TRACKING CLOSE FLYBYS VIA INTERMEDIARY ORBITS

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Abstract: *An analytical solution is presented that captures the bulk of the dynamics of close flybys about oblate celestial bodies. The solution is obtained by perturbations. More specifically, an infinitesimal contact transformation is found that reduces the perturbed hyperbolic Keplerian motion to a hyperbolic Kepler problem with varied angular momentum. It takes into account first order effects of the oblateness coefficient and provides clear improvements over the usual Keplerian solution in the closest approach of the flyby. The examples carried out show that the errors obtained with the new intermediary orbit are several orders of magnitude smaller than those obtained with the Keplerian orbit.*

Keywords: *Perturbed hyperbolic Keplerian motion, J_2-problem, flyby dynamics, perturbation methods.*

1. Introduction

Planetary flybys are commonly described in the Keplerian approximation of hyperbolic orbits. Usually, this assumption is reasonably accurate due to the short times spent in the close vicinity of the visited planet. However, gravitational effects stemming from the non-central character of the mass distribution of the planet, and, in particular, those related to its oblateness, produce a clearly observable departure from the Keplerian trajectory. Due to the essentially hyperbolic dynamics, these modifications of the orbit are more evident in its departure branch. This fact is illustrated in Fig. 1 for the Voyager 2 encounter in August 1981 with Saturn [1], where the Root Sum Square (RSS) errors of predictions based on both the Kepler and the J_2-problem models with respect to ephemeris are shown. The ephemeris model considers the perturbation effects of the Sun, of Saturn's major moons, and of Saturn's J_2 perturbation.

In the case of equatorial flybys, the Keplerian hyperbolas can be replaced by more accurate closed form solutions, which are known to exist when the disturbing effect is restricted to the only influence of the zonal harmonic of the second degree [2, 3]. However, these kinds of analytical solutions are given in implicit form, and, besides, depend on special functions. Therefore, they lack of the immediate insight obtained from the usual formulation in Keplerian elements. Alternatively, explicit analytical solutions of perturbed Keplerian motion that only involve elementary functions are customarily computed by perturbation methods without constraint to the equatorial case. This technique is commonly applied to bounded orbits with an aim on long-term prediction [4, 5], but perturbation solutions of unbounded motion have been computed too [6]. However, an important distinction between these radically different applications of the perturbation approach is that

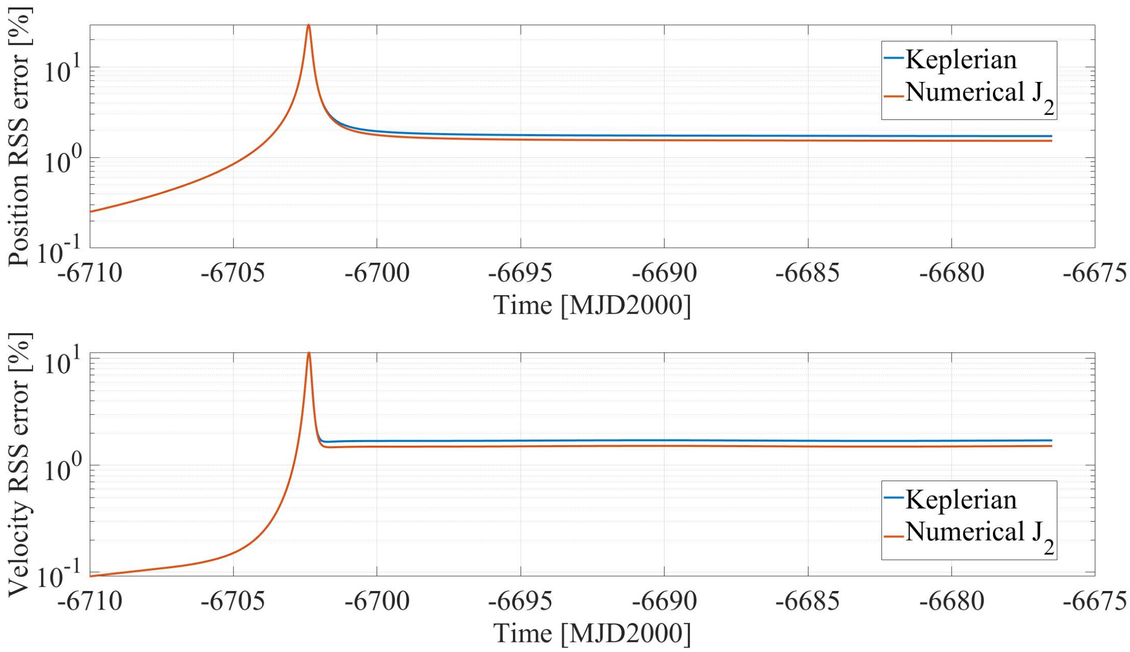

Figure 1. Voyager 2 flyby of Saturn. RSS errors of the Kepler and J_2-problem with respect to an ephemeris model. The error values are shown as relative measures with respect to the ephemeris positions and velocity, at the respective time steps.

boundary conditions, which are mostly irrelevant for perturbations solutions of the elliptic motion, strictly apply in the case of hyperbolic motion. More precisely, at infinity the perturbed Keplerian motion turns into a simple Keplerian hyperbola.

Perturbation solutions of quasi-periodic motion are customarily computed in the action and angle variables of the integrable reference model. This setting has the advantage of preventing the appearance of mixed secular-periodic terms in the transformation equations on which the solution depends upon [7]. However, this characteristic becomes irrelevant in the case of perturbed hyperbolic motion, where the extreme values of the true anomaly are bounded by the asymptotes of the reference Keplerian hyperbola, and, therefore, neither mixed nor secular terms are a concern. In addition, many of the classical intermediary solutions of the artificial satellite problem can be reformulated in polar coordinates [8] and hence are not constrained to the case of negative energies, thus avoiding the need of using two different sets of action-angle variables when dealing with bounded and unbounded orbits by perturbations. In particular, Deprit's radial intermediary, hereafter abbreviated as DRI, has the remarkable advantage of capturing the bulk of the oblateness perturbation while relying only on elementary functions [8]. Because of its simplicity, the use of DRI in orbit propagation applications has been repeatedly encouraged in the case of bounded motion [9, 10, 11].

We explore the use of DRI in the analytical propagation of unbounded perturbed Keplerian motion. Its most effective use is in the *natural* conception. That is, when the intermediary orbit is obtained as the result of an infinitesimal contact transformation that converts the non-integrable model into an integrable one in the new variables. This transformation is derived from a generating function obtained by indefinite integration, which, therefore, is determined only up to an arbitrary "constant". That is, an arbitrary function that does not depend on the integration variable. While ignoring this arbitrary function is usually admissible when dealing with bounded motion [12], this is in no way the case of unbounded orbits, where the particular form of the arbitrary constant must be determined in such way that the associated infinitesimal contact transformation matches the boundary conditions at infinity. The particular choice of the arbitrary constant does not affect the form of the intermediary orbit in the new variables, which remains the same as in the case of bounded motion. On the contrary, the contact transformation from new to original variables is notably modified. Still, a nice feature of the new transformation is that it depends only on periodic functions of the true anomaly. This is in clear contrast with the original transformation in [6], which directly converts the perturbation problem into the Kepler problem in the new variables instead of an intermediary orbit.

In our tests, we checked that the simple replacement of the Keplerian orbit by the solution in new variables stemming from DRI —that is, ignoring the contact transformation— clearly improves flyby descriptions about different solar system bodies. The improvements may reach more than one order of magnitude in the closest approach of the flyby when the complete hyperbolic DRI solution is implemented. Still, the performance of the analytical solution deteriorates for orbits close to parabolic, and fails in the case of parabolic orbits. In this regard, it is worth to remark that the deterioration of analytical perturbation solutions for eccentricities close to the unity is not a peculiarity of hyperbolic-type motion, and, quite on the contrary, happens also to most existing analytical solutions of perturbed elliptic Keplerian motion.

2. DRI for open orbits

The main effects of the noncentral potential of an Earth-like planet on the dynamics of low orbits are a result of the dominant impact of the zonal harmonic of the second degree, whose nondimensional coefficient is commonly denoted J_2 after [13]. Therefore, a gravitational perturbation model in which the gravitational potential is truncated to this term is customarily known as the *main problem* of artificial satellite theory [5], or the J_2-problem.

The J_2-problem problem admits Hamiltonian formulation. When written in polar canonical coordinates (r, θ, R, Θ) —standing for the radius, polar angle, radial velocity, and specific angular momentum, respectively— the Hamiltonian takes the form

$$\mathscr{M} = \mathscr{K} + \mathscr{P}, \tag{1}$$

where $\mathscr{K}$ is the Keplerian

$$\mathscr{K} = \frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2}\right) - \frac{\mu}{r}, \tag{2}$$

in which μ is the central body's gravitational parameter, and $\mathscr{P}$ is the disturbing function

$$\mathscr{P} = -\frac{1}{4}J_2\frac{\Theta^2}{r^2}\frac{R_\oplus^2}{p^2}\frac{p}{r}\left(2 - 3\sin^2 I + 3\sin^2 I\cos 2\theta\right), \tag{3}$$

where $R_\oplus$ is the central body's mean equatorial radius, $p = \Theta^2/\mu$ is the orbit parameter, and $I = \arccos N/\Theta$ is orbital inclination, with N denoting the third component of the angular momentum vector (per unit mass). The latter is constant as long as its conjugate coordinate $\nu = \Omega$, the right ascension of the ascending node, is an ignorable variable in Eq. (1). When written in orbital elements $\mathscr{K} = -\mu/(2a)$, where a is the orbit semimajor axis. Besides, $r = p/(1+e\cos f)$, in which e and f denote the eccentricity and the true anomaly, respectively, whereas $\theta = f + \omega$, with ω denoting the argument of the periapsis.

In addition to N, the J_2-problem accepts the energy as a second integral $\mathscr{M}(r,\theta,R,\Theta;N) = E$. Still, it lacks the necessary third integral that would guarantee the existence of a closed form solution [14]. Nevertheless, for the small values of J_2 of Earth-like bodies, the size of the regions of phase space in which chaos may emerge are so small [15] that, in practice, the search for integrable approximations of the J_2-problem makes full sense.

The search for substitutes of the J_2-problem is a recurrent topic in the literature since the beginning of the space era. Many of them depend on special functions, for instance [16, 17, 18, 19], yet intermediary orbits depending on elementary functions also exist [8, 20]. We focus on DRI,

$$\mathscr{D} = \frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2}\right) - \frac{\mu}{r} - \frac{1}{4}J_2\frac{\Theta^2}{r^2}\frac{R_\oplus^2}{p^2}\left(2 - 3\sin^2 I\right), \tag{4}$$

which enjoys both desired features of being formulated in polar variables —thus applying to both open and closed orbits— as well as being integrable in elementary functions [8]. That DRI is integrable becomes evident from the fact that the coordinate θ is now an ignorable variable of Hamiltonian (4), and, therefore, its conjugate momentum Θ becomes the third needed integral. The solution of DRI can be obtained by the Hamilton-Jacobi method. In particular, the complete Hamiltonian reduction in Delaunay-type hyperbolic variables is detailed in the Appendix.

However, in its *natural* interpretation, DRI is obtained as a result of an infinitesimal contact transformation $\xi_i = \xi_i' + \Delta\xi_i(\xi)$, $i = 1,2,\ldots 6$, in which $\xi = (r,\theta,\nu,R,\Theta,N)$ and ξ_i denotes a particular variable. Hence, Hamiltonian (4) should rather be written in prime variables $\xi' = (r',\theta',\nu',R',\Theta',N')$, and the solution provided by DRI is obtained by plugging the time solution to Eq. (4) into the transformation from prime to original variables. More precisely, the corrections $\Delta\xi_i = J_2\{\xi_i,\mathscr{U}\}$, where the curly brackets denote the Poisson bracket operator, are derived from a generating function $\mathscr{U} = \mathscr{U}(\xi)$ of the canonical variables. Following [21] we write the generating function in the canonical set of Delaunay variables (ℓ,g,h,L,G,H) rather than the polar ones. Namely

$$\mathscr{U} = -G\frac{1}{8}\frac{R_\oplus^2}{p^2}\left\{s^2[3e\sin(f+2g) + 3\sin(2f+2g) + e\sin(3f+2g)] - (6s^2-4)e\sin f\right\} + \mathscr{C}, \tag{5}$$

where $G = \Theta$, $s \equiv \sin I = (1-c^2)^{1/2}$ with $c \equiv \cos I = H/G$ and $H = N$, whereas $\mathscr{C}$ is an integration "constant" —a function of the orbital elements with null derivative with respect to the mean anomaly ℓ, to wit $\mathscr{C} \equiv \mathscr{C}(g,L,G,H)$. Remarkably, Eq. (5) is valid in both cases of bounded and unbounded motion with the caveat that the Keplerian elements used in each distinct case have a different interpretation associated to the particular dynamical regime. The function $\mathscr{C}$ is arbitrary for perturbed elliptic motion and is commonly neglected —yet it is not always the case [12].

On the contrary, $\mathscr{C}$ is no longer arbitrary in the case of open orbits, a case in which the solution must fulfill specific boundary conditions that apply to hyperbolic-type motion. In particular, the perturbed Keplerian motion becomes simply Keplerian at infinity, a place in which the series representing the mean to osculating transformation must, therefore, vanish. For the latter, we follow [6] and apply the boundary conditions $\Delta\xi|_\infty = 0$ to the corrections. After straightforward operations, we obtain,

$$\mathscr{C} = G\frac{1}{4}\frac{R_\oplus^2}{p^2}\left\{(3s^2-2)\eta - \frac{s^2}{e^2}\left[\eta^3\cos 2g + \frac{1}{2}(3e^2-2)\sin 2g\right]\right\}, \tag{6}$$

where, $\eta = -G/L = \sqrt{e^2-1}$. Hence

$$\begin{aligned}
\Delta\ell &= \frac{R_\oplus^2\eta}{32e^4p^2}\Big\{2e^3(3s^2-2)\left[4e\eta + (3e^2-4)\sin f - 4e\sin 2f - e^2\sin 3f\right] + s^2\Big[3e^5\sin(f-2g) \\
&\quad -8(e^2+2)\eta^3\cos 2g - 2(9e^4+8e^2-8)\sin 2g - 3(5e^2+4)e^3\sin(f+2g) \\
&\quad -(e^2-28)e^3\sin(3f+2g) + 18e^4\sin(4f+2g) + 3e^5\sin(5f+2g)\Big]\Big\}, \qquad (7) \\
\Delta g &= \frac{R_\oplus^2}{32e^4p^2}\Big\{-16e^4\eta(6s^2-5) - 2e^3\left[3e^2(17s^2-14) + 4(3s^2-2)\right]\sin f \\
&\quad -8e^4(3s^2-2)\sin 2f - 2e^5(3s^2-2)\sin 3f + 3e^5s^2\sin(f-2g) \\
&\quad +16\eta^3\left[e^2(2s^2-1) - s^2\right]\cos 2g + \left[6e^4(7s^2-4) - 8e^2(7s^2-2) + 16s^2\right]\sin 2g \\
&\quad +3\left[e^2(15s^2-8) - 4s^2\right]e^3\sin(f+2g) + 12(5s^2-2)e^4\sin(2f+2g) \\
&\quad +\left[e^2(19s^2-8) + 28s^2\right]e^3\sin(3f+2g) + 18s^2e^4\sin(4f+2g) + 3s^2e^5\sin(5f+2g)\Big\}, \qquad (8) \\
\Delta h &= \frac{R_\oplus^2}{4p^2}c\Big\{e^{-2}\left[(3\eta^2+1)\sin 2g + 2\eta^3\cos 2g\right] + 3e\sin(f+2g) + 3\sin(2f+2g) \\
&\quad +e\sin(3f+2g) - 6\eta - 6e\sin f\Big\}, \qquad (9) \\
\Delta L &= L\frac{R_\oplus^2}{8\eta^2r^2}\Big\{2e(3s^2-2)\cos f - 3s^2\left[e\cos(f+2g) + 2\cos(2f+2g) + e\cos(3f+2g)\right]\Big\}, \qquad (10) \\
\Delta G &= G\frac{R_\oplus^2}{4p^2}s^2\Big\{e^{-2}[(3\eta^2+1)\cos 2g - 2\eta^3\sin 2g] + 3e\cos(f+2g) + 3\cos(2f+2g) \\
&\quad +e\cos(3f+2g)\Big\}. \qquad (11)
\end{aligned}$$

Note that the corrections in Eqs. (7)–(11) must be evaluated in new variables $\xi_i = \xi_i' + \Delta\xi_i(\xi')$ when computing osculating elements from the time solution of Hamiltonian (4), but in osculating variables $\xi_i' = \xi_i - \Delta\xi_i(\xi)$ when computing the constants on which the solution of Hamiltonian (4) depends on. Recall that the corrections $\Delta\xi_i = J_2\{\xi_i, \mathscr{U}\}$ are accurate to just $\mathscr{O}(J_2)$, thus unavoidably limiting the accuracy of the intermediary solution.

3. Numerical explorations

To asses the validity of DRI in replacing the standard Keplerian orbit in a flyby, we carried out several comparisons for different major bodies of the solar system. We present two meaningful comparisons for the planet Mars, for which we use the physical parameters[1] $\mu = 0.042828\times 10^6$ km^3/s^2,

[1] nssdc.gsfc.nasa.gov/planetary/factsheet/marsfact.html (consulted in May 31, 2022)

$J_2 = 1960.45 \times 10^{-6}$, $R_{\oplus} = 3396.2$ km, and obliquity to orbit of $I_{\rm eq} = 25.19$ degree. From [22], the radius of the sphere of influence (SOI) in which Mars' gravitation dominates the dynamics is $\approx 1.1 \times 10^6$ km or about 324 times Mars' radius. Therefore, our tests will be limited to parts of hyperbolic-type orbits whose distances to Mars clearly lie inside this region.

The first test case is an hyperbolic-type orbit with initial conditions $a = 1298.73$ km, $e = 4$, $I = I_{\rm eq}$, $\Omega = 60^\circ$, $\omega = 90^\circ$, and $M = -16400^\circ$, corresponding to the initial polar variables

$$r = 376{,}948.517\ \text{km}, \quad \theta = -13.71425^\circ, \quad R = -5.76178\ \text{km/s}, \quad \Theta = 28884.81\ \text{km}^2/\text{s},$$

thus starting the simulation from a distance to Mars of about one third of the SOI radius. Figure 2 illustrates the numerically simulated trajectory in the J_2-problem dynamics with a magnification on the flyby event, whose closest approach happens about 500 km over the surface of Mars.

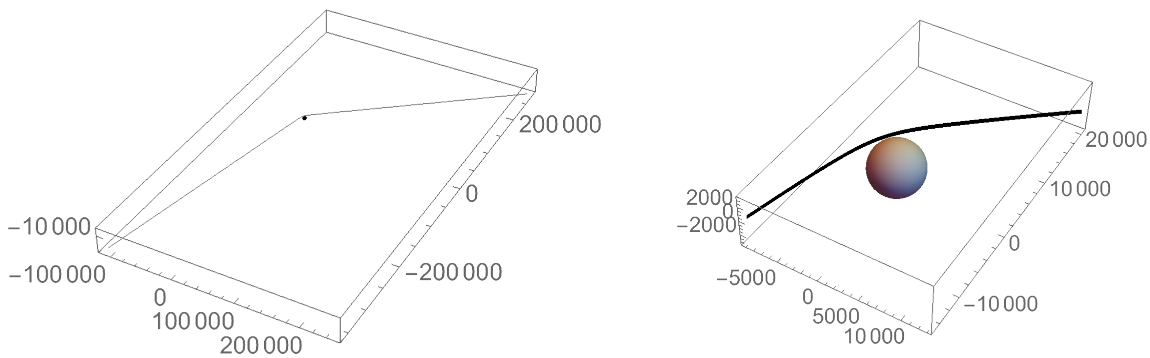

Figure 2. Mars example flyby for $e = 4$. Distances are km.

Figure 3 shows the RSS errors obtained when the true orbit is replaced by different analytical approximations starting from the same initial conditions. The label "Kepler" means the classical Keplerian orbit, which starts with almost vanishing errors due to the negligible effect of J_2 at the initial distance; RSS errors (represented with gray, solid curves) increase slowly when approaching Mars, reach the km level at the flyby, and continue to increase at a much higher rate when departing the planet, reaching about 270 km at the end of the propagation. The label "DRI common" means that the propagation is made using the *common* implementation of DRI. At the beginning of the propagation, where the dynamics remains mostly Keplerian, RSS errors (light gray, dashed) increase faster than in the previous case due to the slightly different mean motion of DRI, yet this situation is amended when the J_2 disturbing effects become more apparent; close to the flyby the RSS errors first balance with, and then improve slightly over those of the Keplerian approach, reaching about 170 km at the end of the propagation. Finally, labels "DRI natural" mean that the propagation is made with the natural version of the intermediary. Errors improve notably in this last case, yet the propagation starts with larger yet insignificant errors than in the previous simulations due to the truncation of the transformation equations to $\mathcal{O}(J_2)$. The errors remain in the meter level during the flyby, and grow slowly in the departure branch, reaching about 200 m at the end of the propagation.

The second test is a quasi-parabolic orbit with initial conditions $a = 879{,}240.0$ km, $e = 1.02$, $I = I_{\rm eq}$, $\Omega = 60^\circ$, $\omega = 90^\circ$, and $M = -6.7^\circ$, corresponding to the initial polar variables

$$r = 86{,}017.0\ \text{km}, \quad \theta = -61.543^\circ, \quad R = -1.06735\ \text{km/s}, \quad \Theta = 19501.96\ \text{km}^2/\text{s},$$

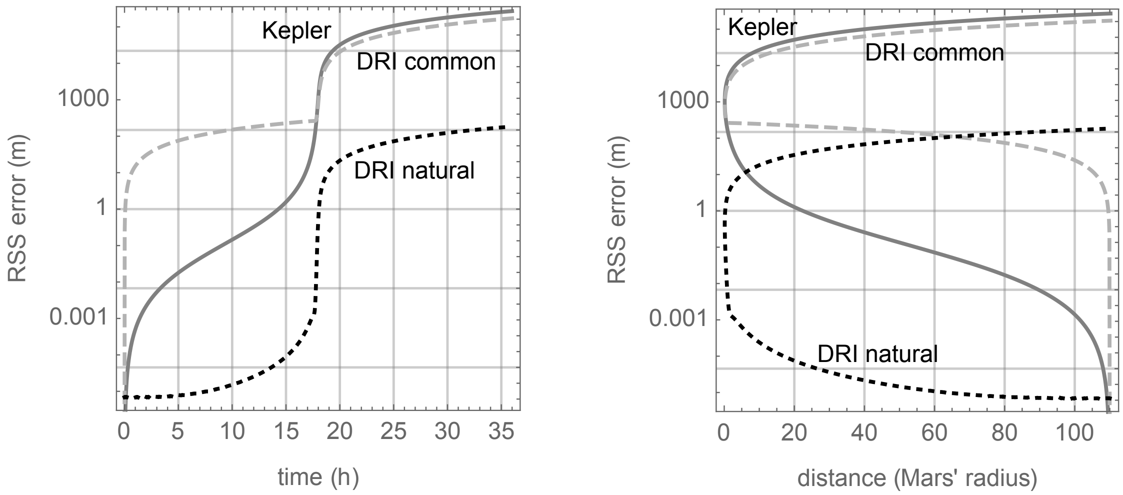

Figure 3. RSS position errors of the flyby in Fig. 2. Note the logarithmic scale.

The numerically simulated trajectory in the J_2-problem dynamics is shown in Fig. 4, where the closest approach now happens about 1000 km over Mars' surface.

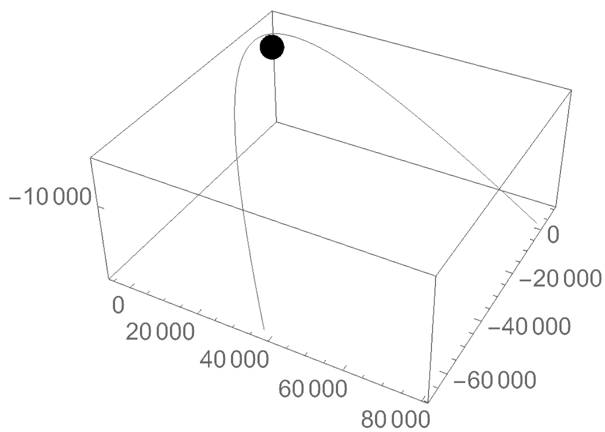

Figure 4. Mars example flyby for a quasi-parabolic orbit ($e = 1.02$). Distances are km.

Now, because $\eta = 0.2$, the divisor $\eta^2 r^2$ in Eq. (10) will reduce the accuracy of the correction to L. Hence, larger errors of DRI predictions should be expected during the flyby, where the radius takes the smallest values. As shown in Fig. 5, that is exactly the case, and the improvements obtained when replacing the Keplerian approximation of the flyby by the natural version of DRI reduce from the several orders of magnitude in the previous example to just one order of magnitude in the current case. Indeed, due to the much longer time spent in the close vicinity of the red planet, the error in the closest approach clearly peaks when using DRI, reaching about 830 m over Mars' surface. After the peak, the DRI errors notably improve, and continuously increase along the departure branch of the quasi-parabolic trajectory at the expected rate, remaining again several orders of magnitude smaller than when using the Keplerian approximation.

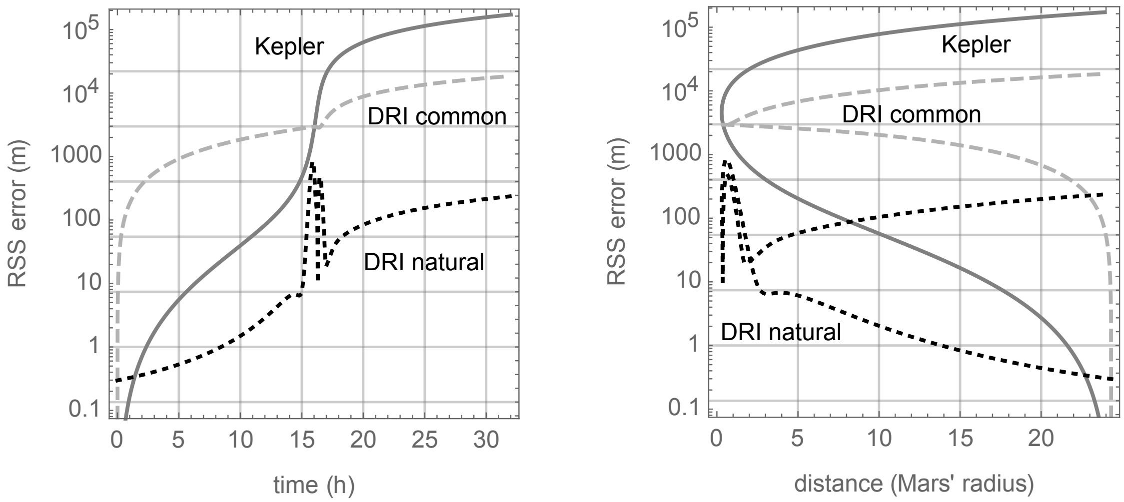

Figure 5. RSS position errors of the quasi-parabolic flyby in Fig. 4. Note the log scale.

4. Conclusions

Clear improvements in flyby simulations about major bodies of the solar system can be obtained when replacing the Keplerian hyperbola by one of the *common* main problem intermediaries in polar coordinates existing in the literature. Among them, DRI stands out for its simplicity and insight into the dynamics. Moreover, the accuracy of the flyby description is dramatically increased when the analytical solution is complemented with the infinitesimal transformation that *naturalizes* the intermediary. As opposite to the case of bounded orbits, extending DRI to higher orders is standard for open orbits, in which case the appearance of mixed terms is not of concern. Still, the increased accuracy is at the unavoidable cost of notably extending the length of the perturbation series that describe the solution. The construction of this kind of improved analytical solution, which would be required in the description of quasi-parabolic flybys, is in progress, and will be reported elsewhere.

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5. References

[1] Stone, E. C. and Miner, E. D. "Voyager 2 Encounter with the Saturnian System." Science, Vol. 215, No. 4532, pp. 499–504, Jan. 1982. doi:10.1126/science.215.4532.499.

[2] Martinusi, V. and Gurfil, P. "Analytical solutions for J_2-perturbed unbounded equatorial orbits." Celestial Mechanics and Dynamical Astronomy, Vol. 115, No. 1, pp. 35–57, Jan. 2013. doi:10.1007/s10569-012-9450-y.

[3] Dang, Z., Luo, J., Shi, P., and Zhang, H. "General Characteristics of the Motion on J_2-Perturbed Equatorial Orbits." Journal of Guidance Control Dynamics, Vol. 42, No. 10, pp. 2319–2324, Oct. 2019. doi:10.2514/1.G004142.

[4] Kozai, Y. "The motion of a close earth satellite." The Astronomical Journal, Vol. 64, pp. 367–377, Nov. 1959. doi:10.1086/107957.

[5] Brouwer, D. "Solution of the problem of artificial satellite theory without drag." The Astronomical Journal, Vol. 64, pp. 378–397, Nov. 1959. doi:10.1086/107958.

[6] Hori, G.-I. "The motion of a hyperbolic artificial satellite around the oblate earth." The Astronomical Journal, Vol. 66, pp. 258–263, Aug. 1961. doi:10.1086/108405.

[7] Kinoshita, H. "First-Order Perturbations of the Two Finite Body Problem." Publications of the Astronomical Society of Japan, Vol. 24, pp. 423–457, 1972.

[8] Deprit, A. "The elimination of the parallax in satellite theory." Celestial Mechanics, Vol. 24, No. 2, pp. 111–153, 1981. doi:10.1007/BF01229192.

[9] Coffey, S. and Alfriend, K. T. "An analytical orbit prediction program generator." Journal of Guidance, Control and Dynamics, Vol. 7, No. 5, pp. 575–581, 1984. doi:10.2514/3.19897.

[10] Gurfil, P. and Lara, M. "Satellite onboard orbit propagation using Deprit's radial intermediary." Celestial Mechanics and Dynamical Astronomy, Vol. 120, No. 2, pp. 217–232, Oct. 2014. doi:10.1007/s10569-014-9576-1.

[11] Lara, M. "LEO intermediary propagation as a feasible alternative to Brouwer's gravity solution." Advances in Space Research, Vol. 56, No. 3, pp. 367–376, Aug. 2015. ISSN 0273-1177. doi:10.1016/j.asr.2014.12.023.

[12] Lara, M., San-Juan, J. F., and López-Ochoa, L. M. "Proper Averaging Via Parallax Elimination (AAS 13-722)." S. B. Broschart, J. D. Turner, K. C. Howell, and F. R. Hoots, editors, "Astrodynamics 2013," Vol. 150 of *Advances in the Astronautical Sciences*, pp. 315–331. American Astronautical Society, Univelt, Inc., P.O. Box 28130, San Diego, California 92198, USA, Jan. 2014.

[13] Merson, R. H. and King-Hele, D. G. "Use of Artificial Satellites to Explore the Earth's Gravitational Field: Results from Sputnik 2 (1957β)." Nature, Vol. 182, No. 4636, pp. 640–641, Sep. 1958. doi:10.1038/182640a0.

[14] Irigoyen, M. and Simó, C. "Non integrability of the J_2 problem." Celestial Mechanics and Dynamical Astronomy, Vol. 55, No. 3, pp. 281–287, Mar. 1993. doi:10.1007/BF00692515.

[15] Simó, C. "Measuring the lack of integrability of the J_2 problem for Earth's satellites." "Predictability, Stability, and Chaos in N-Body Dynamical Systems," Vol. 272 of *NATO Advanced Study Institute (ASI) Series B*, pp. 305–309. Jan. 1991.

[16] Sterne, T. E. "The gravitational orbit of a satellite of an oblate planet." The Astronomical Journal, Vol. 63, pp. 28–40, Jan. 1958. doi:10.1086/107673.

[17] Cid, R. and Lahulla, J. F. “Perturbaciones de corto periodo en el movimiento de un satélite artificial, en función de las variables de Hill.” Publicaciones de la Revista de la Academia de Ciencias de Zaragoza, Vol. 24, pp. 159–165, 1969.

[18] Garfinkel, B. and Aksnes, K. “Spherical Coordinate Intermediaries for an Artificial Satellite.” The Astronomical Journal, Vol. 75, No. 1, pp. 85–91, Feb. 1970. doi:10.1086/110946.

[19] Deprit, A. and Ferrer, S. “Note on Cid’s Radial Intermediary and the Method of Averaging.” Celestial Mechanics, Vol. 40, No. 3-4, pp. 335–343, 1987.

[20] Lara, M. “Earth satellite dynamics by Picard iterations.” arXiv:2205.04310, May 2022.

[21] Lara, M., San-Juan, J. F., and López-Ochoa, L. M. “Delaunay variables approach to the elimination of the perigee in Artificial Satellite Theory.” Celestial Mechanics and Dynamical Astronomy, Vol. 120, No. 1, pp. 39–56, Sep. 2014. doi:10.1007/s10569-014-9559-2.

[22] Farquhar, R. W. “The control and use of libration-point satellites.” Tech. Rep. R-346, Goddard Space Flight Center, Greenbelt, Maryland, Sep. 1970.

[23] Ferrer, S. and Lara, M. “Families of Canonical Transformations by Hamilton-Jacobi-Poincaré Equation. Application to Rotational and Orbital Motion.” Journal of Geometric Mechanics, Vol. 2, No. 3, pp. 223–241, 2010. doi:10.3934/jgm.2010.2.223.

A Complete Hamiltonian reduction of DRI in the case of open orbits

That DRI is integrable becomes evident by noting that Hamiltonian (4) is a quasi-Keplerian system with varied angular momentum. Namely,

$$\mathscr{D}=\frac{1}{2}\left[R^2+\frac{\Gamma(\Theta,N)^2}{r^2}\right]-\frac{\mu}{r},\qquad \Gamma=\Theta\left[1-\frac{1}{2}J_2\frac{R_\oplus^2}{(\Theta^2/\mu)^2}\left(3\frac{N^2}{\Theta^2}-1\right)\right]^{1/2}. \tag{12}$$

Therefore, we can compute the solution by the Hamilton-Jacobi method [23]. We follow analogous steps as those in [10] for the case of bounded orbits. That is, we compute the transformation $(R,\Theta,N)=\partial S/\partial(r,\theta,\nu)$, $(\ell_1,g_1,h_1)=\partial S/\partial(L_1,G_1,H_1)$, stemming from the generating function in separate variables $S=\nu H_1+\theta G_1+W(r,L_1,G_1,H_1)$. We obtain

$$R=\frac{\partial W}{\partial r},\quad \Theta=G_1,\quad N=H_1,\qquad \ell_1=\frac{\partial W}{\partial L_1},\quad g_1=\theta+\frac{\partial W}{\partial G_1},\quad h_1=\nu+\frac{\partial W}{\partial H_1}, \tag{13}$$

and replace them into Eq. (12) to form the Hamilton-Jacobi equation

$$\frac{1}{2}\left[\left(\frac{\partial W}{\partial r}\right)^2+\frac{\Gamma(G_1,H_1)^2}{r^2}\right]-\frac{\mu}{r}=\frac{\mu^2}{2L_1^2},$$

which is solved by indefinite integration. We obtain $W=\Gamma\int_{r_0}^{r}\chi(r;L,G,H)^{1/2}\,\mathrm{d}r$, where the radicand $\chi\geq 0$ is the quadratic form

$$\chi=-\frac{1}{r^2}+2\frac{\mu}{\Gamma^2}\frac{1}{r}+\frac{\mu^2}{L_1^2\Gamma^2}. \tag{14}$$

Then, replacing W into Eq. (13), we trivially obtain

$$R = R(r) \equiv \Gamma\sqrt{\chi}, \tag{15}$$

whereas, after straightforward computations, the non-trivial part of the transformation yields

$$\ell_1 = \frac{\mu^2}{L_1^3}\int_{r_0}^{r}\frac{1}{R(r)}\,\mathrm{d}r, \qquad g_1 = \theta + \Gamma\frac{\partial\Gamma}{\partial G_1}\int_{s_0}^{s}\frac{1}{R(s)}\,\mathrm{d}s, \quad h_1 = \nu + \Gamma\frac{\partial\Gamma}{\partial H_1}\int_{s_0}^{s}\frac{1}{R(s)}\,\mathrm{d}s, \quad s = \frac{1}{r},$$

in which, after replacing in Eq. (12) $\Theta = G_1$, $N = H_1$, from (13), we obtain

$$\frac{\partial\Gamma}{\partial G_1} = \left[3 + \frac{1}{2}J_2\frac{R_\oplus^2}{(G_1^2/\mu)^2}\right]\frac{G_1}{\Gamma} - 2\frac{\Gamma}{G_1}, \qquad \frac{\partial\Gamma}{\partial H_1} = -\frac{3}{2}J_2\frac{R_\oplus^2}{(G_1^2/\mu)^2}\frac{H_1}{\Gamma}. \tag{16}$$

Next, using the auxiliary functions $\tilde{p} = \Gamma^2/\mu$, $\tilde{a} = L_1^2/\mu$, $\tilde{e} = \sqrt{1+\tilde{p}/\tilde{a}} > 1$, we factorize χ in the form

$$\chi = -\frac{1}{r^2} + \frac{2}{\tilde{p}r} + \frac{1}{\tilde{a}\tilde{p}} = \left(\frac{1}{r_\mathrm{P}} - \frac{1}{r}\right)\left(\frac{1}{r} - \frac{1}{r_\mathrm{A}}\right),$$

where $r_\mathrm{P} = \tilde{a}(\tilde{e}-1)$ and $r_\mathrm{A} = -\tilde{a}(\tilde{e}+1)$. Alternatively, replacing $\tilde{a} = \tilde{p}/(\tilde{e}^2-1)$, we obtain $r_\mathrm{P} = \tilde{p}/(\tilde{e}+1)$ and $r_\mathrm{A} = -\tilde{p}/(\tilde{e}-1)$.

Now, the integration in the radius is achieved by a change of variable to the hyperbolic anomaly υ using the relation $r = \tilde{a}(\tilde{e}\cosh\upsilon - 1)$, whereas the integration in the inverse of the radius is solved by the usual change to the true anomaly $s = (1+\tilde{e}\cos\varphi)/\tilde{p}$; note from the latter that $\chi = (\tilde{e}/\tilde{p})^2\sin^2\varphi$. For the lower limit of the integrals $r_0 = 1/s_0 = r_\mathrm{P}$, which is the minimum distance to the origin, we obtain,

$$\int_{r_0}^{r}\frac{1}{R(r)}\,\mathrm{d}r = \frac{\tilde{a}^{3/2}}{\mu^{1/2}}(\tilde{e}\sinh\upsilon - \upsilon), \qquad \int_{s_0}^{s}\frac{1}{R(s)}\,\mathrm{d}s = -\frac{1}{\Gamma}\varphi. \tag{17}$$

Hence, the transformation

$$R = \frac{\Gamma}{\tilde{p}}\tilde{e}\sin\varphi, \quad \Theta = G, \quad N = H, \qquad \ell_1 = \tilde{e}\sinh\upsilon - \upsilon, \quad g_1 = \theta - \frac{\partial\Gamma}{\partial G_1}\varphi, \quad h_1 = \nu - \frac{\partial\Gamma}{\partial H_1}\varphi,$$

completely reduces Hamiltonian (12) to $\mathscr{D} = \mu^2/(2L_1^2)$, from which we obtain the solution in hyperbolic Delaunay variables

$$\ell_1 = \ell_1(t_0) + nt, \quad g_1 = g_1(t_0), \quad h_1 = h_1(t_0), \quad L_1 = L_1(t_0), \quad G_1 = G_1(t_0), \quad H_1 = H_1(t_0),$$

where $n = \partial\mathscr{D}/\partial L_1 = -\mu^2/L_1^3$.

The sequence for making the transformation from hyperbolic Delaunay elements to polar variables and vice versa are summarized in Algorithms 1 and 2, respectively.

Algorithm 1 $(\ell_1, g_1, h_1, L_1, G_1, H_1) \longrightarrow (r, \theta, \nu, R, \Theta, N)$

Make $N = H_1$, $\Theta = G_1$, and compute Γ from the last of Eq. (12).
Evaluate $\tilde{a} = L_1^2/\mu$, $\tilde{p} = \Gamma^2/\mu$, and hence $\tilde{e} = (1 + \tilde{p}/\tilde{a})^{1/2}$.
Solve υ from $\ell_1 = \tilde{e} \sinh \upsilon - \upsilon$.
Compute $\varphi = 2 \tan^{-1} \left[\sqrt{(\tilde{e}+1)/(\tilde{e}-1)} \tanh \frac{1}{2}\upsilon\right]$.
Evaluate $r = \tilde{p}/(1 + \tilde{e}\cos\varphi)$, and $R = (\Gamma/\tilde{p})\tilde{e}\sin\varphi$.
Compute $\theta = g_1 + [(G_1/\Gamma)(3 - \varepsilon) - 2\Gamma/G_1]\,\varphi$, with $\varepsilon = -\frac{1}{2} J_2 R_\oplus^2 \mu^2/G_1^4$.
Compute $\nu = h_1 + 3\varepsilon(H_1/\Gamma)\varphi$.

Algorithm 2 $(r, \theta, \nu, R, \Theta, N) \longrightarrow (\ell_1, g_1, h_1, L_1, G_1, H_1)$

Make $H_1 = N$, $G_1 = \Theta$; evaluate Γ and $E = \mathscr{D}$ from Eq. (12).
Compute $L_1 = -\mu/\sqrt{2E}$; evaluate $\tilde{a} = L_1^2/\mu$, $\tilde{p} = \Gamma^2/\mu$, and hence $\tilde{e} = (1 + \tilde{p}/\tilde{a})^{1/2}$
Solve $\tilde{e}\cos\varphi = -1 + \tilde{p}/r$, $\tilde{e}\sin\varphi = \tilde{p}R/\Gamma$, for φ.
Compute $\upsilon = 2 \tanh^{-1} \left[\sqrt{(\tilde{e}-1)/(\tilde{e}+1)} \tan \frac{1}{2}\varphi\right]$.
Evaluate $\ell_1 = \tilde{e} \sinh \upsilon - \upsilon$
Compute $g_1 = \theta - [(G_1/\Gamma)(3 - \varepsilon) - 2\Gamma/G_1]\,\varphi$, with $\varepsilon = -\frac{1}{2} J_2 R_\oplus^2 \mu^2/G_1^4$.
Compute $h_1 = \nu - 3\varepsilon(H_1/\Gamma)\varphi$.