Pinning Control of Hypergraphs

Pietro De Lellis, *Senior Member, IEEE*, Fabio Della Rossa, Francesco Lo Iudice, *Member, IEEE*, and Davide Liuzza

***Abstract*—A standard assumption in control of network dynamical systems is that its nodes interact through pairwise interactions, which can be described by means of a directed graph. However, in several contexts, multibody, directed interactions may occur, thereby requiring the use of directed hypergraphs rather then digraphs. For the first time, we propose a strategy, inspired by the classic pinning control on graphs, that is tailored for controlling network systems coupled through a directed hypergraph. By drawing an analogy with signed graphs, we provide sufficient conditions for controlling the network onto the desired trajectory provided by the pinner, and a dedicated algorithm to design the control hyperedges.**

***Index Terms*—Pinning control, higher-order interactions, directed hypergraphs, consensus and synchronization, simplicial complexes, networks.**

I. Introduction

IN THE last decades, network science made strides towards understanding how to analyze and control the behavior of complex dynamical systems composed by several interconnected units [1], [2]. A landmark achievement from the research community has been the discovery of how collective behaviors such as consensus and synchronization may spontaneously emerge or be induced within the system [3], [4]. Uncovering these fundamental mechanisms has been paramount in a plethora of applications, which include robotic flocking [5], [6], control of epidemics [7], [8], [9], migration dynamics [10], [11], and control of power grids [12]. In all these diverse contexts, it was observed how the collective behaviors emerging in the system were the result of the individual dynamics taking place at each unit, combined with the effect of their mutual interactions.

Manuscript received 21 July 2022; revised 30 September 2022; accepted 21 October 2022. Date of publication 26 October 2022; date of current version 14 November 2022. The work of Pietro De Lellis was supported by the Research Project PRIN 2017 "Advanced Network Control of Future Smart Grids" funded by the Italian Ministry of University and Research (2020–2023). The work of Francesco Lo Iudice was supported in part by the Research Project PRIN 2017 "Advanced Network Control of Future Smart Grids" funded by the Italian Ministry of University and Research (2020–2023) and in part by the Research Grant "BIOMASS" funded by the University of Naples Federico II—"Finanziamento della Ricerca di Ateneo (FRA)—Linea B." Recommended by Senior Editor J. Daafouz. *(Corresponding author: Pietro De Lellis.)*

Pietro De Lellis and Francesco Lo Iudice are with the Department of Electrical Engineering and Information Technology, University of Naples Federico II, 80125 Naples, Italy (e-mail: pietro.delellis@unina.it).

Fabio Della Rossa is with the Department of Electronics, Information, and Bioengineering, Politecnico di Milano, 20133 Milan, Italy.

Davide Liuzza is with the Department of Engineering, University of Sannio, 82100 Benevento, Italy.

Digital Object Identifier 10.1109/LCSYS.2022.3217093

The study of network dynamical systems has typically relied on the assumption that interactions between the units are pairwise, whereby a directed graph was the natural framework to describe the system: each unit was a node in the graph, whereas a link from node j to node i implied an influence of the dynamics of unit j on the dynamics of unit i. However, the exclusive focus on pairwise interactions is a fundamental limitation of classic network descriptions, as in many empirical systems, such as neuronal networks, opinion dynamics, social contagion, and control of epidemics, we observe many-body interactions [13], [14], [15], [16].

Hypergraphs and simplicial complexes are instead the ideal framework to encapsulate the higher-order interactions taking place in complex systems, and can be viewed as generalizations of the concept of graphs, as they consider also many-body interactions encompassing more than two nodes [17]. Recently, local conditions for the emergence of a synchronous behaviour have been derived in [18], consensus-like dynamics have been studied [19], [20], [21], and a series of rich and novel behaviors were observed in networks of Kuramoto oscillators coupled through simplicial structures [22], [23], [24], suitable to describe undirected interactions.

Considering multi-body interactions is also crucial towards control applications, where directed interactions should be considered. For instance, in pinning control a virtual node, the pinner, exerts a control action that is proportional to the difference between its state and that of a unit in the system [25], [26], [27]. The hidden assumption here is that the pinner can measure the state of a given network node, so that this interaction can be well represented by a directed edge from the pinner to the controlled node. However, due to limitations on sensing and actuation, in several applications it is not possible to measure the state of a given node, but rather a function of the state of a group of nodes. For instance, in feedback control of microbial consortia, fluorescence cannot always be measured at the level of the single node (i.e., the single cell) but rather an aggregated measurement of the fluorescence of groups of cells is obtained [28]. Similarly, limits on the actuation imply that the same control signal has to be injected on more nodes (i.e., more cells).

In this letter, we propose to model these kinds of control actions in terms of directed hyperedges from the pinner (the tail of the hyperedge) to a subset of nodes (the heads of the hyperedge). This modeling approach allows to encompass cases where a group of nodes is subject to the same control action. The manuscript contributions are as follow:

- we generalize the concept of pinning control to consider the presence of higher-order interactions by using the formalism of directed hypergraphs [29],

- we show how dynamics over a directed hypergraphs can be studied over an equivalent signed graph, and build a mapping between the two formulations.

This cogent mapping further allowed us to

- show that the standard conditions for pinning control on digraphs do not trivially extend in the presence of higher-order interactions,
- derive sufficient conditions on the hypergraph structure, on the hyperedges connecting the pinner with the controlled network, and on the individual dynamics so that the higher-order network is asymptotically controlled to the trajectory of the pinner, and
- introduce a heuristic algorithm that, when the network topology is known, allows to select the pinning hyperedges to control the network onto the desired trajectory.

II. Mathematical Preliminaries

A. Matrices and Vector Fields

Given a positive scalar n, we denote by I_n the identity matrix of size n, by $\mathbb{1}_n$ and 0_n the vectors of all ones and all zeros in $\mathbb{R}^n$, respectively. The canonical basis for $\mathbb{R}^n$ is $\{\mathfrak{e}_1, \dots, \mathfrak{e}_n\}$, where $\mathfrak{e}_i$ is the i-th column of I_n. Given a matrix $M \in \mathbb{R}^{n\times n}$, M^{T} is its transpose, $M_{\mathrm{sym}} = (M + M^{\mathrm{T}})/2$ its symmetric part, $\det(M)$ its determinant, and $M > 0$ ($M \geq 0$) means that it is positive (semi-)definite.

Given two matrices $M_1 \in \mathbb{R}^{a\times b}$ and $M_2 \in \mathbb{R}^{c\times d}$, we denote $(M_1 \otimes M_2) \in \mathbb{R}^{ac\times bd}$ their Kronecker product [30], and, when they have the same number of columns ($b = d$), we denote $[M_1; M_2] \in \mathbb{R}^{(a+c)\times b}$ their vertical concatenation, whereas when they have the same number of rows ($a = c$), we denote $[M_1, M_2] \in \mathbb{R}^{a\times(b+d)}$ their horizontal concatenation. When M_1 and M_2 are square and have the same size ($a = b = c = d$), we denote $M_1 \odot M_2$ their bialternate product [31].

Lemma 1 [32]: Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of $M \in \mathbb{R}^{n\times n}$. The eigenvalues of $2A \odot I_n$ are $\lambda_{(i-1)n+j} = \lambda_i + \lambda_j$, $i, j = 1, \dots, n$.

In what follows, given a matrix $M \in \mathbb{R}^{n\times n}$, we sort its eigenvalues $\lambda_1(M), \dots, \lambda_n(M)$ so that their real part is in ascending order, that is, $\Re(\lambda_1(M)) \leq \dots \leq \Re(\lambda_n(M))$.

Definition 1 [33]: A vector field $\phi : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is QUAD if, for any $P = P^{\mathrm{T}} > 0$, there exist a positive definite diagonal matrix Δ such that, for all $y, z \in \mathbb{R}^n$, $t \in \mathbb{R}_{\geq 0}$

$$(y - z)^{\mathrm{T}} P(f(y, t) - f(z, t)) \leq (y - z)^{\mathrm{T}} \Delta(y - z). \tag{1}$$

Globally Lipschitz vector fields and differentiable vector fields with bounded Jacobian fulfil (1) [33], whereas equivalences with the Demidovich condition and boundedness of the matrix measure of the Jacobian can be found in [34].

B. Directed Hypergraphs [29]

A directed hypergraph $\mathscr{H}$ is a pair $(\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{v_1, \dots, v_N\}$ is the set of nodes, and $\mathcal{E} = \{\varepsilon_1, \dots, \varepsilon_M\}$ is the set of directed hyperedges; the i-th directed hyperedge ε_i of $\mathscr{H}$ is an ordered pair $(\mathcal{T}(\varepsilon_i), \mathcal{H}(\varepsilon_i))$ of (possibly empty) disjoint subsets of the hypergraph nodes. Namely, the ordered subsets $\mathcal{T}(\varepsilon_i)$ and $\mathcal{H}(\varepsilon_i)$ of $\mathcal{V}$, are the set of tails and heads of the hyperedge ε_i, with cardinality $|\mathcal{T}(\varepsilon_i)|$ and $|\mathcal{H}(\varepsilon_i)|$, respectively, and such that $\mathcal{T}(\varepsilon_i) \cap \mathcal{H}(\varepsilon_i) = \emptyset$. The functions $\mathfrak{t}(\varepsilon, i)$ and $\mathfrak{h}(\varepsilon, j)$ associate to the i-th tail and j-th head of a hyperedge $\varepsilon \in \mathcal{E}$ the corresponding labels in $\mathcal{V}$, respectively. Furthermore, given two node subsets $\mathcal{V}_1, \mathcal{V}_2 \subseteq \mathcal{V}$, we denote $\mathcal{E}^{\mathcal{V}_1,\mathcal{V}_2} = \{\varepsilon \in \mathcal{E} : \mathcal{V}_1 \subseteq \mathcal{T}(\varepsilon) \wedge \mathcal{V}_2 \subseteq \mathcal{H}(\varepsilon)\}$; with a slight abuse of notation, when a subset is a singleton, we will refer to it by its only element, e.g., if $\mathcal{V}_1 = \{v_j\}$ we write $\mathcal{E}^{j,\mathcal{V}_2}$ in place of $\mathcal{E}^{\{v_j\},\mathcal{V}_2}$. Finally, we denote $\mathcal{E}^{\cdot,j} = \{\varepsilon_i \in \mathcal{E} : v_j \in \mathcal{H}(\varepsilon_i)\}$ as the subset of hyperedges having v_j as a head.

C. Signed Graphs [35]

A weighted signed graph $\mathscr{S}$ is defined by the triple $\{\mathcal{V}, \mathcal{E}, \mathcal{W}\}$, where $\mathcal{V}$ is the set of nodes, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges, and the function $\mathcal{W} : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ associates 0 to each pair $(i, j) \in \mathcal{V} \times \mathcal{V}$ that is not in $\mathcal{E}$, and a non-zero weight to each edge in $\mathcal{E}$. Different from standard weighted digraphs, also negative weights can be associated to edges. The adjacency matrix associated to $\mathscr{S}$ is such that its ij-th entry a_{ij} is equal to the weight $\mathcal{W}(i, j)$ associated to edge (i, j). The Laplacian matrix for signed graphs has been defined as $L = D - A$, where $D = \mathrm{diag}\{d_1^{\mathrm{out}}, \dots, d_{|\mathcal{V}|}^{\mathrm{out}}\}$, with $d_i^{\mathrm{out}} = \sum_{j=1}^{|\mathcal{V}|} a_{ij}$ being the out-degree of node i. By definition, L is zero row-sum, which implies that $0 \in \mathrm{spec}(L)$, and that $\mathbb{1}_N$ is its associated (right) eigenvector.

Definition 2 [36]: Given a matrix $S \in \mathbb{R}^{(|\mathcal{V}|-1)\times|\mathcal{V}|}$, whose rows are an orthonormal basis for the orthogonal complement of $\mathrm{span}(\mathbb{1}_{|\mathcal{V}|})$, matrix $\overline{L} = SLS^{\mathrm{T}}$, which belongs to $\mathbb{R}^{(|\mathcal{V}|-1)\times(|\mathcal{V}|-1)}$, is called the *reduced Laplacian* of the signed graph $\mathscr{S}$.

The superimposition of two signed graphs $\mathscr{S}_1 = \{\mathcal{V}, \mathcal{E}_1, \mathcal{W}_1\}$ and $\mathscr{S}_2 = \{\mathcal{V}, \mathcal{E}_2, \mathcal{W}_2\}$ is the graph $\mathscr{S} = \mathscr{S}_1 \oplus \mathscr{S}_2 = \{\mathcal{V}, \mathcal{E}, \mathcal{W}\}$, where $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$, and $\mathcal{W}(i, j) = \mathcal{W}_1(i, j) + \mathcal{W}_2(i, j)$.

III. Pinning Control of Hypergraphs

A. Controlled Network on a Directed Hypergraph

Let us consider a network dynamical system on a hypergraph $\mathscr{H}_c = \{\mathcal{V}_c, \mathcal{E}_c\}$, which we call the controlled hypergraph, where $\mathcal{V}_c = \{v_1, \dots, v_N\}$ is the set of controlled nodes, and $\mathcal{E}_c = \{\varepsilon_1, \dots, \varepsilon_N\}$ the set of directed hyperedges of $\mathscr{H}_c$. We associate to a node $v_i \in \mathcal{V}_c$ a state variable $x_i \in \mathbb{R}^n$, and to a hyperedge $\varepsilon \in \mathcal{E}_c$ a tail state matrix $x_\varepsilon^\tau = [x_{\mathfrak{t}(\varepsilon,1)}, \dots, x_{\mathfrak{t}(\varepsilon,|\mathcal{T}(\varepsilon)|)}]$ and a head state matrix $x_\varepsilon^h = [x_{\mathfrak{h}(\varepsilon,1)}, \dots, x_{\mathfrak{h}(\varepsilon,|\mathcal{H}(\varepsilon)|)}]$, with $x_{\mathfrak{t}(\varepsilon,i)}$ and $x_{\mathfrak{h}(\varepsilon,j)}$ being the state of the i-th tail and of the j-th head of ε, respectively. The node dynamics are given by

$$\dot{x}_i = f(x_i, t) + \sum_{\varepsilon\in\mathcal{E}_c^{\cdot,i}} \sigma_\varepsilon(x_\varepsilon^\tau \alpha_\varepsilon - x_\varepsilon^h \beta_\varepsilon) + u_i, \tag{2}$$

where $f : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is the continuous and differentiable vector field describing the individual dynamics, σ_ε is the coupling gain associated to the hyperedge ε; $\alpha_\varepsilon = [(\alpha_\varepsilon)_{\mathfrak{t}(\varepsilon,1)}, \dots, (\alpha_\varepsilon)_{\mathfrak{t}(\varepsilon,|\mathcal{T}(\varepsilon)|)}]^{\mathrm{T}}$ and $\beta_\varepsilon = [(\beta_\varepsilon)_{\mathfrak{h}(\varepsilon,1)}, \dots, (\beta_\varepsilon)_{\mathfrak{h}(\varepsilon,|\mathcal{H}(\varepsilon)|)}]^{\mathrm{T}}$ are the (ordered) vectors stacking the weights associated to the tails and heads of ε, respectively, such that $\alpha_\varepsilon^{\mathrm{T}} \mathbb{1}_{|\mathcal{T}(\varepsilon)|} = \beta_\varepsilon^{\mathrm{T}} \mathbb{1}_{|\mathcal{H}(\varepsilon)|} = 1$.

The hyperdiffusive coupling protocol in (2) implies that each head of a hyperedge receives a signal which is the difference between a convex combination of the states of the tails and a convex combination of the states of the heads of the hyperedge. This protocol is a synchronization noninvasive coupling functions, according to the definition given in [18]. Indeed, when all the states are identical, the summation in (2) would be naught. Furthermore, we note that, if $\mathscr{H}_c$ were a digraph, one would have

$$\sum_{\varepsilon\in\mathcal{E}_c^{:,i}} \sigma_\varepsilon(x_\varepsilon^\tau\alpha_\varepsilon - x_\varepsilon^h\beta_\varepsilon) = \sum_{j\in\mathcal{N}_{\text{in}}(i)} \sigma_{ij}(x_j - x_i),$$

with $\mathcal{N}_{\text{in}(i)}$ being the in-neighborhood of i, that corresponds to the diffusive coupling protocol for networks on digraphs.

The goal of the control input u_i is to steer the dynamics of network (2) onto the trajectory of an additional node v_s, the pinner, sharing the same individual dynamics of the controlled nodes, and whose state is denoted $x_s \in \mathbb{R}^n$. The desired trajectory is defined as the solution of

$$\dot{x}_s = f(x_s, t), \tag{3}$$

where $x_s(0) = x_s^0$ is the pinner's initial condition.

The enlarged network system that includes the controlled nodes and the pinner can be described in terms of an augmented hypergraph $\mathscr{H} = \{\mathcal{V}, \mathcal{E}\}$, of which $\mathscr{H}_c$ is a proper sub-hypergraph. More specifically, the node set $\mathcal{V}$ also includes the pinner, that is, $\mathcal{V} = \mathcal{V}_c \cup \{v_s\}$. Note that the dynamics of the pinner, by definition (3), are not influenced by those of the controlled nodes, and therefore it is a node that cannot be a head of any hyperedge, that is, $\mathcal{E}$ is such that $s \notin \bigcup_{\varepsilon\in\mathcal{E}} \mathcal{H}(\varepsilon)$. We can then decompose $\mathcal{E}$ as

$$\mathcal{E} = \mathcal{E}_c \cup \bigcup_{i=1}^{N} \mathcal{E}^{s,i},$$

where $\mathcal{E}^{s,i}$ is the set of hyperedges having the pinner v_s as a tail, and node v_i as a head. We define the set $\mathcal{P}$ of the pinned nodes, that is, the nodes who are directly influenced by the pinner, as $\mathcal{P} = \{i \in \mathcal{V} : \mathcal{E}^{s,i} \neq \emptyset\}$. Accordingly,

$$u_i = \begin{cases} \sum_{\varepsilon\in\mathcal{E}^{s,i}} k_\varepsilon(x_\varepsilon^\tau\alpha_\varepsilon - x_\varepsilon^h\beta_\varepsilon), & i \in \mathcal{P}, \\ 0, & \text{otherwise}, \end{cases} \tag{4}$$

where k_ε is the control gain associated to the hyperedge ε. Note that the pinner's state x_s is one of the elements of x_ε^τ.

Definition 3: We call a hyperdiffusive protocol *homogeneous* when $\alpha_\varepsilon = \mathbb{1}_{|\mathcal{T}(\varepsilon)|}/|\mathcal{T}(\varepsilon)|$ and $\beta_\varepsilon = 1_{|\mathcal{H}(\varepsilon)|}/|\mathcal{H}(\varepsilon)|$.

B. Control Objective and Pinning Error Dynamics

As in pinning control of graphs, we define the pinning error of node i as $e_i = x_i - s$, and the network pinning error as $e = [e_1; \cdots; e_N]$. The goal of the pinning control action u_i, defined in (4), is to steer the dynamics of the nodes in the controlled network toward the pinner's trajectory, that is,

$$\lim_{t\to+\infty} \|e(t)\| = 0. \tag{5}$$

From (2)–(3), the pinning error dynamics can be written as

$$\dot{e}_i = f(x_i, t) - f(x_s, t) + \sum_{\varepsilon\in\mathcal{E}_c^{:,i}} \sigma_\varepsilon(x_\varepsilon^\tau\alpha_\varepsilon - x_\varepsilon^h\beta_\varepsilon) + u_i. \tag{6}$$

Note that, by summing and subtracting the state of the pinner to each element in the summation in (6), one obtains

$$\dot{e}_i = f(x_i, t) - f(x_s, t) + \sum_{\varepsilon\in\mathcal{E}_c^{:,i}} \sigma_\varepsilon(e_\varepsilon^\tau\alpha_\varepsilon - e_\varepsilon^h\beta_\varepsilon) + u_i, \tag{7}$$

where e_ε^τ and e_ε^h are the matrices stacking the error vectors of the tails and heads of ε, respectively.

IV. Main Results

Here, we show that the dynamics taking place on directed hypergraphs can be studied on an equivalent network on a signed graph. We start by showing that, given any directed hyperedge $\varepsilon \in \mathcal{E}$, the following proposition holds:

Proposition 1: Given any edge $\varepsilon \in \mathcal{E}$, for all $i \in \mathcal{H}(\varepsilon)$

$$x_\varepsilon^\tau\alpha_\varepsilon - x_\varepsilon^h\beta_\varepsilon = \sum_{j\in\mathcal{T}(\varepsilon)} \big((\alpha_\varepsilon)_j(x_j - x_i)\big) - \sum_{j\in\mathcal{H}(\varepsilon)} \big((\beta_\varepsilon)_j(x_j - x_i)\big) \tag{8}$$

Proof: Equations (8) can be obtained by summing and subtracting $(x_\varepsilon^h)_i$ to the left-hand side, and considering that $\alpha_\varepsilon^{\mathrm{T}}\mathbb{1}_{|\mathcal{T}(\varepsilon)|} = \beta_\varepsilon^{\mathrm{T}}\mathbb{1}_{|\mathcal{H}(\varepsilon)|} = 1$. ■

From Proposition 1, it follows that the presence of a directed hyperedge ε of weight σ_ε is equivalent to having positive incoming edges from all its tails to all its heads and negative edges between any two heads. In particular, one has that the edge from one of its tails, say i, to any of its heads has weight $(\alpha_\varepsilon)_i\sigma_\varepsilon$, whereas the edge from one of its heads, say j, to any of its other heads is $(\beta_\varepsilon)_j\sigma_\varepsilon$. To formally show this, let us introduce $z = [z_1; \ldots; z_{N+1}]$, where $z_i = x_i$ for $i = 1, \ldots, N$, and $z_{N+1} = x_s$, which stacks the states of the nodes of the controlled network and that of the pinner. Proposition 1 implies that the following equivalence hold:

Proposition 2: The dynamics (2)-(4) on the hypergraph $\mathscr{H}$ can be equivalently written as

$$\dot{z}_i = f(z_i, t) + \sum_{j=1}^{N+1} a_{ij}(z_j - z_i), \tag{9}$$

where a_{ij} is the ij-th entry of the adjacency matrix A of the signed graph $\mathscr{S} = \{\mathcal{V}, \mathcal{E}_s, \mathcal{W}\}$, defined as

$$a_{ij} = \sum_{\varepsilon\in\mathcal{E}_c^{j,i}} (\alpha_\varepsilon)_j\sigma_\varepsilon - \sum_{\varepsilon\in\mathcal{E}_c^{:,\{i,j\}}} (\beta_\varepsilon)_j\sigma_\varepsilon + \mathcal{I}_{\mathcal{P}}(i)\left(\sum_{\varepsilon\in\mathcal{E}^{\{s,j\},i}} (\alpha_\varepsilon)_j k_\varepsilon - \sum_{\varepsilon\in\mathcal{E}_c^{s,\{i,j\}}} (\beta_\varepsilon)_j k_\varepsilon\right), \tag{10}$$

with $\mathcal{I}_{\mathcal{P}}(i) = 1$ if $i \in \mathcal{P}$, and zero otherwise.

Proof: From Proposition 1, and by direct comparison between (2)-(4) and (9)-(10), the thesis follows. ■

Fig. 1 gives a graphical illustration of this equivalence.

Next, let us consider the Laplacian matrix L associated to A, and let us sort its eigenvalues in ascending order by their real part, so that $\Re(\lambda_1(L)) \leq \cdots \leq \Re(\lambda_{N+1}(L))$. Since the pinner's dynamics are independent of those of the other nodes, L can be written in block-triangular form as

$$L = \begin{bmatrix} M & \xi \\ 0_N^{\mathrm{T}} & 0 \end{bmatrix}. \tag{11}$$

Lemma 2: If $\Re(\lambda_2(L)) > 0$, then there exist symmetric positive definite matrices P and Q such that

$$-PM - M^{\mathrm{T}}P \leq -Q. \tag{12}$$

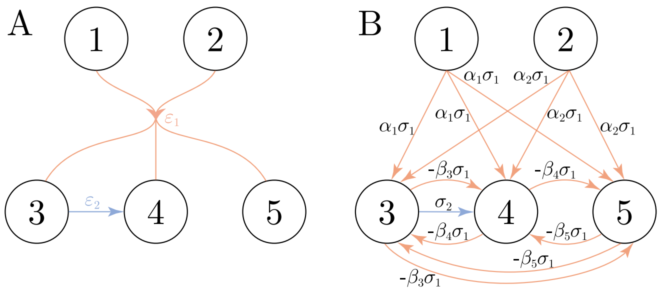
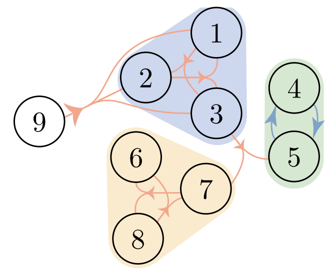


Fig. 1. A hypergraph (panel A) and its equivalent signed graph (panel B).



$$L=\begin{bmatrix} 5/6 & -2/3 & 5/6 & 0 & 0 & 0 & 0 & 0 & -1 \\ -2/3 & 5/6 & 5/6 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1/6 & -1/6 & 4/3 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 3/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1/2 & -1/2 & 0 \\ 0 & 0 & -1 & 0 & 1/2 & 1/2 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & -1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Fig. 2. Sample hypergraph $\mathscr{H}$, where node 9 is the pinner, and its associated Laplacian matrix L.

Proof: Since L is zero row-sum, $0 \in \mathrm{spec}(L)$, with associated eigenvector $\mathbb{1}_N$. From the hypotheses, all the other eigenvalues will have a positive real part. From the block-triangular decomposition (11), $\mathrm{spec}(L) = \mathrm{spec}(M) \cup \{0\}$, which implies all the eigenvalues of M have positive real part, whereby $-M$ is Hurwitz, and the thesis follows. ■

We are now ready to state the following result:

Theorem 1: If f is QUAD, $\Re(\lambda_2) > 0$, and

$$\lambda_{\min}\big((PM)_{\mathrm{sym}}\big) > \delta_{\max}, \tag{13}$$

where P is defined in (12) and $\delta_{\max} = \max_i \Delta_{ii}$, then network (2)-(4) is asymptotically pinning controlled.

Proof: Let us consider the Lyapunov function candidate

$$V = \frac{1}{2} e^{\mathrm{T}} (P \otimes I_n) e, \tag{14}$$

Using Proposition 2, error dynamics (7) can be rewritten as

$$\dot{e}_i = f(x_i, t) - f(x_s, t) - \sum_{j=1}^{N} m_{ij} e_j, \tag{15}$$

where m_{ij} is the ij-th entry of matrix M in (11). In compact form, one can write $\dot{e} = F(x, x_s, t) - (M \otimes I_n)e$ where $F(x, x_s, t) = [f(x_1, t) - f(x_s, t); \cdots ; f(x_N, t) - f(x_s, t)]$. We can then differentiate V, thereby obtaining

$$\dot{V} = e^{\mathrm{T}}(P \otimes I_n)\dot{e} = W_1 + W_2, \tag{16}$$

where $W_1 = \sum_{i=1}^{N} e_i^{\mathrm{T}} P(f(x_i, t) - f(x_s, t))$ and $W_2 = -e^{\mathrm{T}} (PM \otimes I_n)e$.

A) Bounding W_1: Since the vector field f is QUAD, we then have that

$$(x_i - x_s)^{\mathrm{T}} P(f(x_i, t) - f(x_s, t)) \leq (x_i - x_s)^{\mathrm{T}} \Delta (x_i - x_s)$$

for all $i = 1, \ldots, N$, which implies

$$W_1 \leq e^{\mathrm{T}} (I_N \otimes \Delta) e \leq \delta_{\max} e^{\mathrm{T}} e. \tag{17}$$

B) Bounding W_2: Note that $W_2 = -e^{\mathrm{T}}((PM)_{\mathrm{sym}} \otimes I_n)e$. From Lemma 2, all the eigenvalues of $(PM)_{\mathrm{sym}}$ are positive, whereby we have

$$W_2 \leq \lambda_{\min}\big((PM)_{\mathrm{sym}}\big) e^{\mathrm{T}} e. \tag{18}$$

C) Bounding $\dot{V}$: Combining (17) and (18) yields

$$\dot{V} \leq (\delta_{\max} - \lambda_{\min}\big((PM)_{\mathrm{sym}}\big)) e^{\mathrm{T}} e, \tag{19}$$

From (13), the thesis follows. ■

One of the main results in pinning control over standard, positively weighted digraphs is that, if the coupling between any two nodes is sufficiently high, then it is sufficient to pin just one node in each root strongly connected component (RSCC) in the graph, so that there is a path from the pinner to any other network node [37]. This result does not trivially extend to hypegraphs, as the existence of a hyperpath from the pinner to all other nodes does not guarantee that $\Re(\lambda_2(L)) > 0$, as shown in the following paradigmatic example.

Example 1: Let us consider the hypergraph $\mathscr{H}$ illustrated in Fig. 2. The controlled hypergraph has one RSCC (the nodes highlighted in blue) pinned by a hyperedge with one tail, corresponding to the pinner node, and three heads, one for each of the nodes of the RSCC. Note that the Laplacian L associated to $\mathscr{S}$ has a 0 eigenvalue with multiplicity 2 (its associated eigenvectors are $\mathbb{1}_9$ and the eigenvector v, whose i-th element is equal to any scalar η if i belongs to the strongly connected component (SCC) in yellow, to $-\eta$ if it belongs to the SCC in green, and to zero otherwise). This implies that $\Re(\lambda_2) = 0$, and the assumptions of Theorem 1 do not hold.

This simple counterexample shows that the results on standard graphs do not trivially extend to hypergraphs. Indeed, the existence of a path from the pinner to all the other nodes is not sufficient anymore to enforce pinning controllability. In what follows, we are going to obtain a corollary that provides a sufficient condition, which can be algorithmically tested, to check whether it is possible to guarantee pinning control of a hypergraph just by increasing the control and coupling gains. The recursive nature of the condition will allow to identify the critical hyperedges that may prevent pinning controllability.

To start with, let us rewrite equations (2) and (4) as

$$\dot{x}_i = f(x_i) + \sigma \sum_{\varepsilon \in \mathcal{E}_c^{:,i}} \varsigma_\varepsilon (x_\varepsilon^\tau \alpha_\varepsilon - x_\varepsilon^h \beta_\varepsilon) + u_i, \tag{20}$$

$$u_i = \begin{cases} \sigma \sum_{\varepsilon \in \mathcal{E}^{s,i}} c_\varepsilon (x_\varepsilon^\tau \alpha_\varepsilon - x_\varepsilon^h \beta_\varepsilon), & i \in \mathcal{P}, \\ 0, & \text{otherwise}, \end{cases} \tag{21}$$

where $\sigma > 0$ represents the overall coupling strength between interacting nodes, and $\varsigma_\varepsilon = \sigma_\varepsilon/\sigma$ and $c_\varepsilon = k_\varepsilon/\sigma$. Next, let us define $\mathcal{E}^- = \{(i, j) : a_{ij} < 0\}$, $\mathcal{E}^+ = \{(i, j) : a_{ij} > 0\}$, and $\mathcal{W}^+(i, j) = \mathcal{W}(i, j)$ if $(i, j) \in \mathcal{E}^+$, whereas $\mathcal{W}^+(i, j) = 0$ otherwise, and let us denote by $\mathscr{S}^+ = \{\mathcal{V}, \mathcal{E}^+, \mathcal{W}^+\}$ the subgraph of $\mathscr{S}$ that only contains the positive edges of $\mathscr{S}$. We can then compute the binary variable b with the iterative procedure described in Algorithm 1.

Note that Algorithm 1 leverages the continuity of the eigenvalues and Lemma 1, which allow to compute $\delta^\star$ as the

Algorithm 1 Computation of Parameter b

1: Initialization: $b = 1$, $\mathscr{S}_{\text{curr}} = \mathscr{S}^{+}$, and $\mathcal{E}_1 = \mathcal{E}^{-}$.
2: If the Laplacian matrix of $\mathscr{S}^{+}$ has 0 as a simple eigenvalue, set $\mathscr{S}_{\text{curr}} = \mathscr{S}^{+}$ and go to the next step. Otherwise, set $b = 0$ and terminate the algorithm.
3: Set $\mathscr{S}_1 = \mathscr{S}_{\text{curr}}$, and take randomly an edge $(i, j) \in \mathcal{E}_1$. Then, set $\mathscr{S}_2 = \{\mathcal{V}, \mathcal{E}_2, \mathcal{W}_2\}$, with $\mathcal{E}_2 = \{(i, j) \cup (j, i)\}$, $\mathcal{W}_2(i, j) = -\delta\sigma_{ij}$ and $\mathcal{W}_2(j, i) = -\delta\sigma_{ji}$, with $\sigma_{ij} = -a_{ij}$ if $-a_{ij} < 0$, whereas $\sigma_{ij} = 0$ otherwise, and $\sigma_{ji} = -a_{ji}$ if $a_{ji} < 0$, whereas $\sigma_{ji} = 0$ otherwise. Set $\mathcal{E}_1 = (\mathcal{E}_1 \cup \mathcal{E}_2) \setminus \mathcal{E}_2$.
4: Compute $\delta^\star$ as

$$\min_{\delta > 0} \delta \text{ s.t. } \det(2(\bar{L} + \delta \bar{L}_2) \odot I_{n-1}) = 0, \tag{22}$$

with $\bar{L}$ and $\bar{L}_2$ the reduced Laplacian of $\mathscr{S}_{\text{curr}}$ and $\mathscr{S}_2$
5: If $\delta^\star \leq 1$, set $b = 0$ and terminate the algorithm. If $\delta^\star > 1$ and $\mathcal{E}_1 = \emptyset$, terminate the algorithm with $b = 1$. Otherwise, update $\mathscr{S}_{\text{curr}}$ to $\mathscr{S}_1 \oplus \mathscr{S}_2$, and go to step 3.

normalized weight at which an eigenvalue (or a pair of complex eigenvalues) would cross the imaginary axes. Therefore, when the algorithm outputs $b = 1$, the Laplacian matrix associated to $\mathscr{S}$ has a simple 0 eigenvalue, and all its remaining eigenvalues have positive real part.

Corollary 1: If f is QUAD and $b = 1$, then there exists a finite σ in (20)-(21) such that network (2)-(4) is asymptotically controlled to the pinner's trajectory.

Proof: Since $b = 1$, then $\Re(\lambda_2) > 0$, and thus a positive definite symmetric matrix P exists such that $\lambda_{\min}((PM)_{\text{sym}}) > 0$. As (20)-(21) ensure that increasing σ linearly increases $\lambda_{\min}((PM)_{\text{sym}})$, it is always possible to find a sufficiently large σ such that $\lambda_{\min}((PM)_{\text{sym}}) > \delta_{\max}$. From Theorem 1, the thesis follows. ■

Remark 1: From Step 2 of Algorithm 1, assuming $b = 1$ in Corollary 1 means that $\mathscr{S}^{+}$ is such that there exists a directed spanning tree rooted in s, which is a necessary and sufficient condition for having a simple 0 eigenvalue in the Laplacian associated to $\mathscr{S}^{+}$ and all other eigenvalues with positive real part. The next steps of the algorithm are used to check that the addition of negative edges does not change this property, and does not yield another eigenvalue to have 0 or negative real part. In other words, Algorithm 1 identifies these critical negative edges in $\mathscr{S}$, and therefore the hyperedges in $\mathcal{H}$ that have a detrimental effect for control. In the next section, we provide a heuristic then provides a pinning hyperedge selection that counteracts these effects.

V. Heuristic for Pinning Hyperedge Selection

Here, we propose a heuristic for control design, whose iterative structure is reported in Algorithm 2, and which is based on Algorithm 1, that we devised to compute the variable b. Specifically, Steps 1 and 2 of Algorithm 2 leverage a structural condition for the existence in $\mathscr{S}^{+}$ of a directed spanning tree from the pinner, so that a necessary condition for having $b = 1$ is fulfilled (namely, Step 2 of Algorithm 1). Algorithm 2 is then used to identify the critical negative edges, and to subsequently add hyperedges in one of the SCCs of their endpoints (Steps 3-4). Note that the heuristic stops adding hyperedges only when $b = 1$, that is, when we are guaranteed

Algorithm 2 Heuristic for Pinning Hyperedge Selection

1: Perform the condensation of the digraph $\mathscr{S}^{+}$
2: For each RSCC of $\mathscr{S}^{+}$, add a pinning hyperedge whose heads are all in that RSCC. Update $\mathcal{H}$ and $\mathscr{S}^{+}$ accordingly.
3: Run Algorithm 1. If $b = 1$, terminate the algorithm, otherwise store the negative edge e^{-} in $\mathcal{E}^{-}$ that changed b to 0 in Step 5 of Algorithm 1.
4: Pick one of the two nodes that e^{-} connects, say i, and add a pinning hyperedge with heads only in the SCC to which i belongs to (including i). Update $\mathcal{H}$ and $\mathscr{S}^{+}$ accordingly, and return to step 1.

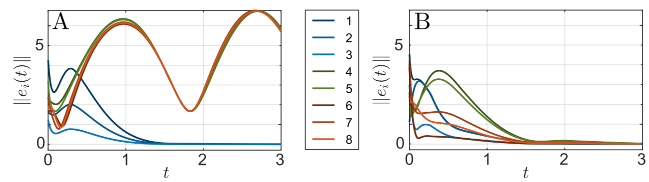


Fig. 3. Dynamics of the pinning error norm for a controlled network of 8 Chua systems when the hypergraph is that in Fig. 2 (panel A) and when the pinning hyperedge whose heads are node 6, 7, and 8 is added (panel B).

by Corollary 1 that we can find a sufficiently large coupling strength σ to control the hypergraph to the pinner's trajectory. In what follows, we demonstrate the effectiveness of this heuristic in the following illustrative application.

A. Pinning Control of Chua's Circuits

In the general equation (2), we consider as individual dynamics the well-known Chua's circuit [38]. Namely, we set $f(x_i, t) = [9(x_{i2} - R(x_{i1})); x_{i1} - x_{i2} + x_{i3}; -100/7\ x_{i2}]$, where $R(x_{i1}) = m_1 x_{i1} + (m_0 - m_1)(|x_{i1} + 1| - |x_{i1} - 1|)/2$, with $m_0 = -1/7$ and $m_1 = 2/7$ selected so that, in the absence of noise and coupling, the dynamics exhibits the double scroll chaotic attractor. Initially, the hypergraph through which the nodes are coupled is that in Fig. 2, where node 9 is the pinner, $\sigma_\varepsilon = 30$ for all ε, and the hyperdiffusive protocol is homogeneous (see Definition 3). Fig. 3A shows that the pinning error norm does not converge to zero. Consistently, Algorithm 1 yields $b = 0$, and Step 3 of Algorithm 2 identifies $(5, 7)$ as a critical negative edge in the equivalent signed graph. Following Step 4, we then add a pinning hyperedge to the multibody group to which node 7 belongs (that is, a hyperedge whose tail is the pinner, node 9, and whose heads are nodes 6, 7, and 8) and go back to Steps 1-2. Running again Algorithm 1, we obtain $b = 1$ and terminate the pinning hyperedge selection. Then, as the Chua's circuit is QUAD [2], both assumptions of Corollary 1 are fulfilled and, consistently, Fig. 3B shows that the pinning error norm converges to zero.

We emphasize that the heuristic not only achieves the control goal, but also strongly reduces the number of hyperedges required to control the network when compared to the case of a random addition of pinning hyperedges to those connecting the pinner with the RSCCs. Indeed, when tested on 100 randomly generated uniform hypergraphs of $N = 100$ nodes, the heuristic required only adding 3.2 ± 2.9 hyperedges, whereas a random selection yielded 31.0 ± 20.6 hyperedges.

VI. Conclusion

In this letter, we have introduced for the first time pinning control in networks of units coupled through a directed hypergraph. By establishing a fundamental equivalence between dynamics over hypergraphs with those on a special class of signed graphs, we have derived sufficient conditions for pinning controllability, which generalize those on standard digraphs. Different from what is observed on graphs, the existence of a spanning tree from the pinner to all other nodes is not sufficient for pinning control. Accordingly, we have introduced an algorithm that is capable of identifying the critical negative edges in the equivalent signed graphs that may hinder pinning control, which is then used by our heuristic for control design to select the pinning hyperedges.

There are several directions along which this letter could be extended. For instance, the design of nonlinear higher-order pinning inputs could be considered to enhance control performance. Moreover, as in standard graphs [39], decentralized estimation and control strategies could be sought to avoid the need of knowing the network structure, thus enhancing scalability. Finally, the impact of possible mismatches in the individual dynamics of the units on the convergence of the pinning error should be investigated.

References

[1] D. D. Šiljak, *Decentralized Control of Complex Systems*. Boston, MA, USA: Academic, 1991.

[2] F. D. Rossa and P. De Lellis, "Synchronization and pinning control of stochastic coevolving networks," *Annu. Rev. Control*, vol. 53, pp. 147–160, Jan. 2022.

[3] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1520–1533, Sep. 2004.

[4] L. M. Pecora and T. L. Carroll, "Master stability functions for synchronized coupled systems," *Phys. Rev. Lett.*, vol. 80, no. 10, p. 2109, 1998.

[5] H. G. Tanner, A. Jadbabaie, and G. J. Pappas, "Flocking in fixed and switching networks," *IEEE Trans. Autom. Control*, vol. 52, no. 5, pp. 863–868, May 2007.

[6] L. E. Beaver and A. A. Malikopoulos, "An overview on optimal flocking," *Annu. Rev. Control*, vol. 51, pp. 88–99, Jan. 2021.

[7] C. Nowzari, V. M. Preciado, and G. J. Pappas, "Analysis and control of epidemics: A survey of spreading processes on complex networks," *IEEE Control Syst. Mag.*, vol. 36, no. 1, pp. 26–46, Feb. 2016.

[8] F. D. Rossa et al., "A network model of Italy shows that intermittent regional strategies can alleviate the COVID-19 epidemic," *Nat. Commun.*, vol. 11, no. 1, pp. 1–9, 2020.

[9] L. Zino and M. Cao, "Analysis, prediction, and control of epidemics: A survey from scalar to dynamic network models," *IEEE Circuits Syst. Mag.*, vol. 21, no. 4, pp. 4–23, 4th Quart., 2021.

[10] F. Simini, M. C. González, A. Maritan, and A.-L. Barabási, "A universal model for mobility and migration patterns," *Nature*, vol. 484, no. 7392, pp. 96–100, 2012.

[11] P. De Lellis, M. R. Marín, and M. Porfiri, "Modeling human migration under environmental change: A case study of the effect of sea level rise in Bangladesh," *Earth's Future*, vol. 9, no. 4, 2021, Art. no. e2020EF001931.

[12] F. Dörfler, J. W. Simpson-Porco, and F. Bullo, "Electrical networks and algebraic graph theory: Models, properties, and applications," *Proc. IEEE*, vol. 106, no. 5, pp. 977–1005, May 2018.

[13] E. Ganmor, R. Segev, and E. Schneidman, "Sparse low-order interaction network underlies a highly correlated and learnable neural population code," *Proc. Nat. Acad. Sci. United States America*, vol. 108, no. 23, pp. 9679–9684, 2011.

[14] Z. Li, Z. Deng, Z. Han, K. Alfaro-Bittner, B. Barzel, and S. Boccaletti, "Contagion in simplicial complexes," *Chaos Solitons Fractals*, vol. 152, Nov. 2021, Art. no. 111307.

[15] P. Cisneros-Velarde and F. Bullo, "Multigroup SIS epidemics with simplicial and higher order interactions," *IEEE Trans. Control Netw. Syst.*, vol. 9, no. 2, pp. 695–705, Jun. 2022.

[16] F. Battiston et al., "Networks beyond pairwise interactions: Structure and dynamics," *Phys. Rep.*, vol. 874, pp. 1–92, Aug. 2020.

[17] G. Bianconi, *Higher-Order Networks*. Cambridge, U.K.: Cambridge Univ. Press, 2021.

[18] L. V. Gambuzza et al., "Stability of synchronization in simplicial complexes," *Nat. Commun.*, vol. 12, pp. 1–13, Feb. 2021.

[19] L. Neuhäuser, R. Lambiotte, and M. T. Schaub, "Consensus dynamics and opinion formation on hypergraphs," in *Higher-Order Systems*. Cham, Switzerland: Springer, 2022, pp. 347–376.

[20] R. Sahasrabuddhe, L. Neuhäuser, and R. Lambiotte, "Modelling nonlinear consensus dynamics on hypergraphs," *J. Phys. Complexity*, vol. 2, no. 2, 2021, Art. no. 25006.

[21] L. DeVille, "Consensus on simplicial complexes: Results on stability and synchronization," *Chaos*, vol. 31, no. 2, 2021, Art. no. 23137.

[22] X. Dai et al., "*D*-dimensional oscillators in simplicial structures: Odd and even dimensions display different synchronization scenarios," *Chaos Solitons Fractals*, vol. 146, May 2021, Art. no. 110888.

[23] A. P. Millán, J. J. Torres, and G. Bianconi, "Explosive higher-order Kuramoto dynamics on simplicial complexes," *Phys. Rev. Lett.*, vol. 124, no. 21, 2020, Art. no. 218301.

[24] C. Xu and P. S. Skardal, "Spectrum of extensive multiclusters in the Kuramoto model with higher-order interactions," *Phys. Rev. Res.*, vol. 3, no. 1, 2021, Art. no. 13013.

[25] X. F. Wang and G. Chen, "Pinning control of scale-free dynamical networks," *Physica A*, vol. 310, nos. 3–4, pp. 521–531, 2002.

[26] M. Frasca, A. Buscarino, A. Rizzo, and L. Fortuna, "Spatial pinning control," *Phys. Rev. Lett.*, vol. 108, no. 20, 2012, Art. no. 204102.

[27] P. DeLellis, F. Garofalo, and F. Lo Iudice, "The partial pinning control strategy for large complex networks," *Automatica*, vol. 89, pp. 111–116, Mar. 2018.

[28] D. Salzano, D. Fiore, and M. Di Bernardo, "Ratiometric control of cell phenotypes in monostrain microbial consortia," *J. Roy. Soc. Interface*, vol. 19, no. 192, 2022, Art. no. 20220335.

[29] G. Gallo, G. Longo, S. Pallottino, and S. Nguyen, "Directed hypergraphs and applications," *Discr. Appl. Math.*, vol. 42, nos. 2–3, pp. 177–201, 1993.

[30] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge, U.K.: Cambridge Univ. Press, 2012.

[31] W. Govaerts, Y. A. Kuznetsov, and B. Sijnave, "Implementation of Hopf and double-Hopf continuation using bordering methods," *ACM Trans. Math. Softw.*, vol. 24, no. 4, pp. 418–436, 1998.

[32] C. Stéphanos, "Sur une extension du calcul des substitutions linéaires," *J. de Mathématiques pures et Appliquées*, vol. 5, no. 6, pp. 73–128, 1900.

[33] P. DeLellis, M. Di Bernardo, and G. Russo, "On QUAD, Lipschitz, and contracting vector fields for consensus and synchronization of networks," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 58, no. 3, pp. 576–583, Mar. 2011.

[34] A. Davydov, S. Jafarpour, and F. Bullo, "Non-Euclidean contraction theory for robust nonlinear stability," *IEEE Trans. Autom. Control*, early access, Jun. 17, 2022, doi: 10.1109/TAC.2022.3183966.

[35] S. Ahmadizadeh, I. Shames, S. Martin, and D. Nešić, "On Eigenvalues of Laplacian matrix for a class of directed signed graphs," *Linear Algebra Appl.*, vol. 523, pp. 281–306, Jun. 2017.

[36] G. F. Young, L. Scardovi, and N. E. Leonard, "A new notion of effective resistance for directed graphs—Part I: Definition and properties," *IEEE Trans. Autom. Control*, vol. 61, no. 7, pp. 1727–1736, Jul. 2016.

[37] W. Lu, X. Li, and Z. Rong, "Global stabilization of complex networks with digraph topologies via a local pinning algorithm," *Automatica*, vol. 46, no. 1, pp. 116–121, 2010.

[38] L. O. Chua, M. Komuro, and T. Matsumoto, "The double scroll family," *IEEE Trans. Circuits Syst.*, vol. 33, no. 11, pp. 1072–1118, Nov. 1986.

[39] A. Di Meglio, P. De Lellis, and M. Di Bernardo, "Decentralized gain adaptation for optimal pinning scontrollability of complex networks," *IEEE Contr. Syst. Lett.*, vol. 4, no. 1, pp. 253–258, Jan. 2020.

Open Access funding provided by 'Università degli Studi di Napoli "Federico II"' within the CRUI CARE Agreement