{F}^{-1}\right)^S_s \left(\mathbf{F}^{-1}\right)^R_r B_R - \frac{\langle \mathbf{F}^{-T}\mathbf{B}, \mathbf{F}^{-T}\mathbf{N}\rangle_g}{\langle \mathbf{F}^{-T}\mathbf{N}, \mathbf{F}^{-T}\mathbf{N}\rangle_g} \left(\mathbf{F}^{-1}\right)^S_s \left(\mathbf{F}^{-1}\right)^R_r N_R \right\}$$

$$= \left\{ B^S - \frac{\langle \mathbf{FB}, \mathbf{FT} \rangle_g}{\langle \mathbf{FT}, \mathbf{FT} \rangle_g} T^S \right\} J^{-2} \frac{\|\mathbf{FT}\|^2}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} ; \tag{60}$$

Hence, introducing the pullback metric tensor in contravariant form, i.e., $g^{\star\, RS} = g^{rs} \left(\mathbf{F}^{-1}\right)_s^S \left(\mathbf{F}^{-1}\right)_r^R$, and multiplying both the sides of the above equation by T_S, the following relationship is found

$$g^{\star\, RS} B_R T_S - \frac{\langle \mathbf{F}^{-T}\mathbf{B}, \mathbf{F}^{-T}\mathbf{N} \rangle_g}{\langle \mathbf{F}^{-T}\mathbf{N}, \mathbf{F}^{-T}\mathbf{N} \rangle_g} g^{\star\, RS} N_R T_S = -\frac{\langle \mathbf{FB}, \mathbf{FT} \rangle_g}{\langle \mathbf{FT}, \mathbf{FT} \rangle_g} J^{-2} \frac{\|\mathbf{FT}\|^2}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} ; \tag{61}$$

Exploiting the equality $\langle \mathbf{FT}, \mathbf{FT} \rangle = \|\mathbf{FT}\|^2$, one can write

$$\langle \mathbf{F}^{-T}\mathbf{B}, \mathbf{F}^{-T}\mathbf{T} \rangle_g - \frac{\langle \mathbf{F}^{-T}\mathbf{B}, \mathbf{F}^{-T}\mathbf{N} \rangle_g}{\langle \mathbf{F}^{-T}\mathbf{N}, \mathbf{F}^{-T}\mathbf{N} \rangle_g} \langle \mathbf{F}^{-T}\mathbf{N}, \mathbf{F}^{-T}\mathbf{T} \rangle_g = -J^{-2} \frac{\langle \mathbf{FB}, \mathbf{FT} \rangle_g}{\langle \mathbf{F}^{-T}\mathbf{N}, \mathbf{F}^{-T}\mathbf{N} \rangle_g} ; \tag{62}$$

or equivalently

$$\langle \mathbf{F}^{-T}\mathbf{B}, \mathbf{F}^{-T}\mathbf{T} \rangle_g \langle \mathbf{F}^{-T}\mathbf{N}, \mathbf{F}^{-T}\mathbf{N} \rangle_g - \langle \mathbf{F}^{-T}\mathbf{B}, \mathbf{F}^{-T}\mathbf{N} \rangle_g \langle \mathbf{F}^{-T}\mathbf{N}, \mathbf{F}^{-T}\mathbf{T} \rangle_g = -J^{-2} \langle \mathbf{FB}, \mathbf{FT} \rangle_g ; \tag{63}$$

Starting from this last equation, by permuting the edge vectors further relationships can be generated, namely

$$\langle \mathbf{F}^{-T}\mathbf{T}, \mathbf{F}^{-T}\mathbf{N} \rangle_g \langle \mathbf{F}^{-T}\mathbf{B}, \mathbf{F}^{-T}\mathbf{B} \rangle_g - \langle \mathbf{F}^{-T}\mathbf{T}, \mathbf{F}^{-T}\mathbf{B} \rangle_g \langle \mathbf{F}^{-T}\mathbf{B}, \mathbf{F}^{-T}\mathbf{N} \rangle_g = -J^{-2} \langle \mathbf{FT}, \mathbf{FN} \rangle_g ; \tag{64}$$

$$\langle \mathbf{F}^{-T}\mathbf{B}, \mathbf{F}^{-T}\mathbf{N} \rangle_g \langle \mathbf{F}^{-T}\mathbf{T}, \mathbf{F}^{-T}\mathbf{T} \rangle_g - \langle \mathbf{F}^{-T}\mathbf{B}, \mathbf{F}^{-T}\mathbf{T} \rangle_g \langle \mathbf{F}^{-T}\mathbf{T}, \mathbf{F}^{-T}\mathbf{N} \rangle_g = -J^{-2} \langle \mathbf{FB}, \mathbf{FN} \rangle_g ; \tag{65}$$

Analogous relationships can be generated by considering the transpose inverse tangent map, hence substituting $\mathbf{F}$ with $\mathbf{F}^{-T}$, $\mathbf{F}^{-T}$ with $\mathbf{F}$ and J with J^{-1}.

(v) It is worth noting that, if we differentiate the squared norm $N_A N^A = 1$, we obtain

$$\frac{\partial N_A}{\partial X_E} N^A = 0 ; \tag{66}$$

As well known, the directional derivative of the face normal along the normal direction vanishes, namely

$$\frac{\partial N_A}{\partial X_E} N^E = 0 ; \tag{67}$$

Moreover, if we prolong smoothly the vector fields $\mathbf{T}$, $\mathbf{B}$ and $\mathbf{N}$ defined along a border edge within a tubular neighborhood, we obtain

$$\frac{\partial B_A}{\partial X_E} N^E = 0 ; \; \frac{\partial B_A}{\partial X_E} B^E = 0 ; \; \frac{\partial T_A}{\partial X_E} N^E = 0 ; \; \frac{\partial T_A}{\partial X_E} B^E = 0 ; \tag{68}$$

(vi) By describing the evolution of Frenet–Serret and Darboux frame vectors along a curve drawn over a surface (such frames differ in a rigid rotation around the edge tangent axis), one obtains

$$\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} \cdot \mathbf{T} = 0 ; \quad \frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s} \cdot \mathbf{B} = 0 ; \quad \frac{\mathrm{d}\mathbf{N}}{\mathrm{d}s} \cdot \mathbf{N} = 0 ; \tag{69}$$

where s denotes as usual the curvilinear abscissa of the edge (intrinsic representation) and symbol $\mathbf{a} \cdot \mathbf{b} \equiv \langle \mathbf{a}, \mathbf{b} \rangle$ indicates the conventional scalar product in the ambient space $\mathcal{R}^3$, with the usual Euclidean metrics. If we consider the first relationship, we can write it equivalently as

$$T^A \frac{\partial T^E}{\partial X^A} T_E = [M_{L\,\|}]_E^A \frac{\partial T^E}{\partial X^A} = 0 ; \tag{70}$$

where the total derivative with respect to the abscissa s was expressed as the directional derivative along the tangent in the ambient space $\mathcal{R}^3$. Of course, for the existence of the gradient of $\mathbf{T}$ a smooth prolongation of the tangent $\mathbf{T}$ in an open neighborhood of the curve is required.

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(Received: July 19, 2022; revised: August 25, 2022; accepted: September 13, 2022)