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Third-gradient continua: nonstandard equilibrium equations and selection of work conjugate variables

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Abstract
This paper outlines the variational derivation of the Lagrangian equilibrium equations for the third-gradient materials, stemming from the minimization of the total potential energy functional, and the selection of suitable dual variables to represent the inner work in the Eulerian configuration. Volume, face, edge and wedge contributions were provided through integration by parts of the inner virtual work and by repeated applications of the divergence theorem extended to embedded submanifolds with codimension one and two. Detailed expressions were provided for the contact pressures and the edge loading, revealing the complex dependence on the face normals and on the mean curvature. Relationships were specified among the Lagrangian (hyper-)stress tensors of rank lower or equal to four, and their Eulerian counterparts.

Keywords
Third-gradient materials, hyperstresses, principle of virtual work, boundary conditions

1. Introduction

In a historical perspective, once that the expert of bridge construction Navier had provided rigorous formulations for the equilibrium problem, continuum mechanics could grow up with the works of Gabrio Piola [1,2] and of Augustin Cauchy [3,4]. These last authors became representatives of two diverse approaches: Piola based the postulation of mechanics on the principle of virtual work, while Cauchy on the balance law of forces and of moments of forces. The *querelle* concerning the most appropriate and effective way to formulate new models in continuum mechanics began immediately after the publication of Piola's works [5–7]. Among the others, Ernst Hellinger in his famous article for the Encyklopädie der mathematische dated 1913 (see [8–10]), established continuum mechanics on the basis of the principle of virtual work: his list of open problems included the generalization of Piola's approach to higher gradient continua.

In recent studies [11–14], the equilibrium equations for the second-gradient materials were derived in a fully variational approach. A viable strategy was drawn to transform such equations from the Lagrangian configuration to the Eulerian configuration, based on remarkable transformation formulae for the tangent and normal to the boundary edges and on a novel formulation for the divergence

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theorem applied to curved surfaces with border, relating material and spatial expressions, which generalizes Piola's bulk transformation. In this paper, the investigation was extended to the third-gradient continua, governed by energy densities which depend on the derivatives of the placement map up to the third order (see, e.g., [15–20]). The approach pursued herein stems from the minimization of the total energy functional: as an alternative, recourse can be made to the postulation scheme for continuum mechanics based on the principle of virtual work, leading to the same results, in which a representation of adequate order can be selected for the inner virtual work regarded as a distribution over the set of virtual displacements. For such a class of materials, the inner work depends on a new ingredient, the *triple stress*, a fourth rank tensor whose entries are dimensionally equivalent to a pressure multiplied by a length squared (see, e.g., [21–23]). In this way, the deformable body becomes capable of bearing not only double forces over the boundary faces and loading distributed along the border edges, but also concentrated forces prescribed at the *wedges*. Such wedges or corner points are located at the intersection of at least three boundary edges, which represent discontinuity loci for the edge tangent. Moreover, the third-gradient modelling allows one to represent several static quantities (or covectors) only apparently exotic entering the equilibrium problem: such generalized loading can make work over the boundary surface versus the first and second normal derivatives of the placement map, and, along the border edges, versus the directional derivative of the placement map along its normals.

The third-gradient modelling represents a wide framework in which several problems of solid and fluid mechanics have been suitably addressed, and many others, already known or still to appear, could find an adequate location. We can mention, for instance, quasicrystals and their dynamic behaviour [24], the dislocation mechanics, the surface instability of homogeneously strained bodies [25], the behaviour of fluid interfaces [26,27], and recently mechanics of bone tissue [28]. Moreover, important insights are expected for multiscale phenomena rooted into the micro or nano structure of engineered materials, implying the existence of a boundary layer and the dependence of the mechanical response on an inner length scale (see, e.g., [29–35]), which are being favouring the development of non conventional numerical models [36–40] and of hybrid numerical experimental strategies [41–47]. Unfortunately, existence, uniqueness and stability results for such a class of materials have not been achieved yet, differently from second-gradient continua (see, e.g., [48–50]). However, several studies recognized common predictive capabilities and formulation similarities among the higher gradient models and other generalized approaches, such as the nonlocal elasticity and peridynamics (see, e.g., [51–53]).

In the recent literature, also the *synthesis* of metamaterials is attracting more and more interest (see, e.g., [32,54–57]). In fact, once the desired mechanical behaviour at the macroscale is specified, the synthesis problem can be regarded as an inverse problem [58–60], seeking the architecture at the microscale apt to generate after homogenization such a macroscopic behaviour [61–68]. For third-gradient continua, the synthesis problem has not yet been addressed. One of the aims of this paper consists of precisely stating the mathematical properties of the macroscopic behaviour assumed as a target, in order to permit the inverse modelling of the microstructure originating such a behaviour. A huge comprehension of the microstructural ingredients giving rise to the third-gradient behaviour can feed novel theories for damage and plasticity, in which higher gradient dependence of deformation energy and Rayleigh functionals [69–74] play a relevant role. Moreover, we expect that also the phase field theory, aiming to regularize the fracture propagation problems, will soon include second- and third-gradient modelling to properly describe the elastic part of the deformation [75]. This opinion is motivated by the need to properly accommodate edge and point contact forces [76], in the presence of fracture processes with geometric nonlinearity.

This paper was organized as follows. In sections 2 and 3, the Lagrangian equilibrium equations for the third-gradient materials were derived through a variational approach, revealing a significant complexity of the diverse contributions and the presence of nonstandard boundary conditions. Section 4 was devoted to alternative expressions for the equilibrium equations which turn out to be especially meaningful, emphasizing the dependence on the normals and on the mean curvature. In section 5, novel transformation formulae were specified, between Lagrangian and Eulerian work conjugate variables, in particular among Piola and Cauchy hyperstress tensors up to the fourth rank. Section 6 outlines some research perspectives for which the present results represent important, although intermediate, achievements. Appendix 1 gathers the basic properties of the surface and edge projectors.

1.1. Notation

Recourse will be made to index, componentwise notation for the involved equations, although sometimes the relevant matrix or tensorial expressions will be reported (see [77]). Classical syntaxis of tensor algebra will be adopted (see [78,79]), with the Einstein convention on the implicit sum of repeated indices. In tensor calculus, to distinguish valences acting on Lagrangian vectors from those specifying Eulerian spaces, e.g., as in F^a_A, the former will be indicated by uppercase letters, i.e., $A, B, \ldots$, the latter by lowercase ones, $a, b, \ldots$. The Lagrangian gradient will be denoted by symbols $\nabla \equiv \partial/\partial X^A$, with the obvious extension to kth F^a_A order gradients as $\nabla^{(k)} = \nabla\,\nabla^{(k-1)}$; seldom recourse will be made to symbol DIV to denote the divergence operator.

2. Third-gradient energy

In some works of Gabrio Piola (see, e.g., [80]), the equilibrium problem of a continuous medium was investigated with reference to general expressions of the deformation energy density, depending not only on the local deformation gradient $\mathbf{F} = \partial\boldsymbol{\chi}/\partial\mathbf{X}$ but also on its higher order spatial derivatives $\nabla^{(k)}\mathbf{F}$, with $k \geqslant 1$. In this paper, we considered energy densities in the form $\mathrm{W}(\mathbf{F}, \nabla\mathbf{F}, \nabla^{(2)}\mathbf{F})$. Accordingly, the placement map $\boldsymbol{\chi}(\mathbf{X})$, defined in the Lagrangian or material domain $\Omega_\star \subset \mathcal{R}^3$ and valued in the Eulerian or deformed domain $\Omega \subset \mathcal{R}^3$ (both equipped with a basis of mutually orthogonal unit vectors), must be sufficiently smooth with its derivatives up to the third order, and ensure a suitable trace regularity (see, e.g., [81]). Assuming for the Jacobian determinant the condition $J = \det(\mathbf{F}) > 0$, the same regularity must be guaranteed for the inverse placement map: hence, the deformation process establishes a diffeomorphism between the reference and the current domain, regarded as differential submanifolds with boundary [12,82]. The objectivity of the energy can be ensured by prescribing a dependence on the right Cauchy–Green tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ and on its gradient $\nabla\mathbf{C}$ and $\nabla^{(2)}\mathbf{C}$ (or on $\mathbf{C}^{-1}$, $\nabla\mathbf{C}^{-1}$, and $\nabla^{(2)}\mathbf{C}^{-1}$), as illustrated in Fortune and Vallee [83] and Auffray et al. [84]. To attain the equilibrium configuration of a deformable body, the minimum of the following energy functional is sought:

$$\hat{\boldsymbol{\chi}} = \arg\min_K \left\{ \mathcal{E}^{\mathrm{TOT}}(\boldsymbol{\chi}) = \int_{\Omega_\star} \mathrm{W}\big(\mathbf{F}, \nabla\mathbf{F}, \nabla^{(2)}\mathbf{F}\big)\, d\Omega_\star - \mathcal{E}^{\mathrm{EXT}}\big(\boldsymbol{\chi}, \nabla\boldsymbol{\chi}, \nabla^{(2)}\boldsymbol{\chi}\big) \right\}, \tag{1}$$

where symbol $\mathcal{E}^{\mathrm{EXT}}$ denotes the external work, and the feasible functional subset K must ensure the sufficient regularity incorporating the essential boundary conditions on the placement map and on its first- and second-order normal derivatives. The volume integral of the stored energy density in equation (1) will be denoted by symbol $\mathcal{E}^{\mathrm{DEF}}$. It is worth emphasizing that the stored energy density is expressed in terms of $\mathbf{F}$ instead of $\mathbf{C}$ (and relevant gradients) to make the formulation simpler and permit the analytical manipulation. When prescribing the stationarity condition, the first variation of the above functional, discussed in Fedele [12], can be evaluated herein with reference to such a more general energy density, as follows

$$\begin{aligned}
&\delta \int_{\Omega_\star} \mathrm{W}\big(\mathbf{F}, \nabla\mathbf{F}, \nabla^{(2)}\mathbf{F}\big)\, d\Omega_\star = \\
&= \int_{\Omega_\star} \left(\frac{\partial \mathrm{W}}{\partial \mathbf{F}} : \delta\mathbf{F} + \frac{\partial \mathrm{W}}{\partial \nabla\mathbf{F}} \,\vdots\, \delta(\nabla\mathbf{F}) + \frac{\partial \mathrm{W}}{\partial \nabla^{(2)}\mathbf{F}} :: \delta\big(\nabla^{(2)}\mathbf{F}\big) \right) d\Omega_\star = \\
&= \underbrace{\int_{\Omega_\star} \frac{\partial \mathrm{W}}{\partial F^i{}_A}\, \delta F^i{}_A \, d\Omega_\star}_{=\delta\mathcal{E}^{\mathrm{DEF}}_I} + \underbrace{\int_{\Omega_\star} \frac{\partial \mathrm{W}}{\partial F^i{}_{A,B}}\, \delta F^i{}_{A,B} \, d\Omega_\star}_{=\delta\mathcal{E}^{\mathrm{DEF}}_{II}} + \underbrace{\int_{\Omega_\star} \frac{\partial \mathrm{W}}{\partial F^i{}_{A,BC}}\, \delta F^i{}_{A,BC} \, d\Omega_\star}_{=\delta\mathcal{E}^{\mathrm{DEF}}_{III}}
\end{aligned} \tag{2}$$

where symbols $:$, $\vdots$, and $::$ denote the usual double dot product, the triple, and quadruple contraction, respectively (see, e.g., [77]). The contributions of the external virtual work $\delta\mathcal{E}^{\mathrm{EXT}}$ will be specified elsewhere. The first two addends $\delta\mathcal{E}^{\mathrm{DEF}}_I$ and $\delta\mathcal{E}^{\mathrm{DEF}}_{II}$ were integrated in Fedele [12,13]. Now let us focus on the term involving the third gradient of the placement map, above specified as $\delta\mathcal{E}^{\mathrm{DEF}}_{III}$. Considering that $\delta F^i_{A,BC} = \partial^3\delta\chi^i/\partial X^A\partial X^B\partial X^C$, the above equation can be further reduced. Moreover, due to the Schwarz theorem, one has $F^i_{A,BC} = F^i_{A,CB} = F^i_{C,AB} = F^i_{C,BA} = F^i_{B,CA} = F^i_{B,AC}$ (3! permutations) for each i, this tensor is totally symmetric in the Lagrangian valences, and the comma can be omitted without ambiguity. By commuting the partial derivative and the first variation, integrating by parts one has

$$\begin{aligned}\delta\mathcal{E}^{\mathrm{DEF}}_{III} &= \int_{\Omega_\star} \frac{\partial \mathrm{W}}{\partial F^i_{ABC}} \delta F^i_{ABC}\, d\Omega_\star = \int_{\Omega_\star} \frac{\partial \mathrm{W}}{\partial F^i_{ABC}} \frac{\partial}{\partial X^C}\left(\delta F^i_{AB}\right) d\Omega_\star = \\ &= \int_{\Omega_\star} \left[\frac{\partial}{\partial X^C}\left(\frac{\partial \mathrm{W}}{\partial F^i_{ABC}} \delta F^i_{AB}\right) - \frac{\partial}{\partial X^C}\left(\frac{\partial \mathrm{W}}{\partial F^i_{ABC}}\right)\delta F^i_{AB}\right] d\Omega_\star\end{aligned} \tag{3}$$

Applying the Gauss–Ostrogradsky divergence theorem to the first addend, and denoting the boundary surface as $\Sigma_\star \equiv \partial\Omega_\star$, one obtains

$$\delta\mathcal{E}^{\mathrm{DEF}}_{III} = \underbrace{\int_{\Sigma_\star} \frac{\partial \mathrm{W}}{\partial F^i_{ABC}} \delta F^i_{AB} N_C\, d\Sigma_\star}_{(\diamond)} \underbrace{- \int_{\Omega_\star} \frac{\partial}{\partial X^C}\left(\frac{\partial \mathrm{W}}{\partial F^i_{ABC}}\right)\delta F^i_{AB}\, d\Omega_\star}_{(\ddagger)} \tag{4}$$

while in the volume integral the test function δF^i_{AB} can be further reduced as follows

$$\begin{aligned}(\ddagger) &= - \int_{\Omega_\star} \frac{\partial}{\partial X^C}\left(\frac{\partial \mathrm{W}}{\partial F^i_{ABC}}\right) \frac{\partial}{\partial X^B}\frac{\partial}{\partial X^A}\delta\chi^i\, d\Omega_\star = \\ &= - \int_{\Sigma_\star} \frac{\partial}{\partial X^C}\left(\frac{\partial \mathrm{W}}{\partial F^i_{ABC}}\right) \frac{\partial}{\partial X^A}\delta\chi^i N_B\, d\Sigma_\star + \\ &\quad + \int_{\Omega_\star} \frac{\partial}{\partial X^B}\frac{\partial}{\partial X^C}\left(\frac{\partial \mathrm{W}}{\partial F^i_{ABC}}\right) \frac{\partial}{\partial X^A}\delta\chi^i\, d\Omega_\star =\end{aligned} \tag{5}$$

In equation (5), the last contribution can easily be integrated by parts with respect to the partial derivative left, obtaining, through the divergence theorem, another term referred to the boundary surface plus a not reducible volume integral, namely

$$\begin{aligned} &= - \int_{\Sigma_\star} \frac{\partial}{\partial X^C}\left(\frac{\partial \mathrm{W}}{\partial F^i_{ABC}}\right)\delta F^i_{A} N_B\, d\Sigma_\star + \\ &\quad + \int_{\Omega_\star} \frac{\partial}{\partial X^A}\left[\frac{\partial}{\partial X^B}\frac{\partial}{\partial X^C}\left(\frac{\partial \mathrm{W}}{\partial F^i_{ABC}}\right)\delta\chi^i\right] d\Omega_\star + \\ &\quad - \int_{\Omega_\star} \frac{\partial}{\partial X^A}\left[\frac{\partial}{\partial X^B}\frac{\partial}{\partial X^C}\left(\frac{\partial \mathrm{W}}{\partial F^i_{ABC}}\right)\right]\delta\chi^i\, d\Omega_\star \\ &= \underbrace{- \int_{\Sigma_\star} \frac{\partial}{\partial X^C}\left(\frac{\partial \mathrm{W}}{\partial F^i_{ABC}}\right)\frac{\partial}{\partial X^A}\delta\chi^i N_B\, d\Sigma_\star}_{=(\Delta)} + \\ &\quad + \int_{\Sigma_\star} \frac{\partial}{\partial X^B}\frac{\partial}{\partial X^C}\left(\frac{\partial \mathrm{W}}{\partial F^i_{ABC}}\right)\delta\chi^i N_A\, d\Sigma_\star + \\ &\quad - \int_{\Omega_\star} \frac{\partial}{\partial X^A}\frac{\partial}{\partial X^B}\frac{\partial}{\partial X^C}\left(\frac{\partial \mathrm{W}}{\partial F^i_{ABC}}\right)\delta\chi^i\, d\Omega_\star\end{aligned} \tag{6}$$

To reduce the boundary term (Δ) of equation (6), a specific strategy was proposed by Paul Germain (see, e.g., [22,85]). In fact, by utilizing the relationship $\delta^C_A = [M_\parallel]^C_A + [M_\perp]^C_A$ involving complementary surface projectors [12] (see Appendix 1), the gradient of the virtual placement map in Lagrangian form can be additively decomposed into a normal and and a tangential contribution, namely

$$
\begin{aligned}
(\Delta) = & -\int_{\Sigma_\star} \frac{\partial}{\partial X^C}\left(\frac{\partial \mathrm{W}}{\partial F^i_{\ ABC}}\right) N_B \delta^D_A \left[\frac{\partial}{\partial X^D}\delta\chi^i\right] d\Sigma_\star = \\
= & -\int_{\Sigma_\star} \frac{\partial}{\partial X^C}\left(\frac{\partial \mathrm{W}}{\partial F^i_{\ ABC}}\right) N_B \left([M_\|]^D_A + [M_\perp]^D_A\right) \left[\frac{\partial}{\partial X^A}\delta\chi^i\right] d\Sigma_\star = \\
= & \underbrace{-\int_{\Sigma_\star} \frac{\partial}{\partial X^C}\left(\frac{\partial \mathrm{W}}{\partial F^i_{\ ABC}}\right) N_B [M_\perp]^D_A \left[\frac{\partial}{\partial X^D}\delta\chi^i\right] d\Sigma_\star}_{(\perp)} \\
& \underbrace{-\int_{\Sigma_\star} \frac{\partial}{\partial X^C}\left(\frac{\partial \mathrm{W}}{\partial F^i_{\ ABC}}\right) N_B [M_\|]^D_A \left[\frac{\partial}{\partial X^D}\delta\chi^i\right] d\Sigma_\star}_{(\|)}
\end{aligned}
\tag{7}
$$

The addend $(\perp)$ with the normal projector includes the normal derivative of the virtual placement and cannot be further reduced. In fact, recalling that $[\mathbf{M}_\perp]^D_A = N^D N_A$, from equation (7) one can write

$$
\begin{aligned}
(\perp) = & -\int_{\Sigma_\star} \frac{\partial}{\partial X^C}\left(\frac{\partial \mathrm{W}}{\partial F^i_{\ ABC}}\right) N_B N_A N^D \left[\frac{\partial}{\partial X^D}\delta\chi^i\right] d\Sigma_\star = \\
= & -\int_{\Sigma_\star} \frac{\partial}{\partial X^C}\left(\frac{\partial \mathrm{W}}{\partial F^i_{\ ABC}}\right) N_B N_A \left[N^D \frac{\partial}{\partial X^D}\delta\chi^i\right] d\Sigma_\star = \\
= & -\int_{\Sigma_\star} \frac{\partial}{\partial X^C}\left(\frac{\partial \mathrm{W}}{\partial F^i_{\ ABC}}\right) N_B N_A \left[\frac{\partial}{\partial N}\delta\chi^i\right] d\Sigma_\star
\end{aligned}
\tag{8}
$$

On the contrary, surface term $(\|)$ in equation (7) including the tangential projector $\mathbf{M}_\|$ can be further reduced. Exploiting the idempotence of the projector (see Appendix 1), namely, by the property $[M_\|]^D_A = [M_\|]^E_A [M_\|]^D_E$, we can duplicate it and utilize the ambient variables for both the divergence operator and the virtual placement map, thus avoiding the intrinsic representation of the surface. Through integration by parts one obtains

$$
\begin{aligned}
(\|) = & -\int_{\Sigma_\star} \frac{\partial}{\partial X^C}\left(\frac{\partial \mathrm{W}}{\partial F^i_{\ ABC}}\right) N_B [M_\|]^E_A \frac{\partial}{\partial X^D}\delta\chi^i [M_\|]^D_E d\Sigma_\star = \\
= & -\int_{\Sigma_\star} \left\{ \frac{\partial}{\partial X^D}\left[\frac{\partial}{\partial X^C}\left(\frac{\partial \mathrm{W}}{\partial F^i_{\ ABC}}\right) N_B [M_\|]^E_A \delta\chi^i\right] + \right. \\
& \left. -\frac{\partial}{\partial X^D}\left[\frac{\partial}{\partial X^C}\left(\frac{\partial \mathrm{W}}{\partial F^i_{\ ABC}}\right) N_B [M_\|]^E_A\right] \delta\chi^i \right\} [M_\|]^D_E d\Sigma_\star
\end{aligned}
\tag{9}
$$

The first addend in equation (9), once attained the format suitable for the surface divergence theorem (see e.g. [84,86]), can be transported to the border edge $L_\star \equiv \partial\Sigma_\star = \partial\partial\Omega_\star$ as follows

$$
\begin{aligned}
= & -\int_{L_\star} \frac{\partial}{\partial X^C}\left(\frac{\partial \mathrm{W}}{\partial F^i_{\ ABC}}\right) N_B B_A \delta\chi^i \, dL_\star + \\
& + \int_{\Sigma_\star} [M_\|]^D_E \frac{\partial}{\partial X^D}\left\{\frac{\partial}{\partial X^C}\left(\frac{\partial \mathrm{W}}{\partial F^i_{\ ABC}}\right) N_B [M_\|]^E_A\right\} \delta\chi^i \, d\Sigma_\star.
\end{aligned}
\tag{10}
$$

In the first addend, we have set $[M_\|]^E_A B_E = B_A$, since the edge normal vector is tangent to the boundary face. The second addend, to be discussed later, must be interpreted as the surface divergence of a tangential vector field $\mathrm{DIV}_{\|\Sigma_\star}(\mathbf{v}_\|)$, where $\mathbf{v}$ herein is the triple stress with two indices already contracted and one (Lagrangian) valence left free for the projector. The expressions within the volume and over the boundary surface derived above must be added to their counterparts provided by the first- and the second-gradient contributions of the energy variation (see [11–13]).

3. Nonstandard boundary conditions

Let us consider the surface integral in equation (4) marked by ($\diamond$). Through the same strategy utilized above, resting on an additive decomposition by complementary projectors (see Appendix 1), we can manage the partial derivatives of the virtual placement one at a time, obtaining

$$\begin{aligned}(\diamond) &= +\int_{\Sigma_\star} \frac{\partial \mathrm{W}}{\partial F^i_{ABC}} N_C \left[\frac{\partial}{\partial X^B}\frac{\partial}{\partial X^A}\delta\chi^i\right] d\Sigma_\star = \\ &= +\int_{\Sigma_\star} \frac{\partial \mathrm{W}}{\partial F^i_{ABC}} N_C \delta^E_B \left[\frac{\partial}{\partial X^E}\frac{\partial}{\partial X^A}\delta\chi^i\right] d\Sigma_\star = \\ &= +\int_{\Sigma_\star} \frac{\partial \mathrm{W}}{\partial F^i_{ABC}} N_C \left([M_\perp]^E_B + [M_\parallel]^E_B\right) \left[\frac{\partial}{\partial X^E}\frac{\partial}{\partial X^A}\delta\chi^i\right] d\Sigma_\star =\end{aligned} \tag{11}$$

Recalling the expression of the normal projector (see [12,84] and Appendix 1), one finds

$$\begin{aligned}&= \underbrace{+\int_{\Sigma_\star} \frac{\partial \mathrm{W}}{\partial F^i_{ABC}} N_C N^E N_B \left[\frac{\partial}{\partial X^E}\frac{\partial}{\partial X^A}\delta\chi^i\right] d\Sigma_\star}_{=(\triangleright)} + \\ &\quad + \int_{\Sigma_\star} \frac{\partial \mathrm{W}}{\partial F^i_{ABC}} N_C [M_\parallel]^E_B \left[\frac{\partial}{\partial X^E}\frac{\partial}{\partial X^A}\delta\chi^i\right] d\Sigma_\star =\end{aligned} \tag{12}$$

Considering the Schwarz theorem on the permutation of the mixed derivatives, we can further manipulate the first addend in equation (12) obtaining

$$\begin{aligned}(\triangleright) &= +\int_{\Sigma_\star} \frac{\partial \mathrm{W}}{\partial F^i_{ABC}} N_B N_C N^E \underbrace{\left([M_\perp]^R_A + [M_\parallel]^R_A\right)}_{=\delta^R_A} \frac{\partial}{\partial X^R}\left[\frac{\partial}{\partial X^E}\delta\chi^i\right] d\Sigma_\star = \\ &= +\int_{\Sigma_\star} \frac{\partial \mathrm{W}}{\partial F^i_{ABC}} N_B N_C N^E N^R N_A \frac{\partial}{\partial X^R}\left[\frac{\partial}{\partial X^E}\delta\chi^i\right] d\Sigma_\star + \\ &\quad - \int_{\Sigma_\star} \frac{\partial \mathrm{W}}{\partial F^i_{ABC}} N_B N_C N^E [M_\parallel]^R_A \frac{\partial}{\partial X^R}\left[\frac{\partial}{\partial X^E}\delta\chi^i\right] d\Sigma_\star = \\ &= +\int_{\Sigma_\star} \frac{\partial \mathrm{W}}{\partial F^i_{ABC}} N_B N_C N_A \frac{\partial}{\partial N}\left[\frac{\partial}{\partial N}\delta\chi^i\right] d\Sigma_\star + \\ &\quad + \int_{\Sigma_\star} \frac{\partial \mathrm{W}}{\partial F^i_{ABC}} N_B N_C N^E [M_\parallel]^R_A \frac{\partial}{\partial X^R}\left[\frac{\partial}{\partial X^E}\delta\chi^i\right] d\Sigma_\star =\end{aligned} \tag{13}$$

We can recognize the second order directional derivative along the normal of the virtual placement map, namely $(\partial^2/\partial N^2)\delta\chi^i$, which was not included among the second-gradient equations (see [14]). The dual covector represents a triple inner force acting over the boundary face. Grouping the residual contributions, namely the last addends in equations (12) and (13) which include the tangential projectors, exploiting idempotence and integrating by parts one has

$$\begin{aligned}
&+\int_{\Sigma_\star} \frac{\partial}{\partial X^R}\left\{ \frac{\partial \mathrm{W}}{\partial F^i_{\ ABC}} N_B N_C N^E\, [M_\|]^{A'}_A\, \frac{\partial}{\partial X^E}\,\delta\chi^i \right\} [M_\|]^R_{A'}\, d\Sigma_\star +\\
&-\int_{\Sigma_\star} \frac{\partial}{\partial X^R}\left\{ \frac{\partial \mathrm{W}}{\partial F^i_{\ ABC}} N_B N_C N^E\, [M_\|]^{A'}_A \right\} \frac{\partial}{\partial X^E}\,\delta\chi^i\, [M_\|]^R_{A'}\, d\Sigma_\star +\\
&+\int_{\Sigma_\star} [M_\|]^E_{E'}\, \frac{\partial}{\partial X^E}\left\{ \frac{\partial \mathrm{W}}{\partial F^i_{\ ABC}} N_C [M_\|]^{E'}_B\, \frac{\partial}{\partial X^A}\,\delta\chi^i \right\} d\Sigma_\star +\\
&-\int_{\Sigma_\star} [M_\|]^E_{E'}\, \frac{\partial}{\partial X^E}\left\{ \frac{\partial \mathrm{W}}{\partial F^i_{\ ABC}} N_C [M_\|]^{E'}_B \right\} \frac{\partial}{\partial X^A}\,\delta\chi^i\, d\Sigma_\star =
\end{aligned} \tag{14}$$

At this point, the surface divergence theorem can be applied to the first and third term in equation (14). Resulting $[M_\|]^R_{A'}[M_\|]^{A'}_A B_R = B_A$, one finds for the first two addends

$$\begin{aligned}
&+\int_{L_\star} \frac{\partial \mathrm{W}}{\partial F^i_{\ ABC}} N_B N_C B_A N^E \left[\frac{\partial}{\partial X^E}\,\delta\chi^i\right] dL_\star +\\
&-\int_{\Sigma_\star} [M_\|]^R_{A'}\, \frac{\partial}{\partial X^R}\left(\frac{\partial \mathrm{W}}{\partial F^i_{\ ABC}} N_B N_C N^E [M_\|]^{A'}_A \right) \left[\frac{\partial}{\partial X^E}\,\delta\chi^i\right] d\Sigma_\star =
\end{aligned} \tag{15}$$

Moreover, further decomposing the Lagrangian gradient of the virtual placement into a normal and a tangential component, we can write

$$\begin{aligned}
&=+\int_{L_\star} \frac{\partial \mathrm{W}}{\partial F^i_{\ ABC}} N_B N_C B_A N^E \left[\frac{\partial}{\partial X^E}\,\delta\chi^i\right] dL_\star +\\
&-\int_{\Sigma_\star} [M_\|]^R_{A'}\, \frac{\partial}{\partial X^R}\left(\frac{\partial \mathrm{W}}{\partial F^i_{\ ABC}} N_B N_C N^E [M_\|]^{A'}_A \right) [M_\perp]^S_E \left[\frac{\partial}{\partial X^S}\,\delta\chi^i\right] d\Sigma_\star +\\
&-\int_{\Sigma_\star} [M_\|]^R_{A'}\, \frac{\partial}{\partial X^R}\left(\frac{\partial \mathrm{W}}{\partial F^i_{\ ABC}} N_B N_C N^E [M_\|]^{A'}_A \right) [M_\|]^S_E \left[\frac{\partial}{\partial X^S}\,\delta\chi^i\right] d\Sigma_\star =\\
&=+\int_{L_\star} \frac{\partial \mathrm{W}}{\partial F^i_{\ ABC}} N_B N_C B_A \left[\frac{\partial}{\partial N}\,\delta\chi^i\right] dL_\star +\\
&-\int_{\Sigma_\star} [M_\|]^R_{A'}\, \frac{\partial}{\partial X^R}\left(\frac{\partial \mathrm{W}}{\partial F^i_{\ ABC}} N_B N_C N^E [M_\|]^{A'}_A \right) N_E \left[\frac{\partial}{\partial N}\,\delta\chi^i\right] d\Sigma_\star +\\
&\underbrace{-\int_{\Sigma_\star} [M_\|]^R_{A'}\, \frac{\partial}{\partial X^R}\left(\frac{\partial \mathrm{W}}{\partial F^i_{\ ABC}} N_B N_C N^E [M_\|]^{A'}_A \right) [M_\|]^S_E \left(\frac{\partial}{\partial X^S}\,\delta\chi^i\right) d\Sigma_\star}_{=(\boxtimes)}
\end{aligned} \tag{16}$$

The first two contributions in the last equality of equation (16) cannot be further reduced including the first normal derivative of the placement map. For the last addend one has

$$\begin{aligned}
(\boxtimes) = &-\int_{\Sigma_\star} [M_\|]^S_{E'}\, \frac{\partial}{\partial X^S}\left\{ [M_\|]^R_{A'}\, \frac{\partial}{\partial X^R}\left(\frac{\partial \mathrm{W}}{\partial F^i_{\ ABC}} N_B N_C N^E [M_\|]^{A'}_A \right)\right.\\
&\left. [M_\|]^{E'}_E\, \delta\chi^i \right\} d\Sigma_\star +\\
&+\int_{\Sigma_\star} [M_\|]^S_{E'}\, \frac{\partial}{\partial X^S}\left\{ [M_\|]^R_{A'}\, \frac{\partial}{\partial X^R}\left(\frac{\partial \mathrm{W}}{\partial F^i_{\ ABC}} N_B N_C N^E [M_\|]^{A'}_A \right) [M_\|]^{E'}_E \right\} \delta\chi^i\, d\Sigma_\star =\\
= &-\int_{L_\star} [M_\|]^R_{A'}\, \frac{\partial}{\partial X^R}\left(\frac{\partial \mathrm{W}}{\partial F^i_{\ ABC}} N_B N_C N^E [M_\|]^{A'}_A \right) B_E\, \delta\chi^i\, dL_\star +\\
&+\int_{\Sigma_\star} [M_\|]^S_{E'}\, \frac{\partial}{\partial X^S}\left\{ [M_\|]^R_{A'}\, \frac{\partial}{\partial X^R}\left(\frac{\partial \mathrm{W}}{\partial F^i_{\ ABC}} N_B N_C N^E [M_\|]^{A'}_A \right) [M_\|]^{E'}_E \right\} \delta\chi^i\, d\Sigma_\star
\end{aligned} \tag{17}$$

Analogously, being $[M_{\|}]^{E}_{E'}[M_{\|}]^{E'}_{B}\,B_E = B_B$, for the last two addends in equation (14) one has

$$\underbrace{+\int_{L_\star}\frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}}N_C B_B\,\frac{\partial}{\partial X^A}\,\delta\chi^i\,dL_\star +}_{=(\odot)} \underbrace{-\int_{\Sigma_\star}[M_{\|}]^{E}_{E'}\frac{\partial}{\partial X^E}\left[\frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}}N_C[M_{\|}]^{E'}_{B}\right]\frac{\partial}{\partial X^A}\,\delta\chi^i\,d\Sigma_\star =}_{=(^\circ)} \tag{18}$$

To reduce the terms in equation (18), let us observe that a pair of linear operators can be defined also along the edge $L_\star$ projecting vectors of the ambient space onto complementary subspaces: we will refer to them as tangential and normal edge projectors (marked by the subscript L), namely, $[M_{L\|}]^E_A = T^E\,T_A$ and $[M_{L\perp}]^E_A = B^E B_A + N^E N_A$, being $\delta^E_A = [M_{L\|}]^E_A + [M_{L\perp}]^E_A$ (see Appendix 1). Hence, one can write

$$\begin{aligned}(\odot)+(^\circ) = &+\int_{L_\star}\frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}}N_C B_B\underbrace{\left([M_{L\|}]^E_A + [M_{L\perp}]^E_A\right)}_{=\delta^E_A}\frac{\partial}{\partial X^E}\,\delta\chi^i\,dL_\star + \\ &-\int_{\Sigma_\star}[M_{\|}]^{S}_{E'}\frac{\partial}{\partial X^S}\left[\frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}}N_C[M_{\|}]^{E'}_{B}\right]\underbrace{\left([M_{\|}]^E_A + [M_{\perp}]^E_A\right)}_{=\delta^E_A}\frac{\partial}{\partial X^E}\,\delta\chi^i\,d\Sigma_\star =\end{aligned} \tag{19}$$

Paying attention to the simultaneous presence of surface and edge projectors (the latter marked by subscript L, see Appendix 1), we find

$$\begin{aligned}= &+\int_{L_\star}\frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}}N_C B_B[M_{L\|}]^E_A\frac{\partial}{\partial X^E}\,\delta\chi^i\,dL_\star + \\ &+\int_{L_\star}\frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}}N_C B_B[M_{L\perp}]^E_A\frac{\partial}{\partial X^E}\,\delta\chi^i\,dL_\star + \\ &-\int_{\Sigma_\star}[M_{\|}]^{S}_{E'}\frac{\partial}{\partial X^S}\left[\frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}}N_C[M_{\|}]^{E'}_{B}\right][M_{\|}]^E_A\frac{\partial}{\partial X^E}\,\delta\chi^i\,d\Sigma_\star + \\ &-\int_{\Sigma_\star}[M_{\|}]^{S}_{E'}\frac{\partial}{\partial X^S}\left[\frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}}N_C[M_{\|}]^{E'}_{B}\right][M_{\perp}]^E_A\frac{\partial}{\partial X^E}\,\delta\chi^i\,d\Sigma_\star =\end{aligned} \tag{20}$$

By representing as usual the normal face projector through the face normals, namely $[M_\perp]^E_A = N^E N_A$, and expressing the projectors along the border edges (with subscript L) in terms of the mutually orthogonal unit vectors $\boldsymbol{B}$, $\boldsymbol{T}$, $\boldsymbol{N}$, one can write

$$\begin{aligned}= &+\int_{L_\star}\frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}}N_C B_B[M_{L\|}]^E_A\frac{\partial}{\partial X^E}\,\delta\chi^i\,dL_\star + \\ &+\int_{L_\star}\frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}}N_C B_B B^E B_A\,\frac{\partial}{\partial X^E}\,\delta\chi^i\,dL_\star + \\ &+\int_{L_\star}\frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}}N_C B_B N^E N_A\,\frac{\partial}{\partial X^E}\,\delta\chi^i\,dL_\star + \\ &-\int_{\Sigma_\star}[M_{\|}]^{S}_{E'}\frac{\partial}{\partial X^S}\left[\frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}}N_C[M_{\|}]^{E'}_{B}\right][M_{\|}]^E_A\frac{\partial}{\partial X^E}\,\delta\chi^i\,d\Sigma_\star + \\ &-\int_{\Sigma_\star}[M_{\|}]^{S}_{E'}\frac{\partial}{\partial X^S}\left[\frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}}N_C[M_{\|}]^{E'}_{B}\right]N_A\left[\frac{\partial}{\partial N}\,\delta\chi^i\right]d\Sigma_\star =\end{aligned} \tag{21}$$

where the edge contribution with the normal edge projector was split into two since $[\mathbf{M}_{L\perp}] = \mathbf{B} \otimes \mathbf{B} + \mathbf{N} \otimes \mathbf{N}$. Hence, exploiting idempotence of the projectors in the first and fourth row of equation (21), rearranging terms one finds

$$\begin{aligned}
&= + \int_{L_\star} \frac{\partial \mathrm{W}}{\partial F^i_{ABC}} B_A B_B N_C \left[\frac{\partial}{\partial B} \delta\chi^i \right] dL_\star + \\
&+ \int_{L_\star} \frac{\partial \mathrm{W}}{\partial F^i_{ABC}} N_A B_B N_C \left[\frac{\partial}{\partial N} \delta\chi^i \right] dL_\star + \\
&- \int_{\Sigma_\star} [M_\parallel]^S_{E'} \frac{\partial}{\partial X^S} \left(\frac{\partial \mathrm{W}}{\partial F^i_{ABC}} N_C [M_\parallel]^{E'}_B \right) N_A \left[\frac{\partial}{\partial N} \delta\chi^i \right] d\Sigma_\star + \\
&\underbrace{+ \int_{L_\star} \frac{\partial \mathrm{W}}{\partial F^i_{ABC}} B_B N_C [M_{L\parallel}]^{A'}_A \frac{\partial \delta\chi^i}{\partial X^E} [M_{L\parallel}]^E_{A'} dL_\star}_{=(\bowtie)} + \\
&\underbrace{- \int_{\Sigma_\star} [M_\parallel]^S_{E'} \frac{\partial}{\partial X^S} \left[\frac{\partial \mathrm{W}}{\partial F^i_{ABC}} N_C [M_\parallel]^{E'}_B \right] [M_\parallel]^{A'}_A \frac{\partial \delta\chi^i}{\partial X^E} [M_\parallel]^E_{A'} d\Sigma_\star}_{=(\propto)}
\end{aligned} \tag{22}$$

In the last equality of equation (22), the first three addends, namely edge and surface integrals including the directional derivatives of the virtual placement along the relevant normals, cannot be further reduced. The last two integrals, including tangential projectors (along the edge and over the face), must be integrated by parts to achieve the format suitable for the divergence theorem. For these last two addends one can write

$$\begin{aligned}
(\bowtie) + (\propto) &= + \int_{L_\star} \frac{\partial}{\partial X^E} \left\{ \frac{\partial \mathrm{W}}{\partial F^i_{ABC}} B_B N_C [M_{L\parallel}]^{A'}_A \delta\chi^i \right\} [M_{L\parallel}]^E_{A'} dL_\star + \\
&- \int_{L_\star} [M_{L\parallel}]^E_{A'} \frac{\partial}{\partial X^E} \left\{ \frac{\partial \mathrm{W}}{\partial F^i_{ABC}} B_B N_C [M_{L\parallel}]^{A'}_A \right\} \delta\chi^i \, dL_\star + \\
&- \int_{\Sigma_\star} \frac{\partial}{\partial X^E} \left\{ [M_\parallel]^S_{E'} \frac{\partial}{\partial X^S} \left(\frac{\partial \mathrm{W}}{\partial F^i_{ABC}} N_C [M_\parallel]^{E'}_B \right) [M_\parallel]^{A'}_A \delta\chi^i \right\} [M_\parallel]^E_{A'} d\Sigma_\star + \\
&+ \int_{\Sigma_\star} \frac{\partial}{\partial X^E} \left\{ [M_\parallel]^S_{E'} \frac{\partial}{\partial X^S} \left(\frac{\partial \mathrm{W}}{\partial F^i_{ABC}} N_C [M_\parallel]^{E'}_B \right) [M_\parallel]^{A'}_A \right\} \delta\chi^i [M_\parallel]^E_{A'} d\Sigma_\star =
\end{aligned} \tag{23}$$

At this stage, we can apply the divergence theorem to the first and third addends of equation (23). Differently from the first-gradient theory, in which only the volume boundary is considered, and from the second-gradient approach, in which contributions relevant to the border edges are included, here the differential "border" of a curved edge $L_\star$ is involved, i.e., $\partial L_\star = \partial\partial\partial\Omega_\star$. Such a discrete set is constituted of the end wedges, separating from each other the contiguous regular edges: the tangential space is spanned by the edge tangent ($\mathbf{M}_{L\parallel} = \mathbf{T} \otimes \mathbf{T}$, see Appendix 1), and the divergence theorem for the first addend along the border edge $L_\star$ is reduced to the fundamental theorem of calculus. Hence, one can write

$$\begin{aligned}
&= + \sum_r \left[\left(\frac{\partial \mathrm{W}}{\partial F^i_{ABC}} N_C B_B T_A \right) \delta\chi^i \right]^{P_{\star r}}_{P_{\star(r-1)}} + \\
&- \int_{L_\star} T^E T_{A'} \frac{\partial}{\partial X^E} \left\{ \frac{\partial \mathrm{W}}{\partial F^i_{ABC}} N_C B_B T^{A'} T_A \right\} \delta\chi^i \, dL_\star + \\
&- \int_{L_\star} [M_\parallel]^E_{E'} \frac{\partial}{\partial X^E} \left(\frac{\partial \mathrm{W}}{\partial F^i_{ABC}} N_C [M_\parallel]^{E'}_B \right) B_A \delta\chi^i \, dL_\star + \\
&+ \int_{\Sigma_\star} [M_\parallel]^E_{A'} \frac{\partial}{\partial X^E} \left\{ [M_\parallel]^S_{E'} \frac{\partial}{\partial X^S} \left(\frac{\partial \mathrm{W}}{\partial F^i_{ABC}} N_C [M_\parallel]^{E'}_B \right) [M_\parallel]^{A'}_A \right\} \delta\chi^i \, d\Sigma_\star
\end{aligned} \tag{24}$$

where $P_{\star r}$ and $P_{\star(r-1)}$ denote the wedges located at the ends of the same regular part of the edge $L_\star$ (possibly numbered as rth), and the square brackets denote as usual the expression $[f]_a^b = f(b) - f(a)$.

Let us notice that, if we differentiate the squared norms $T_A T^A = 1$ and $N_A N^A = 1$, we obtain

$$\frac{\partial T_A}{\partial X_E} T^A = 0; \quad \frac{\partial N_A}{\partial X_E} N^A = 0;$$

Hence, by the product rule, for equation (24) one finds

$$\begin{aligned} &-\int_{L_\star} T^E \frac{\partial}{\partial X^E} \left\{ \frac{\partial \mathrm{W}}{\partial F^i_{ABC}} N_C B_B T_A T^{A'} \right\} T_{A'} \delta\chi^i \, dL_\star = \\ &= -\int_{L_\star} T^E \frac{\partial}{\partial X^E} \left\{ \frac{\partial \mathrm{W}}{\partial F^i_{ABC}} N_C B_B T_A \right\} \delta\chi^i \, dL_\star \end{aligned} \tag{25}$$

Equivalent expressions can be written also for the contributions in equation (16)

$$\begin{aligned} &-\int_{\Sigma_\star} [M_\|]^R_{A'} \frac{\partial}{\partial X^R} \left(\frac{\partial \mathrm{W}}{\partial F^i_{ABC}} N_B N_C N^E [M_\|]^{A'}_A \right) N_E \left[\frac{\partial}{\partial N} \delta\chi^i \right] d\Sigma_\star = \\ &= -\int_{\Sigma_\star} [M_\|]^R_{A'} \frac{\partial}{\partial X^R} \left(\frac{\partial \mathrm{W}}{\partial F^i_{ABC}} N_B N_C [M_\|]^{A'}_A \right) \left[\frac{\partial}{\partial N} \delta\chi^i \right] d\Sigma_\star \end{aligned} \tag{26}$$

and in equation (17)

$$\begin{aligned} &-\int_{L_\star} [M_\|]^R_{A'} \frac{\partial}{\partial X^R} \left(\frac{\partial \mathrm{W}}{\partial F^i_{ABC}} N_B N_C N^E [M_\|]^{A'}_A \right) B_E \delta\chi^i \, dL_\star + \\ &+ \int_{\Sigma_\star} [M_\|]^S_{E'} \frac{\partial}{\partial X^S} \left\{ [M_\|]^R_{A'} \frac{\partial}{\partial X^R} \left(\frac{\partial \mathrm{W}}{\partial F^i_{ABC}} N_B N_C N^E [M_\|]^{A'}_A \right) \right. \\ &\left. [M_\|]^{E'}_E \right\} \delta\chi^i \, d\Sigma_\star = \\ &= -\int_{L_\star} \frac{\partial \mathrm{W}}{\partial F^i_{ABC}} N_B N_C \frac{\partial N^E}{\partial X^A} B_E \delta\chi^i \, dL_\star + \\ &+ \int_{\Sigma_\star} [M_\|]^S_{E'} \frac{\partial}{\partial X^S} \left\{ \frac{\partial \mathrm{W}}{\partial F^i_{ABC}} N_B N_C \frac{\partial N^{E'}}{\partial X^A} \right\} \delta\chi^i \, d\Sigma_\star \end{aligned} \tag{27}$$

resulting

$$[M_\|]^S_{A'} \frac{\partial N^E}{\partial X^S} [M_\|]^{A'}_A = \frac{\partial N^E}{\partial X^A}$$

The contributions to the Lagrangian inner virtual work provided so far, related exclusively to the third gradient, are listed in what follows for the reader convenience. In the volume $\Omega_\star$ we found (equation (6))

$$-\int_{\Omega_\star} \frac{\partial}{\partial X^A} \frac{\partial}{\partial X^B} \frac{\partial}{\partial X^C} \left(\frac{\partial \mathrm{W}}{\partial F^i_{ABC}} \right) \delta\chi^i \, d\Omega_\star \tag{28}$$

Over the boundary surface $\Sigma_\star$, from equations (6) and (17) through (27), (10), (23), (8), (21), (15) through (26) and (13), one has

$$
\begin{aligned}
&+\int_{\Sigma_\star}\frac{\partial}{\partial X^B}\frac{\partial}{\partial X^C}\left(\frac{\partial \mathrm{W}}{\partial F^i_{ABC}}\right)N_A\,\delta\chi^i\,d\Sigma_\star+\\
&+\int_{\Sigma_\star}[M_\|]^S_{E'}\,\frac{\partial}{\partial X^S}\left\{\frac{\partial \mathrm{W}}{\partial F^i_{ABC}}N_BN_C\,\frac{\partial N^{E'}}{\partial X^A}\right\}\delta\chi^i\,d\Sigma_\star+\\
&+\int_{\Sigma_\star}[M_\|]^D_E\,\frac{\partial}{\partial X^D}\left\{\frac{\partial}{\partial X^C}\left(\frac{\partial \mathrm{W}}{\partial F^i_{ABC}}\right)N_B\,[M_\|]^E_A\right\}\delta\chi^i\,d\Sigma_\star+\\
&+\int_{\Sigma_\star}[M_\|]^E_{A'}\,\frac{\partial}{\partial X^E}\left\{[M_\|]^S_{E'}\,\frac{\partial}{\partial X^S}\left(\frac{\partial \mathrm{W}}{\partial F^i_{ABC}}N_C[M_\|]^{E'}_B\right)[M_\|]^{A'}_A\right\}\delta\chi^i\,d\Sigma_\star+\\
&-\int_{\Sigma_\star}\frac{\partial}{\partial X^C}\left(\frac{\partial \mathrm{W}}{\partial F^i_{ABC}}\right)N_BN_A\,\frac{\partial\delta\chi^i}{\partial N}\,d\Sigma_\star+\\
&-\int_{\Sigma_\star}[M_\|]^E_{E'}\,\frac{\partial}{\partial X^E}\left\{\frac{\partial \mathrm{W}}{\partial F^i_{ABC}}N_C[M_\|]^{E'}_B\right\}N_A\,\frac{\partial\delta\chi^i}{\partial N}\,d\Sigma_\star+\\
&-\int_{\Sigma_\star}[M_\|]^R_{A'}\,\frac{\partial}{\partial X^R}\left\{\frac{\partial \mathrm{W}}{\partial F^i_{ABC}}N_BN_C[M_\|]^{A'}_A\right\}\frac{\partial\delta\chi^i}{\partial N}\,d\Sigma_\star+\\
&+\int_{\Sigma_\star}\frac{\partial \mathrm{W}}{\partial F^i_{ABC}}N_AN_BN_C\,\frac{\partial^2\delta\chi^i}{\partial N^2}\,d\Sigma_\star
\end{aligned}
\tag{29}
$$

Along the border edge $L_\star$, from equations (17) through (27), (10), (24) through (25), (22) and (16), we have

$$
\begin{aligned}
&-\int_{L_\star}\frac{\partial \mathrm{W}}{\partial F^i_{ABC}}N_BN_C\,\frac{\partial N^E}{\partial X^A}B_E\,\delta\chi^i\,dL_\star+\\
&-\int_{L_\star}\frac{\partial}{\partial X^C}\left(\frac{\partial \mathrm{W}}{\partial F^i_{ABC}}\right)B_AN_B\,\delta\chi^i\,dL_\star+\\
&-\int_{L_\star}[M_\|]^E_{E'}\,\frac{\partial}{\partial X^E}\left(\frac{\partial \mathrm{W}}{\partial F^i_{ABC}}N_C[M_\|]^{E'}_B\right)B_A\,\delta\chi^i\,dL_\star+\\
&-\int_{L_\star}T^E\,\frac{\partial}{\partial X^E}\left(\frac{\partial \mathrm{W}}{\partial F^i_{ABC}}T_AB_BN_C\right)\delta\chi^i\,dL_\star+\\
&+\int_{L_\star}\frac{\partial \mathrm{W}}{\partial F^i_{ABC}}B_AB_BN_C\,\frac{\partial\delta\chi^i}{\partial B}\,dL_\star+\\
&+\int_{L_\star}\frac{\partial \mathrm{W}}{\partial F^i_{ABC}}(B_AN_B+B_BN_A)N_C\,\frac{\partial\delta\chi^i}{\partial N}\,dL_\star
\end{aligned}
\tag{30}
$$

At the wedges $P_\star$, through equation (24) one has

$$
+\sum_r\left[\left(\frac{\partial \mathrm{W}}{\partial F^i_{ABC}}\,T_AB_BN_C\right)\delta\chi^i\right]^{P_{\star,r}}_{P_{\star,r-1}}
\tag{31}
$$

Due to the complexity of the above scenario, some remarks are provided as follows.

1. Over the volume $\Omega_\star$, equation (28), the differential operator derived herein, $-\mathrm{DIV}(\mathrm{DIV}(\mathrm{DIV}(\cdot)))$, is in agreement with the $+\mathrm{DIV}(\mathrm{DIV}(\cdot)))$ operator found for the second-gradient materials, which in turn generalizes the $-\mathrm{DIV}(\cdot)$ equilibrium operator utilized for Cauchy's continua. Let us notice the alternating sign endowing such operators at increasing the gradient order.

2. Over the boundary surface $\Sigma_\star$, equation (29), we can recognize a contribution depending linearly on the face normal N_A after double contraction of the Lagrangian *triple stress* with the differential operator DIV(DIV($\cdot$)): such a term must be added to the Cauchy traction, analogously to the term provided by the second-gradient contribution (with alternating signs). It represents a force per unit surface, generating work versus the virtual placement $\delta\chi^i$. Moreover, this force is endowed by other two addends, a surface divergence and a double surface divergence: the argument of the former is the tensor $(\partial/\partial X^C)(\partial \mathrm{W}/\partial F^i_{ABC})N_B$ (one Lagrangian valence left free), for the latter tensor $(\partial \mathrm{W}/\partial F^i_{ABC})N_C$, with two free (Lagrangian) valences. In addition, another term can be noticed, which originally was expressed as a double divergence, but then resulted drastically simplified due to the contraction with a valence of the outer projector. All these contributions generate work versus the virtual placement map.
3. Over the surface $\Sigma_\star$, equation (29), it can easily be recognized a *triple force*, namely, the Eulerian covector $(\partial \mathrm{W}/\partial F^i_{ABC})N_A N_B N_C$, dimensionally a force per unit surface multiplied by a squared length, working versus the second normal derivative of the placement map $\partial^2\delta\chi^i/\partial N^2$ (with dimension one over a length).
4. Over the surface $\Sigma_\star$, equation (29), we can recognize also three contributions working versus the first normal derivative of the virtual placement map $\partial\delta\chi^i/\partial N$ (non-dimensional), which must be added to the *double force* of the second-gradient materials. Such contributions, which are dimensionally equivalent to a pressure multiplied by a length (to a work per unit surface), include the Eulerian covectors (at varying *i*) $\partial/\partial X^C(\partial \mathrm{W}/\partial F^i_{ABC})N_B N_A$, and other two tensors resulting from a surface divergence, which differ from each other in one (Lagrangian) valence, namely, $(\partial \mathrm{W}/\partial F^i_{ABC})N_C N_B$ and $(\partial \mathrm{W}/\partial F^i_{ABC})N_C$.
5. Along the border edge $L_\star$, equation (30), having codimension two w.r.t. the ambient space $\mathcal{R}^3$, the tangent space is generated by the edge tangent, while its orthogonal complement is spanned by the linear combination of the face normal $\mathbf{N}$ and the edge normal $\mathbf{B}$. The corresponding edge integrals include contributions work conjugate to $\partial\delta\chi^i/\partial B$ and $\partial\delta\chi^i/\partial N$, which are analogous to the edge (inner) contributions of the second-gradient materials, dimensionally equivalent to a force (i.e. a work per unit length): those are the Eulerian covectors $(\partial \mathrm{W}/\partial F^i_{ABC})B_A B_B N_C$ and $2(\partial \mathrm{W}/\partial F^i_{ABC})B_{(A}N_{B)}N_C$, respectively.
6. Along the edge $L_\star$, equation (30), other four contributions can be recognized, dimensionally equal to a force per unit length, working versus $\delta\chi^i$: the second addend is a covector analogous to the edge force of the second-gradient materials (one Lagrangian valence of the triple stress is contracted by the divergence operator), while the remaining terms originate from surface divergence operators.
7. Finally, the difference of concentrated forces can be recognized, evaluated at the *wedge* points $P_{\star r}$ and $P_{\star(r-1)}$, equation (31), which correspond to the ends of the *r*th regular part of the edge, working versus the virtual placement map at the same locations. Since the sum of all the contributions at varying *r* must be computed, at each wedge $P_{\star r}$ the difference of terms belonging to contiguous regular edges is evaluated.

Equations (28)–(31) were expressed in the same form they were derived through integration by parts and repeated applications of the divergence theorem, and correspond to a perspective "surface-driven". If we intend to list such contributions edge by edge as done in Eugster et al. [11] and Fedele [13] for the second-gradient continua, or wedge by wedge, we must consider that along each edge two contributions must be summed, provided from contiguous surfaces with their borders differently oriented, and the same occurs at each wedge with reference to the edges (at least three) having in common that point. Such a discussion and relevant calculations are beyond the scopes of this paper and will be adequately addressed elsewhere.

4. Dancing with the normals

For several surface and edge integrals listed above, it is possible to provide alternative expressions which are particularly meaningful' revealing in detail how the projectors and the surface divergence operators act onto the Lagrangian *triple stress*. Preliminarily, let us consider the surface divergence of the (surface) tangential projector (see also [12,13]). By expressing the surface tangential projector as $[M_{\|}]^E_{E'} = \delta^E_{E'} - N^E N_{E'}$, we obtain

$$
\begin{aligned}
[M_{\|}]^E_{E'} \frac{\partial}{\partial X^E}\left([M_{\|}]^{E'}_B\right) &= \left(\delta^E_{E'} - N^E N_{E'}\right)\frac{\partial}{\partial X^E}\left(\delta^{E'}_B - N^{E'} N_B\right) = \\
&= \delta^E_{E'} \underbrace{\frac{\partial \delta^{E'}_B}{\partial X^E}}_{=0} + \delta^E_{E'} \frac{\partial}{\partial X^E}\left(-N^{E'} N_B\right) + \\
&+ \left(-N^E N_{E'}\right) \underbrace{\frac{\partial \delta^{E'}_B}{\partial X^E}}_{=0} + \left(-N^E N_{E'}\right)\frac{\partial}{\partial X^E}\left(-N^{E'} N_B\right) = \\
&= -\frac{\partial N^{E'}}{\partial X^{E'}} N_B - \underbrace{N^E \frac{\partial N_B}{\partial X^E}}_{=0} - N_{E'} \underbrace{N^E \frac{\partial}{\partial X^E}\left(N^{E'}\right)}_{=0} N_B + \\
&+ N_{E'} \underbrace{N^E \frac{\partial}{\partial X^E}(N_B)}_{=0} N^{E'} = + \frac{2}{R_m} N_B
\end{aligned}
\tag{32}
$$

where symbol $(1/R_m)$ denotes the mean curvature of the boundary face (see [12,86]). The relationship provided by equation (32) may be useful to simplify a few contributions to the inner work.

(i) With reference to equation (30), let us consider the edge term

$$
\begin{aligned}
&+ \int_{L_\star} [M_{\|}]^E_{E'} \frac{\partial}{\partial X^E}\left\{\frac{\partial \mathrm{W}}{\partial F^i_{ABC}} N_C [M_{\|}]^{E'}_B\right\} B_A \delta\chi^i \, dL_\star = \\
&= + \int_{L_\star} [M_{\|}]^E_{E'} \frac{\partial}{\partial X^E}\left\{\frac{\partial \mathrm{W}}{\partial F^i_{ABC}} N_C\right\} [M_{\|}]^{E'}_B B_A \delta\chi^i \, dL_\star + \\
&+ \int_{L_\star} \left\{\frac{\partial \mathrm{W}}{\partial F^i_{ABC}} N_C B_A\right\} [M_{\|}]^E_{E'} \frac{\partial}{\partial X^E}\left([M_{\|}]^{E'}_B\right) \delta\chi^i \, dL_\star =
\end{aligned}
\tag{33}
$$

Utilizing the above relationship equation (32) and exploiting the projector idempotence one obtains

$$
\begin{aligned}
&= + \int_{L_\star} \underbrace{[M_{\|}]^E_B}_{=\delta^E_B - N^E N_B} \frac{\partial}{\partial X^E}\left\{\frac{\partial \mathrm{W}}{\partial F^i_{ABC}} N_C\right\} B_A \, \delta\chi^i \, dL_\star + \\
&\quad + \int_{L_\star} \frac{2}{R_m}\left(\frac{\partial \mathrm{W}}{\partial F^i_{ABC}} N_C B_A N_B\right) \delta\chi^i \, dL_\star = \\
&= + \int_{L_\star} \frac{\partial}{\partial X^B}\left\{\frac{\partial \mathrm{W}}{\partial F^i_{ABC}} N_C\right\} B_A \, \delta\chi^i \, dL_\star + \\
&\quad - \int_{L_\star} \frac{\partial}{\partial N}\left\{\frac{\partial \mathrm{W}}{\partial F^i_{ABC}} N_C\right\} B_A N_B \, \delta\chi^i \, dL_\star + \\
&\quad + \int_{L_\star} \frac{2}{R_m} \frac{\partial \mathrm{W}}{\partial F^i_{ABC}} N_C B_A N_B \delta\chi^i \, dL_\star =
\end{aligned}
\tag{34}
$$

Due to the fact that the normal derivative of the normal vanishes [12], i.e.

$$
N^E \frac{\partial N_C}{\partial X^E} = \frac{\partial N_C}{\partial N} = 0
$$

applying the product rule to the first addend one obtains

$$
\begin{aligned}
&= + \int_{L_\star} \left\{ \frac{\partial}{\partial X^B} \left(\frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}} \right) N_C B_A + \frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}} \frac{\partial N_C}{\partial X^B} B_A + \right. \\
&\left. - \frac{\partial}{\partial N} \left(\frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}} \right) N_C B_A N_B + \frac{2}{R_m} \frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}} N_C B_A N_B \right\} \delta \chi^i \, dL_\star
\end{aligned}
\tag{35}
$$

(ii) Let us consider over the boundary face $\Sigma_\star$, equation (29), the expressions

$$
\begin{aligned}
&+ \int_{\Sigma_\star} [M_\|]^R_{A'} \frac{\partial}{\partial X^R} \left(\frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}} N_B N_C [M_\|]^{A'}_A \right) \frac{\partial \delta \chi^i}{\partial N} \, d\Sigma_\star + \\
&+ \int_{\Sigma_\star} [M_\|]^E_{E'} \frac{\partial}{\partial X^E} \left(\frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}} N_C [M_\|]^{E'}_B \right) N_A \frac{\partial \delta \chi^i}{\partial N} \, d\Sigma_\star =
\end{aligned}
\tag{36}
$$

Exploiting equation (32) one finds

$$
\begin{aligned}
&= + \int_{\Sigma_\star} [M_\|]^R_A \frac{\partial}{\partial X^R} \left(\frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}} N_B N_C \right) \frac{\partial \delta \chi^i}{\partial N} \, d\Sigma_\star + \\
&+ \int_{\Sigma_\star} \frac{2}{R_m} \left[\frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}} N_A N_B N_C \right] \frac{\partial \delta \chi^i}{\partial N} \, d\Sigma_\star + \\
&+ \int_{\Sigma_\star} [M_\|]^E_B \frac{\partial}{\partial X^E} \left(\frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}} N_C \right) N_A \frac{\partial \delta \chi^i}{\partial N} \, d\Sigma_\star + \\
&+ \int_{\Sigma_\star} \frac{2}{R_m} \left[\frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}} N_C N_B N_A \right] \frac{\partial \delta \chi^i}{\partial N} \, d\Sigma_\star =
\end{aligned}
\tag{37}
$$

Differentiating by the product rule the functional groups within parentheses in the first and third addends and summing all the contributions, we obtain

$$
\begin{aligned}
&= + \int_{\Sigma_\star} \frac{\partial \delta \chi^i}{\partial N} \, d\Sigma_\star \left\{ [M_\|]^R_A \frac{\partial}{\partial X^R} \left(\frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}} \right) N_B N_C + . \right. \\
&+ [M_\|]^E_B \frac{\partial}{\partial X^E} \left(\frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}} \right) N_C N_A + \\
&+ \frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}} \left[\frac{\partial N_B}{\partial X^A} N_C + \frac{\partial N_C}{\partial X^A} N_B + \frac{\partial N_C}{\partial X^B} N_A \right] + \\
&\left. + \frac{4}{R_m} \frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}} N_C N_B N_A \right\} =
\end{aligned}
\tag{38}
$$

where we exploited the relationship

$$
[M_\|]^R_A \frac{\partial N_C}{\partial X^R} = \left(\delta^R_A - N^R N_A \right) \frac{\partial N_C}{\partial X^R} = \frac{\partial N_C}{\partial X^A}
\tag{39}
$$

Hence, by expressing the tangential projectors in terms of normals one finds

$$
\begin{aligned}
&= + \int_{\Sigma_\star} \frac{\partial \delta\chi^i}{\partial N}\, d\Sigma_\star \Bigg\{ \frac{\partial}{\partial X^A}\left(\frac{\partial \mathrm{W}}{\partial F^i_{\ ABC}}\right) N_B N_C - \frac{\partial}{\partial N}\left(\frac{\partial \mathrm{W}}{\partial F^i_{\ ABC}}\right) N_B N_C N_A + \\
&+ \frac{\partial}{\partial X^B}\left(\frac{\partial \mathrm{W}}{\partial F^i_{\ ABC}}\right) N_C N_A - \frac{\partial}{\partial N}\left(\frac{\partial \mathrm{W}}{\partial F^i_{\ ABC}}\right) N_C N_A N_B + \\
&+ \frac{\partial \mathrm{W}}{\partial F^i_{\ ABC}}\left[\frac{\partial N_B}{\partial X^A} N_C + \frac{\partial N_C}{\partial X^A} N_B + \frac{\partial N_C}{\partial X^B} N_A\right] + \frac{4}{R_m}\,\frac{\partial \mathrm{W}}{\partial F^i_{\ ABC}} N_C N_B N_A \Bigg\} =
\end{aligned}
\tag{40}
$$

Rearranging terms and taking into account the total symmetry of the *triple stress*, finally one obtains

$$
\begin{aligned}
&= + \int_{\Sigma_\star} \frac{\partial \delta\chi^i}{\partial N}\, d\Sigma_\star \Bigg\{ \frac{\partial}{\partial X^A}\left(\frac{\partial \mathrm{W}}{\partial F^i_{\ ABC}}\right) N_B N_C + \frac{\partial}{\partial X^B}\left(\frac{\partial \mathrm{W}}{\partial F^i_{\ ABC}}\right) N_C N_A \\
&- 2\,\frac{\partial}{\partial N}\left(\frac{\partial \mathrm{W}}{\partial F^i_{\ ABC}}\right) N_A N_B N_C + \\
&+ \frac{\partial \mathrm{W}}{\partial F^i_{\ ABC}}\left[\frac{\partial N_B}{\partial X^A} N_C + \frac{\partial N_C}{\partial X^A} N_B + \frac{\partial N_C}{\partial X^B} N_A\right] + \frac{4}{R_m}\,\frac{\partial \mathrm{W}}{\partial F^i_{\ ABC}} N_C N_B N_A \Bigg\} = \\
&= + \int_{\Sigma_\star} \frac{\partial \delta\chi^i}{\partial N}\, d\Sigma_\star \Bigg\{ 2\,\frac{\partial}{\partial X^A}\left(\frac{\partial \mathrm{W}}{\partial F^i_{\ ABC}}\right) N_B N_C - 2\,\frac{\partial}{\partial N}\left(\frac{\partial \mathrm{W}}{\partial F^i_{\ ABC}}\right) N_A N_B N_C + \\
&+ 3\,\frac{\partial \mathrm{W}}{\partial F^i_{\ ABC}}\,\frac{\partial N_B}{\partial X^A} N_C + \frac{4}{R_m}\,\frac{\partial \mathrm{W}}{\partial F^i_{\ ABC}} N_C N_B N_A \Bigg\}
\end{aligned}
\tag{41}
$$

(iii) We can operate in analogous way on other two surface contributions, equation (29), in which the work is generated versus the virtual placement map. The covectors correspond in turn to a surface divergence and to a double surface divergence, namely

$$
\begin{aligned}
&+ \int_{\Sigma_\star} [M_\|]^D_{\ E}\,\frac{\partial}{\partial X^D}\left\{\frac{\partial}{\partial X^C}\left(\frac{\partial \mathrm{W}}{\partial F^i_{\ ABC}}\right) N_B\, [M_\|]^E_{\ A}\right\} \delta\chi^i\, d\Sigma_\star + \\
&- \int_{\Sigma_\star} [M_\|]^S_{\ A'}\,\frac{\partial}{\partial X^S}\left\{ [M_\|]^E_{\ E'}\,\frac{\partial}{\partial X^E}\left(\frac{\partial \mathrm{W}}{\partial F^i_{\ ABC}} N_C [M_\|]^{E'}_{\ B}\right) [M_\|]^{A'}_{\ A}\right\} \delta\chi^i\, d\Sigma_\star =
\end{aligned}
\tag{42}
$$

Differentiating by the product rule the expressions within parentheses, we obtain

$$
\begin{aligned}
&= + \int_{\Sigma_\star} [M_\|]^D_{\ A}\,\frac{\partial}{\partial X^D}\left\{\frac{\partial}{\partial X^C}\left(\frac{\partial \mathrm{W}}{\partial F^i_{\ ABC}}\right) N_B\right\} \delta\chi^i\, d\Sigma_\star + \\
&+ \int_{\Sigma_\star} \frac{2}{R_m}\,\frac{\partial}{\partial X^C}\left(\frac{\partial \mathrm{W}}{\partial F^i_{\ ABC}}\right) N_B N_A\, \delta\chi^i\, d\Sigma_\star + \\
&\underbrace{- \int_{\Sigma_\star} [M_\|]^S_{\ A}\,\frac{\partial}{\partial X^S}\left\{ [M_\|]^E_{\ E'}\,\frac{\partial}{\partial X^E}\left(\frac{\partial \mathrm{W}}{\partial F^i_{\ ABC}} N_C [M_\|]^{E'}_{\ B}\right)\right\} \delta\chi^i\, d\Sigma_\star}_{=(\diamond)} + \\
&\underbrace{- \int_{\Sigma_\star} \frac{2}{R_m}\,[M_\|]^E_{\ E'}\,\frac{\partial}{\partial X^E}\left(\frac{\partial \mathrm{W}}{\partial F^i_{\ ABC}} N_C [M_\|]^{E'}_{\ B}\right) N_A\, \delta\chi^i\, d\Sigma_\star}_{=(\square)}
\end{aligned}
\tag{43}
$$

For the sake of clarity, let us develop the last two addends separately. Recalling that $\partial N_C/\partial N = 0$, one finds

$$(\diamond) = -\int_{\Sigma_\star} \delta\chi^i \, d\Sigma_\star [M_\parallel]^S_A \frac{\partial}{\partial X^S}\Bigg\{ \underbrace{[M_\parallel]^E_B}_{=\delta^E_B - N^E N_B} \frac{\partial}{\partial X^E}\left(\frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}} N_C\right) + \\ + \frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}} N_C N_B \frac{2}{R_m}\Bigg\} = \\ = -\int_{\Sigma_\star} \delta\chi^i \, d\Sigma_\star [M_\parallel]^S_A \frac{\partial}{\partial X^S}\Bigg\{\frac{\partial}{\partial X^B}\left(\frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}}\right) N_C + \\ + \frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}} \frac{\partial N_C}{\partial X^B} - N_B N_C \frac{\partial}{\partial N}\left(\frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}}\right) + \frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}} N_C N_B \frac{2}{R_m}\Bigg\} = \tag{44}$$

Utilizing the relationship $[M_\parallel]^S_A = \delta^S_A - N^S N_A$, the above surface integral can be split into two, namely the contribution multiplying δ^S_A

$$= -\int_{\Sigma_\star} \delta\chi^i \, d\Sigma_\star \Bigg\{\frac{\partial}{\partial X^A}\frac{\partial}{\partial X^B}\left(\frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}}\right) N_C + \frac{\partial}{\partial X^B}\left(\frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}}\right)\frac{\partial N_C}{\partial X^A} + \\ + \frac{\partial}{\partial X^A}\left(\frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}}\right)\frac{\partial N_C}{\partial X^B} + \frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}}\frac{\partial}{\partial X^A}\left(\frac{\partial N_C}{\partial X^B}\right) + \\ - \frac{\partial}{\partial X^A}(N_B N_C)\frac{\partial}{\partial N}\left(\frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}}\right) - N_B N_C \frac{\partial}{\partial X^A}\frac{\partial}{\partial N}\left(\frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}}\right) + \\ + \frac{\partial}{\partial X^A}\left(\frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}}\right) N_C N_B \frac{2}{R_m} + \frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}}\frac{\partial}{\partial X^A}\left(N_C N_B \frac{2}{R_m}\right)\Bigg\} + \tag{45}$$

and that multiplying $N^S N_A$

$$+ \int_{\Sigma_\star} \delta\chi^i \, d\Sigma_\star \Bigg\{N_A \frac{\partial}{\partial N}\frac{\partial}{\partial X^B}\left(\frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}}\right) N_C + \frac{\partial}{\partial X^B}\left(\frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}}\right) N_A \underbrace{\frac{\partial N_C}{\partial N}}_{=0} + . \\ + N_A \frac{\partial}{\partial N}\left(\frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}}\right)\frac{\partial N_C}{\partial X^B} + \frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}} N_A N^S \frac{\partial^2 N_C}{\partial X^S X^B} + \\ - N_A \underbrace{\frac{\partial}{\partial N}(N_B N_C)}_{=0}\frac{\partial}{\partial N}\left(\frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}}\right) - N_B N_C N_A \frac{\partial}{\partial N}\frac{\partial}{\partial N}\left(\frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}}\right) + \\ + N_A \frac{\partial}{\partial N}\left(\frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}}\right) N_C N_B \frac{2}{R_m} + \frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}} N_A \frac{\partial}{\partial N}\left(N_C N_B \frac{2}{R_m}\right)\Bigg\} = \tag{46}$$

where the vanishing terms include the normal derivatives of the normal, namely $\partial N_C/\partial N = 0$, explicitly or after permuting the mixed derivatives. For the readers' convenience, only equation (46) is reported herein in a more compact form, namely

$$+ \int_{\Sigma_\star} \delta\chi^i \, d\Sigma_\star \Bigg\{N_A N_C \frac{\partial}{\partial X^B}\frac{\partial}{\partial N}\left(\frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}}\right) + \\ + N_A \frac{\partial}{\partial N}\left(\frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}}\right)\frac{\partial N_C}{\partial X^B} - N_B N_C N_A \frac{\partial}{\partial N}\frac{\partial}{\partial N}\left(\frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}}\right) + \\ + N_A N_C N_B \frac{\partial}{\partial N}\left(\frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}}\right)\frac{2}{R_m} - \frac{\partial \mathrm{W}}{\partial F^i{}_{ABC}} N_A N_C N_B \frac{\partial}{\partial N}\left(\frac{2}{R_m}\right)\Bigg\} \tag{47}$$

being $2/R_m = -\,\partial N^E/\partial X^E$. The remaining term of equation (43) can be modified analogously, namely

$$
\begin{aligned}
(\square) = & -\int_{\Sigma_\star} \delta\chi^i \, d\Sigma_\star \frac{2}{R_m} N_A \left\{ [M_\|]^E_B \frac{\partial}{\partial X^E}\left(\frac{\partial \mathrm{W}}{\partial F^i_{ABC}} N_C\right) + \right. \\
& \left. \frac{\partial \mathrm{W}}{\partial F^i_{ABC}} N_C N_B \frac{2}{R_m} \right\} = \\
= & -\int_{\Sigma_\star} \delta\chi^i \, d\Sigma_\star \frac{2}{R_m} N_A \left\{ \frac{\partial}{\partial X^B}\left(\frac{\partial \mathrm{W}}{\partial F^i_{ABC}} N_C\right) + \right. \\
& \left. - N_B \frac{\partial}{\partial N}\left(\frac{\partial \mathrm{W}}{\partial F^i_{ABC}} N_C\right) + \frac{\partial \mathrm{W}}{\partial F^i_{ABC}} N_C N_B \frac{2}{R_m} \right\} = \\
= & -\int_{\Sigma_\star} \delta\chi^i \, d\Sigma_\star \left\{ \frac{2}{R_m} N_A N_C \frac{\partial}{\partial X^B}\left(\frac{\partial \mathrm{W}}{\partial F^i_{ABC}}\right) + \frac{2}{R_m} N_A \frac{\partial \mathrm{W}}{\partial F^i_{ABC}} \frac{\partial N_C}{\partial X^B} + \right. \\
& \left. - \frac{2}{R_m} N_A N_B N_C \frac{\partial}{\partial N}\left(\frac{\partial \mathrm{W}}{\partial F^i_{ABC}}\right) + \frac{\partial \mathrm{W}}{\partial F^i_{ABC}} N_A N_B N_C \frac{4}{R_m^2} \right\}
\end{aligned}
\tag{48}
$$

Especially for the last contributions to the inner work over the boundary surface $\Sigma_\star$ at point (iii), including as covectors the surface divergence and the double surface divergence, the use of such detailed expressions, useful for a better understanding or possibly for a cross validation, becomes prohibitive in the practice and the initial compact form seems preferable.

5. Dual variables

We derived the inner virtual work equations through the first variation of the Lagrangian energy functional $\mathcal{E}^{\mathrm{DEF}}$ in the reference configuration $\Omega_\star$: integrating by parts, through the divergence theorem we provided volume, surface, edge and wedge contributions. So far, we utilized the partial derivatives of the energy density with respect to the first, second and third Lagrangian gradients of the placement map, namely, F^i_A, F^i_{AB}, F^i_{ABC}, without making more explicit their role. At this stage, we can analyse in depth these terms by setting

$$
P^A_{1\,i} = \frac{\partial \mathrm{W}}{\partial F^i_A}\,; \qquad P^{AB}_{2\,i} = \frac{\partial \mathrm{W}}{\partial F^i_{AB}}\,; \qquad P^{ABC}_{3\,i} = \frac{\partial \mathrm{W}}{\partial F^i_{ABC}} \ : \tag{49}
$$

where symbol $P^A_{1\,i}(\mathbf{X})$ represents a stress-like tensor, referred to as first Piola–Kirchhoff stress, with a leg in the Eulerian configuration and another one in the Lagrangian configuration. $P^{AB}_{2\,i}(\mathbf{X})$ and $P^{ABC}_{2\,i}(\mathbf{X})$ represent the third and fourth rank tensors, one time covariant (Eulerian index) and two and three times contravariant (Lagrangian indices), respectively. We will refer to them in turn as Piola *double* and *triple stress*. Subscripts 1, 2, and 3 remark that the relevant tensors were provided by differentiation of the energy density w.r.t. the first, the second, and third placement gradients, respectively. At this point, we can everywhere substitute the symbols $P^A_{1\,i}$, $P^{AB}_{2\,i}$, and $P^{ABC}_{3\,i}$, without any ambiguity. For instance, the Lagrangian inner virtual work in equation (2) can now be expressed as follows (see e.g. [11,12])

$$
\begin{aligned}
& \delta\mathcal{E}^{\mathrm{DEF}} = \int_{\Omega_\star} P^A_{1\,i} \delta F^i_A \, d\Omega_\star + \int_{\Omega_\star} P^{AB}_{2\,i} \delta F^i_{AB} \, d\Omega_\star + \int_{\Omega_\star} P^{ABC}_{3\,i} \delta F^i_{ABC} \, d\Omega_\star; \\
& \delta F^i_A = \frac{\partial \delta\chi^i}{\partial X^A}\,; \quad \delta F^i_{AB} = \frac{\partial^2 \delta\chi^i}{\partial X^A \partial X^B}\,; \quad \delta F^i_{ABC} = \frac{\partial^3 \delta\chi^i}{\partial X^A \partial X^B \partial X^C}\,;
\end{aligned}
\tag{50}
$$

It must be underlined that the analytical developments through the integration by parts were made possible by the above representation of the internal work, including δF^i_A, δF^i_{AB}, and δF^i_{ABC}. By a trivial change of variables, the same functional equation (50) can be referred to the Eulerian configuration Ω: however, while the Lagrangian virtual work represents the first variation of an energy functional, the Eulerian form does not, since the integration volume changes along the deformation process.

The Eulerian counterpart of the virtual placement map can be defined through the composition with the inverse mapping, namely $\delta\mathbf{c}(\mathbf{x}) = \delta\boldsymbol{\chi} \circ \boldsymbol{\chi}^{-1}(\mathbf{x})$. Accordingly, we can express the first and, in a sequence, the second (spatial) gradients of the Eulerian test function through their Lagrangian counterparts [12,14] namely

$$\delta D^i_j = \frac{\partial}{\partial x^j}\delta\chi^i(\mathbf{x}) = \frac{\partial \delta\chi^i}{\partial X^A}\frac{\partial X^A}{\partial x^j} = \delta F^i_A\left(\mathbf{F}^{-1}\right)^A_j; \quad \left(\delta F^i_A = F^j_A \delta D^i_j\right) \tag{51}$$

and

$$\begin{aligned}\delta D^i_{jk} &= \frac{\partial^2}{\partial x^k x^j}\delta\chi^i(\mathbf{x}) = \frac{\partial}{\partial x^k}\left(\frac{\partial}{\partial x^j}\delta\chi^i(\mathbf{x})\right) = \\ &= \frac{\partial}{\partial x^k}\left(\left(\mathbf{F}^{-1}\right)^A_j \delta F^i_A\right) = \frac{\partial}{\partial x^k}\left(\mathbf{F}^{-1}\right)^A_j \delta F^i_A + \left(\mathbf{F}^{-1}\right)^A_j \frac{\partial}{\partial x^k}\delta F^i_A = \\ &= \frac{\partial}{\partial x^k}\left(\mathbf{F}^{-1}\right)^A_j \delta F^i_A + \left(\mathbf{F}^{-1}\right)^A_j \delta F^i_{AB}\left(\mathbf{F}^{-1}\right)^B_k\end{aligned} \tag{52}$$

Rearranging equations (51) and (52), first one can write

$$\left(\mathbf{F}^{-1}\right)^A_j\left(\mathbf{F}^{-1}\right)^B_k \delta F^i_{AB} = \delta D^i_{kj} - \frac{\partial}{\partial x^k}\left(\mathbf{F}^{-1}\right)^L_j \delta F^i_L \tag{53}$$

Recalling that, by differentiating the relationship $F^j_A(\mathbf{F}^{-1})^L_j = \delta^L_A$, one obtains (see e.g. [84])

$$F^j_A \frac{\partial}{\partial x^k}\left(\mathbf{F}^{-1}\right)^L_j = -\frac{\partial}{\partial x^k}\left(F^j_A\right)\left(\mathbf{F}^{-1}\right)^L_j \qquad (\dagger)$$

Equation (52) can be written as follows, after multiplying both sides by $F^j_A F^k_B$

$$\begin{aligned}\delta F^i_{AB} &= F^j_A F^k_B \delta D^i_{kj} - F^k_B \underbrace{F^j_A \frac{\partial}{\partial x^k}\left(\mathbf{F}^{-1}\right)^L_j}_{\text{via}\,(\dagger)} \delta F^i_L = \\ &= F^j_A F^k_B \delta D^i_{kj} + F^k_B \frac{\partial}{\partial x^k}\left(F^j_A\right)\left(\mathbf{F}^{-1}\right)^L_j \delta F^i_L = \\ &= F^j_A F^k_B \delta D^i_{kj} + F^j_{AC} \underbrace{F^k_B\left(\mathbf{F}^{-1}\right)^C_k}_{=\delta^C_B}\left(\mathbf{F}^{-1}\right)^L_j \delta F^i_L = \\ &= F^j_A F^k_B \delta D^i_{kj} + F^j_{AB}\left(\mathbf{F}^{-1}\right)^L_j \delta F^i_L = F^j_A F^k_B \delta D^i_{kj} + F^j_{AB}\delta D^i_j\end{aligned} \tag{54}$$

Thereafter, let us compute the Eulerian third gradient of the placement map. Differentiating equation (52) by the product rule, one obtains

$$\begin{aligned}\delta D^i_{jkl} &= \frac{\partial^3}{\partial x^l x^k x^j}\delta\chi^i(\mathbf{x}) = \frac{\partial}{\partial x^l}\left(\frac{\partial}{\partial x^k}\frac{\partial}{\partial x^j}\delta\chi^i(\mathbf{x})\right) = \\ &= \frac{\partial}{\partial x^l}\left(\frac{\partial}{\partial x^k}\left(\mathbf{F}^{-1}\right)^A_j \delta F^i_A + \left(\mathbf{F}^{-1}\right)^A_j \delta F^i_{AB}\left(\mathbf{F}^{-1}\right)^B_k\right) = \\ &= \frac{\partial}{\partial x^l}\frac{\partial}{\partial x^k}\left(\mathbf{F}^{-1}\right)^A_j \delta F^i_A + \frac{\partial}{\partial x^k}\left(\mathbf{F}^{-1}\right)^A_j \frac{\partial}{\partial x^l}\delta F^i_A + \frac{\partial}{\partial x^l}\left(\mathbf{F}^{-1}\right)^A_j \delta F^i_{AB}\left(\mathbf{F}^{-1}\right)^B_k + \\ &\quad + \left(\mathbf{F}^{-1}\right)^A_j \frac{\partial}{\partial x^l}\delta F^i_{AB}\left(\mathbf{F}^{-1}\right)^B_k + \left(\mathbf{F}^{-1}\right)^A_j \delta F^i_{AB}\frac{\partial}{\partial x^l}\left(\mathbf{F}^{-1}\right)^B_k = \\ &= \frac{\partial^2}{\partial x^l \partial x^k}\left(\mathbf{F}^{-1}\right)^A_j \delta F^i_A + \frac{\partial}{\partial x^k}\left(\mathbf{F}^{-1}\right)^A_j \delta F^i_{AB}\left(\mathbf{F}^{-1}\right)^B_l + \\ &\quad + \frac{\partial}{\partial x^l}\left(\mathbf{F}^{-1}\right)^A_j \delta F^i_{AB}\left(\mathbf{F}^{-1}\right)^B_k + \left(\mathbf{F}^{-1}\right)^A_j \delta F^i_{ABC}\left(\mathbf{F}^{-1}\right)^C_l\left(\mathbf{F}^{-1}\right)^B_k + \\ &\quad + \left(\mathbf{F}^{-1}\right)^A_j \delta F^i_{AB}\frac{\partial}{\partial x^l}\left(\mathbf{F}^{-1}\right)^B_k\end{aligned} \tag{55}$$

Grouping the contributions including δF^i_{AB}, one finds

$$\delta D^i_{jkl} = \frac{\partial^2}{\partial x^l \partial x^k}\left(\mathbf{F}^{-1}\right)^A_j \delta F^i_A + \delta F^i_{AB}\left\{\frac{\partial}{\partial x^k}\left(\mathbf{F}^{-1}\right)^A_j\left(\mathbf{F}^{-1}\right)^B_l + \right. \\ + \frac{\partial}{\partial x^l}\left(\mathbf{F}^{-1}\right)^A_j\left(\mathbf{F}^{-1}\right)^B_k + \left.\left(\mathbf{F}^{-1}\right)^A_j \frac{\partial}{\partial x^l}\left(\mathbf{F}^{-1}\right)^B_k\right\} + \\ + \delta F^i_{ABC}\left(\mathbf{F}^{-1}\right)^A_j\left(\mathbf{F}^{-1}\right)^B_k\left(\mathbf{F}^{-1}\right)^C_l \tag{56}$$

We intend to express the Lagrangian third gradient of virtual placement map as a function of the Eulerian gradients. To this purpose, let us multiply both the sides of the above equation by the product $F^j_{A'}F^k_{B'}F^l_{C'}$, so that one has $(\mathbf{F}^{-1})^A_j F^j_{A'} = \delta^A_{A'}$, etc. Hence, one can write

$$\delta F^i_{A'B'C'} = +\delta D^i_{jkl} F^j_{A'}F^k_{B'}F^l_{C'} + \\ - \frac{\partial^2}{\partial x^l \partial x^k}\left(\mathbf{F}^{-1}\right)^A_j F^p_A \delta D^i_p F^j_{A'}F^k_{B'}F^l_{C'} + \\ - \delta F^i_{AB}\left\{\frac{\partial}{\partial x^k}\left(\mathbf{F}^{-1}\right)^A_j\left(\mathbf{F}^{-1}\right)^B_l F^j_{A'}F^k_{B'}F^l_{C'} + \right. \\ + \frac{\partial}{\partial x^l}\left(\mathbf{F}^{-1}\right)^A_j\left(\mathbf{F}^{-1}\right)^B_k F^j_{A'}F^k_{B'}F^l_{C'} + \\ + \left.\left(\mathbf{F}^{-1}\right)^A_j \frac{\partial}{\partial x^l}\left(\mathbf{F}^{-1}\right)^B_k F^j_{A'}F^k_{B'}F^l_{C'}\right\} = \tag{57}$$

Preliminarily, we observe that the terms within curly brackets in equation (57) can be developed by utilizing equation (†) and simplifying the products

$$-\delta F^i_{AB}\left\{-\frac{\partial}{\partial x^k}F^j_{A'}\left(\mathbf{F}^{-1}\right)^A_j\left(\mathbf{F}^{-1}\right)^B_l F^k_{B'}F^l_{C'} + \right. \\ - \frac{\partial}{\partial x^l}F^j_{A'}\left(\mathbf{F}^{-1}\right)^A_j\left(\mathbf{F}^{-1}\right)^B_k F^k_{B'}F^l_{C'} + \\ \left. -\left(\mathbf{F}^{-1}\right)^A_j \frac{\partial}{\partial x^l}F^k_{B'}\left(\mathbf{F}^{-1}\right)^B_k F^j_{A'}F^l_{C'}\right\} = \\ = -\delta F^i_{AB}\left\{-F^j_{A'N}\left(\mathbf{F}^{-1}\right)^N_k\left(\mathbf{F}^{-1}\right)^A_j \delta^B_{C'}F^k_{B'} + \right. \\ \left. -F^j_{A'N}\left(\mathbf{F}^{-1}\right)^N_l\left(\mathbf{F}^{-1}\right)^A_j \delta^B_{B'}F^l_{C'} - \delta^A_{A'}F^k_{B'N}\left(\mathbf{F}^{-1}\right)^N_l\left(\mathbf{F}^{-1}\right)^B_k F^l_{C'}\right\} = \\ = -\delta F^i_{AB}\left\{-F^j_{A'N}\delta^N_{B'}\left(\mathbf{F}^{-1}\right)^A_j \delta^B_{C'} + \right. \\ \left. -F^j_{A'N}\delta^N_{C'}\left(\mathbf{F}^{-1}\right)^A_j \delta^B_{B'} - \delta^A_{A'}F^k_{B'N}\delta^N_{C'}\left(\mathbf{F}^{-1}\right)^B_k\right\} = \\ = -\delta F^i_{AB}\left\{-F^j_{A'B'}\left(\mathbf{F}^{-1}\right)^A_j \delta^B_{C'} - F^j_{A'C'}\left(\mathbf{F}^{-1}\right)^A_j \delta^B_{B'} - \delta^A_{A'}F^k_{B'C'}\left(\mathbf{F}^{-1}\right)^B_k\right\} \tag{58}$$

Recalling that, from equation (†), the following expression for the Eulerian derivative becomes available (see [84])

$$\frac{\partial}{\partial x^k}\left(\mathbf{F}^{-1}\right)^A_j = -\left(\mathbf{F}^{-1}\right)^R_j F^m_{RS}\left(\mathbf{F}^{-1}\right)^S_k\left(\mathbf{F}^{-1}\right)^A_m \tag{59}$$

for the term with the second derivatives in equation (57) one finds by the product rule

$$
\begin{aligned}
&\frac{\partial}{\partial x^l}\frac{\partial}{\partial x^k}\left(\mathbf{F}^{-1}\right)^A_j F^j_{A'} F^k_{B'} F^l_{C'} F^p_A \delta D^i_p = \\
&= -\frac{\partial}{\partial x^l}\left(\mathbf{F}^{-1}\right)^R_j F^m_{RS}\left(\mathbf{F}^{-1}\right)^S_k\left(\mathbf{F}^{-1}\right)^A_m F^j_{A'} F^k_{B'} F^l_{C'} F^p_A \delta D^i_p + \\
&- \left(\mathbf{F}^{-1}\right)^R_j \frac{\partial}{\partial x^l} F^m_{RS}\left(\mathbf{F}^{-1}\right)^S_k\left(\mathbf{F}^{-1}\right)^A_m F^j_{A'} F^k_{B'} F^l_{C'} F^p_A \delta D^i_p + \\
&- \left(\mathbf{F}^{-1}\right)^R_j F^m_{RS} \frac{\partial}{\partial x^l}\left(\mathbf{F}^{-1}\right)^S_k\left(\mathbf{F}^{-1}\right)^A_m F^j_{A'} F^k_{B'} F^l_{C'} F^p_A \delta D^i_p + \\
&- \left(\mathbf{F}^{-1}\right)^R_j F^m_{RS}\left(\mathbf{F}^{-1}\right)^S_k \frac{\partial}{\partial x^l}\left(\mathbf{F}^{-1}\right)^A_m F^j_{A'} F^k_{B'} F^l_{C'} F^p_A \delta D^i_p =
\end{aligned}
\tag{60}
$$

The multiple products of deformation gradients and their inverses can easily be simplified obtaining

$$
\begin{aligned}
&= + \frac{\partial}{\partial x^l} F^j_{A'}\left(\mathbf{F}^{-1}\right)^R_j F^m_{RS}\delta^S_{B'}\left(\mathbf{F}^{-1}\right)^A_m F^l_{C'} F^p_A \delta D^i_p + \\
&\quad - \left(\mathbf{F}^{-1}\right)^R_j F^m_{RST}\left(\mathbf{F}^{-1}\right)^T_l \delta^S_{B'}\left(\mathbf{F}^{-1}\right)^A_m F^j_{A'} F^l_{C'} F^p_A \delta D^i_p + \\
&\quad + \delta^R_{A'} F^m_{RS} \frac{\partial}{\partial x^l} F^k_{B'}\left(\mathbf{F}^{-1}\right)^S_k\left(\mathbf{F}^{-1}\right)^A_m F^l_{C'} F^p_A \delta D^i_p + \\
&\quad - \delta^R_{A'} F^m_{RS}\delta^S_{B'} \frac{\partial}{\partial x^l}\left(\mathbf{F}^{-1}\right)^A_m F^l_{C'} F^p_A \delta D^i_p =
\end{aligned}
\tag{61}
$$

where recourse was made to equation (†). Thereafter, one can write

$$
\begin{aligned}
&= + F^j_{A'U}\left(\mathbf{F}^{-1}\right)^U_l\left(\mathbf{F}^{-1}\right)^R_j F^m_{RB'}\delta^p_m F^l_{C'}\delta D^i_p + \\
&\quad - \delta^R_{A'} F^m_{RST}\delta^T_{C'}\delta^S_{B'}\delta^p_m \delta D^i_p + \\
&\quad + \delta^R_{A'} F^m_{RS} F^k_{B'Q}\left(\mathbf{F}^{-1}\right)^Q_l\left(\mathbf{F}^{-1}\right)^S_k \delta^p_m F^l_{C'}\delta D^i_p + \\
&\quad - \delta^R_{A'} F^m_{RS}\delta^S_{B'} \frac{\partial}{\partial x^l}\left(\mathbf{F}^{-1}\right)^A_m F^l_{C'} F^p_A \delta D^i_p = \\
&= + F^j_{A'C'}\left(\mathbf{F}^{-1}\right)^R_j F^m_{RB'}\delta D^i_m - F^m_{A'B'C'}\delta D^i_m + \\
&\quad + F^m_{A'S} F^k_{B'C'}\left(\mathbf{F}^{-1}\right)^S_k \delta D^i_m - F^m_{A'B'} \frac{\partial}{\partial x^l}\left(\mathbf{F}^{-1}\right)^A_m F^l_{C'} F^p_A \delta D^i_p =
\end{aligned}
\tag{62}
$$

Let us notice that the last term in equation (62), after inserting the expression for the Eulerian derivative of the inverse matrix components (from equation 59), can be simplified as follows (note that sign minus disappeared)

$$
\begin{aligned}
&+ F^m_{A'B'}\left(\mathbf{F}^{-1}\right)^T_m\left(\mathbf{F}^{-1}\right)^V_l F^s_{TV}\left(\mathbf{F}^{-1}\right)^A_s F^l_{C'} F^p_A \delta D^i_p = \\
&+ F^m_{A'B'}\left(\mathbf{F}^{-1}\right)^T_m \delta^V_{C'} F^s_{TV}\delta^p_s \delta D^i_p = + F^m_{A'B'}\left(\mathbf{F}^{-1}\right)^T_m F^p_{TC'}\delta D^i_p
\end{aligned}
$$

At the light of the above developments, the contribution with the second derivatives in equation (60) becomes

$$
\begin{aligned}
&\frac{\partial}{\partial x^l}\frac{\partial}{\partial x^k}\left(\mathbf{F}^{-1}\right)^A_j F^j_{A'} F^k_{B'} F^l_{C'} F^p_A \delta D^i_p = \\
&= + F^j_{A'C'}\left(\mathbf{F}^{-1}\right)^R_j F^p_{RB'}\delta D^i_p - F^p_{A'B'C'}\delta D^i_p + F^k_{B'C'}\left(\mathbf{F}^{-1}\right)^S_k F^p_{A'S}\delta D^i_p + \\
&+ F^m_{A'B'}\left(\mathbf{F}^{-1}\right)^T_m F^p_{TC'}\,\delta D^i_p = \\
&= \delta D^i_p\Big\{ + F^j_{A'C'}\left(\mathbf{F}^{-1}\right)^R_j F^p_{RB'} - F^p_{A'B'C'} + \\
&+ F^k_{B'C'}\left(\mathbf{F}^{-1}\right)^S_k F^p_{A'S} + F^m_{A'B'}\left(\mathbf{F}^{-1}\right)^T_m F^p_{TC'}\Big\} = \\
&= \delta D^i_p\Big\{ - F^p_{A'B'C'} + \left(\mathbf{F}^{-1}\right)^R_j\left[F^j_{A'C'}F^p_{RB'} + F^j_{B'C'}F^p_{A'R} + F^j_{A'B'}F^p_{RC'}\right]\Big\}
\end{aligned}
\tag{63}
$$

On the basis of equations (58) and (63), the Lagrangian third gradient of the placement map in equation (57) can now be expressed as

$$
\begin{aligned}
\delta F^i_{A'B'C'} = &+ \delta D^i_{jkl}F^j_{A'}F^k_{B'}F^l_{C'} + \\
&- \delta D^i_p\Big\{ - F^p_{A'B'C'} + \left(\mathbf{F}^{-1}\right)^R_j\left[F^j_{A'C'}F^p_{RB'} + F^j_{B'C'}F^p_{A'R} + F^j_{A'B'}F^p_{RC'}\right]\Big\} + \\
&- \delta F^i_{AB}\Big\{ - F^j_{A'B'}\left(\mathbf{F}^{-1}\right)^A_j\delta^B_{C'} - F^j_{A'C'}\left(\mathbf{F}^{-1}\right)^A_j\delta^B_{B'} - \delta^A_{A'}F^k_{B'C'}\left(\mathbf{F}^{-1}\right)^B_k\Big\} =
\end{aligned}
\tag{64}
$$

but the Eulerian second-gradient D^i_{kl} still does not appear explicitly. Recalling from equation (54) that $\delta F^i_{AB} = F^l_A F^k_B \delta D^i_{kl} + F^l_{AB}\delta D^i_l$, we can write

$$
\begin{aligned}
\delta F^i_{A'B'C'} = &+ \delta D^i_{jkl}F^j_{A'}F^k_{B'}F^l_{C'} + \\
&- \delta D^i_p\Big\{ - F^p_{A'B'C'} + \left(\mathbf{F}^{-1}\right)^R_j\left[F^j_{A'C'}F^p_{RB'} + F^j_{B'C'}F^p_{A'R} + F^j_{A'B'}F^p_{RC'}\right]\Big\} + \\
&- \underbrace{\left[F^l_A F^r_B \delta D^i_{rl} + F^l_{AB}\delta D^i_l\right]}_{=\delta F^i_{AB}}\Big\{ - F^j_{A'B'}\left(\mathbf{F}^{-1}\right)^A_j\delta^B_{C'} + \\
&- F^j_{A'C'}\left(\mathbf{F}^{-1}\right)^A_j\delta^B_{B'} - \delta^A_{A'}F^k_{B'C'}\left(\mathbf{F}^{-1}\right)^B_k\Big\} =
\end{aligned}
\tag{65}
$$

Grouping the coefficients of the Eulerian gradients, we find

$$
\begin{aligned}
\delta F^i_{A'B'C'} = &+ \delta D^i_{jkl}F^j_{A'}F^k_{B'}F^l_{C'} + \\
&- \delta D^i_p\Big\{ - F^p_{A'B'C'} + \left(\mathbf{F}^{-1}\right)^R_j\left[F^j_{A'C'}F^p_{RB'} + F^j_{B'C'}F^p_{A'R} + F^j_{A'B'}F^p_{RC'}\right] + \\
&+ F^p_{AB}\left[- F^j_{A'B'}\left(\mathbf{F}^{-1}\right)^A_j\delta^B_{C'} - F^j_{A'C'}\left(\mathbf{F}^{-1}\right)^A_j\delta^B_{B'} - \delta^A_{A'}F^k_{B'C'}\left(\mathbf{F}^{-1}\right)^B_k\right]\Big\} \\
&- \delta D^i_{kl}\Big\{ - F^l_A F^k_B F^j_{A'B'}\left(\mathbf{F}^{-1}\right)^A_j\delta^B_{C'} - F^l_A F^k_B F^j_{A'C'}\left(\mathbf{F}^{-1}\right)^A_j\delta^B_{B'} + \\
&- F^l_A F^k_B \delta^A_{A'}F^j_{B'C'}\left(\mathbf{F}^{-1}\right)^B_j\Big\} =
\end{aligned}
\tag{66}
$$

and then

$$
\begin{aligned}
= &+ \delta D^i_{jkl}F^j_{A'}F^k_{B'}F^l_{C'} + \\
&- \delta D^i_p\Big\{ - F^p_{A'B'C'} + \left(\mathbf{F}^{-1}\right)^R_j\left[F^j_{A'C'}F^p_{RB'} + F^j_{B'C'}F^p_{A'R} + F^j_{A'B'}F^p_{RC'}\right] + \\
&+ \left[- F^j_{A'B'}\left(\mathbf{F}^{-1}\right)^A_j F^p_{AC'} - F^j_{A'C'}\left(\mathbf{F}^{-1}\right)^A_j F^p_{AB'} - F^p_{A'B}F^k_{B'C'}\left(\mathbf{F}^{-1}\right)^B_k\right]\Big\} \\
&- \delta D^i_{rl}\Big\{ - F^r_B F^j_{A'B'}\,\delta^l_j\delta^B_{C'} - F^r_B F^j_{A'C'}\,\delta^l_j\delta^B_{B'} - F^l_A\,\delta^A_{A'}\,F^j_{B'C'}\delta^r_j\Big\} =
\end{aligned}
\tag{67}
$$

In the above equation, the contributions within the square brackets turn out to be equal opposite and cancel out. Despite the cumbersome calculations, finally a compact form is attained

$$
\begin{aligned}
\delta F^{i}_{A'B'C'} = & + \delta D^{i}_{jkl} F^{j}_{A'} F^{k}_{B'} F^{l}_{C'} - \delta D^{i}_{p} \left\{ -F^{p}_{A'B'C'} \right\} \\
& - \delta D^{i}_{rl} \left\{ -F^{r}_{C'} F^{l}_{A'B'} - F^{r}_{B'} F^{l}_{A'C'} - F^{l}_{A'} F^{r}_{B'C'} \right\} = \\
= & + \delta D^{i}_{p} F^{p}_{A'B'C'} + \delta D^{i}_{rl} \left\{ + F^{r}_{C'} F^{l}_{A'B'} + F^{r}_{B'} F^{l}_{A'C'} + F^{l}_{A'} F^{r}_{B'C'} \right\} + \\
& + \delta D^{i}_{jkl} F^{j}_{A'} F^{k}_{B'} F^{l}_{C'}
\end{aligned}
\tag{68}
$$

The Lagrangian inner virtual work of equation (50), once referred to the spatial configuration, must equal its Eulerian counterpart, expressed through properly selected dual variables.

Through equations (51), (54), and (68), one finds

$$
\begin{aligned}
\delta \mathcal{E}^{\mathrm{DEF}} = & \int_{\Omega} J^{-1} P^{A}_{1\,i} \delta F^{i}_{A}\, d\Omega + \int_{\Omega} J^{-1} P^{AB}_{2\,i} \delta F^{i}_{AB}\, d\Omega + \int_{\Omega} J^{-1} P^{ABC}_{3\,i} \delta F^{i}_{ABC}\, d\Omega = \\
= & \int_{\Omega} J^{-1} P^{A}_{1\,i} \underbrace{F^{j}_{A} \delta D^{i}_{j}}_{\delta F^{i}_{A}} d\Omega + \\
& + \int_{\Omega} J^{-1} P^{AB}_{2\,i} [F^{j}_{A} F^{k}_{B} \delta D^{i}_{kj} + F^{j}_{AB} \underbrace{(\mathbf{F}^{-1})^{L}_{j} \delta F^{i}_{L}}_{\delta D^{i}_{j}}] d\Omega + \\
& + \int_{\Omega} J^{-1} P^{ABC}_{3\,i} [+ \delta D^{i}_{jkl} F^{j}_{A} F^{k}_{B} F^{l}_{C} + \delta D^{i}_{p} F^{p}_{ABC} + \\
& + \delta D^{i}_{rl} \{ + F^{r}_{C} F^{l}_{AB} + F^{r}_{B} F^{l}_{AC} + F^{l}_{A} F^{r}_{BC} \}] d\Omega = \\
= & \int_{\Omega} T^{j}_{1\,i} \delta D^{i}_{j}\, d\Omega + \int_{\Omega} T^{jk}_{2\,i} \delta D^{i}_{jk}\, d\Omega + \int_{\Omega} T^{jkl}_{3\,i} \delta D^{i}_{jkl}\, d\Omega
\end{aligned}
\tag{69}
$$

It is worth emphasizing that the Eulerian test functions appearing in the last row, namely δD^{i}_{j}, δD^{i}_{jk}, and δD^{i}_{jkl}, result from the transformation of the Lagrangian test functions δF^{i}_{A}, δF^{i}_{AB}, and δF^{i}_{ABC}, respectively, when referred to the current configuration Ω. On the basis of equation (69), it is now possible to specify relationships between the Eulerian and the Lagrangian dual quantities, distinguishing first-, second-, and third-gradient contributions, namely

$$
\begin{aligned}
& T^{j}_{1\,i} = J^{-1} P^{A}_{1\,i} F^{j}_{A} + J^{-1} P^{AB}_{2\,i} F^{j}_{AB} + J^{-1} P^{ABC}_{3\,i} F^{j}_{ABC}; \\
& T^{jk}_{2\,i} = J^{-1} P^{AB}_{2\,i} F^{j}_{A} F^{k}_{B} + J^{-1} P^{ABC}_{3\,i} \left(F^{j}_{C} F^{k}_{AB} + F^{j}_{B} F^{k}_{AC} + F^{k}_{A} F^{j}_{BC} \right); \\
& T^{jkl}_{3\,i} = J^{-1} P^{ABC}_{3\,i} F^{j}_{A} F^{k}_{B} F^{l}_{C};
\end{aligned}
\tag{70}
$$

It is worth underlying that the Lagrangian triple stress tensor $P^{ABC}_{3\,i}$ affects all the Eulerian stress tensor of order lower or equal, namely $T^{j}_{1\,i}$, $T^{jk}_{2\,i}$, and $T^{jkl}_{3\,i}$. The same occurs for $P^{AB}_{2\,i}$, affecting $T^{j}_{1\,i}$, $T^{jk}_{2\,i}$, and for $P^{A}_{1\,i}$, related exclusively to $T^{j}_{1\,i}$. This circumstance is a consequence of the mutual relationships among the Lagrangian gradient of the virtual placement map, namely δF^{i}_{A}, δF^{i}_{AB}, and δF^{i}_{ABC}, and their Eulerian counterparts, i.e. δD^{i}_{j}, δD^{i}_{jk}, and δD^{i}_{jkl}. Such a pyramidal, top-down architecture is typical of Nth gradient formulations, and gives rise to a peculiar structure of the equilibrium equations at increasing the gradient order. Once selected the above work conjugate pairs, the mathematical structure of the governing equations remains the same in the Eulerian and in the Lagrangian configuration, and the integration by parts can be carried out once in an abstract setting, without duplicating the procedure.

6. Closing remarks and future prospects

In this paper, the equilibrium equations for the third-gradient materials were derived by a fully variational approach, stemming from the minimization of the total potential energy functional. As expected, the cumbersome analytical calculations and the complexity of the nonstandard equilibrium conditions turned out to be significantly increased with respect to those relevant to the second-gradient continua. The adopted strategy, resting on the integration by parts, on the use of complementary projectors for the boundary faces and for their border edges, and on the divergence theorem for embedded submanifolds, has allowed us to specify volume, surface, edge, and wedge conditions in a very elegant and clear fashion. The inner virtual work turned out to depend not only on the stress tensor but also on double and triple (hyper-)stress tensors, of third and fourth rank, respectively, dimensionally equivalent to a pressure multiplied in turn by a length and by a length squared. It is worth pointing out that, in the present third-gradient approach, the stored energy W depends on F^i_A, F^i_{AB}, and F^i_{ABC}, and the number of parameters entering such an energy and affecting Piola stresses P^A_{1i}, P^{AB}_{2i}, and P^{ABC}_{3i} is very large: this circumstance makes it urgent to develop *ad hoc* homogenization procedures.

The different contributions to the inner virtual work over the boundary face included not only the virtual placement map, but also its first and second normal derivatives: along the edges, with codimension two, virtual work was generated versus the placement derivatives along both the face and the edge normals. Detailed expressions were provided for the contact pressure and the edge force, revealing the complex dependence on the face normals and on the mean curvature of the boundary surface. In addition, the inner virtual work was expressed in the Eulerian form by pairs of work conjugate variables: remarkable transformations among Piola (material) and Cauchy (spatial) hyperstresses with rank lower or equal to four were specified, revealing coupling among the Lagrangian and the Eulerian tensors of different orders. These results represent an important step to develop the transformation of the third-gradient equilibrium equations from the Lagrangian configuration to the Eulerian configuration.

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Appendix 1

Properties of surface and edge projectors

In this appendix the basic properties of the surface and edge projectors are briefly recalled (see also [12]). As well known, at each point of the same curved surface $\Sigma_\star \equiv \partial\Omega_\star$, a pair of complementary linear operators can be defined, apt to project any vector of the ambient space onto the tangential and normal spaces, referred to as the (Lagrangian) tangential and normal projectors and denoted by symbols $[M_\|]^A_B$ and $[M_\perp]^A_B$, respectively. Such projectors possess the following noteworthy properties (in both index and matrix notation):

$$\begin{aligned}
&[M_\|]^A_B + [M_\perp]^A_B = \delta^A_B; \quad \mathbf{M}_\| + \mathbf{M}_\perp = \mathbf{1};\\
&[M_\perp]^C_A = N^C N_A; \quad [\mathbf{M}_\perp] = \mathbf{N} \otimes \mathbf{N};\\
&[M_\|]^C_A = \delta^C_A - N^C N_A; \quad [\mathbf{M}_\|] = \mathbf{1} - \mathbf{N} \otimes \mathbf{N};\\
&[M_\|]^A_B [M_\|]^B_C = [M_\|]^A_C; \quad \mathbf{M}^2_\| = \mathbf{M}_\|;\\
&[M_\perp]^A_B [M_\perp]^B_C = [M_\perp]^A_C; \quad \mathbf{M}^2_\perp = \mathbf{M}_\perp;
\end{aligned} \tag{71}$$

where the Kronecker symbol $\delta^A_B = \mathbf{1} = g^A_B$ represents the unit operator, coincident with the mixed form of the metric tensor. The last two properties are usually referred to as idempotence of the projector.

Analogously, at each point of a border edge $L_\star \equiv \partial\Sigma_\star \equiv \partial\partial\Omega_\star$, which is a unidimensional manifold with codimension two, a pair of complementary linear operators can be defined, apt to project any vector of the ambient space onto the space spanned by the tangent vector $\mathbf{T}$, and onto its orthogonal complement, spanned by any linear combination of the face normal $\mathbf{N}$ and of the edge normal $\mathbf{B}$. Such projectors will be denoted by symbols $[M_{L\,\|}]^A_B$ and $[M_{L\,\perp}]^A_B$, respectively, marked by subscript L. One has:

$$\begin{aligned}
&[M_{L\|}]^E_A = T^E\,T_A\,; \quad \mathbf{M}_{L\,\|} = \mathbf{T}\otimes\mathbf{T}\,;\\
&[M_{L\perp}]^E_A = B^E\,B_A + N^E\,N_A\,; \quad \mathbf{M}_{L\,\perp} = \mathbf{B}\otimes\mathbf{B} + \mathbf{N}\otimes\mathbf{N}\,;\\
&\delta^E_A = [M_{L\|}]^E_A + [M_{L\perp}]^E_A = T^E\,T_A + B^E\,B_A + N^E\,N_A\,;\\
&\mathbf{1} = \mathbf{M}_{L\|} + \mathbf{M}_{L\perp} = \mathbf{T}\otimes\mathbf{T} + \mathbf{B}\otimes\mathbf{B} + \mathbf{N}\otimes\mathbf{N}
\end{aligned} \tag{72}$$

It is worth noting that, along a border edge, the edge tangent vector and the edge normal lie over the plane tangent to the boundary face at that point. By formulae:

$$[M_{\|}]^E_A T^A = T^E; \quad [M_{\|}]^E_A B^A = B^E; \tag{73}$$