Commun. Math. Phys.
Digital Object Identifier (DOI) https://doi.org/10.1007/s00220-023-04637-5

Communications in **Mathematical Physics**

Unitarity of Minimal W-Algebras and Their Representations I

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Received: 9 August 2022 / Accepted: 5 January 2023

Abstract: We begin a systematic study of unitary representations of minimal W-algebras. In particular, we classify unitary minimal W-algebras and make substantial progress in classification of their unitary irreducible highest weight modules. We also compute the characters of these modules.

Contents

Published online: 25 January 2023

1. Introduction

In the present paper we study unitarity of minimal W-algebras and of their representations. Minimal W-algebras are the simplest conformal vertex algebras among the simple vertex algebras $W_k(\mathfrak{g}, x, f)$, constructed in [18,20], associated to a datum $(\mathfrak{g}, x, f)$ and $k \in \mathbb{R}$. Here $\mathfrak{g} = \mathfrak{g}_{\bar 0} \oplus \mathfrak{g}_{\bar 1}$ is a basic Lie superalgebra, i.e. $\mathfrak{g}$ is simple, its even part $\mathfrak{g}_{\bar 0}$ is a reductive Lie algebra and $\mathfrak{g}$ carries an even invariant non-degenerate supersymmetric bilinear form $(.|.)$, x is an ad-diagonalizable element of $\mathfrak{g}_{\bar 0}$ with eigenvalues in $\frac{1}{2}\mathbb{Z}$, $f \in \mathfrak{g}_{\bar 0}$ is such that $[x, f] = -f$ and the eigenvalues of $ad\,x$ on the centralizer $\mathfrak{g}^f$ of f in $\mathfrak{g}$ are non-positive, and $k \neq -h^\vee$, where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$. The most important examples are provided by x and f to be part of an sl_2 triple $\{e, x, f\}$, where $[x, e] = e$, $[x, f] = -f$, $[e, f] = x$. In this case $(\mathfrak{g}, x, f)$ is called a *Dynkin datum*. Recall that $W_k(\mathfrak{g}, x, f)$ is the unique simple quotient of the universal W-algebra, denoted by $W^k(\mathfrak{g}, x, f)$, which is freely strongly generated by elements labeled by a basis of the centralizer of f in $\mathfrak{g}$ [20].

We proved in [16, Lemma 7.3] that if ϕ is a conjugate linear involution of $\mathfrak{g}$ such that

$$\phi(x) = x, \quad \phi(f) = f \text{ and } \overline{(\phi(a)|\phi(b))} = (a|b), \; a, b \in \mathfrak{g}, \tag{1.1}$$

then ϕ induces a conjugate linear involution of the vertex algebra $W^k(\mathfrak{g}, x, f)$, which descends to $W_k(\mathfrak{g}, x, f)$.

We also proved in [16, Proposition 7.4] that if ϕ is a conjugate linear involution of $W_k(\mathfrak{g}, x, f)$, this vertex algebra carries a non-zero ϕ-invariant Hermitian form $H(\cdot, \cdot)$ for all $k \neq -h^\vee$ if and only if $(\mathfrak{g}, x, f)$ is a Dynkin datum; moreover, such H is unique, up to a real constant factor, and we normalize it by the condition $H(\mathbf{1}, \mathbf{1}) = 1$. A module M for a vertex algebra V is called *unitary* if there is a conjugate linear involution ϕ of V such that there is a positive definite ϕ-invariant Hermitian form on M. The vertex algebra V is called unitary if the adjoint module is.

For some levels k the vertex algebra $W_k(\mathfrak{g}, x, f)$ is trivial, i.e. isomorphic to $\mathbb{C}$; then it is trivially unitary. Another easy case is when $W_k(\mathfrak{g}, x, f)$ "collapses" to the affine part. In both cases we will say that k is *collapsing level*.

In the case of a Dynkin datum let $\mathfrak{g}^\natural$ be the centralizer of the sl_2 subalgebra $\mathfrak{s} = span\,\{e, x, f\}$ in $\mathfrak{g}_{\bar 0}$; it is a reductive subalgebra. If ϕ satisfies the first two conditions in (1.1), it fixes e, x, f, hence $\phi(\mathfrak{g}^\natural) = \mathfrak{g}^\natural$. It is easy to see that unitarity of $W_k(\mathfrak{g}, x, f)$ implies, when k is not collapsing, that $\phi_{|[\mathfrak{g}^\natural, \mathfrak{g}^\natural]}$ is a compact involution.

In the present paper we consider only *minimal* data $(\mathfrak{g}, x, f)$, defined by the property that for the $ad\, x$-gradation $\mathfrak{g} = \bigoplus_{j\in\frac{1}{2}\mathbb{Z}} \mathfrak{g}_j$ one has

$$\mathfrak{g}_j = 0 \text{ if } |j| > 1, \text{ and } \mathfrak{g}_{-1} = \mathbb{C}f. \tag{1.2}$$

In this case $(\mathfrak{g}, x, f)$ is automatically a Dynkin datum. The corresponding W-algebra is called *minimal*. The element $f \in \mathfrak{g}$ is a root vector attached to a root $-\theta$ of $\mathfrak{g}$, and we shall normalize the invariant bilinear form on $\mathfrak{g}$ by the usual condition $(\theta|\theta) = 2$, which is equivalent to $(x|x) = \frac{1}{2}$. Recall that the dual Coxeter number $h^\vee$ of $\mathfrak{g}$ is half of the eigenvalue of its Casimir element of $\mathfrak{g}$, attached to the bilinear form $(.|.)$. We shall denote by $W_k^{\min}(\mathfrak{g})$ the minimal W-algebra, corresponding to $\mathfrak{g}$ and $k \neq -h^\vee$, and by $W^k_{\min}(\mathfrak{g})$ the corresponding universal W-algebra.

We proved in [16, Proposition 7.9] that, if $W_k^{\min}(\mathfrak{g})$ is unitary and k is not a collapsing level, then the parity of $\mathfrak{g}$ is compatible with the $ad\, x$-gradation, i.e. the parity of the whole subspace $\mathfrak{g}_j$ is $2j \mod 2$.

It follows from [18], [20] that for each basic simple Lie superalgebra $\mathfrak{g}$ there is at most one minimal Dynkin datum, compatible with parity, and the complete list of the $\mathfrak{g}$ which admit such a datum is as follows:

$$\begin{gathered} sl(2|m) \text{ for } m \geq 3, \quad psl(2|2), \quad spo(2|m) \text{ for } m \geq 0, \\ osp(4|m) \text{ for } m > 2 \text{ even}, \quad D(2,1;a) \text{ for } a \in \mathbb{C}, \quad F(4), \quad G(3). \end{gathered} \tag{1.3}$$

The even part $\mathfrak{g}_{\bar{0}}$ of $\mathfrak{g}$ in this case is isomorphic to the direct sum of the reductive Lie algebra $\mathfrak{g}^\natural$ and $\mathfrak{s} \cong sl_2$.

One of our conjectures (see Conjecture 4 in Sect. 8)[1] states that any unitary $W^k_{\min}(\mathfrak{g})$-module descends to $W_k^{\min}(\mathfrak{g})$. In fact, it is tempting to conjecture that for any conformal vertex algebra V any unitary V-module descends to the simple quotient of V.

It turns out (cf. Proposition 7.2) that a conjugate linear involution of the universal minimal W-algebra $W^k_{\min}(\mathfrak{g})$ at non-collapsing level k is necessarily induced by a conjugate linear involution ϕ of $\mathfrak{g}$. Moreover, by Proposition 8.9, if $W^k_{\min}(\mathfrak{g})$ admits a unitary highest weight module and k is not collapsing, then $\mathfrak{g}^\natural$ has to be semisimple. As explained above, the involution ϕ of $\mathfrak{g}$ must be *almost compact*, according to the following definition.

Definition 1.1. A conjugate linear involution ϕ on $\mathfrak{g}$ is called almost compact if

(i) ϕ fixes e, x, f;
(ii) ϕ is a compact conjugate linear involution of $\mathfrak{g}^\natural$.

Indeed (i) is equivalent to the first two requirements in (1.1), and the third requirement in (1.1) follows from Lemma 3.1 in Sect. 3.

So, in order to study unitarity of highest weight modules, it is not restrictive to assume that the conjugate linear involution of $W^k_{\min}(\mathfrak{g})$ is induced by an almost compact conjugate linear involution of $\mathfrak{g}$.

We prove in Sects. 3 and 4 that an almost compact conjugate linear involution ϕ exists for all $\mathfrak{g}$ from the list (1.3), except that a must lie in $\mathbb{R}$ in case of $D(2,1;a)$, and is essentially unique.

[1] See Note added in proof.

It was shown in [20] that the central charge of $W_k^{\min}(\mathfrak{g})$ equals

$$c(k) = \frac{k\,d}{k+h^\vee} - 6k + h^\vee - 4, \text{ where } d = \operatorname{sdim}\mathfrak{g}. \tag{1.4}$$

Here is another useful way to write this formula:

$$c(k) = 7h^\vee + d - 4 - 12\sqrt{} - 6\frac{(k+h^\vee - \sqrt{})^2}{k+h^\vee}, \text{ where } \sqrt{} = \sqrt{\frac{d\,h^\vee}{6}}. \tag{1.5}$$

Recall that the most important superconformal algebras in conformal field theory are the simple minimal W-algebras or are obtained from them by a simple modification:

(a) $W_k^{\min}(spo(2|N))$ is the Virasoro vertex algebra for $N = 0$, the Neveu-Schwarz vertex algebra for $N = 1$, the $N = 2$ vertex algebra for $N = 2$, and becomes the $N = 3$ vertex algebra after tensoring with one fermion; it is the Bershadsky-Knizhnik algebra for $N > 3$;
(b) $W_k^{\min}(psl(2|2))$ is the $N = 4$ vertex algebra;
(c) $W_k^{\min}(D(2,1;a))$ tensored with four fermions and one boson is the big $N = 4$ vertex algebra.

The unitary Virasoro ($N = 0$), Neveu-Schwarz ($N = 1$) and $N = 2$ simple vertex algebras, along with their irreducible unitary modules, were classified in the mid 80s. Up to isomorphism, these vertex algebras depend only on the central charge $c(k)$, given by (1.4). Putting $k = \frac{1}{p} - 1$ in (1.5) in all three cases, we obtain

$$c(k) = 1 - \frac{6}{p(p+1)} \quad \text{for Virasoro vertex algebra,} \tag{1.6}$$

$$c(k) = \frac{3}{2}\left(1 - \frac{8}{p(p+2)}\right) \quad \text{for Neveu-Schwarz vertex algebra,} \tag{1.7}$$

$$c(k) = 3\left(1 - \frac{2}{p}\right) \quad \text{for } N = 2 \text{ vertex algebra.} \tag{1.8}$$

The following theorem is a result of several papers, published in the 80s in physics and mathematics literature, see e.g. [5] for references.

Theorem 1.2. *The complete list of unitary $N = 0, 1,$ and 2 vertex algebras is as follows: either $c(k)$ is given by (1.6), (1.7), or (1.8), respectively, for $p \in \mathbb{Z}_{\geq 2}$, or $c(k) \geq 1, \frac{3}{2}$ or 3, respectively.*

The above three cases cover all minimal W-algebras, associated with $\mathfrak{g}$, such that the eigenspace $\mathfrak{g}_0$ of $ad\,x$ is abelian. Thus, we may assume that $\mathfrak{g}_0$ is not abelian.

In order to study unitarity of the simple minimal W-algebra $W_k^{\min}(\mathfrak{g})$, one needs to consider the more general framework of representation theory of universal minimal W-algebras $W^k_{\min}(\mathfrak{g})$. Of course, unitarity of $W^k_{\min}(\mathfrak{g})$ is equivalent to that of $W_k^{\min}(\mathfrak{g})$. It is therefore natural to study unitarity of irreducible $W^k_{\min}(\mathfrak{g})$-modules. For that purpose, we take, in Sect. 6, a long detour to develop a general theory of invariant Hermitian forms on modules over the vertex algebra of free bosons, which will be eventually applied to our main object of interest. As a byproduct we obtain a field theoretic version of the *Fairlie construction*, which yields explicit models of unitary representations of the Virasoro algebra for certain values of the highest weight (cf. [17, 3.4], Example 6.9).

We consider in Sect. 9 the free field realization $\Psi : W^k_{\min}(\mathfrak{g}) \to \mathcal{V}^k = V^{k+h^\vee}(\mathbb{C}x) \otimes V^{\alpha_k}(\mathfrak{g}^\natural) \otimes F(\mathfrak{g}_{1/2})$ introduced in [20] (here $V^\gamma(\mathfrak{a})$ denotes the universal affine vertex algebra associated to the Lie algebra $\mathfrak{a}$ and to a 2-cocycle γ, α_k is the 2-cocycle defined in (7.24), and $F(\mathfrak{g}_{1/2})$ is the fermionic vertex algebra "attached" to $\mathfrak{g}_{1/2}$). Let $M(\mu)$ be the Verma module of highest weight $\mu \in \mathbb{C}$ for the bosonic vertex algebra $V^{k+h^\vee}(\mathbb{C}x)$ and consider the $\mathcal{V}^k$-module $N(\mu) = M(\mu) \otimes V^{\alpha_k}(\mathfrak{g}^\natural) \otimes F(\mathfrak{g}_{1/2})$. Applying to $N(\mu)$ results from Sect. 6, we obtain in Proposition 9.2 a generalization of the Fairlie construction to universal minimal W-algebras.

The conformal vertex algebras $(W^k_{\min}(\mathfrak{g}), L)$ and $(\mathcal{V}^k, \widehat{L}(0))$ (see (6.29)) both admit Hermitian invariant forms $H(\cdot,\cdot)_W$ and $H(\cdot,\cdot)_{free}$, respectively. Unfortunately, the embedding Ψ is not conformal, i.e., $\Psi(L) \neq \widehat{L}(0)$, in particular Ψ is not an isometry (which was erroneously claimed in [14]). So, though the vertex algebra $\mathcal{V}^k$ is unitary, this does not imply the unitarity of $W^k_{\min}(\mathfrak{g})$. A few explicit computations suggest the following conjecture, which we were unable to prove.

Conjecture 1. *For each $w \in W^k_{\min}(\mathfrak{g})$, $H(w,w)_W \geq H(\Psi(w),\Psi(w))_{free}$. In particular if $\mathcal{V}^k$ is unitary, then $W^k_{\min}(\mathfrak{g})$ is unitary.*

We start the study of unitary modules over minimal W-algebras in Sect. 8 by introducing the irreducible highest weight $W^k_{\min}(\mathfrak{g})$-modules $L^W(\nu, \ell_0)$ with highest weight (ν, ℓ_0), where ν is a real weight of $\mathfrak{g}^\natural$ and $\ell_0 \in \mathbb{R}$ is the minimal eigenvalue of L_0. We prove that $L^W(\nu, \ell_0)$ admits a ϕ-invariant nondegenerate Hermitian form (unique up to normalization), see Lemma 8.1. In Sect. 8 we also determine necessary conditions for the unitarity of $L^W(\nu, \ell_0)$. Part of the necessary conditions is displayed in Proposition 8.5. They say that unitarity of $L^W(\nu, \ell_0)$ implies that the levels $M_i(k)$ of the affine Lie algebras $\widehat{\mathfrak{g}}^\natural_i$ in $W^k_{\min}(\mathfrak{g})$ (given in Table 2, Sect. 7), where $\mathfrak{g}^\natural_i$ are the simple components of $\mathfrak{g}^\natural$, are non-negative integers, ν is dominant integral of levels $M_i(k)$, and the inequality (1.9) below holds. Proposition 8.8 provides a further necessary condition, which says that (1.9) must be an equality when ν is an "extremal" weight. See Theorem 1.3 (1) below for a precise statement.

In Sect. 10, using the generalization of the Fairlie construction, developed in Sect. 9, we prove a partial converse result: if $M_i(k) + \chi_i \in \mathbb{Z}_+$, where χ_i are negative integers, displayed in Table 2, and ν is dominant integral weight for $\mathfrak{g}^\natural$ which is not extremal, then the $W^k_{\min}(\mathfrak{g})$-module $L^W(\nu, \ell_0)$ is unitary for l_0 sufficiently large, see Proposition 10.2.

In Sect. 11 we prove our central Theorem 11.1, which claims that actually Proposition 10.2 holds for l_0 satisfying the inequality (1.9), provided that ν is not extremal. This is established by the following construction. Let $\widehat{\mathfrak{g}}$ be the affinization of $\mathfrak{g}$. We introduce in (11.4) a highest weight module $\overline{M}(\widehat{\nu}_h)$ over $\widehat{\mathfrak{g}}$, whose highest weight $\widehat{\nu}_h$ depends on $h \in \mathbb{C}$, with the following two properties

(1) $\overline{M}(\widehat{\nu}_h)$ is irreducible, except possibly for an explicit set J of values of h.
(2) For the quantum Hamiltonian reduction functor H_0, the $W^k_{\min}(\mathfrak{g})$-module $H_0(\overline{M}(\widehat{\nu}))$ admits a Hermitian form, depending polynomially on h.

Using the irreducibility theorem by Arakawa [2], we deduce that $H_0(\overline{M}(\widehat{\nu}_h)) = L^W(\nu, \ell(h))$ for $h \notin J$, where $\ell(h)$ is defined by (11.45). It turns out that, miraculously, if $h \in J$, then $\ell(h)$ does not satisfy (1.9). Moreover $L^W(\nu, \ell_0)$ is unitary for $l_0 \gg 0$. By continuity, the determinant of the Hermitian form on $L^W(\nu, \ell_0)$ is positive if the inequality (1.9) holds. See Theorem 1.3 (2) below for a precise statement.

Let us state our main results. First of all, if $\mathfrak{g} = sl(2|m)$ with $m \geq 3$ or $osp(4|m)$ with $m \geq 2$ even, then none of the $W^k_{\min}(\mathfrak{g})$-modules $L^W(\nu, \ell_0)$ are unitary for a non-collapsing level k. For the remaining $\mathfrak{g}$ from the list (1.3) the Lie algebra $\mathfrak{g}^\natural$ is semisimple (actually simple, except for $\mathfrak{g} = D(2, 1; a)$, when $\mathfrak{g}^\natural = sl_2 \oplus sl_2$). Let $\theta_i^\vee$ be the coroots of the highest roots θ_i of the simple components $\mathfrak{g}_i^\natural$ of $\mathfrak{g}^\natural$. Let $2\rho^\natural$ be the sum of positive roots of $\mathfrak{g}^\natural$, and let ξ be a highest weight of the $\mathfrak{g}^\natural$-module $\mathfrak{g}_{-1/2}$ (this module is irreducible, except for $\mathfrak{g} = psl(2|2)$ when it is $\mathbb{C}^2 \oplus \mathbb{C}^2$). Let ν be a dominant integral weight for $\mathfrak{g}^\natural$ and $l_0 \in \mathbb{R}$. We prove the following theorem.

Theorem 1.3. *Let $L^W(\nu, \ell_0)$ be an irreducible highest weight $W^k_{\min}(\mathfrak{g})$-module for $\mathfrak{g} = psl(2|2)$, $spo(2|m)$ with $m \geq 3$, $D(2, 1; a)$, $F(4)$ or $G(3)$.*

(1) This module can be unitary only if the following conditions hold:
(a) $M_i(k)$ are non-negative integers,
(b) $\nu(\theta_i^\vee) \leq M_i(k)$ for all i,
(c)

$$l_0 \geq \frac{(\nu|\nu + 2\rho^\natural)}{2(k + h^\vee)} + \frac{(\xi|\nu)}{k + h^\vee}((\xi|\nu) - k - 1), \tag{1.9}$$

and equality holds in (1.9) if $\nu(\theta_i^\vee) > M_i(k) + \chi_i$ for $i = 1$ or 2.

(2) This module is unitary if the following conditions hold:
(a) $M_i(k) + \chi_i \in \mathbb{Z}_+$ for all i,
(b) $\nu(\theta_i^\vee) \leq M_i(k) + \chi_i$ for all i (i.e. ν is not extremal),
(c) inequality (1.9) holds.

Conjecture 2. *The modules $L^W(\nu, \ell_0)$ are unitary if ν is extremal and $l_0 = R.H.S.$ of (1.9). In other words, the necessary conditions of unitarity in Theorem 1.3 (1) are sufficient.*

We were able to prove this conjecture only for $\mathfrak{g} = psl(2|2)$ and $spo(2|3)$, obtaining thereby a complete classification of unitary simple highest weight $W^k_{\min}(\mathfrak{g})$-modules in these two cases. Note that papers [3,4,21] respectively claim (without proof) these results.

Since $\nu = 0$ is extremal iff k is collapsing, we obtain the following complete classification of minimal simple unitary W-algebras:

Theorem 1.4. *The simple minimal W-algebra $W^{\min}_{-k}(\mathfrak{g})$ with $k \neq h^\vee$ and $\mathfrak{g}_0$ non-abelian is non-trivial unitary if and only if*

(1) $\mathfrak{g} = sl(2|m)$, $m \geq 3$, $k = 1$ (in this case the W-algebra is a free boson);
(2) $\mathfrak{g} = psl(2|2)$, $k \in \mathbb{N} + 1$;
(3) $\mathfrak{g} = spo(2|3)$, $k \in \frac{1}{4}(\mathbb{N} + 2)$;
(4) $\mathfrak{g} = spo(2|m)$, $m > 4$, $k \in \frac{1}{2}(\mathbb{N} + 1)$;
(5) $\mathfrak{g} = D(2, 1; \frac{m}{n})$, $k \in \frac{mn}{m+n}\mathbb{N}$, where $m, n \in \mathbb{N}$ are coprime, $k \neq \frac{1}{2}$;
(6) $\mathfrak{g} = F(4)$, $k \in \frac{2}{3}(\mathbb{N} + 1)$;
(7) $\mathfrak{g} = G(3)$, $k \in \frac{3}{4}(\mathbb{N} + 1)$.

This result, along with all known results on unitarity of vertex algebras, leads to the following general conjecture.

Conjecture 3. *A CFT type vertex operator algebra admitting a invariant Hermitian form and having a unitary module is unitary.*

In the final Sect. 14 we provide character formulas for all unitary $W^k_{\min}(\mathfrak{g})$-modules $L^W(\nu, \ell_0)$, which are obtained by applying the quantum Hamiltonian reduction to the corresponding irreducible highest weight modules over the affinization $\widehat{\mathfrak{g}}$ of $\mathfrak{g}$. There are two cases to consider. In the first case, called massive (or typical), when inequality (1.9) is strict, this character formula is easy to prove (see the proof of Proposition 11.5), which leads to the character formula (14.5). In the second case, called massless (or atypical), when the inequality (1.9) is equality, there is a general KW-formula for maximally atypical tame integrable $\widehat{\mathfrak{g}}$-modules, conjectured in [19] and proved in [7] for all $\mathfrak{g}$ in question, except for $\mathfrak{g} = D(2, 1; \frac{m}{n})$, $\nu \neq 0$, which leads to the character formula (14.6). Character formulas were also given in [4] (resp. [21]) for the $N = 4$ superconformal algebra (resp. for $W^k_{\min}(spo(2|3))$, hence for the $N = 3$ superconformal algebra). The proofs given in these papers are incomplete since they assume that their list of singular vectors is complete and that in the usual argument of inclusion-exclusion of Verma modules subsingular vectors cancel out. Their formulas for both massive and massless representations coincide with (14.5) and (14.6), respectively.

In our next paper of this series we will study unitarity of twisted representations of minimal W-algebras.

Throughout the paper the base field is $\mathbb{C}$, and $\mathbb{Z}_+$ and $\mathbb{N}$ stand for the set of non negative and positive integers, respectively.

2. Setup

2.1. Basic Lie superalgebras. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a basic finite-dimensional Lie superalgebra over $\mathbb{C}$ as in (1.3). Choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}_{\bar{0}}$. It is a maximal ad-diagonalizable subalgebra of $\mathfrak{g}$, for which the root space decomposition is of the form

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha, \tag{2.1}$$

where $\Delta \subset \mathfrak{h}^* \setminus \{0\}$ is the set of roots. In all cases, except for $\mathfrak{g} \cong psl(2|2)$, the root spaces have dimension 1. In the case $\mathfrak{g} = psl(2|2)$ one can achieve this property by embedding in $pgl(2|2)$ and replacing (2.1) by the root space decomposition with respect to a Cartan subalgebra of $pgl(2|2)$, which we will do.

Let Δ^+ be a subset of positive roots and $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ be the corresponding set of simple roots. We will denote by $\Pi_{\bar{0}}$, $\Pi_{\bar{1}}$, the sets of even and odd simple roots, respectively. For each $\alpha \in \Delta^+$ choose $X_\alpha \in \mathfrak{g}_\alpha$ and $X_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that $(X_\alpha|X_{-\alpha}) = 1$, and let $h_\alpha = [X_\alpha, X_{-\alpha}]$. Let $e_i = X_{\alpha_i}$, $f_i = X_{-\alpha_i}$, $i = 1, \ldots, r$. The set $\{e_i, f_i, h_{\alpha_i} \mid i = 1, \ldots, r\}$ generates $\mathfrak{g}$, and satisfies the following relations

$$[e_i, f_j] = \delta_{ij} h_{\alpha_i}, \quad [h_{\alpha_i}, e_j] = (\alpha_i|\alpha_j) e_j, \quad [h_{\alpha_i}, f_j] = -(\alpha_i|\alpha_j) f_j. \tag{2.2}$$

The Lie superalgebra $\tilde{\mathfrak{g}}$ on generators $\{e_i, f_i, h_{\alpha_i} \mid i = 1, \ldots, r\}$ subject to relations (2.2) is a (infinite-dimensional) $\mathbb{Z}$-graded Lie algebra, where the grading is defined by $\deg h_{\alpha_i} = 0$, $\deg e_i = -\deg f_i = 1$, with a unique $\mathbb{Z}$-graded maximal ideal, and $\mathfrak{g}$ is the quotient of $\tilde{\mathfrak{g}}$ by this ideal. We assume that $(\alpha_i|\alpha_j) \in \mathbb{R}$ for all $\alpha_i, \alpha_j \in \Pi$.

2.2. Conjugate linear involutions and real forms. In the above setting, given a collection of complex numbers $\Lambda = \{\lambda_1, \dots, \lambda_r\}$ such that $\lambda_i \in \sqrt{-1}\mathbb{R}$ if α_i is an odd root and $\lambda_i \in \mathbb{R}$ if α_i is an even root, we can define an antilinear involution $\omega_\Lambda : \mathfrak{g} \to \mathfrak{g}$ setting

$$\omega_\Lambda(e_i) = \lambda_i f_i, \quad \omega_\Lambda(f_i) = \bar{\lambda}_i^{-1} e_i, \quad \omega_\Lambda(h_{\alpha_i}) = -h_{\alpha_i}, \ 1 \le i \le r. \tag{2.3}$$

Since ω_Λ preserves relations (2.2), it induces an antilinear involution of $\tilde{\mathfrak{g}}$, and, since ω_Λ preserves the $\mathbb{Z}$-grading of $\tilde{\mathfrak{g}}$, it preserves its unique maximal ideal, hence it induces an antilinear involution of $\mathfrak{g}$.

Set $\sigma_\alpha = -1$ if α is an odd negative root and $\sigma_\alpha = 1$ otherwise, so that $(X_\alpha | X_{-\alpha}) = \sigma_\alpha$. Let

$$\xi_\alpha = \begin{cases} sgn(\alpha|\alpha) & \text{if } \alpha \text{ is an even root,} \\ 1 & \text{if } \alpha \text{ is an odd root.} \end{cases}$$

Then in [8, (4.13), (4.15)] it is proven (using results from [9]), that one can choose root vectors X_α in such a way that

$$\omega_\Lambda(X_\alpha) = -\sigma_\alpha \xi_\alpha \lambda_\alpha X_{-\alpha}, \tag{2.4}$$

where

$$\lambda_\alpha = \prod_i (-\xi_{\alpha_i} \lambda_i)^{n_i} \text{ for } \alpha = \sum_{i=1}^r n_i \alpha_i. \tag{2.5}$$

We shall call this a *good choice* of root vectors.

2.3. Invariant Hermitian forms on vertex algebras. Let V be a conformal vertex algebra with conformal vector $L = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ (see [16] for the definition and undefined notation). Let ϕ be a conjugate linear involution of V. A Hermitian form $H(\ .\ ,\ .\ )$ on V is called ϕ-*invariant* if, for all $a \in V$, one has [16]

$$H(v, Y(a,z)u) = H(Y(A(z)a, z^{-1})v, u), \quad u, v \in V. \tag{2.6}$$

Here the linear map $A(z) : V \to V((z))$ is defined by

$$A(z) = e^{zL_1} z^{-2L_0} g, \tag{2.7}$$

where

$$g(a) = e^{-\pi\sqrt{-1}(\frac{1}{2}p(a)+\Delta_a)} \phi(a), \quad a \in V, \tag{2.8}$$

Δ_a stands for the L_0-eigenvalue of a, and

$$p(a) = \begin{cases} 0 \in \mathbb{Z} & \text{if } a \in \mathfrak{g}_{\bar{0}}, \\ 1 \in \mathbb{Z} & \text{if } a \in \mathfrak{g}_{\bar{1}}. \end{cases}$$

3. The Almost Compact Conjugate Linear Involution of $\mathfrak{g}$

From now on we let $\mathfrak{g}$ be a basic simple finite-dimensional Lie superalgebra such that

$$\mathfrak{g}_{\bar{0}} = \mathfrak{s} \oplus \mathfrak{g}^{\natural}. \tag{3.1}$$

where $\mathfrak{s} \cong sl_2$ and $\mathfrak{g}^{\natural}$ is the centralizer of $\mathfrak{s}$ in $\mathfrak{g}$.

This corresponds to consider $\mathfrak{g}$ as in Table 2 of [20]. We will also assume that $\mathfrak{g}^{\natural}$ is not abelian; this condition rules out $\mathfrak{g} = spo(2|m),\ m = 0, 1, 2$. The explicit list is given in the leftmost column of Table 1. Note that $sl(2|1)$ and $osp(4|2)$ are missing there since $sl(2|1) \cong spo(2|2)$ and $osp(4|2) \cong D(2, 1; a)$ with $a = 1, -2$ or $-\frac{1}{2}$.

First, we prove the simple lemma mentioned in the Introduction, which states that the first two conditions of (1.1) imply the third one.

Lemma 3.1. *Let $\mathfrak{g}$ be a simple Lie superalgebra with an invariant supersymmetric bilinear form $(.|.)$, let $x \in \mathfrak{g}$, and let ϕ be a conjugate linear involution of $\mathfrak{g}$, such that*

$$(x|x) \textit{ is a non-zero real number, and } \phi(x) = x. \tag{3.2}$$

Then

$$\overline{(\phi(a)|\phi(b))} = (a|b),\ \textit{for all } a, b \in \mathfrak{g}. \tag{3.3}$$

Proof. Note that $\overline{(\phi(a)|\phi(b))}$ is an invariant supersymmetric bilinear form as well, hence it is proportional to $(a|b)$ since $\mathfrak{g}$ is simple. Due to (3.2) these two bilinear forms coincide. $\square$

We now discuss the existence of an almost compact involution of $\mathfrak{g}$ (see Definition 1.1).

Proposition 3.2. *For any sl_2-triple $\mathfrak{s} = \{e, x, f\}$, such that $[e, f] = x, [x, e] = e, [x, f] = -f$, and (3.1) holds, an almost compact involution exists.*

Proof. Choose a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}_{\bar{0}}$. We observe that if we prove the existence of an almost compact involution ϕ for a special choice of $\{e, x, f\}$, then an almost compact involution exists for any choice of the sl_2-triple. Indeed, if $\{e', x', f'\}$ is another sl_2-triple, then there is an inner automorphism ψ of $\mathfrak{s}$ mapping $\{e, x, f\}$ to $\{e', x', f'\}$, which extends to an inner automorphism of $\mathfrak{g}$. Therefore $\phi' = \psi\phi\psi^{-1}$ is an almost compact involution for $\{e', x', f'\}$. The construction of $\{e, x, f\}$ and ϕ and the verification of properties (i)–(iii) in Definition 1.1 will be done in four steps:

(1) make a suitable choice of positive roots for $\mathfrak{g}$ with respect to $\mathfrak{t}$;
(2) define ϕ by specializing (2.3);
(3) construct $\{e, f, x\}$ and verify that $\phi(f) = f, \phi(x) = x, \phi(e) = e$;
(4) check that ϕ is a compact involution for $\mathfrak{g}^{\natural}$;

Step 1. We need some preparation. Let $\Delta^{\natural}$ be the set of roots of $\mathfrak{g}^{\natural}$ with respect to the Cartan subalgebra $\mathfrak{t} \cap \mathfrak{g}^{\natural}$. Let $\{\pm\theta\}$ be the $\mathfrak{t} \cap \mathfrak{s}$-roots of $\mathfrak{s}$. Then $R_{\bar{0}} = \{\pm\theta\} \cup \Delta^{\natural}$ is the set of roots of $\mathfrak{g}_{\bar{0}}$ with respect to $\mathfrak{t}$.

Let R be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{t}$, let R^+ be the subset of positive roots whose corresponding set of simple roots $S = \{\alpha_1, \ldots, \alpha_r\}$ is displayed in Table 1.

Note that θ is the highest root of R.

Table 1. Simple roots, invariant form, and highest root of $\mathfrak{g}$

$\mathfrak{g}$	S	$(.\|.)$	θ
$psl(2\|2)$	$\{\epsilon_1-\delta_1, \delta_1-\delta_2, \delta_2-\epsilon_2\}$	$(\epsilon_i\|\epsilon_j)=\delta_{i,j}=-(\delta_i\|\delta_j)$ $(\epsilon_i\|\delta_j)=0$	$\epsilon_1-\epsilon_2$
$sl(2\|m), m>2$	$\{\epsilon_1-\delta_1, \delta_1-\delta_2, \dots, \delta_m-\epsilon_2\}$	$(\epsilon_i\|\epsilon_j)=\delta_{i,j}=-(\delta_i\|\delta_j)$ $(\epsilon_i\|\delta_j)=0$	$\epsilon_1-\epsilon_2$
$osp(4\|m), m>2$	$\{\epsilon_1-\epsilon_2, \epsilon_2-\delta_1, \delta_1-\delta_2, \dots, \delta_{m-1}-\delta_m, 2\delta_m\}$	$(\epsilon_i\|\epsilon_j)=\delta_{i,j}=-(\delta_i\|\delta_j)$ $(\epsilon_i\|\delta_j)=0$	$\epsilon_1+\epsilon_2$
$spo(2\|2m+1), m\geq 1$	$\{\delta_1-\epsilon_1, \epsilon_1-\epsilon_2, \dots, \epsilon_{m-1}-\epsilon_m, \epsilon_m\}$	$(\epsilon_i\|\epsilon_j)=-\frac{1}{2}\delta_{i,j}, (\delta_1\|\delta_1)=\frac{1}{2}, (\epsilon_i\|\delta_1)=0$	$2\delta_1$
$spo(2\|2m), m\geq 3$	$\{\delta_1-\epsilon_1, \epsilon_1-\epsilon_2, \dots, \epsilon_{m-1}-\epsilon_m, \epsilon_{m-1}+\epsilon_m\}$	$(\epsilon_i\|\epsilon_j)=-\frac{1}{2}\delta_{i,j}, (\delta_1\|\delta_1)=\frac{1}{2}, (\epsilon_i\|\delta_1)=0$	$2\delta_1$
$D(2,1;a)$	$\{\epsilon_1-\epsilon_2-\epsilon_3, 2\epsilon_2, 2\epsilon_3\}$	$(\epsilon_1\|\epsilon_1)=\frac{1}{2}, (\epsilon_2\|\epsilon_2)=\frac{-1}{2(1+a)}, (\epsilon_3\|\epsilon_3)=\frac{-a}{2(1+a)}$ $(\epsilon_1\|\epsilon_2)=(\epsilon_1\|\epsilon_3)=(\epsilon_2\|\epsilon_3)=0$	$2\epsilon_1$
$F(4)$	$\{\frac{1}{2}(\delta_1-\epsilon_1-\epsilon_2-\epsilon_3), \epsilon_3, \epsilon_2-\epsilon_3, \epsilon_1-\epsilon_2\}$	$(\epsilon_i\|\epsilon_j)=-\frac{2}{3}\delta_{i,j}, (\delta_1\|\delta_1)=2$ $(\epsilon_i\|\delta_1)=0$	δ_1
$G(3)$	$\{\delta_1+\epsilon_3, \epsilon_1, \epsilon_2-\epsilon_1\}$	$(\epsilon_i\|\epsilon_j)=\frac{1-3\delta_{i,j}}{4}, (\delta_1\|\delta_1)=\frac{1}{2}$ $(\epsilon_i\|\delta_1)=0,\ \epsilon_1+\epsilon_2+\epsilon_3=0$	$2\delta_1$

Step 2. Define

$$\Lambda_0=\{\lambda_1,\dots,\lambda_r\},\quad \lambda_i=\begin{cases}-sgn(\alpha_i|\alpha_i) & \text{if } \alpha_i \text{ is even,}\\ -\sqrt{-1} & \text{if } \alpha_i \text{ is odd.}\end{cases} \tag{3.4}$$

Set $\phi=\omega_{\Lambda_0}$ (see (2.3)).
Step 3. Consider a good choice of root vectors X_α for Λ_0. Set

$$x=\frac{\sqrt{-1}}{2}(X_\theta-X_{-\theta}), e=\frac{1}{2}(X_\theta+X_{-\theta}+\sqrt{-1}h_\theta), f=\frac{1}{2}(X_\theta+X_{-\theta}-\sqrt{-1}h_\theta). \tag{3.5}$$

If $\theta=\sum_{i=1}^r m_i\alpha_i$, then, by our special choice of Δ^+, we have either $m_i=2$ for exactly one odd simple root α_i, or $m_i=m_j=1$ for exactly two odd distinct simple roots α_i,α_j (this corresponds to the fact that R^+ is distinguished, in the terminology of [8]). By (2.4) we have

$$\phi(X_\theta)=-(\sqrt{-}1)^2X_{-\theta}=X_{-\theta}. \tag{3.6}$$

Since $h_\theta=\sum_{i=1}^r m_ih_{\alpha_i}$ and $\phi(h_{\alpha_i})=-h_{\alpha_i}$, it is clear from (3.5) that ϕ fixes e,f,x. One checks directly that $\{e,f,x\}$ is an sl_2-triple.
Step 4. Endow $\mathfrak{g}$ with the $\mathbb{Z}$-grading

$$\mathfrak{g}=\bigoplus_{i\in\mathbb{Z}}\mathfrak{q}_i \tag{3.7}$$

which assigns degree 0 to $h\in\mathfrak{t}$ and to e_i and f_i if α_i is even, and degree 1 to e_i and degree -1 to f_i, if α_i is odd.

A direct check on Table 1 shows that $\mathfrak{q}_0=\mathfrak{g}^\natural$. Recall from [8, Proposition 4.5] that the fixed points of ϕ in $\mathfrak{q}_0$ are a compact form of $\mathfrak{q}_0$ if and only if $\lambda_i(\alpha_i|\alpha_i)<0$ for all $\alpha_i\in S\setminus S_1$. Step 4 now follows from (3.4). □

4. Explicit Expressions for Almost Compact Real Forms

In this section we exhibit explicitly an almost compact involution ϕ in each case and discuss its uniqueness. If ϕ is an almost compact involution of $\mathfrak{g}$, we denote by $\mathfrak{g}^{ac}$ the corresponding real form (the fixed point set of ϕ). We can define $\mathfrak{g}^{ac}$ by specifying a real form $\mathfrak{g}^{ac}_{\bar{0}}$ of $\mathfrak{g}_{\bar{0}}$ and a real form $\mathfrak{g}^{ac}_{\bar{1}}$ of $\mathfrak{g}_{\bar{1}}$.

(1) $\mathfrak{g} = spo(2|m)$. Then $\mathfrak{g}_{\bar{0}} = sl_2 \oplus so_m$ and $\mathfrak{g}_{\bar{1}} = \mathbb{C}^2 \otimes \mathbb{C}^m$ as $\mathfrak{g}_{\bar{0}}$-module. We set

$$\mathfrak{g}^{ac}_{\bar{0}} = sl_2(\mathbb{R}) \oplus so_m(\mathbb{R}), \quad \mathfrak{g}^{ac}_{\bar{1}} = \mathbb{R}^2 \otimes \mathbb{R}^m.$$

Explicitly, let B be a non-degenerate $\mathbb{R}$-valued bilinear form of the superspace $\mathbb{R}^{2|m}$ with matrix $\left(\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & 0 & 0 \\ \hline 0 & 0 & I_m \end{array}\right)$. Then for $\mathfrak{g} = spo(2|m)$ we have:

$$\mathfrak{g}^{ac} = \{A \in sl(m|n; \mathbb{R}) \mid B(Au, v) + (-1)^{p(A)p(u)} B(u, Av) = 0\}.$$

(2) $\mathfrak{g} = psl(2|2)$. Let H be a $\mathbb{C}$-valued non-degenerate sesquilinear form on the superspace $\mathbb{C}^{2|2}$ whose matrix is $diag(\sqrt{-1}, -\sqrt{-1}, 1, 1)$. Set

$$\tilde{\mathfrak{g}}^{ac} = \{A \in sl(2|2; \mathbb{C}) \mid H(Au, v) + (-1)^{p(A)p(u)} H(u, Av) = 0\}.$$

Then

$$\mathfrak{g}^{ac} = \tilde{\mathfrak{g}}^{ac}/\mathbb{R}\sqrt{-1}I.$$

Explicitly, we have $\mathfrak{g}_{\bar{0}} = sl_2 \oplus sl_2$ and $\mathfrak{g}_{\bar{1}} = \left\{ \left(\begin{array}{c|c} 0 & B \\ \hline C & 0 \end{array}\right) \mid B, C \in M_{2,2}(\mathbb{C}) \right\}$ as a $\mathfrak{g}_{\bar{0}}$-module. Then

$$\tilde{\mathfrak{g}}^{ac}_{\bar{0}} = \left\{ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array}\right) \mid A \in su(1,1), D \in su_2 \right\},$$

$$\tilde{\mathfrak{g}}^{ac}_{\bar{1}} = \left\{ \left(\begin{array}{cc|c} 0 & 0 & u \\ 0 & 0 & v \\ \hline \sqrt{-1}\bar{u}^t & -\sqrt{-1}\bar{v}^t & 0 \end{array}\right) \mid u, v \in \mathbb{C}^2 \right\}.$$

(3) $\mathfrak{g} = D(2,1;a)$. Then $\mathfrak{g}_{\bar{0}} = sl_2 \oplus sl_2 \oplus sl_2 = so(4,\mathbb{C}) \oplus sl_2$ and $\mathfrak{g}_{\bar{1}} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^4 \otimes \mathbb{C}^2$ as $\mathfrak{g}_{\bar{0}}$-module. We set

$$\mathfrak{g}^{ac}_{\bar{0}} = so(4,\mathbb{R}) \oplus span_{\mathbb{R}}\{e, f, x\}, \quad \mathfrak{g}^{ac}_{\bar{1}} = \mathbb{R}^4 \otimes \mathbb{R}^2.$$

To get an explicit realization, consider the contact Lie superalgebra (see [11] for more details)

$$K(1,4) = \mathbb{C}[t, \xi_1, \xi_2, \xi_3, \xi_4]$$

where t is an even variable and ξ_i, $1 \le i \le 4$, are odd variables. Introduce on the associative superalgebra $K(1,4)$ a $\mathbb{Z}$-grading by letting

$$\deg' t = 2, \quad \deg' \xi_i = 1,$$

and the bracket

$$\{F, G\} = (2 - \sum_{i=1}^{4} \xi_i \partial_i) F \partial_t G - \partial_t F (2 - \sum_{i=1}^{4} \xi_i \partial_i) G + \sum_{i=1}^{4} (-1)^{p(F)} \partial_i F \partial_i G,$$

where $\partial_i = \partial_{\xi_i}$. This is a $\mathbb{Z}$-graded Lie superalgebra with compatible grading $\deg F = \deg' F - 2$. We have

$$K(1,4) = \bigoplus_{j \geq -2} K(1,4)_j,$$

where

$K(1,4)_{-2} = \mathbb{C}1,$ $\quad K(1,4)_{-1} = span_{\mathbb{C}}(\xi_i \mid 1 \leq i \leq 4),$
$K(1,4)_0 = span_{\mathbb{C}}(\xi_i \xi_j, t \mid 1 \leq i, j \leq 4),$ $\quad K(1,4)_1 = \mathfrak{g}_1' \oplus \mathfrak{g}_1''$, where
$\mathfrak{g}_1' = span_{\mathbb{C}}(t\xi_i \mid 1 \leq i \leq 4),$ $\quad \mathfrak{g}_1'' = span_{\mathbb{C}}(\xi_i \xi_j \xi_k \mid 1 \leq i, j, k \leq 4).$

Note that $span_{\mathbb{C}}(\xi_i \xi_j \mid 1 \leq i, j \leq 4) = \Lambda^2 \mathbb{C}^4 \cong so(4, \mathbb{C})$, that $\mathfrak{g}_1'$ is isomorphic to the standard representation $\mathbb{C}^4$ of $so(4, \mathbb{C})$ and that $\mathfrak{g}_1''$ is isomorphic to $\Lambda^3 \mathbb{C}^4$, so that $K(1,4)_1 = \mathbb{C}^4 \oplus \mathbb{C}^4$ as $so(4, \mathbb{C})$-module. Also notice that $\{\mathfrak{g}_1', \mathfrak{g}_1'\} = \mathbb{C}t^2$, $\{\mathfrak{g}_1'', \mathfrak{g}_1''\} = 0$. Fix now a copy $\tilde{\mathfrak{g}}_b$ of an $so(4, \mathbb{C})$-module $\mathbb{C}^4$ in $\mathbb{C}^4 \oplus \mathbb{C}^4$, depending on a constant $b \in \mathbb{R}$, as follows. Set, for $1 \leq i \leq 4$,

$$a_i = t\xi_i + b\hat{\xi}_i, \text{ where } \hat{\xi}_i = (-1)^{i+1} \prod_{j \neq i} \xi_j,$$

and define

$$\tilde{\mathfrak{g}}_b = \sum_{i=1}^{4} \mathbb{C} a_i.$$

Let $b \in \mathbb{R}$. Note that, setting $\xi = \xi_1 \xi_2 \xi_3 \xi_4$, we have

$$\{t\xi_i + b\hat{\xi}_i, t\xi_j + b\hat{\xi}_j\} = \delta_{ij}(-t^2 + 2b\xi).$$

Hence, if we set

$$e = -t^2 + 2b\xi, \quad f = -1, \quad x = t/2,$$

then $\{e, x, f\}$ is an sl_2-triple. Set

$$\mathfrak{g}^{ac} = \mathbb{R}.1 \oplus \left(\sum_{i=1}^{4} \mathbb{R}\xi_i \right) \oplus \left(\sum_{i,j=1}^{4} \mathbb{R}\xi_i \xi_j \oplus \mathbb{R}\tfrac{t}{2} \right) \oplus \left(\sum_{i=1}^{4} \mathbb{R}a_i \right) \oplus \mathbb{R}(-t^2 + 2b\xi).$$

Then $\mathfrak{g}^{ac}$ is an almost compact form of $D(2, 1; \frac{1+b}{1-b})$. To prove this, it suffices to calculate the Cartan matrix for a choice of Chevalley generators of the complexification of $\mathfrak{g}^{ac}$. Fix a Cartan subalgebra in $\mathfrak{g}^\natural = so(4, \mathbb{C})$ as the span of $v_2 = -\sqrt{-1}\xi_1 \xi_2$, $v_3 = -\sqrt{-1}\xi_3 \xi_4$. Set $v_1 = t$; then $\{v_1, v_2, v_3\}$ is a basis of a Cartan subalgebra of $\mathfrak{g}$. Let $\{\epsilon_1, \epsilon_2, \epsilon_3\}$ be the dual basis to $\{v_1, v_2, v_3\}$. One can choose $\{\alpha_1 = \epsilon_2 - \epsilon_1, \alpha_2 = \epsilon_1 - \epsilon_3, \alpha_3 = \epsilon_1 + \epsilon_3\}$ as a set of simple roots. The associated Chevalley generators are

$$
\begin{array}{lll}
e_1=-\sqrt{-1}a_1+a_2 & e_2=\xi_1\xi_3+\xi_2\xi_4+\sqrt{-1}(\xi_1\xi_4-\xi_2\xi_3) & e_3=\xi_1\xi_3-\xi_2\xi_4-\sqrt{-1}(\xi_1\xi_4+\xi_2\xi_3)\\
f_1=\sqrt{-1}\xi_1+\xi_2 & f_2=\xi_1\xi_3+\xi_2\xi_4-\sqrt{-1}(\xi_1\xi_4-\xi_2\xi_3) & f_3=\xi_1\xi_3-\xi_2\xi_4+\sqrt{-1}(\xi_1\xi_4+\xi_2\xi_3)\\
h_1=-2v_1+2v_2+2b\,v_3 & h_2=4v_1-4v_3 & h_3=4v_1+4v_3
\end{array}
$$

and the corresponding Cartan matrix, normalized as in [11], is $\begin{pmatrix} 0 & 1 & \frac{1+b}{1-b} \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$. Hence $a=\frac{1+b}{1-b}$ and therefore all $a\neq -1$ occur in this construction. Since this subalgebra is 17-dimensional, it is isomorphic to $D(2,1;a)$.

Remark 4.1. Note that $a=0$ for $b=-1$. In this case, $D(2,1;0)$ contains a 11-dimensional solvable ideal generated by f_1, which is spanned by h_1 and the root vectors relative to roots having α_1 in their support. If we replace a_i by a_i/b and h_1 by h_1/b, and let b tend to $+\infty$, we recover also the Lie superalgebra of derivations of $psl(2|2)$, and its almost compact real form.

(4) $\mathfrak{g}=G(3)$. Then $\mathfrak{g}_{\bar 0}=sl_2\oplus G_2$ and $\mathfrak{g}_{\bar 1}=\mathbb{C}^2\otimes L_{\min}$, where $L_{\min}$ is the complex 7-dimensional irreducible representation of G_2, and we let

$$\mathfrak{g}^{ac}_{\bar 0}=sl_2(\mathbb{R})\oplus G_{2,0},\quad \mathfrak{g}^{ac}_{\bar 1}=\mathbb{R}^2\otimes L_{\min,0}.$$

where $G_{2,0}$ is the real compact form of G_2 and $L_{\min,0}$ is the real 7-dimensional irreducible representation of $G_{2,0}$ whose complexification is $L_{\min}$.

(5) $\mathfrak{g}=F(4)$. Then $\mathfrak{g}_{\bar 0}=sl_2\oplus so_7$ and $\mathfrak{g}_{\bar 1}=\mathbb{C}^2\otimes spin_7$, where $spin_7$ is the complex spinor representation of so_7, and we let

$$\mathfrak{g}^{ac}_{\bar 0}=sl_2(\mathbb{R})\oplus so_7(\mathbb{R}),\quad \mathfrak{g}^{ac}_{\bar 1}=\mathbb{R}^2\otimes spin(\mathbb{R}^7),$$

where $spin(\mathbb{R}^7)$ is the spinor representation of the compact group $so_7(\mathbb{R})$.

It is proved in [11, Proposition 5.3.2] that in both cases (4) and (5) $\mathfrak{g}^{ac}=\mathfrak{g}^{ac}_{\bar 0}\oplus\mathfrak{g}^{ac}_{\bar 1}$ is an almost compact form of $\mathfrak{g}$.

4.1. Uniqueness of the almost compact involution.

Proposition 4.2. *An almost compact involution is uniquely determined up to a sign by its action on $\mathfrak{g}_0$, provided that the $\mathfrak{g}_0$-module $\mathfrak{g}_{1/2}$ is irreducible.*

Proof. If there are two different extensions of the compact involution, then their ratio ψ, say, is identical on $\mathfrak{g}_0$, hence, by Schur's lemma, ψ acts as a scalar on $\mathfrak{g}_{-1/2}$. Since $\phi(f)=f$, we conclude that this scalar is ± 1. □

It remains to discuss the cases $\mathfrak{g}=sl(2|m)$, $m\geq 3$, and $psl(2|2)$, since in all other cases of Table 1 the $\mathfrak{g}_0$-module $\mathfrak{g}_{1/2}$ is irreducible. In this cases $\mathfrak{g}$ is of type I, that is $\mathfrak{g}_{\bar 1}=\mathfrak{g}^+_{\bar 1}\oplus\mathfrak{g}^-_{\bar 1}$ where $\mathfrak{g}^{\pm}_{\bar 1}$ are contragredient irreducible $\mathfrak{g}_{\bar 0}$-modules and $[\mathfrak{g}^{\pm}_{\bar 1},\mathfrak{g}^{\pm}_{\bar 1}]=0$. Let δ_λ be the linear map on $\mathfrak{g}$ defined by setting

$$\delta_{\lambda|\mathfrak{g}_{\bar 0}}=Id,\quad \delta_{\lambda|\mathfrak{g}^+_{\bar 1}}=\lambda Id,\quad \delta_{\lambda|\mathfrak{g}^-_{\bar 1}}=\lambda^{-1}Id. \tag{4.1}$$

Then δ_λ is an automorphism of $\mathfrak{g}$ for any $\lambda\in\mathbb{C}$. Suppose that ϕ' is another conjugate almost compact linear involution such that $\phi'_{|\mathfrak{g}_{\bar 0}}=\phi$. Then $\phi'=\phi\circ\gamma$ with γ an automorphism of ϕ such that $\gamma_{|\mathfrak{g}_{\bar 0}} = Id$. If $\mathfrak{g} = sl(2|m)$, by [22, Lemmas 1 and 2], we have $\gamma = \delta_\lambda$. Since $\phi(\mathfrak{g}_{\bar 1}^+) = \mathfrak{g}_{\bar 1}^-$ and $(\phi')^2 = Id$ we have that $\lambda \in \mathbb{R}$. If $\mathfrak{g} = psl(2|2)$, then γ belongs to a three-parameter family of automorphisms explicitly described in [8, §4.6], and contained in $SL(2,\mathbb{C})$. This $SL(2,\mathbb{C})$ is the group of automorphisms of $\mathfrak{g}$ corresponding to the Lie algebra sl_2 of outer derivations of $\mathfrak{g}$.

Remark 4.3. Note that if ϕ is an almost compact involution, then

$$\tilde\phi(a) = (-1)^{2j}\phi(a),\ a \in \mathfrak{g}_j$$

is again an almost compact involution.

5. The Bilinear Form $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}_{-1/2}$

Let $\mathfrak{s} = \{e, x, f\}$ be an sl_2-triple as in Proposition 3.2. Consider the following symmetric bilinear forms on $\mathfrak{g}_{-1/2}$ and $\mathfrak{g}_{1/2}$ respectively:

$$\langle u, v\rangle = (e|[u, v]),\ u, v \in \mathfrak{g}_{-1/2}, \tag{5.1}$$

$$\langle u, v\rangle_{ne} = (f|[u, v]),\ u, v \in \mathfrak{g}_{1/2}. \tag{5.2}$$

Note that, since $[f, \mathfrak{g}_{-1/2}] = 0$, we have

$$\langle [e, u], [e, v]\rangle_{ne} = -\tfrac{1}{2}\langle u, v\rangle,\ u, v \in \mathfrak{g}_{-1/2}. \tag{5.3}$$

We want to prove the following

Proposition 5.1. *We can choose an almost compact involution such that the bilinear form $\langle .,.\rangle$ is positive definite on $\mathfrak{g}^{ac} \cap \mathfrak{g}_{-1/2}$. In particular, the Hermitian form $\langle\phi(u), v\rangle$ (resp. $\langle\phi(u), v\rangle_{ne}$) is positive definite (resp, negative definite) on $\mathfrak{g}^{ac} \cap \mathfrak{g}_{-1/2}$ (resp. $\mathfrak{g}^{ac} \cap \mathfrak{g}_{1/2}$).*

The proof requires a detailed analysis of the action of an almost compact involution on $\mathfrak{g}_{-1/2}$. Define structure constants $N_{\alpha,\beta}$ for a good choice of root vectors (see Sect. 2.2) by the relation

$$[X_\alpha, X_\beta] = N_{\alpha,\beta}X_{\alpha+\beta}.$$

Observe that $\{X_{-\theta}, X_\theta, \frac{1}{2}h_\theta\}$ is a sl_2-triple in $\mathfrak{s}$. Let

$$\mathfrak{g} = \mathbb{C}X_\theta \oplus \tilde{\mathfrak{g}}_{1/2} \oplus \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_{-1/2} \oplus \mathbb{C}X_{-\theta}$$

be the decomposition into $ad\ \frac{1}{2}h_\theta$ eigenspaces. By the sl_2 representation theory, $ad\ X_{\pm\theta} : \tilde{\mathfrak{g}}_{\mp 1/2} \to \tilde{\mathfrak{g}}_{\pm 1/2}$ is an isomorphism of $\mathfrak{g}^\natural$-modules. Moreover, by our choice of R^+ in Sect. 3, the roots of $\tilde{\mathfrak{g}}_{-1/2}$ (resp. $\tilde{\mathfrak{g}}_{1/2}$) are precisely the negative (resp. positive) odd roots. In particular, the map $\alpha \mapsto -\theta + \alpha$ defines a bijection between the positive and negative odd roots. We shall need the following properties.

Lemma 5.2. *For a positive odd root α we have*

$$N_{-\theta,\alpha}N_{\theta,\alpha-\theta} = 1, \tag{5.4}$$

$$N_{-\theta,\alpha}^2 = 1. \tag{5.5}$$

In particular $N_{\theta,\alpha}$ is real.

Proof. Relation (5.4) is proven in [8, Lemma 4.3]. Equation (5.5) follows from [8, (4.8)], noting that the $-\theta$-string through α has length 1. $\square$

Arguing as in Proposition 3.2, we can assume in the proof of Proposition 5.1 that $\{e, x, f\}$ is the sl_2-triple defined in (3.5); $ad\,x$ defines on $\mathfrak{g}$ a minimal grading

$$\mathfrak{g} = \mathbb{C}f \oplus \mathfrak{g}_{-1/2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{1/2} \oplus \mathbb{C}e. \tag{5.6}$$

Set, for an odd root $\alpha \in R^+$

$$u_\alpha = X_\alpha + \sqrt{-1}N_{-\theta,\alpha}X_{\alpha-\theta}. \tag{5.7}$$

Note that

$$\begin{aligned}[x, u_\alpha] =& \tfrac{\sqrt{-1}}{2}[X_\theta - X_{-\theta}, X_\alpha + \sqrt{-1}N_{-\theta,\alpha}X_{\alpha-\theta}]\\ =&\tfrac{1}{2}N_{-\theta,\alpha}N_{\theta,\alpha-\theta}X_\alpha - \tfrac{\sqrt{-1}}{2}N_{-\theta,\alpha}X_{\alpha-\theta} = -\tfrac{1}{2}u_\alpha,\end{aligned}$$

hence $\{u_\alpha \mid \alpha \in R^+, \alpha \text{ odd}\}$ is a basis of $\mathfrak{g}_{-1/2}$.

Lemma 5.3. *If α is a positive odd root then*

$$\phi(u_\alpha) = -N_{-\theta,\alpha}u_{\theta-\alpha}. \tag{5.8}$$

Proof. By (2.4), $\phi(X_\alpha) = -\sqrt{-1}X_{-\alpha}$ if α is an odd positive root, hence, by (5.5), since $N_{-\theta,\alpha}$ is real,

$$\begin{aligned}\phi(u_\alpha) &= \phi(X_\alpha + \sqrt{-1}N_{-\theta,\alpha}X_{\alpha-\theta}) = -(\sqrt{-1}X_{-\alpha} + N_{-\theta,\alpha}X_{\theta-\alpha})\\ &= -N_{-\theta,\alpha}(X_{\theta-\alpha} + \sqrt{-1}N^{-1}_{-\theta,\alpha}X_{-\alpha}).\end{aligned} \tag{5.9}$$

Note that, since $\phi(x) = x$, $\phi(u_\alpha)$ has to belong to $\mathfrak{g}_{-1/2}$. This forces

$$N_{-\theta,\alpha}N_{-\theta,\theta-\alpha} = 1, \tag{5.10}$$

and (5.9) becomes (5.8). $\square$

Proof of Proposition 5.1. Set $v_\alpha = \frac{1}{2}(u_\alpha + \phi(u_\alpha)) + \frac{\sqrt{-1}}{2}(u_\alpha - \phi(u_\alpha))$, where α runs over the positive odd roots. It is clear that $v_\alpha \in \mathfrak{r}$. We want to prove that the vectors v_α form an orthogonal basis of $\mathfrak{r}$. We need two auxiliary computations:

$$[e, u_\alpha] = \sqrt{-1}X_\alpha + N_{-\theta,\alpha}X_{\alpha-\theta}, \tag{5.11}$$

$$\langle u_\alpha, u_\beta\rangle = -(N_{-\theta,\alpha} + N_{-\theta,\beta})\delta_{\theta-\alpha,\beta}. \tag{5.12}$$

To prove (5.11) use (5.4):

$$\begin{aligned}[e, u_\alpha] &= \tfrac{1}{2}[X_\theta + X_{-\theta} + \sqrt{-1}h_\theta, X_\alpha + \sqrt{-1}N_{-\theta,\alpha}X_{\alpha-\theta}]\\ &= \tfrac{1}{2}(\sqrt{-1}N_{-\theta,\alpha}N_{\theta,\alpha-\theta}X_\alpha + N_{-\theta,\alpha}X_{\alpha-\theta} + \sqrt{-1}X_\alpha + N_{-\theta,\alpha}X_{\alpha-\theta})\\ &= \sqrt{-1}X_\alpha + N_{-\theta,\alpha}X_{\alpha-\theta}.\end{aligned}$$

To prove (5.12) use (5.11):

$$\begin{aligned}\langle u_\alpha, u_\beta\rangle &= (e|[u_\alpha, u_\beta]) = ([e, u_\alpha]|u_\beta)\\ &= (\sqrt{-1}X_\alpha + N_{-\theta,\alpha}X_{\alpha-\theta}|X_\beta + \sqrt{-1}N_{-\theta,\beta}X_{\beta-\theta}) =\\ &= \sigma_{\alpha-\theta}N_{-\theta,\alpha}\delta_{\theta-\alpha,\beta} - \sigma_\alpha N_{-\theta,\beta}\delta_{\theta-\alpha,\beta}\\ &= -(N_{-\theta,\alpha} + N_{-\theta,\beta})\delta_{\theta-\alpha,\beta}.\end{aligned}$$

Set

$$M_{\alpha,\beta} = -(N_{-\theta,\alpha} + N_{-\theta,\beta}).$$

Then, using (5.12)

$$\begin{aligned}&\langle v_\alpha, v_\beta\rangle\\ &= \langle \tfrac{1+\sqrt{-1}}{2}u_\alpha - \tfrac{1-\sqrt{-1}}{2}N_{-\theta,\alpha}u_{\theta-\alpha}, \tfrac{1+\sqrt{-1}}{2}u_\beta - \tfrac{1-\sqrt{-1}}{2}N_{-\theta,\beta}u_{\theta-\beta}\rangle\\ &= \tfrac{\sqrt{-1}}{2}\langle u_\alpha, u_\beta\rangle - \tfrac{1}{2}N_{-\theta,\alpha}\langle u_{\theta-\alpha}, u_\beta\rangle\\ &\quad - \tfrac{1}{2}N_{-\theta,\beta}\langle u_\alpha, u_{\theta-\beta}\rangle - \tfrac{\sqrt{-1}}{2}N_{-\theta,\alpha}N_{-\theta,\beta}\langle u_{\theta-\alpha}, u_{\theta-\beta}\rangle\\ &= \tfrac{\sqrt{-1}}{2}M_{\alpha,\beta}\delta_{\theta-\alpha,\beta} - \tfrac{1}{2}N_{-\theta,\alpha}M_{\theta-\alpha,\beta}\delta_{\alpha,\beta} - \tfrac{1}{2}N_{-\theta,\beta}M_{\alpha,\theta-\beta}\delta_{\theta-\alpha,\theta-\beta}\\ &\quad - \tfrac{\sqrt{-1}}{2}N_{-\theta,\alpha}N_{-\theta,\beta}M_{\theta-\alpha,\theta-\beta}\delta_{\alpha,\theta-\beta}.\end{aligned}$$

Therefore by (5.4) and (5.10)

$$\langle v_\alpha, v_\beta\rangle = 2\delta_{\alpha,\beta}.$$

In particular, the restriction of $\langle\cdot,\cdot\rangle$ to $\mathfrak{g}^{ac}\cap\mathfrak{g}_{-1/2}$ is positive definite. The final claim follows immediately from (5.3), using that $[e, \mathfrak{g}_{-1/2}] = \mathfrak{g}_{1/2}$. □

6. A General Theory of Invariant Hermitian Forms on Modules Over the Vertex Algebra of Free Boson and the Fairlie Construction

Consider the infinite dimensional Heisenberg Lie algebra $\mathcal{H} = (\mathbb{C}[\tau, \tau^{-1}]\otimes\mathbb{C}a)\oplus\mathbb{C}K$ with K central and bracket

$$[\tau^n\otimes a, \tau^m\otimes a] = \delta_{n,-m}nK.$$

Let $\mathcal{H}_0 = \mathbb{C}a + \mathbb{C}K$, and, given $\mu\in\mathbb{C}$, define $\mu^*\in\mathcal{H}_0^*$ by $\mu^*(a) = \mu$, $\mu^*(K) = 1$. Let $M(\mu)$ be the Verma module for the Lie algebra $\mathcal{H}$ with highest weight μ^*. Let v_μ be a highest weight vector, i.e. $(\tau^n\otimes a)v_\mu = 0$ for $n > 0$, $hv_\mu = \mu^*(h)v_\mu$ for $h\in\mathcal{H}_0$. It is well known that $M(0)$ carries a simple vertex algebra structure, called the vertex algebra of free boson, which we denote by $V^1(\mathbb{C}a)$, and that $M(\mu)$ is a simple module over the vertex algebra $V^1(\mathbb{C}a)$. Moreover, $V^1(\mathbb{C}a)$ is the universal enveloping vertex algebra of the nonlinear Lie conformal algebra $R = \mathbb{C}[T]\otimes\mathbb{C}a$ with λ-bracket

$$[a_\lambda a] = \lambda.$$

We introduce conformal weight Δ on $V^1(\mathbb{C}a)$ by letting $\Delta_a = 1$, and for $v\in V^1(\mathbb{C}a)$ we write the corresponding quantum field as $Y(v, z) = \sum_{j\in\mathbb{Z}} v_j z^{-j-\Delta_v}$.

Fix $t \in \mathbb{C}$ and set

$$L(t) = \frac{1}{2} :aa: + tTa \in V^1(\mathbb{C}a). \tag{6.1}$$

It is an energy-momentum element for all t. Set $H(t) = L(t)_0 = \frac{1}{2} :aa:_0 - ta_0$. Since $a_0 = 0$ as operator on $V^1(\mathbb{C}a)$, $H(t) = \frac{1}{2} :aa:_0$. (Note that the conformal weights on $V^1(\mathbb{C}a)$ are the eigenvalues of $H(t) = H(0)$).

If $b \in V^1(\mathbb{C}a)$, write b_n^μ for $b_n^{M(\mu)}$. By the -1-st product identity

$$:aa:_0^\mu = 2\sum_{j\in\mathbb{N}} a_{-j}^\mu a_j^\mu + (a_0^\mu)^2.$$

In particular

$$:aa:_0^\mu v_\mu = \mu^2 v_\mu. \tag{6.2}$$

On the other hand, by the commutator formula,

$$\frac{1}{2}[:aa:_0^\mu, a_j^\mu] = \frac{1}{2}\sum_r \binom{1}{r}(:aa:_{(r)}a)_j = (Ta)_j^\mu + a_j^\mu = -ja_j^\mu. \tag{6.3}$$

Recall that a basis of $M(\mu)$ is

$$\left\{(a_{-j_1}^\mu)^{i_1}\cdots(a_{-j_r}^\mu)^{i_r}.v_\mu \mid j_1 > \cdots > j_r > 0\right\}. \tag{6.4}$$

Let $M(\mu)_n$ be the eigenspace for the energy operator $H(t)$ corresponding to the eigenvalue $n + \frac{1}{2}\mu^2 - t\mu$. Since

$$\frac{1}{2} :aa:_0^\mu + t(Ta)_0^\mu = \frac{1}{2} :aa:_0^\mu - ta_0^\mu$$

and $[a_0^\mu, a_{-j}^\mu] = 0$ for all j, it follows from (6.2) and (6.3) that

$$M(\mu)_n = span\{(a_{-j_1}^\mu)^{i_1}\cdots(a_{-j_r}^\mu)^{i_r}.v_\mu \mid \sum_s i_s j_s = n\}.$$

Thus

$$M(\mu) = \oplus_{n\in\mathbb{Z}_+} M(\mu)_n.$$

This shows that $M(\mu)$ is a positive energy $V^1(\mathbb{C}a)$-module, i.e. real parts of the eigenvalues of $H(t)$ are bounded below. Moreover its minimal energy subspace is

$$M(\mu)_0 = \mathbb{C}v_\mu.$$

Lemma 6.1.

$$e^{zL(t)_1} = e^{zL(0)_1}\prod_{n=1}^\infty e^{-\frac{2t}{n}z^n a_n}. \tag{6.5}$$

Proof. Identify $V^1(\mathbb{C}a)$ with the polynomial algebra in infinitely many variables using (6.4) with $\mu = 0$:

$$\mathcal{P} = \mathbb{C}[a_{-1}, a_{-2}, \dots, a_{-n}, \dots].$$

Since $L(t)_1\mathbf{1} = 0$ and $a_n\mathbf{1} = 0$ if $n > 0$, both $L(t)_1$ and a_n for $n > 0$ act as derivations of the algebra $\mathcal{P}$ under our identification. It follows that both sides of (6.5) are automorphisms of $\mathcal{P}$. It is therefore enough to check the equality only on the generators a_{-n}.

We need the following formulae:

$$[a_\lambda L(t)] = \lambda a + t\lambda^2\mathbf{1}, \tag{6.6}$$
$$[a_n, L(t)_1] = na_{n+1} + \delta_{n,-1}2tI, \tag{6.7}$$
$$[a_n, a_{-m}] = \delta_{n,m}nI. \tag{6.8}$$

Applying these formulae we find

$$e^{-\frac{2t}{n}z^n a_n}(a_{-m}) = e^{-2tz^n\frac{\partial}{\partial a_{-n}}}(a_{-m}) = a_{-m} - \delta_{n,m}2tz^m I. \tag{6.9}$$

It follows that

$$e^{zL(0)_1}\prod_{n=1}^{\infty} e^{-\frac{2t}{n}z^n a_n}(a_{-m}) = e^{zL(0)_1}(a_{-m} - 2tz^m\mathbf{1}) = e^{zL(0)_1}a_{-m} - 2tz^m I. \tag{6.10}$$

To conclude we only need to check that, if $n \geq 1$, then

$$L(t)_1^n(a_{-m}) = L(0)_1^n a_{-m} - 2n!\delta_{n,m}tI.$$

We prove this by induction on n. If $n = 1$ the formula reads

$$L(t)_1(a_{-m}) = L(0)_1 a_{-m} - 2\delta_{1,m}tI.$$

Using (6.7) with $t = 0$ we see that the latter formula is equivalent to

$$L(t)_1(a_{-m}) = ma_{-m+1} - 2\delta_{1,m}tI,$$

which is just (6.7).

If $n > 1$ and $m = 1$, then

$$L(t)_1^n(a_{-1}) = L(t)_1^{n-1}L(t)_1(a_{-1}) = L(t)_1^{n-1}(-2t) = 0 = L(0)_1^n(a_{-1}).$$

If $n > 1$ and $m > 1$, then

$$\begin{aligned} L(t)_1^n(a_{-m}) =& L(t)_1^{n-1}L(t)_1(a_{-m}) = L(t)_1^{n-1}(ma_{-m+1}) \\ =& L(0)_1^{n-1}(ma_{-m+1}) - 2(n-1)!m\delta_{n-1,m-1}tI) \\ =& L(0)_1^n a_{-m} - 2n!\delta_{n,m}tI. \end{aligned}$$

$\square$

Let ϕ be the conjugate linear involution of the vector space $\mathbb{C}a$ defined by $\phi(a) = -a$. Assume from now on that $t \in \sqrt{-1}\mathbb{R}$. This assumption is necessary since, in order to apply the results of [16], we need to assume $\phi(L(t)) = L(t)$. Set (cf. (2.7))

$$A(z,t) = e^{zL(t)_1} z^{-2H(0)} g, \tag{6.11}$$

where g is defined in (2.8). Let $\pi_Z : V^1(\mathbb{C}a) \to Zhu_{H(0)}(V^1(\mathbb{C}a))$ be the canonical projection to the Zhu algebra (see e.g. [16, Section 2]). Let ω be the conjugate linear anti-homomorphism of $Zhu_{H(0)}(V^{k+h^\vee}(\mathbb{C}x))$ defined by

$$\omega(\pi_Z(v)) = \pi_Z(A(1,t)v)$$

It is proven in [16, Proposition 6.1] that ω is indeed well-defined.

Lemma 6.2.

$$\omega(\pi_Z(a))v_\mu = (\mu - 2t)v_\mu \tag{6.12}$$

Proof. By Lemma 6.1, since $g(a) = a$ and $L(0)_1 a = 0$,

$$\begin{aligned}\omega(\pi_Z(a))v_\mu =&(A(1,t)a)_0^\mu v_\mu = (e^{L(t)_1}a)_0^\mu v_\mu = (e^{L(0)_1}(a) - 2t\mathbf{1})_0^\mu v_\mu = a_0^\mu v_\mu - 2tv_\mu \\ =&(\mu - 2t)v_\mu.\end{aligned}$$

□

Recall from [16, Definition 6.4] that if V is a conformal vertex algebra and ϕ is a conjugate linear involution of V, a Hermitian form $H(\ .\ ,\ .\ )$ on a V-module M is called ϕ-invariant if, for all $v \in V$, $m_1, m_2 \in M$

$$(m_1, Y_M(a,z)m_2) = (Y_M(A(z)a, z^{-1})m_1, m_2).$$

By abuse of terminology, we shall call $H(\cdot,\cdot)$ an L-invariant Hermitian form, where L is the conformal vector of V. If $\mu \in \mathbb{C}$ we denote by $\Re(\mu)$ and $\Im(\mu)$ the real and imaginary part of μ, respectively.

Proposition 6.3. *There is a non-zero $L(t)$-invariant Hermitian form on $M(\mu)$ if and only if $t = \sqrt{-1}\Im(\mu)$.*

Proof. Let $(\cdot,\cdot)$ be the unique Hermitian form on $\mathbb{C}v_\mu$ such that $(v_\mu, v_\mu) = 1$. By Proposition 6.7 of [16], there is a non-zero $L(t)$-invariant Hermitian form on $M(\mu)$ if and only if $(\cdot,\cdot)$ is an ω-invariant Hermitian form on $\mathbb{C}v_\mu$. By Lemma 6.2, that is equivalent to

$$\mu = (v_\mu, a_0 v_\mu) = (v_\mu, \pi_Z(a)v_\mu) = (\omega(\pi_Z(a))v_\mu, v_\mu) = \overline{\mu - 2t}.$$

Thus

$$-2\bar{t} = 2\sqrt{-1}\Im(\mu),$$

hence the statement. □

We denote by H_μ the unique $L(\sqrt{-1}\Im(\mu))$-invariant Hermitian form on $M(\mu)$ such that $H_\mu(v_\mu, v_\mu) = 1$.

Lemma 6.4. *If* $m, m' \in M(\mu)$, *then*

$$H_\mu(m, a_n^\mu m') = H_\mu(a_{-n}^\mu m, m') + \delta_{n,0} 2\sqrt{-1}\Im(\mu) H_\mu(m, m').$$

Proof. By invariance of the Hermitian form,

$$\begin{aligned} H_\mu(m, a_n m') =& Res_z z^n H_\mu(m, Y^\mu(a, z)m') \\ =& Res_z z^n H_\mu(Y^\mu(A(z)a, z^{-1})m, m') \\ =& Res_z z^n H_\mu(Y^\mu(e^{zL(t)_1} z^{-2L(t)_0} g(a), z^{-1})m, m') \\ =& Res_z z^{n-2} H_\mu(Y^\mu(e^{zL(0)_1} a - 2\sqrt{-1}\Im(\mu) z\mathbf{1}, z^{-1})m, m') \\ =& Res_z z^{n-2} H_\mu(Y^\mu(a - 2\sqrt{-1}\Im(\mu) z\mathbf{1}, z^{-1})m, m'). \end{aligned}$$

The last two steps follow by (6.10) and the fact that $L(0)_1 a = 0$. As

$$Y^\mu(a, z^{-1}) = \sum_r a_r^\mu z^{r+1}, \ Y^\mu(\mathbf{1}, z^{-1}) = \sum_r \delta_{r,0} I z^r,$$

we get the result. □

It is now easy to compute the invariant form in the basis (6.4):

$$\begin{aligned} & H_\mu\left((a_{-j_1}^\mu)^{i_1} \cdots (a_{-j_r}^\mu)^{i_r}.v_\mu, (a_{-j_1'}^\mu)^{i_1'} \cdots (a_{-j_{r'}'}^\mu)^{i_{r'}'}.v_\mu\right) \\ & = H_\mu\left((a_{j_{r'}'}^\mu)^{i_{r'}'} \cdots (a_{j_1'}^\mu)^{i_1'} (a_{-j_1}^\mu)^{i_1} \cdots (a_{-j_r}^\mu)^{i_r}.v_\mu, v_\mu\right). \end{aligned} \tag{6.13}$$

It follows that the basis is orthogonal and

$$\left\| (a_{-j_1}^\mu)^{i_1} \cdots (a_{-j_r}^\mu)^{i_r}.v_\mu \right\|_\mu = \prod_s i_s! j_s^{i_s}$$

In particular the form is positive definite and its values on the chosen basis do not depend on μ.

Let $\mu \in \mathbb{C}$ and $t \in \sqrt{-1}\mathbb{R}$. Let $M(\mu, t)^\vee$ be the conjugate dual of $M(\mu)$ with action given by, for $b \in V^1(\mathbb{C}a)$, $m \in M(\mu)$, $f \in M(\mu, t)^\vee$,

$$(Y^{M(\mu,t)^\vee}(b, z) f)(m) = f(Y^\mu(A(t, z)b, z^{-1})m),$$

where $A(z, t)$ is defined by (2.7), (2.8).

Using the $L(\sqrt{-1}\Im(\mu))$-invariant form on $M(\mu)$ (see (6.13)), we can identify $M(\mu)$ and $M(\mu, t)^\vee$ (as vector spaces) by indentifying m with $f_m : m' \mapsto H_\mu(m', m)$.

We now want to describe explicitly the action of $V^1(\mathbb{C}a)$ under this identification. We need the following result:

Lemma 6.5. *If* $t \in \sqrt{-1}\mathbb{R}$, *then*

$$z^{2H(0)} e^{z^n a_n} z^{-2H(0)} = e^{z^{-n} a_n} \tag{6.14}$$

$$e^{tz^n a_n} g = g e^{t(-z)^n a_n} \tag{6.15}$$

Proof. If $b \in V^1(\mathbb{C}a)$ then,

$$\begin{aligned} e^{2zH(0)}e^{z^n a_n}e^{-2zH(0)}b &= z^{-2\Delta_b}\sum_r \frac{1}{r!}z^{2\Delta_b-2rn}z^{nr}a_n^r b \\ &= \sum_r \frac{1}{r!}z^{-rn}a_n^r b = e^{z^{-n}a_n}b. \end{aligned}$$

For the second formula note that

$$g(a_n^r b) = (-1)^{\Delta_b-nr}\phi(a_n^r b) = (-1)^{\Delta_b-nr}(-1)^r a_n^r\phi(b) = (-1)^{-nr}(-1)^r a_n^r g(b)$$

so, since t is purely imaginary,

$$e^{tz^n a_n}g(b) = \sum_r \frac{1}{r!}t^r z^{nr}a_n^r g(b) = \sum_r \frac{1}{r!}(-1)^{-nr}z^{nr}g(t^r a_n^r b) = e^{t(-z)^n a_n}b.$$

$\square$

Proposition 6.6. *If $m \in M(\mu)$ and $f_m \in M(\mu,t)^\vee$ is defined by $f_m(m') = H_\mu(m',m)$, then*

$$Y^{M(\mu,t)^\vee}(b,z)f_m = f_{Y^\mu(\prod_{n=1}^\infty e^{\frac{2(-t+\sqrt{-1}\Im(\mu))}{n}(-z)^{-n}a_n}b,z)m}$$

In particular the fields

$$Y^{\mu,t}(b,z) := Y^\mu\left(\prod_{n=1}^\infty e^{\frac{2(-t+\sqrt{-1}\Im(\mu))}{n}(-z)^{-n}a_n}b, z\right) \tag{6.16}$$

define a $V^1(\mathbb{C}a)$-module structure on $M(\mu)$.

Proof. By definition,

$$\begin{aligned} (Y^{M(\mu,t)^\vee}(b,z)f_m)(m') =& H_\mu(Y^\mu(A(t,z)b,z^{-1})m',m) \\ =& (Y^\mu(e^{zL(t)_1}z^{-2L(t)_0}g(b),z^{-1})m',m) \end{aligned}$$

Using (6.5) we can write

$$e^{z(L(t))_1} = e^{zL(0)_1}\prod_{n=1}^\infty e^{-\frac{2t}{n}z^n a_n} = e^{zL(\sqrt{-1}\Im(\mu))_1}\prod_{n=1}^\infty e^{-\frac{2(t-\sqrt{-1}\Im(\mu))}{n}z^n a_n}$$

so, by Lemma 6.5,

$$\begin{aligned} &(Y^{M(\mu,t)^\vee}(b,z)f_m)(m') \\ =& H_\mu(Y^\mu(e^{zL(\sqrt{-1}\Im(\mu))_1}\prod_{n=1}^\infty e^{-\frac{2(t-\sqrt{-1}\Im(\mu))}{n}z^n a_n}z^{-2L(t)_0}g(b),z^{-1})m',m) \\ =& H_\mu(Y^\mu(e^{zL(\sqrt{-1}\Im(\mu))_1}z^{-2L(t)_0}g\prod_{n=1}^\infty e^{\frac{2(-t+\sqrt{-1}\Im(\mu))}{n}(-z)^{-n}a_n}b,z^{-1})m',m) \end{aligned}$$

Since the form H_μ is $L(\sqrt{-1}\Im(\mu))$-invariant, we find that

$$(Y^{M(\mu,t)^\vee}(b,z)f_m)(m') = H_\mu(m', Y^\mu(\prod_{n=1}^{\infty} e^{\frac{2(-t+\sqrt{-1}\Im(\mu))}{n}(-z)^{-n}a_n}b,z)m)$$
$$= f_{Y^\mu(\prod_{n=1}^{\infty} e^{\frac{2(-t+\sqrt{-1}\Im(\mu))}{n}(-z)^{-n}a_n}b,z)m}(m').$$

□

To simplify notation write $\mathbf{a} = (a_{-1}, a_{-2}, \dots)$. If I is an infinite sequence $(i_1, i_2, \dots)$, with $i_j \in \mathbb{Z}_+$ almost all zero, then set $\mathbf{a}^I = \prod_{r=1}^{\infty} a_{-r}^{j_r}$. We can regard $b \in V^1(\mathbb{C}a)$ as a polynomial $b(\mathbf{a})$. More precisely, we write

$$b(\mathbf{a}) = \sum_I c_I \mathbf{a}^I \mathbf{1},\ c_I \in \mathbb{C}.$$

We also set

$$\rho(z) = (z\mathbf{1}_0, z^2\mathbf{1}_0, z^3\mathbf{1}_0, \dots) = (zI, z^2 I, z^3 I, \dots).$$

Lemma 6.7. *Write $Y^{\mu,t}(b,z) = \sum_{r\in\mathbb{Z}} b_r^{\mu,t} z^{-r-\Delta_b}$. Then*

$$b_r^{\mu,t} = \left(b(\mathbf{a}^{shift})\right)_r^\mu,\ \textit{where } \mathbf{a}^{shift} = \mathbf{a} + 2(-t+\sqrt{-1}\Im(\mu))\rho(-1).$$

Proof. Since $b_r^{\mu,t} = Res_z z^{r+\Delta_b-1}(Y^{\mu,t}(b,z))$, we need to check that

$$Res_z z^{r+\Delta_b-1}(Y^{\mu,t}(b,z)) = b(\mathbf{a} + (-t+\sqrt{-1}\Im(\mu))\rho(-1))_r^\mu.$$

It is enough to check this for $b = \mathbf{a}^I\mathbf{1}$. Using (6.9), we can write

$$\prod_{n=1}^{\infty} e^{\frac{2(-t+\sqrt{-1}\Im(\mu))}{n}(-z)^{-n}a_n}\mathbf{a}^I\mathbf{1} = (\mathbf{a} + 2(-t+\sqrt{-1}\Im(\mu))\rho(-z^{-1}))^I\mathbf{1}.$$

It follows that

$$Y^{\mu,t}(b,z) = Y^\mu(b(\mathbf{a} + 2(-t+\sqrt{-1}\Im(\mu))\rho(-z^{-1}), z)$$

hence we need to check that

$$Res_z z^{r+\Delta_{\mathbf{a}^I}-1}(Y((\mathbf{a} + 2(-t+\sqrt{-1}\Im(\mu))\rho(-z^{-1}))^I\mathbf{1}\otimes v, z))$$
$$= \left((\mathbf{a}^{shift})^I\mathbf{1}\otimes v\right)_r.$$

Indeed, setting $t_0 = 2(-t+\sqrt{-1}\Im(\mu))$ and letting q be the number of j such that $i_j \neq 0$,

$$Y^\mu((\mathbf{a} + t_0\rho(-z^{-1}))^I\mathbf{1}, z))$$
$$= \sum_s \sum_{j_1\le i_1,\dots,j_q\le i_q} \left(\prod_{p=1}^q \binom{i_p}{j_p}\left(\frac{t_0}{(-z)^p}\right)^{i_p-j_p}\right)\mathbf{a}^J\mathbf{1}_s z^{-s-\sum_{p=1}^q pj_p}$$
$$= \sum_s \sum_{j_1\le i_1,\dots,j_q\le i_q} \left(\prod_{p=1}^q (-1)^{p(i_p-j_p)} t_0^{i_p-j_p}\binom{i_p}{j_p}\right)\mathbf{a}^J\mathbf{1}_s z^{-s-\Delta_{\mathbf{a}^I\mathbf{1}}}$$

so

$$\begin{aligned}
Res_z z^{r+\Delta_{\mathbf{a}^I\mathbf{1}}-1}&(Y^{\mu}(\mathbf{a}+t_0\rho(-z^{-1}))^I\mathbf{1},z))\\
&=\sum_{j_1\le i_1,\dots,j_q\le i_q}\left(\prod_{p=1}^{q}((-1)^p t_0)^{i_p-j_p}\binom{i_p}{j_p}\right)\mathbf{a}^J\mathbf{1}_r\\
&=\left((\mathbf{a}+t_0\rho(-1))^I\mathbf{1}\right)_r,
\end{aligned}$$

as wished. □

In particular,

$$a_r^{\mu,t}=(a_{-1}\mathbf{1})_r^{\mu,t}=a_r^{\mu}-2(-t+\sqrt{-1}\Im(\mu))\delta_{r,0}I \tag{6.17}$$

so

$$a_0^{\mu,t}=(\mu-2(-t+\sqrt{-1}\Im(\mu)))I=(\overline{\mu}+2t)I.$$

Hence we have an isomorphism of $V^1(\mathbb{C}a)$-modules

$$M(\mu,t)^{\vee}\cong M(\overline{\mu}+2t). \tag{6.18}$$

Let $M[\mu,t]$ denote the vector space $M(\mu)$ equipped with the $V^1(\mathbb{C}a)$-module structure given by $b\mapsto Y^{\mu,t}(b,z)$ so that

$$M[\mu,t]\simeq M(\mu,t)^{\vee}\simeq M(\overline{\mu}+2t).$$

Let $\Upsilon_{\mu,t}:M[\mu,t]\to M(\overline{\mu}+2t)$ denote such an isomorphism. By (6.17), $\Upsilon_{\mu,t}(v_\mu)\in\mathbb{C}v_{\overline{\mu}+2t}$. We can therefore normalize $\Upsilon_{\mu,t}$ so that $v_\mu\mapsto v_{\overline{\mu}+2t}$. It follows from (6.17) that, if $j_1\ge j_2\ge\cdots\ge j_r$,

$$\Upsilon_{\mu,t}(a^{\mu}_{-j_1}\cdots a^{\mu}_{-j_r}v_\mu)=\Upsilon_{\mu,t}(a^{\mu,t}_{-j_1}\cdots a^{\mu,t}_{-j_r}v_\mu)=a^{\overline{\mu}+2t}_{-j_1}\cdots a^{\overline{\mu}+2t}_{-j_r}v_{\overline{\mu}+2t}.$$

Note that, by (6.13),

$$H_{\overline{\mu}+2t}(\Upsilon_{\mu,t}(m),\Upsilon_{\mu,t}(m'))=H_\mu(m,m'). \tag{6.19}$$

Moreover

$$\begin{aligned}
Y^{\mu,t+s}(b,z)&=Y^{\mu}(\prod_{n=1}^{\infty}e^{\frac{2(-t-s+\sqrt{-1}\Im(\mu))}{n}(-z)^{-n}a_n}b,z)\\
&=Y^{\mu,t}(\prod_{n=1}^{\infty}e^{\frac{-2s}{n}(-z)^{-n}a_n}b,z),
\end{aligned} \tag{6.20}$$

and, if $m\in M(\mu)$ and $m'\in M(\overline{\mu}+2s)$,

$$\begin{aligned}
H_{\overline{\mu}+2s}(\Upsilon_{\mu,s}Y^{\mu,t}(b,z)m,m')&=H_{\overline{\mu}+2s}(\Upsilon_{\mu,s}Y^{\mu,s+(t-s)}(b,z)m,m')\\
&=H_{\overline{\mu}+2s}(\Upsilon_{\mu,s}Y^{\mu,s}(\prod_{n=1}^{\infty}e^{\frac{-2(t-s)}{n}(-z)^{-n}a_n}b,z)m,m')\\
&=H_{\overline{\mu}+2s}(Y^{\overline{\mu}+2s}(\prod_{n=1}^{\infty}e^{\frac{-2(t-s)}{n}(-z)^{-n}a_n}b,z)\Upsilon_{\mu,s}(m),m')\\
&=H_{\overline{\mu}+2s}(Y^{\overline{\mu}+2s,t+s-\sqrt{-1}\Im(\mu)}(b,z)\Upsilon_{\mu,s}(m),m').
\end{aligned}$$

It follows that

$$\Upsilon_{\mu,s}Y^{\mu,t}(b,z)=Y^{\overline{\mu}+2s,t+s-\sqrt{-1}\Im(\mu)}(b,z)\Upsilon_{\mu,s} \tag{6.21}$$

In particular, if μ is real,

$$\Upsilon_{\mu,s}Y^{\mu,t}(b,z)=Y^{\mu+2s,t+s}(b,z)\Upsilon_{\mu,s} \tag{6.22}$$

Lemma 6.8. *If $m,m'\in M(\mu)$ and $b\in V^1(\mathbb{C}a)$, then*

$$H_\mu(m,Y^{\mu,t}(b,z)m')=H_\mu(Y^{\mu,s}(A(-\sqrt{-1}\Im(\mu)+t+s,z)b,z^{-1})m,m'). \tag{6.23}$$

In particular, if b is quasiprimary for $L(-\sqrt{-1}\Im(\mu)+t+s)$, then

$$H_\mu(m,b_n^{\mu,t}m')=H_\mu(g(b)_{-n}^{\mu,s}m,m'). \tag{6.24}$$

Proof. We first prove that

$$H_\mu(m,Y^{\mu,t}(b,z)m')=H_\mu(Y^{\mu,t}(A(-\sqrt{-1}\Im(\mu)+2t,z)b,z^{-1})m,m'). \tag{6.25}$$

Indeed,

$$\begin{aligned}
H_\mu(m,Y^{\mu,t}(b,z)m')&=H_{\overline{\mu}+2t}(\Upsilon_{\mu,t}(m),\Upsilon_{\mu,t}(Y^{\mu,t}(b,z)m'))\\
&=H_{\overline{\mu}+2t}(\Upsilon_{\mu,t}(m),Y^{\overline{\mu}+2t}(b,z)\Upsilon_{\mu,t}(m'))\\
&=H_{\overline{\mu}+2t}(Y^{\overline{\mu}+2t}(A(-\sqrt{-1}\Im(\mu)+2t,z)b,z^{-1})\Upsilon_{\mu,t}(m),\Upsilon_{\mu,t}(m'))\\
&=H_{\overline{\mu}+2t}(\Upsilon_{\mu,t}(Y^{\mu,t}(A(-\sqrt{-1}\Im(\mu)+2t,z)b,z^{-1})m),\Upsilon_{\mu,t}(m')),
\end{aligned}$$

so (6.25) follows.

To prove (6.23) write

$$H_\mu(m,Y^{\mu,t}(b,z)m')=H_\mu(m,Y^{\mu,s}(\prod_{n=1}^{\infty}e^{\frac{-2(t-s)}{n}(-z)^{-n}a_n}b,z)m').$$

By (6.25), setting $s_0=-\sqrt{-1}\Im(\mu)+2s$,

$$\begin{aligned}
H_\mu(m,Y^{\mu,t}(b,z)m')&=H_\mu(Y^{\mu,s}(A(s_0,z)\prod_{n=1}^{\infty}e^{\frac{-2(t-s)}{n}(-z)^{-n}a_n}b,z^{-1})m,m')\\
&=H_\mu(Y^{\mu,s}(e^{zL(s_0)_1}z^{-2L(s_0)_0}g\prod_{n=1}^{\infty}e^{\frac{-2(t-s)}{n}(-z)^{-n}a_n}b,z^{-1})m,m')\\
&=H_\mu(Y^{\mu,s}(e^{zL(s_0)_1}\prod_{n=1}^{\infty}e^{\frac{-2(t-s)}{n}z^na_n}z^{-2L(s_0)_0}g(b),z^{-1})m,m').
\end{aligned}$$

Since, if $p\in\sqrt{-1}\mathbb{R}$,

$$e^{z(L(p))_1}=e^{zL(0)_1}\prod_{n=1}^{\infty}e^{-\frac{2p}{n}z^na_n},$$

we find that

$$
\begin{aligned}
&H_\mu(m, Y^{\mu,s}(b,z)m') \\
&= H_\mu(Y^{\mu,s}(e^{zL(0)_1}\prod_{n=1}^{\infty} e^{-\frac{2(-\sqrt{-1}\Im(\mu)+s+t)}{n}z^n a_n} z^{-2L(0)_0} g(b), z^{-1})m, m') \\
&= H_\mu(Y^{\mu,s}(e^{zL(-\sqrt{-1}\Im(\mu)+s+t)_1} z^{-2L(0)_0} g(b), z^{-1})m, m') \\
&= H_\mu(Y^{\mu,s}(e^{zL(0)_1}\prod_{n=1}^{\infty} e^{-\frac{2(-\sqrt{-1}\Im(\mu)+s+t)}{n}z^n a_n} z^{-2L(0)_0} g(b), z^{-1})m, m') \\
&= H_\mu(Y^{\mu,s}(A(-\sqrt{-1}\Im(\mu)+s+t, z)b, z^{-1})m, m').
\end{aligned}
$$

□

Example 6.9 (The Fairlie construction). Since $L(s) = \frac{1}{2}a_{-1}^2\mathbf{1}+sa_{-2}\mathbf{1}$, by (6.17) we have

$$
\begin{aligned}
L(s)_n^{\mu,t} &= \tfrac{1}{2}(a_{-1}+2t-2\sqrt{-1}\Im(\mu))^2\mathbf{1}_n + s(a_{-2}-2t+2\sqrt{-1}\Im(\mu))\mathbf{1}_n \\
&= \tfrac{1}{2}a_{-1}^2\mathbf{1}_n + 2(t-\sqrt{-1}\Im(\mu))a_{-1}\mathbf{1}_n + 2(t-\sqrt{-1}\Im(\mu))^2\mathbf{1}_n \\
&\quad + sa_{-2}\mathbf{1}_n - 2s(t-\sqrt{-1}\Im(\mu))\mathbf{1}_n \\
&= \tfrac{1}{2} :aa:_n + 2(t-\sqrt{-1}\Im(\mu))a_n + s(Ta)_n \\
&\quad + 2(t-\sqrt{-1}\Im(\mu))(t-\sqrt{-1}\Im(\mu)-s)\mathbf{1}_n.
\end{aligned}
$$

In other words

$$
\begin{aligned}
L(s)_n^{\mu,t} &= \tfrac{1}{2} :aa:_n^{\mu} + s(Ta)_n^{\mu} + 2(t-\sqrt{-1}\Im(\mu))a_n^{\mu} + 2(t-\sqrt{-1}\Im(\mu)) \\
&\quad (t-\sqrt{-1}\Im(\mu)-s)\mathbf{1}_n^{\mu}
\end{aligned}
\tag{6.26}
$$

In particular, if $\mu\in\mathbb{R}$, we have

$$
L(s)_n^{\mu,t} = \tfrac{1}{2} :aa:_n^{\mu} + s(Ta)_n^{\mu} + 2ta_n^{\mu} + 2(t^2-st)\mathbf{1}_n^{\mu}, \tag{6.27}
$$

and, setting $s=2t$, (6.27) becomes

$$
L(s)_n^{\mu,s/2} = \tfrac{1}{2} :aa:_n^{\mu} + s(Ta)_n^{\mu} + sa_n^{\mu} - \tfrac{1}{2}s^2\mathbf{1}_n^{\mu} = L(s)_n^{\mu} + sa_n^{\mu} - \tfrac{1}{2}s^2\mathbf{1}_n^{\mu}. \tag{6.28}
$$

By the -1-st product identity,

$$
:aa:_n^{\mu} = \begin{cases} \sum_{j\in\mathbb{Z}} a_{-j}^{\mu}a_{j+n}^{\mu} & \text{if } n\neq 0, \\ 2\sum_{j\in\mathbb{N}} a_{-j}^{\mu}a_j^{\mu} + (a_0^{\mu})^2 & \text{if } n=0. \end{cases}
$$

Moreover, $(Ta)_n^{\mu} = -(n+1)a_n^{\mu}$, hence, substituting in (6.28), we obtain

$$
L(s)_n^{\mu,s/2} = \frac{1}{2}\sum_{j\in\mathbb{Z}} a_{-j}^{\mu}a_{j+n}^{\mu} - sna_n^{\mu} \text{ if } n\neq 0,
$$

while

$$
L(s)_0^{\mu,s/2} = \sum_{j\in\mathbb{N}} a_{-j}^{\mu}a_j^{\mu} + \frac{\mu^2-s^2}{2}I.
$$

Since $b \mapsto Y^{\mu,s/2}(b,z)$ gives a $V^1(\mathbb{C}a)$-module structure to $M(\mu)$ and

$$[L(s)_\lambda L(s)] = (T+2\lambda)L(s) + \frac{\lambda^3}{12}(1-12s^2),$$

by the Borcherds commutator formula,

$$[L(s)_n^{\mu,s/2}, L(s)_m^{\mu,s/2}] = (n-m)L(s)_{n+m}^{\mu,s/2} + \frac{n^3-n}{12}(1-12s^2)\delta_{n,-m}.$$

Finally, since $L(s)$ is quasiprimary for $L(s)$ and $g(L(s)) = L(s)$, by (6.24) we have

$$H_\mu(m, L(s)_n^{\mu,s/2}m') = H_\mu(L(s)_{-n}^{\mu,s/2}m, m').$$

We now extend the previous analysis of invariant Hermitian forms on bosons to the case of the vertex algebra $V^1(\mathbb{C}a) \otimes V$ where V is a conformal vertex algebra.

Let $\widehat{L}$ be the conformal vector of V. Set

$$\widehat{L}(s) = L(s) + \widehat{L}. \tag{6.29}$$

If M is a V-module, then $M(\mu) \otimes M$ is a $V^1(\mathbb{C}a) \otimes V$-module and, if M is equipped with a $\widehat{L}$-invariant form $(\,.\,,.\,)$, then $H_\mu(\,.\,,.\,) \otimes (\,.\,,.\,)$ is a $\widehat{L}(\sqrt{-1}\Im(\mu))$-invariant form on $M(\mu) \otimes M$ that we keep denoting by $H_\mu(\,.\,,.\,)$.

The arguments developed in this section for $V^1(\mathbb{C}a)$ can be carried out in the same way in the more general setting of the vertex algebra

$$V^1(\mathbb{C}a) \otimes V, \tag{6.30}$$

where V is any conformal vertex algebra. In particular, we have

Proposition 6.10. *If $b \in V^1(\mathbb{C}a) \otimes V$ and M is a V-module, then the fields*

$$Y^{\mu,t}(b,z) = Y^\mu\Big(\prod_{n=1}^{\infty} e^{\frac{2(-t+\sqrt{-1}\Im(\mu))}{n}(-z)^{-n}a_n} b, z\Big)$$

define a $V^1(\mathbb{C}a) \otimes V$-module structure on $M(\mu) \otimes M$.

As before, we can regard $b \in V^1(\mathbb{C}a) \otimes V$ as a polynomial $b(\mathbf{a})$ with values in V. More precisely, we write

$$b(\mathbf{a}) = \sum_I \mathbf{a}^I \otimes c_I,\ \ c_I \in V.$$

The following is the generalization of Lemma 6.7. The proof is the same.

Lemma 6.11. *Write $Y^{\mu,t}(b,z) = \sum_{r\in-\Delta_b+\mathbb{Z}} b_r^{\mu,t} z^{-r-\Delta_b}$. Then*

$$b_r^{\mu,t} = b(\mathbf{a} + 2(-t+\sqrt{-1}\Im(\mu))\rho(-1))_r^\mu.$$

7. Minimal W-Algebras

7.1. λ-brackets and conjugate linear involutions. Let, as before, $\mathfrak{g}$ be a basic classical Lie superalgebra, and $x \in \mathfrak{g}$ be an element, for which $ad\,x$ is diagonalizable with eigenvalues in $\frac{1}{2}\mathbb{Z}$, the $ad\,x$-gradation of $\mathfrak{g}$ satisfies (1.2) with some $f \in \mathfrak{g}_{-1}$ and is compatible with the parity of $\mathfrak{g}$. Then for some $e \in \mathfrak{g}_1$, $\{e, x, f\}$ is an sl_2-triple as in Proposition 3.2, i.e. (3.1) holds with $\mathfrak{g}^\natural$ the centralizer of f in $\mathfrak{g}$. Recall that the invariant bilinear form $(.|.)$ on $\mathfrak{g}$ is normalized by the condition $(x|x) = \frac{1}{2}$, and we have the orthogonal direct sum of ideals

$$\mathfrak{g}_0 = \mathbb{C}x \oplus \mathfrak{g}^\natural. \tag{7.1}$$

Choose a Cartan subalgebra $\mathfrak{h}^\natural$ of $\mathfrak{g}^\natural$, so that, by (7.1), $\mathfrak{h} = \mathbb{C}x \oplus \mathfrak{h}^\natural$ is a Cartan subalgebra of $\mathfrak{g}_0$ (and of $\mathfrak{g}$).

Let

$$\mathfrak{g}^\natural = \bigoplus_{i=0}^{s} \mathfrak{g}_i^\natural \tag{7.2}$$

be the decomposition of $\mathfrak{g}^\natural$ into the direct sum of ideals, where $\mathfrak{g}_0^\natural$ is the center and the $\mathfrak{g}_i^\natural$ are simple for $i > 0$. Let $h^\vee$ be the dual Coxeter number of $\mathfrak{g}$, and denote by $\bar{h}_i^\vee$ half of the eigenvalue of the Casimir element of $\mathfrak{g}_i^\natural$ with respect to $(.|.)_{|\mathfrak{g}_i^\natural \times \mathfrak{g}_i^\natural}$, when acting on $\mathfrak{g}_i^\natural$. Note that $\bar{h}_0^\vee = 0$.

In [18] the authors introduced (as a special case of a more general construction) the universal *minimal* W-algebra $W^k_{\min}(\mathfrak{g})$, whose simple quotient is $W_k^{\min}(\mathfrak{g})$, attached to the grading (5.6). This is a vertex algebra strongly and freely generated by elements L, $J^{\{v\}}$ where v runs over a basis of $\mathfrak{g}^\natural$, $G^{\{u\}}$ where u runs over a basis of $\mathfrak{g}_{-1/2}$, with the following λ-brackets ([20, Theorem 5.1]): L is a Virasoro element (conformal vector) with central charge $c(k)$ given by (1.4), $J^{\{u\}}$ are primary of conformal weight 1, $G^{\{v\}}$ are primary of conformal weight $\frac{3}{2}$, and

$$[J^{\{u\}}{}_\lambda G^{\{v\}}] = G^{\{[u,v]\}} \qquad \text{for } u \in \mathfrak{g}^\natural,\ v \in \mathfrak{g}_{-1/2}, \tag{7.3}$$

$$[J^{\{u\}}{}_\lambda J^{\{v\}}] = J^{\{[u,v]\}} + \lambda\beta_k(u|v) \qquad \text{for } u, v \in \mathfrak{g}^\natural, \tag{7.4}$$

where

$$\beta_k(u, v) = \delta_{i,j}(k + \tfrac{h^\vee - \bar{h}_i^\vee}{2})(u|v), \quad u \in \mathfrak{g}_i^\natural,\ v \in \mathfrak{g}_j^\natural,\ i, j \geq 0. \tag{7.5}$$

Furthermore, the most explicit formula for the λ-bracket between the $G^{\{u\}}$ is given in [1, (1.1)] and in [20, Theorem 5.1 (e)]. We will need both formulas:

$$\begin{aligned}[G^{\{u\}}{}_\lambda G^{\{v\}}] = & -2(k + h^\vee)\langle u, v\rangle L + \langle u, v\rangle \sum_{\alpha=1}^{\dim \mathfrak{g}^\natural} :J^{\{u^\alpha\}} J^{\{u_\alpha\}}: + \\ & \sum_{\gamma=1}^{\dim \mathfrak{g}_{1/2}} :J^{\{[u,w^\gamma]^\natural\}} J^{\{[w_\gamma,v]^\natural\}}: + 2(k+1)\partial J^{\{[[e,u],v]^\natural\}} \\ & + 4\lambda \sum_i \frac{p(k)}{k_i} J^{\{[[e,u],v]_i^\natural\}} + 2\lambda^2 \langle u, v\rangle p(k)\mathbf{1},\end{aligned} \tag{7.6}$$

$$(G^{\{u\}}{}_\lambda G^{\{v\}}) = -2(k+h^\vee)\langle u,v\rangle L + \langle u,v\rangle \sum_{\alpha=1}^{\dim \mathfrak{g}^\natural} :J^{\{u^\alpha\}}J^{\{u_\alpha\}}: +$$
$$2\sum_{\alpha,\beta}\langle [u_\alpha,u],[v,u^\beta]\rangle :J^{\{u^\alpha\}}J^{\{u_\beta\}}: +2(k+1)(\partial+2\lambda)J^{\{[[e,u],v]^\natural\}}$$
$$+2\lambda\sum_{\alpha,\beta}\langle [u_\alpha,u],[v,u^\beta]\rangle J^{\{[u^\alpha,u_\beta]\}} + 2\lambda^2\langle u,v\rangle p(k)\mathbf{1}, \tag{7.7}$$

where $\{u_\alpha\}$ and $\{u^\alpha\}$ (resp. $\{w_\gamma\}$, $\{w^\gamma\}$) are dual bases of $\mathfrak{g}^\natural$ (resp. $\mathfrak{g}_{1/2}$) with respect to $(.|.)$ (resp. with respect to $\langle\cdot,\cdot\rangle_{\rm ne}$), $a\mapsto a_i^\natural$ (resp. $a\mapsto a^\natural$) for $a\in\mathfrak{g}_0$ is the orthogonal projection to $\mathfrak{g}_i^\natural$ (resp $\mathfrak{g}^\natural$), $p(k)$ is the monic quadratic polynomial proportional to (7.28), introduced in [1, Table 4], and thoroughly investigated in [15], and $k_i = k+\frac{1}{2}(h^\vee-\bar h_i^\vee)$, $i=1,\dots,s$ (see Table 2 below for the values of $\bar h_i^\vee$).

The following proposition is a special case of [16, Lemma 7.3], in view of Lemma 3.1.

Proposition 7.1. *Let ϕ be a conjugate linear involution of $\mathfrak{g}$ such that $\phi(f)=f$, $\phi(x)=x$, $\phi(e)=e$. Then the map*

$$\phi(J^{\{u\}}) = J^{\{\phi(u)\}},\quad \phi(G^{\{v\}}) = G^{\{\phi(v)\}},\quad \phi(L)=L,\quad u\in\mathfrak{g}^\natural,\ v\in\mathfrak{g}_{-1/2} \tag{7.8}$$

extends to a conjugate linear involution of the vertex algebra $W^k_{\min}(\mathfrak{g})$.

The following result is a sort of converse to Proposition 7.1.

Proposition 7.2. *Assume that $k\in\mathbb{R}$ is non-collapsing. Let ψ be a conjugate linear involution of $W^k_{\min}(\mathfrak{g})$. Then there exists a conjugate linear involution ϕ of $\mathfrak{g}$ satisfying (1.1) such that ψ is the conjugate linear involution induced by ϕ.*

Proof. If $a,b\in\mathfrak{g}^\natural$, define $\phi(a)$ by

$$\psi(J^{\{a\}}) = J^{\{\phi(a)\}}.$$

Then

$$\psi([J^{\{a\}}{}_\lambda J^{\{b\}}]) = \psi(J^{\{[a,b]\}}) + \lambda\overline{\beta_k(a,b)} = J^{\{\phi([a,b])\}} + \lambda\overline{\beta_k(a,b)} \tag{7.9}$$
$$[J^{\{\phi(a)\}}{}_\lambda J^{\{\phi(b)\}}] = J^{\{[\phi(a),\phi(b)]\}} + \lambda\beta_k(\phi(a),\phi(b)) \tag{7.10}$$

Since ψ is a vertex algebra conjugate linear automorphism, (7.9) equals (7.10), so that ϕ is a conjugate linear involution of $\mathfrak{g}^\natural$, and we have

$$\overline{\beta_k(a,b)} = \beta_k(\phi(a),\phi(b)). \tag{7.11}$$

Since k is not collapsing, relations (7.22), (7.28) and (7.11) imply that

$$\overline{(a|b)} = (\phi(a)|\phi(b)) \text{ for } a,b\in\mathfrak{g}^\natural. \tag{7.12}$$

We now prove that there is a unique extension of ϕ to a conjugate linear automorphism of $\mathfrak{g}$ fixing e, x, and f. Note that $\phi(\mathfrak{g}_{-1/2})\subset\mathfrak{g}_{-1/2}$ and that $\mathfrak{g}_{1/2}=[e,\mathfrak{g}_{-1/2}]$. In particular, setting $\phi(x)=x$, $\phi(f)=f$, $\phi(e)=e$, $\phi(u)=[e,\phi(v)]$ for $u\in\mathfrak{g}_{1/2}$, $u=[e,v]$, $v\in\mathfrak{g}_{-1/2}$, we extend ϕ to a conjugate linear bijection $\mathfrak{g}\to\mathfrak{g}$. In particular, ϕ is unique. It remains to prove that it is a conjugate linear automorphism. Note first that, by (7.1), equation (7.12) holds for $a, b \in \mathfrak{g}_0$. Consider elements

$$g = \alpha\, e + u + a + v + \beta\, f + \gamma\, x, \quad g' = \alpha'\, e + u' + a' + v' + \beta'\, f + \gamma'\, x,$$

where $\alpha, \alpha', \beta, \beta', \gamma, \gamma' \in \mathbb{C},\ u, u' \in \mathfrak{g}_{1/2},\ v, v' \in \mathfrak{g}_{-1/2},\ a, a' \in \mathfrak{g}^\natural$. Then

$$\begin{aligned}
&\phi([g, g']) = \\
&\phi([e, v'] + \alpha\beta' x - \alpha\gamma' e + [u, u'] + [u, a'] + [u, v'] + \beta'[u, f] \\
&- \tfrac{1}{2}\gamma' u + [a, u'] + [a, a'] + [a, v'] + \alpha'[v, e] + [v, u'] + [v, a'] + [v, v'] \\
&+ \tfrac{1}{2}\gamma' v - \beta\alpha' x + \beta[f, u'] + \beta\gamma' f + \gamma\alpha' e + \tfrac{1}{2}\gamma u' - \tfrac{1}{2}\gamma v' - \beta'\gamma f), \qquad (7.13)\\
&[\phi(g), \phi(g')] = \\
&\bar\alpha[e, \phi(v')] + \bar\alpha\bar\beta' x - \bar\alpha\bar\gamma' e + [\phi(u), \phi(u')] + [\phi(u), \phi(a')] + [\phi(u), \phi(v')] + \bar\beta'[\phi(v), f] \\
&- \tfrac{1}{2}\bar\gamma'\phi(u) + [\phi(a), \phi(u')] + [\phi(a), \phi(a')] + [\phi(a), \phi(v')] + \bar\alpha'[\phi(v), e] + [\phi(v), \phi(u')] \\
&+ [\phi(v), \phi(a')] + [\phi(v), \phi(v')] + \tfrac{1}{2}\bar\gamma'\phi(v) - \bar\beta\bar\alpha' x + \beta[f, \phi(u')] + \beta\gamma' f + \bar\gamma\bar\alpha e \\
&+ \tfrac{1}{2}\bar\gamma\phi(u') - \tfrac{1}{2}\bar\gamma\phi(v') - \bar\beta'\bar\gamma f). \qquad (7.14)
\end{aligned}$$

Hence (7.13) equals (7.14), provided the following equalities hold

$$\phi([u, u']) = [\phi(u), \phi(u')], \qquad (7.15)$$
$$\phi([u, a']) = [\phi(u), \phi(a')], \qquad (7.16)$$
$$\phi([u, v']) = [\phi(u), \phi(v')], \qquad (7.17)$$
$$\phi([v, v']) = [\phi(v), \phi(v')], \qquad (7.18)$$
$$\phi([v, a']) = [\phi(v), \phi(a')], \qquad (7.19)$$
$$\phi([u, f]) = [\phi(u), f]. \qquad (7.20)$$

Relation (7.3) implies at once (7.19). To prove (7.18) note that $[v, v'] = \langle v, v'\rangle f$, so it is enough to prove that $\langle\phi(v), \phi(v')\rangle = \overline{\langle v, v'\rangle}$. By (7.7),

$$G^{\{\phi(v)\}}{}_{3/2}G^{\{\phi(v')\}} = 4p(k)\langle\phi(v), \phi(v')\rangle\mathbf{1} = \phi(4p(k)\langle v, v'\rangle\mathbf{1}) = 4p(k)\overline{\langle v, v'\rangle}\mathbf{1}.$$

Since $p(k) \neq 0$ (k is not collapsing) and k is real, we have the claim.

Now we prove (7.20). Here and in the following we write $u = [e, v],\ v \in \mathfrak{g}_{-1/2}$. Then

$$\begin{aligned}
\phi([u, f]) &= \phi([[e, v], f]) = -\phi([x, v]) \\
&= \tfrac{1}{2}\phi(v) = -[x, \phi(v)] = [[e, \phi(v)], f] = [\phi(u), f].
\end{aligned}$$

Next we prove (7.17). We have to prove that

$$\phi([[e, v], v']) = [[e, \phi(v)], \phi(v')].$$

By (7.6)

$$G^{\{\phi(v)\}}{}_{1/2}G^{\{\phi(v')\}} = \sum_i \tfrac{p(k)}{k_i} J^{\{[[e,\phi(v)],\phi(v')]^\natural_i\}}.$$

On the other hand

$$\psi(G^{\{v\}}{}_{1/2}G^{\{v'\}}) = \sum_i \tfrac{p(k)}{k_i} J^{\{\phi([[e,v],v']_i^\natural)\}}.$$

Since ϕ is an automorphism of $\mathfrak{g}^\natural$ there is a permutation $i \mapsto i'$ such that $\phi(\mathfrak{g}_i^\natural) = \mathfrak{g}_{i'}^\natural$. It follows that $\phi([[e,v],v']_i^\natural) = \phi([[e,v],v'])_{i'}^\natural$ hence $[[e,\phi(v)],\phi(v')]_{i'}^\natural = \phi([[e,v],v']_i^\natural)$ for all i, and also

$$\begin{aligned}[][[e,\phi(v)],\phi(v')]^\natural &= \sum_{i'}[[e,\phi(v)],\phi(v')]_{i'}^\natural = \sum_i \phi([[e,v],v']_i^\natural)\\ &= \sum_{i'} \phi([[e,v],v'])_{i'}^\natural = \phi([[e,v],v'])^\natural.\end{aligned}$$

To conclude we have to check the x-component:

$$\begin{aligned}(x|[[e,\phi(v)],\phi(v')]) &= ([x,[e,\phi(v)]]|\phi(v')) = \tfrac{1}{2}([e,\phi(v)]|\phi(v'))\\ &= \tfrac{1}{2}\langle\phi(v),\phi(v')\rangle = \tfrac{1}{2}\overline{\langle v,v'\rangle}.\end{aligned}$$

Since (1.1) holds on $\mathfrak{g}_0$, we have

$$(x|\phi([[e,v],v']) = \overline{(\phi(x)|[[e,v],v'])} = \overline{(x|[[e,v],v'])} = \tfrac{1}{2}\overline{\langle v,v'\rangle}.$$

Next, we prove (7.16). We have

$$\begin{aligned}\phi([[e,v],a']) &= \phi([e,[v,a']]) = [e,\phi([v,a'])] = [e,[\phi(v),\phi(a')]]\\ &= [[e,\phi(v)],\phi(a')] = [\phi(u),\phi(a')].\end{aligned}$$

Next, we prove (7.15). Consider $u = [e,v],\ u' = [e,v'],\ v,v' \in \mathfrak{g}_{-1/2}$.

$$\phi([u,u']) = \phi([[e,v],[e,v']]) = \phi([e,[[e,v],v']]) = [e,\phi([[e,v],v'])].$$

By (7.17), we obtain

$$\phi([u,u']) = [e,[\phi([e,v]),\phi(v')])] = [e,[\phi(u),\phi(v')]] = [\phi(u),\phi(u')].$$

It remains to check that

$$(\phi(a)|\phi(b)) = \overline{(a|b)}$$

for $a,b \in \mathfrak{g}$. We already observed that this relation holds for $a,b \in \mathfrak{g}_0$ and it is obvious that $(\phi(e)|\phi(f)) = \overline{(e|f)}$. We now compute for $u \in \mathfrak{g}_{1/2}$, $v' \in \mathfrak{g}_{-1/2}$,

$$(\phi(u)|\phi(v')) = ([e,\phi(v)]|\phi(v')) = \langle\phi(v),\phi(v')\rangle = \overline{\langle v,v'\rangle} = \overline{(u|v')}.$$

$\square$

By Proposition 3.2 there is a conjugate linear involution ϕ on $\mathfrak{g}$ such that $\phi(x) = x$, $\phi(f) = f$ and $(\mathfrak{g}^\natural)^\phi$ is a compact real form of $\mathfrak{g}^\natural$, hence, by Proposition 7.1, ϕ induces a conjugate linear involution of the vertex algebra $W^k_{\min}(\mathfrak{g})$, and descends to a conjugate linear involution of its unique simple quotient $W_k^{\min}(\mathfrak{g})$, which we again denote by ϕ.

By [16, Proposition 7.4 (b)], $W^k_{\min}(\mathfrak{g})$ admits a unique ϕ-invariant Hermitian form $H(\cdot,\cdot)$ such that $H(\mathbf{1},\mathbf{1}) = 1$. Recall that if $k + h^\vee \neq 0$ then the kernel of $H(\cdot,\cdot)$ is the unique maximal ideal of $W^k_{\min}(\mathfrak{g})$, hence $H(\cdot,\cdot)$ descends to a non-degenerate ϕ-invariant Hermitian form on $W_k^{\min}(\mathfrak{g})$, which we again denote by $H(\cdot,\cdot)$.

We need to fix notation for affine vertex algebras. Let $\mathfrak{a}$ be a Lie superalgebra equipped with a nondegenerate invariant supersymmetric bilinear form B. The universal affine vertex algebra $V^B(\mathfrak{a})$ is the universal enveloping vertex algebra of the Lie conformal superalgebra $R = (\mathbb{C}[T]\otimes\mathfrak{a})\oplus\mathbb{C}$ with λ-bracket given by

$$[a_\lambda b] = [a,b] + \lambda B(a,b),\ \ a,b\in\mathfrak{a}.$$

In the following, we shall say that a vertex algebra V is an affine vertex algebra if it is a quotient of some $V^B(\mathfrak{a})$. If $\mathfrak{a}$ is simple Lie algebra, we denote by $(.|.)^\mathfrak{a}$ the normalized invariant bilinear form on $\mathfrak{a}$, defined by the condition $(\alpha|\alpha)^\mathfrak{a} = 2$ for a long root α. Then $B = k(.|.)^\mathfrak{a}$, and we simply write $V^k(\mathfrak{a})$. If $k \neq -h^\vee$, then $V^k(\mathfrak{a})$ has a unique simple quotient, which will be denoted by $V_k(\mathfrak{a})$.

Let ψ be a conjugate linear involution of $\mathfrak{a}$ such that $(\psi(x)|\psi(y)) = \overline{(x|y)}$. By [16, §5.3] there exists a unique ψ-invariant Hermitian form $H_\mathfrak{a}$ on $V^k(\mathfrak{a})$. The kernel of $H_\mathfrak{a}$ is the maximal ideal of $V^k(\mathfrak{a})$, hence $H_\mathfrak{a}$ descends to $V_k(\mathfrak{a})$.

7.2. Some numerical information. Recall the decomposition (7.2) of the Lie algebra $\mathfrak{g}^\natural$, and that we assume that $\mathfrak{g}^\natural$ is not abelian, i.e. $s \geq 1$ in (7.2). Let θ_i be the highest root of the simple component $\mathfrak{g}_i^\natural$ for $i > 0$. Set

$$M_i(k) = \frac{2}{u_i}\left(k + \frac{h^\vee - \bar{h}_i^\vee}{2}\right),\quad i \geq 0, \tag{7.21}$$

where

$$u_i = \begin{cases} 2 & \text{if } i = 0,\\ (\theta_i|\theta_i) & \text{if } i > 0.\end{cases}$$

Let $(.|.)_i^\natural$ denote the invariant bilinear form on $\mathfrak{g}_i^\natural$, normalized by the condition $(\theta_i|\theta_i)_i^\natural = 2$ for $i > 0$, and let $(.|.)_0^\natural = (.|.)_{|\mathfrak{g}_0^\natural\times\mathfrak{g}_0^\natural}$. Note that, for $i > 0$, $(a|b)_i^\natural = \delta_{i,j}\frac{(\theta_i|\theta_i)}{2}(a|b)$, hence, formula (7.5) can be written as

$$\begin{aligned}\beta_k(a,b) &= \delta_{i,j} M_i(k)\frac{(\theta_i|\theta_i)}{2}(a|b) &(7.22)\\ &= \delta_{i,j} M_i(k)(a|b)_i^\natural \qquad \text{for } a\in\mathfrak{g}_i^\natural,\ b\in\mathfrak{g}_j^\natural,\ i,j\geq 0. &(7.23)\end{aligned}$$

In other words, the vertex subalgebra of $W^k_{\min}$ generated by $J^{\{a\}}$, $a\in\mathfrak{g}^\natural$, is $\bigotimes_{i\geq 0} V^{M_i(k)}(\mathfrak{g}_i^\natural)$.

Table 2. Numerical information

$\mathfrak{g}$	$\mathfrak{g}^\natural$	u_i	$h^\vee$	$\bar{h}_i^\vee$	$M_i(k)$	χ_i
$sl(2\|m), m>2$	$\mathbb{C}\oplus sl_m$	$2,-2$	$2-m$	$0,-m$	$k-\frac{m-2}{2}, -k-1$	$1-m/2,-1$
$psl(2\|2)$	sl_2	-2	0	-2	$-k-1$	-1
$osp(4\|m), m>2$	$sl_2\oplus sp_m$	$2,-4$	$2-m$	$2,-m-2$	$k-\frac{m}{2}, -\frac{1}{2}k-1$	$-m/2,-1$
$spo(2\|3)$	sl_2	$-1/2$	$1/2$	$-1/2$	$-4k-2$	-2
$spo(2\|m), m>4$	so_m	-1	$2-m/2$	$1-m/2$	$-2k-1$	-1
$D(2,1;a)$	$sl_2\oplus sl_2$	$-\frac{2}{1+a}, -\frac{2a}{1+a}$	0	$-\frac{2}{1+a}, -\frac{2a}{1+a}$	$-(1+a)k-1, -\frac{1+a}{a}k-1$	$-1,-1$
$F(4)$	so_7	$-4/3$	-2	$-10/3$	$-\frac{3}{2}k-1$	-1
$G(3)$	G_2	$-2/3$	$-3/2$	-3	$-\frac{4}{3}k-1$	-1

Closely related to the vertex algebra $W^k_{\min}(\mathfrak{g})$ is the universal affine vertex algebra $V^{\alpha_k}(\mathfrak{g}_0)$ (see [20, (5.16)]), where

$$\alpha_k(a,b) = ((k+h^\vee)(a|b) - \tfrac{1}{2}\kappa_{\mathfrak{g}_0}(a,b)), \tag{7.24}$$

and where $\kappa_{\mathfrak{g}_0}$ denotes the Killing form of $\mathfrak{g}_0$. Note that

$$\alpha_k(a,b) = \delta_{i,j}(k+h^\vee - \bar{h}_i^\vee)(a|b) \text{ if } a\in\mathfrak{g}_i^\natural,\ b\in\mathfrak{g}_j^\natural,\ i,j\geq 0.$$

We have another formula for the cocycle α_k, closely related to (7.23):

$$\begin{aligned}\alpha_k(a,b) &= \delta_{i,j}\frac{2}{(\theta_i|\theta_i)}\left(k+h^\vee-\bar{h}_i^\vee\right)(a|b)_i^\natural \\ &= \delta_{i,j}(M_i(k)+\chi_i)(a|b)_i^\natural \text{ for } a\in\mathfrak{g}_i^\natural, b\in\mathfrak{g}_j^\natural,\ i,j\geq 0,\end{aligned} \tag{7.25}$$

where

$$\chi_i = \frac{h^\vee - \bar{h}_i^\vee}{u_i},\ i\geq 0. \tag{7.26}$$

The relevant data for computing the $M_i(k)$ and χ_i are collected in Table 2, where their explicit values are also displayed. Note that $M_0(k) = k+\frac{1}{2}h^\vee$.

As in the Introduction, denote by $\xi\in(\mathfrak{h}^\natural)^*$ a highest weight of the $\mathfrak{g}^\natural$-module $\mathfrak{g}_{-1/2}$.

Lemma 7.3. *For $i\geq 1$ we have*

$$\chi_i = -\xi(\theta_i^\vee), \tag{7.27}$$

with the exception of χ_1 for $\mathfrak{g} = osp(4|m)$.

Proof. The weights ξ are restrictions to $\mathfrak{h}^\natural$ of the maximal odd roots of $\mathfrak{g}$; they are listed in Table 3, together with the maximal roots θ_i. Relation (7.27) is then checked directly using the data in Tables 1, 2, 3. □

Recall from [1] that a level k is *collapsing* for $W_k^{\min}(\mathfrak{g})$ if $W_k^{\min}(\mathfrak{g})$ is a subalgebra of the simple affine vertex algebra $V_{\beta_k}(\mathfrak{g}^\natural)$.

We summarize in the following result the content of Theorem 3.3 and Proposition 3.4 of [1] relevant to our setting. We say that an ideal in $\mathfrak{g}^\natural$ is a *component* of $\mathfrak{g}^\natural$ if it is simple or 1-dimensional.

Table 3. Highest odd roots and highest roots of $\mathfrak{g}^\natural$

$\mathfrak{g}$	Highest odd roots	θ_i
$sl(2\|m), m>2$	$\epsilon_1-\delta_m, \delta_1-\epsilon_2$	$\delta_1-\delta_m$
$psl(2\|2)$	$\epsilon_1-\delta_2, \delta_1-\epsilon_2$	$\delta_1-\delta_2$
$osp(4\|m), m>2$	$\epsilon_1+\delta_1$	$\epsilon_1-\epsilon_2, 2\delta_1$
$spo(2\|3)$	$\delta_1+\epsilon_1$	ϵ_1
$spo(2\|m), m>4$	$\delta_1+\epsilon_1$	$\epsilon_1+\epsilon_2$
$D(2,1;a)$	$\epsilon_1+\epsilon_2+\epsilon_3$	$2\epsilon_2, 2\epsilon_3$
$F(4)$	$\frac{1}{2}(\delta_1+\epsilon_1+\epsilon_2+\epsilon_3)$	$\epsilon_1+\epsilon_2$
$G(3)$	$\delta_1+\epsilon_1+\epsilon_2$	$\epsilon_1+2\epsilon_2$

Theorem 7.4. *Let $\mathfrak{g}$ be a basic Lie superalgebra from Table 2. Assume $k\neq -h^\vee$. Let $p(k)$ be the monic quadratic polynomial in k, proportional to*

$$\begin{cases} M_1(k)M_2(k) & \text{if } \mathfrak{g}^\natural \text{ has two components,}\\ M_1(k)(k+\frac{\bar h_1^\vee}{2}+1) & \text{otherwise.}\end{cases} \tag{7.28}$$

Then

(1) k is collapsing if and only if $p(k)=0$.
(2) If $\mathfrak{g}^\natural$ is simple then
  (a) $W_k^{\min}(\mathfrak{g})=\mathbb{C}$ if and only if $M_1(k)=0$;
  (b) if $k=-\frac{\bar h_1^\vee}{2}-1$, then $W_k^{\min}(\mathfrak{g})\cong V_{M_1(k)}(\mathfrak{g}^\natural)$.
(3) If $\mathfrak{g}=D(2,1;a)$ and k is collapsing, then $W_k^{\min}(\mathfrak{g})=V_{M_j(k)}(\mathfrak{g}_j^\natural)$, with $j\neq i$ if $M_i(k)=0$.

Remark 7.5. If $M_i(k)\in\mathbb{Z}_+$ for all $i\geq 1$, $\mathfrak{g}\neq osp(4|m)$ and $M_i(k)<-\chi_i$ for some $i\geq 1$, then k is a collapsing level (or critical). This is clear by looking at Table 2.

8. Necessary Conditions for Unitarity of Modules Over $W^k_{\min}(\mathfrak{g})$

We assume that $\mathfrak{g}$ is from the list (1.3); in particular, $\mathfrak{g}^\natural$ is a reductive Lie algebra. We parametrize the highest weight modules for $W^k_{\min}(\mathfrak{g})$ following Sect. 7 of [20]. Let $\mathfrak{h}^\natural$ be a Cartan subalgebra of $\mathfrak{g}^\natural$, and choose a triangular decomposition $\mathfrak{g}^\natural=\mathfrak{n}_-^\natural\oplus\mathfrak{h}^\natural\oplus\mathfrak{n}_+^\natural$. For $\nu\in(\mathfrak{h}^\natural)^*$ and $l_0\in\mathbb{C}$, let $L^W(\nu,\ell_0)$ (resp. $M^W(\nu,\ell_0)$) denote the irreducible highest weight (resp. Verma) $W^k_{\min}(\mathfrak{g})$-module with highest weight (ν,ℓ_0) and highest weight vector v_{ν,ℓ_0}. This means that one has

$$\begin{aligned}&J_0^{\{h\}}v_{\nu,\ell_0}=\nu(h)v_{\nu,\ell_0} \text{ for } h\in\mathfrak{h}^\natural,\quad L_0v_{\nu,\ell_0}=l_0v_{\nu,\ell_0},\\ &J_n^{\{u\}}v_{\nu,\ell_0}=G_n^{\{v\}}v_{\nu,\ell_0}=L_nv_{\nu,\ell_0}=0 \text{ for } n>0, u\in\mathfrak{g}^\natural,\ v\in\mathfrak{g}_{-1/2},\\ &J_0^{\{u\}}v_{\nu,\ell_0}=0 \text{ for } u\in\mathfrak{n}_+^\natural.\end{aligned}$$

Let ϕ is an almost compact conjugate linear involution of $\mathfrak{g}$ (see Definition 1.1); in particular, the fixed points set $\mathfrak{g}^\natural_\mathbb{R}$ of $\phi_{|\mathfrak{g}^\natural}$ is a compact Lie algebra (the adjoint group is compact). Set $\mathfrak{h}^\natural_\mathbb{R}=\mathfrak{g}^\natural_\mathbb{R}\cap\mathfrak{h}^\natural$. Recall that $\nu\in(\mathfrak{h}^\natural_\mathbb{R})^*$ is said to be purely imaginary if $\nu(\mathfrak{h}^\natural_\mathbb{R})\subset\sqrt{-1}\mathbb{R}$. It is well-known that if α is a root of $\mathfrak{g}^\natural$ and ν is purely imaginary then $\nu(\alpha)\in\mathbb{R}$.

Lemma 8.1. *Assume that $l_0 \in \mathbb{R}$ and that ν is purely imaginary. Then $L^W(\nu, \ell_0)$ admits a unique ϕ-invariant nondegenerate Hermitian form $H(.,.)$ such that $H(v_{\nu,\ell_0}, v_{\nu,\ell_0}) = 1$.*

Proof. It is enough to show that the Verma module $M^W(\nu, \ell_0)$ admits a ϕ-invariant Hermitian form H such that $H(v_{\nu,\ell_0}, v_{\nu,\ell_0}) = 1$. Fix a basis $\{v_i \mid i \in I\}$ of $\mathfrak{g}_{-1/2}$ and a basis $\{u_i \mid i \in J\}$ of $\mathfrak{n}^\natural_-$. Set $A^{\{i\}} = J^{\{u_i\}}$ if $i \in J$, $A^{\{i\}} = G^{\{v_i\}}$ if $i \in I$, and $A^{\{0\}} = L$. Then

$$\mathcal{B} = \left\{ \left(A^{\{1\}}_{-m_1}\right)^{b_1} \cdots \left(A^{\{s\}}_{-m_s}\right)^{b_s} v_{\nu,\ell_0} \right\}$$

where $b_i \in \mathbb{Z}_+$, $b_i \le 1$ if $i \in I$, $m_i > 0$ or $m_i = 0$ when $i \in J$, is a basis of $M^W(\nu, \ell_0)$.

Define the conjugate-linear map $F : M \to \mathbb{C}$ by setting $F(v_{\nu,\ell_0}) = 1$ and $F(v) = 0$ if $v \in \mathcal{B}$, $v \ne v_{\nu,\ell_0}$.

If $v \in M^W(\nu, \ell_0)$, $m > 0$, and $u \in \mathfrak{g}^\natural$, then

$$(J^{\{u\}}_m F)(v) = -F(J^{\{\phi(u)\}}_{-m} v) = 0.$$

Similarly we see that, if $u \in \mathfrak{g}_{-1/2}$, then

$$(G^{\{u\}}_m F)(v) = (L_m F)(v) = 0.$$

On the other hand, if $u \in \mathfrak{n}_{0+}$, then, since $\phi(u) \in \mathfrak{n}_{0-}$,

$$(J^{\{u\}}_0 F)(v) = -F(J^{\{\phi(u)\}}_0 v) = 0.$$

If $h \in \mathfrak{h}^\natural_{\mathbb{R}}$, then, since $\nu(h)$ is purely imaginary,

$$(J^{\{h\}}_0 F)(v_{\nu,\ell_0}) = -F(J^{\{\phi(h)\}}_0 v_{\nu,\ell_0}) = -F(J^{\{h\}}_0 v_{\nu,\ell_0}) = \nu(h) F(v_{\nu,\ell_0}),$$

and, if $v \in \mathcal{B}$, $v \ne v_{\nu,\ell_0}$, then

$$(J^{\{h\}}_0 F)(v) = -F(J^{\{\phi(h)\}}_0 v) = -F(J^{\{h\}}_0 v) = 0.$$

It follows that $J^{\{h\}}_0 F = \nu(h) F$ for all $h \in \mathfrak{h}^\natural$. Finally, since $l_0 \in \mathbb{R}$,

$$(L_0 F)(v_{\nu,\ell_0}) = F(L_0 v_{\nu,\ell_0}) = l_0 F(v_{\nu,\ell_0}),$$

and, if $v \in \mathcal{B}$, $v \ne v_{\nu,\ell_0}$, then

$$(L_0 F)(v) = F(L_0 v) = 0.$$

so $L_0 F = l_0 F$. It follows that there is a $W^k_{\min}(\mathfrak{g})$-module map $\beta : M^W(\nu, \ell_0) \to M^W(\nu, \ell_0)^\vee$ mapping v_{ν,ℓ_0} to F. Define a Hermitian form on $M^W(\nu, \ell_0)$ by setting

$$H(m, m') = \beta(m')(m).$$

Let us check that this form is ϕ-invariant: write Y^{ν,ℓ_0} for the field $Y^{M^W(\nu,\ell_0)}$ and $\check{Y}^{\nu,\ell_0}$ for the field $Y^{M^W(\nu,\ell_0)^\vee}$. Then

$$H(m, Y^{\nu,\ell_0}(u, z) m') = \beta(Y^{\nu,\ell_0}(u, z) m')(m) = \check{Y}^{\nu,\ell_0}(u, z) \beta(m')(m)$$

$$= \beta(m')(Y^{\nu,\ell_0}(A(z)u, z^{-1})m),$$

so

$$H(m, Y^{\nu,\ell_0}(u,z)m') = H(Y^{\nu,\ell_0}(A(z)u, z^{-1})m, m').$$

□

Definition 8.2. The $W_k^{\min}(\mathfrak{g})$-module $L^W(\nu, \ell_0)$ is called unitary if the Hermitian form $H(\cdot,\cdot)$ is positive definite. The vertex algebra $W_k^{\min}(\mathfrak{g})$ is called unitary if its adjoint module is unitary.

As usual, we denote $\|u\| = H(u,u)$, $u \in L^W(\nu, \ell_0)$. In order to obtain necessary conditions for unitarity of $L^W(\nu, \ell_0)$ we compute $||G^{\{v\}}_{-1/2} v_{\nu,\ell_0}||$.

Lemma 8.3. *Let, as before, ξ be a highest weight of the $\mathfrak{g}^\natural$-module $\mathfrak{g}_{-1/2}$, and fix a highest weight vector $v \in \mathfrak{g}_{-1/2}$. Then*

$$\|G^{\{v\}}_{-1/2} v_{\nu,\ell_0}\|^2 = (-2(k+h^\vee)l_0 + (\nu|\nu+2\rho^\natural) - 2(k+1)(\xi|\nu) + 2(\xi|\nu)^2)\langle\phi(v), v\rangle. \tag{8.1}$$

Proof. To prove (8.1) we observe that, since $g(G^{\{v\}}) = G^{\{\phi(v)\}}$ and $G^{\{v\}}$ is primary,

$$\begin{aligned} H(G^{\{v\}}_{-1/2} v_{\nu,\ell_0}, G^{\{v\}}_{-1/2} v_{\nu,\ell_0}) &= H(G^{\{\phi(v)\}}_{1/2} G^{\{v\}}_{-1/2} v_{\nu,\ell_0}, v_{\nu,\ell_0}) \\ &= H([G^{\{\phi(v)\}}_{1/2}, G^{\{v\}}_{-1/2}] v_{\nu,\ell_0}, v_{\nu,\ell_0}). \end{aligned}$$

Using Borcherds' commutator formula

$$[G^{\{\phi(v)\}}_{1/2}, G^{\{v\}}_{-1/2}] = \sum_j \binom{1}{j} (G^{\{\phi(v)\}}{}_{(j)} G^{\{v\}})_0,$$

and formula (7.7) with $u = \phi(v)$ we obtain

$$\begin{aligned} [G^{\{\phi(v)\}}_{1/2}, G^{\{v\}}_{-1/2}] = & -2(k+h^\vee)\langle\phi(v), v\rangle L_0 + \langle\phi(v), v\rangle \sum_{\alpha=1}^{\dim \mathfrak{g}^\natural} :J^{\{u^\alpha\}} J^{\{u_\alpha\}}:_0 + \\ & 2\sum_{\alpha,\beta} \langle[u_\alpha, \phi(v)], [v, u^\beta]\rangle :J^{\{u^\alpha\}} J^{\{u_\beta\}}:_0 + 2(k+1) J_0^{\{[[e_\theta,\phi(v)],v]^\natural\}} \\ & + 2\sum_{\alpha,\beta} \langle[u_\alpha, \phi(v)], [v, u^\beta]\rangle J_0^{\{[u^\alpha, u_\beta]\}}. \end{aligned} \tag{8.2}$$

By the -1-st product identity,

$$:J^{\{u^\alpha\}} J^{\{u_\beta\}}:_0 = \sum_{j\in\mathbb{Z}_+} (J^{\{u^\alpha\}}_{-j-1} J^{\{u_\beta\}}_{j+1} + J^{\{u_\beta\}}_{-j} J^{\{u^\alpha\}}_j),$$

hence

$$:J^{\{u^\alpha\}} J^{\{u_\beta\}}:_0 v_{\nu,\ell_0} = J_0^{\{u_\beta\}} J_0^{\{u^\alpha\}} v_{\nu,\ell_0}. \tag{8.3}$$

We choose the basis $\{u_\alpha\}$ so that $\{u_\alpha\} = \{u_\gamma \mid u_\gamma \in \mathfrak{g}^\natural_\gamma\} \cup \{u_i \mid 1 \le i \le \operatorname{rank} \mathfrak{g}^\natural\}$ with $\{u_i\}$ a basis of $\mathfrak{h}^\natural$. Then $u^\gamma \in \mathfrak{g}^\natural_{-\gamma}$. It follows that

$$H(J_0^{\{u_\gamma\}} J_0^{\{u^{\gamma'}\}} v_{\nu,\ell_0}, v_{\nu,\ell_0}) \neq 0 \Rightarrow \gamma = \gamma'. \tag{8.4}$$

Since

$$[[e_\theta, \phi(v)], v]^\natural = \sum_{\gamma\in\Delta^\natural} ([[e_\theta, \phi(v)], v]|u_\gamma) u^\gamma + \sum_i ([[e_\theta, \phi(v)], v]|u_i) u^i,$$

we see that

$$\begin{aligned} &H(J_0^{\{[[e_\theta,\phi(v)],v]^\natural\}} v_{\nu,\ell_0}, v_{\nu,\ell_0}) \\ &\quad = \sum_i ([[e_\theta, \phi(v)], v]|u_i)\nu(u_i) = \sum_i (e_\theta|[\phi(v), [v, u_i]])\nu(u_i). \end{aligned} \tag{8.5}$$

We assume that $v \in \mathfrak{g}_\xi$. Then (8.5) yields

$$H(J_0^{\{[[e_\theta,\phi(v)],v]^\natural\}} v_{\nu,\ell_0}, v_{\nu,\ell_0}) = -\sum_i (e_\theta|[\phi(v), v])\xi(u_i)\nu(u^i) = -\langle\phi(v), v\rangle(\xi|\nu). \tag{8.6}$$

From (8.4) we see that $(J_0^{\{[u^{\gamma'},u_\gamma]\}} v_{\nu,\ell_0}, v_{\nu,\ell_0}) = 0$ unless $\gamma' = \gamma$. Clearly $J_0^{\{[u^i,u_j]\}} = 0$ for all i, j. Combining (8.2), (8.4), (8.6) we find

$$\begin{aligned} &H([G_{1/2}^{\{\phi(v)\}}, G_{-1/2}^{\{v\}}] v_{\nu,\ell_0}, v_{\nu,\ell_0}) \\ &\quad = -2(k+h^\vee)\langle\phi(v), v\rangle l_0 + \langle\phi(v), v\rangle(\nu|\nu+2\rho^\natural) - 2(k+1)\langle\phi(v), v\rangle(\xi|\nu) \\ &\qquad + 2\sum_{\alpha,\beta} \langle[u_\alpha, \phi(v)], [v, u^\beta]\rangle H(J_0^{\{u^\alpha\}} J_0^{\{u_\beta\}} v_{\nu,\ell_0}, v_{\nu,\ell_0}). \end{aligned} \tag{8.7}$$

Recall that ϕ is a compact involution of $\mathfrak{g}^\natural$, thus

$$\phi(h_\alpha) = -h_\alpha \text{ for all } \alpha \in \Delta^\natural. \tag{8.8}$$

(As usual h_α stands for the element of $\mathfrak{h}^\natural$ corresponding to α in the identification of $\mathfrak{h}^\natural$ with $(\mathfrak{h}^\natural)^*$ via $(.|.)$). It follows that $[h_\alpha, \phi(v)] = -\xi(h_\alpha)\phi(v)$, so the weight of $\phi(v)$ is $-\xi$. In particular, since v is a highest weight vector for the $\mathfrak{g}^\natural$-module $\mathfrak{g}_{-1/2}$, we have

$$\begin{aligned} &\sum_{\alpha,\beta} \langle[u_\alpha, \phi(v)], [v, u^\beta]\rangle H(J_0^{\{u^\alpha\}} J_0^{\{u_\beta\}} v_{\nu,\ell_0}, v_{\nu,\ell_0}) = \\ &\sum_{i,j} \langle[u_i, \phi(v)], [v, u^j]\rangle H(J_0^{\{u^i\}} J_0^{\{u_j\}} v_{\nu,\ell_0}, v_{\nu,\ell_0}) \\ &\qquad + \sum_{\gamma<0} \langle[u_\gamma, \phi(v)], [v, u^\gamma]\rangle H(J_0^{\{u^\gamma\}} J_0^{\{u_\gamma\}} v_{\nu,\ell_0}, v_{\nu,\ell_0}) \\ &\quad = \sum_{i,j} \xi(u_i)\nu(u^i)\xi(u^j)\nu(u_j)\langle\phi(v), v\rangle. \end{aligned} \tag{8.9}$$

Substituting (8.9) into (8.7) we obtain

$$\begin{aligned} H(G^{\{v\}}_{-1/2}v_{\nu,\ell_0}, G^{\{v\}}_{-1/2}v_{\nu,\ell_0}) =& -2(k+h^\vee)\langle\phi(v), v\rangle l_0 + \langle\phi(v), v\rangle(\nu|\nu+2\rho^\natural) \\ &- 2(k+1)\langle\phi(v), v\rangle(\xi|\nu) + 2(\xi|\nu)^2\langle\phi(v), v\rangle, \end{aligned}$$

as claimed. □

Remark 8.4. Let $v \in \mathfrak{g}_{-1/2}$ be as in Lemma 8.3 and u a root vector for the root θ_i. Then

$$\|J^{\{u\}}_{-1}G^{\{v\}}_{-1/2}v_{\nu,\ell_0}\|^2 = ((\theta_i|\xi+\nu)(\phi(u)|u) - \beta_k(\phi(u), u))\|G^{\{v\}}_{-1/2}v_{\nu,\ell_0}\|^2.$$

Indeed,

$$\begin{aligned} &H(J^{\{u\}}_{-1}G^{\{v\}}_{-1/2}v_{\nu,\ell_0}, J^{\{u\}}_{-1}G^{\{v\}}_{-1/2}v_{\nu,\ell_0}) \\ &= -H(G^{\{\phi(v)\}}_{1/2}J^{\{\phi(u)\}}_{1}J^{\{u\}}_{-1}G^{\{v\}}_{-1/2}v_{\nu,\ell_0}, v_{\nu,\ell_0}) \\ &= -H(G^{\{\phi(v)\}}_{1/2}[J^{\{\phi(u)\}}_{1}, J^{\{u\}}_{-1}]G^{\{v\}}_{-1/2}v_{\nu,\ell_0}, v_{\nu,\ell_0}) \\ &= (\theta_i|\xi+\nu)(\phi(u)|u)H(G^{\{v\}}_{-1/2}v_{\nu,\ell_0}, G^{\{v\}}_{-1/2}v_{\nu,\ell_0}) \\ &\quad - \beta_k(\phi(u), u)H(G^{\{v\}}_{-1/2}v_{\nu,\ell_0}, G^{\{v\}}_{-1/2}v_{\nu,\ell_0}). \end{aligned}$$

Let $P^+ \subset (\mathfrak{h}^\natural)^*$ be the set of dominant integral weights for $\mathfrak{g}^\natural$ and let

$$P^+_k = \left\{\nu \in P^+ \mid \nu(\theta_i^\vee) \le M_i(k) \text{ for all } i \ge 1\right\}. \tag{8.10}$$

Recall that $\xi \in (\mathfrak{h}^\natural)^*$ is a highest weight of the $\mathfrak{g}^\natural$-module $\mathfrak{g}_{-1/2}$. Introduce the following number

$$A(k,\nu) = \frac{(\nu|\nu+2\rho^\natural)}{2(k+h^\vee)} + \frac{(\xi|\nu)}{k+h^\vee}((\xi|\nu)-k-1). \tag{8.11}$$

Proposition 8.5. *Assume that $k+h^\vee \neq 0$. If the $W^k_{\min}(\mathfrak{g})$-module $L^W(\nu,\ell_0)$ is unitary, then $M_i(k) \in \mathbb{Z}_+$ for all $i \ge 1$, $\nu \in P^+_k$, and*

$$\ell_0 \ge A(k,\nu). \tag{8.12}$$

Proof. In order to prove that $M_i(k) \in \mathbb{Z}_+$ for all $i \ge 1$ and $\nu \in P^+_k$, it is enough to observe that, if $L^W(\nu,\ell_0)$ is a unitary module over $W^k_{\min}(\mathfrak{g})$, then, in particular, $V^{\beta_k}(\mathfrak{g}^\natural)v_{\nu,\ell_0}$ is a unitary module over $V^{\beta_k}(\mathfrak{g}^\natural)$, hence $\nu \in P^+_k$ [12], which is non-empty if and only if $M_i(k) \in \mathbb{Z}_+$ for all $i \ge 1$.

To prove the second claim recall that, by Proposition 5.1, the Hermitian form $\langle\phi(.), .\rangle$ is positive definite on $\mathfrak{g}_{-1/2}$. Since $k+h^\vee < 0$, we obtain from (8.1) that

$$\ell_0 \ge \frac{(\nu|\nu+2\rho^\natural)}{2(k+h^\vee)} - \frac{(k+1)}{k+h^\vee}(\xi|\nu) + \frac{(\xi|\nu)^2}{k+h^\vee} = A(k,\nu),$$

as claimed. □

Consider the short exact sequence

$$0 \to I^k \to W^k_{\min}(\mathfrak{g}) \to W_k^{\min}(\mathfrak{g}) \to 0.$$

If a $W^k_{\min}(\mathfrak{g})$-module $L^W(\nu, \ell_0)$ is unitary, then, restricted to the subalgebra $V^{\beta_k}(\mathfrak{g}^\natural)$ it is unitary, hence a direct sum of irreducible integrable highest weight $\widehat{\mathfrak{g}}^\natural$-modules of levels $M_i(k),\ i \geq 1$. But it is well known that all these modules descend to $V_{\beta_k}(\mathfrak{g}^\natural)$. Also, all these modules are annihilated by the elements

$$(J^{\{e_{\theta_i}\}}_{(-1)})^{M_i(k)+1}\mathbf{1}, \quad i \geq 1. \tag{8.13}$$

Let $\widetilde{I}^k \subset I^k$ be the ideal of $W^k_{\min}(\mathfrak{g})$ generated by the elements (8.13), and let $\widetilde{W}_k^{\min} = W^k_{\min}/\widetilde{I}^k$. We thus obtain

Proposition 8.6. *If the $W^k_{\min}(\mathfrak{g})$-module $L^W(\nu, \ell_0)$ is unitary, then it descends to $\widetilde{W}_k^{\min}(\mathfrak{g})$.*

Note that a unitary $W^k_{\min}(\mathfrak{g})$-module descends to $W_k^{\min}(\mathfrak{g})$ if and only if

$$\widetilde{I}^k = I^k. \tag{8.14}$$

Conjecture 4. [2] *Equality (8.14) holds for all unitary vertex algebras $W^k_{\min}(\mathfrak{g})$. Consequently, any unitary $W^k_{\min}(\mathfrak{g})$-module descends to $W_k^{\min}(\mathfrak{g})$.*

Definition 8.7. An element $\nu \in P^+_k$ is called an *extremal weight* if $\nu + \xi$ doesn't lie in P^+_k.

Proposition 8.8. *If $L^W(\nu, \ell_0)$ is unitary and ν is an extremal weight, then*

$$\ell_0 = A(k, \nu).$$

Proof. Let u be a root vector for ξ. Then $G^{\{u\}}_{-1/2}v_{\nu,\ell_0}$ is a singular vector for $V^{\beta_k}(\mathfrak{g}^\natural)$. Since $L^W(\nu, \ell_0)$ is unitary, all vectors that are singular for $V^{\beta_k}(\mathfrak{g}^\natural)$ should have weight in P^+_k. By the assumption, we have $G^{\{u\}}_{-1/2}v_{\nu,\ell_0} = 0$, hence the norm of this vector is 0, and we can apply (8.1). □

In the setting of the above proposition, note that ν is extremal iff $\nu(\theta_i^\vee) > M_i(k) + \chi_i$ for some i. Moreover, k is collapsing iff $M_i(k) + \chi_i < 0$ (cf. Remark 7.5).

Proposition 8.9. *(a) For $k \neq -1$, $W^k_{\min}(sl(2|m))$, $m \geq 3$, has no unitary highest weight modules. In particular, $W_k^{\min}(sl(2|m))$, $m \geq 3$, is unitary if and only if $k = -1$ and this W-algebra collapses to the free boson.*

(b) The W-algebra $W^k_{\min}(osp(4|m))$, $m > 2$, has no unitary highest weight modules for all k.

Proof. (a) Let $\mathfrak{g} = sl(2|m)$. Then $\mathfrak{g}_0^\natural = \mathbb{C}\varpi$, where $\varpi = \left(\begin{array}{cc|c} m/2 & 0 & 0 \\ 0 & m/2 & 0 \\ \hline 0 & 0 & I_m \end{array}\right)$, and $(a|b) = str(ab)$. By Theorem 7.4, the collapsing levels are $k = -1$ and $k = m/2 - 1$.

If $k = -1$ then $M_0(-1) = -m/2$, $M_1(-1) = 0$ and $W_k^{\min}(\mathfrak{g})$ is the Heisenberg vertex algebra $M(\mathbb{C}\varpi) = V^{-m/2}(\mathbb{C}\varpi) = V_{-m/2}(\mathbb{C}\varpi)$ and this vertex algebra is unitary.

[2] See Note added in proof.

If $k = m/2-1$ then $M_0(m/2-1) = 0$, $M_1(m/2-1) = -m/2$ and $W_k^{\min}(sl(2|m)) = V_{-m/2}(sl(m))$ which has no unitary highest weight modules.

Assume that k is not collapsing. Let ψ be a conjugate linear involution of $W^k_{\min}(sl(2|m))$ such that $L^W(\nu, \ell_0)$ has a positive definite ψ-invariant Hermitian form H, normalized by the condition $H(v_{\nu,\ell_0}, v_{\nu,\ell_0}) = 1$. By Proposition 7.2, the involution ψ is induced by an involution ψ on $\mathfrak{g}$ satisfying (1.1). This implies that $\psi(\varpi) = \zeta\varpi$ with $|\zeta| = 1$.

The vertex algebra $V^{k-m/2-1}(\mathbb{C}\varpi) \otimes V^{-k-1}(sl(m))$ embeds in $W^k_{\min}(sl(2|m))$. In particular, $(V^{k-m/2-1}(\mathbb{C}\varpi) \otimes V^{-k-1}(sl(m))).v_{\nu,\ell_0}$ is a unitary module. This implies that $\psi_{|sl(m)}$ corresponds to a compact real form of $sl(m)$ and $-k-1 \in \mathbb{Z}_+$. Using the formulas given in [16, §5.3] we have

$$\begin{aligned} 0 \le H(J^{\{\varpi\}}v_{\nu,\ell_0}, J^{\{\varpi\}}v_{\nu,\ell_0}) &= H(-J_1^{\{\psi(\varpi)\}}J_{-1}^{\{\varpi\}}v_{\nu,\ell_0}, v_{\nu,\ell_0}) \\ &= -(k-m/2-1)\zeta^{-1}(\varpi|\varpi) \\ &= -\zeta^{-1}(k-m/2-1)(m^2/2-m). \end{aligned}$$

Therefore $\zeta = 1$, so that

$$\psi(\varpi) = \varpi. \tag{8.15}$$

Note that

$$[\varpi, u] = \pm\tfrac{m}{2}u, \quad u \in \mathfrak{g}_{-1/2}. \tag{8.16}$$

Write $\mathfrak{g}_{-1/2} = \mathfrak{g}^+_{-1/2} \oplus \mathfrak{g}^-_{-1/2}$ for the corresponding eigenspace decomposition. Since $\psi(\varpi) = \varpi$, we have $\psi(\mathfrak{g}^\pm_{-1/2}) = \mathfrak{g}^\pm_{-1/2}$. Since the form $\langle .,\ .\rangle$ is $\mathfrak{g}^\natural$-invariant, we have

$$\langle \mathfrak{g}^+, \mathfrak{g}^+\rangle = \langle \mathfrak{g}^-, \mathfrak{g}^-\rangle = 0.$$

It follows that, if $u \in \mathfrak{g}_{-1/2}$,

$$\langle \psi(u), u\rangle = 0. \tag{8.17}$$

Observe now that by [1], since k is not collapsing, the image of $G^{\{u\}}$ in $W_k^{\min}(\mathfrak{g})$ is non-zero if $u \neq 0$. We observe that, since $g(G^{\{u\}}) = G^{\{\psi(u)\}}$ and $G^{\{u\}}$ is primary, for $n \in \frac{1}{2} + \mathbb{Z}_+$

$$\begin{aligned} H(G^{\{u\}}_{-n}v, G^{\{u\}}_{-n}v) &= H(G_n^{\{\psi(u)\}}G^{\{u\}}_{-n}, v) \\ &= H([G_n^{\{\psi(u)\}}, G^{\{u\}}_{-n}]v, v) \end{aligned} \tag{8.18}$$

for any $v \in L^W(\nu, \ell_0)$. Using Borcherds' commutator formula

$$[G_n^{\{\psi(u)\}}, G^{\{u\}}_{-n}] = \sum_j \binom{n+\frac{1}{2}}{j}(G^{\{\psi(u)\}}{}_{(j)}G^{\{u\}})_0,$$

and combining formulas (7.7) and (8.17), we obtain

$$[G_n^{\{\psi(u)\}}, G^{\{u\}}_{-n}] = -2(k+h^\vee)\langle\psi(u), u\rangle L_0 + \langle\psi(u), u\rangle \sum_{\alpha=1}^{\dim\mathfrak{g}^\natural} :J^{\{u^\alpha\}}J^{\{u_\alpha\}}:_0 +$$

$$
\begin{aligned}
&2\sum_{\alpha,\beta}\langle[u_\alpha,\psi(u)],[u,u^\beta]\rangle : J^{\{u^\alpha\}}J^{\{u_\beta\}} :_0 +4n(k+1)J_0^{\{[[e_\theta,\psi(u)],u]^\natural\}}\\
&\quad+(2n+1)\sum_{\alpha,\beta}\langle[u_\alpha,\psi(u)],[u,u^\beta]\rangle J_0^{\{[u^\alpha,u_\beta]\}}+(2n^2-\tfrac{1}{2})p(k)\langle\psi(u),u\rangle\\
&=2\sum_{\alpha,\beta}\langle[u_\alpha,\psi(u)],[u,u^\beta]\rangle : J^{\{u^\alpha\}}J^{\{u_\beta\}} :_0 +4n(k+1)J_0^{\{[[e_\theta,\psi(u)],u]^\natural\}}\\
&\quad+(2n+1)\sum_{\alpha,\beta}\langle[u_\alpha,\psi(u)],[u,u^\beta]\rangle J_0^{\{[u^\alpha,u_\beta]\}}. \qquad (8.19)
\end{aligned}
$$

Now we compute (8.18) for $v\in L^W(\nu,\ell_0)_{\ell_0}$. As in the proof of Lemma 8.3, using (8.4), (8.6) with ψ instead of ϕ, we find that (8.19) becomes, with the notation of the proof of Lemma 8.3,

$$
\begin{aligned}
H([G_n^{\{\psi(u)\}},G_{-n}^{\{u\}}]v,v)&=(2n+1)\sum_{\alpha,\beta}\langle[u_\alpha,\psi(u)],[u,u^\beta]\rangle H(J_0^{\{u^\alpha\}}J_0^{\{u_\beta\}}v,v)\\
&=(2n+1)\sum_{i,j}\langle[u_i,\psi(u)],[u,u^j]\rangle H(J_0^{\{u^i\}}J_0^{\{u_j\}}v,v) \quad (8.20)\\
&+(2n+1)\sum_{\gamma\in\Delta^\natural}\langle[u_\gamma,\psi(u)],[u,u^\gamma]\rangle H(J_0^{\{u^\gamma\}}J_0^{\{u_\gamma\}}v,v). \quad (8.21)
\end{aligned}
$$

Recall that ψ ia a compact involution of $[\mathfrak{g}^\natural,\mathfrak{g}^\natural]$, hence, by (8.8), $\psi(u_\gamma)\in\mathfrak{g}^\natural_{-\gamma}$, so that, for some constant b we have

$$\langle[u_\gamma,\psi(u)],[u,u^\gamma]\rangle=b\langle\psi([u,u^\gamma]),[u,u^\gamma]\rangle,$$

so, by (8.17), the summand (8.21) is zero.

The summand (8.20) vanishes since $\langle[u_i,\psi(u)],[u,u^j]\rangle$ is a multiple of $\langle\psi(u),u\rangle=0$. This shows that $Y^{L^W(\nu,\ell_0)}(G^{\{u\}},z)v=0$. By relation (7.3), $G_n^{\{u\}}A_mv=0$ with $A\in V^{\beta_k}(\mathfrak{g}^\natural)$ for all n,m, hence, since $G^{\{u\}}$ is primary,

$$Y^{L^W(\nu,\ell_0)}(G^{\{u\}},z)L^W(\nu,\ell_0)=0.$$

Hence $G^{\{u\}}$ lies in a proper ideal of $W^k_{\min}(\mathfrak{g})$, contradicting the fact that, since the level is not collapsing, $G^{\{u\}}$ is non zero in $W_k^{\min}(\mathfrak{g})$.

(b) For $\mathfrak{g}=osp(4|m)$, the conditions of Proposition 8.5 imply $k-m/2\in\mathbb{Z}_+$, $-\frac{1}{2}k-1\in\mathbb{Z}_+$. These relations are never satisfied at the same time. □

Proposition 8.10. *Non-trivial unitary irreducible highest weight $W^k_{\min}(\mathfrak{g})$-modules with $k\neq-h^\vee$ may exist only in the following cases*

(1) $\mathfrak{g}=sl(2|m)$, $m\geq3$, $k=-1$ *(then $W_k^{\min}=W^k_{\min}$ is a free boson);*
(2) $\mathfrak{g}=psl(2|2)$, $-k\in\mathbb{N}+1$*;*
(3) $\mathfrak{g}=spo(2|3)$, $-k\in\frac{1}{4}(\mathbb{N}+2)$*;*
(4) $\mathfrak{g}=spo(2|m)$, $m>4$, $-k\in\frac{1}{2}(\mathbb{N}+1)$*;*
(5) $\mathfrak{g}=D(2,1;\frac{m}{n})$, $-k\in\frac{mn}{m+n}\mathbb{N}$, $m,n\in\mathbb{N}$ *are coprime,* $k\neq-\frac{1}{2}$*;*
(6) $\mathfrak{g}=F(4)$, $-k\in\frac{2}{3}(\mathbb{N}+1)$*;*

(7) $\mathfrak{g} = G(3)$, $-k \in \frac{3}{4}(\mathbb{N}+1)$.

Proof. By Proposition 8.9, we may assume that $\mathfrak{g}$ is not one of the Lie superalgebras $sl(2|m)$ woth $m \geq 3$ or $osp(4|m)$ with $m > 2$. The remaining cases are treated, using only the easy necessary conditions $M_i = M_i(k) \in \mathbb{Z}_+$ for all i. In all cases, except for $\mathfrak{g} = D(2, 1; a)$, the condition $M_i \in \mathbb{Z}_+$ is obviously equivalent to the condition on k, given in the statement of the proposition.

Consider the remaining case $\mathfrak{g} = D(2, 1; a)$. By this we mean the contragredient Lie superalgebra with Cartan matrix $\begin{pmatrix} 0 & 1 & a \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$. By Proposition 8.5, we need to find the values of a such that $M_i = M_i(k)$, $i = 1, 2$, from Table 2 are non-negative integers. These conditions imply that

$$k = -\tfrac{M_1+1}{a+1} \text{ and } k = -\tfrac{(M_2+1)a}{a+1}. \text{ where } M_1, M_2 \in \mathbb{Z}_+. \tag{8.22}$$

Equating these two expressions for k, we obtain $a = \frac{M_1+1}{M_2+1}$ is a positive rational number. Inserting this in either of the expressions (8.22) for k, we obtain

$$k = -\tfrac{(M_1+1)(M_2+1)}{(M_1+1)+(M_2+1)},$$

proving the claim. ($k = -1/2$ corresponds to the trivial $D(2, 1; 1)$-module.) □

Definition 8.11. Given $\mathfrak{g}$ in the above list, we call the corresponding set of values of $k \neq -h^\vee$ the *unitarity range* of $W^k_{\min}(\mathfrak{g})$.

Remark 8.12. For $\mathfrak{g} = D(2, 1; a)$, there are actually three possible choices of the minimal root. We now describe how the unitarity range depends on this choice. We choose $\{2\epsilon_1, 2\epsilon_2, 2\epsilon_3\}$ as the set of positive roots in $\mathfrak{g}_{\bar{0}}$: hence, if $-\theta$ is a minimal root, then $\theta = 2\epsilon_i$ for some $1 \leq i \leq 3$. The bilinear form $(.|.)$, displayed in Table 1, corresponds to the choice $\theta = 2\epsilon_1$, so that $(2\epsilon_1|2\epsilon_1) = 2$. If we choose $\theta = 2\epsilon_2$, then the bilinear form $(.|.)$ is given by

$$(\epsilon_1|\epsilon_1) = -\tfrac{1+a}{2},\ (\epsilon_2|\epsilon_2) = \tfrac{1}{2},\ (\epsilon_3|\epsilon_3) = \tfrac{a}{2},\ (\epsilon_1|\epsilon_2) = (\epsilon_1|\epsilon_3) = (\epsilon_2|\epsilon_3) = 0.$$

We have $M_1(k) = -\frac{1}{1+a}k - 1$, $M_2(k) = \frac{1}{a}k - 1$. Then $a = -\frac{m}{m+n}$, $m, n \in \mathbb{N}$, m and n are coprime (i. e. $a \in \mathbb{Q}$, $-1 < a < 0$) and in turn $k \in -\frac{mn}{m+n}\mathbb{N}$. If we choose $\theta = 2\epsilon_3$, then the bilinear form $(.|.)$ is given by

$$(\epsilon_1|\epsilon_1) = -\tfrac{a+1}{2a},\ (\epsilon_2|\epsilon_2) = \tfrac{1}{2a},\ (\epsilon_3|\epsilon_3) = \tfrac{1}{2},\ (\epsilon_1|\epsilon_2) = (\epsilon_1|\epsilon_3) = (\epsilon_2|\epsilon_3) = 0.$$

We have $M_1(k) = -\frac{a}{1+a}k - 1$, $M_2(k) = ak - 1$. Then $a = -\frac{m+n}{m}$, $m, n \in \mathbb{N}$, m and n are coprime (i. e. $a \in \mathbb{Q}$, $a < -1$) and in turn $k \in -\frac{mn}{m+n}\mathbb{N}$.

Recall that one obtains isomorphic superalgebras of the family $D(2, 1; a)$, $a \neq 0, -1$, under the action of the group S_3, generated by the transformations $a \mapsto 1/a$, $a \mapsto -1-a$. These transformations permute transitively the domains $\mathbb{Q}_{>0}$, $\mathbb{Q}_{>-1} \cap \mathbb{Q}_{<0}$ and $\mathbb{Q}_{<-1}$, which correspond to the above three cases.

Corollary 8.13. *If k is from the unitarity range for $W^k_{\min}(\mathfrak{g})$, then $k + h^\vee$ is a negative rational number.*

9. Free Field Realization of Minimal W-Algebras

Let $\Psi : W^k_{\min}(\mathfrak{g}) \to \mathcal{V}^k = V^{k+h^\vee}(\mathbb{C}x) \otimes V^{\alpha_k}(\mathfrak{g}^\natural) \otimes F(\mathfrak{g}_{1/2})$ be the free field realization introduced in [20, Theorem 5.2]; it is explicitly given on the generators of $W^k_{\min}(\mathfrak{g})$ by

$$J^{\{b\}} \mapsto b + \frac{1}{2} \sum_{\alpha \in S_{1/2}} : \Phi^\alpha \Phi_{[u_\alpha, b]} : (b \in \mathfrak{g}^\natural), \tag{9.1}$$

$$G^{\{v\}} \mapsto \sum_{\alpha \in S_{1/2}} : [v, u_\alpha]\Phi^\alpha : -(k+1) \sum_{\alpha \in S_{1/2}} (v|u_\alpha) T\Phi^\alpha$$
$$+ \frac{1}{3} \sum_{\alpha,\beta \in S_{1/2}} : \Phi^\alpha \Phi^\beta \Phi_{[u_\beta,[u_\alpha,v]]} : (v \in \mathfrak{g}_{-1/2}), \tag{9.2}$$

$$L \mapsto \frac{1}{2(k+h^\vee)} \sum_{\alpha \in S_0} : u_\alpha u^\alpha : + \frac{k+1}{k+h^\vee} Tx + \frac{1}{2} \sum_{\alpha \in S_{1/2}} : (T\Phi^\alpha)\Phi_\alpha : . \tag{9.3}$$

Recall that $F(\mathfrak{g}_{1/2})$ is the universal enveloping vertex algebra of the (non-linear) Lie conformal superalgebra $\mathbb{C}[T] \otimes \mathfrak{g}_{1/2}$ with $[a_\lambda b] = \langle a, b\rangle_{ne}\mathbf{1}$, $a, b \in \mathfrak{g}_{1/2}$, and $\{\Phi_\alpha\}_{\alpha \in S_{1/2}}$, $\{\Phi^\alpha\}_{\alpha \in S_{1/2}}$ are dual bases of $\mathfrak{g}_{1/2}$ with respect to $\langle ., .\rangle_{ne}$.

We now apply the results of Sect. 6 to $V^{k+h^\vee}(\mathbb{C}x)$. By Corollary 8.13 unitarity of $W^k_{\min}(\mathfrak{g})$ implies $k + h^\vee < 0$. Hence, using the normalization

$$a = \sqrt{-1}\frac{\sqrt{2}}{\sqrt{|k+h^\vee|}}x, \tag{9.4}$$

we have $V^{k+h^\vee}(\mathbb{C}x) = V^1(\mathbb{C}a)$, since, by (7.24), $\alpha_k(x, x) = \frac{1}{2}(k+h^\vee)$, hence $\alpha_k(a, a) = 1$

Recall that in Proposition 5.1 we proved that one can choose an almost compact involution ϕ of $\mathfrak{g}$ that fixes pointwise the sl_2-triple $\{e, x, f\}$ in such a way that the Hermitian form $\langle \phi(u), v\rangle_{ne}$ on $\mathfrak{g}_{1/2}$ is negative definite. This conjugate linear involution induces a conjugate linear involution of $W^k_{\min}(\mathfrak{g})$ and of $V^{\alpha_k}(\mathfrak{g}_0) \otimes F(\mathfrak{g}_{1/2})$ as well, both denoted again by ϕ. It is readily checked, using (9.1), (9.2), and (9.3), that

$$\Psi(\phi(v)) = \phi(\Psi(v)) \quad \text{for all } v \in W^k_{\min}(\mathfrak{g}). \tag{9.5}$$

Since $\phi(x) = x$, we see that $\phi(a) = -a$. The conformal vector of the vertex algebra $\mathcal{V}^k$ is

$$L_{free} = \frac{1}{2} : aa : + L_{\mathfrak{g}^\natural} + L_F, \tag{9.6}$$

where

$$L_{\mathfrak{g}^\natural} = \tfrac{1}{2(k+h^\vee)} \sum_{\alpha \in S^\natural} : u_\alpha u^\alpha :, \quad L_F = \tfrac{1}{2} \sum_{\alpha \in S_{1/2}} : (T\Phi^\alpha)\Phi_\alpha : .$$

Here $\{u_\alpha\}_{\alpha \in S^\natural}$ and $\{u^\alpha\}_{\alpha \in S^\natural}$ are dual bases of $\mathfrak{g}^\natural$ with respect to the bilinear form $(.|.)$ restricted to $\mathfrak{g}^\natural$. Recall that $L_{\mathfrak{g}^\natural}$ is the conformal vector of $V^{\alpha_k}(\mathfrak{g}^\natural)$ and L_F is the conformal vector of $F(\mathfrak{g}_{1/2})$. Let

$$s_k = \sqrt{-1}\frac{(k+1)}{\sqrt{2|k+h^\vee|}}. \tag{9.7}$$

It follows from (9.3) and (9.6) that

$$\Psi(L) = L(s_k) + \widehat{L} = L_{free} + s_k T(a), \tag{9.8}$$

where $\widehat{L} = L_{\mathfrak{g}^\natural} + L_F$, and $L(s) = \frac{1}{2} :aa: + sTa$, $\widehat{L}(s) = L(s) + \widehat{L}$, cf. (6.1) and (6.29), respectively.

Note that $\mathcal{V}^k = V^1(\mathbb{C}a) \otimes V$, where $V = V^{\alpha_k}(\mathfrak{g}^\natural) \otimes F(\mathfrak{g}_{1/2})$, and $\Psi(L) = \widehat{L}(s)$ (cf. (6.28), (6.29)).

Given $\mu \in \mathbb{C}$, let $M(\mu)$ be the irreducible $V^1(\mathbb{C}a)$-module with highest weight μ, and consider the $\mathcal{V}^k$-module

$$N(\mu) = M(\mu) \otimes V.$$

Recall that V carries a ϕ-invariant Hermitian form $H_{\mathfrak{g}^\natural} \otimes H_F$, which is positive definite. Recall also that, by Proposition 6.3, the $V^1(\mathbb{C}a)$-module $M(\mu)$ carries a unique $L(t)$-invariant Hermitian form, provided that $t = \sqrt{-1}\Im(\mu)$, which is positive definite. This Hermitian form, normalized by the condition that the norm of the highest weight vector equals 1, was denoted by H_μ. Hence we have a ϕ-invariant positive definite Hermitian form $H_\mu(\,.\,,\,.\,) \otimes H_{\mathfrak{g}^\natural}(\,.\,,\,.\,) \otimes H_F(\,.\,,\,.\,)$ on $N(\mu)$, which we denote by $(\cdot,\cdot)_\mu$.

It follows from Proposition 6.10 that, restricting the fields $Y^{\mu,t}(-,z)$ from $\mathcal{V}^k$ to $\Psi(W^k_{\min}(\mathfrak{g}))$, one equips $N(\mu)$ with a structure of a $W^k_{\min}(\mathfrak{g})$-module. We now explicitly describe this action of the generators of $W^k_{\min}(\mathfrak{g})$ on $N(\mu)$.

Proposition 9.1. *For $b \in W^k_{\min}(\mathfrak{g})$, write*

$$Y^{\mu,t}(\Psi(b),z) = \sum_{n\in -\Delta_b+\mathbb{Z}} b_n^{\mu,t} z^{-n-\Delta_b},$$

and let $\mu \in \mathbb{R}$. Then

$$L_n^{\mu,t} = \Psi(L)_n^\mu + 2ta_n^\mu + 2(t^2 - st)\mathbf{1}_n^\mu, \tag{9.9}$$

$$(J^{\{u\}})_n^{\mu,t} = \Psi(J^{\{u\}})_n^\mu, \ \ u \in \mathfrak{g}^\natural, \tag{9.10}$$

$$(G^{\{v\}})_n^{\mu,t} = \Psi(G^{\{v\}})_n^\mu + 2t\sqrt{-1}\sqrt{2|k+h^\vee|}(\Phi_{[e,v]})_n^\mu, \ \ v \in \mathfrak{g}_{-1/2}. \tag{9.11}$$

Furthermore, if $m, m' \in N(\mu)$, then

$$(m, L_n^{\mu,t} m')_\mu = (L_{-n}^{\mu,s-t} m, m')_\mu, \tag{9.12}$$

$$(m, (J^{\{u\}})_n^{\mu,t} m')_\mu = -((J^{\{\phi(u)\}})_{-n}^{\mu,s-t} m, m')_\mu, \tag{9.13}$$

$$(m, (G^{\{v\}})_n^{\mu,t} m')_\mu = ((G^{\{\phi(v)\}})_{-n}^{\mu,s-t} m, m')_\mu. \tag{9.14}$$

Proof. We already noted that $\Psi(L) = \widehat{L}(s)$. By (6.27),

$$\begin{aligned}\Psi(L)_n^{\mu,t} &= (L(s) + \widehat{L})_n^{\mu,t} = \tfrac{1}{2} :aa:_n^\mu + sTa_n^\mu + 2ta_n^\mu + 2(t^2 - st)\mathbf{1}_n + \widehat{L}_n^\mu \\ &= \widehat{L}(s)_n^\mu + 2ta_n^\mu + 2(t^2 - st)\mathbf{1}_n^\mu.\end{aligned}$$

If $u \in \mathfrak{g}^\natural$, then $\Psi(J^{\{u\}}) \in V^{\alpha_k}(\mathfrak{g}^\natural) \otimes F(\mathfrak{g}_{1/2})$, hence, by Lemma 6.11, $(J^{\{u\}})_n^{\mu,t} = \Psi(J^{\{u\}})_n^\mu$. Finally, if $v \in \mathfrak{g}_{-1/2}$,

$$[v, u_\alpha] = 2([v,u_\alpha]|x)x + [v,u_\alpha]^\natural = \sqrt{-1}\frac{\sqrt{|k+h^\vee|}}{\sqrt{2}}(v|u_\alpha)a + [v,u_\alpha]^\natural,$$

where $u^\natural$ is the orthogonal projection of u onto $\mathfrak{g}^\natural$ with respect to $(\,.\,|\,.\,)$. Since

$$[e, v] = \sum_\alpha \langle [e, v], u_\alpha\rangle_{ne} u^\alpha = \sum_\alpha (f|[[e, v], u_\alpha])u^\alpha = \sum_\alpha (f|[e, [v, u_\alpha]])u^\alpha$$
$$= \sum_\alpha ([[f, e], v]|u_\alpha)u^\alpha = -\sum_\alpha ([x, v]|u_\alpha)u^\alpha = \tfrac{1}{2}\sum_\alpha (v|u_\alpha)u^\alpha,$$

we can write

$$\Psi(G^{\{v\}}) = \sqrt{-1}\sqrt{2|k+h^\vee|} : a\Phi_{[e,v]} : + \sum_{\alpha\in S_{1/2}} : [v, u_\alpha]^\natural \Phi^\alpha : -2(k+1)T\Phi_{[e,v]}$$
$$+ \frac{1}{3}\sum_{\alpha,\beta\in S_{1/2}} : \Phi^\alpha\Phi^\beta\Phi_{[u_\beta,[u_\alpha,v]]} : .$$

Set

$$G^{\{v\}} = \sum_{\alpha\in S_{1/2}} : [v, u_\alpha]^\natural \Phi^\alpha : + \frac{1}{3}\sum_{\alpha,\beta\in S_{1/2}} : \Phi^\alpha\Phi^\beta\Phi_{[u_\beta,[u_\alpha,v]]} : .$$

so that

$$(G^{\{v\}})^{\mu,t}_n = \sqrt{-1}\sqrt{2|k+h^\vee|} : a\Phi_{[e,v]} :^\mu_n + 2t\sqrt{-1}\sqrt{2|k+h^\vee|}(\Phi_{[e,v]})^\mu_n$$
$$- 2(k+1)(T\Phi_{[e,v]})^\mu_n + {G^{\{v\}}}^\mu_n.$$

Thus,

$$(G^{\{v\}})^{\mu,t}_n = \Psi(G^{\{v\}})^\mu_n + 2t\sqrt{-1}\sqrt{2|k+h^\vee|}(\Phi_{[e,v]})^\mu_n. \tag{9.15}$$

For proving (9.12), (9.13), and (9.14), it is enough to observe that L, $G^{\{v\}}$, and $J^{\{u\}}$ are quasiprimary for $\widehat{L}(s)$ and apply (6.24). We use the fact that $\Psi(g(b)) = g(\Psi(b))$ for all $b \in W^k_{\min}(\mathfrak{g})$, where g is defined by (2.8). This follows from (9.5) and the fact that Ψ preserves both parity and conformal weight. □

As an application of Proposition 9.1, we obtain a generalization of the Fairlie construction to minimal W-algebras.

Proposition 9.2. *Set $s = s_k$ (cf. (9.7)) and*

$$L^{\mu,s/2}_n = \Psi(L)^\mu_n + sa^\mu_n + \tfrac{|s|^2}{2}\mathbf{1}^\mu_n = \Psi(L)^\mu_n + \frac{k+1}{k+h^\vee}x^\mu_n - \tfrac{(k+1)^2}{4(k+h^\vee)}\mathbf{1}^\mu_n,$$
$$(G^{\{v\}})^{\mu,s/2}_n = \Psi(G^{\{v\}})^\mu_n - (k+1)(\Phi_{[e,v]})^\mu_n,$$
$$(J^{\{u\}})^{\mu,s/2}_n = \Psi(J^{\{u\}})^\mu_n.$$

The fields

$$Y^{\mu,s}(L, z) = \sum_{n\in\mathbb{Z}} L^{\mu,s}_n z^{-n-2},$$
$$Y^{\mu,s}(G^{\{v\}}, z) = \sum_{n\in 1/2+\mathbb{Z}} (G^{\{v\}})^{\mu,s}_n z^{-n-3/2},$$

$$Y^{\mu,s}(J^{\{u\}},z)=\sum_{n\in\mathbb{Z}}(J^{\{u\}})^{\mu,s}_n z^{-n-1}$$

endow $N(\mu)$ with a $W^k_{\min}(\mathfrak{g})$-module structure. Moreover, the Hermitian form $(\,.\,,\,.\,)_\mu$ on $N(\mu)$ is invariant.

Proof. Plug $t=s/2$ in Proposition 9.1. By (9.12), (9.13), and (9.14), we have

$$(m,L^{\mu,s/2}_n m')_\mu=(L^{\mu,s/2}_{-n}m,m')_\mu,$$
$$(m,(J^{\{u\}})^{\mu,s/2}_n m')_\mu=-((J^{\{\phi(u)\}})^{\mu,s/2}_{-n}m,m')_\mu,$$
$$(m,(G^{\{v\}})^{\mu,s/2}_n m')_\mu=((G^{\{\phi(v)\}})^{\mu,s/2}_{-n}m,m')_\mu.$$

thus the representations $N(\mu)$ acquire a $W^k_{\min}(\mathfrak{g})$-module structure and the Hermitian form $(\cdot,\cdot)_\mu$ is ϕ-invariant. □

10. Sufficient Conditions for Unitarity of Modules Over $W^k_{\min}(\mathfrak{g})$

Due to the Proposition 8.9 (a), we may assume in this section that $\mathfrak{g}\neq sl(2|m)$ and $osp(4|m)$, $m>2$. Then, in particular, $\mathfrak{g}^\natural=\oplus_{i\geq1}\mathfrak{g}^\natural_i$ is the decomposition of $\mathfrak{g}^\natural$ into simple ideals, and the χ_i are given by (7.27).

Proposition 10.1. *Assume that $k+h^\vee\neq0$. Then there exists a unitary module $L^W(\nu,\ell_0)$ over $W^k_{\min}(\mathfrak{g})$ if and only if $M_i(k)\in\mathbb{Z}_+$ for all i and $\nu\in P^+_k$.*

Proof. One implication has been already proven in Proposition 8.5. To show that the converse implication also holds, assume $M_i(k)\in\mathbb{Z}_+$ for all i. Recall (see (7.25)) that the cocycle α_k is given by

$$\alpha_{k\,|\mathfrak{g}^\natural_i\times\mathfrak{g}^\natural_i}=(M_i(k)+\chi_i)(.|.)^\natural_i.$$

Assume first that $M_i(k)+\chi_i\in\mathbb{Z}_+$ for all i. Then the simple quotient $V_{\alpha_k}(\mathfrak{g}^\natural)$ of $V^{\alpha_k}(\mathfrak{g}^\natural)$ is unitary, since it is an integrable $\widehat{\mathfrak{g}}^\natural$-module [11]. Next, the vertex algebra $F(\mathfrak{g}_{1/2})$ is unitary due to Proposition 5.1 and [16, §5.1]. Finally, the $V^1(\mathbb{C}a)$-module $M(s)$, where s is given by (9.7), is unitary by the observation following Lemma 6.4.

Consider the unitary $W^k_{\min}(\mathfrak{g})$-module $M(s)\otimes V_{\alpha_k}(\mathfrak{g}^\natural)\otimes F(\mathfrak{g}_{1/2})$, and its submodule

$$U=\Psi(W^k_{\min}(\mathfrak{g})).(v_s\otimes\mathbf{1}\otimes\mathbf{1}).$$

Since the Hermitian form $H_s(\,.\,,\,.)$ is $\widehat{L}(s)$-invariant and $\Psi(L)=\widehat{L}(s)$, we see that U admits a ϕ-invariant Hermitian positive definite form, thus U is a unitary highest weight module for $W^k_{\min}(\mathfrak{g})$.

Now we look at the missing cases, where there is i such that $0\leq M_i(k)<-\chi_i$, described in Remark 7.5. Assume first that $\mathfrak{g}^\natural$ is simple. If $\chi_1=-1$ then the only possible value is $M_1(k)=0$, so, $W_k^{\min}(\mathfrak{g})=\mathbb{C}$, by Theorem 7.4 (1) (a). In the case of $\mathfrak{g}=spo(2|3)$ one should consider the cases $M_1(k)=1$ and $M_1(k)=0$: in the former case $k=-\frac{h_1^\vee}{2}-1$, hence Theorem 7.4 (1) (b) applies and $W_k^{\min}(spo(2|3))=V_1(sl(2))$, whereas in the latter case $k+h^\vee=0$. If $\mathfrak{g}^\natural$ is semisimple but not simple, then $\mathfrak{g}=D(2,1;a)$. In this case we have to consider only the case in which either $M_1(k)$ or $M_2(k)$ is zero. If $M_1(k)=0$ (resp. $M_2(k)=0$) then, by Theorem 7.4 (2), $W_k^{\min}(D(2,1;a))=V_{M_2(k)}(sl(2))$ (resp. $=V_{M_1(k)}(sl(2))$). □

We now generalize the construction given in the proof of Proposition 10.1 to provide families of unitary representations. For $\nu \in P_k^+$ introduce the following number

$$B(k,\nu) = \frac{(\nu|\nu+2\rho^\natural)}{2(k+h^\vee)} - \frac{(k+1)^2}{4(k+h^\vee)}. \tag{10.1}$$

Proposition 10.2. *Assume that $k+h^\vee \neq 0$ and $M_i(k)+\chi_i \in \mathbb{Z}_+$ for all $i > 0$. If $\nu \in P^+$ is such that $\nu(\theta_i^\vee) \leq M_i(k)+\chi_i$ for all $i > 0$ (then $\nu \in P_k^+$) and*

$$\ell_0 \geq B(k,\nu), \tag{10.2}$$

then $L^W(\nu,\ell_0)$ is a unitary $W^k_{\min}(\mathfrak{g})$-module.

Proof. Let $L^\natural(\nu)$ be the irreducible highest weight $V^{\alpha_k}(\mathfrak{g}^\natural)$-module of highest weight ν and let v_ν be a highest weight vector. Fix $\mu \in \mathbb{R}$ and set

$$N(\mu,\nu) = \Psi(W^k_{\min}(\mathfrak{g})).(v_{\mu+s} \otimes v_\nu \otimes \mathbf{1}) \subset M(\mu+s) \otimes L^\natural(\nu) \otimes F(\mathfrak{g}_{1/2}),$$

where $s = s_k$ is given by formula (9.7). Note that the Hermitian form $(\cdot,\cdot)_{\mu+s}$ is $\widehat{L}(s)$-invariant. Since $M_i(k)+\chi_i \in \mathbb{Z}_+$ and $\nu(\theta_i^\vee) \leq M_i(k)+\chi_i$ for all i, then $L^\natural(\nu)$ is integrable for $V^{\alpha_k}(\mathfrak{g}^\natural)$, hence unitary [12]. Thus $N(\mu,\nu)$ is a unitary representation of $W^k_{\min}(\mathfrak{g})$.

We now compute the highest weight of $N(\mu,\nu)$. Recall that

$$\Psi(J^{\{h\}}) = h + \frac{1}{2}\sum_{\alpha\in S_{1/2}} :\Phi^\alpha \Phi_{[u_\alpha,h]}: .$$

By the -1-st product identity,

$$:\Phi^\alpha \Phi_{[u_\alpha,h]}:_0 = \sum_{j\in\frac{1}{2}+\mathbb{Z}_+} \left(\Phi^\alpha_{-j}(\Phi_{[u_\alpha,h]})_j - (\Phi_{[u_\alpha,h]})_{-j}\Phi^\alpha_j\right)$$

so

$$\Psi(J^{\{h\}})_0.(v_{\mu+s} \otimes v_\nu \otimes \mathbf{1}) = \nu(h)(v_{\mu+s} \otimes v_\nu \otimes \mathbf{1}).$$

It follows that $N(\mu,\nu) = L^W(\nu,\ell_0)$ for some ℓ_0. We now compute ℓ_0:

$$L_0(v_{\mu+s} \otimes v_\nu \otimes \mathbf{1}) = \left(\frac{\mu^2 - s^2}{2} + \frac{(\nu|\nu+2\rho^\natural)}{2(k+h^\vee)}\right)(v_{\mu+s} \otimes v_\nu \otimes \mathbf{1})$$

so that, using (9.7),

$$\ell_0 = \frac{\mu^2 - s^2}{2} + \frac{(\nu|\nu+2\rho^\natural)}{2(k+h^\vee)} = \frac{\mu^2}{2} - \frac{(k+1)^2}{4(k+h^\vee)} + \frac{(\nu|\nu+2\rho^\natural)}{2(k+h^\vee)}.$$

Hence $\ell_0 \geq B(k,\nu)$. Letting $\mu = 2\sqrt{\ell_0 - B(k,\nu)}$, we see that the module $L^W(\nu,\ell_0) = N(\mu,\nu)$ is unitary. $\square$

11. Unitarity of Minimal W-Algebras and Modules Over Them

The main result of this paper is the following.

Theorem 11.1. *Let $k \neq -h^\vee$, and recall the number $A(k,\nu)$ given by (8.11). If k lies in the unitary range (hence $M_i(k) \in \mathbb{Z}_+$ for $i \geq 1$), then the $W^k_{\min}(\mathfrak{g})$-module $L^W(\nu, \ell_0)$ is unitary for all non extremal $\nu \in \widehat{P}^+_k$ and $\ell_0 \geq A(k,\nu)$.*

Corollary 11.2. *If k lies in the unitary range, then the $W^k_{\min}(\mathfrak{g})$-module $L^W(0, \ell_0)$ is unitary for all $\ell_0 \geq 0$. Consequently, $W_k^{\min}(\mathfrak{g})$ is a unitary vertex algebra if and only if k lies in the unitary range.*

In the rest of this section we give a proof of these results. First, by Proposition 8.9 (a), we may exclude $\mathfrak{g} = sl(2|m),\ m > 2$, from consideration, so that $\mathfrak{g}^\natural$ is semisimple and by Proposition 8.5, conditions $M_i(k) \in \mathbb{Z}_+$ are necessary for unitarity, hence we shall assume that these conditions hold.

Let $\widehat{\mathfrak{g}} = (\mathbb{C}[t,t^{-1}] \otimes \mathfrak{g}) \oplus \mathbb{C}K \oplus \mathbb{C}d$ be the affinization of $\mathfrak{g}$ (with bracket $[t^m \otimes a, t^n \otimes b] = t^{n+m} \otimes [a,b] + \delta_{m,-n} m K(a|b),\ a, b \in \mathfrak{g}$). Let $\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}d$ be its Cartan subalgebra. Define Λ_0 and $\delta \in \widehat{\mathfrak{h}}^*$ setting $\Lambda_0(\mathfrak{h}) = \Lambda_0(d) = \delta(\mathfrak{h}) = \delta(K) = 0$ and $\Lambda_0(K) = \delta(d) = 1$. Let $\widehat{\Delta} \subset \widehat{\mathfrak{h}}^*$ be the set of roots of $\widehat{\mathfrak{g}}$. As a subset of simple roots for $\widehat{\mathfrak{g}}$ we choose $\widehat{\Pi} = \{\alpha_0 = \delta - \theta\} \cup \Pi$, where Π is the set of simple roots for $\mathfrak{g}$ given in Table 1. We denote by $\widehat{\Delta}^+$ the corresponding set of positive roots and by $\widehat{\rho} \in \widehat{\mathfrak{h}}^*$ the corresponding ρ-vector.

For $\nu \in P^+_k$ and $h \in \mathbb{C}$, set

$$\widehat{\nu}_h = k\Lambda_0 + \nu + h\theta \in \widehat{\mathfrak{h}}^*. \tag{11.1}$$

Let $\widehat{\mathfrak{p}}$ be the parabolic subalgebra of $\widehat{\mathfrak{g}}$ with Levi factor $\widehat{\mathfrak{h}} + \mathfrak{g}^\natural$ and the nilradical $\widehat{\mathfrak{u}}_+ = \sum_{\alpha \in \widehat{\Delta}^+ \setminus \Delta^\natural} \widehat{\mathfrak{g}}_\alpha$. Set $\widehat{\mathfrak{u}}_- = \sum_{\alpha \in \widehat{\Delta}^+ \setminus \Delta^\natural} \widehat{\mathfrak{g}}_{-\alpha}$. Let $V^\natural(\nu)$ denote the irreducible $\mathfrak{g}^\natural$-module with highest weight ν and extend the $\mathfrak{g}^\natural$ action to $\widehat{\mathfrak{p}}$ by letting $\widehat{\mathfrak{u}}_+$ act trivially; x, K, and d act by h, k, and 0 respectively. Let $M^\natural(\widehat{\nu}_h)$ be the corresponding generalized Verma module for $\widehat{\mathfrak{g}}$, i.e.

$$M^\natural(\widehat{\nu}_h) = U(\widehat{\mathfrak{g}}) \otimes_{U(\widehat{\mathfrak{p}})} V^\natural(\nu).$$

We denote by $v_{\widehat{\nu}_h}$ a highest weight vector for $M^\natural(\widehat{\nu}_h)$. If $\widehat{\mu} \in \widehat{\mathfrak{h}}^*$ and M is a $\widehat{\mathfrak{g}}$-module, we denote by $M_{\widehat{\mu}}$ the corresponding weight space. Let $\eta_i = \delta - \theta_i$, $1 \leq i \leq s$ (recall that $s = 1$ or 2).

If $\alpha \in \widehat{\Delta}$ is a non-isotropic root, denote by $s_\alpha \in End(\widehat{\mathfrak{h}}^*)$ the corresponding reflection and the group generated by them by $\widehat{W}$. If $\beta \in \widehat{\Delta} \setminus \mathbb{Z}\delta$ is an odd isotropic root, we let r_β denote the corresponding odd reflection. We denote by x_α a root vector attached to $\alpha \in \widehat{\Delta}$. Denote by $w.$ the shifted action of $\widehat{W}$: $w.\lambda = w(\lambda + \widehat{\rho}) - \widehat{\rho}$.

Lemma 11.3. *Let $\widehat{\Pi}'$ be a set of simple roots for $\widehat{\Delta}$. Let M be a $\widehat{\mathfrak{g}}$-module and assume that $m \in M$ is a singular vector with respect to $\widehat{\Pi}'$. If $\alpha_j \in \widehat{\Pi}'$ is an isotropic root and $x_{-\alpha_j} m \neq 0$, then $x_{-\alpha_j} m$ is a singular vector with respect to $r_{\alpha_j}(\widehat{\Pi}')$.*

Proof. Since α_j is odd isotropic, it follows that $x^2_{-\alpha_j} m = 0$. If $r \neq j$ and $(\alpha_r|\alpha_j) = 0$ then $x_{\alpha_r} x_{-\alpha_j} m = x_{-\alpha_j} x_{\alpha_r} m = 0$. If $r \neq j$ and $(\alpha_r|\alpha_j) \neq 0$ then $x_{\alpha_r+\alpha_j} x_{-\alpha_j} m = x_{-\alpha_j} x_{\alpha_r+\alpha_j} m + x_{\alpha_r} m = 0$. □

For $\nu \in P_k^+$ set

$$N_i(k,\nu) = (\widehat{\nu}_h + \widehat{\rho}|\eta_i^\vee). \tag{11.2}$$

Note that $N_i(k,\nu)$ does not depend on h. We will simply write N_i when the dependence on k and ν is clear from the context.

Lemma 11.4. *For $\nu \in P_k^+$ not extremal, we have*

$$N_i(k,\nu) = M_i(k) + \chi_i + 1 - (\nu|\theta_i^\vee) \in \mathbb{N}. \tag{11.3}$$

Moreover, for

$$v_i(h) := x_{-\eta_i}^{N_i} x_{-\alpha_0-\alpha_1} x_{-\alpha_1} v_{\widehat{\nu}_h},$$

the subspace $\sum_i U(\widehat{\mathfrak{g}})v_i(h)$ is a proper submodule of the $\widehat{\mathfrak{g}}$-module $M^\natural(\widehat{\nu}_h)$.

Proof. Note that

$$\begin{aligned}(\widehat{\nu}_h + \widehat{\rho}|\eta_i^\vee) &= \frac{2}{(\theta_i|\theta_i)}(k+h^\vee) - (\nu+\rho|\theta_i^\vee) = \frac{2}{(\theta_i|\theta_i)}(k + \frac{h^\vee - \bar{h}_i^\vee}{2} \\ &\quad + \frac{h^\vee + \bar{h}_i^\vee}{2} - (\nu + \rho^\natural|\theta_i)) \\ &= M_i(k) + \frac{2}{(\theta_i|\theta_i)}(\frac{h^\vee - \bar{h}_i^\vee}{2} + \frac{(\theta_i|\theta_i)}{2}) - (\nu|\theta_i^\vee) \\ &= M_i(k) + \chi_i + 1 - (\nu|\theta_i^\vee).\end{aligned}$$

Since ν is not extremal, $(\widehat{\nu}_h + \widehat{\rho}|\eta_i^\vee) \in \mathbb{N}$.

Recall from Table 1 the set Π of simple roots for $\mathfrak{g}$. Let α_1 be an odd root in Π. A direct (easy) verification shows that $\alpha_0+\alpha_1$ is an odd root and that the set of simple roots $r_{\alpha_0+\alpha_1}(r_{\alpha_1}(\widehat{\Pi}))$ contains both α_0 and $\{\eta_i \mid 1 \le i \le s\}$. Clearly $x_{-\alpha_0-\alpha_1}x_{-\alpha_1}v_{\widehat{\nu}_h} \neq 0$ in $M^\natural(\nu_h)$ so, by Lemma 11.3, $x_{-\alpha_0-\alpha_1}x_{-\alpha_1}v_{\widehat{\nu}_h}$ is a singular vector for the set of simple roots $r_{\alpha_0+\alpha_1}(r_{\alpha_1}(\widehat{\Pi}))$. The weight of this singular vector is, clearly, $\widehat{\nu}_h' = \widehat{\nu}_h - \alpha_0 - 2\alpha_1$. Since the ρ-vector $\widehat{\rho}'$ of $r_{\alpha_0+\alpha_1}(r_{\alpha_1}(\widehat{\Pi}))$ is $\widehat{\rho}+\alpha_0+2\alpha_1$, we see that $(\widehat{\nu}_h'+\widehat{\rho}'|\eta_i^\vee) = (\widehat{\nu}_h+\widehat{\rho}|\eta_i^\vee) = N_i$. Since η_i is a simple root in $r_{\alpha_0+\alpha_1}(r_{\alpha_1}(\widehat{\Pi}))$, we obtain that $x_{-\eta_i}^{N_i}x_{-\alpha_0-\alpha_1}x_{-\alpha_1}v_{\widehat{\nu}_h}$ is a singular vector for the set of simple roots $r_{\alpha_0+\alpha_1}(r_{\alpha_1}(\widehat{\Pi}))$. It follows that $\sum_i U(\widehat{\mathfrak{g}})v_i(h)$ is a proper submodule of $U(\widehat{\mathfrak{g}})x_{-\alpha_0-\alpha_1}x_{-\alpha_1}v_{\widehat{\nu}_h} \subset M^\natural(\widehat{\nu}_h)$. □

Set

$$\overline{M}(\widehat{\nu}_h) = M^\natural(\widehat{\nu}_h)/(\sum_i U(\widehat{\mathfrak{g}})v_i(h)). \tag{11.4}$$

Recall (cf. [13] in the non-super case) that for $\widehat{\mu}, \widehat{\lambda} \in \widehat{\mathfrak{h}}^*$, $\widehat{\mu}$ is said to be linked to $\widehat{\lambda}$ if there exists a sequence of roots $\{\gamma_1, \dots, \gamma_t\} \subset \widehat{\Delta}^+$ and weights $\widehat{\lambda} = \mu_0, \mu_1 \dots, \mu_t = \widehat{\mu}$ such that, for $1 \le r \le t$ one has

- $(\mu_{r-1} + \widehat{\rho}|\gamma_r) = \frac{m_r}{2}(\gamma_r|\gamma_r),\ m_r \in \mathbb{N}$, where $m_r = 1$ if γ_r is an odd isotropic root and m_r is odd if γ_r is an odd non-isotropic root,
- $\mu_r = \mu_{r-1} - m_r\gamma_r$.

The proof of the following proposition is inspired by [7, Section 11]. It also provides a simple proof of Lemma 2 from [10].

Proposition 11.5. *Assume that $\nu \in P_k^+$ is not extremal and that*

$$(\widehat{\nu}_h + \widehat{\rho}|\alpha) \neq \tfrac{n}{2}(\alpha|\alpha) \text{ for all } n \in \mathbb{N} \text{ and } \alpha \in \widehat{\Delta}^+ \setminus \widehat{\Delta}^+(\mathfrak{g}^\natural). \tag{11.5}$$

Then

(i) the module $\overline{M}(\widehat{\nu}_h)$ is irreducible;
(ii) its character is

$$ch\overline{M}(\widehat{\nu}_h) = \sum_{w \in \widehat{W}^\natural} \det(w) ch M(w.\widehat{\nu}_h). \tag{11.6}$$

Proof. We have

(1) $(\widehat{\nu}_h + \widehat{\rho}|\alpha) \neq 0$ for all odd isotropic roots;
(2) $(\widehat{\nu}_h + \widehat{\rho}|\alpha^\vee) \in \mathbb{N}$ for all $\alpha \in \widehat{\Delta}^+(\mathfrak{g}^\natural)$;
(3) $(\widehat{\nu}_h + \widehat{\rho}|\alpha) \neq \frac{n}{2}(\alpha|\alpha)$ for all $n \in \mathbb{N}$ and for all positive roots α of the affinization of $sl_2 = \langle e, f, x\rangle$ and for all non-isotropic odd positive roots.

Indeed, (1), (3) follow from (11.5). To prove (2), first remark that if α is a simple root for $\Delta_+^\natural$, then $\alpha \in \widehat{\Pi}$. It follows that $(\widehat{\rho}|\alpha^\vee) = (\rho^\natural|\alpha^\vee) = 1$. This implies that $(\widehat{\nu}_h + \widehat{\rho}|\alpha^\vee) = (\nu + \rho^\natural|\alpha^\vee) \in \mathbb{N}$ for $\alpha \in \Delta_+^\natural$. Since ν is not extremal, (11.3) gives $(\widehat{\nu}_h + \widehat{\rho}|\eta_i^\vee) \in \mathbb{N}$.

We have

$$ch\,\overline{M}(\widehat{\nu}_h) = \sum_{w \in \widehat{W}^\natural} c(w) ch M(w.\widehat{\nu}_h), \ \ where\ c(w) \in \mathbb{Z}. \tag{11.7}$$

Indeed, if $ch\ M(\widehat{\mu})$ appears in $ch\ \overline{M}(\widehat{\nu}_h)$ then, using the determinant formula proved in [6], and the corresponding Jantzen filtration [10], one shows, as in [13], that there is a sequence of roots $\{\gamma_1, \dots, \gamma_t\} \subset \widehat{\Delta}^+$ linking $\widehat{\mu}$ to $\widehat{\nu}_h$. Properties (1), (3) imply that $\gamma_i \in \widehat{\Delta}^+(\mathfrak{g}^\natural)$ and this yields (11.7). It is clear that $\mathfrak{g}^\natural$ acts locally finitely on $M^\natural(\widehat{\nu}_h)$, hence also on $\overline{M}(\widehat{\nu}_h)$. By (1), $x_{-\alpha_0-\alpha_1}x_{-\alpha_1}v_{\widehat{\nu}_h}$ generates $\overline{M}(\widehat{\nu}_h)$. Since $x_{-\eta_i}^{N_i}(x_{-\alpha_0-\alpha_1}x_{-\alpha_1}v_{\widehat{\nu}_h}) = v_i(h) = 0$ in $\overline{M}(\widehat{\nu}_h)$, $\overline{M}(\widehat{\nu}_h)$ is integrable for $\widehat{\mathfrak{g}}^\natural$, in particular $ch\ \overline{M}(\widehat{\nu}_h)$ is $\widehat{W}^\natural$-invariant. Hence, we obtain $c(w) = \det(w)$; therefore (ii) holds. Since the proof of (ii) didn't use irreducibility, the irreducible quotient of $\overline{M}(\widehat{\nu}_h)$ has the same character, proving (i). □

The following functions $h_{n,\epsilon m}$, $h_{m,\gamma}$ relate singular weights of Verma modules over $\widehat{\mathfrak{g}}$ to those over $W^k_{\min}(\mathfrak{g})$ [20, Remark 7.2]:

$$h_{n,\epsilon m}(k,\nu) = \frac{1}{4(k+h^\vee)}((\epsilon m(k+h^\vee) - n)^2 - (k+1)^2 + 2(\nu|\nu + 2\rho^\natural)), \tag{11.8}$$

$$h_{m,\gamma}(k,\nu) = \frac{1}{4(k+h^\vee)}((2(\nu+\rho^\natural|\gamma) + 2m(k+h^\vee))^2 - (k+1)^2 + 2(\nu|\nu+2\rho^\natural))\,. \tag{11.9}$$

Here $\gamma \in \Delta'$, the set of $\mathfrak{g}^\natural$-weights in $\mathfrak{g}_{-1/2}$, $\epsilon = 2$ (resp. 1) if $0 \in \Delta'$ (resp. $0 \notin \Delta'$), $m, n \in \epsilon^{-1}\mathbb{N}$ and $m - n \in \mathbb{Z}$ in (11.8) and $m \in \frac{1}{2} + \mathbb{Z}_+$ in (11.9).

Lemma 11.6. *Let k be in the unitarity range and let $A(k,\nu)$ be as in (8.11). Assume that ν is not extremal. Then*

$$h_{n,\epsilon m}(k,\nu)\le A(k,\nu), \tag{11.10}$$
$$h_{m,\gamma}(k,\nu)\le A(k,\nu). \tag{11.11}$$

Proof. First we prove (11.10). Plugging (11.8) into (11.10) we get

$$\frac{(\epsilon m(k+h^\vee)-n)^2-(k+1)^2+2(\nu|\nu+2\rho^\natural)}{4(k+h^\vee)}\le\frac{(\nu|\nu+2\rho^\natural)}{2(k+h^\vee)}+\frac{(\xi|\nu)}{k+h^\vee}((\xi|\nu)-k-1),$$

which is equivalent to

$$n-\epsilon m(k+h^\vee)\ge|(k+1)-2(\xi|\nu)|. \tag{11.12}$$

Since $k+h^\vee<0$, it is enough to check (11.12) with $\epsilon m=1,\ n=1/\epsilon$. In the case $(k+1)\le 2(\xi|\nu)$, (11.12) reads

$$1/\epsilon-h^\vee\ge 2(\xi|\nu)-1. \tag{11.13}$$

Looking at the values of $h^\vee$ in Table 2, we see that the L.H.S. of (11.13) is non-negative. Now we prove that $(\xi|\nu)\le 0$. Indeed, from Table 1 we deduce that the restriction of $(.|.)$ to the real span of $\Delta^\natural$ is negative definite. From Tables 1 and 3 one checks that ξ is a linear combination with non-negative coefficients of simple roots of $\mathfrak{g}^\natural$; since ν is dominant, if $\alpha\in\Delta^\natural$ is a simple root then $\nu(\alpha^\vee)\ge 0$, hence $(\nu|\alpha)\le 0$ since $(\alpha|\alpha)<0$. In the case $(k+1)\ge 2(\xi|\nu)$ we have to prove that

$$k+\tfrac{h^\vee}{2}\le(\xi|\nu)+\tfrac{1-\epsilon}{2\epsilon}. \tag{11.14}$$

The non-extremality condition means that $(\nu+\xi)(\theta_i^\vee)\le\frac{2}{(\theta_i|\theta_i)}\left(k+\frac{h^\vee-\bar h_i^\vee}{2}\right)$ or

$$k+\tfrac{h^\vee}{2}\le(\nu+\xi|\theta_i)+\tfrac{\bar h_i^\vee}{2}, \tag{11.15}$$

hence it is enough to prove that

$$(\nu+\xi|\theta_i)+\tfrac{\bar h_i^\vee}{2}\le(\xi|\nu)+\tfrac{1-\epsilon}{2\epsilon}. \tag{11.16}$$

Note that $\theta_i=\xi+\beta_i$, where, as above, β_i is a linear combination with non-negative coefficients of simple roots of $\mathfrak{g}^\natural$. Therefore (11.16) can be written as

$$(\nu|\beta_i)+(\xi|\xi+\beta_i)\le\tfrac{1-\epsilon}{2\epsilon}-\tfrac{\bar h_i^\vee}{2},$$

which is clearly verified, since the left hand side is negative and the right hand side is positive (use the data in Table 2).

Now we prove (11.11). Substituting (11.8) in it we obtain

$$(2(\nu+\rho^\natural|\gamma)+2m(k+h^\vee))^2-((k+1)-2(\xi|\nu))^2\ge 0,$$

which is equivalent to

$$|(2(\nu+\rho^\natural|\gamma)+2m(k+h^\vee)|\ge|(k+1)-2(\xi|\nu)|. \tag{11.17}$$

Table 4. Data employed in the proof of Lemma 11.6

$\mathfrak{g}$	ϵ	$\rho^\natural$	$\max(\rho^\natural\|\gamma)$	$h^\vee$
$psl(2\|2)$	1	$\frac{1}{2}(\delta_1-\delta_2)$	$1/2$	0
$spo(2\|2m), m\geq 3$	1	$(m-1)\epsilon_1+(m-2)\epsilon_2+\ldots+\epsilon_{m-1}$	$(m-1)/2$	$2-m$
$spo(2\|2m+1), m\geq 1$	2	$\frac{2m-1}{2}\epsilon_1+\frac{2m-3}{2}\epsilon_2+\ldots+\frac{1}{2}\epsilon_m$	$(2m-1)/4$	$3/2-m$
$D(2,1;a)$	1	$\epsilon_2+\epsilon_3$	$\frac{1}{2}$	0
$F(4)$	1	$\frac{5}{2}\epsilon_1+\frac{3}{2}\epsilon_2+\frac{1}{2}\epsilon_3$	$3/2$	-2
$G(3)$	2	$2\epsilon_1+3\epsilon_2$	$5/4$	$-3/2$

Recall that, even though $\mathfrak{g}_{-1/2}$ can be reducible as a $\mathfrak{g}^\natural$-module, all irreducible components have the same highest weight ξ. It follows that

$$-(\xi|\nu)=\max_{\gamma\in\Delta'}(\gamma|\nu). \tag{11.18}$$

A direct check on Table 4 shows that

$$2\max_{\gamma\in\Delta'}(\rho^\natural|\gamma)+h^\vee=1. \tag{11.19}$$

Note that, by (11.18) and (11.19)

$$(k+h^\vee)+2(\nu+\rho^\natural|\gamma)\leq(k+1)-2(\xi|\nu). \tag{11.20}$$

Therefore, if $(k+1)\leq 2(\xi|\nu)$ then

$$\begin{aligned}&2(\nu+\rho^\natural|\gamma)+2m(k+h^\vee)\\&\quad=2(\nu+\rho^\natural|\gamma)+(k+h^\vee)+(2m-1)(k+h^\vee)\\&\quad\leq 2(\nu+\rho^\natural|\gamma)+(k+h^\vee)\leq(k+1)-2(\xi|\nu)\leq 0,\end{aligned}$$

and (11.17) reads

$$2(\nu+\rho^\natural|\gamma)+2m(k+h^\vee)\leq(k+1)-2(\xi|\nu),$$

which is clearly true.

Now consider the case

$$(k+1)\geq 2(\xi|\nu),\quad -2(\nu+\rho^\natural|\gamma)-2m(k+h^\vee)\geq 0. \tag{11.21}$$

The inequality (11.17) becomes

$$-2(\nu+\rho^\natural|\gamma)-2m(k+h^\vee)\geq(k+1)-2(\xi|\nu).$$

which is implied by

$$-2(\nu+\rho^\natural|\gamma)-(k+h^\vee)\geq(k+1)-2(\xi|\nu). \tag{11.22}$$

If $\gamma=-\xi$, then the left hand side of (11.22) is

$$-2(\nu+\rho^\natural|-\xi)-(k+h^\vee)=2(\nu|\xi)+h^\vee-1-(k+h^\vee)=2(\nu|\xi)-k-1,$$

hence (11.21) implies that both members of (11.22) are zero.

If $\gamma \neq -\xi$, then (11.22) is equivalent to

$$k+\tfrac{h^\vee}{2} \leq -\tfrac{1}{2}+(\xi|\nu)-(\nu+\rho^\natural|\gamma), \tag{11.23}$$

hence, by (11.15), we are done if we prove that

$$(\nu+\xi|\theta_i)+\tfrac{\bar h_i^\vee}{2} \leq -\tfrac{1}{2}+(\xi|\nu)-(\nu+\rho^\natural|\gamma). \tag{11.24}$$

Remark that, since $\gamma \neq -\xi$, then $\xi-\gamma=\alpha\in\Delta_+^\natural\cup\{0\}$. If $\mathfrak{g}^\natural$ is simple, then $(\nu|\theta_i)\leq(\nu|\xi-\gamma)$, hence

$$(\nu+\xi|\theta_i)+\tfrac{\bar h_i^\vee}{2} \leq (\nu|\xi-\gamma)+(\xi|\theta_i)+\tfrac{\bar h_i^\vee}{2},$$

and therefore (11.22) is implied by

$$(\nu|\xi-\gamma)+(\xi|\theta_i)+\tfrac{\bar h_i^\vee}{2} \leq -\tfrac{1}{2}+(\xi|\nu)-(\nu+\rho^\natural|\gamma),$$

or

$$(\xi|\theta_i)+\tfrac{\bar h_i^\vee}{2} \leq -\tfrac{1}{2}-(\rho^\natural|\gamma). \tag{11.25}$$

The minimum of the left hand side of (11.25) is obtained when $(\rho^\natural|\gamma)$ is maximum, hence, by (11.19), we are left with proving that

$$(\xi|\theta_i)+\tfrac{\bar h_i^\vee}{2} \leq \tfrac{h^\vee}{2}-1. \tag{11.26}$$

This relation is checked using the data in Tables 1, 2, 3. When $\mathfrak{g}^\natural$ is not simple, i.e. $\mathfrak{g}=D(2,1;a)$, relation (11.22) is proven directly. We have $\nu=r\epsilon_2+s\epsilon_3$, $r,s\in\mathbb{Z}_+$ $\gamma=\pm\epsilon_2\pm\epsilon_3,\ \xi=\epsilon_2+\epsilon_3$; if we exclude $\gamma=-\xi$, (11.22) translates into

$$k\leq 0,\quad k\leq -\tfrac{r+1}{1+a},\quad k\leq -\tfrac{(s+1)a}{1+a}, \tag{11.27}$$

according to whether $\gamma=\epsilon_2+\epsilon_3,\ \epsilon_2,\ \epsilon_3$. The non extremality conditions are

$$k\leq -\tfrac{r+2}{1+a},\quad k\leq -\tfrac{(s+2)a}{1+a}, \tag{11.28}$$

so that (11.28) implies (11.27).

We are left with proving (11.17) when both arguments in the absolute values are non-negative, i.e.

$$2(\nu+\rho^\natural|\gamma)+2m(k+h^\vee)\geq(k+1)-2(\xi|\nu). \tag{11.29}$$

We claim that the conditions $2(\nu+\rho^\natural|\gamma)+2m(k+h^\vee)\geq 0$ combined with (11.15) force $m=1/2$ and $\gamma=-\xi$. Taking this fact for granted, (11.29) reads

$$-2(\nu+\rho^\natural|\xi)+h^\vee-1\geq -2(\xi|\nu),$$

which holds by (11.18) and (11.19).

To prove our claim, assume that there is $m>1/2$ such that

$$k\geq\frac{1-2mh^\vee-2(\xi|\nu)-2(\nu+\rho^\natural|\gamma)}{2m-1},$$

or

$$k+\frac{h^\vee}{2}\geq \frac{2-(2m+1)h^\vee-4(\xi|\nu)-4(\nu+\rho^\natural|\gamma)}{2(2m-1)}.$$

Taking (11.15) into account, we are done if we prove that

$$\frac{2-(2m+1)h^\vee-4(\xi|\nu)-4(\nu+\rho^\natural|\gamma)}{2(2m-1)} > (\nu+\xi|\theta_i)+\frac{\bar h_i^\vee}{2}. \tag{11.30}$$

We have

$$\begin{aligned}\text{L. H. S. of (11.30)} &\geq \frac{2-(2m+1)h^\vee-4(\xi|\nu)-4(\nu|\gamma)+2(h^\vee-1)}{2(2m-1)}\\ &=-\frac{h^\vee}{2}+\frac{-4(\xi|\nu)-4(\nu|\gamma)}{2(2m-1)}\geq -\frac{h^\vee}{2}\geq \frac{\bar h_i^\vee}{2}\geq (\nu+\xi|\theta_i)+\frac{\bar h_i^\vee}{2}.\end{aligned} \tag{11.31}$$

The next to last inequality in (11.31) follows from Table 2; more precisely, the strict inequality holds in all cases except for $spo(2|3)$. The last inequality in (11.31) uses that $(\nu+\xi|\theta_i)\leq 0$. For $\mathfrak{g}=spo(2|3)$ the last inequality in (11.31) is strict, hence (11.30) is proven in all cases.

Hence we have necessarily $m=1/2$ in (11.29). We now prove that if

$$2(\nu+\rho^\natural|\gamma)+(k+h^\vee)\geq 0, \tag{11.32}$$

$$(k+1)-2(\xi|\nu)\geq 0, \tag{11.33}$$

hold, then (11.15) implies $\gamma=-\xi$. We proceed case by case.

- $\mathfrak{g}=psl(2|2)$ or $spo(2|3)$. Since $\gamma\in\{0,\pm\xi\}$, relation (11.32) forces $\gamma=-\xi$.
- $\mathfrak{g}=spo(2|m)$, $m>4$. In this case $\nu=\sum_i n_i\epsilon_i, n_1\geq n_2\geq\ldots\geq 0$, $\gamma=\pm\epsilon_i$, $\rho^\natural=\sum_i(\frac{m}{2}-i)\epsilon_i$, $\xi=\epsilon_1$. Then (11.32) reads

$$2\left(\sum_i (n_i+\tfrac{m}{2}-i)|\pm\epsilon_j\right)+k+2-\tfrac{m}{2}\geq 0,$$

or

$$\mp(n_j+\tfrac{m}{2}-j)+k+2-\tfrac{m}{2}\geq 0.$$

Since $k+2-\frac{m}{2}\leq 0$, we have

$$n_j-j+k+2\geq 0.$$

By (11.15)

$$k\leq -\tfrac12 n_1-\tfrac12 n_2-1,$$

therefore

$$0\leq n_j-j+k+2\leq -\tfrac12 n_1-\tfrac12 n_2+n_j-j+1.$$

This relation can be written as

$$0\leq \tfrac{n_j-n_1}{2}+\tfrac{n_j-n_2}{2}-j+1,$$

which holds only if $j=1$, since the n_j are non-increasing half integers. If $j=1$ then $\gamma=-\epsilon_1=-\xi$.

- $\mathfrak{g} = D(2,1;a)$. In this case $\nu = r\epsilon_2 + s\epsilon_3$, $r, s \in \mathbb{Z}_+$, $\gamma = \pm\epsilon_2 \pm \epsilon_3$, $\rho^\natural = \epsilon_2 + \epsilon_3$, $\xi = \epsilon_2 + \epsilon_3$, and in this case (11.32) becomes

$$2((r+1)\epsilon_2 + (s+1)\epsilon_3| \pm \epsilon_2 \pm \epsilon_3) + k \geq 0,$$

which gives

$$\mp (r+1) \mp (s+1)a + (1+a)k \geq 0. \tag{11.34}$$

Condition (11.15) is

$$k \leq ((r+1)\epsilon_2 + (s+1)\epsilon_3|2\epsilon_2) - \tfrac{1}{1+a}, \quad k \leq ((r+1)\epsilon_2 + (s+1)\epsilon_3|2\epsilon_3) - \tfrac{a}{1+a},$$

or

$$(1+a)k \leq -(r+2), \quad (1+a)k \leq -(s+2)a. \tag{11.35}$$

The only possibility to fulfill (11.34) and (11.35) at the same time is to take $\gamma = -\epsilon_2 - \epsilon_3 = -\xi$.

- $\mathfrak{g} = F(4)$. In this case $\nu = n_1\epsilon_1 + n_2\epsilon_2 + n_3\epsilon_3$, $n_1 \geq n_2 \geq n_3 \geq 0$, $\rho^\natural = \frac{5}{2}\epsilon_1 + \frac{3}{2}\epsilon_2 + \frac{1}{2}\epsilon_3$, $\gamma = \frac{1}{2}(\pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3)$, $\xi = \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3)$. Then (11.32) reads

$$-\tfrac{2}{3}(\pm(n_1 + \tfrac{5}{2}) \pm (n_2 + \tfrac{3}{2}) \pm (n_3 + \tfrac{1}{2})) + k - 2 \geq 0. \tag{11.36}$$

By (11.15) we have

$$k \leq -\tfrac{2}{3}(n_1 + n_2) - \tfrac{4}{3}. \tag{11.37}$$

Write now (11.36) using (11.37)

$$\begin{aligned} 0 &\leq -\tfrac{2}{3}(\pm(n_1 + \tfrac{5}{2}) \pm (n_2 + \tfrac{3}{2}) \pm (n_3 + \tfrac{1}{2})) + k - 2 \\ &\leq -\tfrac{2}{3}(\pm(n_1 + \tfrac{5}{2}) \pm (n_2 + \tfrac{3}{2}) \pm (n_3 + \tfrac{1}{2})) - \tfrac{2}{3}(n_1 + n_2) - \tfrac{10}{3} \\ &\leq -\tfrac{2}{3}(\pm n_1 - \tfrac{5}{2} \pm n_2 - \tfrac{3}{2} \pm n_3 - \tfrac{1}{2}) - \tfrac{2}{3}(n_1 + n_2) - \tfrac{10}{3} \\ &= -\tfrac{2}{3}(\pm n_1 \pm n_2) - \tfrac{2}{3}(n_1 + n_2 \pm n_3) - \tfrac{1}{3}. \end{aligned}$$

This inequality holds if and only if the minus sign is taken in all occurrences of $\pm$, i.e. $\gamma = -\xi$.

- $\mathfrak{g} = G(3)$. In this case $\nu = m(\epsilon_1 + \epsilon_2) + n(\epsilon_1 + 2\epsilon_2)$, $m, n \in \mathbb{Z}_+$, $\gamma \in \{0, \pm\epsilon_1, \pm\epsilon_2, \pm(\epsilon_1 + \epsilon_2)\}$, $\rho^\natural = 2\epsilon_1 + 3\epsilon_2$, $\xi = \epsilon_1 + \epsilon_2$. Then (11.32) reads

$$2((m+n+2)\epsilon_1 + (m+2n+3)\epsilon_2|\gamma) + k - \tfrac{3}{2} \geq 0. \tag{11.38}$$

and we can confine ourselves to consider $\gamma \in \{-\epsilon_1, -\epsilon_2, -\epsilon_1 - \epsilon_2\}$. The inequalities corresponding to $\gamma = -\epsilon_1$, $\gamma = -\epsilon_2$ are

$$k + \tfrac{m}{2} - 1 \geq 0, \tag{11.39}$$
$$k + \tfrac{m+3n+1}{2} \geq 0, \tag{11.40}$$

respectively. Relation (11.41) gives

$$k \leq ((m+n+1)\epsilon_1 + (m+2n+1)\epsilon_2|\epsilon_1 + 2\epsilon_2) - \tfrac{3}{4}.$$

or

$$k \le -\tfrac{3}{4}(m+2n) - \tfrac{3}{2} \tag{11.41}$$

Substituting (11.39), (11.40), into (11.41) we obtain

$$0 \le k + \tfrac{m}{2} - 1 \le -\tfrac{1}{4}m - \tfrac{3}{2}n - \tfrac{5}{2}, \tag{11.42}$$

$$0 \le k + \tfrac{m+3n+1}{2} \le -\tfrac{m}{4} - 1, \tag{11.43}$$

respectively. Inequalities (11.42), (11.43) are never verified. Once again we conclude that $\gamma = -\xi$.

□

Let H_0 denote the quantum Hamiltonian reduction functor, from the category $\mathcal{O}$ of $\widehat{\mathfrak{g}}$-modules of level k to the category of $W^k_{\min}(\mathfrak{g})$-modules. Recall that, for a $\widehat{\mathfrak{g}}$-module M, $H_0(M)$ is the zeroth homology of the complex $(M \otimes F(\mathfrak{g}, x, f), d_0)$ defined in [18]. Recall that the functor H_0 maps Verma modules to Verma modules [20, Theorem 6.3] and it is exact [2, Corollary 6.7.3]. By [20, Lemma 7.3 (b)], if M is a highest weight module over $\widehat{\mathfrak{g}}$ of highest weight $\Lambda \in \widehat{\mathfrak{h}}^*$, $H_0(M)$ is either zero or a highest weight module over $W^k_{\min}(\mathfrak{g})$ of highest weight (ν, ℓ) with

$$\nu = \Lambda_{|\mathfrak{h}^\natural}, \quad \ell = \frac{(\Lambda|\Lambda + 2\widehat{\rho})}{2(k+h^\vee)} - \Lambda(x+d). \tag{11.44}$$

Remark 11.7. Let $L(\Lambda)$ denote the irreducible $\widehat{\mathfrak{g}}$-module of highest weight $\Lambda \in \widehat{\mathfrak{h}}^*$. By Arakawa's theorem [2, Main Theorem] $H_0(L(\Lambda))$ is either irreducible or zero, and it is zero if and only if $(\Lambda|\alpha_0) = \frac{n}{2}(\alpha_0|\alpha_0)$, $n \in \mathbb{Z}_+$. In particular, if (11.5) holds, then $H_0(\overline{M}(\widehat{\nu}_h))$ is a non-zero highest weight module of highest weight $(\nu, \ell(h))$, where

$$\ell(h) = \frac{(\widehat{\nu}_h|\widehat{\nu}_h + 2\widehat{\rho})}{2(k+h^\vee)} - h. \tag{11.45}$$

For $\Lambda \in \widehat{\mathfrak{h}}^*$, by a slight abuse of notation, we set $M^W(\Lambda) = H_0(M(\Lambda))$, where $M(\Lambda)$ is the Verma module over $\widehat{\mathfrak{g}}$ of highest weight Λ. Note that $M^W(\Lambda) = M^W(\nu, \ell)$, where ν, ℓ are given by (11.44).

From now on we assume

- k is in the unitarity range;
- $\nu \in P^+_k$;
- $\ell(h) \in \mathbb{R}$.

Lemma 11.8. *Let h, h' be the solutions of the equation $\ell(h) = \ell_0$. If $(\widehat{\nu}_h + \widehat{\rho}|\delta - \theta) = n$, $n \in \mathbb{N}$, then $(\widehat{\nu}_{h'} + \widehat{\rho}|\delta - \theta) \notin \mathbb{N}$.*

Proof. Recalling that

$$\ell(h) = \frac{(\widehat{\nu}_h|\widehat{\nu}_h + 2\widehat{\rho})}{2(k+h^\vee)} - h = \frac{(\nu|\nu + 2\rho^\natural)}{2(k+h^\vee)} + \frac{h(h-k-1)}{k+h^\vee}$$

we see that $h' = k + 1 - h$. If $(\widehat{\nu}_h + \widehat{\rho}|\delta - \theta) = n \in \mathbb{N}$, then

$$((k+h^\vee)\Lambda_0 + h\theta + \nu + \rho|\delta - \theta) = k + 1 - 2h = n,$$

hence $h = (k+1-n)/2$ and $h' = (k+n+1)/2$ so that

$$(\widehat{\nu}_{h'} + \widehat{\rho}|\delta - \theta) = k + 1 - 2h' = -n.$$

□

Theorem 11.9. *If $\ell(h) > A(k,\nu)$, then $H_0(\overline{M}(\widehat{\nu}_h))$ is an irreducible $W^k_{\min}(\mathfrak{g})$-module and its character is*

$$ch\, H_0(\overline{M}(\widehat{\nu}_h)) = \sum_{w\in\widehat{W}^\natural} det(w) ch\, M^W(w.\widehat{\nu}_h). \tag{11.46}$$

Proof. If $\ell(h) > A(k,\nu)$, then, by Lemma 11.6

$$\ell(h) \neq h_{n,\epsilon m}(k,\nu) \quad \text{and} \quad \ell(h) \neq h_{m,\gamma}(k,\nu). \tag{11.47}$$

By [20, Lemma 7.3 (c)], (11.47) implies that $(\widehat{\nu}_h + \widehat{\rho}|\alpha) \neq \frac{n}{2}(\alpha|\alpha)$ for all $\alpha \in \widehat{\Delta}^+ \setminus (\widehat{\Delta}^+(\mathfrak{g}^\natural) \cup \{\delta-\theta\})$. By exchanging h and h' if $h \in \mathbb{N}$ and applying Lemma 11.8, we find that one can choose h so that (11.5) is satisfied. Hence, by Propositions 8.8 and 11.5, $\overline{M}(\widehat{\nu}_h)$ is irreducible. By Remark 11.7, $H_0(\overline{M}(\widehat{\nu}_h))$ is irreducible and non-zero. On the other hand, by Theorem 6.2 of [20], we find that $H^j((\overline{M}(\widehat{\nu}_h) \otimes F(\mathfrak{g},x,f))) = 0$ if $j \neq 0$. Thus, using Euler-Poincaré character, the fact that H_0 maps Verma modules over $\widehat{\mathfrak{g}}$ to Verma modules over $W^k_{\min}(\mathfrak{g})$, and (ii) in Proposition 11.5, we find that (11.46) holds. □

Recall from Sect. 6 the Heisenberg algebra $\mathcal{H}$. Let y be an indeterminate. Define an action of $\mathcal{H}_0 = \mathbb{C}a + \mathbb{C}K$ on $\mathbb{C}[y]$ by letting K act as the identity and a act by multiplication by y. Let $M(y)$ be the corresponding Verma module. This module can be regarded as a $V^1(\mathbb{C}a)$-module by means of the field $Y(a,z)$ defined by setting, for $m \in M(y)$,

$$Y(a,z)m = \sum_{j\in\mathbb{Z}} (\tau^j \otimes a)\cdot m\, z^{-j-1}.$$

Note also that $M(y)$ is free over $\mathbb{C}[y]$ with basis

$$\{(\tau^{-j_1}\otimes a)^{i_1}\cdots(\tau^{-j_r}\otimes a)^{i_r}(1\otimes 1) \mid j_1 > \cdots > j_r > 0\}. \tag{11.48}$$

Recall from Sect. 9 the free field realization $\Psi : W^k_{\min}(\mathfrak{g}) \to \mathcal{V}^k = V^1(\mathbb{C}a) \otimes V^{\alpha_k}(\mathfrak{g}^\natural) \otimes F(\mathfrak{g}_{1/2})$. If $\nu \in P^+_k$ is not extremal, recall that we denoted by $L^\natural(\nu)$ the integrable $V^{\alpha_k}(\mathfrak{g}^\natural)$-module of highest weight ν. We also let v_ν be a highest weight vector of $L^\natural(\nu)$. Then

$$M(y) \otimes L^\natural(\nu) \otimes F(\mathfrak{g}_{1/2})$$

is a $\mathcal{V}^k$-module, hence, by means of Ψ, a $W^k_{\min}(\mathfrak{g})$-module. Set

$$N(y,\nu) = \Psi(W^k_{\min}(\mathfrak{g}))\cdot(1\otimes\mathbb{C}[y]\otimes v_\nu\otimes\mathbf{1}) \subset M(y)\otimes L^\natural(\nu)\otimes F(\mathfrak{g}_{1/2}).$$

Since $M(y)\otimes L^\natural(\nu)\otimes F(\mathfrak{g}_{1/2})$ is free as a $\mathbb{C}[y]$-module, $N(y,\nu)$ is also free. If $\mu \in \mathbb{C}$, set also

$$N(\mu,\nu) = (\mathbb{C}[y]/(y-\mu)) \otimes_{\mathbb{C}[y]} N(y,\nu).$$

By construction $N(\mu,\nu)$ is clearly a highest weight module for $W^k_{\min}(\mathfrak{g})$. As shown in Sect. 10, its highest weight is (ν,ℓ_0) with

$$\ell_0 = \tfrac{1}{2}\mu^2 - s_k\mu + \frac{(\nu|\nu+2\rho^\natural)}{2(k+h^\vee)}.$$

Since we are looking for unitary representations, we will always assume that $\ell_0 \in \mathbb{R}$.

Lemma 11.10. *If $\ell_0 > A(k,\nu)$ then $N(\mu,\nu)$ is an irreducible $W^k_{\min}(\mathfrak{g})$-module.*

Proof. Choose $h \in \mathbb{C}$ such that $\ell_0 = \ell(h)$. By Theorem 11.9, $H_0(\overline{M}(\widehat{\nu}_h))$ is an irreducible $W^k_{\min}(\mathfrak{g})$-module, hence there is an onto map $N(\mu,\nu) \to H_0(\overline{M}(\widehat{\nu}_h)) = L^W(\ell_0,\nu)$. If $\ell_0 \gg 0$, by the proof of Proposition 10.2, $N(\mu,\nu) = L^W(\ell_0,\nu)$. Observe that, since $\ell_0 > A(k,\nu)$, by Lemma 11.6, relations (11.5) hold for our chosen h. It follows from (11.46) that

$$ch\, N(\mu,\nu) = \sum_{w \in \widehat{W}^\natural} det(w) ch\, M^W(w.\widehat{\nu}_h) \text{ for } \ell_0 \gg 0. \tag{11.49}$$

By (11.44), the highest weight of $M^W(w.\widehat{\nu}_h)$ is $(\nu(w,h), \ell_0(w,h))$ where

$$\nu(w,h) = (w.\widehat{\nu}_h)_{|\mathfrak{h}^\natural},\ \ \ell_0(w,h) = \frac{\|w(\widehat{\nu}_h + \widehat{\rho})\|^2 - \|\widehat{\rho}\|^2}{2(k+h^\vee)} - (w.\widehat{\nu}_h)(x+d).$$

Since $w \in \widehat{W}^\natural$, $(w.\widehat{\nu}_h)(x) = h$ and $(w.\widehat{\nu}_h)(d)$ as well as $\nu(w,h)$ do not depend on h. We can therefore write

$$\begin{aligned}
\ell_0(w,h) &= \frac{\|w(\widehat{\nu}_h + \widehat{\rho})\|^2 - \|\widehat{\rho}\|^2}{2(k+h^\vee)} - (w.\widehat{\nu}_0)(d) - h \\
&= \frac{\|(\widehat{\nu}_0 + \widehat{\rho})\|^2 - \|\widehat{\rho}\|^2}{2(k+h^\vee)} - (w.\widehat{\nu}_0)(d+x) + \frac{\|(\widehat{\nu}_h + \widehat{\rho})\|^2 - \|\widehat{\nu}_0 + \widehat{\rho}\|^2}{2(k+h^\vee)} - h \\
&= \frac{\|w(\widehat{\nu}_0 + \widehat{\rho})\|^2 - \|\widehat{\rho}\|^2}{2(k+h^\vee)} - (w.\widehat{\nu}_0)(d+x) + \frac{2h^2 + (h^\vee - 1)h}{2(k+h^\vee)} - h \\
&= \ell_0(w,0) + \frac{2h^2 + (h^\vee - 1)h}{2(k+h^\vee)} - h.
\end{aligned}$$

It follows that

$$ch\, M^W(w.\widehat{\nu}_h) = ch\, M^W(w.\widehat{\nu}_0) e^{(0, \frac{2h^2+(h^\vee-1)h}{2(k+h^\vee)} - h)}$$

and

$$\sum_{w \in \widehat{W}^\natural} det(w) ch\, M^W(w.\widehat{\nu}_h) = \left(\sum_{w \in \widehat{W}^\natural} det(w) ch\, M^W(w.\widehat{\nu}_0) \right) e^{(0, \frac{2h^2+(h^\vee-1)h}{2(k+h^\vee)} - h)}. \tag{11.50}$$

In particular, if $\ell_0 \gg 0$, then

$$ch\, N(\mu,\nu) = \left(\sum_{w \in \widehat{W}^\natural} det(w) ch\, M^W(w.\widehat{\nu}_0) \right) e^{(0, \frac{2h^2+(h^\vee-1)h}{2(k+h^\vee)} - h)}.$$

Since $N(y,\nu)$ is a free $\mathbb{C}[y]$-module, the dimensions of the weight spaces of $N(\mu,\nu)$ do not depend on μ. By (11.50), the coefficents of both sides of (11.49) do not depend on μ. It follows that (11.49) holds for all μ. In particular, if $\ell_0 > A(k,\nu)$, by Theorem 11.9,

$$ch\, N(\mu,\nu) = ch\, H_0(\overline{M}(\widehat{\nu}_h)),$$

hence $N(\mu,\nu) \simeq H_0(\overline{M}(\widehat{\nu}_h))$ is irreducible. $\square$

The lowest energy space of $N(\mu,\nu)$ is $1\otimes 1\otimes V^{\natural}(\nu)\otimes \mathbf{1}$ with L_0 acting by multiplication by ℓ_0. This space admits a ω-invariant Hermitian form hence there exists a ϕ-invariant Hermitian form $H(\cdot,\cdot)$ on $N(\mu,\nu)$.

If $\widehat{\zeta}(y)\in Hom_{\mathbb{C}[y]}(\mathbb{C}[y]\otimes\widehat{\mathfrak{h}},\mathbb{C}[y])$ is a weight of $N(y,\nu)$, fix a basis $\mathcal{B}_{\widehat{\zeta}(y)}$ of $N(y,\nu)_{\widehat{\zeta}(y)}$. Set $\widehat{\zeta}=\widehat{\zeta}(\mu)$. Then $1\otimes\mathcal{B}_{\widehat{\zeta}(y)}$ gives a basis $\mathcal{B}_{\widehat{\zeta}}$ of $N(\mu,\nu)_{\widehat{\zeta}}=(\mathbb{C}[y]/(y-\mu))\otimes_{\mathbb{C}[y]}N(y,\nu)_{\widehat{\zeta}(y)}$. Let $det_{\widehat{\zeta}}(\ell_0)$ be the determinant of the matrix in this basis of the Hermitian form $H(\cdot,\cdot)$ restricted to $N(\mu,\nu)_{\widehat{\zeta}}$. Note that $det_{\widehat{\zeta}}(\ell_0)$ is a polynomial in ℓ_0.

End of proof of Theorem 11.1 and Corollary 11.2. We may assume that the level is not collapsing, so that $M_i(k)+\chi_i\in\mathbb{Z}_+$ by Remark 7.5. Then, by Proposition 10.2, the Hermitian form on $L^W(\nu,\ell_0)$ is positive definite for $\ell_0\gg 0$. By Lemma 11.10, $N(\mu,\nu)=L^W(\nu,\ell_0)$ if $\ell_0=\frac{1}{2}\mu^2-s_k\mu+\frac{(\nu|\nu+2\rho^{\natural})}{2(k+h^{\vee})}>A(k,\nu)$, hence $det_{\widehat{\zeta}}(\ell_0)\neq 0$ for all weights $\widehat{\zeta}$ of $N(\mu,\nu)$. It follows that the Hermitian form is positive definite for $\ell_0>A(k,\nu)$, hence positive semidefinite for $\ell_0=A(k,\nu)$.

Corollary 11.2 follows from Proposition 8.10 and Theorem 11.1 in the case $\nu=0$, since $A(k,0)=0$, and Remark 7.5. □

12. Explicit Necessary Conditions and Sufficient Conditions of Unitarity

Looking for the pairs (ν,ℓ_0), $\nu\in\widehat{P}^+_k$, $\ell_0\in\mathbb{R}$, such that $L^W(\nu,\ell_0)$ is a unitary $W^k_{\min}(\mathfrak{g})$-module for k in the unitarity range, we rewrite for each case (excluding the trivial case (1)) the conditions in terms of the parameters $M_i=M_i(k)$ from Table 2. Namely, we provide the necessary and sufficient conditions of unitarity of $L^W(\nu,\ell_0)$ for a non-extremal weight ν, given by Theorem 11.1, and the necessary condition of unitarity for an extremal weight ν, given by Proposition 8.8. We also provide explicit expressions for the cocycle α_k and the central charge c of L. Recall the invariant bilinear form $(.|.)^{\natural}_i$ on $\mathfrak{g}^{\natural}_i$, introduced in Sect. 7.

12.1. $psl(2|2)$. In this case $\mathfrak{g}^{\natural}=sl(2)$, $M_1\in\mathbb{N}$ and $\alpha_k=(M_1-1)(\,.\,|\,.\,)^{\natural}_1$. If $\nu=r\theta_1/2$, with $r\in\mathbb{Z}_{\geq 0}$ (i.e. ν is dominant integral), and $r\leq M_1-1$, then the necessary and sufficient condition for unitarity is

$$\ell_0\geq\frac{r}{2}.$$

If $r=M_1$, then then necessary condition is $\ell_0=M_1/2$.
The central charge is $c=-6(k+1)=6M_1$.

12.2. $spo(2|3)$. In this case $\mathfrak{g}^{\natural}=sl(2)$, $M_1\in\mathbb{N}$ and $\alpha_k=(M_1-2)(\,.\,|\,.\,)^{\natural}_1$. If $\nu=r\theta_1/2=r\alpha/2$, with $r\in\mathbb{Z}_{\geq 0}, r\leq M_1-2$, then the necessary and sufficient condition for unitarity is

$$\ell_0\geq\frac{r}{4}.$$

If $M_1-1\leq r\leq M_1$, then then necessary condition is $\ell_0=r/4$.
The central charge is $c=-6k-\frac{7}{2}=\frac{3}{2}M_1-\frac{1}{2}$.

12.3. $spo(2|m)$, $m > 4$. In this case $\mathfrak{g}^\natural = so(m)$, $M_1 \in \mathbb{N}$ and $\alpha_k = (M_1 - 1)(\,.\,|\,.\,)_1^\natural$. If ν is dominant integral, $\nu(\theta_1^\vee) \le M_1 - 1$, then the necessary and sufficient condition for unitarity is

$$\ell_0 \ge \frac{(\nu|\nu + 2\rho^\natural)^\natural}{2(M_1 + m - 3)} + \frac{r(M_1 - r - 1)}{2(m + M_1 - 3)} = -\frac{(\nu|\nu + 2\rho^\natural)^\natural - r(2k + r + 2)}{2(2k - m + 4)}, \tag{12.1}$$

where $r = (\omega_1|\nu)^\natural$, and ω_1 is the highest weight of the standard representation of $so(m)$. If $\nu(\theta_1^\vee) = M_1$, the necessary condition is that equality must hold in (12.1).
The central charge is $c = \frac{M_1\left(m^2+6M_1-10\right)}{2(m+M_1-3)} = -\frac{(2k+1)\left(12k-m^2+16\right)}{4k-2m+8}$.

12.4. $D(2, 1; \frac{m}{n})$, $m, n \in \mathbb{N}$, m, n coprime. In this case $\mathfrak{g}^\natural = \mathfrak{g}_1^\natural \oplus \mathfrak{g}_2^\natural$ with $\mathfrak{g}_i^\natural \simeq sl(2)$, and

$$\alpha_k(b, c) = (M_i(k) - 1)(b|c)_i^\natural \text{ if } b, c \in \mathfrak{g}_i^\natural.$$

If $\nu = \frac{r_1}{2}\theta_1 + \frac{r_2}{2}\theta_2$ is dominant integral with $r_i \le M_i(k) - 1$, then the necessary and sufficient condition for unitarity is

$$\ell_0 \ge \frac{2(M_1 + 1)r_2 + 2(M_2 + 1)r_1 + (r_1 - r_2)^2}{4(M_1 + M_2 + 2)} = \frac{2(a + 1)k(ar_2 + r_1) - a(r_1 - r_2)^2}{4(a + 1)^2 k} \tag{12.2}$$

If $r_i = M_i$ for some i, then the necessary condition is that equality must hold in (12.2). The central charge is $c = 6\frac{(M_1+1)(M_2+1)}{M_1+M_2+2} - 3 = -3(1 + 2k)$.

12.5. $F(4)$. In this case $\mathfrak{g}^\natural = so(7)$, $M_1 \in \mathbb{N}$ and $\alpha_k = (M_1 - 1)(\,.\,|\,.\,)^\natural$. If $\nu(\theta_1^\vee) \le M_1 - 1$, then the necessary and sufficient condition for unitarity is

$$\begin{aligned}\ell_0 &\ge \frac{r_1(M_1 + 7) + r_2(M_1 + 4) + r_3(M_1 + 1) + r_1^2 + r_2^2 + r_3^2 - r_1r_2 - r_1r_3 - r_2r_3}{3(M_1 + 4)}\\ &= \frac{r_1(6 - \frac{3}{2}k) + r_2(3 - \frac{3}{2}k) + r_3(-\frac{3}{2}k) + r_1^2 + r_2^2 + r_3^2 - r_1r_2 - r_1r_3 - r_2r_3}{3(3 - \frac{3}{2}k)},\end{aligned} \tag{12.3}$$

where we write $\nu = r_1\epsilon_1 + r_2\epsilon_2 + r_3\epsilon_3$ with ϵ_i as in Table 1. If $\nu(\theta_1^\vee) = M_1$, then the necessary condition is that equality must hold in (12.3).
The central charge is $c = \frac{2M_1(2M_1+11)}{M_1+4} = -\frac{2(k-3)(3k+2)}{k-2}$.

12.6. $G(3)$. In this case $\mathfrak{g}^\natural = G_2$, $M_1 \in \mathbb{N}$ and $\alpha_k = (M_1-1)(\cdot|\cdot)^\natural$. If $\nu(\theta_1^\vee) \le M_1-1$, then the necessary and sufficient condition for unitarity is

$$\ell_0 \ge \frac{r_1(3M_1+1)+r_2(3M_1+7)+3(r_1-r_2)^2}{12(M_1+3)} \\ = \frac{r_1(-2-4k)+r_2(4-4k)+3(r_1-r_2)^2}{8(3-2k)}, \tag{12.4}$$

where we write $\nu = r_1\epsilon_1 + r_2\epsilon_2$ with ϵ_i as in Table 1. If $\nu(\theta_1^\vee) = M_1$, then the necessary condition is that equality must hold in (12.4).
The central charge is $c = \frac{M_1(9M_1+31)}{2(M_1+3)} = \frac{-24k^2+26k+33}{4k-6}$.

13. Unitarity for Extremal Modules Over the $N=3$, $N=4$ and big $N=4$ Superconformal Algebras

A module $L^W(\nu, \ell_0)$ for $W^k_{\min}(\mathfrak{g})$ is called *extremal* if the weight ν is extremal (see Definition 8.7). In this section we give a partial solution of Conjecture 2 for some $\mathfrak{g}$. Namely, $\mathfrak{g}$ will be either $spo(2|3)$, or $psl(2|2)$, or $D(2,1;a)$, so that $W^k_{\min}(\mathfrak{g})$ is related to the $N=3$, $N=4$ and big $N=4$ superconformal algebra, respectively. Recall from [20, Section 8] that in these cases, up to adding a suitable number of bosons and fermions, it is always possible to make the λ-brackets between the generating fields linear, hence the span of their Fourier coefficients gets endowed with a Lie superalgebra structure, called the $N=3$, $N=4$ and big $N=4$ superconformal algebra respectively.

Recall that, by Proposition 8.8, for each extremal weight ν there is at most one ℓ_0 for which the extremal module $L^W(\nu, \ell_0)$ is unitary, hence for each extremal ν it suffices to construct one such unitary module.

13.1. $\mathfrak{g} = spo(2|3)$. Consider $W^k_{\min}(spo(2|3))$ and the Lie conformal superalgebra $R = (\mathbb{C}[\partial] \otimes \mathfrak{a}) \oplus \mathbb{C}K$, where $\mathfrak{a}$ is an 8-dimensional superspace with basis $\tilde{L}, \tilde{G}^\pm, \tilde{G}^0, J^\pm, J^0, \Phi$, where $\tilde{L}, J^\pm, J^0$ are even and $\tilde{G}^\pm, \tilde{G}^0, \Phi$ are odd, and the following λ-brackets

$$[J^0{}_\lambda \tilde{G}^0] = -2\lambda\Phi\,,\ [J^+{}_\lambda \tilde{G}^-] = -2\tilde{G}^0 + 2\lambda\Phi\,,\ [J^-{}_\lambda \tilde{G}^+] = \tilde{G}^0 + \lambda\Phi\,,\ [\tilde{G}^\pm{}_\lambda \tilde{G}^\pm] = 0\,,$$
$$[\tilde{G}^+{}_\lambda \tilde{G}^-] = \tilde{L} + \tfrac{1}{4}(\partial+2\lambda)J^0 - \lambda^2 K\,, [\tilde{G}^+{}_\lambda \tilde{G}^0] = \tfrac{1}{4}(\partial+2\lambda)J^+\,, [\tilde{G}^0{}_\lambda \tilde{G}^0] = \tilde{L} - \lambda^2 K\,,$$
$$[\tilde{G}^-{}_\lambda \tilde{G}^0] = -\tfrac{1}{2}(\partial+2\lambda)J^-\,,\ [\tilde{G}^+{}_\lambda \Phi] = \tfrac{1}{4}J^+\,,\ [\tilde{G}^-{}_\lambda \Phi] = \tfrac{1}{2}J^-\,,\ [\tilde{G}^0{}_\lambda \Phi] = -\tfrac{1}{4}J^0$$
$$[\Phi_\lambda \Phi] = -K, [J^+_\lambda J^-] = J^0 - 4\lambda K, , [J^0_\lambda J^\pm] = \pm 2J^\pm, [J^0_\lambda J^0] = -8\lambda K,$$
$$[\tilde{L}_\lambda \tilde{L}] = \partial\tilde{L} + 2\lambda\tilde{L} - \tfrac{\lambda^3}{2}K.$$

Furthermore $\tilde{G}^\pm, \tilde{G}^0, J^\pm, J^0, \Phi$ are primary for $\tilde{L}$ of conformal weight $\frac{3}{2}, \frac{3}{2}, 1, 1, \frac{1}{2}$, respectively.

The $N=3$ superconformal algebra $\mathcal{W}^k_{N=3}$ is $V(R)/(K-(k+\frac{1}{2})\mathbf{1})$, where $V(R)$ is the universal enveloping vertex algebra of R. Let F_Φ be the fermionic vertex algebra generated by an odd element Φ, with λ-braket $[\Phi_\lambda \Phi] = -(k+\frac{1}{2})\mathbf{1}$. Then there is a conformal vertex algebra embedding

$$\mathcal{W}^k_{N=3} \hookrightarrow W^k_{\min}(spo(2|3)) \otimes F_\Phi$$

given by (cf [20, §8.5])

$$\tilde{L} \mapsto L - \tfrac{1}{2k+1} :\partial\Phi\Phi:, \; \tilde{G}^+ \mapsto \tfrac{\sqrt{-1}}{\sqrt{k+1/2}} G^+ - \tfrac{1}{4k+2} :J^+\Phi:,$$
$$\tilde{G}^- \mapsto \tfrac{-\sqrt{-1}}{\sqrt{k+1/2}} G^- - \tfrac{1}{2k+1} :J^-\Phi:, \; \tilde{G}^0 \mapsto \tfrac{-\sqrt{-1}}{\sqrt{k+1/2}} G^0 + \tfrac{1}{4k+2} :J^0\Phi: .$$
$$\Phi \mapsto \Phi, \; J^\pm \mapsto J^\pm, \; J^0 \mapsto J^0.$$

Extend the conjugate linear involution ϕ to $W^k_{\min}(spo(2|3)) \otimes F_\Phi$ setting $\phi(\Phi) = -\Phi$. Recall from [16] that the unique ϕ-invariant Hermitian form on F_Φ is positive definite. Also recall that the tensor product of invariant Hermitian forms is still invariant; in particular if we prove that $L^W(\nu, \ell_0) \otimes F_\Phi$ is unitary for $\mathcal{W}^k_{N=3}$, then $L^W(\nu, \ell_0)$ is a unitary $W^k_{\min}(spo(2|3))$-module. Recall that, for $a, b \in V(R)$, the modes of a, b have a Lie superalgebra structure given by

$$[a_r, b_s] = \sum_{j \in \mathbb{Z}_+} \binom{\Delta_a + r - 1}{j} (a_{(j)}b)_{r+s}.$$

Observe that the span $\mathcal{L}$ of $\tilde{L}_n, \tilde{G}^\pm_m, \tilde{G}^0_m, J^\pm_n, J^0_n, \Phi_m, K, n \in \mathbb{Z}, m \in \frac{1}{2} + \mathbb{Z}$, is a Lie superalgebra. If M (resp. M') are modules for $\mathcal{W}^k_{N=3}$ (resp. $\mathcal{W}^{k'}_{N=3}$), then $M \otimes M'$ inherits an action of $\mathcal{L}$ which makes $M \otimes M'$ a $\mathcal{W}^{k+k'+\frac{1}{2}}_{N=3}$-module. Clearly, if both M, M' are unitary, then $M \otimes M'$ is unitary. The argument used in the next proposition generalizes the one used for the oscillator representation of the Virasoro algebra in [17, §3.4].

Proposition 13.1. *Let $M_1 = -4k - 2 \in \mathbb{N}$. Then the extremal $W^k_{\min}(spo(2|3))$-modules $L^W(\frac{M_1-1}{2}\alpha, \frac{M_1-1}{4})$, $L^W(\frac{M_1}{2}\alpha, \frac{M_1}{4})$ are both unitary, where α is the simple root of $\mathfrak{g}^\natural = sl_2$.*

Proof. To make the argument more transparent we make explicit the dependence on k, so we write $L(k, \nu, \ell_0)$ for the $W^k_{\min}(spo(2|3))$-module $L^W(\nu, \ell_0)$. Recall that $\nu = r\alpha/2$.

We proceed by induction on M_1. The base case $M_1 = 1$ corresponds to the collapsing level $k = -3/4$, when $W^{\min}_{-3/4}(spo(2|3)) = V_1(sl_2)$. Recall that $V_1(sl_2)$ has only two irreducible modules N_1 and N_2, which are both unitary and have highest weights $\nu = 0$ and $\nu = \alpha/2$ respectively. Recall from § 12.2 that if $M_1 - 1 \le r \le M_1$, then the necessary condition for unitarity is $\ell_0 = M_1/4$. Hence N_1 and N_2 are $L(-3/4, 0, 0)$ and $L(-3/4, \alpha/2, 1/4)$. Set $k_1 = -\frac{M_1+1}{4}$. Assume by induction that $L(k_1, \frac{M_1-2}{2}\alpha, \frac{M_1-2}{4})$ and $L(k_1, \frac{M_1-1}{2}\alpha, \frac{M_1-1}{4})$ are unitary. Then $M = L(k_1, \frac{M_1-2}{2}\alpha, \frac{M_1-2}{4}) \otimes F_\Phi$ is unitary for $\mathcal{W}^{k_1}_{N=3}$ and $M' = L(-3/4, \alpha/2, 1/4) \otimes F_\Phi$ is unitary for $\mathcal{W}^{-3/4}_{N=3}$. Therefore $M \otimes M'$ is unitary for $\mathcal{W}^{k_2}_{N=3}$, $k_2 = k_1 - \frac{3}{4} + \frac{1}{2} = -\frac{M_1+1}{4} - \frac{3}{4} + \frac{1}{2} = -\frac{M_1}{4} - \frac{1}{2} = k$. In particular, the $W^k_{\min}(spo(2|3))$-module generated by $v_{\frac{M_1-2}{2}\alpha, \frac{M_1-2}{4}} \otimes \mathbf{1} \otimes v_{\frac{\alpha}{2}, \frac{1}{4}} \otimes \mathbf{1}$ is unitary, and its weight is $(\frac{M_1-1}{2}\alpha, \frac{M_1-1}{4})$, as required.

Repeating this argument with $L(k_1, \frac{M_1-1}{2}\alpha, \frac{M_1-1}{4}) \otimes F_\Phi$ proves the unitarity of $L(k, \frac{M_1}{2}\alpha, \frac{M_1}{4})$. $\square$

13.2. $\mathfrak{g} = psl(2|2)$. We choose strong generators $J^0, J^{\pm}, G^{\pm}, \bar{G}^{\pm}, L$ for $W^k_{\min}(psl(2|2))$ as in [20, §8.4]. We can choose the generators so that, if ϕ is the almost compact involution corresponding to the real form described in Sect. 4, then

$$\phi(L) = L,\ \phi(J^+) = -J^-,\ \phi(J^0) = -J^0,\ \phi(G^+) = \bar{G}^-,\ \phi(G^-) = \bar{G}^+. \tag{13.1}$$

The λ-brackets among these generators are linear, hence their Fourier coefficients span the $N = 4$ superconformal algebra. It is therefore enough to prove unitarity of the extremal module $L^W(\theta_1/2, 1/2)$ at level $k = -2$, since all the other extremal modules at level $k < -2$ are obtained by iterated tensor product of $L^W(\theta_1/2, 1/2)$.

The unitarity of $L^W(\theta_1/2, 1/2)$ is proved by constructing this module as a submodule of a manifestly unitary module. This is achieved by using the free field realization of $W^{-2}_{\min}(psl(2|2))$ given in [3], in terms of four bosonic fields and four fermionic fields, which we now describe. Let $\mathcal{F}$ be the vertex algebra generated by four even fields a^i, $1 \le i \le 4$ and four odd fields b^i, $1 \le i \le 4$ with λ-bracket

$$[a_{i\,\lambda} a_j] = \delta_{ij}\lambda,\ [b_{i\,\lambda} b_j] = \delta_{ij},\ [a_{i\,\lambda} b_j] = 0.$$

There is an homomorphism $FFR : W^{-2}_{\min}(psl(2|2)) \to \mathcal{F}$ given by

$$\begin{aligned}
L &\mapsto \tfrac{1}{2}\sum_{i=1}^{4}(:a^i a^i: + :Tb^i b^i:)\\
J^+ &\mapsto -\tfrac{1}{2} :b^1b^3: -\tfrac{1}{2}\sqrt{-1} :b^1b^4: -\tfrac{1}{2}\sqrt{-1} :b^2b^3: +\tfrac{1}{2} :b^2b^4:\\
J^- &\mapsto \tfrac{1}{2} :b^1b^3: -\tfrac{1}{2}\sqrt{-1} :b^1b^4: -\tfrac{1}{2}\sqrt{-1} :b^2b^3: -\tfrac{1}{2} :b^2b^4:\\
J^0 &\mapsto -\sqrt{-1} :b^1b^2: -\sqrt{-1} :b^3b^4:\\
G^+ &\mapsto \tfrac{1}{2} :(a^1 + \sqrt{-1}a^2)(b^3 + \sqrt{-1}b^4): -\tfrac{1}{2} :(a^3 + \sqrt{-1}a^4)(b^1 + \sqrt{-1}b^2):\\
G^- &\mapsto \tfrac{1}{2} :(a^1 + \sqrt{-1}a^2)(b^1 - \sqrt{-1}b^2): +\tfrac{1}{2} :(a^3 + \sqrt{-1}a^4)(b^3 - \sqrt{-1}b^4):\\
\bar{G}^+ &\mapsto \tfrac{1}{2} :(a^1 - \sqrt{-1}a^2)(b^1 + \sqrt{-1}b^2): +\tfrac{1}{2} :(a^3 - \sqrt{-1}a^4)(b^3 + \sqrt{-1}b^4):\\
\bar{G}^- &\mapsto \tfrac{1}{2} :(a^1 - \sqrt{-1}a^2)(b^3 - \sqrt{-1}b^4): -\tfrac{1}{2} :(a^3 - \sqrt{-1}a^4)(b^1 - \sqrt{-1}b^2):
\end{aligned}$$

We define a conjugate linear involution ψ on $\mathcal{F}$ by

$$a_i \mapsto -a_i,\ b_i \mapsto -b_i$$

so that, according to [16, §5.1,5.2], there is a ψ-invariant positive definite Hermitian form $H_{\mathcal{F}}$ on $\mathcal{F}$. It is clear from (13.1) that $\psi \circ FFR = FFR \circ \phi$. Using FFR we can define an action of $W^{-2}_{\min}(psl(2|2))$ on $\mathcal{F}$. Since $H_{\mathcal{F}}$ is invariant with respect to the conformal vector $FFR(L)$, it follows that $\mathcal{F}$ is a unitary $W^{-2}_{\min}(psl(2|2))$-module. An easy calculation shows that $v = b^1 + \sqrt{-1}b^2$ is a singular vector for $W^{-2}_{\min}(psl(2|2))$, thus v generates a unitary highest weight representation $L^W(\nu, \ell_0)$ of $W^{-2}_{\min}(psl(2|2))$. Clearly $FFR(L)_0 v = \frac{1}{2}v$, while $J^0 v = v$, hence $\nu = \frac{1}{2}\theta_1$ and $\ell_0 = \frac{1}{2}$. This proves that the highest weight module corresponding the extremal weight $\nu = \frac{1}{2}\theta_1$ is indeed unitary.

13.3. $\mathfrak{g} = D(2,1;\frac{m}{n})$. In this case we are able to prove unitarity only in the very special case when either $m = 1$ of $n = 1$.

If $n = 1$, then the unitarity range is $\{-\frac{m}{m+1}N \mid N \in \mathbb{N}\}$. Take $N = 1$ and observe that $W_{\min}^{-\frac{m}{m+1}}(D(2,1;m))$ collapses to $V_{m-1}(sl(2))$. In this case there is only one extremal weight $\nu = \frac{m-1}{2}\alpha_2$, which gives rise to a unitary representation since it is integrable. The case $m = 1$ is dealt with in a similar way, switching the roles of α_2, α_3.

14. Characters of the Irreducible Unitary $W^k_{\min}(\mathfrak{g})$-Modules

Recall that, for $\Lambda \in \widehat{\mathfrak{h}}^*$, we denoted by $M^W(\Lambda)$ the Verma module $M^W(\nu,\ell)$, where (ν,ℓ) is given by (11.44). It follows from [20, (6.11)], that

$$ch\, M^W(\Lambda) = e^{\nu} q^{\ell} F^{NS}(q), \tag{14.1}$$

where $q = e^{(0,1)}$ and

$$F^{NS}(q) = \prod_{n=1}^{\infty} \frac{\prod_{\alpha\in\Delta_{1/2}}(1+q^{n-\frac{1}{2}}e^{-\alpha})}{(1-q^n)^{rank\mathfrak{g}^\natural+1}\prod_{\alpha\in\Delta_+^\natural}((1-q^{n-1}e^{-\alpha})(1-q^n e^{\alpha}))}. \tag{14.2}$$

In particular,

$$ch\, M^W(\widehat{\nu}_h) = e^{\nu} q^{\ell(h)} F^{NS}(q), \tag{14.3}$$

where $\ell(h)$ is given by (11.45).

The characters of unitary $W^k_{\min}(\mathfrak{g})$-modules $L^W(\nu,\ell_0)$ are computed by applying the quantum Hamiltonian reduction to the irreducible highest weight $\widehat{\mathfrak{g}}$-modules $L(\widehat{\nu}_h)$, where $\nu \in P^+_k$ and $\ell_0 = \ell(h)$, and using the argument in the proof of Theorem 11.9, which is based on Remark 11.7. There are two cases to consider in computation of their characters. First, if the weight $\widehat{\nu}_h$ is typical, i.e. conditions (11.5) hold, then $ch\, L(\widehat{\nu}_h)$ is given by the R.H.S. of (11.6), by Proposition 11.5.

The second case occurs when the weight $\widehat{\nu}_h$ satisfies the condition

$$(\widehat{\nu}_h + \widehat{\rho}|\alpha) = 0 \text{ for all } \alpha \in \Pi_{\bar{1}},$$

where $\Pi_{\bar{1}}$ denotes the set of simple isotropic roots of $\mathfrak{g}$. Then the weight $\widehat{\nu}_h$ is maximally atypical, and $L(\widehat{\nu}_h)$ is integrable, hence the following formula is a special case of [7, Formula (14)] if $\mathfrak{g} \neq D(2,1;\frac{m}{n})$ and of [7, Section 6.1] if $\mathfrak{g} = D(2,1;\frac{m}{n})$ and $\nu = 0$:

$$e^{\widehat{\rho}} R\, ch\, L(\widehat{\nu}_h) = \sum_{w\in\widehat{W}^\natural} det(w) w \frac{e^{\widehat{\nu}_h+\widehat{\rho}}}{\prod_{\beta\in\Pi_{\bar{1}}}(1+e^{-\beta})}, \tag{14.4}$$

where R equals the character of the Verma module $M(0)$ over $\widehat{\mathfrak{g}}$ with highest weight 0.

Theorem 14.1. *Let k be in the unitary range and let $\nu \in P^+_k$. Let $L^W(\nu,\ell_0)$ be a unitary irreducible $W^k_{\min}(\mathfrak{g})$-module. Choose h so that $\ell(h) = \ell_0$ and let, as before,*

$$\widehat{\nu}_h = k\Lambda_0 + \nu + h\theta.$$

(i) If $\ell_0 > A(k,\nu)$, then

$$chL^W(\nu,\ell_0) = \sum_{w\in\widehat{W}^\natural} det(w)chM^W(w.\widehat{\nu}_h). \tag{14.5}$$

(ii) If $\ell_0 = A(k,\nu)$, and $\nu = 0$ if $\mathfrak{g} = D(2,1;\frac{m}{n})$, then

$$chL^W(\nu,\ell_0) = \sum_{w\in\widehat{W}^\natural}\sum_{\gamma\in\mathbb{Z}_+\Pi_{\bar 1}} (-1)^\gamma det(w)chM^W(w.(\widehat{\nu}_h-\gamma)), \tag{14.6}$$

where $\Pi_{\bar 1} = \{\gamma_1,\gamma_2,\dots\}$ is the set of isotropic simple roots for $\mathfrak{g}$, and for $\gamma = n_1\gamma_1+\cdots$, we write $(-1)^\gamma = (-1)^{n_1+\cdots}$.

Proof. Formula (14.5) follows from (11.46). Formula (14.6) follows from (14.4) by applying quantum Hamiltonian reduction to the $\widehat{\mathfrak{g}}$-module $L(\widehat{\nu}_h)$. In order to use (14.4), write explicitly the relation $\ell_0 = \ell(h) = A(k,\nu)$. We have

$$\begin{aligned}&\frac{(k\Lambda_0+h\theta+\nu|k\Lambda_0+h\theta+\nu+2h^\vee\Lambda_0+2\rho)}{2(k+h^\vee)} - h\\ &= \frac{(\nu|\nu+2\rho^\natural)}{2(k+h^\vee)} + \frac{(\xi|\nu)}{k+h^\vee}((\xi|\nu)-k-1),\end{aligned}$$

or

$$h(h-1-k) = (\xi|\nu)((\xi|\nu)-k-1).$$

Hence either $h = (\xi|\nu)$ or $h = 1+k-(\xi|\nu)$. We observe that if $\alpha\in\Pi_{\bar 1}$, then, restricted to $\mathfrak{h}^\natural$, it coincides with $-\xi$, hence $(\xi|\nu) = -(\alpha|\nu)$, and also $(\theta|\alpha) = 1$. Therefore, for $h = (\xi|\nu)$ we have

$$(\widehat{\nu}_h+\widehat{\rho}|\alpha) = ((k+h^\vee)\Lambda_0+(\xi|\nu)\theta+\nu+\rho|\alpha) = (\xi|\nu)+(\alpha|\nu) = 0.$$

Hence we may apply (14.4). Note that $H_0(L(\widehat{\nu}_h)) \neq 0$ since $(\widehat{\nu}_h|\alpha_0) < 0$, so that we can apply Remark 11.7. □

Remark 14.2. It is still an open problem whether in the case $\mathfrak{g} = D(2,1;\frac{m}{n})$ formula (14.4) holds for an arbitrary $\nu\in P_k^+$.

Remark 14.3. For the $N = 4$ superconformal algebra, formula (14.5) appears, in a different form, in [4, formula (14)], where it has been derived in a non-rigorous way. To establish a dictionary to match the two formulas first observe that a parameter y occurs in the formulas of [4] corresponding to an extra $U(1)$-symmetry that we do not consider, hence, to compare the formulas, we set $y = 1$. Next recall that in this case $\widehat{W}^\natural$ is of type $A_1^{(1)}$, hence its elements are of the form $u_i = \underbrace{s_0s_1\cdots}_{i\text{ factors}}$ or $u_i' = \underbrace{s_1s_0\cdots}_{i\text{ factors}}$ (set $u_0 = u_0' = Id$). In the notation of [4], the pairs (a_n,b_n) corresponding to the α-series (resp. β-series) in formula (12) of [4] match exactly the pairs (ν,ℓ) given in (11.44) for the weight $\Lambda = u_i.\widehat{\nu}_h$ (resp. $\Lambda = u_i'.\widehat{\nu}_h$). The factor $F^{NS}(\theta,1)$ translates precisely to (14.3) according to the dictionary

$$e^{\delta_1-\delta_2} \leftrightarrow e^{\sqrt{-1}\theta}.$$

The character formula (14.6) corresponds to the formula (26) in [4] for the character of "massless" representations. To show this, we first remark that, if $\gamma \in \mathbb{Z}_+\Pi_{\bar{1}}$, then

$$M^W(w.(\widehat{\nu}_h - \gamma)) = M^W(\nu, \ell),$$

where (ν, ℓ) is given by (11.44). In particular

$$\begin{aligned}
\ell &= \frac{(w.(\widehat{\nu}_h - \gamma)|w.(\widehat{\nu}_h - \gamma) + 2\widehat{\rho})}{2(k+h^\vee)} - (w.(\widehat{\nu}_h - \gamma))(x+d)\\
&= \frac{||\widehat{\nu}_h - \gamma + \widehat{\rho}||^2 - ||\widehat{\rho}||^2}{2(k+h^\vee)} - (w.(\widehat{\nu}_h - \gamma))(x+d)\\
&= \frac{||\widehat{\nu}_h + \widehat{\rho}||^2 - ||\widehat{\rho}||^2}{2(k+h^\vee)} - w.(\widehat{\nu}_h)(x+d) + w(\gamma)(x+d)\\
&= \ell(h) + (\widehat{\nu}_h + \widehat{\rho})(x+d) - w(\widehat{\nu}_h + \widehat{\rho})(x+d) + w(\gamma)(x+d),
\end{aligned}$$

hence, using formula (14.1),

$$ch\, M^W(w.(\widehat{\nu}_h - \gamma)) = q^{\ell(h)} F^{NS}(q) e^{(w.\widehat{\nu}_h)_{|\mathfrak{h}^\natural}} q^{(\widehat{\nu}_h + \widehat{\rho} - w(\widehat{\nu}_h + \widehat{\rho}))(x+d)} e^{-(w\gamma)_{|\mathfrak{h}^\natural}} q^{w(\gamma)(x+d)},$$

and we obtain that

$$\sum_{\gamma \in \mathbb{Z}_+\Pi_{\bar{1}}} (-1)^\gamma ch\, M^W(w.(\widehat{\nu}_h - \gamma)) = q^{\ell(h)} F^{NS}(q) \frac{e^{(w.(\widehat{\nu}_h))_{|\mathfrak{h}^\natural}} q^{(\widehat{\nu}_h + \widehat{\rho} - w(\widehat{\nu}_h + \widehat{\rho}))(x+d)}}{\prod_{\alpha \in \Pi_{\bar{1}}} (1 + e^{-w(\alpha)_{|\mathfrak{h}^\natural}} q^{w(\alpha)(x+d)})}. \tag{14.7}$$

Since θ is orthogonal to $(\widehat{\mathfrak{h}^\natural})^*$ (where $\widehat{\mathfrak{h}^\natural} = \mathbb{C}K + \mathbb{C}d + \mathfrak{h}^\natural$), we can apply the formulas of [12, Chapter 6] to $\widehat{\mathfrak{g}}^\natural$ and its Weyl group. Since, in our case, $\widehat{\nu}_h + \widehat{\rho} = k\Lambda_0 + (h - \frac{1}{2})\theta + (r + \frac{1}{2})\eta_1$, $r \in \frac{1}{2}\mathbb{Z}_+$, we have, for $m \in \mathbb{Z}$,

$$(s_0 s_1)^m (\widehat{\nu}_h + \widehat{\rho}) = k\Lambda_0 + (h - \tfrac{1}{2})\theta + (r - km + \tfrac{1}{2})\eta_1 - (m(-km + 2r + 1))\delta.$$

and, if $\alpha \in \Pi_{\bar{1}}$,

$$(s_0 s_1)^m(\alpha) = \alpha + m\delta.$$

Since $s_1 = s_{\eta_1}$, it follows that

$$\begin{aligned}
&((s_0 s_1)^m.\widehat{\nu}_h)_{|\mathfrak{h}^\natural} = (r - km)\eta_1,\ ((s_1(s_0 s_1)^m).\widehat{\nu}_h)_{|\mathfrak{h}^\natural} = -(r - km + 1)\eta_1,\\
&\widehat{\nu}_h + \widehat{\rho} - (s_0 s_1)^m(\widehat{\nu}_h + \widehat{\rho})(x+d) = \widehat{\nu}_h + \widehat{\rho} - s_1(s_0 s_1)^m(\widehat{\nu}_h + \widehat{\rho})(x+d)\\
&= (m(-km + 2r + 1)) = -km^2 + (2r + 1)m,
\end{aligned}$$

and

$$\begin{aligned}
&(s_0 s_1)^m(\alpha)_{|\mathfrak{h}^\natural} = -\tfrac{1}{2}\eta_1,\ s_1(s_0 s_1)^m(\alpha)_{|\mathfrak{h}^\natural} = \tfrac{1}{2}\eta_1,\\
&(s_0 s_1)^m(\alpha)(x+d) = s_1(s_0 s_1)^m(\alpha)(x+d) = (m + \tfrac{1}{2}).
\end{aligned}$$

Substituting (14.7) into (14.6), recalling that $M_1(k) = -k - 1$, we obtain

$ch\, L^W(r\eta_1, \ell_0)$

$$= q^{\ell_0} F^{NS}(q) \sum_{m\in\mathbb{Z}} \left(\frac{e^{(r+m(M_1(k)+1))\eta_1}}{(1+e^{\frac{1}{2}\eta_1} q^{m+\frac{1}{2}})^2} - \frac{e^{-(r+m(M_1(k)+1)+1)\eta_1}}{(1+e^{-\frac{1}{2}\eta_1} q^{m+\frac{1}{2}})^2} \right) q^{m^2(M_1(k)+1)+(2r+1)m}$$

which, under our dictionary, corresponds to formula (26) of [4] in the NS sector.

For $W^k_{\min}(spo(2|3))$, formula (14.5) appears (with a non-rigorous proof) in [21, formula (4.3)]. Again, in this case $\widehat{W}^{\natural}$ is of type $A_1^{(1)}$and its elements are of the form u_i or u'_i (notation as above). The pairs (l_n, h_n) displayed in [21, (4.2.a),(4.2.b)], corresponding to the A-series (resp. B-series), match exactly the pairs (ν, ℓ) given in (11.44) for the weight $\Lambda = u_i.\widehat{\nu}_h$ (resp. $\Lambda = u'_i.\widehat{\nu}_h$). The denominator $F^{NS}(q, z)$ in [21, (3.15.i)] translates precisely to (14.3) according to the dictionary

$$e^{\epsilon_1} \leftrightarrow z.$$

In the massless case, the character formula (14.6) corresponds to formula (4.6.1) in [21], hence Theorem 14.1 provides a proof of it, since formula (14.4) holds in this case, due to [7, Subsection 12.3].

Note added in proof: Conjecture 4 has been proved in a joint paper with Drazen Adamović.

Acknowledgements. V.K. is partially supported by the Stephen Berenson mathematical exploration Fund and the Simons collaboration Grant. The authors thank Drazen Adamović, Thomas Creutzig and Anne Taormina for correspondence. The authors are grateful to Maria Gorelik for many very useful and enlightening discussions.

Funding Open access funding provided by Università degli Studi di Roma La Sapienza within the CRUI-CARE Agreement.

Data Availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interests The authors have no competing interests to declare that are relevant to the content of this article.

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Communicated by Y. Kawahigashi