DESIGN OF TRACKING SYSTEMS

INCORPORATING MULTIVARIABLE PLANTS

by

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SYNOPSIS

The methodology for the design of error-actuated digital set-point tracking controllers proposed by Porter and co-workers has emerged as a result of the pursuit of effective and practical solutions to the problem of designing digital control systems for unknown, dynamically complex multivariable plants with measurable outputs. In this thesis, such digital set-point tracking controllers and the resulting digital set-point tracking systems are enriched to embrace plants with unmeasurable outputs and plants with more outputs than manipulated inputs.

In the study of the latter plants, the novel concepts of limit tracking (ie, the tracking exhibited by plants with more outputs than inputs) is introduced and an associated methodology for the design of self-selecting controllers is proposed. Such controllers involve the selection of different set-point tracking controllers to control the most critical subset of plant outputs based upon the developed rigorous theoretical foundations for the limit-tracking systems. In such foundations, the classification of linear multivariable plants into Class I and Class II plants based upon their steady-state transfer function matrices facilitates the assessment of the feasibility of limit-tracking systems. Furthermore, the associated order-reduction technique simplifies the problem of deciding the minimum numbers of different subsets of plant outputs to be controlled by corresponding set-point tracking controllers. In addition, the

dynamical properties of limit-tracking systems are also investigated using the phase-plane method and a methodology for the design of supervisory self-selecting controllers is proposed so as to prevent the occurrence of dynamical peculiarities such as limit-cycle oscillations which might happen in limit-tracking systems.

The effectiveness of all the proposed methodologies and techniques is illustrated by examples, and the robustness properties of set-point tracking systems and limit-tracking systems in the face of plant variations and unknown disturbances are tested. Finally, self-selecting controllers are designed for a nonlinear gas-turbine engine and their practical effectiveness is clearly demonstrated.

CONTENTS

PART I

INTRODUCTION

CHAPTER 1

INTRODUCTION

1.1 Introduction

The design of tracking systems, where the plant output is expected to track or follow the command input, has been an important issue in control engineering and therefore investigated for a long time. For Multi-Input/Multi-Output (MIMO) linear multivariable plants with highly interactive dynamics, the classical design methodologies developed for Single-Input/Single-Output (SISO) plants presented difficulties in finding practical solutions to the problem of designing tracking systems. Therefore, a few multivariable design methodologies for set-point tracking systems have been suggested. However, most of these techniques perpetuated the obvious shortcomings such as the heavy reliance upon reasonably accurate plant models and the non-practical requirement such as the accessibility for measurement of all the state variables. At the same time, the design algorithms involved are usually conceptually and computationally complex and therefore control engineers experience difficulty in producing reasonable closed-loop performance unless they are experts in the particular design methodology. Furthermore, the analogue nature of most of such techniques hindered their application in Direct Digital Control, which is becoming even more popular with the fast development of digital microprocessors and digital electronics. Finally, another major difficulty with

most of such techniques is that the controllers are unrobust in the presence of plant uncertainties such as parameter variations or unmodelled dynamics. The main reason for these drawbacks is the lack of practical considerations in developing these design methodologies.

Therefore, the emergence of robust controller synthesis methodologies which are conceptually and computationally simple, free from the reliance upon accurate plant models, suitable for Direct Digital Control, and utilise only input-output measurements of the plant has been long-waited. The error-actuated tunable digital set-point tracking controllers appeared as a masterpiece. The synthesis of such controllers utilises only data obtained from direct input-output measurements in the time domain. The controllers perform effectively the control action by measuring on-line error signals between plant outputs and command inputs. Such controllers can reject unmeasurable disturbance inputs whilst simultaneously causing the plant outputs to track command inputs. Therefore, such controllers naturally assume that the outputs from the plants under control are directly available for control purposes and are expected to demonstrate excellent set-point tracking performance for plants with measurable outputs. The successful application of such controllers has extended from distillation columns to nuclear power reactors. The robustness of such controllers to plant uncertainties and unknown disturbances has also been verified during these application stages.

However, in case the plant outputs which are required to be controlled are unmeasurable (for example, in aero gas-turbine engines, the in-flight thrust is normally unavailable for control purpose), the above assumption is invalid. In such cases, the error-actuated tunable digital set-point tracking controllers need to be enriched so as to embrace linear multivariable plants with unmeasurable outputs.

So far, the objective of set-point tracking systems is to cause all of the plant outputs to track their corresponding set-point commands. Such an objective is attainable by incorporating as many integrators as the number of outputs provided that the plant meets the fundamental requirement of functional controllability. In order to satisfy this necessary condition, it is evident that the number of manipulated inputs must not be less than the number of controlled outputs. However, in case plants have more controlled outputs than manipulated inputs, they fail to meet these requirements. Therefore, set-point tracking systems incorporating as many integrators as the number of outputs do not work properly in such cases. It is then obvious that, when control engineers face such functionally uncontrollable plants, they can either choose an appropriate subset of plant outputs and design a set-point tracking controller for only this subset so as to meet as many control requirements as possible or give up designing a controller. In the former case, it might happen in some plants such as gas-turbine engines that some of the uncontrolled outputs violate the engine operational limits whilst the controlled plant outputs are tracking their corresponding

set-point commands. In order to overcome such problems it is necessary to create a new design methodology for tracking systems which is based upon a new concept of tracking and enables all the plant outputs to be under control. In such tracking systems, it is desirable that — although not all the plant outputs can track their corresponding set-point commands — as many outputs as possible track their corresponding set-point commands whilst none of them violate the operational limits of the plant.

Self-selecting controllers were born under such circumstances for plants with more controlled outputs than manipulated inputs. Such a controller incorporates a number of set-point tracking controllers and works by selecting different set-point tracking controllers to control the most critical subset of outputs, which usually changes with time as both set-point commands and plant outputs change. The usual criterion for choosing which outputs to control at any time is either a highest-wins, lowest-wins, or highest-wins/lowest-wins strategy. In this context, 'highest-wins' or 'lowest-wins' refers to the instantaneous error between the set-point and the corresponding plant outputs. Therefore, it is required that the steady states of tracking systems incorporating self-selecting controllers and m-input/p-output plants (m < p) are such that set-point tracking occurs for the most critical m out of p outputs and that the remaining p-m outputs stay between upper and lower limits with a certain safety margin. In the case of lowest-wins strategies, these p-m outputs remain under the control of set-point commands corresponding to the

upper limits on the outputs, ie nonnegative errors are obtained for such channels and considered to be safe. Therefore, the tracking exhibited by entire sets of plant outputs can be considered to be *limit tracking* in the sense that none of the outputs exceeds its corresponding set-point command, ie its limit value.

Although self-selecting controllers are giving good results in practical applications such as the control of gas turbines, systems incorporating such controllers have not been properly understood yet. Especially, the proper understanding of the steady states of limit-tracking systems is very important so that it not only offers the possibility of the application of such systems to general multivariable plants but also provides the foundations for the dynamical analysis. Furthermore, due to the selection of different controllers depending upon set-point commands and plant outputs, limit-tracking systems change their structures discontinuously, ie they are variable-structure systems. Therefore, even though each control loop produces stable behaviour when considered separately, the stability of the complete system is not guaranteed. This justifies the necessity of the careful investigation of the dynamical properties of limit-tracking systems.

In summary, to cope with the cases in which plants have unmeasurable outputs or more outputs than inputs, set-point tracking systems need to be enriched both conceptually and methodologically.

1.2 Review of multivariable feedback control systems

One of the principal objectives of feedback control is to synthesise control systems with desirable properties in the case of imperfect knowledge of the dynamical characteristics of the controlled plant (Hosoe (1987)). As a result of this objective, feedback control is fundamentally robust since the controller works so as to cause the control deviation to be zero against the variation of signals and plant characteristics. In this sense, feedback control is different from open-loop control methodologies such as Pontryagin's Maximum Principle (Pontryagin et al (1962)).

It was not until the 1930s that the significance of feedback control was understood clearly by using the Laplace transform and associated frequency-response techniques. Nyquist (1932), who is the creator of the Nyquist frequency-domain stability criterion, showed analytically the trade-off between stability and large loop gain in feedback control systems; Hazen (1934), who investigated the performance characteristics of servomechanisms; and Black (1934), who proposed large loop gains for the design of feedback amplifiers, are among contributors to the progress of automatic control in this early period. The ideas of Nyquist and Black formed the basis of robust controller design for feedback amplifiers developed by Bode (1945). The classical automatic control theory (for example, Truxall (1955)) was then joined by the root-locus method presented by Evans (1948).

For the control of MIMO multivariable plants, state-space (or state-variable) methods appeared in the 1960s which have their basis in Linear System Theory (for example, Kalman (1963), Zadeh and Desoer (1963)). These include as controller design methodologies for linear multivariable plants the Linear Quadratic Regulator (LQR) (Kalman (1960a), Kalman et al (1969)), the Decoupling Controller (Falb and Wolovich (1967), Gilbert (1969)), the Modal Controller (Porter and Crossley (1972)), the Pole-Assignment Controller (Wonham (1967), Kimura (1975)), together with the Observer (Luenberger (1966)) or the Kalman Filter (Kalman (1960b), Arimoto and Porter (1973)) as the measurement tools for inaccessible state variables. However, whilst the theoretical development of these state-space methods was undertaken with enthusiasm in the 1960s to the 1970s, the response from industry was cool and applications had not widely spread. This is explained by considering a few of the difficulties associated with the use of such methods.

(i) The design is implemented by a linear state-variable feedback law. It is necessary to have access to all the states of the plant. This difficulty can be overcome by the introduction of an observer or a Kalman filter to estimate inaccessible states using input-output data from the plant model. But this creates additional problems such as the increase of controller complexity, the difference between the true states and the estimate states during the transient stage (Patel and Munro (1982)), and the degradation of robustness in case of the LQR (Doyle and Stein (1979)).

(ii) A suitable choice of the performance criterion (eg, in case of the LQR, the weighting matrices Q and R) is difficult to find for industrial processes (Patel and Munro (1982)).

(iii) A model which describes the whole plant (including not only the essential part but also the non-essential part) is necessary. It is difficult to design controllers for plants about which little or nothing is known (Foss (1973)).

(iv) It is difficult to relate the closed-loop responses of the control system to the plant's physical characteristics, and there is little room for empirical knowledge to play an important role (Kimura (1978)).

(v) The problem of parameter variation was not well-formulated in the theory. Although such robustness issues were commonly referred to as sensitivity design problems, no robust controller design algorithms were available (Kimura (1978)).

(vi) The steady-state characteristics of control systems were neglected. For example, the LQR can treat only impulsive disturbances (Kimura (1978)).

The SISO approach in the frequency-domain had been referred to as "classical control theory" during the reign in the 1960s of the so-called "modern control theory" of the state-space methods, but the MIMO approach in the frequency domain was proposed by Rosenbrock (1969), (1970) in the late 1960s. This approach was aimed at the approximate decoupling of multivariable plants by using the Inverse Nyquist Array. Then, it was expected that the classical SISO approach could be

applied to each separate input-output pair. MacFarlane (1970), Belletrutti and MacFarlane (1971), and MacFarlane and Postlethwaite (1977) developed the multivariable theory of Generalised Nyquist-Stability Condition and Characteristic Loci. These methodologies overcame some of the difficulties of state-space methods such as (i), (ii), and (iv). However, the design algorithms relied heavily on the interpretation of graphs in the frequency domain, which becomes delicate and complex in the case of high-order plants. Furthermore, robustness was another problem with these approaches since the plant variation was not well taken into account.

The study of tracking control problems (or servomechanism problems) for multivariable plants advanced through the improvement of steady-state characteristics by means of state-space methods. Johnson (1968) considered the condition on plant matrices A, B, C, and D to make the LQR effective in the presence of constant disturbances. Then, different forms of conditions on plant matrices were obtained by Porter and Power (1970), and Power and Porter (1970) in regard to the controllability of the closed-loop systems incorporating integral feedback, by Porter (1971) in regard to the LQR with integral feedback, and by Davison and Smith (1971) in regard to the pole-placement control. Furthermore, in regard to state-plus-integral feedback, it was pointed out in Kwakernaak and Sivan (1972) that

"in case the number of integral variables is equal to

that of manipulated inputs, it can be shown, by a

slight extension of the argument *of Power and Porter*

(1970) involving the controllability canonical form of

the plant, that necessary and sufficient conditions

for the existence of asymptotically stable control law

are that

(C-i)   the plant is stabilisable; and

(C-ii) the open-loop plant transfer function matrix

has no zeros at the origin."

Thus, the significance of integral action was illustrated in the state space. A few years later, Francis and Wonham (1975), and Davison (1976a) solved the tracking control problems for a more general class of external signals and compensators so that asymptotic tracking or regulation occurs independently of disturbances and plant parameter variations. In this sense, such control was called robust control.

A by-product of the study of tracking control problems is the deeper understanding of the zeros of multivariable systems (or loosely termed, multivariable zeros (Sain and Schrader (1990))). Since Rosenbrock (1970) provided the definitions of multivariable zeros such as decoupling zeros and transmission zeros (zeros of a transfer-function matrix), the issues of multivariable zeros prompted numerous investigations. In addition, various concepts involving multivariable zeros (for example, system zeros, invariant zeros) were introduced

(Rosenbrock (1973), (1974), Kouvaritakis and MacFarlane (1976), Pugh (1977)). In case a system is controllable and observable, the set of system zeros, the set of invariant zeros, and the set of transmission zeros all coincide; in other cases, they do not. Among such sets of zeros, the transmission zeros — which are physically associated with the transmission-blocking properties of plants — drew much attention because of their close relation with functional controllability (Rosenbrock (1970)) and the performance of feedback controllers. It is stated in connection with the non-minimum phase characteristics of transmission zeros in Porter and Jones (1985c) that

"The effectiveness of feedback controllers for linear multivariable plants is crucially constrained by the location in the complex plane of the transmission zeros of such plants. In particular, the presence of transmission zeros in the right half of the complex plane leads to closed-loop instability whilst the presence of transmission zeros at the origin of the complex plane leads to functional controllability."

The methods for the computation of transmission zeros were also keenly investigated (Wolovich (1973), Davison and Wang (1974), Laub and Moore (1978), Porter (1979)). However, these methods are not applicable for unknown plants since they require detailed knowledge of the open-loop plant dynamics in either

state-space or transfer-function matrix form. In such cases, the time-domain identification of transmission zero locations of asymptotically stable multivariable plants was reported by Porter and Jones (1985b). In their approach, the step-response matrix of the plant was utilised. Furthermore, Porter and Jones (1985c) later extended this procedure to the time-domain identification of non-minimum phase characteristics of such plants.

After the pioneering work regarding robust feedback design by Bode (1945) in which the differential sensitivity function was introduced, robustness issues had been investigated in the context of sensitivity analysis. However, many of the problems were considered to have already been solved and therefore not much attention was aimed at robustness issues in the 1960s. For example, it is stated by Kouvaritakis et al (1982) regarding the conference (the Symposium on Sensitivity Analysis) held in Yugoslavia in 1964 that

"Such was the enthusiasm and optimism of this era that, of the whole range of topics considered, only a few, such as large parameter variations and structural sensitivity of various functional block decompositions, were deemed not to have been resolved."

Exceptionally, the robust controller design methodology which inherited Bode's idea was presented by Horowitz (1963) during this era. In the 1970s, the criticism of modern control theory

(for example, Foss (1973)) and the unsatisfactory spread of multivariable controllers to industry seemed to cause control theoreticians to review the practicality of modern control methodologies. Thus, robustness issues began to be re-considered in the context of control theory.

In the field of LQG theory (which includes the LQR and the Kalman filter), the attempt was made to obtain robustness in the presence of model uncertainties by the loop-transfer recovery approach (Lehtomaki et al (1981)). However, the drawbacks such as the aforementioned (i) to (iv) still remain.

Next, by considering the transfer function gain/phase limitations in the face of unstructured uncertainties, the importance of loop shaping in the frequency domain was pointed out by Doyle and Stein (1981). Zames (1981) presented the new H^{∞} norm to measure the robustness of closed-loop feedback systems and proposed such an H^{∞} norm of the transfer function from the disturbance to the controlled variables as the minimised criterion for the robust controller design. This meant the appearance of a new criterion which succeeded the quadratic mean error used by the LQR/LQG. Since then, these approaches have been favoured and widely investigated by theoreticians partly because of their theoretical formality and depth (for example, Doyle et al (1989)). However, the aforementioned drawbacks (iii) and (iv) still hold. Furthermore, it is considered that the controller is high-order and complex, that the way to choose free design parameters is not given, and that the result is too conservative.

There were also some attempts towards the development of robust control in the frequency domain. Horowitz's approach evolved into the quantitative feedback theory (QFT) for multivariable plants (Yaniv and Horowitz (1986)). One of the others is the robust Nyquist array methodology, in which closed-loop system stability is examined by using the Gershgorin bands in the face of plant variations (Arkun et al (1984)). However, no systematic compensator design procedure was presented. Another simple robust controller design for unknown multivariable plants in which the plant dynamics are approximated by a first-order lag for SISO systems was proposed by Owens and Chotai (1983), (1984). The main drawbacks of this method in the continuous-time case are:

(i)   The plant must be minimum phase.
(ii)   The real closed-loop system must be stable for all
       high gains.

The controller design methodologies reviewed above are strongly based upon definite plant models in either the time domain or the frequency domain. Therefore, such methodologies can be called "model-based control" in the sense that they are firmly constrained by the models and that the design cannot proceed without models (Kimura (1987)). Since such models have their own fixed-structure (for example, state-space form or transfer-function matrix), once the type of structure is decided, the plant is characterised by a set of model parameters whose number relates to the plant's dynamical

complexity. Then, the design loses the direct connection with the explicit information from the plant such as input/output data and the design procedure results in a standardised numerical computation. However, here is a trap in which modern control theory is often caught — ie even though there never exists a perfect model of a real physical plant, modern control theory depends too heavily upon the model and it lacks a careful concern for the imperfection of the model. Under such circumstances, "model-free control" in the sense that controller design is free from such constraints as model type and model order prompted much attention. In model-free control, it is desired that controller design positively utilises the direct input/output data from the plant, thus keeping the direct connection with such explicit information during the controller design stage. Therefore, attempts were naturally made to extend conventional tunable PI/PID controllers from SISO to MIMO multivariable plants. Such controllers not only use directly measured input-output physical data from the plant thus preventing themselves from falling into a model-related trap but also are robust in the face of possible plant variations.

The multivariable tuning regulators, in which the plant to be controlled is assumed to be linear, time-invariant and open-loop asymptotically stable but no other assumptions such as known plant order or minimum-phase behaviours are needed, were proposed by Davison (1976b). After some simple "off-line" tests are performed on the plant, the controller is then obtained by tuning "on-line" a scalar positive parameter in the

same manner as in the SISO tuning method (for example, Ziegler and Nichols (1942)). However, such regulators yielded rather poor closed-loop performance when applied to a commercial heat exchanger (Davison et al (1980)) although it was assumed that the plant had already been speeded up by using some type of heuristic output control, eg, proportional and derivative output control. In order to improve the regulator performance in respect of fast responses and low-interaction, a parameter optimisation technique was introduced into this approach (Davison and Ferguson (1981)). Whilst Davison's approach uses only the integral of error (ie I-controller), multivariable PI-controllers in which the error between command input and plant output is also used were proposed to speed up the transient responses of the closed-loop systems (Penttinen and Koivo (1980)). However, such controllers exist only for the restricted class of plants with first Markov parameters of maximal rank. Furthermore, an important common drawback of these methods is the fact that they are only concerned with the design of analogue controllers.

In order to overcome such difficulties of I/PI-controllers, new approaches to the design of tunable analogue/digital set-point tracking controllers for unknown multivariable plants were presented by Porter (1981), (1982a). In the former approach (Porter (1981)), the proportional controller matrix involved the inverse (or right inverse) of the plant steady-state transfer-function matrix and positive scalar tuning parameters were used. Furthermore, this approach was extended to plants with unmeasurable outputs by using measurement matrices which

involve steady-state transfer-function matrices for both measurable and unmeasurable outputs (Porter and Bradshaw (1983)). In the latter approach (Porter (1982a)), the decoupling theory of Falb and Wolovich (1967) was used to obtain initial non-undershooting responses, the proportional controller matrix involved the inverse of the plant decoupling matrix, and the tuning parameters became positive diagonal matrices. However, this would still need mathematical models of plants. Therefore, step-response matrices were introduced to the proportional part of the controller (Porter and Jones (1984a)) since such matrices are easily determined from off-line open-loop tests performed on the plant. Furthermore, the controller was rendered Proportional, Integral, and Derivative (ie PID-controller) so as to improve transient responses and the step-response matrices were used also in the derivative part of the controller (Porter and Jones (1985a)). The extensions of these types of PI/PID controllers were reported by Porter (1982b) for time-delayed plants, by Porter and Jones (1984b) for plants with Lur'e-type nonlinearities, by Porter and Boddy (1988) for open-loop unstable plants, and by Porter and Khaki-Sedigh (1990) for type-one plants. The robustness of the controllers in the face of plant variations was assessed by Porter and Khaki-Sedigh (1989). The problem of the extension of non-undershooting controllers to plants with unmeasurable outputs was tackled by Porter and Yamane (1989).

Another approach to model-free control is Model Predictive Control (MPC) (García et al (1989)) in which impulse-response coefficients or step-response coefficients are used to predict

the effect of future actions of the manipulated inputs on the outputs. Since such prediction is carried out over a certain moving horizon, the future set-point commands over this horizon must be known. Furthermore, since a constrained optimisation problem must be solved at each time instant, the computational efforts involved are complex. Therefore, although MPC is applicable in the process industry where large and powerful computational capability is available and the required transient-response time is of the order of minutes, this approach is not suitable for the plants such as aero gas-turbines where the computation has to be carried out by microprocessors and the required transient-response time is of the order of seconds.

In case plants have more controlled outputs than manipulated inputs, the condition of functional controllability is not satisfied. Therefore, systems incorporating as many integrators as the number of inputs do not work properly. In such cases, asymptotic tracking for all of the plant outputs has to be abandoned and, alternatively, only the most critical output or subset of outputs can be integrally controlled. Then, such subsets change with time as both set-point commands and output change. Therefore, different controllers are selected to control such most critical subsets and the controller switching occurs when the controlled subset changes. This working principle of so-called self-selecting controllers is so simple that the jet engine hydromechanical/electronical fuel controller has incorporated this principle to guarantee safe engine operation. Glattfelder et al (1980) dealt with

microcomputer—based self—selecting controllers which incorporate 'highest—wins' and 'lowest—wins' gates to keep control signals within a certain range. The self—selecting controllers based upon lowest—wins strategies for SIMO and MIMO plants were discussed by Foss (1981b) as the multivariable limit controller and applied to a gas—turbine engine. However, this approach was not general since the binary lowest—wins strategies constrained the way to select critical subsets of signals. Jones et al (1988) presented more general self—selecting multivariable PI controllers which can be considered as extensions of tunable set—point tracking controllers (Porter and Jones (1984a)). This approach was also applied to gas—turbine engines successfully (Jones et al (1988), (1990)). However, the problems such as the existence of steady states, the minimum numbers of different controllers, etc are unresolved.

Due to controller switching, systems incorporating self—selecting controllers are variable—structure discontinuous dynamical systems. The first analysis of the stability of self—selecting control systems based upon lowest—wins strategies was presented by Foss (1981a), (1981b). In this analysis, discontinuous systems were transformed into continuous systems with nonlinear elements, and describing—function criteria or passivity criteria were used to assess the stability of the complete systems. These criteria were also used to assess the stability of control systems with nonlinearity such as saturation and antireset—windup circuits (Glattfelder and Schaufelberger (1983), Glattfelder et al

(1988)). However, this approach is not in general effective for the analysis of self-selecting control systems which are untransformable.

Variable-structure systems, which are discontinuous dynamical systems and described by differential equations with discontinuous right-hand sides, have prompted many investigations. Utkin (1977), (1978) and Emelyanov (1987) are among the contributors. The existence of sliding modes is recognised as one of the typical characteristics of such systems. Filippov (1964) gave a definition of the solution of the equations of motion of such systems and studied the properties of these solutions. If various non-idealities such as hysteresis, delay, and dynamical non-idealities (which are present in a real sliding mode) are made to tend to zero, this limiting process leads to the same equations that result from Filippov's method. Filippov's trajectories can therefore be considered as the ideal representation of the trajectories obtained in real systems, thus indicating one of the reasons for the wide use of Filippov's method in studies of variable-structure systems (Utkin (1978)). However, it was shown by Porter and Yamane (1990) that Filippov's solution concept is not enough for self-selecting control systems and that dynamical peculiarities such as sliding motion or limit-cycle oscillations can occur even in the case of a very simple first-order plant.

1.3 Objectives

The central objective of this thesis is to provide a pragmatic means to design tracking systems incorporating multivariable plants. Such tracking systems incorporate as core elements digital set-point tracking PI/PID controllers or self-selecting PI/PID controllers. The digital set-point tracking controllers are to be designed for plants with measurable outputs or with unmeasurable outputs. In such plants, the number of inputs and the number of outputs are equal. For plants with more outputs than inputs, theoretical foundations for the analysis of tracking systems incorporating such plants are to be constructed and effective procedures are to be developed that assess the feasibility of tracking system design. Using the developed procedures, self-selecting controllers are to be designed for such plants. In order to obtain enhanced stability of the closed-loop control systems, supervisory self-selecting controllers are to be proposed whilst it is to be shown that dynamical peculiarities such as limit-cycle oscillations might occur in self-selecting control systems. Finally, the robustness of tracking systems is to be assessed.

The design of tracking systems is to be characterised by the following practical guidelines:

1) Procedures should be developed that assess the feasibility of tracking system design for multivariable plants;

2) Controllers should be applicable to plants with measurable outputs, or with unmeasurable outputs, or with more outputs

than inputs, as long as the assessment 1) is feasible;

3) Controllers should be simple, easy to tune, and preferably digital;

4) Only plant input/output data should be used in the design (ie design should be free from a heavy reliance upon accurate plant models and state should be regarded as a mathematical abstraction);

5) Control laws should use only such input/output data;

6) Procedures should be provided that identify the data used in the controllers;

7) Controllers should be robust (ie the controllers should function in the face of unknown disturbances and plant variations).

1.4 Outline of the thesis

This thesis consists of six parts and a few appendices. In Part I (Chapter 1), an introduction to the problems involved in the design of tracking systems incorporating complex multivariable plants is given. A review of multivariable feedback control systems and an outline of the objectives of this thesis are also given.

In Part II (Chapters 2 and 3), methodologies for the design of set-point tracking systems are presented. Block-diagonalisation transforms are utilised to show the asymptotic properties of closed-loop digital control systems

incorporating linear multivariable plants with measurable outputs and tunable digital set-point tracking controllers (Chapter 2). Such controllers are then enriched to embrace linear multivariable plants with unmeasurable outputs by the inclusion of associated pre-filters (Chapter 3). In order to circumvent the need for detailed mathematical models of the plants, it is shown that the design of these controllers can be achieved using only data obtained from open-loop step-response tests performed on the plants. The excellent tracking performance of the resulting set-point tracking systems is demonstrated by the presentation of simulation results for a highly interactive gas-turbine engine.

In Part III (Chapters 4 to 8), tracking systems incorporating linear multivariable plants with more controlled outputs than manipulated inputs are discussed. In Chapter 4, after pointing out that set-point tracking systems incorporating such plants fail to operate properly, a more general tracking concept (ie undertracking and overtracking which are expressed by sets of inequalities) is introduced to characterise such general tracking systems. Then, the classification of linear multivariable plants into Class I and Class II plants is carried out in the context of convex analysis. Thus, the theoretical foundations for the design of controllers for such plants with more outputs than inputs are provided. In Chapter 5, the problems regarding the steady states of tracking systems incorporating self-selecting controllers (which themselves consist of a number of set-point point tracking controllers) are presented in the context of lowest-wins strategies, and the

tracking exhibited by entire sets of plant outputs is called limit tracking. Next, by investigating the facial structure of the resulting limit-tracking systems, a novel order-reduction technique is developed that decides the minimum numbers of different subsets of plant outputs to be controlled by corresponding set-point tracking controllers. Thus, a new synthesis approach to limit-tracking systems is given. In Chapter 6, this new synthesis approach to limit-tracking systems obtained from the steady-state analysis underlies the methodologies for the design of digital self-selecting controllers. A block-diagonalisation transform is utilised to show the asymptotic properties of separate closed-loop systems. Implementation issues in regard to the integration of separate controllers are considered. The excellent limit-tracking performance of closed-loop control systems is demonstrated by the presentation of simulation results for a highly interactive gas-turbine engine. In Chapter 7, to enhance closed-loop stability of self-selecting control systems, theoretical foundations for the dynamical analysis of such systems are constructed and methodologies for the design of supervisory self-selecting controllers are presented. It is shown that three operational modes and two assessment blocks form such supervisory controllers and that enhanced stability can be achieved using this controller for the case in which the non-supervisory controller causes limit-cycle oscillations.

In Part IV (Chapters 8 and 9), the robustness of tracking systems is assessed in the face of unknown disturbances and plant variations. The effect of controller parameters of

supervisory self-selecting controllers on tracking performance is also studied.

In Part V (Chapter 10), as a case study, a digital self-selecting controller is designed for a nonlinear model of a gas-turbine engine and the results of nonlinear simulation are presented.

In Part VI (Chapter 11), the principal features of the developed design methodologies are reviewed and discussed, the important results are summarised, and recommendations for further work in this field are provided.

Finally, in Appendix 1, a procedure to perform Open-loop tests on plants is described. In Appendix 2, proofs of various theorems and propositions stated in this thesis are given. In Appendices 3 and 4, models of an aero gas-turbine engine are given. In Appendix 5, the analysis of tracking systems (presented in Part III) is related to well-known Linear Programming problems and the difference between the two approaches is explained. In Appendix 6, the problem in regard to the dynamical properties of self-selecting control systems is illustrated by applying the phase-plane method to a simple example and showing the dynamical peculiarities of closed-loop control systems.

PART II

DESIGN OF SET-POINT TRACKING SYSTEMS

INCORPORATING LINEAR MULTIVARIABLE PLANTS

CHAPTER 2

DESIGN OF TUNABLE DIGITAL SET-POINT TRACKING

PID CONTROLLERS FOR LINEAR

MULTIVARIABLE PLANTS WITH MEASURABLE OUTPUTS

2.1 Introduction

In this chapter, the design of controllers for unknown open-loop asymptotically stable linear multivariable plants is considered. In order to circumvent the need for mathematical models of linear multivariable plants expressed in either state-space or transfer-function matrix form, the proportional, integral, and derivative controller matrices embodied in the tunable digital PID controllers proposed must be directly obtainable from open-loop tests performed on the asymptotically stable plants. These controllers must ensure that the resulting closed-loop systems are asymptotically stable and that satisfactory set-point tracking behaviour occurs. Furthermore, in the case of nearly all practical systems, there exist uncertainties such as plant variations and unknown disturbances. The effects of these uncertainties must also be taken into account. Therefore, the controller design problem is discussed in this chapter and the robustness properties of the controllers in the face of such uncertainties are considered in Chapter 8.

It is shown that the proportional, integral, and derivative controller matrices used in these PID controllers can be directly determined from open-loop step-response tests performed on plants (Appendix 1). The proportional and derivative controller matrices are chosen as the inverse of the open-loop step-response matrix, which is itself derived from the classical decoupling theory of Falb and Wolovich (1967). This choice is made in order to exploit the initial interactions within the plant and thus to cause set-point tracking to occur without initial interaction or under-shoot (Mita and Yoshida (1981)). The integral controller matrix is chosen as the inverse of the open-loop steady-state transfer-function matrix in order to exploit the final interactions within the plant. Thus, provided only that the plants satisfy the fundamental condition of Porter and Power (1970) and Power and Porter (1970) for the preservation of stabilisability in the presence of integral action, such error-actuated controllers can be readily designed for unknown multivariable plants.

A block-diagonalisation transformation is used to investigate the asymptotic properties of closed-loop systems under the action of such PID controllers. The closed-loop plant matrix is decomposed into three sub-matrices, using the block-diagonalisation transformation of Kokotović (1975), and it is shown that the basic design criterion for asymptotic stability and set-point tracking can be satisfied in terms of the characteristic roots of the sub-matrices.

The effectiveness of such a tunable controller is illustrated by designing, for a highly interactive gas-turbine engine, a tunable digital set-point tracking PID controller which exhibits excellent set-point tracking characteristics and corresponding minimal loop-interactions.

2.2 Analysis

The linear multivariable plants under consideration are assumed to be governed on the continuous-time set $T = [0,+\infty)$ by state and output equations of the respective forms

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad (2.1)$$

and

$$y(t) = Cx(t) , \qquad (2.2)$$

where the state vector $x(t) \in R^n$, the input vector $u(t) \in R^m$, the output vector $y(t) \in R^m$, the plant matrix $A \in R^{n \times n}$ whose eigenvalues all lie in the open left-half plane C^-, the input matrix $B \in R^{n \times m}$, and $C \in R^{m \times n}$ is the output matrix. Furthermore, it is assumed that the introduction of integral action preserves stabilisability and therefore that (Porter and Power (1970), Power and Porter (1970))

$$\text{rank } G = m , \qquad (2.3)$$

where the plant transfer-function matrix

$$G(s) = C(sI-A)^{-1}B \tag{2.4}$$

and the steady-state transfer-function matrix

$$G = G(0) = -CA^{-1}B \in R^{m \times m} \tag{2.5}$$

are known from open-loop tests performed on the plant (Appendix 1).

Finally, it is assumed that input-output decoupling is achievable and therefore that (Falb and Wolovich (1967))

$$\text{rank } F = m \ , \tag{2.6}$$

where the decoupling matrix

$$F = \begin{bmatrix} c_1^T A^{d_1} B \\ \cdots \cdots \\ \cdots \cdots \\ \cdots \cdots \\ c_m^T A^{d_m} B \end{bmatrix} \in R^{m \times m} \tag{2.7}$$

and the d_i (i=1,2,...,m) and the c_i^T (i=1,2,...,m) are, respectively, the decoupling indices of the plant (Falb and Wolovich (1967)) and the rows of the output matrix. In the case of such plants, it is important to note that

$$F = \lim_{t \to 0} \Lambda^{-1}(t)H(t) \tag{2.8}$$

and

$$F^{-1} = \lim_{t \to 0} H^{-1}(t)\Lambda(t) \ , \qquad (2.9)$$

where

$$\Lambda(t) = \text{diag}\{t^{d_1+1}/(d_1+1)!,\ldots,t^{d_m+1}/(d_m+1)!\} \qquad (2.10)$$

and

$$H(t) = C\Lambda^{-1}(e^{At}-I_n)B \qquad (2.11)$$

is the plant step-response matrix.

In order to design error-actuated set-point tracking PID controllers for linear multivariable plants governed by state and output equations of the respective forms (2.1) and (2.2), it is convenient to consider the behaviour of such plants on the discrete-time set $T_T = \{0,T,2T,\ldots,kT,\ldots\}$. This behaviour is governed by state and output equations of the respective forms (Kwakernaak and Sivan (1972))

$$x_{k+1} = \Phi x_k + \Psi u_k \qquad (2.12)$$

and

$$y_k = \Gamma x_k \ , \qquad (2.13)$$

where $x_k = x(kT) \in R^n$, $u_k = u(kT) \in R^m$, $y_k = y(kT) \in R^m$,

$$\Phi = \exp(AT) \; , \tag{2.14}$$

$$\Psi = \int_0^T \exp(At)B \; dt \; , \tag{2.15}$$

$$\Gamma = C \; , \tag{2.16}$$

and $T \in R^+$ is the sampling period.

The block diagram of the digital control system is shown in Fig 2.1. The set-point tracking error-actuated tunable digital PID controller is governed on the discrete-time set T_T by a control-law equation of the form

$$u_k = TK_1 e_k + TK_2 z_k + K_3 (e_k - e_{k-1}). \tag{2.17}$$

This controller is required to generate a piecewise-constant control input vector $u(t) = u_k$, $t \in [kT,(k+1)T)$, $kT \in T_T$, so as to cause the output vector $y(t)$ to track any constant set-point vector $v \in R^m$ on T_T, in the sense that the error vector $e_k = v - y_k \in R^m$ assumes the steady-state value

$$\lim_{k \to +\infty} e_k = \lim_{k \to +\infty} (v - y_k) = 0 \tag{2.18}$$

for arbitrary initial conditions. In equation (2.17), the digital integral-of-error vector

$$z_k = z_{k-1} + Te_{k-1} \in R^m \quad , \tag{2.19}$$

and the controller matrices $K_1 \in R^{m \times m}$, $K_2 \in R^{m \times m}$, and $K_3 \in R^{m \times m}$.

It follows from equations (2.12), (2.13), (2.17), and (2.19) that discrete-time tracking systems incorporating such plants and such controllers are governed on T_T by state and output equations of the respective forms

$$
\begin{bmatrix} x_{k+1} \\ z_{k+1} \\ f_{k+1} \end{bmatrix} = \begin{bmatrix} \Phi - T\Psi K_1 \Gamma - \Psi K_3 \Gamma & , & T\Psi K_2 & , & -\Psi K_3 \\ -T\Gamma & , & I_m & , & 0 \\ -\Gamma & , & 0 & , & 0 \end{bmatrix} \begin{bmatrix} x_k \\ z_k \\ f_k \end{bmatrix} + \begin{bmatrix} T\Psi K_1 + \Psi K_3 \\ TI_m \\ I_m \end{bmatrix} v
$$

(2.20)

and

$$
y_k = [\ \Gamma\ ,\ 0\ ,\ 0\ ] \begin{bmatrix} x_k \\ z_k \\ f_k \end{bmatrix} , \qquad (2.21)
$$

where $f_k = e_{k-1} \in R^m$ is the stored error vector.

Therefore, provided only that T, K_1, K_2, and K_3 are such that all the eigenvalues of the closed-loop plant matrix in equation (2.20) lie in the open unit disc D^-,

$$
\lim_{k \to \infty} \Delta z_k = \lim_{k \to \infty} \{z_{k+1} - z_k\} = 0 \qquad (2.22)
$$

and therefore

$$\lim_{k \to \infty} e_k = 0 \tag{2.23}$$

so that set-point tracking occurs.

The closed-loop characteristic equation can be readily expressed in the form (Porter and Jones (1985a))

$$\phi_c(z) = \phi_1(z)\phi_2(z)\phi_3(z) \tag{2.24}$$

by invoking the block-diagonalisation procedure of Kokotović (1975), and the response characteristics of the closed-loop system can accordingly thus be elucidated. The asymptotic properties of tracking systems under the action of such controllers can be characterised in terms of the eigenstructure of the closed-loop plant matrix, which involves the decomposition of this matrix into three sub-systems based on the explicitly invertible block diagonalisation transform (Kokotović (1975)).

This block-diagonalisation procedure transforms the matrix triple incorporated in equations of the form

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} A_{11} & , & A_{12} \\ A_{21} & , & A_{22} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(k) \tag{2.25}$$

and

$$y(k) = [C_1 , C_2] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \tag{2.26}$$

where $x_1(k) \in R^{n_1}$, $x_2(k) \in R^{n_2}$, $A_{ij} \in R^{n_i \times n_j}$ $(i,j=1,2)$,
$B_1 \in R^{n_1 \times m}$, $B_2 \in R^{n_2 \times m}$, $C_1 \in R^{m \times n_1}$, and $C_2 \in R^{m \times n_2}$ into the
block-diagonal form incorporated in the equations

$$\begin{bmatrix} \chi_1(k+1) \\ \chi_2(k+1) \end{bmatrix} = \begin{bmatrix} A_{11} & , & 0 \\ 0 & , & A_{22} \end{bmatrix} \begin{bmatrix} \chi_1(k) \\ \chi_2(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(k) \qquad (2.27)$$

and

$$y(k) = [\ C_1 \ , \ C_2 \ ] \begin{bmatrix} \chi_1(k) \\ \chi_2(k) \end{bmatrix} \qquad . \qquad (2.28)$$

The state vectors in these equations are related by the linear
state transformation (Kokotović (1975))

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = W \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix} \qquad (2.29)$$

where

$$W = \begin{bmatrix} I_{n_1} & , & M \\ -L & , & I_{n_2} - LM \end{bmatrix} \in R^{(n_1+n_2) \times (n_1+n_2)} , \qquad (2.30)$$

$\chi_1(k) \in R^{n_1}$, $\chi_2(k) \in R^{n_2}$, $A_{ij} \in R^{n_i \times n_j}$ $(i,j=1,2)$, $B_1 \in R^{n_1 \times m}$,
$B_2 \in R^{n_2 \times m}$, $C_1 \in R^{m \times n_1}$, $C_2 \in R^{m \times n_2}$, $L \in R^{n_2 \times n_1}$,and $M \in R^{n_1 \times n_2}$.

Thus, if L and M satisfy the matrix Riccati equations (Kokotovic (1975))

$$A_{21} + LA_{11} - A_{22}L - LA_{12}L = 0 \tag{2.31}$$

and

$$(A_{11} - A_{12}L)M - M(A_{22} + LA_{12}) + A_{12} = 0 , \tag{2.32}$$

it follows from equations (2.25), (2.27), and (2.29) that

$$A_{11} = A_{11} - A_{12}L \tag{2.33}$$

and

$$A_{22} = A_{22} + LA_{12} \qquad . \tag{2.34}$$

The asymptotic properties of the discrete-time closed-loop tracking system can now be readily determined by regarding T as a perturbation parameter in equations (2.20) and (2.21). Thus, by regarding in equation (2.25)

$$A_{11} = \begin{bmatrix} \Phi - T\Psi K_1\Gamma - \Psi K_3\Gamma & , & T\Psi K_2 \\ -T\Gamma & , & I_m \end{bmatrix} , \tag{2.35}$$

$$A_{12} = \begin{bmatrix} -\Psi K_3 \\ 0 \end{bmatrix} , \tag{2.36}$$

$$A_{21} = [-\Gamma , 0] , \tag{2.37}$$

and

$$A_{22} = 0 \qquad\qquad , \qquad (2.38)$$

the solution of equations (2.31) and (2.32) can be readily obtained by using power series expansions in T. This involves the definition of matrices L_1 and L_2 such that

$$L = [\ L_1\ ,\ L_2\ ] \qquad (2.39)$$

where

$$L_1 = L_{10} + L_{11}T + \ldots \ldots \qquad (2.40)$$

$$L_2 = L_{20} + L_{21}T + \ldots \ldots \qquad (2.41)$$

in which $L_{1i} \in R^{n_2 \times n_3}$, $L_{2i} \in R^{n_2 \times n_4}$, (i=0,1,2,......). Therefore, it is clear from equations (2.31), (2.35), (2.36), (2.37), (2.38), and (2.39) that on isolating coefficients

$$L = [\ C\ ,\ 0\ ] + O(T) \qquad (2.42)$$

and therefore from equations (2.33) and (2.34) that

$$A_{11} = \begin{bmatrix} \Phi - T\Psi K_1\Gamma & , & T\Psi K_2 \\ -T\Gamma & , & I_m \end{bmatrix} \qquad (2.43)$$

and

$$A_{22} = -TCBK_3 + O(T^2) \qquad . \qquad (2.44)$$

The matrix A_{11} in equation (2.43) is now block-diagonalised, again by regarding T as a perturbation parameter in equation (2.43) and by regarding in equation (2.25)

$$\bar{A}_{11} = \Phi - T\Psi K_1 \Gamma \qquad , \qquad (2.45)$$

$$\bar{A}_{12} = T\Psi K_2 \qquad , \qquad (2.46)$$

$$\bar{A}_{21} = -T\Gamma \qquad , \qquad (2.47)$$

and

$$\bar{A}_{22} = I_m \qquad . \qquad (2.48)$$

In addition, the matrix $\bar{L}$ is defined in the power-series form

$$\bar{L} = \bar{L}_0 + T\bar{L}_1 + T^2\bar{L}_2 + \ldots \qquad (2.49)$$

In equations (2.45), (2.46), (2.47), (2.48), and (2.49), the overbar has been used to distinguish between the two explicit stages of the block-diagonalisation procedure.

Therefore, it is clear from equations (2.31), (2.45), (2.46), (2.47), (2.48), and (2.49) that on isolating coefficients

$$\overline{L} = CA^{-1} + T(CA^{-1}BK_1CA^{-1} + CA^{-1}BK_2CA^{-2} - C/2) + O(T^2).$$

$$(2.50)$$

Hence, it follows from (2.33), (2.34) and (2.50) that

$$\overline{A}_{11} = I_n + TA + T^2A^2/2 - T^2BK_1C - T^2BK_2CA^{-1} + O(T^3)$$

$$(2.51)$$

and

$$\overline{A}_{22} = I_m - T^2CA^{-1}BK_2 + O(T^3) \qquad . \qquad (2.52)$$

Thus, it is evident from equations (2.43), (2.44), (2.51), and (2.52) that the characteristic polynomials as expressed in equation (2.24) are

$$\phi_1(z) = \mid zI_n - I_n - TA - T^2A^2/2 + T^2BK_1C$$
$$+ T^2BK_2CA^{-1} + O(T^3) \mid , \qquad (2.53)$$

$$\phi_2(z) = \mid zI_m - I_m - T^2CA^{-1}BK_2 + O(T^3) \mid , \qquad (2.54)$$

and

$$\phi_3(z) = \mid zI_m + TCBK_3 + O(T^2) \mid \qquad . \qquad (2.55)$$

2.3 Synthesis

It is clear that tracking will occur in the sense of equation (2.23) provided only that the set of closed-loop characteristic roots

$$Z_c = Z_1 \cup Z_2 \cup Z_3 \subset D^- \qquad (2.56)$$

where D^- is the open unit disc and the sets of characteristic roots Z_1, Z_2, and Z_3 are, respectively, the roots of the characteristic polynomials as expressed in equation (2.24).

Therefore, in case

$$K_1 = H(T)^{-1}\Lambda(T)\Pi \qquad , \qquad (2.57)$$

where $H(T)$ is given by equation (2.11) and

$$\Pi = \text{diag}\{\pi_1, \pi_2, \ldots, \pi_m\} \quad , \quad \pi_i \in R^+ \ (i=1,2,\ldots,m), \qquad (2.58)$$

$$K_2 = G(0)^{-1}\Sigma \qquad , \qquad (2.59)$$

where $G(0)$ is given by equation (2.5) and

$$\Sigma = \text{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_m\} \quad , \quad \sigma_i \in R^+ \ (i=1,2,\ldots,m), \qquad (2.60)$$

and

$$K_3 = H^{-1}(T)\Lambda(T)\Delta \qquad , \qquad (2.61)$$

where H(T) is given by equation (2.11) and

$$\Delta = \text{diag}\{\delta_1, \delta_2, \ldots, \delta_m\} \ , \ \delta_i \in R^+ \ (i=1,2,\ldots,m), \qquad (2.62)$$

it follows from equations (2.24), (2.53), (2.54), (2.55), (2.57), (2.59), and (2.61) that

$$Z_1 = \{z \in C : | zI_n - I_n - TA + O(T^2) | = 0\} \ , \qquad (2.63)$$

$$Z_2 = \{z \in C : | zI_m - I_m + T^2\Sigma + O(T^3) | = 0 \} \ , \qquad (2.64)$$

and

$$Z_3 = \{z \in C : | zI_m + O(T) | = 0 \} \qquad . \qquad (2.65)$$

These expressions indicate that, provided T is sufficiently small, all the closed-loop characteristic roots lie within the open unit disc. This follows since the open-loop plant is asymptotically stable on the continuous-time set $T = [0, +\infty)$ and since $T^2\Sigma$ is a positive diagonal matrix. The introduction of error-actuated digital set-point tracking PID controllers governed by equations (2.17), (2.57), (2.59) and (2.61) accordingly ensures that set-point tracking occurs when the sampling time $T \in (0, T^*]$, where $T^* = T^*(\Pi, \Sigma, \Delta)$ can be readily obtained by simple "on-line" tuning (Porter and Jones (1985a)).

Furthermore, it follows from equations (2.20) and (2.21) that the output from the initially quiescent plant after the first sampling interval under error-actuated digital PID control is

$$y(T) = [TH(T)K_1 + H(T)K_3]v \qquad . \qquad (2.66)$$

It is evident from equations (2.57), (2.61), and (2.66) that

$$y(T) = [T\Lambda(T)\Pi + \Lambda(T)\Delta]v \qquad (2.67)$$

and therefore that set-point tracking occurs when $T \in (0, T^*]$ with no initial interaction since $\Lambda(T)\Pi$ and $\Lambda(T)\Delta$ are diagonal matrices for all $T \in R^+$. Moreover, it follows from (2.67) that the ith element (i=1,2,....,m) of the output vector $y(T)$ is given by

$$y_i(T) = [T\lambda_i(T)\pi_i + \lambda_i(T)\delta_i]v_i \qquad (2.68)$$

where $\lambda_i(T)$, π_i, and δ_i are the elements of the diagonal matrices $\Lambda(T)$, Π, and Δ, respectively. It is thus evident from equations (2.10) and (2.68) that

$$y_i(T) = [T^{d_i+2}\pi_i/(d_i+1)! + T^{d_i+1}\delta_i/(d_i+1)!]v_i \qquad (2.69)$$

where d_i is the decoupling index (Falb and Wolovich (1967)) associated with the ith channel (i=1,2,...,m). Equation (2.69) indicates that the presence of derivative action in the error-actuated digital PID controller "speeds up" the

closed-loop response by reducing the effective decoupling index associated with the ith channel from d_i to d_i-1 ($i=1,2,..,m$). Indeed, this is directly reflected by equation (2.68) where the scalar $[T\lambda_i(T)\pi_i+\lambda_i(T)\delta_i]$ represents the proportion of the set-point which has been achieved after the first sampling period.

The proportional, integral, and derivative controller matrices K_1, K_2, and K_3 given by equations (2.57), (2.59), and (2.61), respectively, can all be directly determined from the step-response matrix $H(t)$. This is the case since it follows from equation (2.11) that

$$G(0) = \lim_{t\to\infty} H(t) = -CA^{-1}B \qquad (2.70)$$

because the open-loop plant is asymptotically stable and therefore has a bounded step-response matrix. Furthermore, since the expressions (2.57) and (2.61) for the proportional and derivative controller matrices, respectively, involve the inverse of the initial step-response matrix of the open-loop plant $H(T)$, it is clear that the sampling period must be selected so that the minimum singular value of $H(T)$ ($\sigma_{min}[H(T)]$) is not small, so that $H(T)$ is well-conditioned.

2.4 Illustrative example

The use of these methods can be conveniently illustrated by designing a tunable digital set-point tracking PID controller for the linear model of the F100 engine obtained at

Intermediate condition (Appendix 3).

The plant has five measurable outputs and five manipulated inputs and is governed by state and output equations of the forms (A3.8) and (A3.10). The elements of the open-loop step-response matrix of the plant are obtained by "off-line" open-loop tests (Appendix 1) and are shown in Figs 2.2 to 2.6. It is clear from these figures that the plant is highly interactive. Furthermore, the corresponding plot of the minimum singular values $(\sigma_{min}[H(t)])$ of the step-response matrix shown in Fig 2.7 indicates that the plant is minimum phase (Porter and Jones (1985c)) and that $G(0)$ is well-conditioned since $\sigma_{min}[H(+\infty)]$ is not small.

Therefore, it is found from Figs 2.2 to 2.6 that

$$
H(0.05) = \begin{bmatrix}
0.63349E{-}03 & 1.2999 & -0.13554 \\
0.17616E{-}04 & -0.80181E{-}01 & -0.99195E{-}03 \\
0.23170E{-}03 & 0.15636 & 0.30845E{-}02 \\
0.11637E{-}04 & -0.18878 & 0.42304E{-}03 \\
0.60822E{-}04 & -0.86794E{-}02 & -0.66965E{-}04
\end{bmatrix}
$$

$$
\begin{bmatrix}
0.28229E{-}01 & -3.0846 \\
-0.61116 & -1.9583 \\
0.21946E{-}01 & -1.5823 \\
-0.96205E{-}03 & -0.19460E{-}01 \\
0.75545E{-}03 & 0.61871E{-}01
\end{bmatrix} \qquad (2.71)
$$

and

$$
G(0) = \begin{bmatrix}
0.37904 & 1238.8 & -28.508 \\
0.30777 & 660.79 & -2.8675 \\
0.20602E-01 & -39.863 & 0.25947 \\
0.15944E-02 & -12.168 & 0.38479E-01 \\
0.90309E-01 & 210.94 & -1.7403
\end{bmatrix}
$$

$$
\begin{bmatrix}
-9.2619 & -57.405 \\
-25.646 & -46.221 \\
-0.76283 & -6.8275 \\
-0.33542E-01 & -0.44527 \\
2.2101 & 12.248
\end{bmatrix} . \qquad (2.72)
$$

The excellent set-point tracking behaviour of the plant under the action of the resulting error-actuated PID controller tuned such that T = 0.05 sec, $\Lambda(0.05)\Pi$ = diag{0.05, 0.2, 0.1, 0.1, 0.1}, Σ = diag{50, 50, 50, 50, 100}, and $\Lambda(0.05)\Delta$ = $0.001I_5$, is shown in Figs 2.8 and 2.9. In this case, the set-point vector for the measurable outputs is v = [126, 93.4, 14.5, 1.78, $1.97]^T$ so that the thrust change is 500 lb. It is evident from these figures that the response of the gas-turbine engine consists of a fast approach to the desired values with minimum interaction between the five channels and that the corresponding manipulated variables exhibit no practically undesirable characteristics.

2.5 Conclusion

In this chapter, a block-diagonalisation transformation has been used to exhibit the asymptotic properties of discrete-time closed-loop tracking systems incorporating asymptotically stable linear multivariable plants under the action of digital PID controllers. The controller parameters have been chosen so

that set-point tracking is achieved without initial interaction or undershoot. It has been shown that the design of error-actuated digital controllers, which ensure that set-point tracking behaviour of the closed-loop system occurs, can be readily effected even though the detailed dynamical properties of the processes involved are unknown.

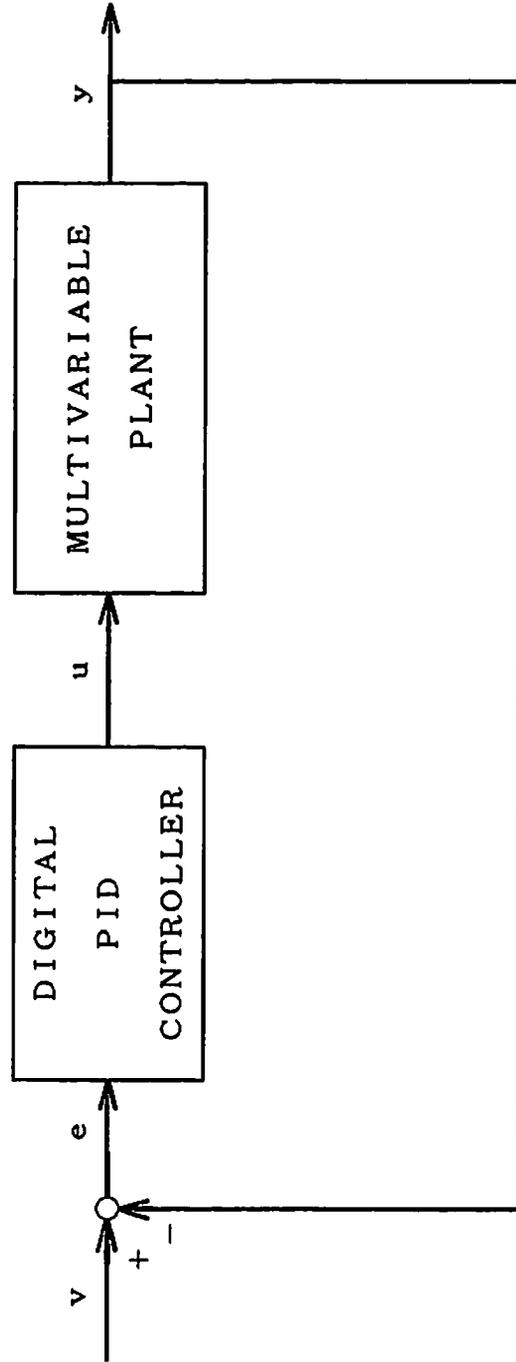Finally, the effectiveness of these methodologies has been illustrated by designing a digital set-point tracking controller for a highly interactive gas-turbine engine.

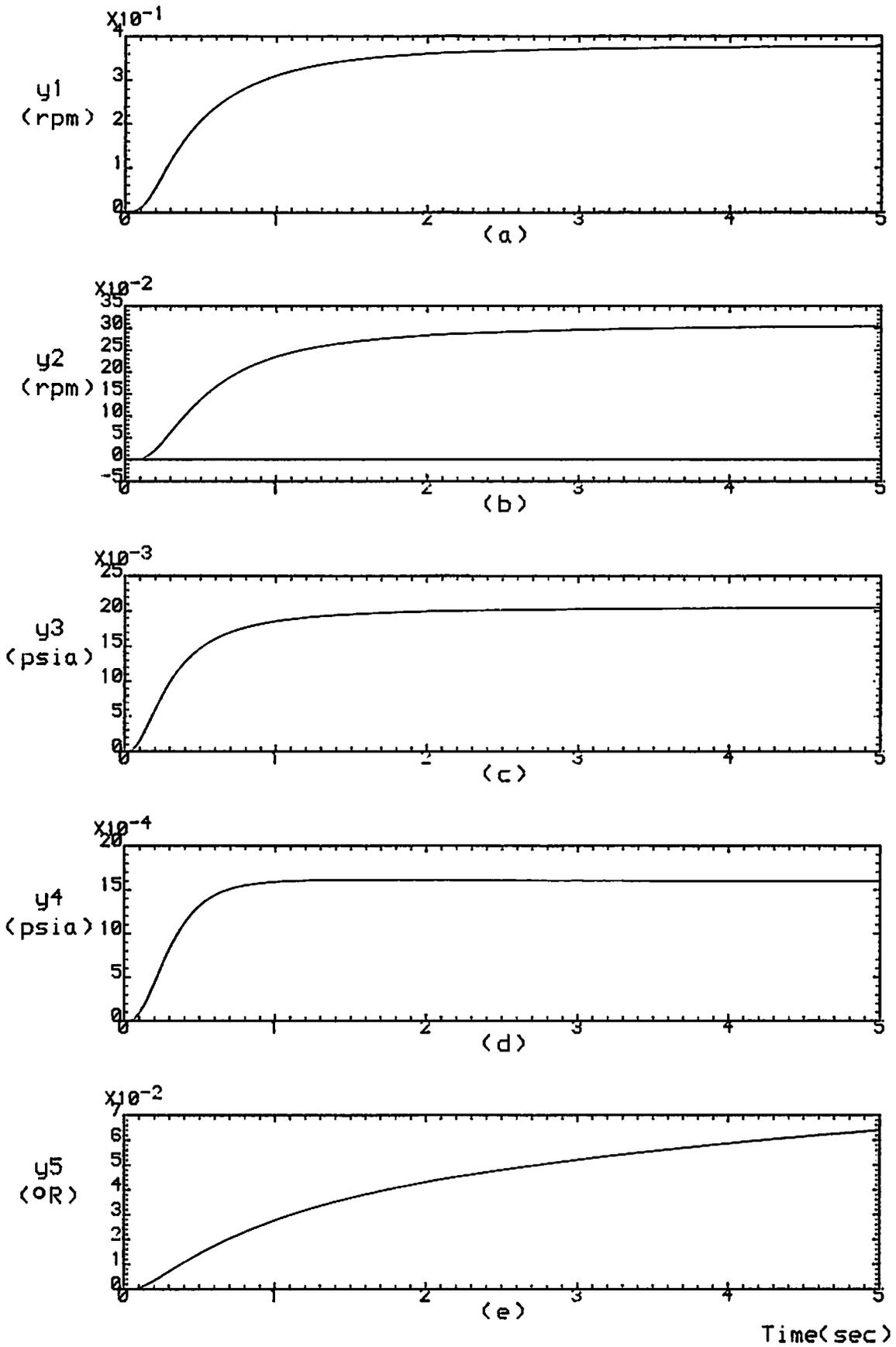Fig 2.1   Block diagram of digital control system

Fig 2.2 Open-loop step-responses of F100 engine
Measurable outputs when u=[1 0 0 0 0]
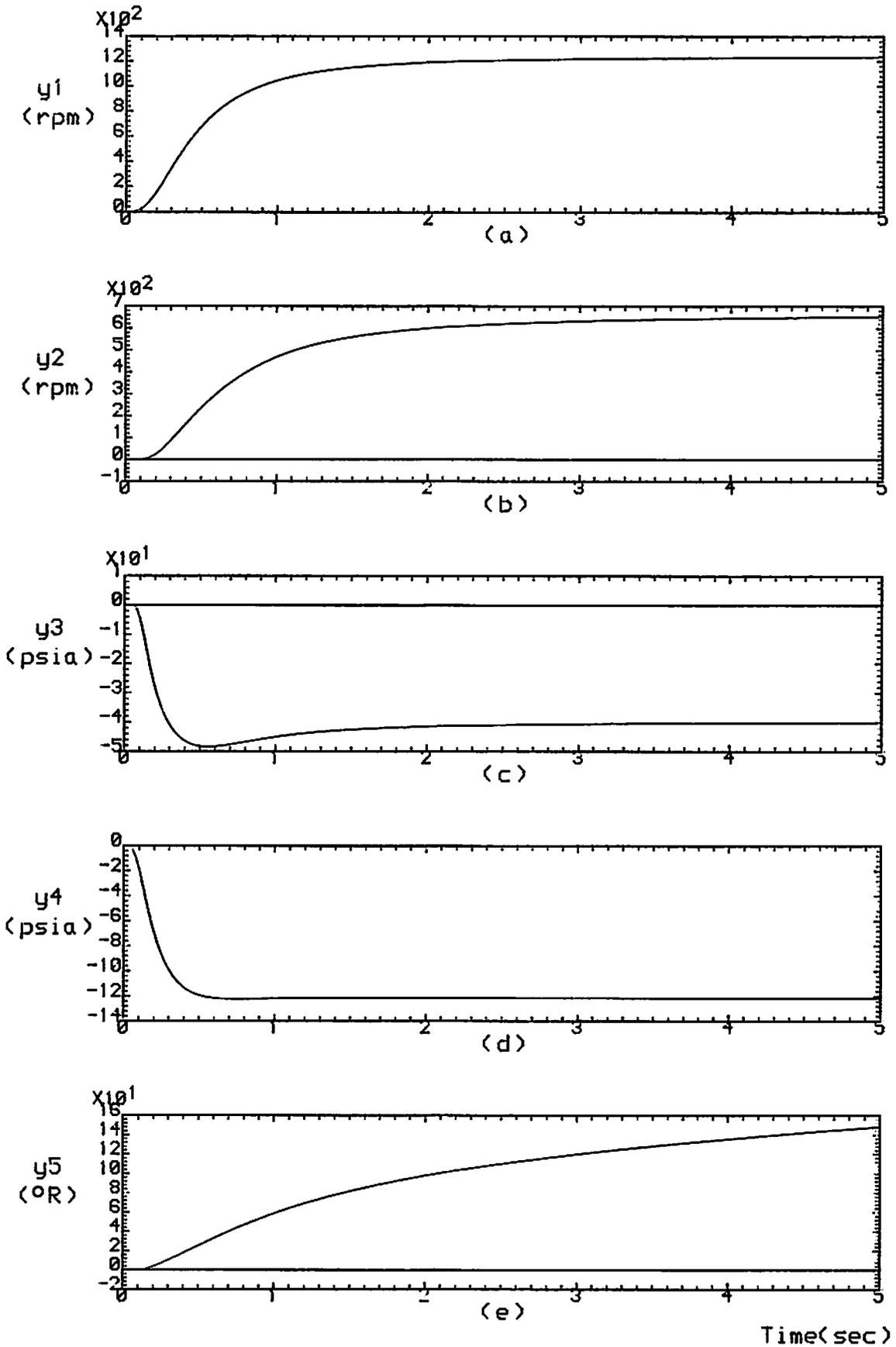
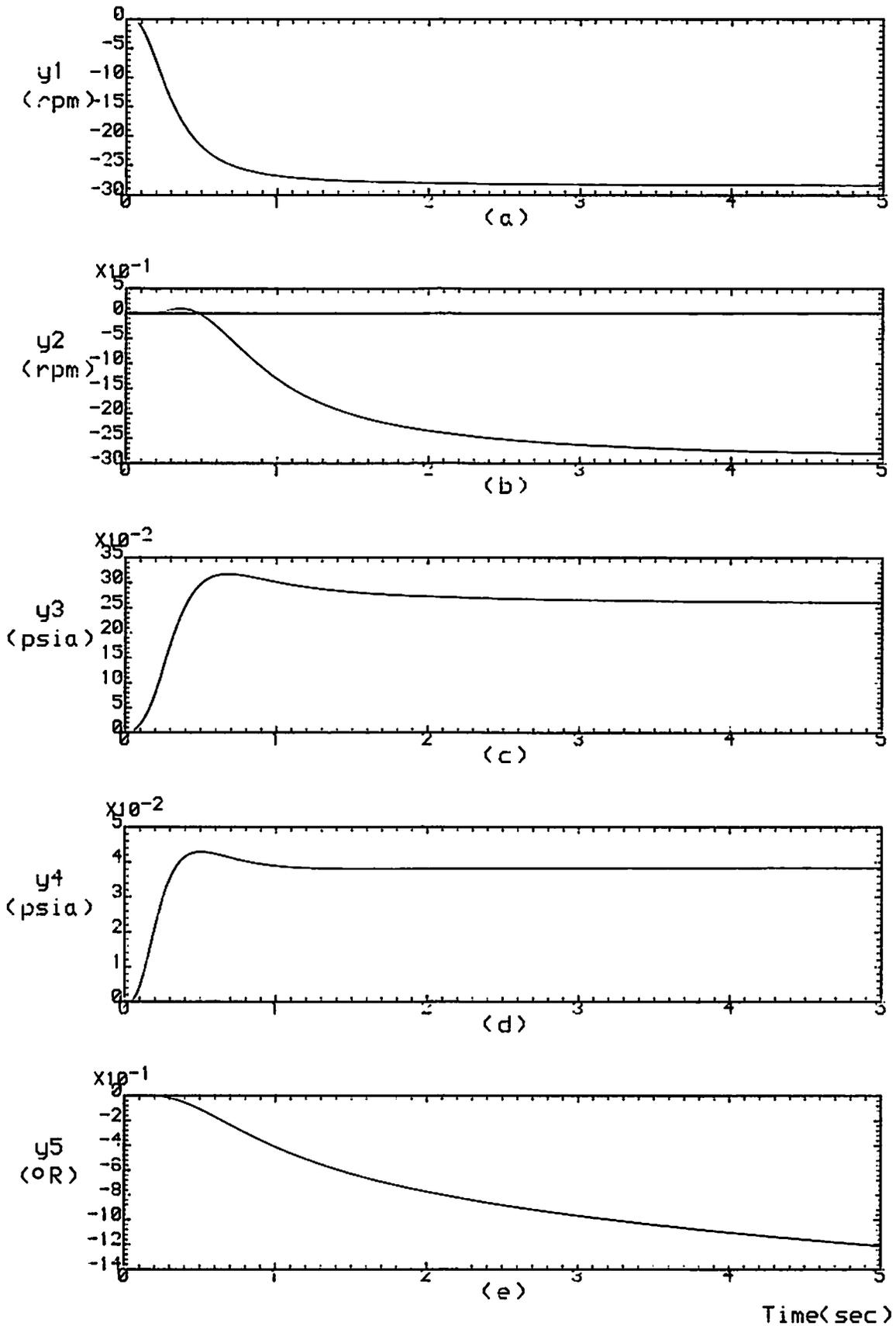Fig 2.3 Open-loop step-responses of F100 engine
Measurable outputs when u=[0 1 0 0 0]

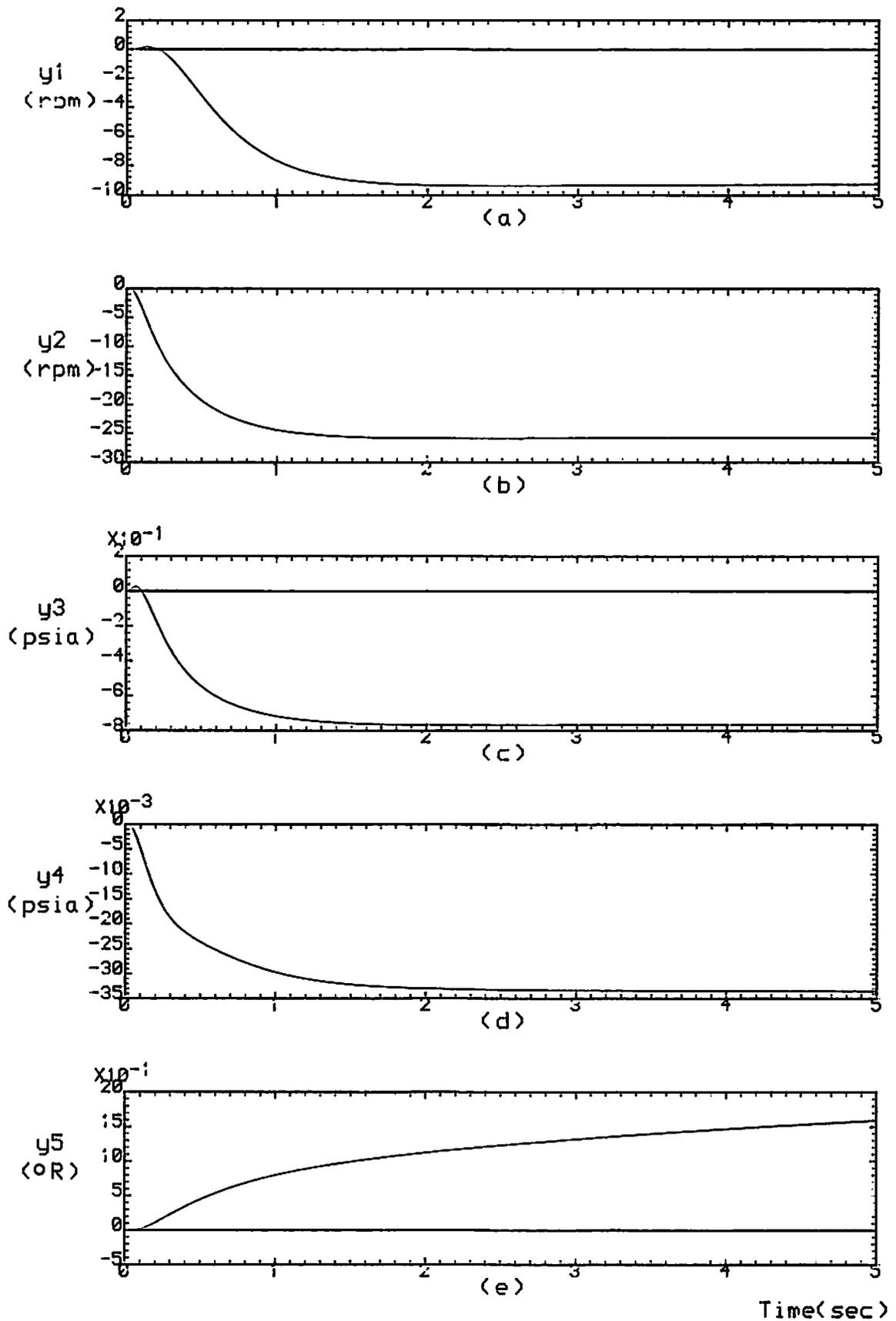Fig 2.4 Open-loop step-responses of F100 engine
Measurable outputs when u=[0 0 1 0 0]

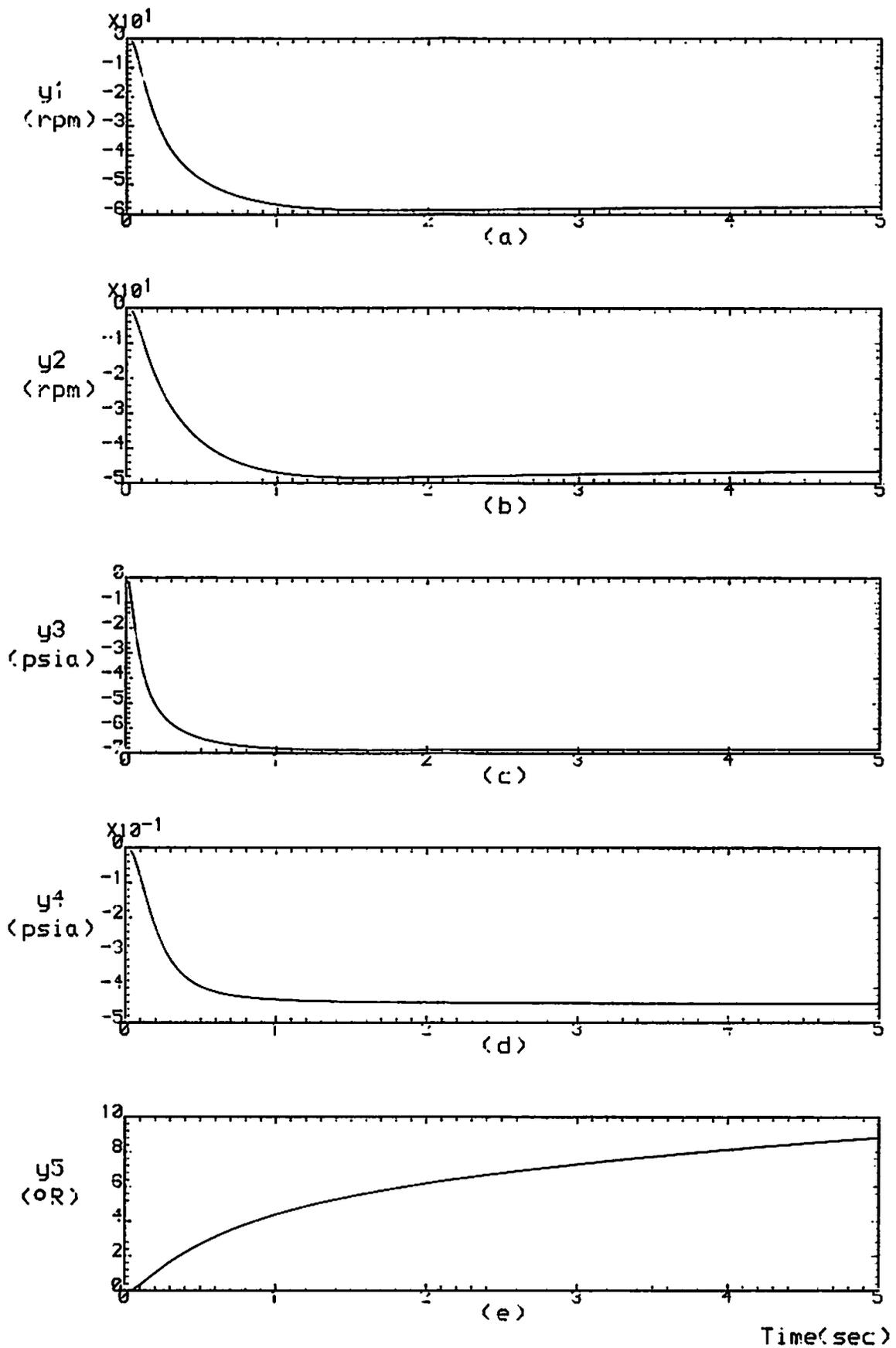Fig 2.5   Open-loop step-responses of F100 engine
Measurable outputs when u=[0 0 0 1 0]

Fig 2.6 Open-loop step-responses of F100 engine
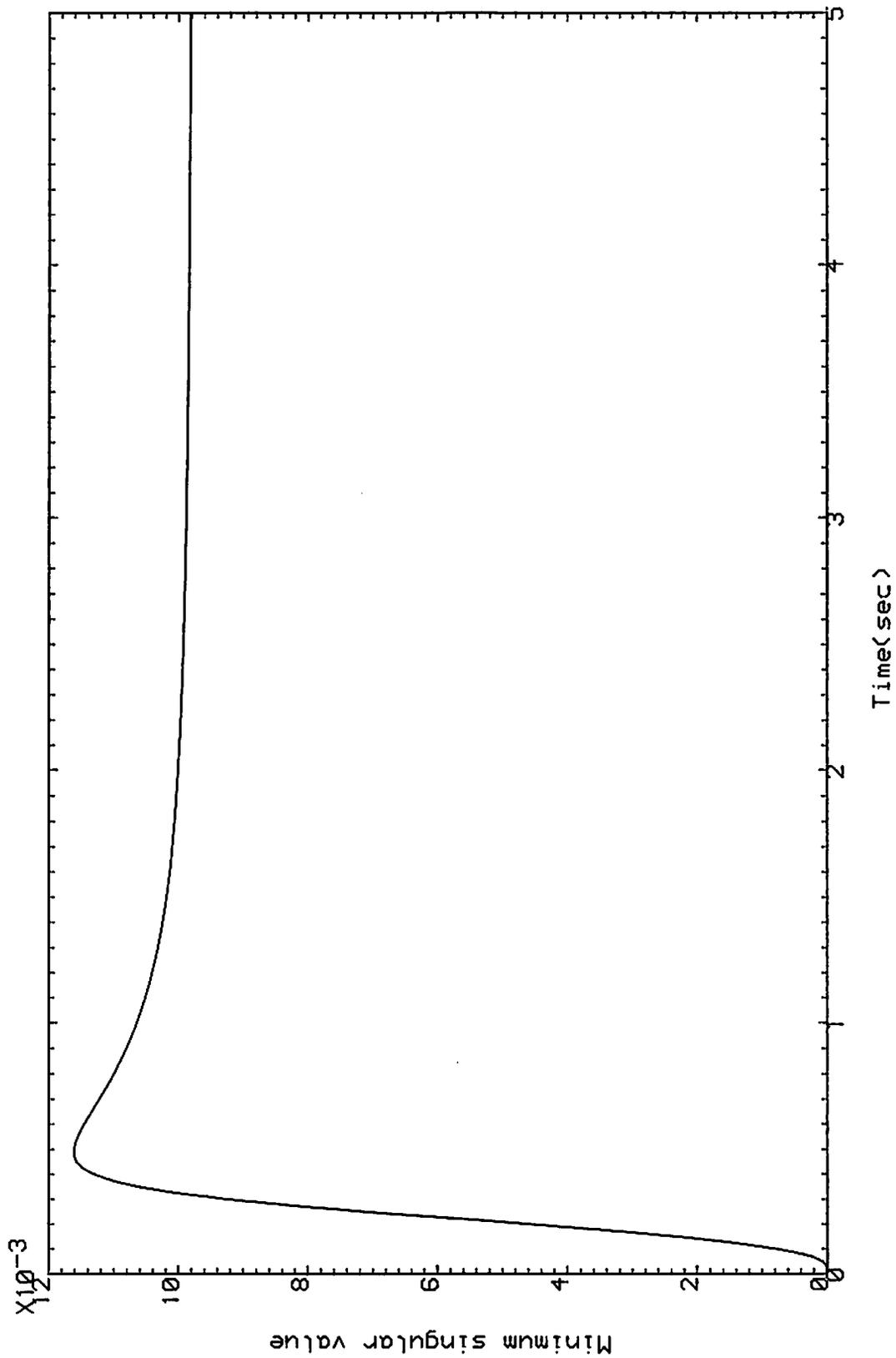Measurable outputs when u=[0 0 0 0 1]

Fig. 2.7   Minimum singular value plot of the plant step-response matrix
Plant: F100 engine 33 states model with 5 inputs and
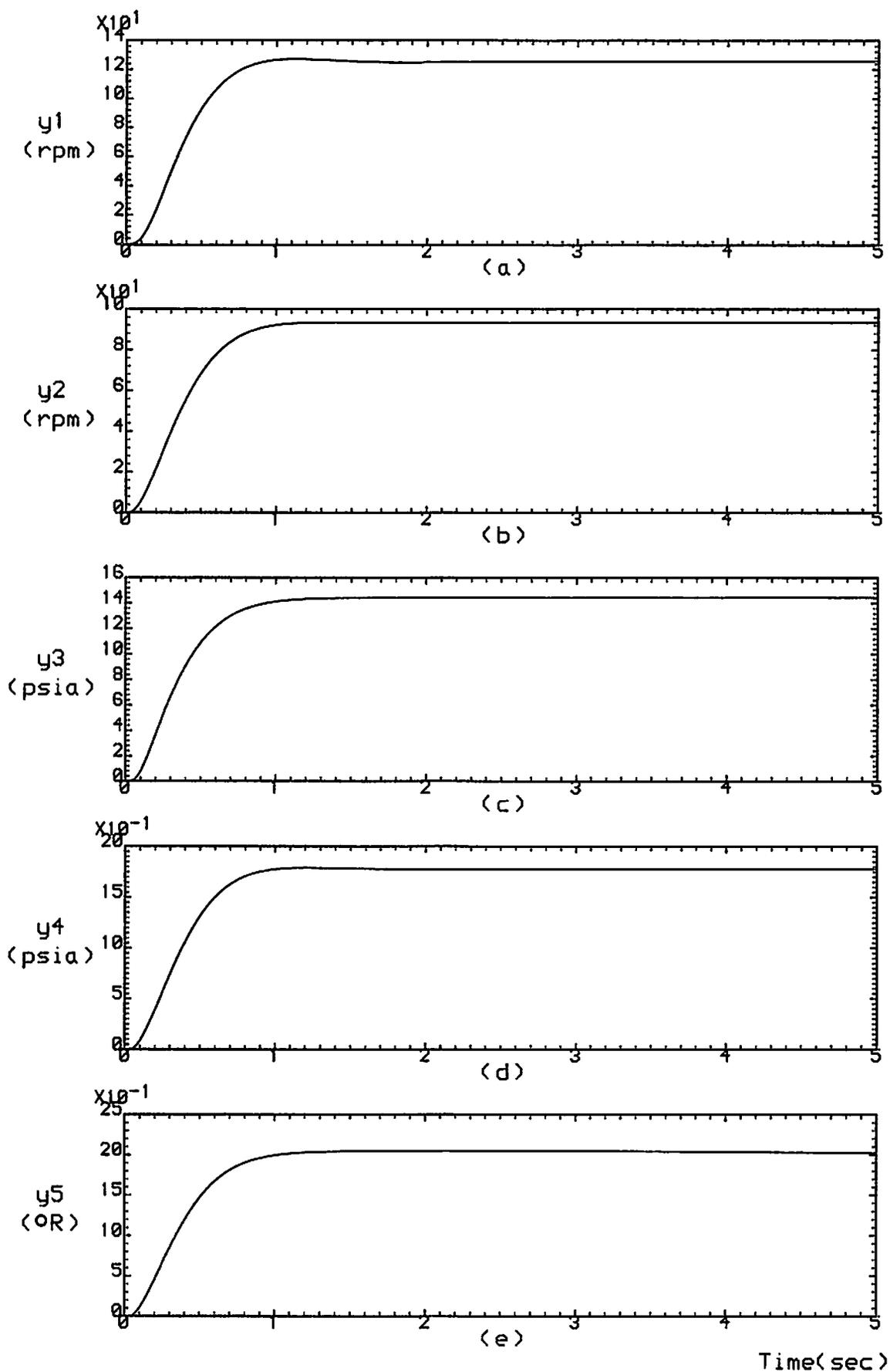5 measurable outputs

Fig 2.8 Measurable outputs of F100 engine
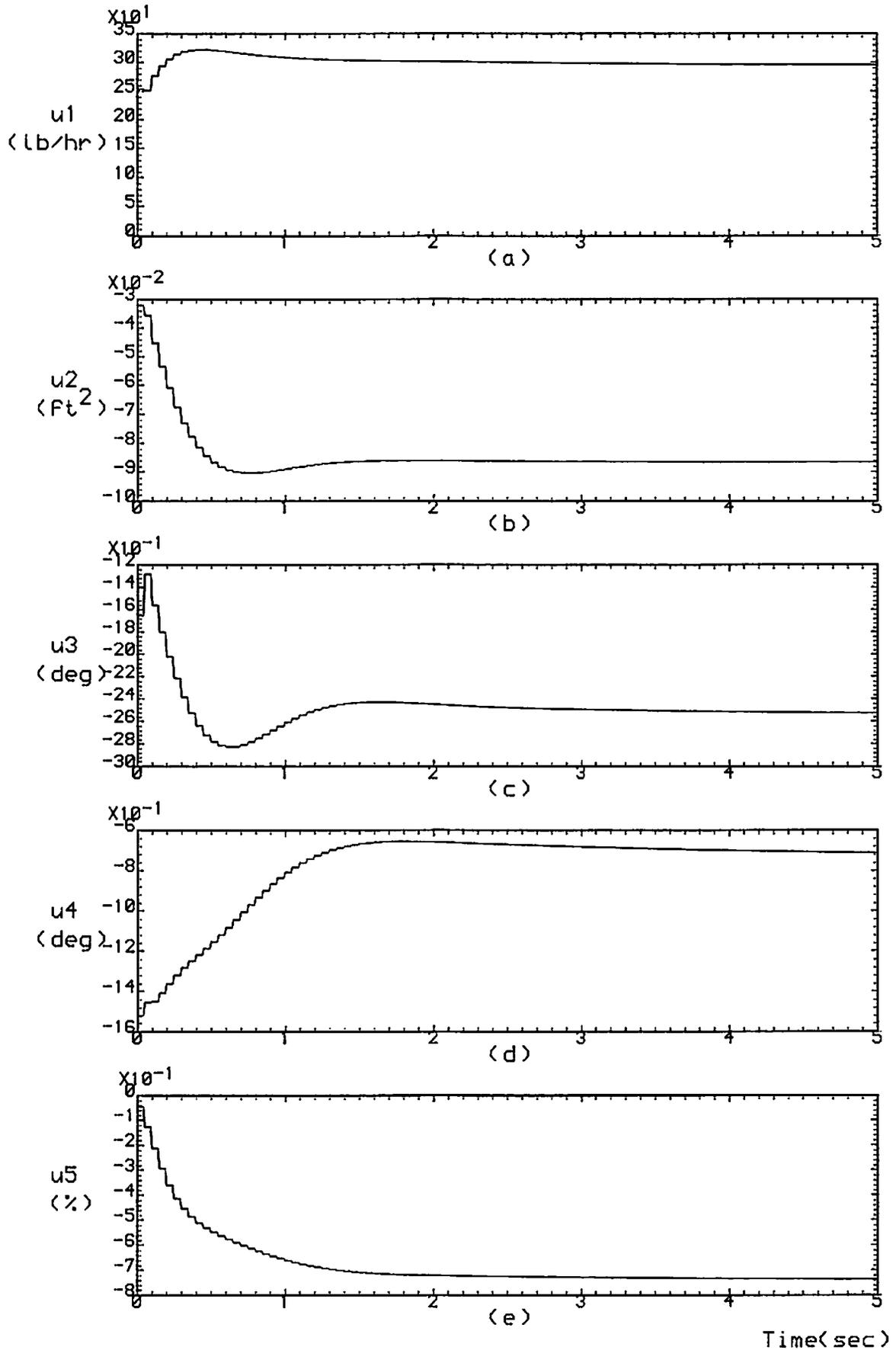under digital PID control

Fig 2.9 Manipulated variables of F100 engine
under digital PID control

CHAPTER 3

DESIGN OF TUNABLE DIGITAL SET-POINT TRACKING

PID/PRE-FILTER CONTROLLERS FOR LINEAR

MULTIVARIABLE PLANTS WITH UNMEASURABLE OUTPUTS

3.1 Introduction

In designing the tunable digital set-point tracking PID
controllers proposed in Chapter 2, it was assumed that the
outputs from the plants under control are directly available
for control purposes. However, in many technologically
important applications such as gas turbines, the plant outputs
which are required to be controlled are unmeasurable so that
this assumption is invalid. Therefore, in this chapter, the
tunable digital set-point tracking PID controllers of Chapter 2
are enriched by the inclusion of pre-filters so as to embrace
linear multivariable plants with unmeasurable outputs. It is
noted that the robustness properties of the resulting
controllers are considered in Chapter 8.

It is shown that the pre-filter matrices, together with the
proportional, integral, and derivative controller matrices
embodied in the resulting PID/Pre-filter controllers, can be
determined from open-loop step-response tests performed on
plants (Appendix 1). The proportional and derivative
controller matrices are chosen as the inverse of the open-loop
step-response matrix for unmeasurable outputs, which is itself
derived from the classical decoupling theory of Falb and

Wolovich (1967). This choice is made in order to exploit the initial interactions within the plant and thus to cause set-point tracking to occur without initial interaction or under-shoot (Mita and Yoshida (1981)). The integral controller matrix is chosen as the inverse of the open-loop steady-state transfer-function matrix for measurable outputs in order to exploit the final interactions within the plant. Finally, the pre-filter matrix which converts the set-point commands for unmeasurable outputs into set-point commands for measurable outputs is designed to achieve set-point tracking for unmeasurable outputs. Although the use of the step-response matrix for unmeasurable outputs in controller matrices implies that off-line measurements of such outputs is necessary in the design stage, the design procedure is free from on-line measurements of such outputs. Thus, provided only that the plants satisfy the fundamental condition of Porter and Power (1970) and Power and Porter (1970) for the preservation of stabilisability in the presence of integral action, such error-actuated controllers can be readily designed for unknown multivariable plants with unmeasurable outputs.

A block-diagonalisation transformation is used to investigate the asymptotic properties of closed-loop systems under the action of such PID/Pre-filter controllers. The closed-loop plant matrix is decomposed into three sub-matrices, using the block-diagonalisation transformation of Kokotovic (1975), and it is shown that the basic design criterion for stability and set-point tracking can be satisfied in terms of the characteristic roots of these sub-matrices.

The effectiveness of such a tunable controller is illustrated by designing a tunable digital set-point tracking PID/Pre-filter controller for a highly interactive gas-turbine engine with five measurable outputs (which are not to be directly regulated but are available for control purposes) and five unmeasurable outputs (which are to be directly regulated but are not available for control purposes). It is shown that the proportional and derivative controller matrices include the inverse of the step-response matrix for unmeasurable outputs. Therefore, as long as such data are available in the controller design stage, the controller ensures the initial non-interaction or non-under-shooting and the final set-point tracking of unmeasurable outputs. This direct action in respect of the unmeasurable outputs forms the distinctive feature of such tunable PID/Pre-filter controllers.

3.2 Analysis

The linear multivariable plants under consideration are assumed to be governed on the continuous-time set $T = [0,+\infty)$ by state, output, and measurement equations of the respective forms

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad , \qquad (3.1)$$

$$w(t) = Ex(t) \qquad , \qquad (3.2)$$

and

$$y(t) = Cx(t) \qquad , \qquad (3.3)$$

where the state vector $x(t) \in R^n$, the input vector $u(t) \in R^m$, the unmeasurable plant output vector $w(t) \in R^m$, the measurable output vector $y(t) \in R^m$, the plant matrix $A \in R^{n \times n}$ whose eigenvalues all lie in the open left-half plane C^-, the input matrix $B \in R^{n \times m}$, the unmeasurable output matrix $E \in R^{m \times n}$, and the measurable output matrix $C \in R^{m \times n}$. Furthermore, it is assumed that the introduction of integral action preserves stabilisability and therefore that (Porter and Power (1970), Power and Porter (1970))

$$\text{rank } G_w = \text{rank } G_y = m .\qquad(3.4)$$

Here, the steady-state transfer function matrices

$$G_w = G_w(0) = -EA^{-1}B \in R^{m \times m}\qquad(3.5)$$

and

$$G_y = G_y(0) = -CA^{-1}B \in R^{m \times m}\qquad(3.6)$$

where the plant transfer-function matrices

$$G_w(s) = E(sI-A)^{-1}B\qquad(3.7)$$

and

$$G_y(s) = C(sI-A)^{-1}B \quad . \tag{3.8}$$

Finally, it is assumed that input-output decoupling is achievable and therefore that (Falb and Wolovich (1967))

$$\text{rank } F_w = m \tag{3.9}$$

where the decoupling matrix

$$F_w = \begin{bmatrix} e_1^T A^{d_1} B \\ \cdots\cdots \\ \cdots\cdots \\ \cdots\cdots \\ e_m^T A^{d_m} B \end{bmatrix} \in R^{m \times m} \tag{3.10}$$

and the d_i (i=1,2,...,m) and the e_i^T (i=1,2,...,m) are, respectively, the decoupling indices of the plant (Falb and Wolovich (1967)) and the rows of the unmeasurable output matrix. In the case of such plants, it is important to note that

$$F_w = \lim_{t \to 0} A_w^{-1}(t) H_w(t) \tag{3.11}$$

and

$$F_w^{-1} = \lim_{t \to 0} H_w^{-1}(t) A_w(t) \tag{3.12}$$

where

$$A_w(t) = \text{diag}\{t^{d_1+1}/(d_1+1)!,\ldots,t^{d_m+1}/(d_m+1)!\} \tag{3.13}$$

and

$$H_w(t) = EA^{-1}(e^{At}-I_n)B \tag{3.14}$$

is the plant step-response matrix for unmeasurable outputs. Similarly,

$$H_y(t) = CA^{-1}(e^{At}-I_n)B \tag{3.15}$$

is the plant step-response matrix for measurable outputs. This obviously indicates that, although the measurement of unmeasurable outputs is not necessary in the on-line operation of the controller, such measurement is necessary in the off-line design stage of the controller.

In order to design error-actuated set-point tracking PID controllers for linear multivariable plants governed by state, output, and measurement equations of the respective forms (3.1), (3.2), and (3.3), it is convenient to consider the behaviour of such plants on the discrete-time set $T_T = \{0,T,2T,\ldots,kT,\ldots\}$. This behaviour is governed by state, output, and measurement equations of the respective forms (Kwakernaak and Sivan (1972))

$$x_{k+1} = \Phi x_k + \Psi u_k , \tag{3.16}$$

$$w_k = \Xi x_k , \tag{3.17}$$

and

$$y_k = \Gamma x_k \ , \qquad (3.18)$$

where $x_k = x(kT) \in R^n$, $u_k = u(kT) \in R^m$, $w_k = w(kT) \in R^m$, $y_k = y(kT) \in R^m$,

$$\Phi = \exp(AT) \ , \qquad (3.19)$$

$$\Psi = \int_0^T \exp(At)B \ dt \ , \qquad (3.20)$$

$$\Xi = E \ , \qquad (3.21)$$

$$\Gamma = C \ , \qquad (3.22)$$

and $T \in R^+$ is the sampling period. Furthermore, in designing such controllers, it is necessary to introduce pre-filters which generate the set-point vector for measurable outputs $v \in R^m$ from the set-point vector for unmeasurable outputs $r \in R^m$ in accordance with equation

$$v = Jr \ , \qquad (3.23)$$

where the pre-filter matrix $J \in R^{m \times m}$ is to be determined. Thus, if the measurable output vector is caused to track its set-point vector in the sense that

$$\lim_{k \to +\infty} (v - y_k) = 0 \ , \qquad (3.24)$$

it follows from equations (3.23) and (3.24) that

$$\lim_{k \to +\infty} (Jr - y_k) = 0 \tag{3.25}$$

and therefore that

$$\lim_{k \to +\infty} (Jr - G_y G_w^{-1} w_k) = 0 \tag{3.26}$$

in view of equations (3.5) and (3.6). Therefore, if such pre-filters are chosen such that

$$J = G_y G_w^{-1} \in R^{m \times m} , \tag{3.27}$$

it follows from equations (3.26) and (3.27) that

$$\lim_{k \to +\infty} (r - w_k) = 0 \tag{3.28}$$

so that the unmeasurable output vector is caused to track its set-point vector in the steady state. It is thus evident (as indicated in the block diagram shown in Fig 3.1) that the essential function of digital PID/Pre-filter controllers for plants with unmeasurable output vectors is to cause the measurable output vectors to track their set-point vectors in the sense of equation (3.24), where the set-point vectors for the measurable output vectors are generated from the set-point vectors for the unmeasurable set-point vectors in accordance with equations (3.23) and (3.27).

The state and output equations of such plants under the action of error-actuated digital PID/Pre-filter controllers governed on the dicrete-time set $T_T = \{0,T,2T,\ldots,kT,\ldots\}$ by control-law equations of the forms

$$u_k = TK_1 e_k + TK_2 z_k + K_3(e_k - e_{k-1}) \qquad (3.29)$$

clearly assume the respective forms

$$\begin{bmatrix} x_{k+1} \\ z_{k+1} \\ f_{k+1} \end{bmatrix} = \begin{bmatrix} \Phi - T\Psi K_1 \Gamma - \Psi K_3 \Gamma & , & T\Psi K_2 & , & -\Psi K_3 \\ -T\Gamma & , & I_m & , & 0 \\ -\Gamma & , & 0 & , & 0 \end{bmatrix} \begin{bmatrix} x_k \\ z_k \\ f_k \end{bmatrix} + \begin{bmatrix} T\Psi K_1 + \Psi K_3 \\ TI_m \\ I_m \end{bmatrix} v$$

$$(3.30)$$

and

$$w_k = [\ \Xi\ ,\ 0\ ,\ 0\ ] \begin{bmatrix} x_k \\ z_k \\ f_k \end{bmatrix} , \qquad (3.31)$$

In equation (3.29), the error vector $e_k = v - y_k \in R^m$, the stored error-vector $f_k = e_{k-1} \in R^m$, the set-point vector for measurable outputs $v \in R^m$, the digital integral-of-error vector

$$z_k = z_{k-1} + Te_{k-1} \in R^m , \qquad (3.32)$$

and the controller matrices $K_1 \in R^{m \times m}$, $K_2 \in R^{m \times m}$, and $K_3 \in R^{m \times m}$. Therefore, provided only that T, K_1, K_2, and K_3 are such that all the eigenvalues of the closed-loop plant matrix

in equation (3.30) lie in the open unit disc D^-,

$$\lim_{k \to \infty} \Delta z_k = \lim_{k \to \infty} \{z_{k+1} - z_k\} = 0 \qquad (3.33)$$

and consequently

$$\lim_{k \to \infty} e_k = 0 \qquad (3.34)$$

so that set-point tracking occurs in the sense of equation (3.24).

The characteristic equation of the closed-loop plant matrix in equation (3.30) can be readily expressed in the form

$$\phi_c(z) = \phi_1(z)\phi_2(z)\phi_3(z) \qquad (3.35)$$

by invoking the block-diagonalisation procedure of Kokotovic (1975), and the response characteristics of the closed-loop system can accordingly be elucidated. The asymptotic properties of tracking systems under the action of such controllers can be characterised in terms of the eigenstructure of the closed-loop plant matrix, which involves the decomposition of this matrix into three sub-systems based on the explicitly invertible block diagonalisation transform (Kokotovic (1975)).

This block-diagonalisation procedure transforms the matrix triple incorporated in equations of the form

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} A_{11} & , & A_{12} \\ A_{21} & , & A_{22} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(k) \qquad (3.36)$$

and

$$y(k) = [\ C_1\ ,\ C_2\ ] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \qquad (3.37)$$

where $x_1(k) \in R^{n_1}$, $x_2(k) \in R^{n_2}$, $A_{ij} \in R^{n_i \times n_j}$ $(i,j=1,2)$, $B_1 \in R^{n_1 \times m}$, $B_2 \in R^{n_2 \times m}$, $C_1 \in R^{m \times n_1}$, and $C_2 \in R^{m \times n_2}$ into the block-diagonal form incorporated in the equations

$$\begin{bmatrix} \chi_1(k+1) \\ \chi_2(k+1) \end{bmatrix} = \begin{bmatrix} A_{11} & , & 0 \\ 0 & , & A_{22} \end{bmatrix} \begin{bmatrix} \chi_1(k) \\ \chi_2(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(k) \qquad (3.38)$$

and

$$y(k) = [\ C_1\ ,\ C_2\ ] \begin{bmatrix} \chi_1(k) \\ \chi_2(k) \end{bmatrix} \qquad . \qquad (3.39)$$

The state vectors in these equations are related by the linear state transformation (Kokotović (1975))

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = W \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix} \qquad (3.40)$$

where

$$W = \begin{bmatrix} I_{n_1} & , & M \\ -L & , & I_{n_2} - LM \end{bmatrix} \in R^{(n_1+n_2) \times (n_1+n_2)} \, , \qquad (3.41)$$

$\chi_1(k) \in R^{n_1}$, $\chi_2(k) \in R^{n_2}$, $A_{ij} \in R^{n_i \times n_j}$ (i,j=1,2), $B_1 \in R^{n_1 \times m}$, $B_2 \in R^{n_2 \times m}$, $C_1 \in R^{m \times n_1}$, $C_2 \in R^{m \times n_2}$, $L \in R^{n_2 \times n_1}$,and $M \in R^{n_1 \times n_2}$.

Thus, if L and M satisfy the matrix Riccati equations (Kokotovic (1975))

$$A_{21} + LA_{11} - A_{22}L - LA_{12}L = 0 \qquad (3.42)$$

and

$$(A_{11} - A_{12}L)M - M(A_{22} + LA_{12}) + A_{12} = 0. \qquad (3.43)$$

it follows from equations (3.36), (3.38), and (3.40) that

$$A_{11} = A_{11} - A_{12}L \qquad (3.44)$$

and

$$A_{22} = A_{22} + LA_{12} \qquad . \qquad (3.45)$$

The asymptotic properties of the discrete-time closed-loop tracking system can now be readily determined by regarding T as a perturbation parameter in equations (3.30) and (3.31). Thus, by regarding in equation (3.36)

$$A_{11} = \begin{bmatrix} \Phi - T\Psi K_1 \Gamma - \Psi K_3 \Gamma & , & T\Psi K_2 \\ -T\Gamma & , & I_m \end{bmatrix} \qquad , \qquad (3.46)$$

$$A_{12} = \begin{bmatrix} -\Psi K_3 \\ 0 \end{bmatrix} \qquad , \qquad (3.47)$$

$$A_{21} = [-\Gamma , 0] \qquad , \qquad (3.48)$$

and

$$A_{22} = 0 \qquad , \qquad (3.49)$$

the solution of equations (3.42) and (3.43) can be readily obtained by using power series expansions in T. This involves the definition of matrices L_1 and L_2 such that

$$L = [L_1 , L_2] \qquad (3.50)$$

where

$$L_1 = L_{10} + L_{11} T + \ldots \qquad (3.51)$$

$$L_2 = L_{20} + L_{21} T + \ldots \qquad (3.52)$$

in which $L_{1i} \in R^{n_2 \times n_3}$, $L_{2i} \in R^{n_2 \times n_4}$, $(i=0,1,2,\ldots)$. Therefore, it is clear from equations (3.42), (3.46), (3.47), (3.48), (3.49), and (3.50) that on isolating coefficients

$$L = [\ C \ , \ 0 \ ] + O(T) \qquad\qquad (3.53)$$

and therefore from equations (3.44) and (3.45) that

$$A_{11} = \begin{bmatrix} \Phi - T\Psi K_1 \Gamma \ , & T\Psi K_2 \\ -T\Gamma \ , & I_m \end{bmatrix} \qquad\qquad (3.54)$$

and

$$A_{22} = -TCBK_3 + O(T^2) \qquad . \qquad (3.55)$$

The matrix A_{11} in equation (3.43) is now block-diagonalised, by again regarding T as a perturbation parameter in equation (3.54) and by regarding in equation (3.36)

$$\bar{A}_{11} = \Phi - T\Psi K_1 \Gamma \ , \qquad\qquad (3.56)$$

$$\bar{A}_{12} = T\Psi K_2 \ , \qquad\qquad (3.57)$$

$$\bar{A}_{21} = -T\Gamma \ , \qquad\qquad (3.58)$$

and

$$\bar{A}_{22} = I_m \quad . \qquad\qquad (3.59)$$

In addition, the matrix $\bar{L}$ is defined in a power-series form

$$\bar{L} = \bar{L}_0 + T\bar{L}_1 + T^2\bar{L}_2 + \ldots \qquad\qquad (3.60)$$

In equations (3.56), (3.57), (3.58), (3.59), and (3.60), the overbar has been used to distinguish between the two explicit stages of the block-diagonalisation procedure.

Therefore, it is clear from equations (3.42), (3.56), (3.57), (3.58), (3.59), and (3.60) that on isolating coefficients

$$\bar{L} = CA^{-1} + T(CA^{-1}BK_1CA^{-1} + CA^{-1}BK_2CA^{-2} - C/2) + O(T^2).$$

$$(3.61)$$

Hence, it follows from (3.44), (3.45) and (3.61) that

$$\bar{A}_{11} = I_n + TA + T^2A^2/2 - T^2BK_1C - T^2BK_2CA^{-1} + O(T^3)$$

$$(3.62)$$

and

$$\bar{A}_{22} = I_m - T^2CA^{-1}BK_2 + O(T^3) \qquad . \qquad (3.63)$$

Thus, it is evident from equations (3.54), (3.55), (3.62), and (3.63) that the characteristic polynomials as expressed in equation (3.35) are

$$\phi_1(z) = | zI_n - I_n - TA - T^2A^2/2 + T^2BK_1C$$
$$+ T^2BK_2CA^{-1} + O(T^3) | , \qquad (3.64)$$

$$\phi_2(z) = | zI_m - I_m - T^2CA^{-1}BK_2 + O(T^3) | , \qquad (3.65)$$

and

$$\phi_3(z) = | zI_m + TCBK_3 + O(T^2) | . \qquad (3.66)$$

3.3 Synthesis

It is clear that tracking will occur in the sense of equation (3.24) provided only that the set of closed-loop characteristic roots

$$Z_c = Z_1 \cup Z_2 \cup Z_3 \subset D^- \qquad (3.67)$$

where D^- is the open unit disc and the sets of characteristic roots Z_1, Z_2, and Z_3 are, respectively, the roots of the characteristic polynomials as expressed in equation (3.35).

Therefore, in case

$$K_1 = H_w^{-1}(T)A_w(T)\Pi J^{-1} , \qquad (3.68)$$

where $H_y(T)$ is given by equation (3.14) and

$$\Pi = \text{diag}\{\pi_1, \pi_2, \dots, \pi_m\} , \quad \pi_i \in R^+ \ (i=1,2,\dots,m), \qquad (3.69)$$

$$K_2 = G_y^{-1}(0)\Sigma , \qquad (3.70)$$

where $G_w(0)$ is given by equation (3.6) and

$$\Sigma = \text{diag}\{\sigma_1, \sigma_2, \dots, \sigma_m\} , \quad \sigma_i \in R^+ \ (i=1,2,\dots,m), \qquad (3.71)$$

and

$$K_3 = H_w^{-1}(T)\Lambda_w(T)\Delta J^{-1} \, , \qquad (3.72)$$

where $H_w(T)$ is given by equation (3.14) and

$$\Delta = \text{diag}\{\delta_1, \delta_2, \ldots, \delta_m\} \, , \quad \delta_i \in R^+ \, (i=1,2,\ldots,m), \qquad (3.73)$$

it follows from equations (3.35), (3.64), (3.65), (3.66), (3.68), (3.70), and (3.72) that

$$Z_1 = \{z \in C : | \, zI_n - I_n - TA + O(T^2) \, | = 0\} \, , \qquad (3.74)$$

$$Z_2 = \{z \in C : | \, zI_m - I_m + T^2\Sigma + O(T^3) \, | = 0\} \, , \qquad (3.75)$$

and

$$Z_3 = \{z \in C : | \, zI_m + O(T) \, | = 0\} \, . \qquad (3.76)$$

These expressions indicate that, provided T is sufficiently small, all the closed-loop characteristic roots lie within the open unit disc. This follows since the open-loop plant is asymptotically stable on the continuous-time set $T = [0,+\infty)$ and since $T^2\Sigma$ is a positive diagonal matrix. The introduction of error-actuated digital set-point tracking PID controllers governed by equations (3.29), (3.68), (3.70) and (3.72) accordingly ensures that set-point tracking occurs for the

measurable outputs in the sense of equation (3.24) when the sampling time $T \in (0,T^*]$, where $T^* = T^*(\Pi,\Sigma,\Delta)$ can be readily obtained by simple "on-line" tuning (Porter and Jones (1985a)). The presence of pre-filters governed by equations (3.23) and (3.27) then ensures that set-point tracking occurs for the unmeasurable outputs in the sense of equation (3.28).

Furthermore, it follows from equations (3.30) and (3.31) that the unmeasurable output from the initially quiescent plant after the first sampling interval under error-actuated digital PID/Pre-filter control is

$$w(T) = [TH_w(T)K_1 + H_w(T)K_3]v \qquad (3.77)$$

since

$$\Xi\Psi = \int_0^T E\exp(At)B\,dt = H_w(T) \qquad (3.78)$$

in view of equations (3.20) and (3.21). It is thus evident from equations (3.23), (3.68), (3.72), and (3.77) that

$$w(T) = [T\Lambda_w(T)\Pi + \Lambda_w(T)\Delta]r \qquad (3.79)$$

and therefore that set-point tracking occurs when $T \in (0,T^*]$ with initial non-interaction since $\Lambda_w(T)\Pi$ and $\Lambda_w(T)\Delta$ are diagonal matrices for all $T \in R^+$. The pre-filter matrix $J \in R^{m \times m}$ given by equation (3.27), together with the proportional, integral, and derivative controller matrices $K_1 \in R^{m \times m}$, $K_2 \in$

$R^{m \times m}$, and $K_3 \in R^{m \times m}$ given by equations (3.68), (3.70), and (3.72), respectively, can all be obtained from the step-response matrices $H_w(t)$ and $H_y(t)$. This is the case since it follows from equations (3.14) and (3.15) that

$$G_w(0) = \lim_{t \to \infty} H_w(t) = -EA^{-1}B \qquad (3.80)$$

and

$$G_y(0) = \lim_{t \to \infty} H_y(t) = -CA^{-1}B \qquad (3.81)$$

because the open-loop plant is asymptotically stable.

Therefore, tunable digital set-point tracking PID controllers — with associated pre-filters — can be readily designed for linear multivariable plants without the need for mathematical models provided only that the step-response matrices $H_w(t)$ and $H_y(t)$ are obtained from open-loop tests. Then, it is clear that although the controller operation does not require the on-line measurement of unmeasurable outputs, the off-line measurement of such outputs is necessary in the design stage of the controller. It can be considered that the superior operational performance such as initial non-interaction for unmeasurable outputs is obtained in exchange for the effort of measuring such outputs off line.

Finally, since the expressions (3.68) and (3.72) for K_1 and K_3 involve $H_w^{-1}(T)$, it is clear that the sampling period T must be chosen such that the minimum singular value of $H_w(T)$ ($\sigma_{min}[H_w(T)]$) is not small, so that $H_w(T)$ is well-conditioned

(Porter and Jones (1985c)).

3.4 Illustrative example

These general results can be conveniently illustrated by designing a tunable digital set-point tracking PID/Pre-filter controller for the linear model of the F100 gas-turbine engine obtained at Intermediate condition (Appendix 3).

The plant has five measurable outputs, five unmeasurable outputs, and five manipulated inputs and is governed by state, output, and measurement equations of the form (A3.8), (A3.9), and (A3.10). The elements of the plant step-response matrices $H_y(t)$ and $H_w(t)$ are obtained by "off-line" open-loop tests. It is possible to obtain such data in engine running tests which are carried out using altitude test facilities. They are shown in Figs 2.2 to 2.6 and in Figs 3.2 to 3.6, respectively. It is clear from these figures that the plant is highly interactive. Furthermore, the corresponding plots of the minimum singular values ($\sigma_{min}[H_y(t)]$ and $\sigma_{min}[H_w(t)]$) of the step-response matrices shown in Fig 2.7 and Fig 3.7(a),(b) indicate that the plant is minimum phase for the measurable outputs and nonminimum phase for the unmeasurable outputs (Porter and Jones (1985c)). It is evident from Figs 2.2 to 2.6 that $G_y(0)$ is well-conditioned since $\sigma_{min}[H_y(+\infty)]$ is not small. However, attention must be given in order to choose the sampling interval T so as not to use an ill-conditioned $H_w(T)$, since $\sigma_{min}[H_w(+\infty)]$ vanishes once.

It is found from Figs 2.2 to 2.6 and from Figs 3.2 to 3.6 that

$$H_w(0.05) = \begin{bmatrix} 0.20199E-01 & -268.18 & 8.4970 & 0.11445 & -90.294 \\ 0.63298E-04 & 0.27604 & 0.16262 & 0.10948E-02 & -0.10496 \\ 0.17399E-01 & -0.94227 & -0.72047E-02 & 0.27036 & 12.481 \\ -0.12214E-06 & 0.67912E-02 & -0.15042E-02 & 0.11095E-03 & 0.74531E-03 \\ 0.19447E-06 & -0.21704E-03 & -0.91478E-05 & -0.56074E-02 & 0.70200E-02 \end{bmatrix} \quad (3.82)$$

$$G_w(0) = \begin{bmatrix} 0.93215 & -1384.6 & 18.823 & -11.079 & -197.54 \\ 0.53537E-02 & 17.599 & 0.28572 & -0.12785 & -0.78934 \\ 0.12133 & 282.50 & -2.1459 & 2.7500 & 19.700 \\ 0.10422E-04 & 0.26652 & -0.85391E-02 & -0.24992E-03 & 0.24874E-02 \\ -0.20603E-05 & -0.85660E-02 & 0.33076E-04 & -0.68327E-02 & 0.68195E-02 \end{bmatrix} \quad (3.83)$$

and

$$G_y(0) = \begin{bmatrix} 0.37904 & 1238.8 & -28.508 & -9.2619 & -57.405 \\ 0.30777 & 660.79 & -2.8675 & -25.646 & -46.221 \\ 0.20602E-01 & -39.863 & 0.25947 & -0.76283 & -6.8275 \\ 0.15944E-02 & -12.168 & 0.38479E-01 & -0.33542E-01 & -0.44527 \\ 0.90309E-01 & 210.94 & -1.7403 & 2.2101 & 12.248 \end{bmatrix} . \quad (3.84)$$

It is accordingly found from equations (3.27), (3.83), and (3.84) that

$$
J = \begin{bmatrix}
0.25123 & 22.335 & -0.18958 \\
0.18671 & 31.039 & -0.38513 \\
0.28902E-01 & 0.21873 & -0.66287E-01 \\
0.35558E-02 & -0.30482 & -0.54230E-04 \\
0.39481E-02 & 0.38839 & 0.69140
\end{bmatrix}
$$

$$
\begin{bmatrix}
4688.4 & 282.44 \\
1893.0 & 2645.6 \\
57.423 & 31.907 \\
-6.8342 & 5.0751 \\
51.518 & -60.740
\end{bmatrix} . \qquad (3.85)
$$

The excellent set-point tracking behaviour of the plant under the action of the resulting error-actuated PID/Pre-filter controller tuned such that $T = 0.05$ sec, $\Lambda_w(0.05)\Pi = 1.0I_5$, $\Sigma = $ diag$\{50, 50, 50, 50, 100\}$, $\Lambda_w(0.05)\Delta = 0.01I_5$, is shown in Figs 3.8 and 3.9. In this case, the required thrust change is 500 lb so that the set-point vector for the unmeasurable outputs is $r = [500, 0, 0, 0, 0]^T$ whilst the corresponding set-point vector for the measurable outputs is $v = G_y G_w^{-1}[500, 0, 0, 0, 0]^T = [126, 93.4, 14.5, 1.78, 1.97]^T$. It is evident from Figs 3.8 and 3.9 that the response of the gas-turbine engine consists of a fast approach to the desired unmeasurable and measurable outputs with minimum interaction between the five channels. In addition, it is clear from Fig 3.10 that the corresponding manipulated variables exhibit no practically undesirable characteristics. Finally, it is noted that the distinction between the PID/Pre-filter controllers and the PID controllers (Chapter 2) comes from the initial decoupling of unmeasurable outputs. In the case of PID controllers (Chapter 2), the initial transient behaviour of unmeasurable outputs is not considered. Therefore, initial decoupling is not obtained

for such outputs as shown in Fig 3.11 which corresponds to the example in Chapter 2. On the other hand, in the case of the PID/Pre-filter controller, initial decoupling for such outputs is obtained as was shown in Fig 3.8.

3.5 Conclusion

In this chapter, a block-diagonalisation transformation has been used to exhibit the asymptotic properties of discrete-time closed-loop tracking systems incorporating asymptotically stable linear multivariable plants with unmeasurable outputs under the action of digital PID/Pre-filter controllers. It has been shown that the pre-filter matrices, together with the proportional, integral, and derivative controller matrices embodied in the resulting tunable digital PID/Pre-filter controllers, can be determined from open-loop step-response tests thus circumventing the need for detailed mathematical models of complex plants. In order to obtain the step-response data for unmeasurable outputs, it is necessary to measure off-line such outputs in the design stage of the controller. Some effort might be required to do so. However, such measurement is possible for plants such as aero gas-turbines during ground and altitude tests and therefore the efforts involved are compensated by the superior initial transient response for unmeasurable outputs. Finally, the effectiveness of these methodologies has been illustrated by designing a digital set-point tracking controller for a gas-turbine engine with five measurable outputs and five unmeasurable outputs.
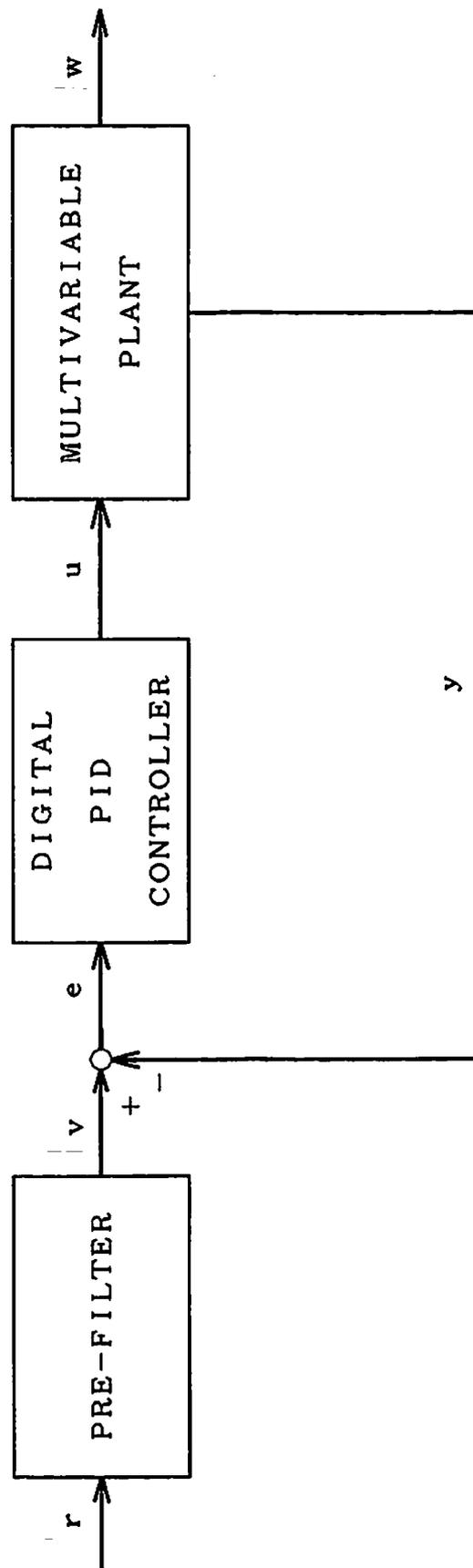
Fig 3.1   Block diagram of digital control system
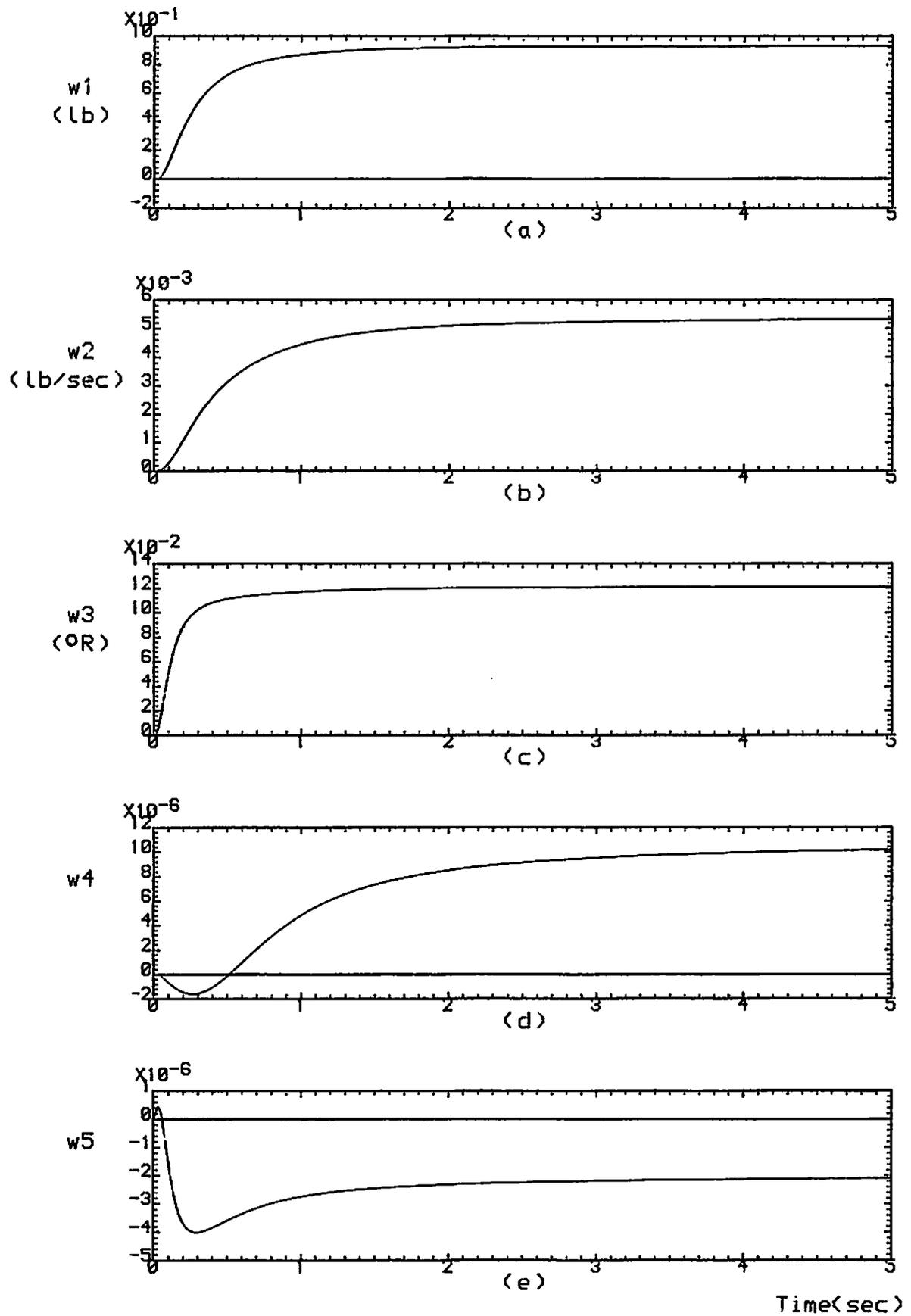for plant with unmeasurable outputs

Fig 3.2 Open-loop step-responses of F100 engine
          Unmeasurable outputs when u=[1 0 0 0 0]

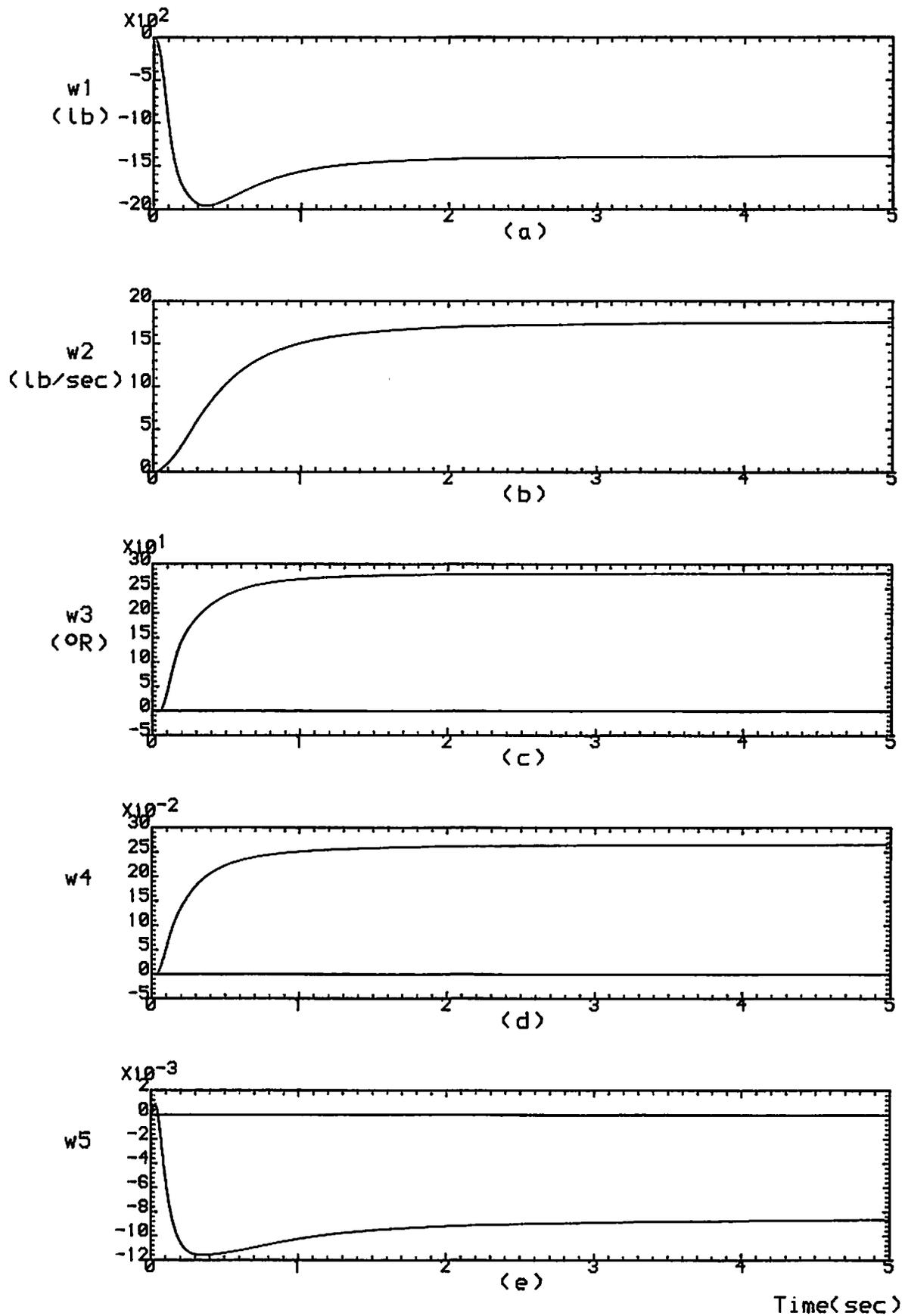Fig 3.3 Open-loop step-responses of F100 engine
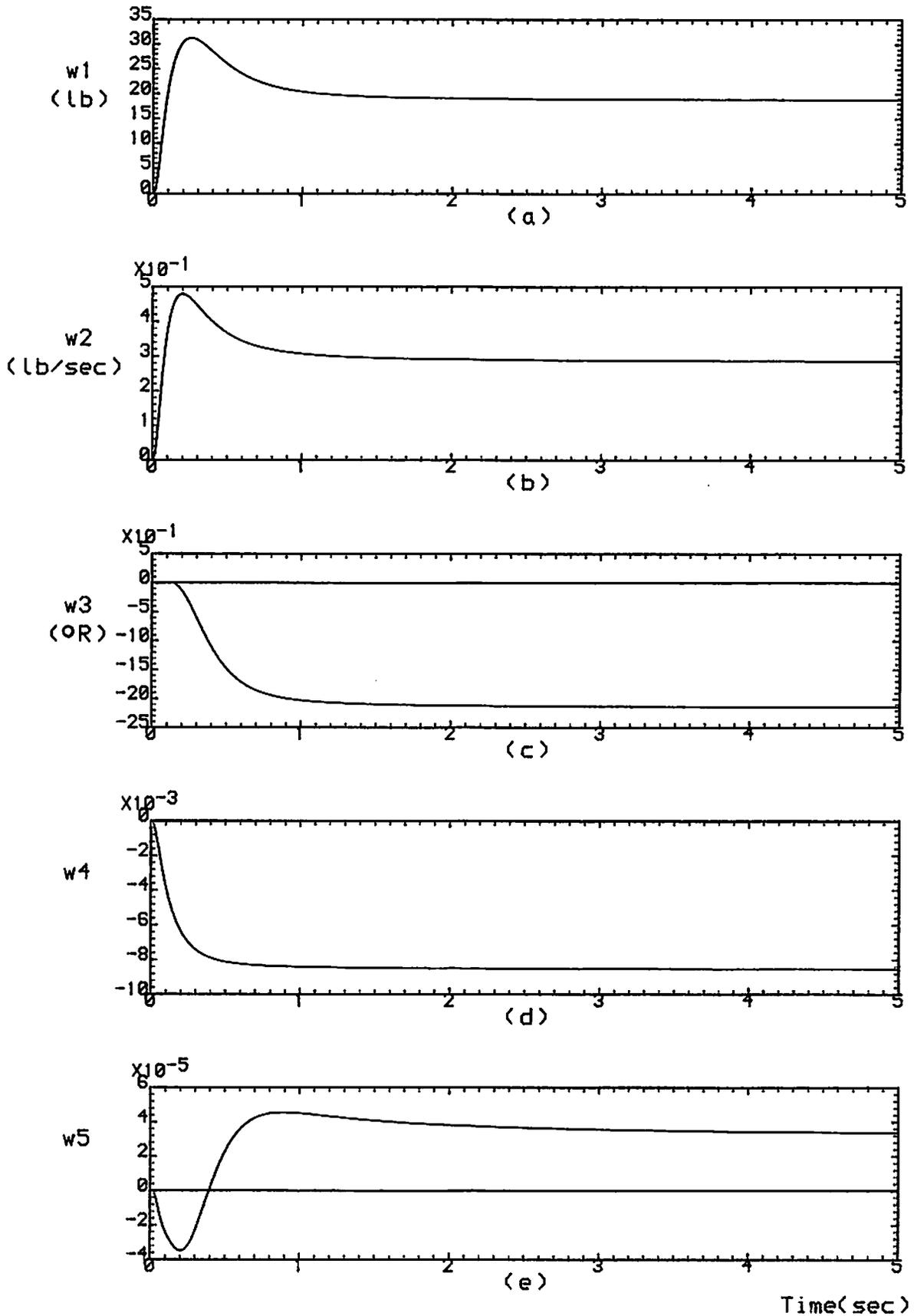   Unmeasurable outputs when u=[0 1 0 0 0]

Fig 3.4 Open-loop step-responses of F100 engine
Unmeasurable outputs when u=[0 0 1 0 0]
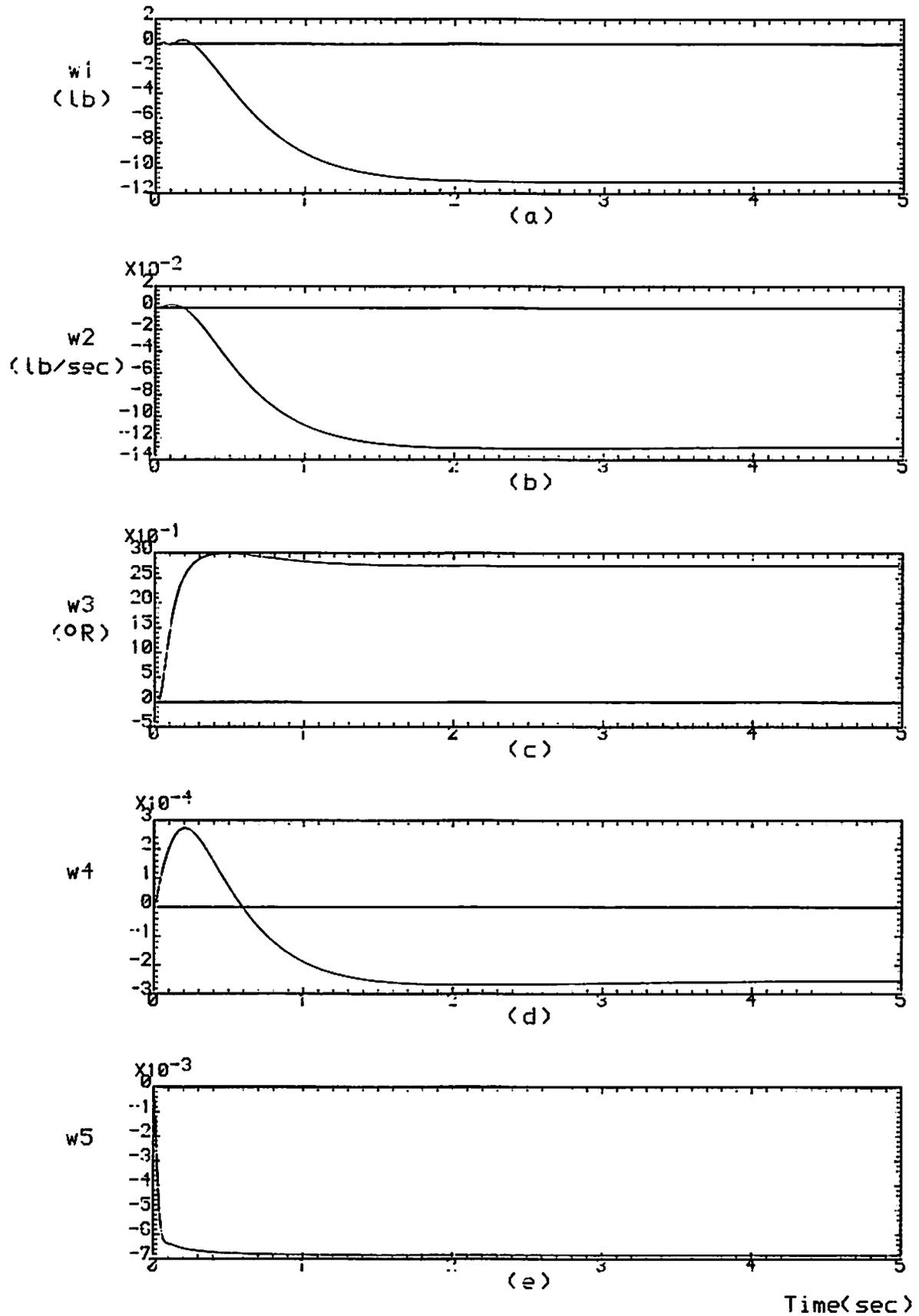
Fig 3.5 Open-loop step-responses of F100 engine
Unmeasurable outputs when u=[0 0 0 1 0]

83

$\times 10^1$

w1
(lb)

0
-5
-10
-15
-20

(a)

$\times 10^{-1}$

w2
(lb/sec)

0
-2
-4
-6
-8
-10

(b)

w3
(°R)

25
20
15
10
5
0

(c)

$\times 10^{-4}$

w4

30
25
20
15
10
5
0

(d)

$\times 10^{-3}$

w5

8
6
4
2
0

(e)

Time(sec)
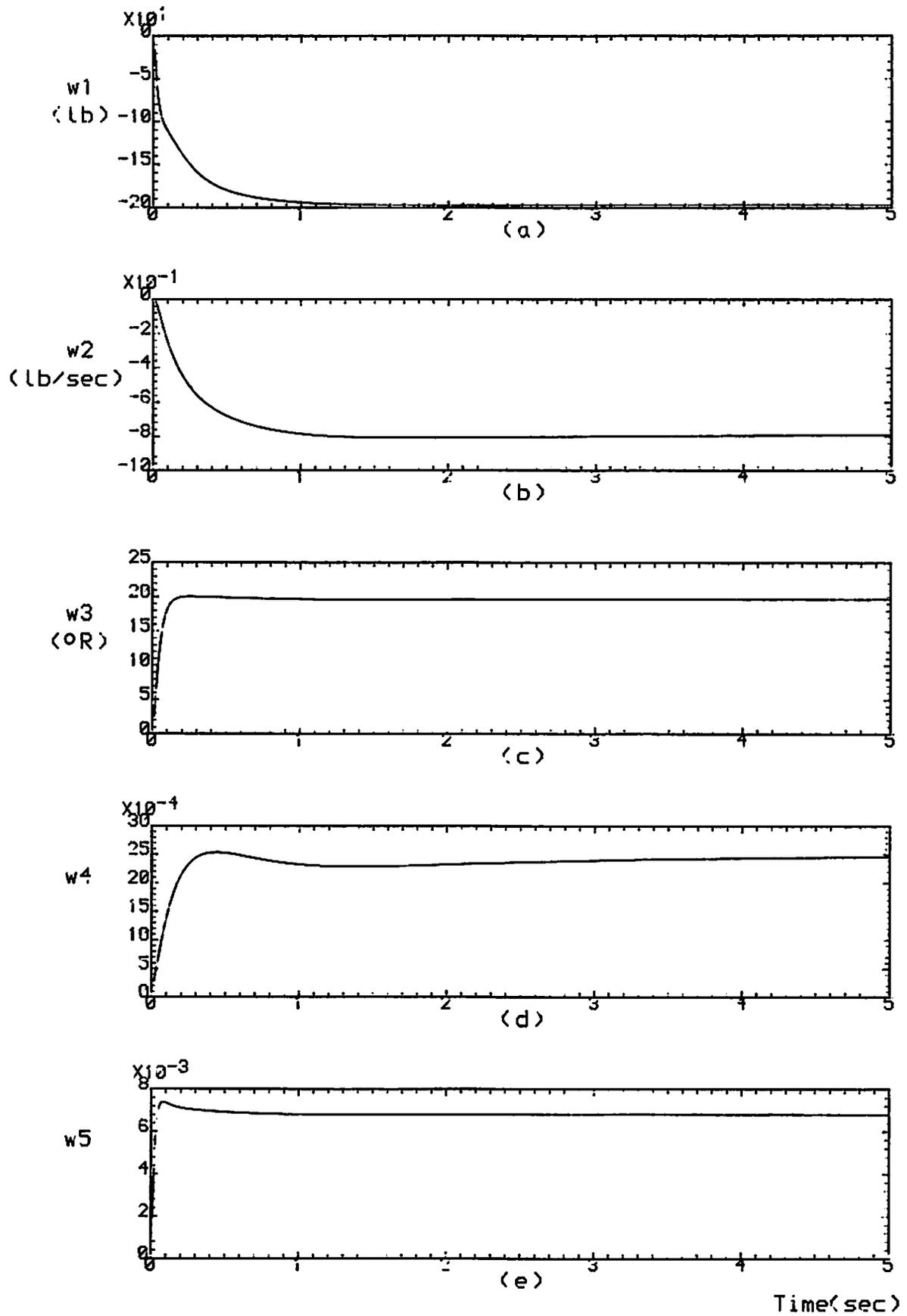
Fig 3.6 Open-loop step-responses of F100 engine
Unmeasurable outputs when u=[0 0 0 0 1]
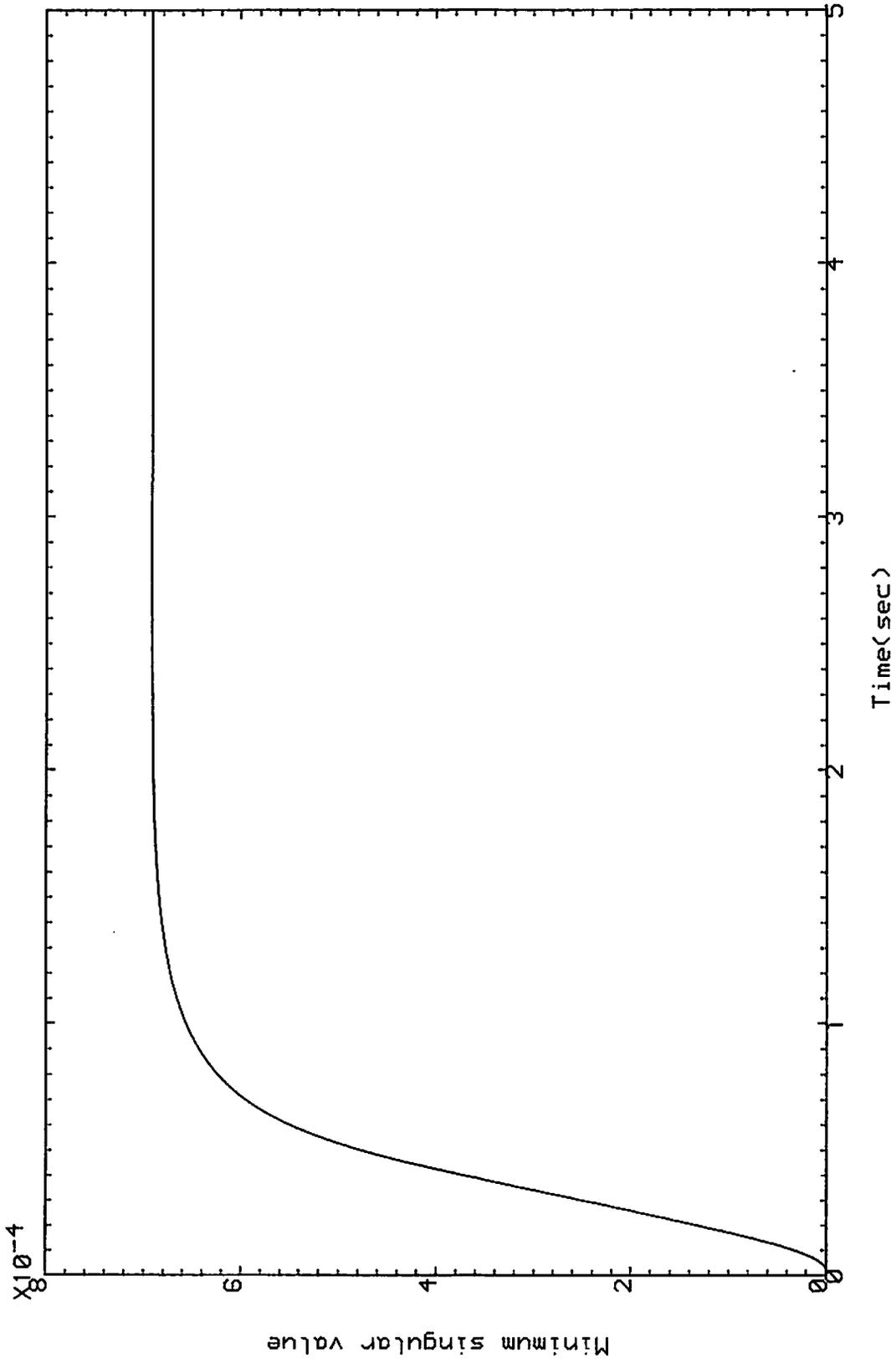
Fig.3.7(a) Minimum singular value plot of the plant step-response matrix
Plant: F100 engine 33 states model with 5 inputs and
5 unmeasurable outputs
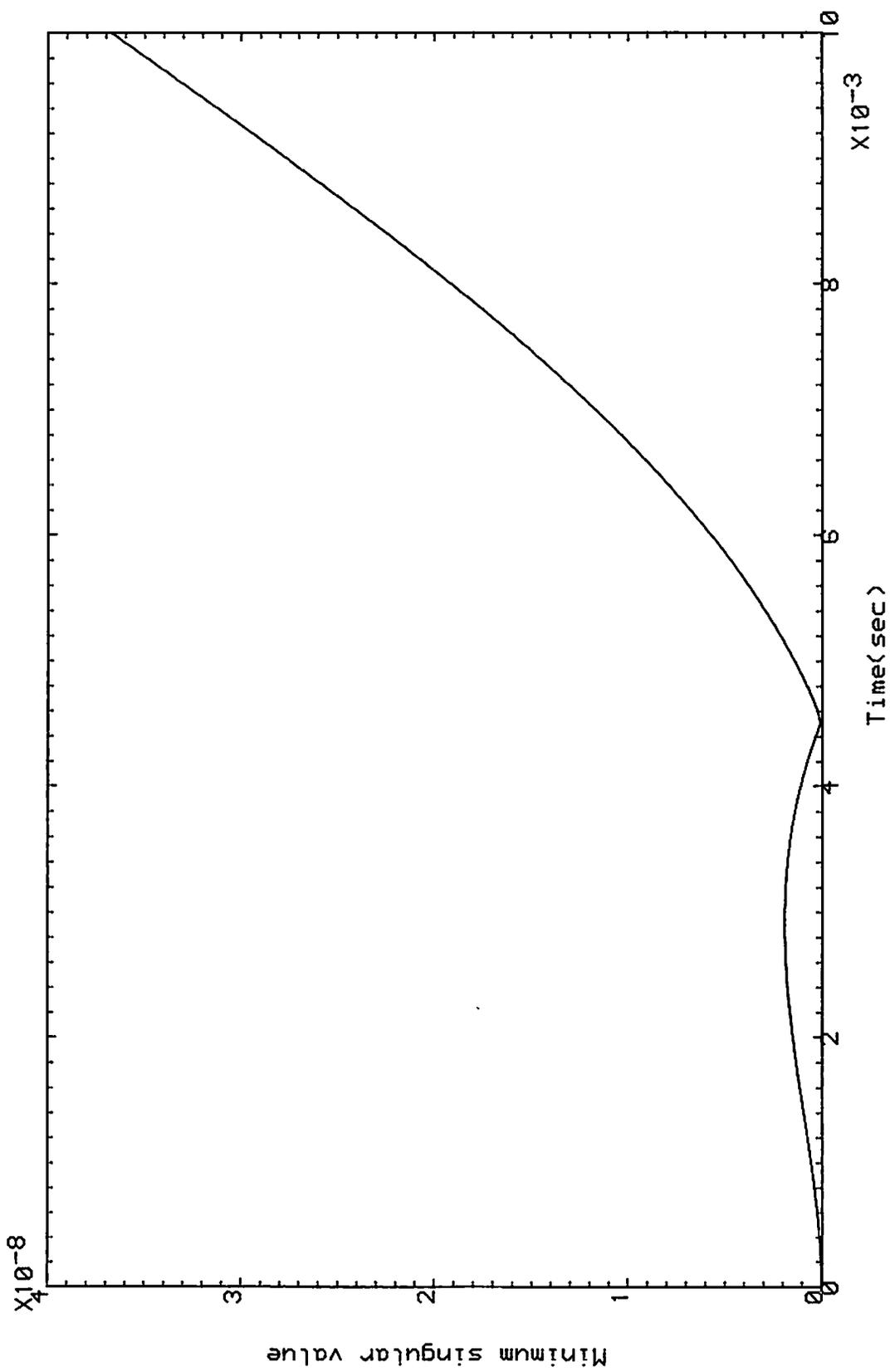
Fig.3.7(b) Minimum singular value plot of the plant step-response matrix
Plant: F100 engine 33 states model with 5 inputs and
5 unmeasurable outputs
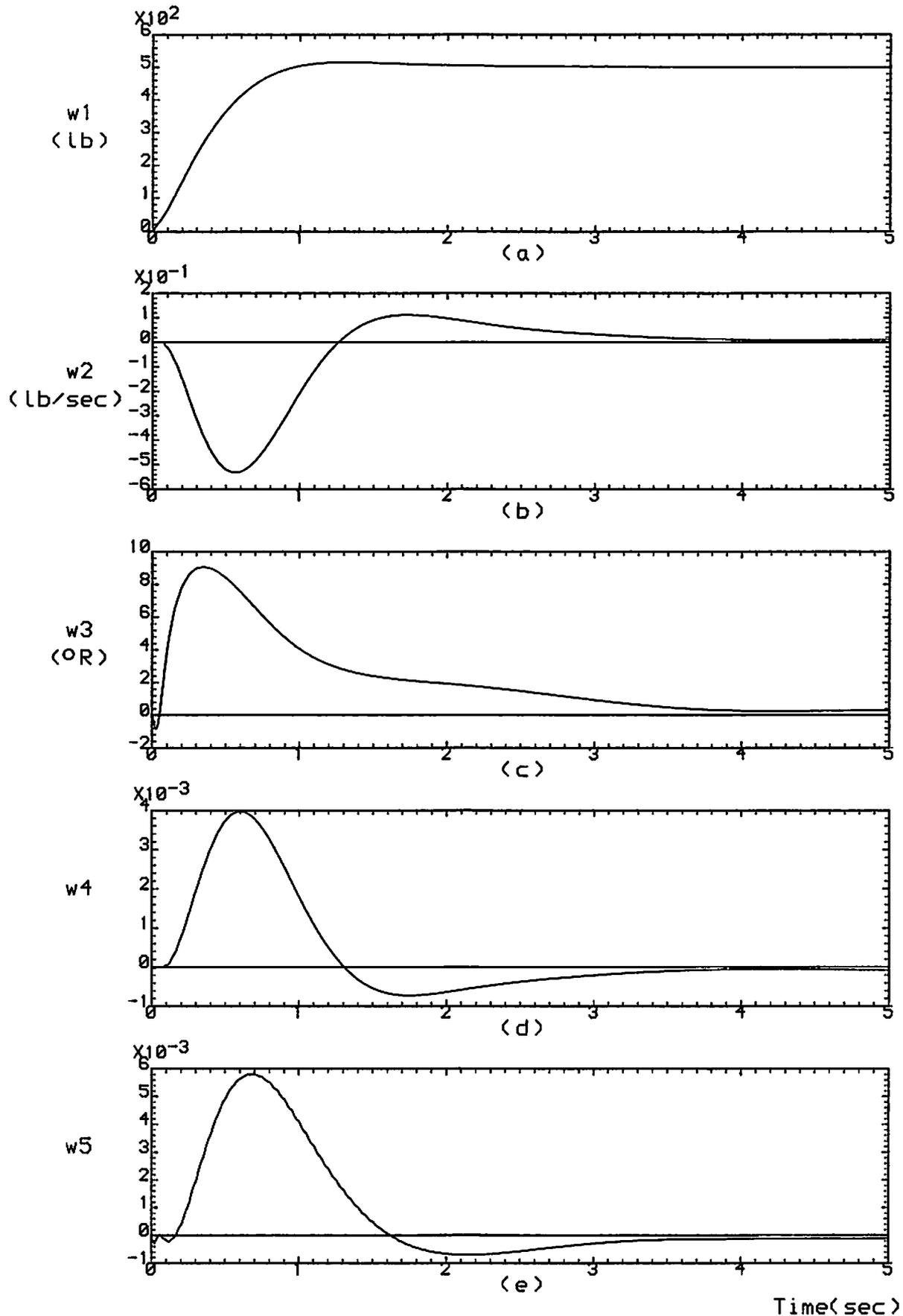
Fig 3.8 Unmeasurable outputs of F100 engine
under digital PID/Pre-Filter control
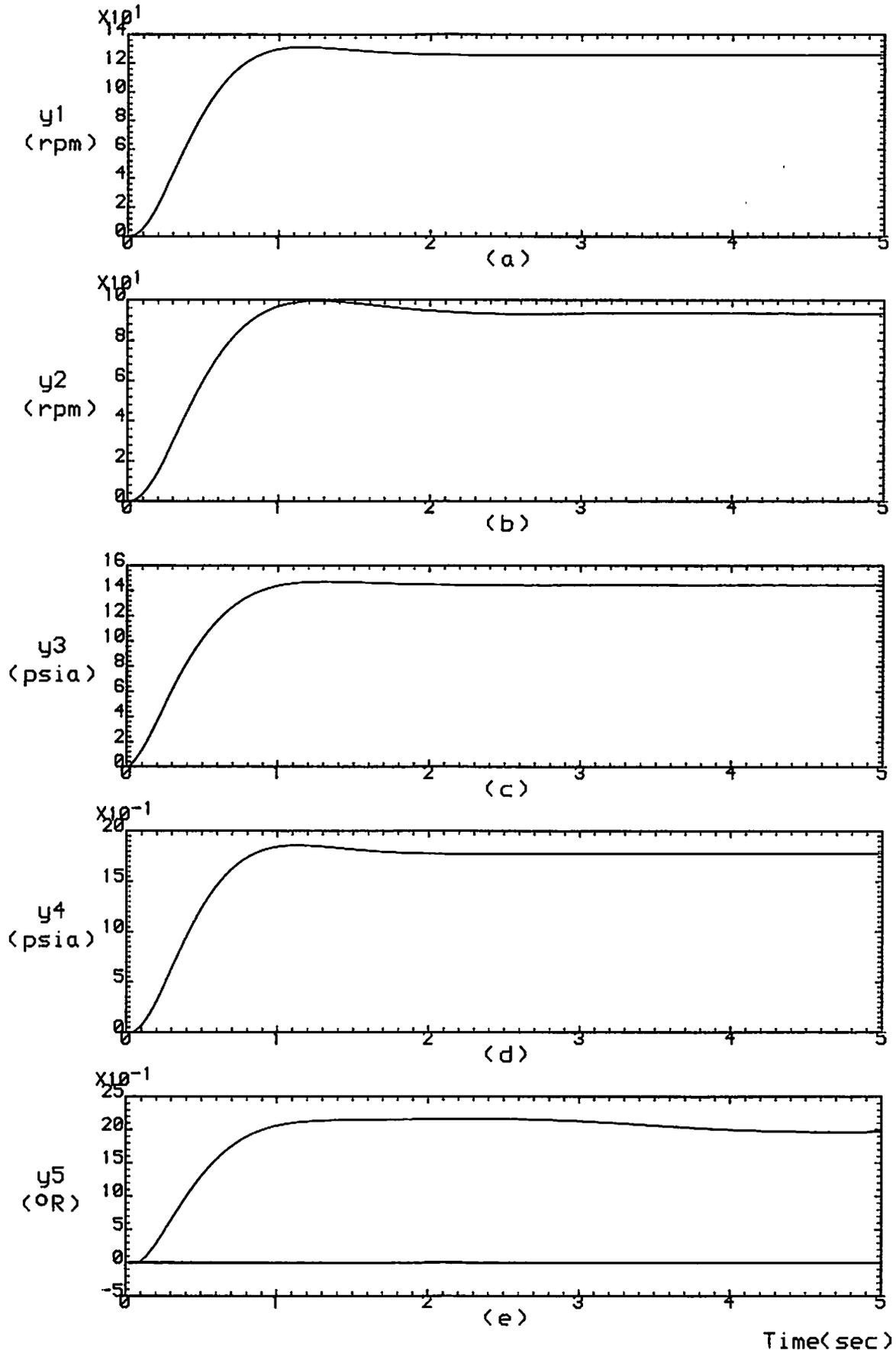
Fig 3.9   Measurable outputs of F100 engine
under digital PID/Pre-Filter control

Fig 3.10 Manipulated variables of F100 engine
under digital PID/Pre-Filter control

Fig 3.11 Unmeasurable outputs of F100 engine
under digital PID control

PART III

DESIGN OF LIMIT—TRACKING SYSTEMS

INCORPORATING LINEAR MULTIVARIABLE PLANTS

CHAPTER 4

GENERALISED CHARACTERISATION OF TRACKING SYSTEMS

AND LINEAR MULTIVARIABLE PLANTS

4.1 Introduction

The methodologies for the design of set-point tracking systems introduced in Part II deal with linear multivariable plants in which the numbers of input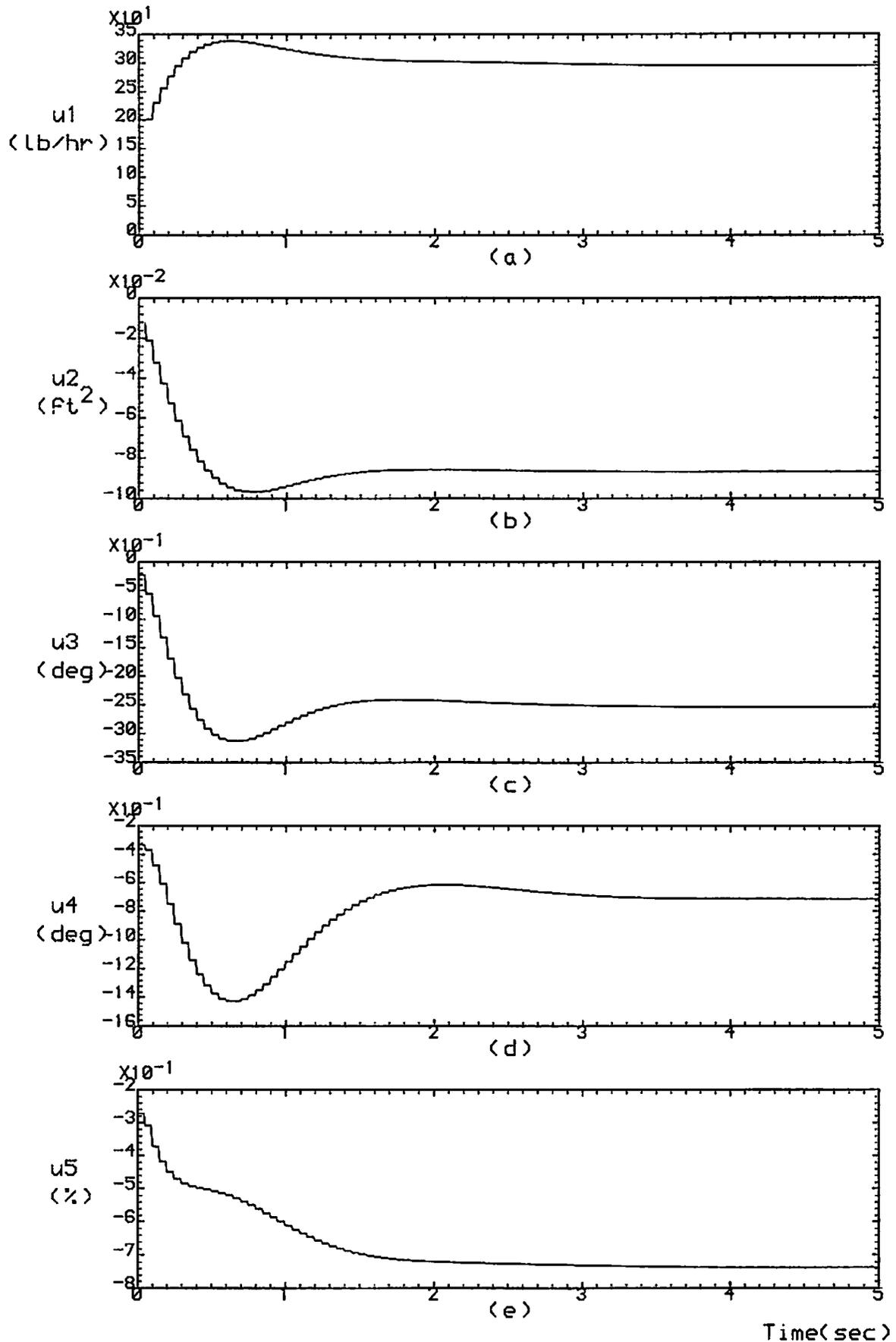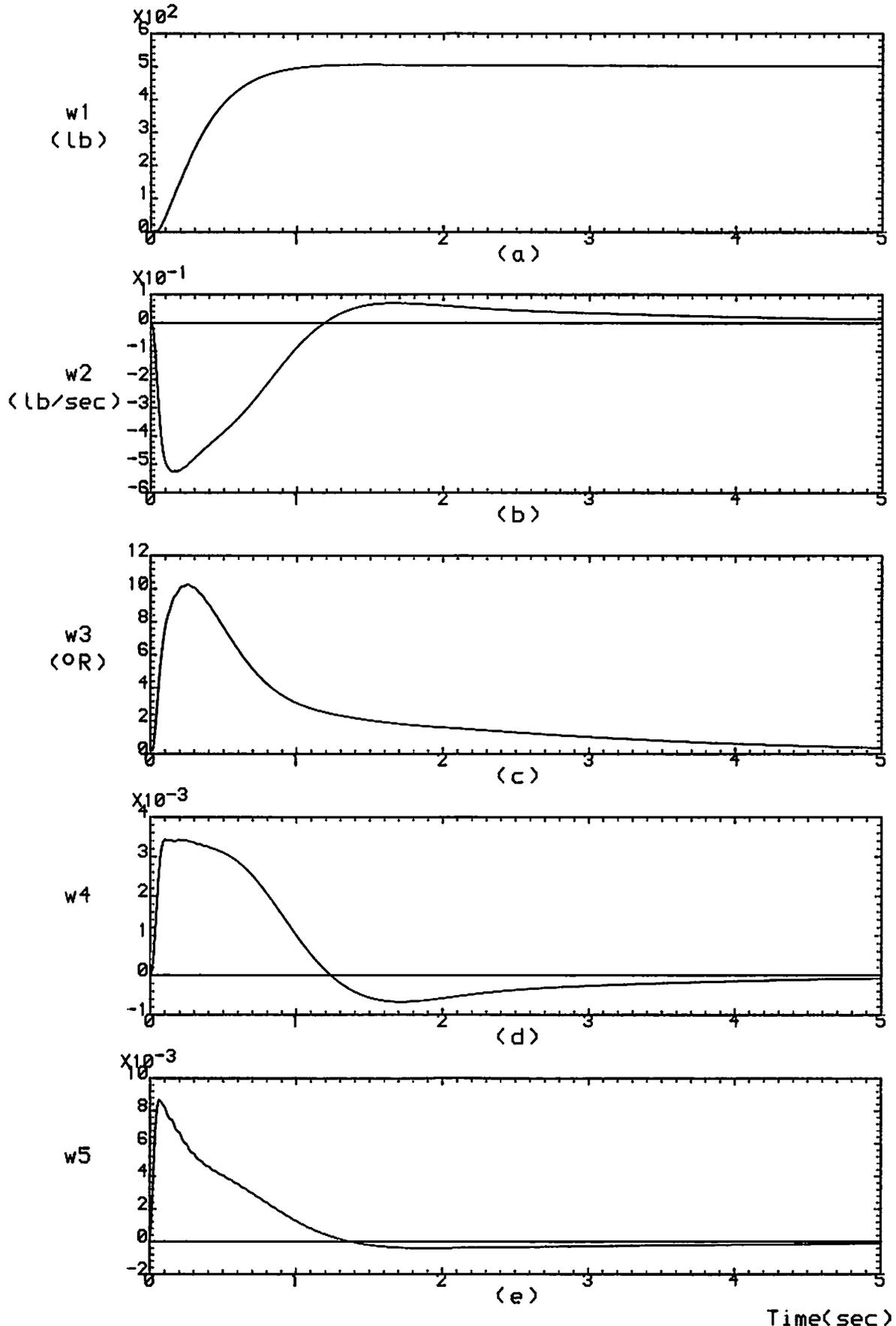s and outputs are equal. Such tracking systems work effectively provided that plants meet the fundamental requirement of functional controllability. Therefore, the number of manipulated inputs has to be not less than that of controlled outputs. However, in case plants have more controlled outputs than manipulated inputs, they fail to meet these requirements. Therefore, set-point tracking systems incorporating as many integrators as the number of outputs do not work properly. In such cases, if control engineers choose an appropriate subset of plant outputs and design a set-point tracking controller for only this subset, it might happen in some plants such as gas-turbine engines that some of the uncontrolled plant outputs violate the engine operational limits whilst the controlled plant outputs are tracking their corresponding set-point commands. Therefore, the need for a more general tracking concept than set-point tracking is evident in order to give a sound basis for the design of controllers for linear multivariable plants with more outputs than inputs.

The creation of such a general tracking concept is carried out by the inclusion of inequalities in tracking conditions. Thus, firstly, the tracking characteristics of linear multivariable plants are expressed by sets of linear inequalities involving the steady-state transfer function matrices of such plants. Such sets of inequalities, which also occur in problems of linear programming, can be investigated very effectively using results from convex analysis (Rockafellar (1970)). In this investigation, undertracking (ie tracking with nonnegative errors) is defined and its characteristics are discussed in terms of vector spaces. Next, it is shown that the possibility of undertracking is characterised by the separation theorem of convex analysis. This leads to the classification of plants and to the presentation of geometrical and analytical features of this classification. Furthermore, tracking characteristics under the action of constant disturbances are also discussed. Finally, illustrative examples explain these concepts. The proofs of Propositions and Theorems are given in Appendix 2. Thus, the foundations for the design of controllers for linear multivariable plants with more outputs than inputs are constructed.

4.2 Problem statement

It is supposed that the asymptotically stable plants under investigation have steady-state transfer function matrices $G \in R^{p \times m}$ which satisfy the equation

$$y = G u = \begin{bmatrix} g_1^T \\ \cdot\ \cdot \\ \cdot\ \cdot \\ \cdot\ \cdot \\ g_p^T \end{bmatrix} u \tag{4.1}$$

where the steady-state input vector $u \in U = R^m$, the steady-state output vector $y \in Y = R^p$, and the positive numbers p and m are arbitrary. Equation (4.1) represents the steady-state input-output relation of an open-loop asymptotically stable plant or a closed-loop system stabilised by appropriate feedback.

In the study of tracking systems, it is important to determine the characteristics of G that are required to make such systems effective for an arbitrary set-point command vector $v \in R^p$. Thus, for example, if rank $G = p \leq m$, it is clear that the input vector $u = G^T[GG^T]^{-1}v$ enables the output to follow any set-point command. However, if rank $G \leq m < p$ or rank $G < p \leq m$, the plant is functionally uncontrollable, the right-inverse of G does not exist, and set-point tracking in the sense that $y = v$ is impossible for arbitrary set-point command vectors. In this case when nonnegative or nonpositive errors can be allowed in the sense that $y \leq v$ or $y \geq v$ (where vector inequalities are interpreted component by component), it may be possible to design tracking systems in this sense which is practically very important. However, the conditions necessary for the plant to make such tracking systems feasible are not clear. Therefore, the investigation is aimed at the case rank $G < p$, although the analysis requires no restrictions

on p and m.

4.3 Characterisation of tracking systems

The general tracking characteristics of linear multivariable plants can be defined by using a vector equality together with vector inequalities (ie sets of equalities/inequalities).

Definition 4.1: Tracking

1 Set-point tracking

   The tracking is said to be set-point tracking if and only if

$$y = G\, u = v \quad . \tag{4.2}$$

2 Undertracking (Tracking with nonnegative errors)

   The tracking is said to be undertracking if and only if

$$y = G\, u \leq v \quad . \tag{4.3}$$

3 Overtracking (Tracking with nonpositive errors)

   The tracking is said to be overtracking if and only if

$$y = G\, u \geq v \quad . \tag{4.4}$$

In this definition, the vector inequalities in equations (4.3) and (4.4) are interpreted component by component and include the case $y = v$. Furthermore, it is clear that 1 implies 2 or 3 in Definition 4.1.

Since the difference between undertracking and overtracking is only in the directions of inequalities, the subsequent investigation is carried out only for undertracking. Firstly, the theory of convexity is used to characterise undertracking in terms of vector spaces. Then, the property of polyhedral convexity is stated and the separation theorem is introduced.

Definition 4.2

1 Set $U_F(v)$ of feasible inputs

$$U_F(v) = \{u \in U : G\,u \leq v\} \tag{4.5}$$

2 Set Y_R of reachable outputs

$$Y_R = \{y \in Y : y = G\,u \,,\, u \in U\} \tag{4.6}$$

3 Set $Y_A(v)$ of admissible outputs

$$Y_A(v) = \{y \in Y : y \leq v\} \tag{4.7}$$

4 Set $Y_F(v)$ of feasible outputs

$$Y_F(v) = Y_R \cap Y_A(v) \tag{4.8}$$

Proposition 4.1

1 $U_F(v)$, $Y_A(v)$ and $Y_F(v) = G(U_F(v))$ are closed polyhedral convex sets.

2 Y_R is a subspace of Y, closed and convex.

Proposition 4.2

1 (i) $U_F(v) = \emptyset$ if and only if (ii) $Y_F(v) = \emptyset$

2 (i) $U_F(v) \neq \emptyset$ if and only if (ii) $Y_F(v) \neq \emptyset$

Proposition 4.2 means that set-theoretical results in U-space and Y-space are equivalent.

Proposition 4.3

$$\forall G, \ U_F(v) \neq \emptyset \ , \ Y_F(v) \neq \emptyset \text{ for } v \geq 0$$

Theorem 4.1: Separation

1 (i) $Y_F(v) \neq \emptyset$ for $v < 0$ if and only if

(ii) there does not exist a hyperplane separating $Y_A(0)$ and Y_R properly.

2 (i) $Y_F(v) = \emptyset$ for $v < 0$ if and only if

(ii) there exists a hyperplane separating $Y_A(0)$ and Y_R properly.

It is clear by Theorem 4.1 that the existence of $Y_F(v)$ or $U_F(v)$ for $v < 0$ depends upon whether there exists a hyperplane separating $Y_A(0)$ and Y_R properly or not. Furthermore, since both $Y_A(0)$ and Y_R are polyhedral convex sets, the following proposition can be stated.

Proposition 4.4

If there exists a hyperplane separating $Y_A(0)$ and Y_R properly, it contains Y_R and does not contain $Y_A(0)$.

4.4 Classification of plants

The results of Theorem 4.1 can be used to classify plants.

Definition 4.3: Classification

1 Class I plant

$$\text{Class I} = \{G : U_F(v) \neq \phi \text{ and } Y_F(v) \neq \phi \text{ for } v < 0\} \qquad (4.9)$$

2 Class II plant

$$\text{Class II} = \{G : U_F(v) = \phi \text{ and } Y_F(v) = \phi \text{ for } v < 0\} \qquad (4.10)$$

Theorem 4.2

1 (i) If $G \in$ Class I, then (ii) $\forall v$, $U_F(v) \neq \phi$ and $Y_F(v) \neq \phi$.

2 (i) If $\exists v$, $U_F(v) = \phi$ and $Y_F(v) = \phi$, then (ii) $G \in$ Class II.

3 (i) If $G \in$ Class II, then (ii) $\forall v < 0$, $U_F(v) = \phi$ and $Y_F(v) = \phi$.

4 (i) If $\exists v < 0$, $U_F(v) \neq \phi$ and $Y_F(v) \neq \phi$, then (ii) $G \in$ Class I.

Theorem 4.2.1 means that, provided the plant belongs to

Class I, undertracking is possible for any set-point command, thus clarifying the importance of Class I plants in tracking systems. Theorem 4.2.2 means that, if undertracking is impossible for any particular set-point command, the plant belongs to Class II. Furthermore, Theorem 4.2.4 means that if undertracking is possible for some negative set-point command, the plant belongs to Class I.

Proposition 4.5

1 $G \in$ Class II if $\exists i \in [1,p]$, $g_i = 0$.

2 $\forall i \in [1,p]$, $g_i \neq 0$ if $G \in$ Class I.

Proposition 4.5 indicates the sufficient condition for Class II plants and the necessary condition for Class I plants.

Theorem 4.3

(i) $G \in$ Class I if and only if

(ii) $\forall i \in [1,p]$, $g_i \neq 0$ and $U_F(0)$ is an m-dimensional convex cone.

Proposition 4.6

$U_F(v)$ is unbounded and dim $U_F(v) = m$ if $G \in$ Class I.

In this section, the steady-state transfer function matrix G has been classified. It follows from Definitions 4.1.2, 4.2, and Theorem 4.2.1 that undertracking is always possible for $G \in$ Class I. So, there always exists an input vector u such that

$$y = G \, u \leq v$$

for $G \in$ Class I with rank $G < p$. The next theorem shows the importance of Class I plants in disturbance rejection.

Theorem 4.4

(i) $G \in$ Class I   if and only if

(ii) $\forall v$, $\forall d_y$, $U_F(v, d_y) \neq \emptyset$ and $Y_F(v, d_y) \neq \emptyset$,

where

the unmeasurable constant disturbance vector $d_y \in R^p$,

$$U_F(v, d_y) = \{u : Gu + d_y \leq v\} , \qquad (4.11)$$

and

$$Y_F(v, d_y) = \{y : y = Gu + d_y , \quad y \leq v\}. \qquad (4.12)$$

Finally, a sufficient condition for $G \in$ Class I is given.

Theorem 4.5

$G \in$ Class I   if $\exists i \in [1,m]$, $g_{c_i} > 0$,

where

$$G = \begin{bmatrix} & | & & | & \\ g_{c_1} & | & \cdots & | & g_{c_m} \\ & | & & | & \end{bmatrix} , \quad g_{c_i} \in R^p. \tag{4.13}$$

4.5 Illustrative examples

The results established in the previous sections can be conveniently illustrated by examples such as gas-turbine engines.

Example 4.1

$$G = \begin{bmatrix} 1 \\ 2 \end{bmatrix} , \quad U = R^1 , \quad Y = R^2$$

$U_F(0) = \{u : u \leq 0\}$

$$Y_R = \{y : y = \begin{bmatrix} 1 \\ 2 \end{bmatrix} u , \quad u \in U\}$$

$Y_A(0) = \{y : y \leq 0\}$

$G > 0$ and $U_F(0)$ is a 1-dimensional half line. Therefore, by Theorem 4.3 or 4.5, $G \in$ Class I. This is also confirmed in Y-space as is shown in Fig 4.1 because there does not exist a

hyperplane (ie a line in the case of R^2) separating $Y_A(0)$ and Y_R properly. Therefore, using Theorem 4.1, $Y_F(v) \neq \emptyset$ for $v < 0$ so that $G \in$ Class I.

Example 4.2

$$G = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad , \quad U = R^1 \quad , \quad Y = R^2$$

$$U_F(0) = \{ 0 \}$$

$$Y_R = \{y : y = \begin{bmatrix} -1 \\ 2 \end{bmatrix} u , \quad u \in U\}$$

$$Y_A(0) = \{y : y \leq 0\}$$

By Theorem 4.3, $G \in$ Class II. This is confirmed in Y-space as is shown in Fig 4.2 because Y_R itself separates $Y_A(0)$ and Y_R properly. Therefore, using Theorem 4.1, $Y_F(v) = \emptyset$ for $v < 0$ so that $G \in$ Class II.

Example 4.3

$$G = \begin{bmatrix} g_1^T \\ g_2^T \\ g_3^T \end{bmatrix} = \begin{bmatrix} -1 & , & 2 \\ 1 & , & 1 \\ 2 & , & -1 \end{bmatrix} \quad , \quad U = R^2 \quad , \quad Y = R^3$$

If such a plant is given, the condition of Theorem 4.5 is not

satisfied and it is not apparent whether $G \in$ Class I or not. Actually, by Theorem 4.3, $G \in$ Class I since $U_F(0)$ is a 2-dimensional convex cone in U-space as is shown in Fig 4.3. It may also be confirmed in Y-space that a plane Y_R penetrates $Y_A(0)$. Thus, there does not exist a hyperplane separating $Y_A(0)$ and Y_R properly.

Example 4.4: Nonlinear F100 engine model at Sea Level

Static/Idle (Appendix 4)

$$G = \begin{bmatrix} g_1^T \\ \cdot\ \cdot \\ \cdot\ \cdot \\ \cdot\ \cdot \\ g_5^T \end{bmatrix}$$

$$= \begin{bmatrix} 2.68035 & , & 102.516 \\ 4.50972 & , & 135.000 \\ 1.02400e\text{-}1 & , & 2.76397 \\ 1.34031e\text{-}3 & , & -2.43161e\text{-}1 \\ -1.21223e\text{-}1 & , & -9.60357 \end{bmatrix} ,$$

$$U = R^2 \quad , \quad Y = R^5$$

$U_F(0)$ is a 2-dimensional convex cone in U-space as is shown in Fig 4.4. Therefore, by Theorem 4.3, $G \in$ Class I. Equations $g_1^T u = 0$ and $g_5^T u = 0$ represent extreme rays of $U_F(0)$.

Example 4.5: Nonlinear F100 engine model at Sea Level

Static/Intermediate (Appendix 4)

$$\frac{\partial \, y_i}{\partial \, u_1} > 0 \quad , \quad i \in [1,5] \quad \text{ie} \quad g_{c_1} > 0,$$

where

$$G = \left[\begin{array}{c|c} & | \\ g_{c_1} & | \; g_{c_2} \\ & | \end{array} \right]$$

$$= \left[\begin{array}{c} g_1^T \\ \cdot \cdot \\ \cdot \cdot \\ \cdot \cdot \\ g_5^T \end{array} \right]$$

$$= \left[\begin{array}{rcr} 2.52760e{-}1 & , & 1451.03 \\ 1.35074e{-}1 & , & -7.04544 \\ 2.22808e{-}2 & , & -4.56090 \\ 2.08151e{-}3 & , & -7.70092 \\ 1.46573e{-}2 & , & 2.96426 \end{array} \right] \quad ,$$

$$U = R^2 \quad , \quad Y = R^5 \; .$$

By Theorem 4.5, $G \in$ Class I. $U_F(0)$ is shown in Fig 4.5. Equations $g_1^T \, u = 0$ and $g_4^T \, u = 0$ represent extreme rays of $U_F(0)$.

4.6 Conclusion

The characterisation of tracking systems in which tracking conditions are expressed by sets of equalities or inequalities, and the classification of linear multivariable plants, have been carried out by applying the theory of convexity. The decisive factor in such a classification is the separating hyperplane in Y-space or the m-dimensional convex cone in U-space. Although only the case of undertracking has been investigated extensively, it is easily confirmed that similar characterisations and classifications are also possible and effective for the case of overtracking. Illustrative examples have shown the effectiveness of the proposed technique. Such a technique provides sound foundations for the design of controllers for linear multivariable plants with more outputs than inputs. It is noted in the classification that the existence of nonempty $U_F(v)$ can be transformed into the existence of nonempty feasible region of linear equation with nonnegativity constraint, which is common in linear programming problems (Appendix 5) and therefore that the linear programming technique might be applied to the classification. However, the results obtained here are geometrically simple and more easily applicable to two- or three-input multivariable plants than linear programming.
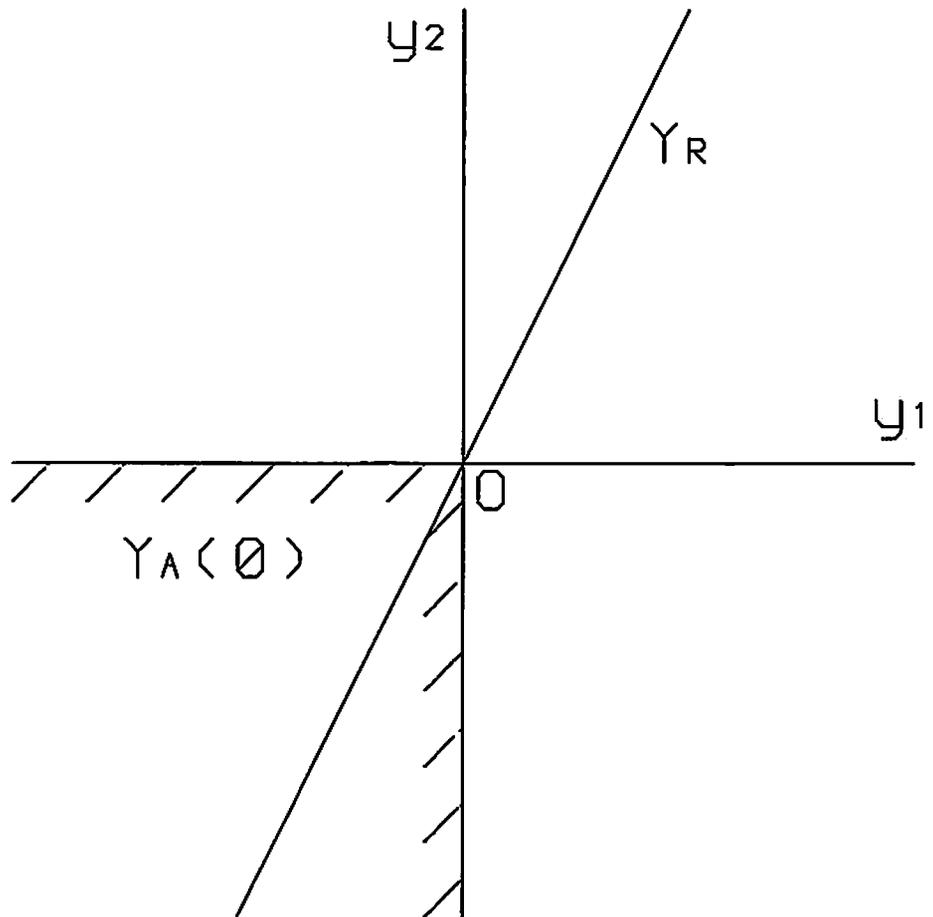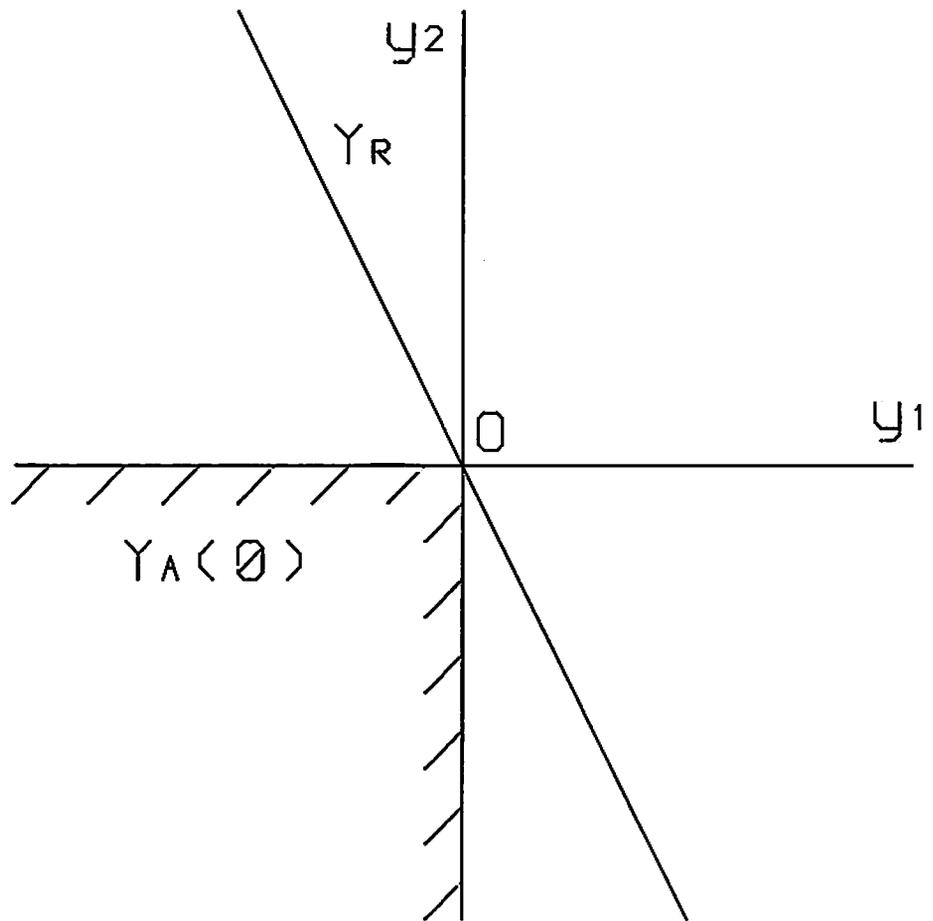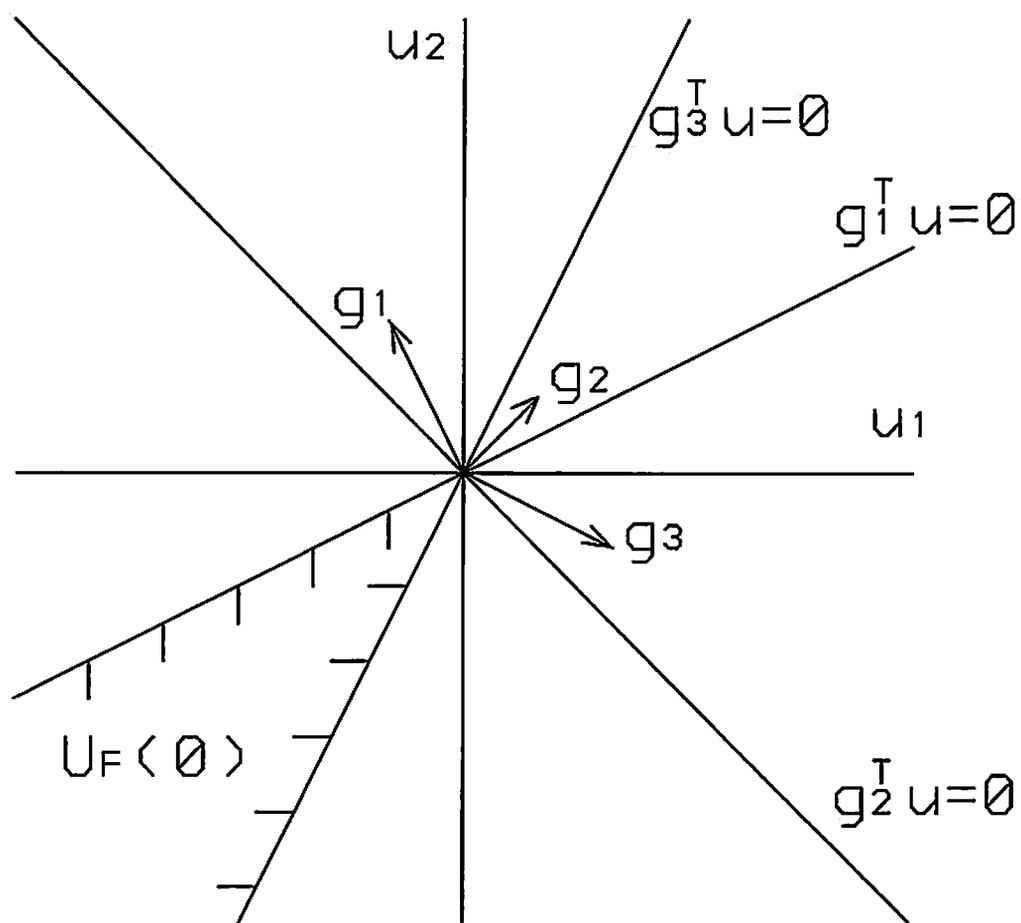
Fig 4.1   Y-space

Fig 4.2   Y-space

Fig 4.3 U-space
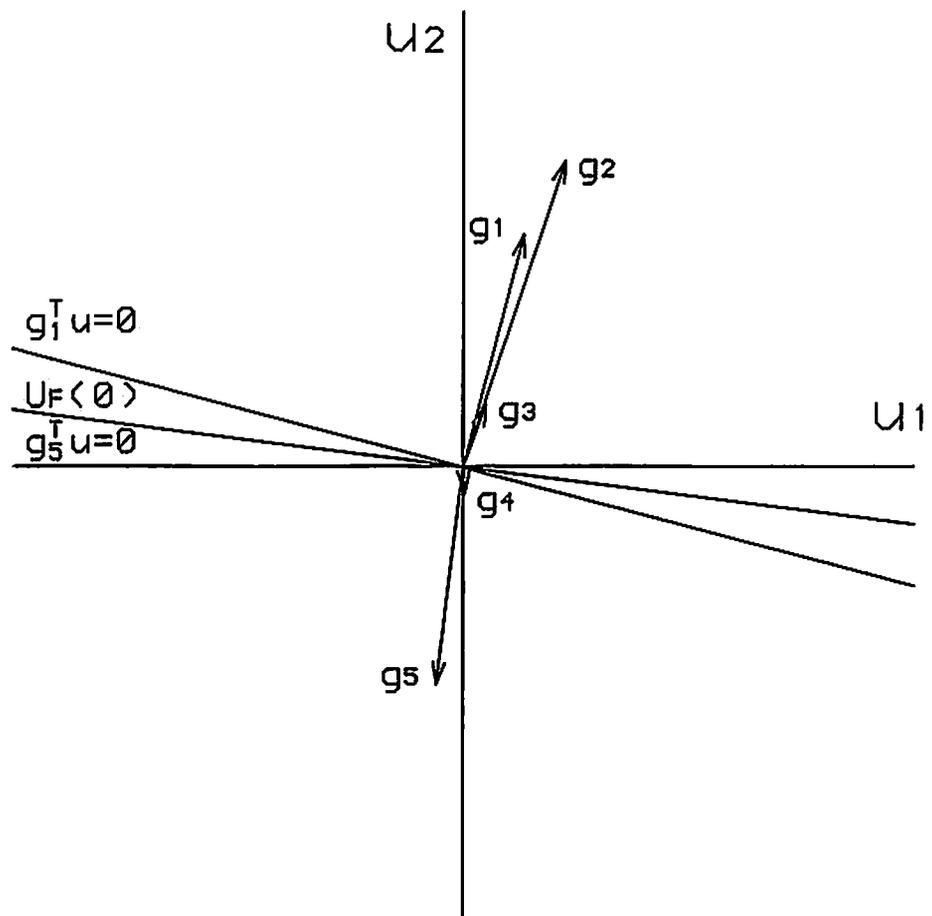
Fig 4.4   U-space
Nonlinear F100 engine model at SLS/Idle
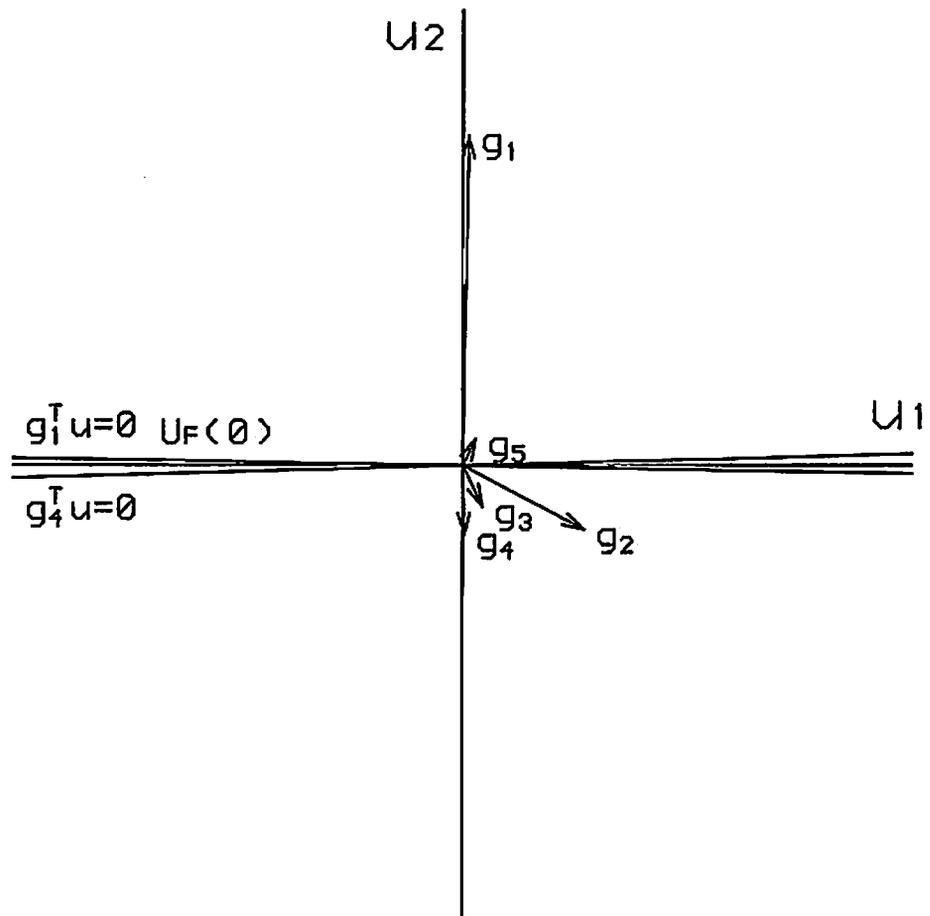
Fig 4.5 U-space
Nonlinear F100 engine model at SLS/Intermediate

CHAPTER 5

SYNTHESIS OF LIMIT-TRACKING SYSTEMS

USING ORDER-REDUCTION TECHNIQUE

5.1 Introduction

Set-point tracking systems fail to operate in case plants have more controlled outputs than manipulated inputs. In such cases, a more general tracking concept than set-point tracking is necessary to design controllers. Therefore, undertracking and overtracking were introduced in Chapter 4 and the properties of tracking systems were discussed rigorously in the context of convex analysis.

It is known that the self-selecting controller is one of the effective solutions to cope with plants with more outputs than inputs. Self-selecting controllers for single-input/multi-output plants were investigated by Foss(1981a), Glattfelder and Schaufelberger (1983), and Glattfelder et al (1980). Although Foss (1981b) extended his approach to multi-input plants, the approach was not general. Jones et al (1988) developed digital self-selecting PI controllers for multi-input/multi-output plants by extending tunable digital set-point tracking controllers (Porter and Jones (1984a)). Successful application of these self-selecting controllers to gas-turbine engines was also reported (Jones et al (1988), (1990)). However, the successful application does not necessarily mean that the entire systems are understood

well.

The self-selecting controllers incorporate a number of set-point tracking controllers for corresponding subsets of plant outputs and exert the control action on the most critical subset of outputs, which usually changes with time as both set-point commands and plant outputs change. The usual criterion for choosing which outputs to control at any time is either a highest-wins, lowest-wins, or highest-wins/lowest-wins strategy. In this context, 'highest-wins' or 'lowest-wins' refers to the instantaneous error between the set-point and the corresponding plant output. Therefore, different controllers are used for different subsets of the outputs and such controllers necessarily embody integral action for m input-output pairs in the case of m-input/p-output plants (m < p).

It is required that the steady states of tracking systems incorporating self-selecting controllers and multivariable plants are such that set-point tracking occurs for the most critical m out of p outputs and that the remaining p-m outputs stay between upper and lower limits with a certain safety margin. In the case of lowest-wins strategies, those p-m outputs remain under the control of set-point commands corresponding to the upper limits on the outputs, ie nonnegative errors are obtained for such channels and considered to be safe. Therefore, the tracking exhibited by entire sets of plant outputs can be considered to be limit tracking in the sense that none of the outputs exceeds its

corresponding set-point command, ie its limit value. Furthermore, systems incorporating self-selecting controllers and linear multivariable plants with more outputs than inputs can accordingly be called limit-tracking systems.

Then, the synthesis problem of limit-tracking systems arises before starting the further design procedure:

1: Is such limit tracking always possible for a given

   m-input/p-output plant and given set-point commands?

2: If the answer to 1 is "No",

   how can such feasibility be assessed?

3: For the plants in which limit tracking is feasible,

   is it necessary to design different controllers for each of

   the $_pC_m$ subsets of the plant outputs?

4: If the answer to 3 is "No",

   what is the minimum number of different controllers to

   enable the self-selecting controller to work properly for

   any set-point command?

5: If the answer to 4 is obtained,

   for what subsets of the plant outputs should such number of

   different controllers be designed?

In order to answer these questions, the characteristics of the

steady states of limit-tracking systems need to be investigated. In this investigation, the results obtained in Chapter 4 are effectively utilised after pointing out that limit-tracking belongs to undertracking. It is noted that, in the sequel, only systems incorporating self-selecting controllers based upon lowest-wins strategies are considered.

Firstly, the facial structure of limit-tracking systems is investigated and the coincident relation is revealed between limit tracking and an extreme point of the nonempty polyhedral convex set $U_F(v)$ (Definition 4.2) which contains no lines. Next, such properties are fully exploited to synthesise limit-tracking systems by giving answers to the above questions. Thus, a new order-reduction technique is developed to decide the minimum numbers of different subsets of plant outputs to be controlled by corresponding set-point tracking controllers. The proofs of Propositions and Theorems are given in Appendix 2.

5.2 Facial structure of limit-tracking systems

Since generally the more outputs that follow the corresponding set-point commands the better it is for the tracking system, there still remains another question concerning the number of equalities such as $y_i = v_i$, $i \in [1,p]$ and inequalities such as $y_j < v_j$, $j \in [1,p]$ that are obtainable in such tracking systems if undertracking is possible.

In the following, limit tracking is defined as the special case of undertracking in which the number of pairs of equal plant outputs and set points is not less than rank G. The tracking systems which accomplish such limit tracking can be called limit-tracking systems.

Definition 5.1: *Limit tracking and limit-tracking input*

The tracking is said to be limit tracking if and only if

$$y_{s_i} = g^T_{s_i} u = v_{s_i} \qquad\qquad i \in [1,k] \qquad\qquad (5.1)$$

$$y_{t_j} = g^T_{t_j} u < v_{t_j} \qquad\qquad j \in [1,p-k] \qquad\qquad (5.2)$$

$$\text{rank } G_s = \text{rank } G \qquad\qquad\qquad\qquad (5.3)$$

where

$$1 \le s_i \ , \quad t_j \le p \ , \qquad\qquad\qquad (5.4)$$

$$G_s = \begin{bmatrix} g^T_{s_1} \\ \vdots \\ \cdots \\ \vdots \\ g^T_{s_k} \end{bmatrix} \ , \qquad\qquad\qquad (5.5)$$

$$k \ge \text{rank } G \ , \qquad\qquad\qquad (5.6)$$

and $g^T_{s_1}$ $i \in [1,k]$ and $g^T_{t_j}$ $j \in [1,p-k]$ are, respectively, the s_ith and t_jth row vectors of the steady-state transfer-function matrix $G \in R^{p \times m}$ of the asymptotically stable plant.

Then, u is called the limit-tracking input.

Definition 4.1.1 implies Definition 5.1. Definition 5.1 implies Definition 4.1.2. It is evident in the case rank $G = m$ that Definition 5.1 satisfies the requirement for the steady states of systems incorporating self-selecting controllers and m-input/p-output plants, since $k \geq m$ in equation (5.6).

Next, the existence of limit tracking is shown for $G \in$ Class I (Definition 4.3) in both the cases rank $G = m$ and rank $G < m$. Thus, the control action of the self-selecting controller is given validity in the sense that the existence of the steady states of closed-loop systems is guaranteed.

Theorem 5.1: Existence theorem

If $G \in$ Class I then

1 there always exists at least one limit-tracking input,

and

2 in case rank $G = m$,

  (i) $u \in$ ext $U_F(v)$ if and only if

  (ii) u is a limit-tracking input,

where $U_F(v)$ is defined in Definition 4.2 and ext • means the set of extreme points of the convex set •.

Proposition 5.1

(i) $G \in$ Class I if and only if (ii) $\bar{G} \in$ Class I,

where rank $G = q < m$ and $\bar{G}$ consists of q linearly independent columns of G in the form,

$$\bar{G} = \begin{bmatrix} \bar{g}_1^T \\ \vdots \\ \vdots \\ \bar{g}_p^T \end{bmatrix} \in R^{p \times q} , \quad \bar{g}_i \in U = R^q , \quad i \in [1,p], \tag{5.7}$$

$$\text{rank } \bar{G} = q. \tag{5.8}$$

It is noted that $G \in$ Class I is a sufficient condition for the existence of a limit-tracking input. For $G \in$ Class II, if $U_F(v)$ is not empty and contains no lines, there exists at least one extreme point, ie one limit-tracking input.

These results are illustrated by Examples 5.1, 5.2 and 5.3.

5.3 Order-reduction technique

In the previous section, important fundamental properties have been established for limit-tracking systems. In this section, the utilisation of such properties in synthesising limit-tracking systems incorporating self-selecting controllers and linear multivariable plants is discussed.

The idea of the self-selecting controller is to exert control action on the most critical subset of the outputs, thus making all the outputs stay at or under certain limit values. In such

tracking systems, the controller necessarily embodies integral action for m input-output pairs in the case of m-input/p-output plants.

In order to preserve the stabilisability of closed-loop systems under integral action, the following condition was given as functional controllability by Porter and Power (1970) and Power and Porter (1970):

$$\text{rank } G^{(i)} = m \qquad (i = 1,\ldots,r)$$

where $G^{(i)} \in R^{m \times m}$ $(i = 1,\ldots,r)$ are the steady-state transfer function matrices for the corresponding subsets of plant outputs and r is the number of controllers/control loops.

This condition requires that rank $G = m$. So, the case rank $G = m < p$ will be discussed in the following.

Given G, Proposition 4.5, Theorems 4.3 and 4.5 can be used to check whether $G \in$ Class I or $G \in$ Class II. If $G \in$ Class II, neither set-point tracking, nor undertracking, nor limit tracking is obtainable for arbitrary set-point commands. So, suppose that $G \in$ Class I.

By Propositions 4.1 and 4.6, $U_F(v)$ forms an m-dimensional unbounded polyhedral convex set. By Theorem 5.1, there always exists at least one limit-tracking input and it coincides with an extreme point of $U_F(v)$. Then, the synthesis of self-selecting controllers and resulting limit-tracking systems can be facilitated by fully exploiting the facial structure of

such sets. The key concept of the developed new approach is the order reduction from m to 1. Thus, the controller structure for the case m ≥ 2 becomes the same as that for the case m = 1. So, the controller structure for the case m = 1 is discussed first of all.

In the case of 1-input/p-output plants, $U_F(v)$ forms a 1-dimensional polyhedral convex set, ie a half-line. A limit-tracking input u_1 (ie an extreme point of $U_F(v)$) is the unique vertex of $U_F(v)$ and is expressed in the form

$$g_{s_1} u_1 = v_{s_1} \quad , \quad s_1 \in [1,p], \quad (5.9)$$

where $G = [g_1, \ldots, g_p]^T$.

Therefore, at least one such index s_1 corresponds to a limit-tracking input. The minimum number of controllers/control loops is p, and p set-point tracking controllers are to be designed. The lowest-wins strategies need to compare p competing signals to determine which output is the most critical among p outputs and to find the corresponding index and controller/control loop.

In the case of m-input/p-output plants (m ≥ 2), the order reduction is carried out by applying the following useful results about the facial structure of the polyhedral convex set $U_F(v)$.

Proposition 5.2

1: A line corresponding to an extreme ray of $U_F(0)$ is the intersection of m-1 hyperplanes and given in the form

$$
\begin{bmatrix} g_{s_1}^T \\ \cdots \\ \cdots \\ \cdots \\ g_{s_{m-1}}^T \end{bmatrix} u = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} , \quad s_i \in [1,p], \quad i \in [1,m-1], \quad (5.10)
$$

where a hyperplane $g_{s_i}^T u = 0$ passes through the origin, a vector g_{s_i} is a normal to such a hyperplane, and the vectors $g_{s_1}, \ldots, g_{s_{m-1}}$ are linearly independent.

2: A line corresponding to an exposed half-line face of $U_F(v)$ is the intersection of m-1 hyperplanes and given in the form

$$
\begin{bmatrix} g_{s_1}^T \\ \cdots \\ \cdots \\ \cdots \\ g_{s_{m-1}}^T \end{bmatrix} u = \begin{bmatrix} v_{s_1} \\ \cdot \\ \cdot \\ \cdot \\ v_{s_{m-1}} \end{bmatrix} , \quad s_i \in [1,p], \quad i \in [1,m-1], \quad (5.11)
$$

where $g_{s_i}^T u = v_{s_i}$ represents a hyperplane, a vector g_{s_i} is a normal to such a hyperplane, and the vectors $g_{s_1}, \ldots, g_{s_{m-1}}$ are linearly independent.

3: There exists a corresponding extreme ray for every exposed half-line face which has the same direction called the extreme direction. Therefore, a line corresponding to an exposed half-line face is parallel to a line corresponding to such an extreme ray.

4: The unique vertex of every exposed half-line face of $U_F(v)$

   is an extreme point, although the number of such faces or

   points is generally unknown.

Let a line corresponding to an exposed half-line face of $U_F(v)$
be given in the equation (5.11). When vectors $g_{s_1}, \ldots, g_{s_{m-1}}$
are removed from $g_1, \ldots, g_p$, as long as every one of all the
remaining p—m+1 vectors is linearly independent of
$g_{s_1}, \ldots, g_{s_{m-1}}$, this line has an intersection with every
hyperplane to which one of the remaining p—m+1 vectors is a
normal. Therefore, p—m+1 intersections (ie candidates for
limit-tracking input) are distributed along this line and at
least one of them is the unique vertex of the exposed half-line
face (ie a limit-tracking input). In this sense, the dimension
of the problem of finding a limit-tracking input has been
reduced from m to 1. Then, such a limit-tracking input u_1 is
given in the form

$$
\begin{bmatrix} g_{s_1}^T \\ \cdots \\ \cdots \\ \cdots \\ g_{s_m}^T \end{bmatrix} u_1 = \begin{bmatrix} v_{s_1} \\ \cdot \\ \cdot \\ \cdot \\ v_{s_m} \end{bmatrix} , \quad s_i \in [1,p], \quad i \in [1,m]. \qquad (5.12)
$$

This means that at least one index s_m among the remaining p—m+1
indices corresponds to a limit-tracking input when $s_1, \ldots, s_{m-1}$
are removed. Therefore, in the case $m \geq 2$ as well as the case
m = 1, the lowest-wins strategies only need to compare p—m+1
competing signals to determine which output is the most
critical among p—m+1 outputs and to find the corresponding

index and controller/control loop. It is noted that finding the indices $s_1, \ldots, s_{m-1}$ of the above exposed half-line face can be replaced by finding such indices of an extreme ray because the line expressed by the equation (5.10) corresponding to an extreme ray is parallel to the line expressed by the equation (5.11). Furthermore, since hyperplanes corresponding to p-m+1 indices must intersect such lines, this is equivalent to finding an extreme ray with unique representation.

Based upon this discussion, the following algorithm follows to obtain the minimum number of subsets of plant outputs to be controlled in the case of m-input/p-output Class I plants (rank $G = m$):

Algorithm 5.1: Order reduction

Step 1:

In the case $m = 1$, go to Step 3. In the case $m \geq 2$, find the extreme rays of $U_F(0)$. Since an extreme ray is determined by m-1 hyperplanes and corresponding normal vectors, an index set of such vectors represents an extreme ray. Let I_i, $i \in [1,k]$ be the index sets of such vectors.

Step 2:

Find I_j^*, $j \in [1,k^*]$ among I_i, $i \in [1,k]$ such that every one of the p-m+1 vectors corresponding to the index set $I \backslash I_j^*$ is linearly independent of all the vectors corresponding to I_j^*, where $I = \{1, \ldots, p\}$. I_j^*, $j \in [1,k^*]$ correspond to extreme rays

_

which have the unique representation. If such I_j^* cannot be found, see (Remark).

(Remark)

Such cases occur very exceptionally when every extreme ray is represented as the intersection of more than m-1 hyperplanes. This means that there exists at least one redundant hyperplane to express every extreme ray. Therefore, if all the redundant hyperplanes (which is one or more) for any one of extreme rays are omitted, it is possible to find I_j^* for this extreme ray. However, this omission means that the tracking of the corresponding output must be abandoned.

Step 3:

In the case m = 1, $I^* = \phi$. In the case m ≥ 2, choose one index set $I^* = \{s_1, \ldots, s_{m-1}\}$ among I_j^*, j ∈ $[1, k^*]$. The designer has the freedom to choose one index set out of k^* sets. If because of the controller specification some particular outputs must always track the set-point commands, the indices of such outputs must also belong to the chosen index set. Then, m-1 outputs $y_{s_1}, \ldots, y_{s_{m-1}}$ are fixed to be always controlled integrally.

Step 4:

Pick one index t_i out of the remaining p-m+1 indices of $I \backslash I^*$ and make subsets Y_i, i ∈ [1, p-m+1] of plant outputs

$$Y_1 \quad = \{y_{s_1}, \ldots, y_{s_{m-1}}, y_{t_1}\}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \qquad\qquad (5.13)$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$Y_{p-m+1} = \{y_{s_1}, \ldots, y_{s_{m-1}}, y_{t_{p-m+1}}\}$$

where

$$I\backslash I^* = \{t_1, \ldots, t_{p-m+1}\}. \qquad\qquad (5.14)$$

The minimum number of subsets of plant outputs to be controlled is $p-m+1$ and $Y_1, \ldots, Y_{p-m+1}$ are such subsets. This means that $p-m+1$ sets of indices $\{s_1, \ldots, s_{m-1}, t_1\}, \ldots, \{s_1, \ldots, s_{m-1}, t_{p-m+1}\}$ represent the candidate controllers for m-input/p-output plants. Therefore, $p-m+1$ set-point tracking controllers are to be designed for these subsets $Y_1, \ldots, Y_{p-m+1}$.

(Algorithm 5.1 end)

The $(m-1)$-dimensional order reduction has been discussed in the sense of finding an extreme ray uniquely represented by $m-1$ hyperplanes and limiting the region of limit-tracking inputs on a corresponding line. It is noted that the lowest-wins strategies only need to compare not vectors but scalar signals even in the case $m \geq 2$ as well as the case $m = 1$. This not only reduces the computational complexity of the implemented controllers but also makes the controller structure simple. The minimum number of scalar signals to be compared is $p-m+1$. If m (the number of plant inputs) increases, it may not be easy to find the extreme rays of $U_F(0)$. However, at least for

m = 2 or 3, this technique is easy and very effective.

These results are illustrated by Examples 5.4, 5.5 and 5.6.

5.4 Illustrative examples

The results obtained in the previous sections can be conveniently illustrated by examples.

Example 5.1: *Same as Example 4.1*

$$G = \begin{bmatrix} 1 \\ 2 \end{bmatrix} , \quad G \in \text{Class I}$$

For $v = [2 \ 2]^T$, $U_F(v) = \{u : u \leq 1\}$ and limit-tracking occurs at an extreme point of $U_F(v)$, ie $u = 1$. Then

$$\begin{cases} 1 & u < 2 \\ 2 & u = 2 \end{cases}$$

and the index 2 corresponds to a limit-tracking input. Furthermore, at least one index 1 or 2 corresponds to a limit-tracking input for any set-point command vector. Thus, at least one equality holds.

Example 5.2: *Same as Example 4.2*

$$G = \begin{bmatrix} -1 \\ 2 \end{bmatrix} , \quad G \in \text{Class II}$$

For $v = [2 \ 2]^T$, $U_F(v) = \{u : -2 \leq u \leq 1\}$ and limit-tracking occurs at either extreme point of $U_F(v)$, ie $u = 1$ or $u = -2$. Then

$$\begin{cases} -1 \quad u < 2 \\ 2 \quad u = 2 \end{cases} \text{, if } u = 1 \quad ,$$

$$\begin{cases} -1 \quad u = 2 \\ 2 \quad u < 2 \end{cases} \text{, if } u = -2 \quad .$$

For $v = [-2 \ -2]^T$, $U_F(v) = \emptyset$ and limit-tracking does not occur.

Example 5.3

$$G = \left[g_{c_1} \Big| g_{c_2} \right] = \begin{bmatrix} g_1^T \\ g_2^T \end{bmatrix} = \begin{bmatrix} 1 & , & 2 \\ 2 & , & 4 \end{bmatrix} , \quad U = R^2 , \quad Y = R^2$$

rank $G = 1 < m$

Since

$$g_{c_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} > 0 ,$$

$G \in$ Class I by Theorem 4.5.

For $v = [2 \ 2]^T$, if u is chosen by the method described in the

proof of Theorem 5.1 Part 2, then

$$\begin{cases} [1 \ 2] \ u = 1 < v_1 \\ [2 \ 4] \ u = 2 = v_2 \end{cases} \quad , \quad u = \begin{bmatrix} 1 - 2 \ \tilde{u} \\ \tilde{u} \end{bmatrix} \quad , \quad \tilde{u} \in R \ .$$

Thus, a line is limit-tracking input and at least one equality $g_2^T \ u = v_2$ is obtained.

Example 5.4: *Same as Example 4.3*

$$G = \begin{bmatrix} -1 \ , \ 2 \\ 1 \ , \ 1 \\ 2 \ ,-1 \end{bmatrix} \quad , \quad U = R^2 \quad , \quad Y = R^3$$

Assume that the output y_2 must always track the corresponding set-point command v_2. The control action must be exerted either on $\{y_1,y_2\}$ or $\{y_2,y_3\}$. Then, *does there exist a limit-tracking input for any set-point command vector?* Generally, the answer is "No". Indeed, as one counter example, assume that $v = [1 \ 3 \ 1]^T$. In case $\{y_1,y_2\}$ is integrally controlled,

$$\begin{cases} y_1 = [-1 \ \ 2] \ u = 1 = v_1 \\ y_2 = [\ 1 \ \ 1] \ u = 3 = v_2 \\ y_3 = [\ 2 \ -1] \ u = 2 \ne v_3 \end{cases} \quad , \quad u = \begin{bmatrix} -1 \ , \ 2 \\ 1 \ , \ 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \ .$$

In case $\{y_2,y_3\}$ is integrally controlled,

$$\begin{cases} y_1 = [-1 \quad 2] \ u = 2 \nleqslant v_1 \\ y_2 = [\ 1 \quad 1] \ u = 3 = v_2 \\ y_3 = [\ 2 \ -1] \ u = 1 = v_3 \end{cases} , \quad u = \begin{bmatrix} 1 & , & 1 \\ 2 & , & -1 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 1 \end{bmatrix} .$$

In both cases, y_1 or y_3 exceeds its corresponding set-point, ie its limit value. Fig 5.1 shows the hyperplanes (ie lines in the case $m = 2$) $g_i^T u = v_i$, $i \in [1,3]$. $U_F(v)$ is clearly the region surrounded by $g_1^T u = 1$ and $g_3^T u = 1$, and the unique extreme point is the intersection of these two lines. This means that y_2 must be released from the control action and that, instead, the control action must be exerted on $\{y_1, y_3\}$. Thus, the question arises: *How can the controller be synthesised systematically?* To answer this question, the proposed controller synthesis based upon the facial structure is illustrated.

It is clear from Figs 4.3 and 5.2 that a line $g_1^T u = v_1$ or $g_3^T u = v_3$ always corresponds to an extreme ray or an exposed half-line face of $U_F(v)$. The algorithm follows:

Step 1: $I_1 = \{1\}$ and $I_2 = \{3\}$.

Step 2: $I\backslash I_1 = \{2,3\}$ and g_2 or g_3 is linearly independent of g_1. $I\backslash I_2 = \{1,2\}$ and g_1 or g_2 is linearly independent of g_3. Thus, $I_1^* = I_1$ and $I_2^* = I_2$.

Step 3: The designer can choose I_1^* or I_2^* as the index set of the permanently controlled variable. The minimum number of subsets Y_i, $i \in [1,2]$ of plant outputs to be controlled is two, and these subsets are

$$Y_1 = \{y_1, y_2\} \text{ and } Y_2 = \{y_1, y_3\} \text{ if } I_1^* \text{ is chosen.}$$

or

$$Y_1 = \{y_3, y_1\} \text{ and } Y_2 = \{y_3, y_2\} \text{ if } I_2^* \text{ is chosen.}$$

If the controller has three subsets $\{y_1, y_2\}$, $\{y_1, y_3\}$, and $\{y_2, y_3\}$, either $\{y_1, y_2\}$ or $\{y_2, y_3\}$ is redundant and the uniqueness of the limit-tracking input is lost in the case of Fig 5.2(a) in the sense that there exist two limit-tracking inputs for one set-point command.

Example 5.5

$$G = \begin{bmatrix} | \\ g_{c_1} & g_{c_2} \\ | \end{bmatrix} = \begin{bmatrix} g_1^T \\ g_2^T \\ g_3^T \end{bmatrix} = \begin{bmatrix} 1 & , & 2 \\ 2 & , & 4 \\ 1 & , & -1 \end{bmatrix} , \quad U = R^2 , \quad Y = R^3$$

$$\text{rank } G = 2 = m < p$$

Since

$$g_{c_1} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} > 0 ,$$

$G \in$ Class I by Theorem 4.5.

One extreme ray of $U_F(0)$ is $g_1^T u = [1 \ 2]u = 0$ or $g_2^T u = [2 \ 4]u = 0$. Another extreme ray is $g_3^T u = [1 \ -1]u = 0$. Thus, $I_1 = \{1\}$, $I_2 = \{2\}$, and $I_3 = \{3\}$.

However, g_1 and g_2 are linearly dependent and represent the same extreme ray,

ie

$$\det \begin{bmatrix} 1 & , & 2 \\ 2 & , & 4 \end{bmatrix} = 0 \ .$$

Therefore, $I_1^* = I_3$ and y_3 must always be controlled and the lowest-wins strategy needs to compare only two signals corresponding to y_1 and y_2. This means that the self-selecting controller can exert the control action either on $Y_1 = \{y_1, y_3\}$ or on $Y_2 = \{y_2, y_3\}$. Then, a limit-tracking input always exists for any set-point command.

Example 5.6: Nonlinear F100 engine model at the same condition

as Example 4.5

Equations $g_1^T u = 0$ corresponding to y_1(Fan speed N_1) and $g_4^T u = 0$ corresponding to y_4(Augmentor pressure P_7) represent extreme rays of $U_F(0)$. Since either extreme ray has a unique representation and p-m+1=4, a minimum of four control loops is required.

If y_1 must always be controlled, the subsets of plant outputs to be controlled are

$$Y_1 = \{y_1, y_2\}, \ Y_2 = \{y_1, y_3\}, \ Y_3 = \{y_1, y_4\}, \ Y_4 = \{y_1, y_5\} \ .$$

If y_4 must always be controlled, the subsets are

$$Y_1 = \{y_4, y_1\}, \quad Y_2 = \{y_4, y_2\}, \quad Y_3 = \{y_4, y_3\}, \quad Y_4 = \{y_4, y_5\}.$$

Thus, lowest-wins strategies need to compare only four scalar signals.

5.5 Conclusion

In this chapter, the characteristics of the steady states of limit-tracking systems have been discussed and a new synthesis approach to limit-tracking systems has been developed. It has been shown that, in the case of Class I linear multivariable plants, limit tracking (ie steady states of systems incorporating self-selecting controllers) always exists and that such self-selecting controllers can be synthesised by the proposed order-reduction technique which utilises the facial structure of limit-tracking systems. Furthermore, it has been shown that the order-reduction technique is based upon the discovery of extreme rays which have a unique representation. Therefore, this technique is effective unless every extreme ray of $U_F(0)$ is represented as the intersection of not less than m hyperplanes in m-dimensional U-space (Algorithm 5.1 Step 2 (Remark)). In fact, by using this technique, the number of controllers can be reduced from $_pC_m$ to p-m+1 (ie order reduction from m to 1) in the case of m-input/p-output plants whilst guaranteeing the existence of steady states of such systems. It is noted that there is no need for the dynamical model of the plant to carry out this approach. The effectiveness of the order-reduction technique has been

illustrated by examples such as gas-turbine engines.

Although the controller synthesis in the case rank $G < m$ has not been discussed, it is possible to modify the order-reduction algorithm so as to incorporate such cases by using $\bar{G}$ which is defined in Proposition 5.1. Furthermore, although only self-selecting controllers based upon lowest-wins strategies have been considered, it is possible to extend the proposed technique to controllers based upon highest-wins strategies or lowest-wins/highest-wins strategies. Finally, it is noted that the limit-tracking input corresponds to a special form of the basic feasible solution of the transformed linear programming problem (Appendix 5).
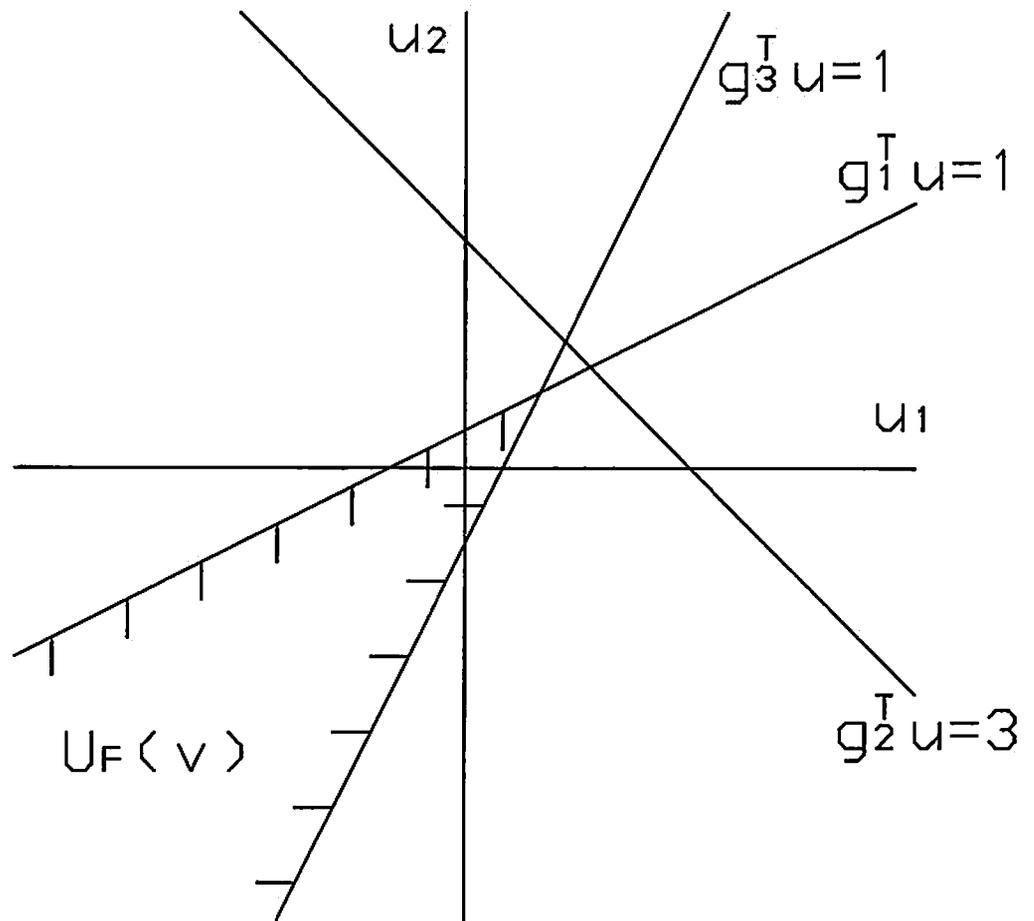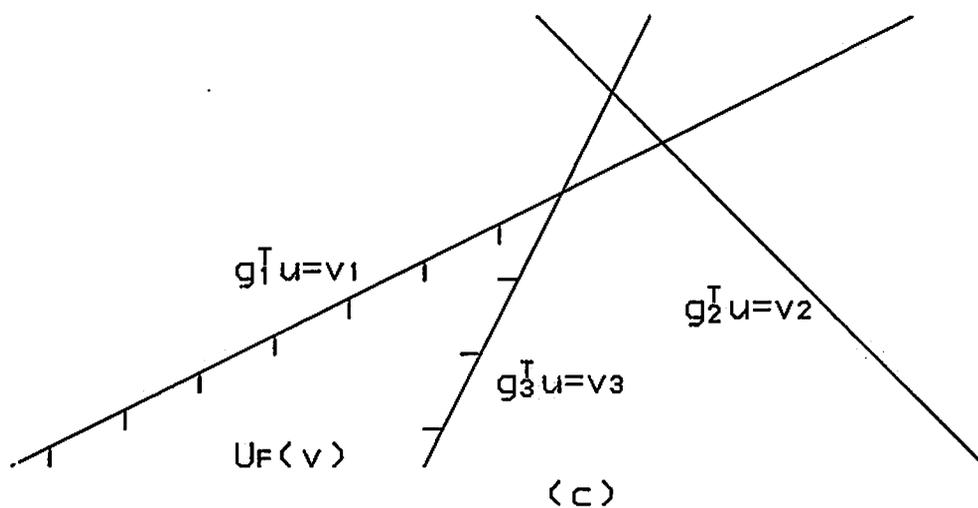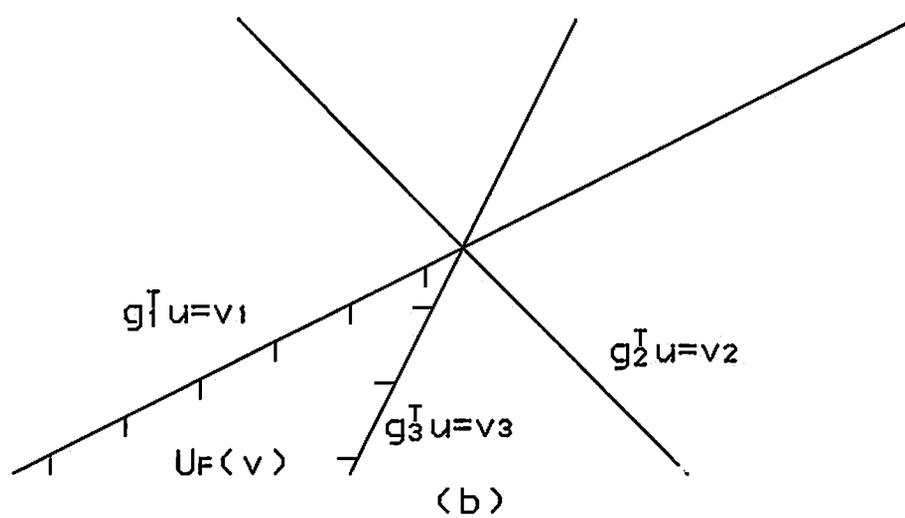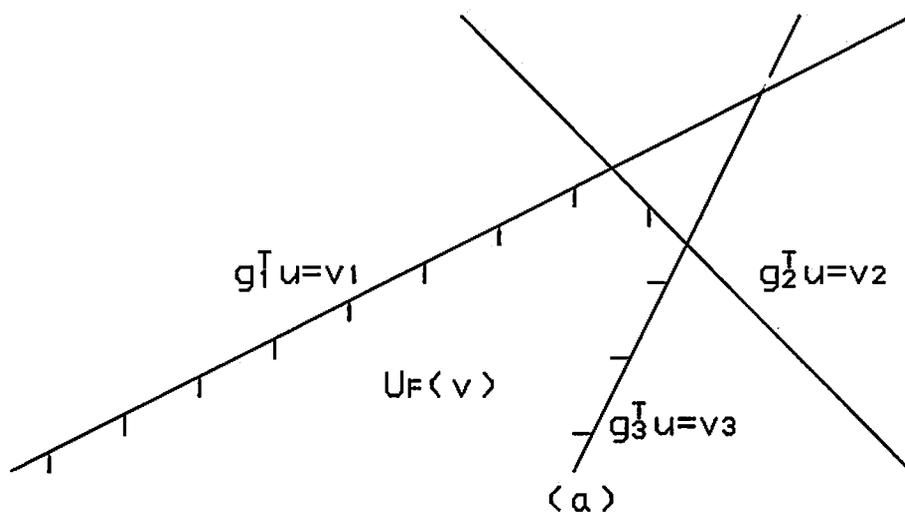
Fig 5.1   U-space

$g_1^T u = v_1$ $g_2^T u = v_2$

$U_F(v)$

$g_3^T u = v_3$

(a)

$g_1^T u = v_1$ $g_2^T u = v_2$

$g_3^T u = v_3$

$U_F(v)$

(b)

$g_1^T u = v_1$ $g_2^T u = v_2$

$g_3^T u = v_3$

$U_F(v)$

(c)

Fig 5.2 U-space

CHAPTER 6

DESIGN OF DIGITAL SELF-SELECTING PID CONTROLLERS

FOR LINEAR MULTIVARIABLE PLANTS

WITH MORE OUTPUTS THAN INPUTS

6.1 Introduction

In this chapter, a methodology for the design of controllers for unknown open-loop asymptotically stable linear multivariable plants with more controlled outputs than manipulated inputs is obtained by using the synthesis technique developed in Chapter 5. Thus, an extension of the tunable set-point tracking PID controllers (Chapter 2) is carried out. This is also an extension of the self-selecting PI controllers (Jones et al (1988)).

In order to circumvent the need for detailed mathematical models of the plants, the design procedure utilises only the data which is directly obtainable from open-loop step-response tests performed on plants (Appendix 1). For such plants, in which the ranks of the steady-state transfer-function matrices are less than the number of outputs, set-point tracking in the sense that the plant outputs track their corresponding set-point commands asymptotically is impossible for arbitrary set-point commands. In order to overcome this problem, a new tracking concept, ie limit tracking (Definition 5.1), is utilised in the design of controllers. It is assumed that the plant belongs to Class I (Definition 4.3), that the controller

incorporates a number of set-point tracking controllers for corresponding subsets of plant outputs, and that one of these controllers is selected at any time to control the most critical subset of outputs based upon lowest-wins strategies (ie the self-selecting controller). This operational principle ensures that, as long as the entire closed-loop system is asymptotically stable, nonnegative errors are obtained in the steady state and none of the plant outputs exceeds its corresponding set-point command. This is practically very useful for plants such as gas-turbine engines in which none of the outputs is allowed to exceed engine operational limits.

By applying the order-reduction technique (Algorithm 5.1) to m-input/p-output plants, the structure of the controllers is decided and therefore p-m+1 subsets of plant outputs which are to be controlled by corresponding set-point tracking controllers are specified. Then, the corresponding parts of the plant can be called the sub-plants and the design of tunable digital set-point tracking PID controllers for such p-m+1 sub-plants is considered.

It is shown that the proportional, integral, and derivative controller matrices used in these PID controllers can be directly determined from open-loop step-response tests performed on plants (Appendix 1). The proportional and derivative controller matrices are chosen as the inverse of the sub-plant open-loop step-response matrix, which is itself derived from the classical decoupling theory of Falb and Wolovich (1967). This choice is made in order to exploit the

initial interactions within the plant and thus to cause set-point tracking to occur without initial interaction or under-shoot (Mita and Yoshida (1981)). The integral controller matrix is chosen as the inverse of the sub-plant open-loop steady-state transfer-function matrix in order to exploit the final interactions within the plant. Thus, provided only that all the sub-plants satisfy the fundamental condition of Porter and Power (1970) and Power and Porter (1970) for the preservation of stabilisability in the presence of integral action, such error-actuated controllers can be readily designed for unknown multivariable plants.

A block-diagonalisation transformation is used to investigate the asymptotic properties of separate closed-loop systems under the action of such PID controllers. The closed-loop sub-plant matrix is decomposed into three sub-matrices, using the block-diagonalisation transformation of Kokotović (1975), and it is thus shown that the basic design criterion for stability and set-point tracking can be satisfied in terms of the characteristic roots of the sub-matrices.

Next, the separate error-actuated digital set-point tracking PID controllers are integrated into the digital self-selecting PID controller. Therefore, implementation problems in regard to this process are discussed. The index set of lowest errors and the loop index of the actually selected loop are decided in lowest-wins strategies. Furthermore, the controller switching logic which gives a good initial transient response of the plant outputs is considered.

Finally, the effectiveness of such a tunable controller is illustrated by designing, for a highly interactive gas-turbine engine, a digital self-selecting PID controller which exhibits excellent limit-tracking characteristics and corresponding minimal loop-interactions.

6.2 Analysis

The linear multivariable Class I plants (Definition 4.3) under consideration are assumed to be governed on the continuous-time set $T = [0,+\infty)$ by state and output equations of the respective forms

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad (6.1)$$

and

$$y(t) = Cx(t) , \qquad (6.2)$$

where the state vector $x(t) \in R^n$, the input vector $u(t) \in R^m$, the output vector $y(t) \in R^p$ (p > m), the plant matrix $A \in R^{n \times n}$ whose eigenvalues all lie in the open left-half plane C^-, the input matrix $B \in R^{n \times m}$, and $C \in R^{p \times n}$ is the output matrix.

The transfer-function matrix is

$$G(s) = C(sI-A)^{-1}B \qquad (6.3)$$

and the steady-state transfer-function matrix

$$G = G(0) = -CA^{-1}B \in R^{m \times m} \tag{6.4}$$

is known from open-loop tests performed on the plant (Appendix 1). It is assumed that

$$\text{rank } G = m \ , \tag{6.5}$$

and therefore that, by applying the order-reduction technique (Algorithm 5.1), p-m+1 subsets and sets of indices of plant outputs to be controlled by corresponding set-point tracking controllers are obtained in the form

$$\left.\begin{aligned}
Y_1 &= \{y_{s_1}, \ldots, y_{s_{m-1}}, y_{t_1}\} \\
&\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
&\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
&\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
Y_r &= \{y_{s_1}, \ldots, y_{s_{m-1}}, y_{t_r}\}
\end{aligned}\right\} \tag{6.6}$$

where r = p-m+1 and the index set of all the control loops is $I_r = \{1,2,\ldots,r\}$. Then, the parts of the plant which correspond to such subsets can be called sub-plants and the sub-output vectors of these sub-plants are

$$y^{(i)}(t) = \begin{bmatrix} y_{s_1}(t) \\ \cdot \\ \cdot \\ \cdot \\ y_{s_{m-1}}(t) \\ y_{t_i}(t) \end{bmatrix} = C^{(i)}x(t) \in R^m \quad (i = 1,2,\ldots,r) \tag{6.7}$$

where each of sub-output matrices $C^{(i)} \in R^{m \times n}$ $(i = 1,2,\ldots,r)$ consists of s_1th,$\ldots$,s_{m-1}th and t_ith rows of the output matrix C.

Next, the design of tunable digital set-point tracking PID controllers for each sub-plant is considered separately. It is assumed that the introduction of integral action for each subset Y_i, $i \in I_r$ preserves stabilisability and therefore that (Porter and Power (1970), Power and Porter (1970))

$$\text{rank } G^{(i)} = m \ , \tag{6.8}$$

where the sub-plant transfer-function matrix

$$G^{(i)}(s) = C^{(i)}(sI-A)^{-1}B \ , \tag{6.9}$$

and the steady-state transfer-function matrix for the subset Y_i

$$G^{(i)} = G^{(i)}(0) = -C^{(i)}A^{-1}B \in R^{m \times m} \tag{6.10}$$

is obtained from equation (6.3).

Furthermore, it is assumed that input-output decoupling is achievable between inputs and the Y_i, $i \in I_r$ and therefore that (Falb and Wolovich (1967))

$$\text{rank } F^{(i)} = m \ , \tag{6.11}$$

where the decoupling matrix

$$F^{(i)} = \begin{bmatrix} c_1^{(i)T}A^{d_1^{(i)}}B \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \\ c_m^{(i)T}A^{d_m^{(i)}}B \end{bmatrix} \in R^{m \times m} \qquad (6.12)$$

and the $d_j^{(i)}$ $(j=1,2,\ldots,m)$ and the $c_j^{(i)T}$ $(j=1,2,\ldots,m)$ are, respectively, the decoupling indices (Falb and Wolovich (1967)) and the rows of the sub-output matrix $C^{(i)}$. In the case of such plants, it is important to note that

$$F^{(i)} = \lim_{t \to 0} A^{(i)-1}(t)H^{(i)}(t) \qquad (6.13)$$

and

$$F^{(i)-1} = \lim_{t \to 0} H^{(i)-1}(t)A^{(i)}(t) , \qquad (6.14)$$

where

$$A^{(i)}(t) = \mathrm{diag}\{t^{d_1^{(i)}+1}/(d_1^{(i)}+1)!,\ldots,t^{d_m^{(i)}+1}/(d_m^{(i)}+1)!\} \qquad (6.15)$$

and

$$H^{(i)}(t) = C^{(i)}A^{-1}(e^{At}-I_n)B \qquad (6.16)$$

is the sub-plant step-response matrix.

In order to design error-actuated digital set-point tracking PID controllers for sub-plants governed by state and output

equations of the respective forms (6.1) and (6.7), it is convenient to consider the behaviour of such plants on the discrete-time set $T_T = \{0,T,2T,...,kT,...\}$. This behaviour is governed by state and output equations of the respective forms (Kwakernaak and Sivan (1972))

$$x_{k+1} = \Phi x_k + \Psi u_k \tag{6.17}$$

and

$$y_k^{(i)} = \Gamma^{(i)} x_k , \tag{6.18}$$

where $x_k = x(kT) \in R^n$, $u_k = u(kT) \in R^m$, $y_k^{(i)} = y^{(i)}(kT) \in R^m$, $i \in I_r$,

$$\Phi = \exp(AT) , \tag{6.19}$$

$$\Psi = \int_0^T \exp(At)B \, dt , \tag{6.20}$$

$$\Gamma^{(i)} = C^{(i)} , \quad i \in I_r , \tag{6.21}$$

and $T \in R^+$ is the sampling period.

Each individual set-point tracking error-actuated tunable digital PID controller is governed on the discrete-time set T_T by a control-law equation of the form

$$u_k = TK_1^{(i)} e_k^{(i)} + TK_2^{(i)} z_k + K_3^{(i)} (e_k^{(i)} - e_{k-1}^{(i)}) , \tag{6.22}$$

where the sub-error vector $e_k^{(i)} = v^{(i)} - y_k^{(i)} \in R^m$, the sub-plant set-point vector $v^{(i)} \in R^m$, the digital integral of sub-error vector $z_k \in R^m$, the controller matrices $K_1^{(i)} \in R^{m \times m}$, $K_2^{(i)} \in R^{m \times m}$, and $K_3^{(i)} \in R^{m \times m}$, and the superscript $^{(i)}$ means that the vectors and the matrices correspond to the subset Y_i, $i \in I_r$. Furthermore, it is assumed that

$$z_{k+1} = z_k + T e_k^{(i)} . \tag{6.23}$$

Hence, it is noted as a whole that the overall set-point vector is

$$v = [v_1, \ldots, v_p]^T \in R^p , \tag{6.24}$$

the overall plant output vector is

$$y_k = y(kT) = [y_1(kT), \ldots, y_p(kT)]^T \in R^p , \tag{6.25}$$

and the overall error vector is

$$e_k = e(kT)$$

$$= [e_1(kT), \ldots, e_p(kT)]^T$$

$$= v - y_k \in R^p . \tag{6.26}$$

Furthermore, it is noted in view of equations (6.6) and (6.7) that

$$v^{(i)} = E^{(i)}v = \begin{bmatrix} v_{s_1} \\ \cdot \\ \cdot \\ \cdot \\ v_{s_{m-1}} \\ v_{t_i} \end{bmatrix} , \qquad (6.27)$$

$$y_k^{(i)} = E^{(i)}y_k = \begin{bmatrix} y_{s_1}(kT) \\ \cdot \\ \cdot \\ \cdot \\ y_{s_{m-1}}(kT) \\ y_{t_i}(kT) \end{bmatrix} , \qquad (6.28)$$

and

$$e_k^{(i)} = E^{(i)}e_k = \begin{bmatrix} e_{s_1}(kT) \\ \cdot \\ \cdot \\ \cdot \\ e_{s_{m-1}}(kT) \\ e_{t_i}(kT) \end{bmatrix} . \qquad (6.29)$$

where $E^{(i)} \in R^{m \times p}$ consists of s_1th,...,s_{m-1}th and t_ith rows of a unit matrix I_p.

It follows from equations (6.17), (6.18), (6.22), and (6.23) that such discrete-time tracking systems are governed on T_I by state and output equations of the respective forms

$$\begin{bmatrix} x_{k+1} \\ z_{k+1} \\ f_{k+1} \end{bmatrix} = \begin{bmatrix} \Phi - T\Psi K_1^{(i)}\Gamma^{(i)} - \Psi K_3^{(i)}\Gamma^{(i)} & , & T\Psi K_2^{(i)} & , & -\Psi K_3^{(i)} \\ -T\Gamma^{(i)} & , & I_m & , & 0 \\ -\Gamma^{(i)} & , & 0 & , & 0 \end{bmatrix} \begin{bmatrix} x_k \\ z_k \\ f_k \end{bmatrix}$$

$$+ \begin{bmatrix} T\Psi K_1^{(i)} + \Psi K_3^{(i)} \\ TI_m \\ I_m \end{bmatrix} v^{(i)} \qquad (6.30)$$

and

$$y_k^{(i)} = [\ \Gamma^{(i)} \ , \ 0 \ , \ 0 \ ] \begin{bmatrix} x_k \\ z_k \\ f_k \end{bmatrix} \qquad , \qquad (6.31)$$

where $f_k = e_{k-1}^{(i)} \in R^m$ is the stored sub-error vector.

Therefore, provided only that T, $K_1^{(i)}$, $K_2^{(i)}$, and $K_3^{(i)}$, where i $\in I_r$, are such that all the eigenvalues of the closed-loop sub-plant matrix in equation (6.30) lie in the open unit disc D^-,

$$\lim_{k \to \infty} \Delta z_k = \lim_{k \to \infty} \{z_{k+1} - z_k\} = 0 \qquad (6.32)$$

and therefore

$$\lim_{k \to \infty} e_k^{(i)} = 0 \qquad (6.33)$$

so that set-point tracking for the subset Y_i occurs.

The closed-loop characteristic equation can be readily expressed in the form (Porter and Jones (1985a))

$$\phi_c^{(i)}(z) = \phi_1^{(i)}(z)\phi_2^{(i)}(z)\phi_3^{(i)}(z) \qquad (6.34)$$

by invoking the block-diagonalisation procedure of Kokotović (1975), and the response characteristics of the closed-loop

system can thus accordingly be elucidated. The asymptotic properties of the tracking system under the action of such controllers can be characterised in terms of the eigenstructure of the closed-loop plant matrix, which involves the decomposition of this matrix into three sub-systems based on the explicitly invertible block diagonalisation transform (Kokotović (1975)).

This block-diagonalisation procedure transforms the matrix triple incorporated in equations of the form

$$
\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} A_{11} & , & A_{12} \\ A_{21} & , & A_{22} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(k) \qquad (6.35)
$$

and

$$
y(k) = [\ C_1 \ , \ C_2\ ] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \qquad (6.36)
$$

where $x_1(k) \in R^{n_1}$, $x_2(k) \in R^{n_2}$, $A_{ij} \in R^{n_i \times n_j}$ (i,j=1,2), $B_1 \in R^{n_1 \times m}$, $B_2 \in R^{n_2 \times m}$, $C_1 \in R^{m \times n_1}$, and $C_2 \in R^{m \times n_2}$ into the block-diagonal form incorporated in the equations

$$
\begin{bmatrix} \chi_1(k+1) \\ \chi_2(k+1) \end{bmatrix} = \begin{bmatrix} A_{11} & , & 0 \\ 0 & , & A_{22} \end{bmatrix} \begin{bmatrix} \chi_1(k) \\ \chi_2(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(k) \qquad (6.37)
$$

and

$$
y(k) = [\ C_1 \ , \ C_2\ ] \begin{bmatrix} \chi_1(k) \\ \chi_2(k) \end{bmatrix} \qquad . \qquad (6.38)
$$

The state vectors in these equations are related by the linear state transformation (Kokotović (1975))

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = W \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \tag{6.39}$$

where

$$W = \begin{bmatrix} I_{n_1} & , & M \\ -L & , & I_{n_2} - LM \end{bmatrix} \in R^{(n_1+n_2) \times (n_1+n_2)} \tag{6.40}$$

and $X_1(k) \in R^{n_1}$, $X_2(k) \in R^{n_2}$, $A_{ij} \in R^{n_i \times n_j}$ (i,j=1,2), $B_1 \in R^{n_1 \times m}$, $B_2 \in R^{n_2 \times m}$, $C_1 \in R^{m \times n_1}$, $C_2 \in R^{m \times n_2}$, $L \in R^{n_2 \times n_1}$,and $M \in R^{n_1 \times n_2}$. It is noted that, although there exists one linear transformation for every sub-plant, the superscript [i] is omitted to simplify the notation.

Thus, if L and M satisfy the matrix Riccati equations (Kokotović (1975))

$$A_{21} + LA_{11} - A_{22}L - LA_{12}L = 0 \tag{6.41}$$

and

$$(A_{11} - A_{12}L)M - M(A_{22} + LA_{12}) + A_{12} = 0 , \tag{6.42}$$

it follows from equations (6.35), (6.37), and (6.39) that

$$\overline{A}_{11} = A_{11} - A_{12}L \qquad\qquad (6.43)$$

and

$$\overline{A}_{22} = A_{22} + LA_{12} \qquad\qquad . \qquad\qquad (6.44)$$

The asymptotic properties of the discrete-time closed-loop tracking system can now be readily determined by regarding T as a perturbation parameter in equations (6.30) and (6.31). Thus, by regarding in equation (6.35)

$$A_{11} = \begin{bmatrix} \Phi - T\Psi K_1^{(i)} \Gamma^{(i)} - \Psi K_3^{(i)} \Gamma^{(i)} & , & T\Psi K_2^{(i)} \\ -T\Gamma^{(i)} & , & I_m \end{bmatrix} , \qquad (6.45)$$

$$A_{12} = \begin{bmatrix} -\Psi K_3^{(i)} \\ 0 \end{bmatrix} \qquad\qquad , \qquad (6.46)$$

$$A_{21} = [-\Gamma^{(i)} , 0] \qquad\qquad , \qquad (6.47)$$

and

$$A_{22} = 0 \qquad\qquad , \qquad (6.48)$$

the solution of equations (6.41) and (6.42) can be readily obtained by using power series expansion in T. This involves the definition of matrices L_1 and L_2 such that

$$L = [L_1 , L_2] \qquad\qquad (6.49)$$

where

$$L_1 = L_{10} + L_{11}T + \ldots\ldots \qquad (6.50)$$

$$L_2 = L_{20} + L_{21}T + \ldots\ldots \qquad (6.51)$$

in which $L_{1i} \in R^{n_2 \times n_3}$, $L_{2i} \in R^{n_2 \times n_4}$, $(i=0,1,2,\ldots\ldots)$. Therefore, it is clear from equations (6.41) and (6.45) to (6.49) that on isolating coefficients

$$L = [\ C^{(i)} \ , \ 0 \ ] + O(T) \qquad (6.52)$$

and therefore from equations (6.43) and (6.44) that

$$A_{11} = \begin{bmatrix} \Phi - T\Psi K_1^{(i)}\Gamma^{(i)} \ , & T\Psi K_2^{(i)} \\ -T\Gamma^{(i)} \ , & I_m \end{bmatrix} \qquad (6.53)$$

and

$$A_{22} = -TC^{(i)}BK_3^{(i)} + O(T^2) \qquad . \qquad (6.54)$$

The matrix A_{11} in equation (6.53) is now block-diagonalised, again by regarding T as a perturbation parameter in equation (6.53) and by regarding in equation (6.35)

$$\bar{A}_{11} = \Phi - T\Psi K_1^{(i)} \qquad , \qquad (6.55)$$

$$\bar{A}_{12} = T\Psi K_2^{(i)} \qquad , \qquad (6.56)$$

$$\bar{A}_{21} = -T\Gamma^{(i)} \qquad , \qquad (6.57)$$

and

$$\bar{A}_{22} = I_m \qquad . \qquad (6.58)$$

In addition, the matrix $\bar{L}$ is defined in the power-series form

$$\bar{L} = \bar{L}_0 + T\bar{L}_1 + T^2\bar{L}_2 + \ldots \qquad (6.59)$$

In equations (6.55) to (6.59), the overbar has been used to distinguish between the two explicit stages of the block-diagonalisation procedure.

Therefore, it is clear from equations (6.41) and (6.55) to (6.59) that on isolating coefficients

$$\bar{L} = C^{(i)}A^{-1} + T(C^{(i)}A^{-1}BK_1^{(i)}C^{(i)}A^{-1}$$
$$+C^{(i)}A^{-1}BK_2^{(i)}C^{(i)}A^{-2} - C^{(i)}/2) + O(T^2). \qquad (6.60)$$

Hence, it follows from (6.43), (6.44) and (6.60) that

$$\bar{A}_{11} = I_n + TA + T^2A^2/2 - T^2BK_1^{(i)}C^{(i)}$$
$$- T^2BK_2^{(i)}C^{(i)}A^{-1} + O(T^3) \qquad (6.61)$$

and

$$\bar{A}_{22} = I_m - T^2C^{(i)}A^{-1}BK_2^{(i)} + O(T^3) . \qquad (6.62)$$

Thus, it is evident from equations (6.53), (6.54), (6.61), and (6.62) that the characteristic polynomials as expressed in equation (6.34) are

$$\phi_1^{(i)}(z) = \mid zI_n - I_n - TA - T^2A^2/2 + T^2BK_1^{(i)}C^{(i)}$$
$$+ T^2BK_2^{(i)}C^{(i)}A^{-1} + O(T^3) \mid , \qquad (6.63)$$

$$\phi_2^{(i)}(z) = \mid zI_m - I_m - T^2C^{(i)}A^{-1}BK_2^{(i)} + O(T^3) \mid , \qquad (6.64)$$

and

$$\phi_3^{(i)}(z) = \mid zI_m + TC^{(i)}BK_3^{(i)} + O(T^2) \mid . \qquad (6.65)$$

6.3 Synthesis

It is clear that tracking will occur in the sense of equation (6.33) provided only that the set of closed-loop characteristic roots

$$Z_c^{(i)} = Z_1^{(i)} \cup Z_2^{(i)} \cup Z_3^{(i)} \subset D^- \qquad (6.66)$$

where D^- is the open unit disc and the sets of characteristic roots $Z_1^{(i)}$, $Z_2^{(i)}$, and $Z_3^{(i)}$ are, respectively, the roots of the characteristic polynomials as expressed in equation (6.34).

Therefore, in case

$$K_1^{(i)} = H^{(i)}(T)^{-1}\Lambda^{(i)}(T)\Pi^{(i)} , \qquad (6.67)$$

$$K_2^{(i)} = G^{(i)}(0)^{-1} \Sigma^{(i)} \ , \qquad (6.68)$$

and

$$K_3^{(i)} = H^{(i)}(T)^{-1} \Lambda^{(i)}(T) \Delta^{(i)} \ , \qquad (6.69)$$

where $H^{(i)}(T)$ and $G^{(i)}(0)$ are given by equations (6.16) and (6.10) respectively,

$$\Pi^{(i)} = \text{diag}\{\pi_{s_1}, \pi_{s_2}, \ldots, \pi_{s_{m-1}}, \pi_{t_i}\} \ , \qquad (6.70)$$

$$\pi_{s_1}, \pi_{s_2}, \ldots, \pi_{s_{m-1}}, \pi_{t_i} \in R^+ \ , \qquad (6.71)$$

$$\Sigma^{(i)} = \text{diag}\{\sigma_{s_1}, \sigma_{s_2}, \ldots, \sigma_{s_{m-1}}, \sigma_{t_i}\} \ , \qquad (6.72)$$

$$\sigma_{s_1}, \sigma_{s_2}, \ldots, \sigma_{s_{m-1}}, \sigma_{t_i} \in R^+ \ , \qquad (6.73)$$

$$\Delta^{(i)} = \text{diag}\{\delta_{s_1}, \delta_{s_2}, \ldots, \delta_{s_{m-1}}, \delta_{t_i}\} \ , \qquad (7.74)$$

and

$$\delta_{s_1}, \delta_{s_2}, \ldots, \delta_{s_{m-1}}, \delta_{t_i} \in R^+ \ , \qquad (6.75)$$

it follows from equations (6.34), (6.63) to (6.65), and (6.67) to (6.69) that

$$Z_1^{(i)} = \{z \in C : |zI_n - I_n - TA + 0(T^2)| = 0\} \ , \qquad (6.76)$$

$$Z_2^{(i)} = \{ z \in C : \mid zI_m - I_m + T^2 \Sigma^{(i)} + O(T^3) \mid = 0 \} \ ,$$

$$(6.77)$$

and

$$Z_3^{(i)} = \{ z \in C : \mid zI_m + O(T) \mid = 0 \} \ .$$
$$(6.78)$$

These expressions indicate that, provided T is sufficiently small, all the closed-loop characteristic roots lie within the open unit disc in each case. This follows since the open—loop plant is asymptotically stable on the continuous—time set $T = [0,+\infty)$ and since $T^2\Sigma^{(i)}$ is a positive diagonal matrix. The introduction of error—actuated digital set-point tracking PID controllers governed by equations (6.22) and (6.67) to (6.69) accordingly ensures that set-point tracking for the subset Y_i occurs when the sampling period $T \in (0,T_i^*]$, where $T_i^* = T_i^*(\Pi^{(i)},\Sigma^{(i)})$ can be readily obtained by simple "on-line" tuning (Porter and Jones (1985a)). Therefore, in case $T \in (0,\min(T_i^*)]$, $i \in I_r$, all the r closed loops are asymptotically stable and set-point tracking of each loop is ensured when considered separately.

Furthermore, it follows from equations (6.30) and (6.31) that the sub-output from the initially quiescent plant after the first sampling interval under error—actuated digital PID control is

$$y^{(i)}(T) = [TH^{(i)}(T)K_1^{(i)} + H^{(i)}(T)K_3^{(i)}]v^{(i)} \ .$$
$$(6.79)$$

It is evident from equations (6.67), (6.69), and (6.79) that

$$y^{(i)}(T) = [TA^{(i)}(T)\Pi^{(i)} + A^{(i)}(T)\Delta^{(i)}]v^{(i)} \qquad (6.80)$$

and therefore that set-point tracking occurs for the subset Y_i when $T \in (0,\min(T_i^*)]$ with no initial interaction since $A^{(i)}(T)\Pi^{(i)}$ and $A^{(i)}(T)\Delta^{(i)}$ are diagonal matrices for all $T \in R^+$.

The proportional, integral, and derivative controller matrices $K_1^{(i)}$, $K_2^{(i)}$, and $K_3^{(i)}$ given by equations (6.67), (6.68), and (6.69), respectively, can all be directly determined from the sub-plant step-response matrix $H^{(i)}(t)$ since it follows from equation (6.16) that

$$G^{(i)}(0) = \lim_{t \to \infty} H^{(i)}(t) = -C^{(i)}A^{-1}B \qquad (6.81)$$

because the open-loop plant is asymptotically stable and therefore has a bounded step-response matrix. Furthermore, since the expressions (6.67) and (6.69) for the proportional and derivative controller matrices, respectively, involve the inverse of the initial sub-plant step-response matrix of the open-loop plant $H^{(i)}(T)$, it is clear that the sampling period must be selected so that the minimum singular value of $H^{(i)}(T)$ ($\sigma_{min}[H^{(i)}(T)]$) is not small, so that $H^{(i)}(T)$ is well-conditioned.

6.4 Implementation of digital self-selecting controllers

The block diagram of the system incorporating the self-selecting controller is shown in Fig 6.1. The individual set-point tracking controllers are designed by the procedure described in the previous sections and then integrated into the self-selecting controller. Then, the selection of the most critical subset of plant outputs and the resulting controller switching are the remaining functions of the self-selecting controller. Therefore, in the following, such functions are discussed in accordance with lowest-wins strategies.

It follows from equation (6.29) that all the sub-error vectors $e_k^{(i)} \in R^m$ (i=1,2,...,r) have m-1 common elements. Therefore, the lowest-win strategies need to compare only the remaining r scalar signals which are not common in $e_k^{(1)}$, $e_k^{(2)}$,..., $e_k^{(r)}$ to determine the control loop. Furthermore, it follows that the index set J(kT) of lowest-errors and the loop index ℓ_k of the actually selected loop are defined on the discrete-time set T_T = {0,T,2T,...,kT,...} by the respective forms

$$J(kT) = \{j : e_{t_j}(kT) = \min_{i \in I_r} e_{t_i}(kT)\} \qquad (6.82)$$

and

$$\ell_k = \ell(kT) \in J(kT) \subset I_r . \qquad (6.83)$$

Therefore, it is clear from equation (6.22) that the self-selecting controller is governed on T_T by equations of the

form

$$u_k = TK_1^{(\ell_k)}e_k^{(\ell_k)} + TK_2^{(\ell_k)}z_k + K_3^{(\ell_k)}(e_k^{(\ell_k)} - e_{k-1}^{(\ell_k)}) \quad (6.84)$$

and

$$z_{k+1} = z_k + Te_k^{(\ell_k)} , \quad (6.85)$$

where the superscript ℓ_k means that the control loop ℓ_k is in action at time kT. Since equations (6.82) and (6.83) decide which controller should be used at each sampling instant, controller switching may occur.

During the controller switching from one control configuration (loop index ℓ_{k-1}) to another ($\ell_k \neq \ell_{k-1}$) at time kT, it is preferable that the input vector changes in a bumpless manner from u_{k-1} to u_k (ie bumpless transfer operation). This can be achieved by resetting the integrator states in equation (6.85) every time there is controller switching, so that the resulting control input vector remains constant ie,

$$u_k = u_{k-1} \quad (6.86)$$

with corresponding effect on the plant. The demerit of this bumpless transfer operation is that, in case the set-point change and the controller switching occur at the same time, the transient response of the plant is not rapid because of the effect of equation (6.86). However, a similar demerit holds for controller switching governed by an equation of the

incremental form

$$u_k = u_{k-1} + TK_1^{(\ell_k)}(e_k^{(\ell_k)} - e_{k-1}^{(\ell_k)}) + T^2 K_2^{(\ell_k)} e_{k-1}^{(\ell_k)}$$
$$+ K_3^{(i)}(e_k^{(\ell_k)} - 2e_{k-1}^{(\ell_k)} + e_{k-2}^{(\ell_k)}) . \qquad (6.87)$$

Thus, assuming that the controller switching from loop index ℓ_1 to ℓ_2 occurs at time kT for the plant operating in steady state (ie $e_k^{(\ell_1)} = e_{k-1}^{(\ell_1)} = e_{k-2}^{(\ell_1)} = 0$), and that

$$v = \begin{cases} v_1 = [v_{1_1}, v_{1_2}, \ldots, v_{1_p}]^T & \text{for } 0 \le t \le (k-1)T \\ \\ v_2 = [v_{2_1}, v_{2_2}, \ldots, v_{2_p}]^T & \text{for } t \ge kT \end{cases}, \qquad (6.88)$$

$$v_1 \ne v_2 , \qquad (6.89)$$

$$v_{1_{s_i}} = v_{2_{s_i}} \qquad (i = 1,2,\ldots,m-1) , \qquad (6.90)$$

$$v_{1_{t_{\ell_2}}} = v_{2_{t_{\ell_2}}} , \qquad (6.91)$$

it follows that

$$y_k^{(\ell_2)} = y_{k-1}^{(\ell_2)} = y_{k-2}^{(\ell_2)} \qquad (6.92)$$

and

$$v_1^{(\ell_2)} = v_2^{(\ell_2)} . \qquad (6.93)$$

Therefore,

$$e_k^{(\ell_2)} - e_{k-1}^{(\ell_2)} = (v_2^{(\ell_2)} - y_k^{(\ell_2)}) - (v_1^{(\ell_2)} - y_{k-1}^{(\ell_2)})$$

$$= 0 \qquad\qquad (6.94)$$

and

$$e_k^{(\ell_2)} - 2e_{k-1}^{(\ell_2)} + e_{k-2}^{(\ell_2)} = v_2^{(\ell_2)} - 2v_1^{(\ell_2)} + v_1^{(\ell_2)}$$

$$= 0 \; . \qquad\qquad (6.95)$$

It follows from equations (6.87), (6.94), and (6.95) that

$$u_k = u_{k-1} + T^2 K_2^{(\ell_2)} e_{k-1}^{(\ell_2)} \; . \qquad\qquad (6.96)$$

It is clear from equation (6.96) that u_k is independent of $e_k^{(\ell_k)}$. Therefore, if $e_k^{(\ell_k)} \neq 0$, it follows that that the proportional and derivative terms of the input are suppressed and that only the integral term contributes to the input change. Therefore, initial decoupling in the sense of equation (6.80) does not occur with such switching logic.

In order to circumvent this problem and obtain initial decoupling, equation (6.87) is modified in the form

$$u_k = u_{k-1} + T K_1^{(\ell_k)}(e_k - e_{k-1}) + T^2 K_2^{(\ell_k)} e_{k-1}$$
$$+ K_3^{(\ell_k)}(e_k - 2e_{k-1} + e_{k-2}) \qquad , \qquad (6.97)$$

where the sub-error vector at time kT is $e_k = e_k^{(\ell_k)}$. Using

this modification, the control input at the controller switching instant under the above conditions (6.88) to (6.93) is given in the form

$$u_k \simeq u_{k-1} + TK_1^{(\ell_k)}e_k + K_3^{(\ell_k)}e_k \qquad (6.98)$$

since $e_{k-1} = e_{k-1}^{(\ell_k-1)} = e_{k-2}^{(\ell_k-1)} \simeq 0$. Thus, the proportional and derivative terms are efficiently utilised at the controller switching instant.

Finally, the self-selecting control law together with the lowest-wins strategy and the controller switching logic is embodied in equations (6.82), (6.83), and (6.97). However, it is evident from equation (6.97) that elements of the error vectors e_k, e_{k-1}, and e_{k-2}, which are compared in the lowest-wins strategy, might have different units. Therefore, it is required that such error vectors are properly scaled so that the bumpless transfer operation is attained. Furthermore, it is noted that such scaling might also be effective in the lowest-wins strategy in equation (6.82).

6.5 Illustrative example

In order to demonstrate the performance characteristics of the digital self-selecting PID controller proposed in the previous sections, such a controller is designed for the two-input three-output linear F100 engine model at Intermediate power condition (Appendix 3). In this case, the manipulated input variables are main burner fuel flow (lb/hr) and nozzle jet area

(ft^2). The output variables are fan speed N_1 (rpm), augmentor pressure P_7 (psia), and fan turbine inlet temperature FTIT (°R).

The design starts with the classification of the plant. The steady-state transfer-function matrix G is obtained from Figs 2.2 and 2.3 in the form

$$G = G(0)$$

$$= \begin{bmatrix} g_{c_1} & | & g_{c_2} \\ & | & \end{bmatrix}$$

$$= \begin{bmatrix} g_1^T \\ g_2^T \\ g_3^T \end{bmatrix}$$

$$= \begin{bmatrix} 0.37904 & 1238.8 \\ 0.15944e\text{-}2 & -12.168 \\ 0.90309e\text{-}1 & 210.94 \end{bmatrix} \qquad\qquad . \qquad (6.99)$$

Since $g_{c_1} > 0$, by Theorem 4.5, it follows that $G \in$ Class I. The input space is shown in Fig 6.2. Clearly, $U_F(0)$ is a 2-dimensional convex cone. Equations $g_1^T u = 0$ corresponding to N_1 and $g_2^T u = 0$ corresponding to P_7 represent extreme rays of $U_F(0)$.

It is evident that either extreme ray has a unique representation and that a minimum of two control loops is required. If N_1 must always be controlled, the subsets of plant outputs to be controlled are (Structure 1)

$$Y_1 = \{N_1, P_7\}, \quad Y_2 = \{N_1, FTIT\} .$$

If P7 must always be controlled, the subsets are (Structure 2)

$$Y_1 = \{P_7, N_1\}, \quad Y_2 = \{P_7, FTIT\} .$$

Therefore, it is possible to design a self-selecting controller for the plant based upon either Structure 1 or Structure 2. The corresponding minimum singular value plots $(\sigma_{min}[H^{(i)}(t)])$ of the sub-plant step-response matrices shown in Figs 6.3(a),(b), 6.4(a),(b), and 6.5 indicate that the plant is nonminimum phase for the output pairs $[N_1,P_7]$ and $[N_1,FTIT]$ and that the plant is minimum phase for the output pair $[P_7,FTIT]$ (Porter and Jones (1985c)). Furthermore, $G(0)$ is well-conditioned since $\sigma_{min}[H^{(i)}(t)]$ is not small. However, attention should be given in order to choose the sampling period T so as not to use an ill-conditioned $H^{(i)}(T)$, since $\sigma_{min}[H^{(i)}(t)]$ vanishes once for $[N_1,P_7]$ and $[N_1,FTIT]$.

It is found from Figs 2.2 and 2.3 that

$$H(0.05) = \begin{bmatrix} 0.63349e{-}3 & 1.2999 \\ 0.11637e{-}4 & -0.18878 \\ 0.60822e{-}4 & -0.86794e{-}2 \end{bmatrix} . \tag{6.100}$$

Firstly, a self-selecting lowest-wins PID controller is designed and tuned based upon Structure 1 such that $T = 0.05$ sec, $\Lambda^{(1)}(0.05)\Pi^{(1)} = \text{diag}\{0.04, 0.1\}$, $\Lambda^{(2)}(0.05)\Pi^{(2)} = \text{diag}\{0.04, 0.02\}$, $\Sigma^{(1)} = \Sigma^{(2)} = 50.0I_2$, and $\Lambda^{(1)}(0.05)\Delta^{(1)} = \Lambda^{(2)}(0.05)\Delta^{(2)} = 0.0005I_2$. The excellent limit tracking and switching behaviour of the plant under the action of the resulting error-actuated controller is shown in Figs 6.6 and 6.7, where the loops show that $P_7(y_2)$ and $FTIT(y_3)$ are controlled in turn whilst $N_1(y_1)$ is permanently controlled.

Next, a self-selecting controller is designed and tuned based upon Structure 2 such that $T = 0.05$ sec, $\Lambda^{(1)}(0.05)\Pi^{(1)} = \text{diag}\{0.1, 0.04\}$, $\Lambda^{(2)}(0.05)\Pi^{(2)} = \text{diag}\{0.1, 0.02\}$, $\Sigma^{(1)} = \Sigma^{(2)} = 50.0I_2$, and $\Lambda^{(1)}(0.05)\Delta^{(1)} = \Lambda^{(2)}(0.05)\Delta^{(2)} = 0.0005I_2$. The excellent limit tracking and switching behaviour of the plant under the action of the resulting error-actuated controller is shown in Figs 6.8 and 6.9, where the loops show that $N_1(y_1)$ and $FTIT(y_3)$ are controlled in turn whilst $P_7(y_2)$ is permanently controlled.

It is noted that, in both cases, the elements of the sub-error vectors which are used in the control-law equation (6.97) and in the lowest-wins strategy equation (6.82) have been scaled so that the steady-state gains of the open-loop plant for the fuel flow are equal. Thus, it follows from equation (6.26) that for $i \in I_r$

$$\lim_{k \to \infty} e_{t_i}(kT) = v_i - \lim_{k \to \infty} y_{t_i}(kT) \ . \tag{6.101}$$

Now, let $g_{c_1}^{t_i}$ be the t_ith element of g_{c_1} in equation (6.99). The output change in steady states for the step change of fuel flow Δu_1 is

$$\lim_{k \to \infty} \Delta y_{t_i}(kT) = g_{c_1}^{t_i} \Delta u_1 \qquad (i = 1,2) . \qquad (6.102)$$

Then, it follows from equations (6.101) and (6.102) that

$$\lim_{k \to \infty} \frac{\partial e_{t_i}(kT)}{\partial u_1} = -\lim_{k \to \infty} \frac{\Delta y_{t_i}(kT)}{\Delta u_1}$$

$$= - g_{c_1}^{t_i} \qquad (i = 1,2) \qquad (6.103)$$

and therefore that

$$\lim_{k \to \infty} \frac{\partial e_{t_i}(kT)}{\partial u_1} \frac{1}{g_{c_1}^{t_i}} = const \qquad (i = 1,2). \qquad (6.104)$$

This implies that, if the elements of the sub-error vectors e_{t_1} and e_{t_2} are multiplied by $1/g_{c_1}^{t_i}$, their effects in steady states are equal. Thus, the elements of the sub-error vectors have been scaled.

6.6 Conclusion

In this chapter, a new methodology for the design of self-selecting PID controllers, which uses the order-reduction

technique (Algorithm 5.1) and corresponds to the extension of tunable set-point tracking controllers (Chapter 2), has been developed.

Firstly, the order-reduction technique (Algorithm 5.1) has been applied to m-input/p-output Class I linear multivariable plants and p-m+1 subsets of plant outputs have been chosen. For these subsets, tunable digital set-point tracking PID controllers have been designed. A block-diagonalisation transformation has been used to exhibit the asymptotic properties of the separate discrete-time closed-loop tracking systems which correspond to these subsets. It has been shown that the proportional, integral, and derivative matrices embodied in such set-point tracking controllers can be readily determined from open-loop test performed on asymptotically stable plants, thus circumventing the need for detailed mathematical models. Next, the implementation of the self-selecting controller using these different set-point tracking controllers has been discussed. The lowest-wins strategies which cause the selection of the most critical subset of outputs, together with the controller switching logic, have been formulated. Finally, the effectiveness of the proposed design methodology has been illustrated by designing self-selecting controllers based upon two different structures for a highly interactive gas-turbine engine.

It is noted that, although the asymptotic stability of separate closed-loop tracking systems has been guaranteed by this design methodology, this does not guarantee the stability of the

complete system. Therefore, it is recommended that the stability and performance of self-selecting controllers be verified in simulation studies before field application.
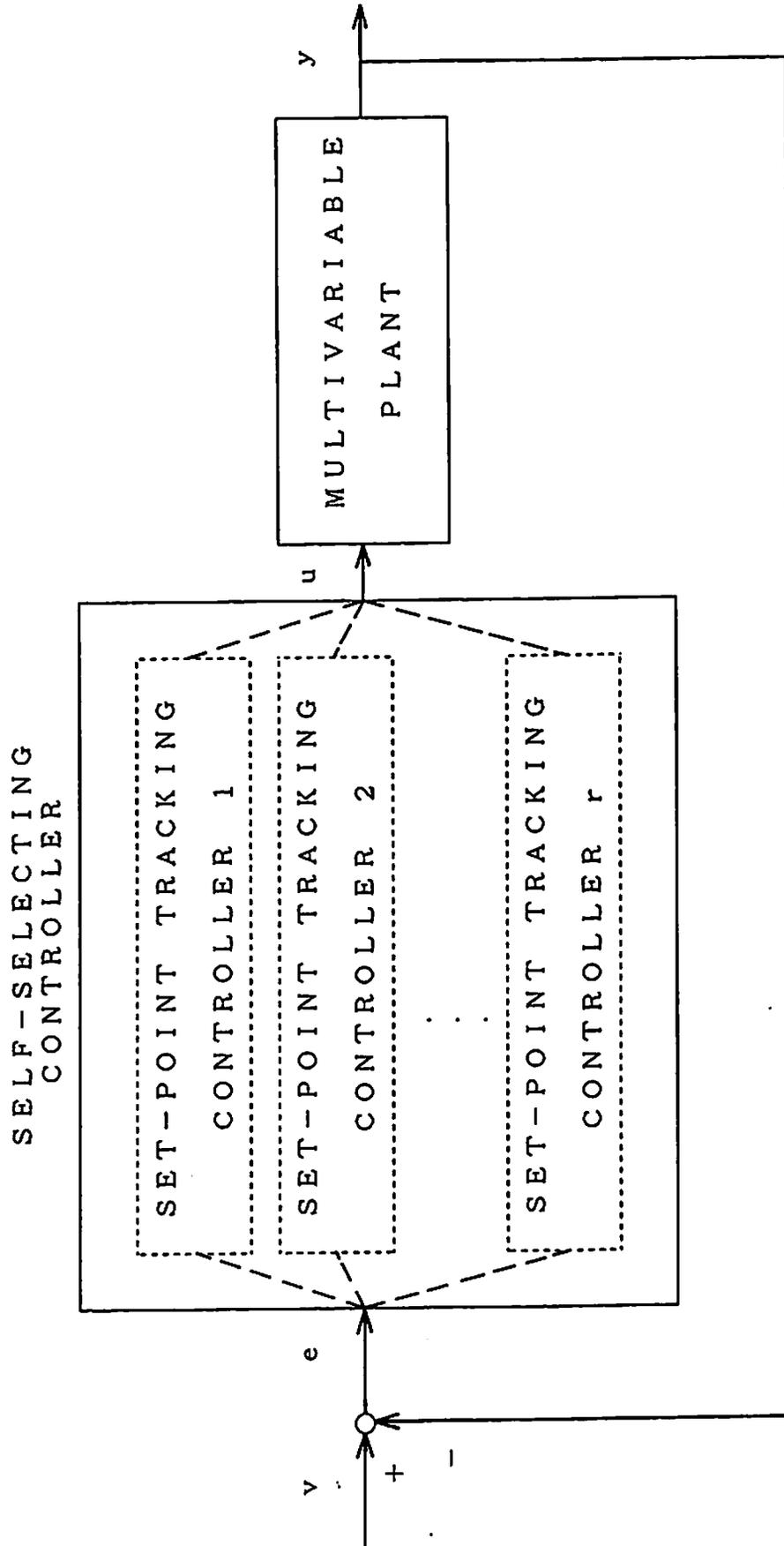
Fig 6.1 Block diagram of system
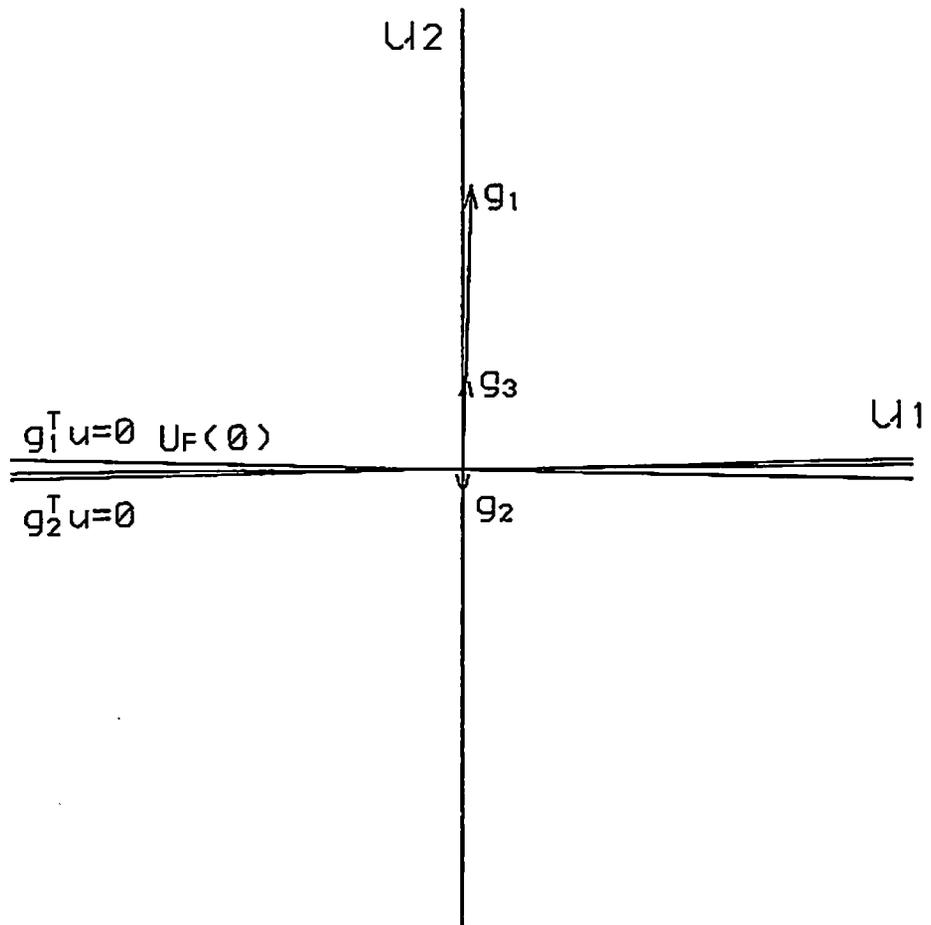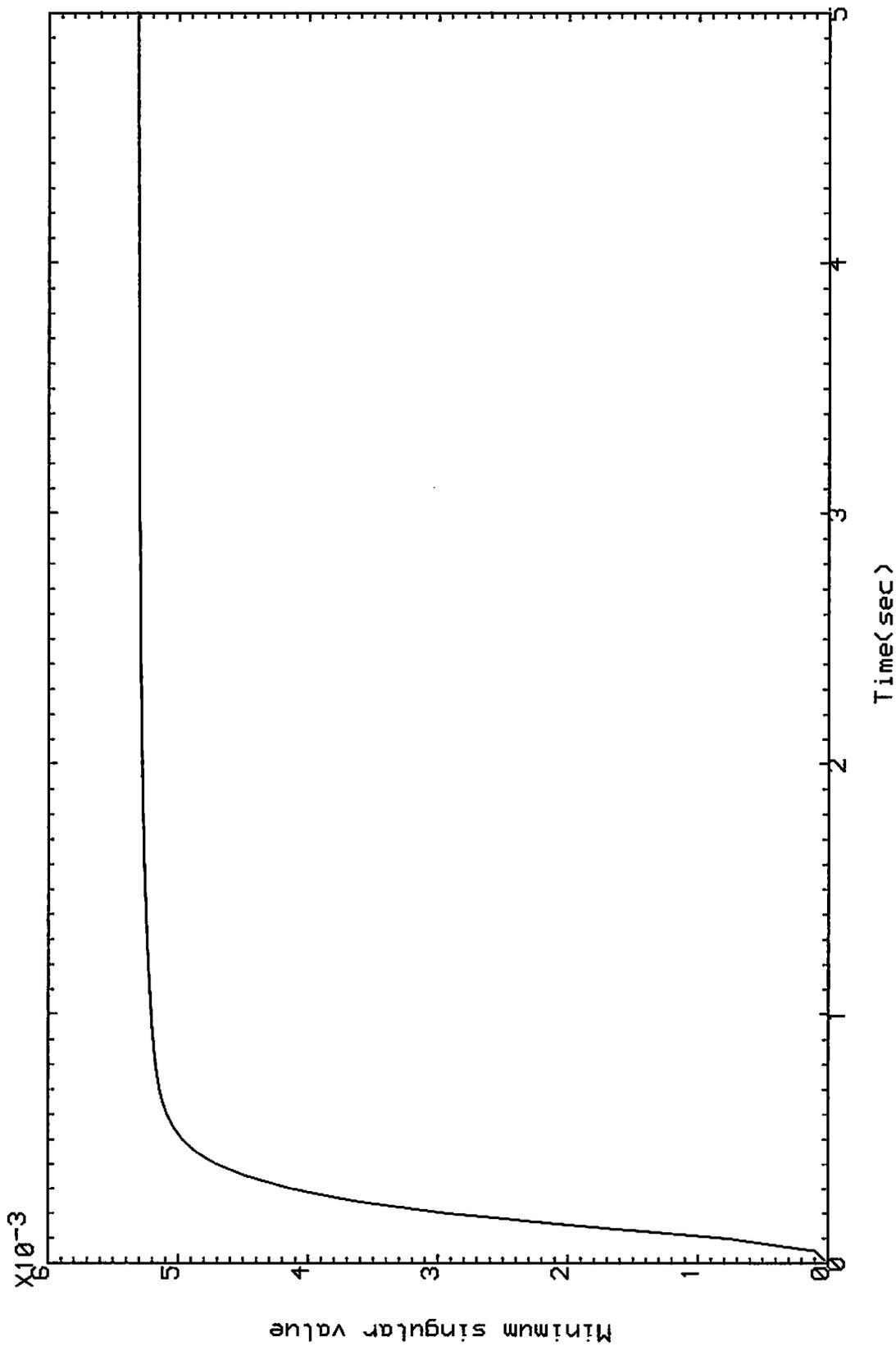incorporating self-selecting controller

Fig 6.2 U-space

Plant: Linear F100 engine model at Intermediate

Fig 6.3(a) Minimum singular value plot of the plant step-response matrix
Plant: F100 engine linear model with 2 inputs and
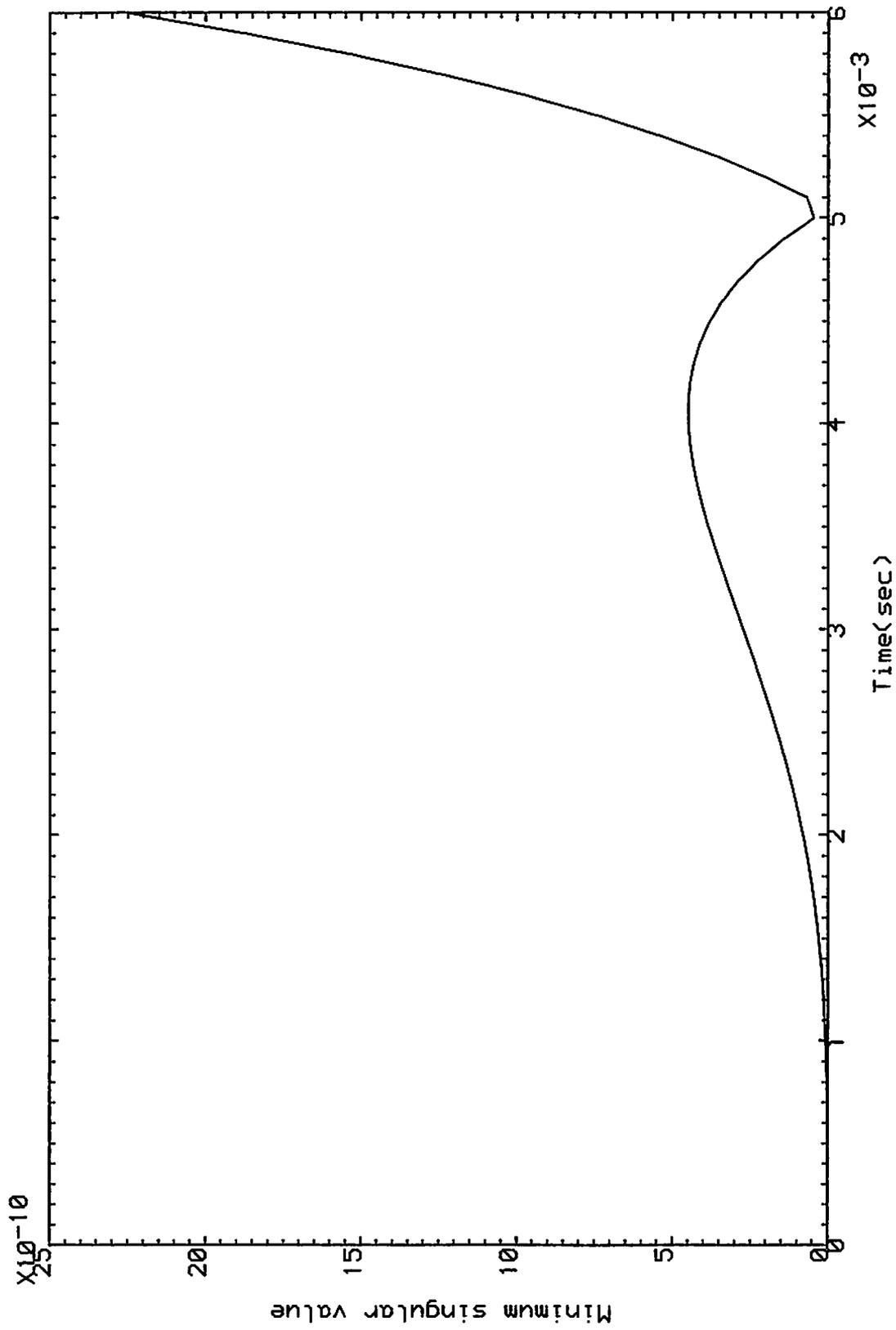2 outputs [N1,P7]

Fig.6.3(b) Minimum singular value plot of the plant step-response matrix
Plant: F100 engine linear model with 2 inputs and
2 outputs [N1,P7]

Fig 6.4(a) Minimum singular value plot of the plant step-response matrix
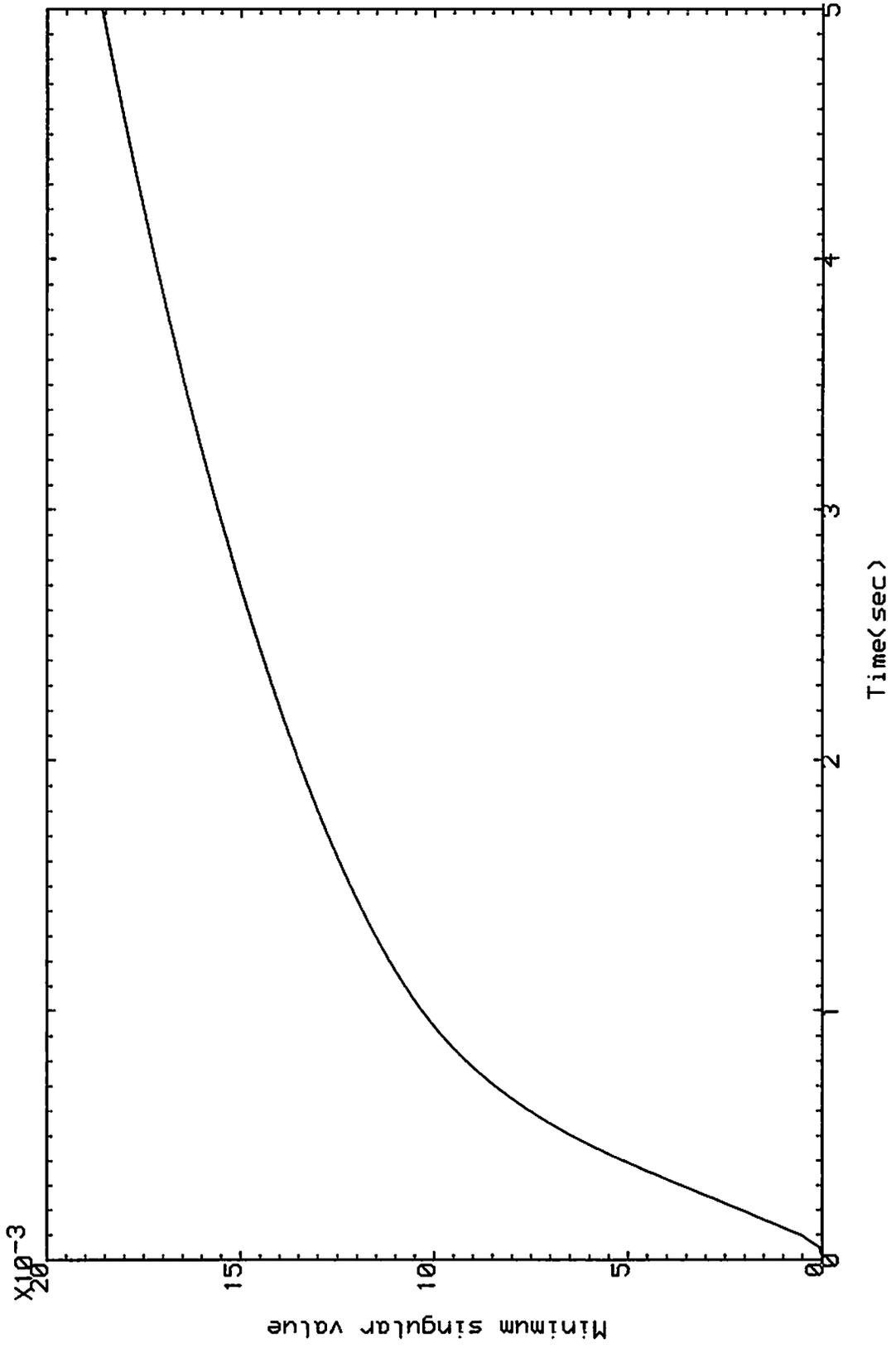Plant: F100 engine linear model with 2 inputs and
2 outputs [N1,FTIT]

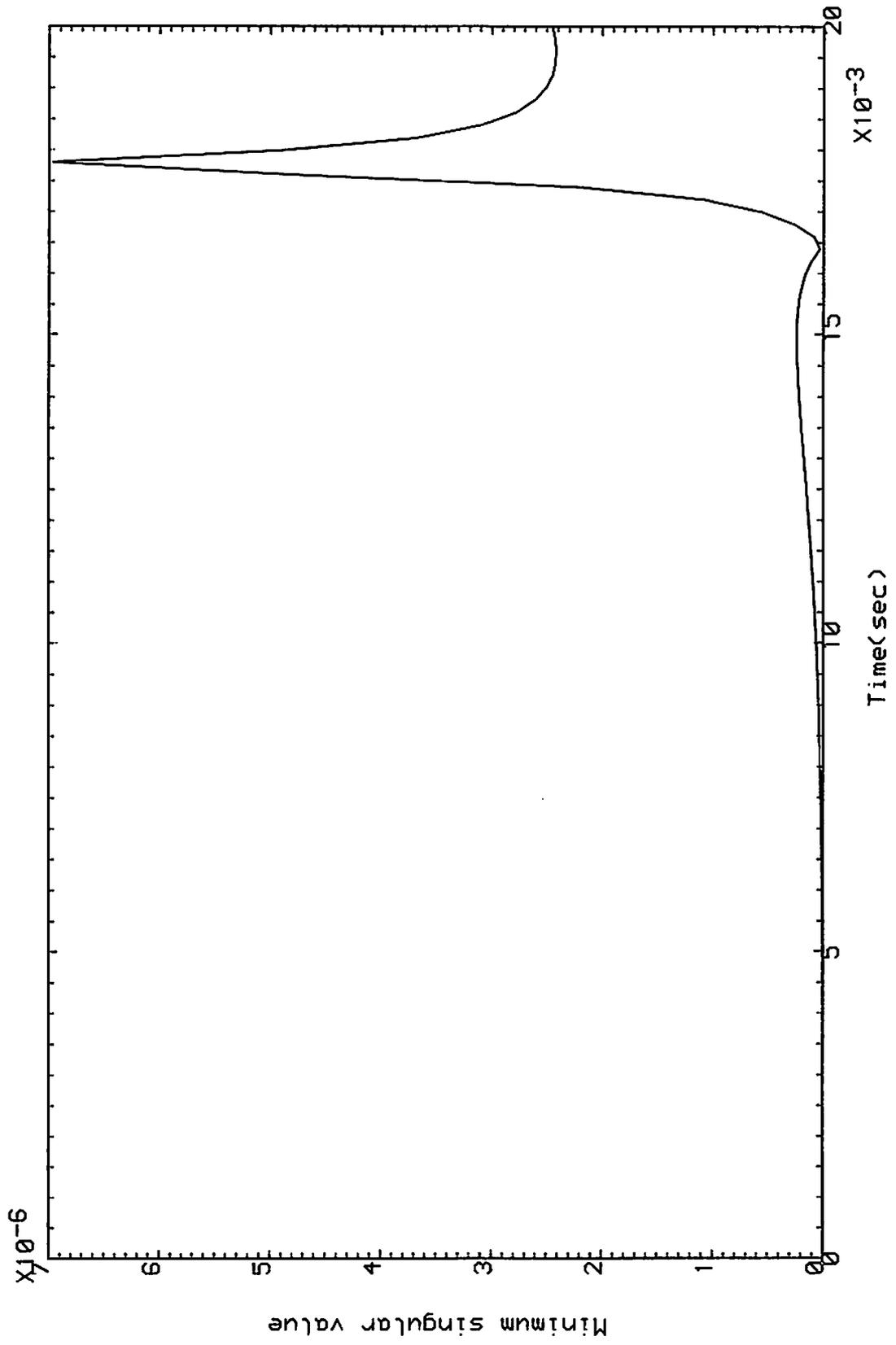Fig 6.4(b) Minimum singular value plot of the plant step-response matrix
Plant: F100 engine linear model with 2 inputs and
2 outputs [N1,FTIT]
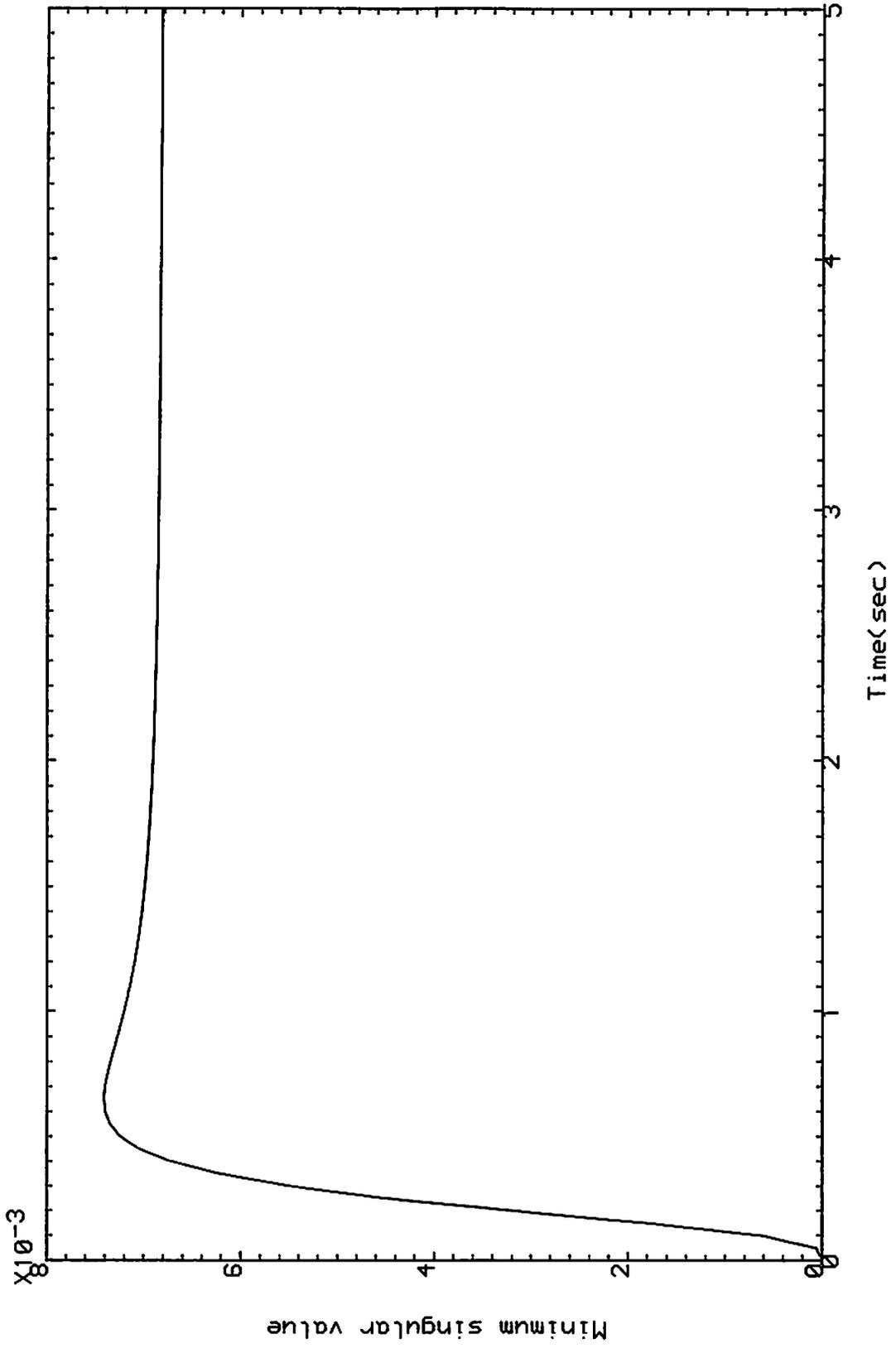
Fig. 6.5 Minimum singular value plot of the plant step-response matrix
Plant: F100 engine linear model with 2 inputs and
2 outputs [P7,FTIT]

Fig 6.6   Responses of F100 engine under digital
self-selecting PID control
Structure 1: [N1,P7] & [N1,FTIT]

Fig 6.7   Manipulated variables of F100 engine
under digital self-selecting PID control
Structure 1 : [N1,P7] & [N1,FTIT]

Fig 6.8 Responses of F100 engine under digital
self-selecting PID control
Structure 2: [P7,N1] & [P7,FTIT]
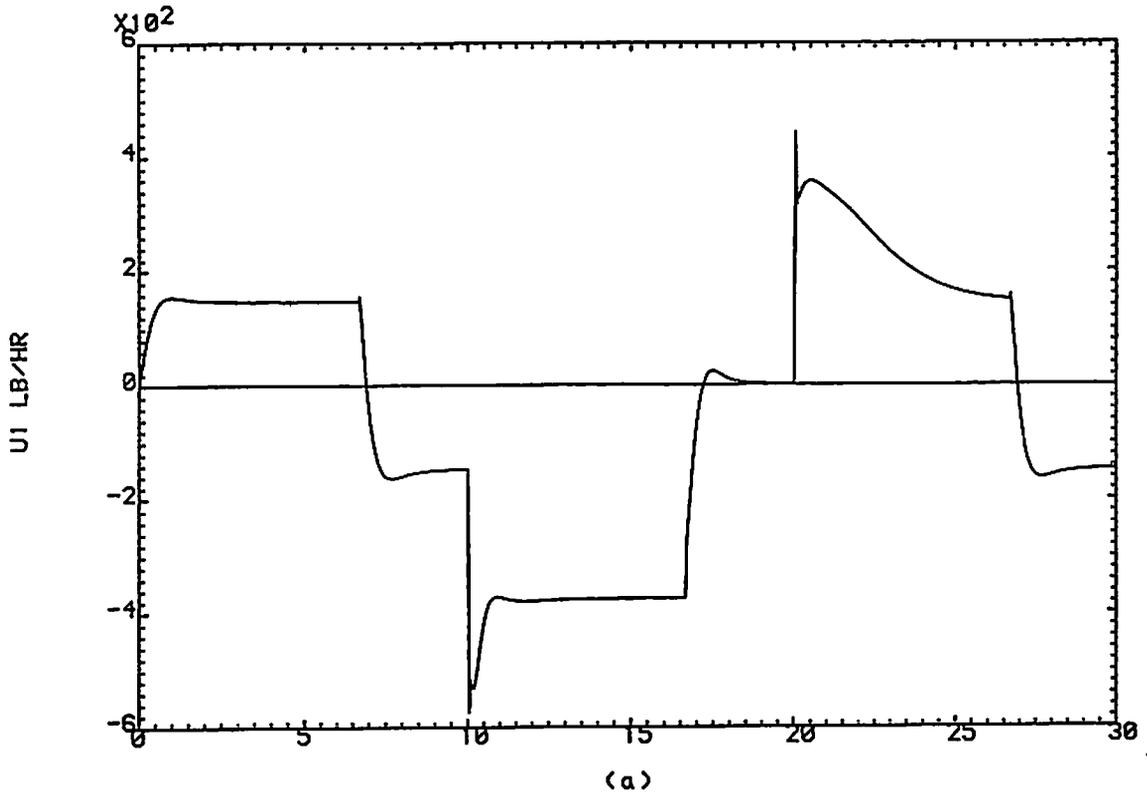
Fig 6.9 Manipulated variables of F100 engine
under digital self-selecting PID control
Structure 2: [P7,N1] & [P7,FTIT]

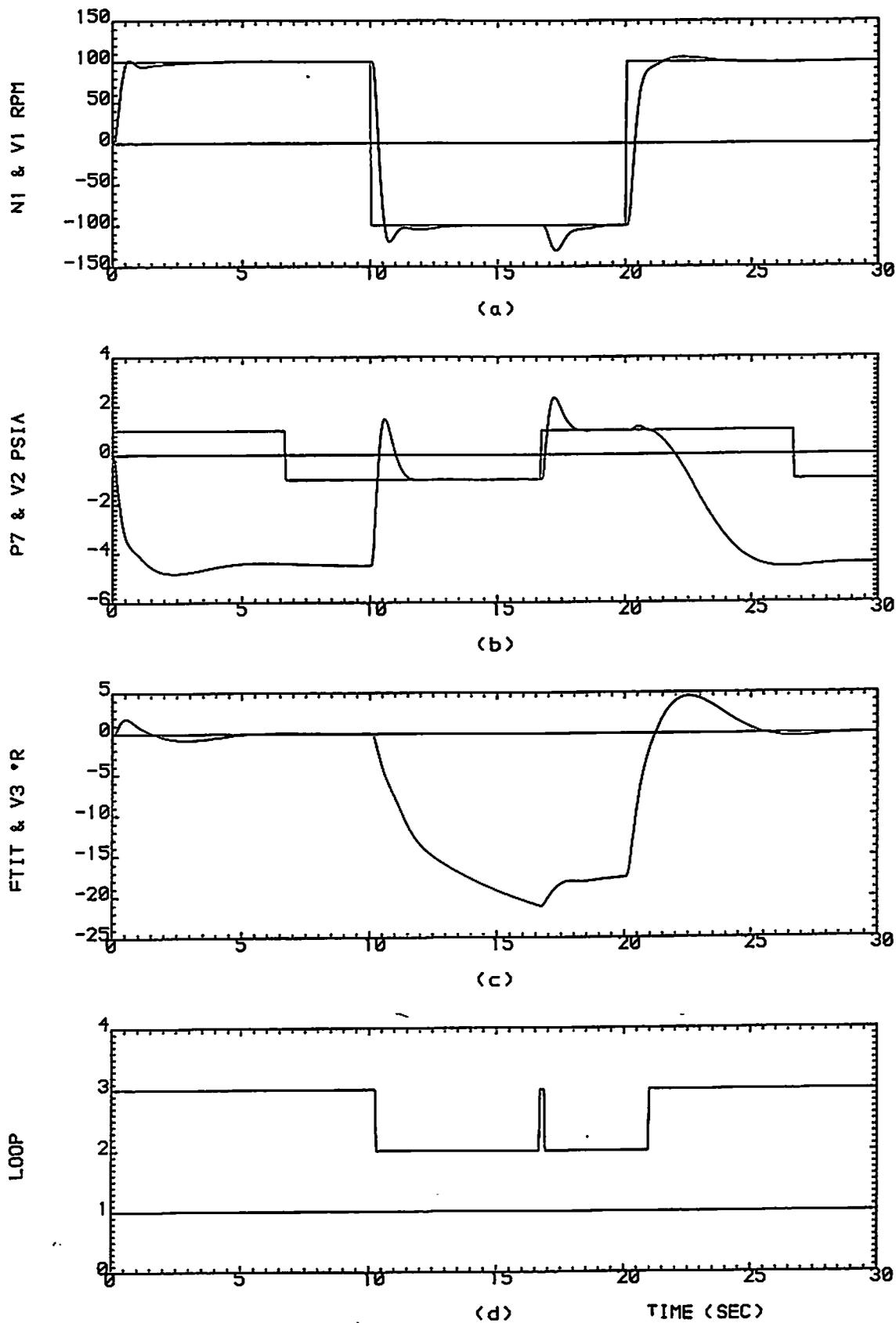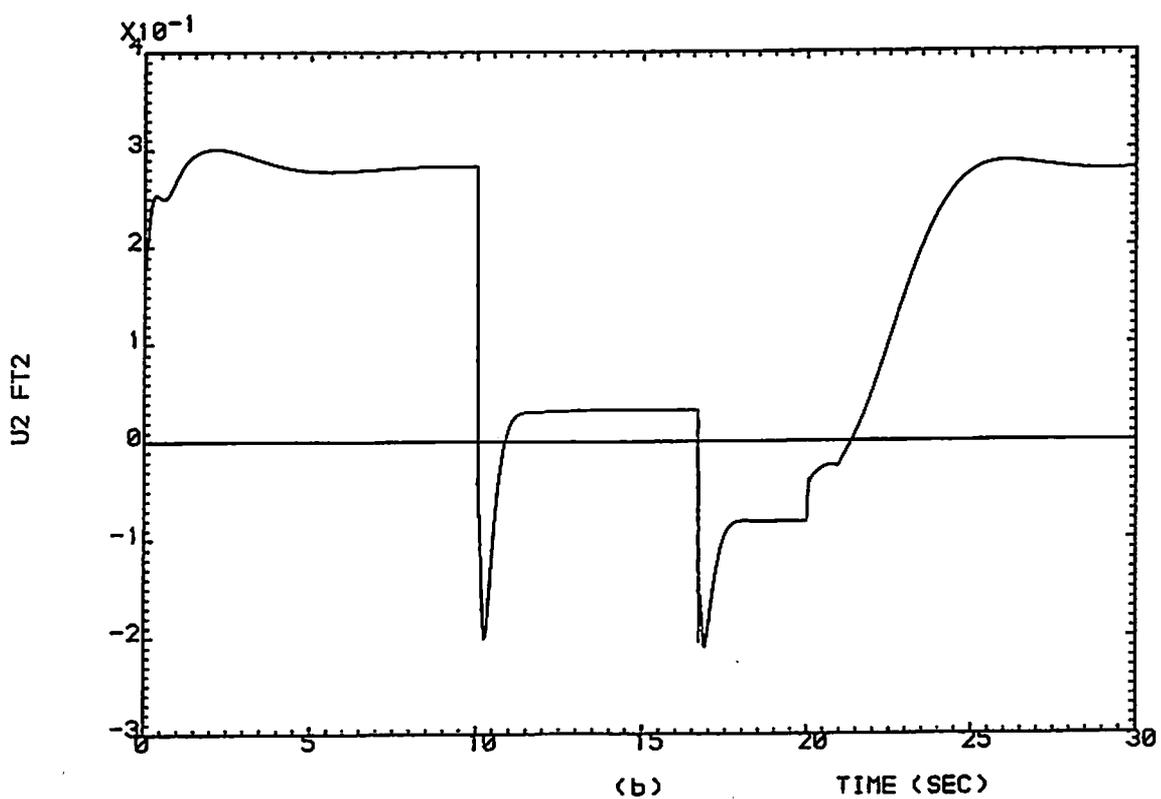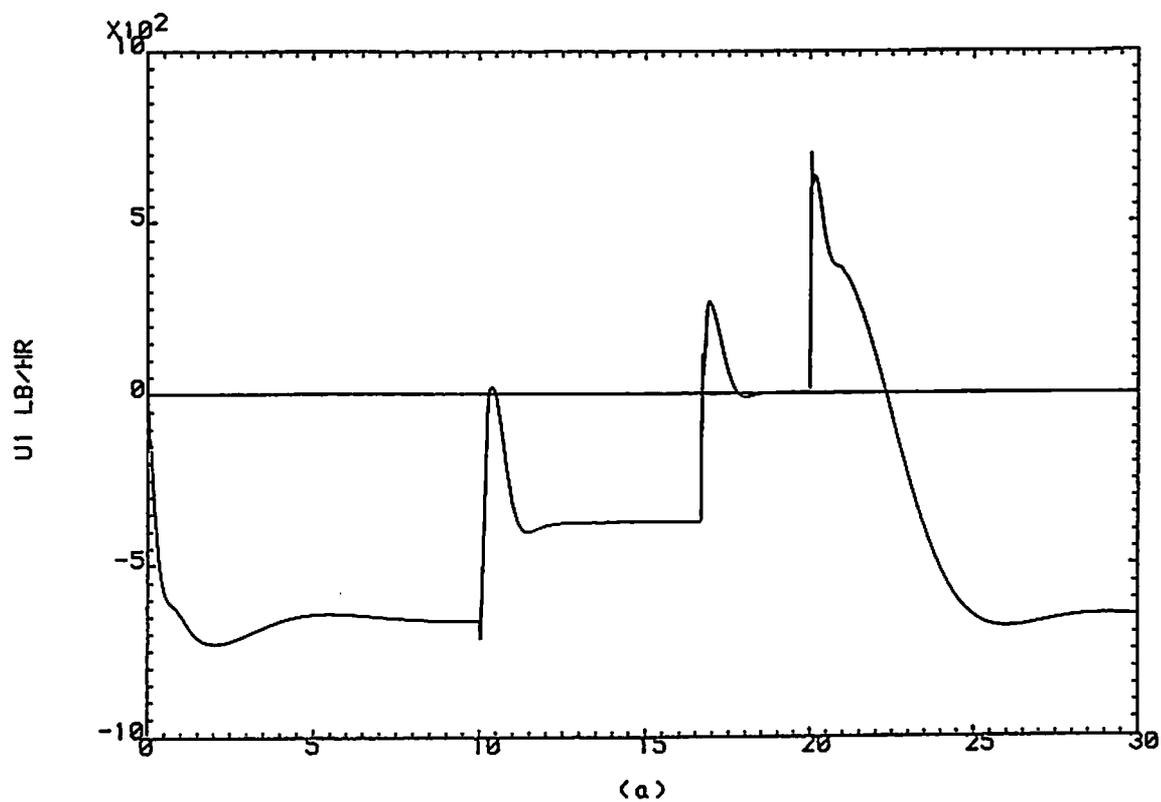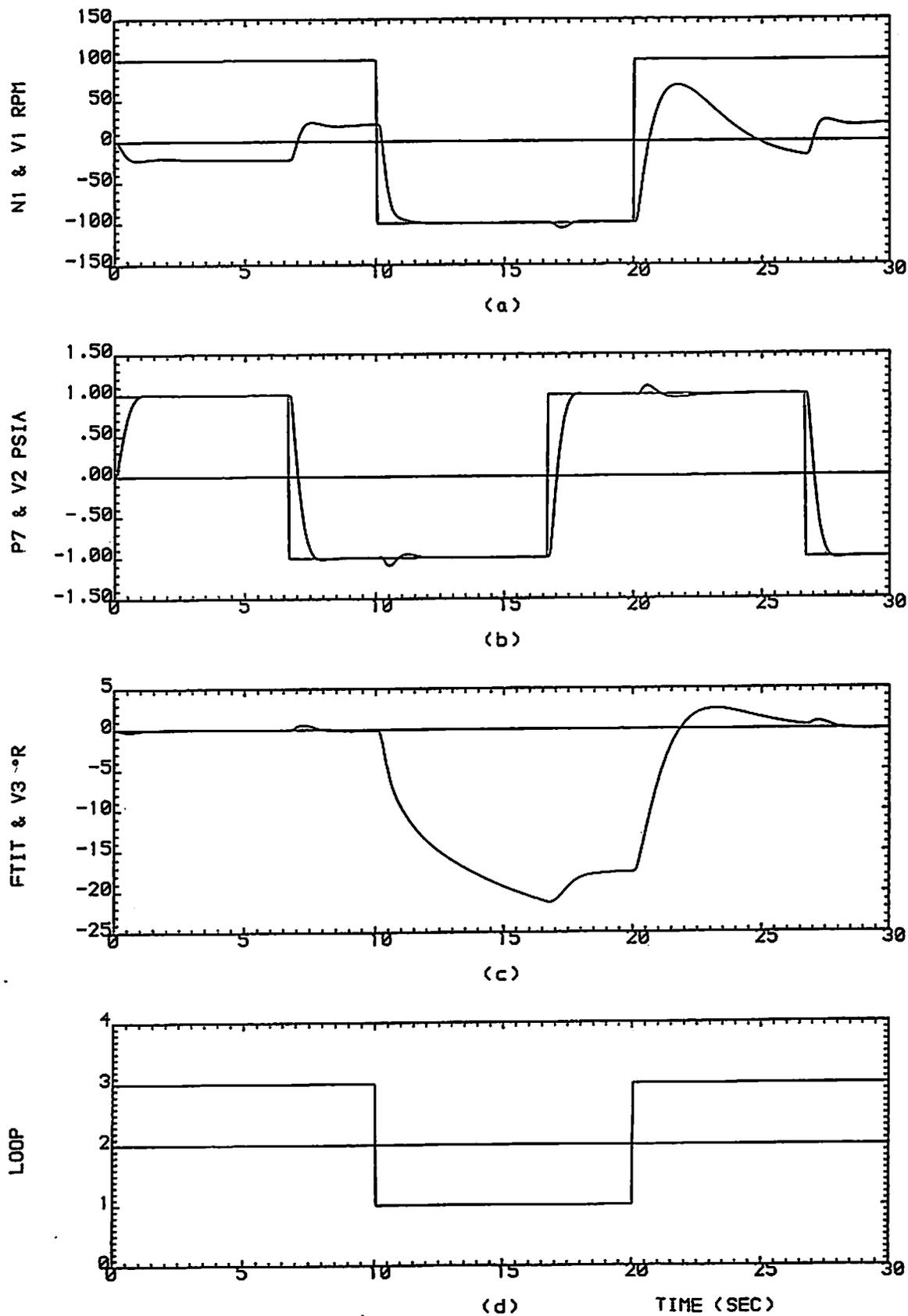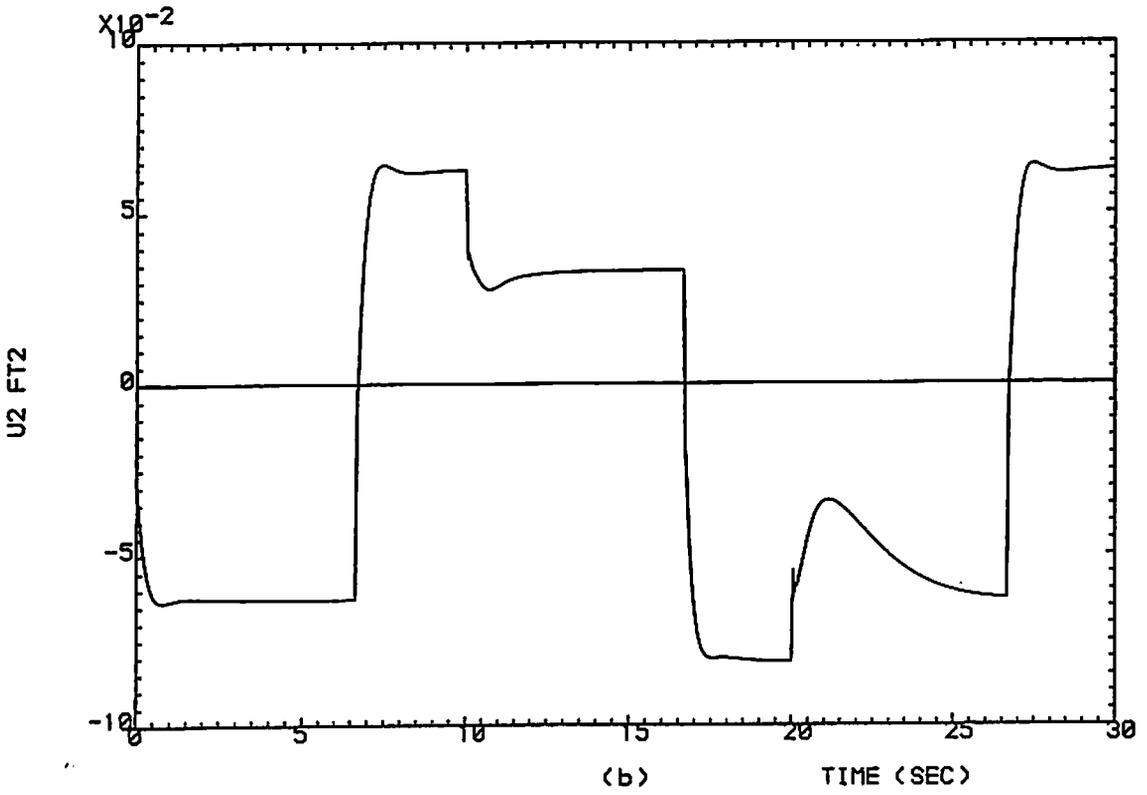CHAPTER 7

DESIGN OF DIGITAL SUPERVISORY SELF-SELECTING CONTROLLERS

FOR LINEAR MULTIVARIABLE PLANTS

WITH MORE OUTPUTS THAN INPUTS

7.1 Introduction

In Appendix 6, some of the dynamical peculiarities of self-selecting control systems are described. Such peculiarities indicate the richness of the possible responses of higher-order multivariable self-selecting control systems and the difficulty of analysing such systems. They thus stimulate and justify the investigation of more powerful controllers which gurantee limit tracking in steady states and produce well-regulated dynamical behaviour of complete self-selecting control systems. Therefore, in this chapter, a new approach to the stability augmentation of self-selecting controllers is considered. Using this approach, it is expected that self-selecting controllers are provided with the enhanced dynamical stability.

Firstly, the dynamical tracking characteristics of variable-structure self-selecting control systems are investigated based upon the approach of Grujić and Porter (1980). Thus, important foundamental properties such as a solution concept, equilibrium states, steady states, asymptotically stable tracking, and perfect/nearly perfect dynamical limit tracking are established, where the proofs of

Propositions and Theorems are given in Appendix 2. Then, a synthesis approach to supervise the operation of digital self-selecting controller by observing error vectors and controller switchings is developed, and the controller synthesised in this approach is called a digital supervisory self-selecting controller. It is shown that the controller has three operational modes (ie Normal mode, Loop-excluded mode, and Loop-fixed mode) and two assessment blocks (ie Tracking assessment and Correct/Incorrect loop assessment). Next, the tracking performance and the stability of complete systems incorporating digital supervisory self-selecting controllers are investigated. Finally, the effectiveness of such supervisory controllers is illustrated by designing a supervisory self-selecting controller for a plant which is simple but which neverthless has shown dynamical peculiarities such as limit-cycle oscillations in Appendix 6. It is shown that the limit-tracking behaviour of the plant under the action of a supervisory self-selecting controller, tuned as before such that limit-cycle oscillations occur, exhibits no limit-cycle oscillations but rather stable dynamical limit-tracking.

7.2 Analysis

The linear multivariable Class I plants (Definition 4.3) under consideration are assumed to be governed on the continuous-time set $T = [0, +\infty)$ by state and output equations of the respective forms

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad (7.1)$$

and

$$y(t) = Cx(t) , \qquad (7.2)$$

where the state vector $x(t) \in R^n$, the input vector $u(t) \in R^m$, the output vector $y(t) \in R^p$ (p > m), the plant matrix $A \in R^{n \times n}$ whose eigenvalues all lie in the open left-half plane C^-, the input matrix $B \in R^{n \times m}$, and the output matrix $C \in R^{p \times n}$.

The transfer-function matrix is

$$G(s) = C(sI-A)^{-1}B \qquad (7.3)$$

and the steady-state transfer-function matrix

$$G = G(0) = -CA^{-1}B \in R^{p \times m} \qquad (7.4)$$

is known from open-loop tests performed on the plant (Appendix 1). It is assumed that

$$\text{rank } G = m , \qquad (7.5)$$

and therefore that, by applying the order-reduction technique (Algorithm 5.1), p-m+1 subsets of plant outputs to be controlled by corresponding set-point tracking controllers are obtained in the form

$$Y_1 = \{y_{s_1}, \ldots, y_{s_{m-1}}, y_{t_1}\}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$Y_r = \{y_{s_1}, \ldots, y_{s_{m-1}}, y_{t_r}\}$$

$$\quad (7.6)$$

Here, $r = p-m+1$, the index set of all the control loops is $I_r = \{1,2,\ldots,r\}$, the index set of all the outputs $I = \{1,2,\ldots,p\}$, the index set of permanently controlled outputs $I^* = \{s_1,s_2,\ldots,s_{m-1}\}$, and the index set of intermittently controlled outputs $I\backslash I^* = \{t_1,t_2,\ldots,t_r\}$. Then, the parts of the plant which correspond to such subsets can be called sub-plants. The corresponding sub-output vectors of these sub-plants are

$$y^{(i)}(t) = \begin{bmatrix} y_{s_1}(t) \\ \cdot \\ \cdot \\ \cdot \\ y_{s_{m-1}}(t) \\ y_{t_i}(t) \end{bmatrix} = C^{(i)}x(t) \in R^m \quad (i = 1,2,\ldots,r)$$

$$\quad (7.7)$$

where each of sub-output matrices $C^{(i)} \in R^{m \times n}$ $(i=1,2,\ldots,r)$ consists of s_1th,$\ldots$,s_{m-1}th and t_ith rows of the output matrix C.

Furthermore, it is assumed that the introduction of integral action for each subset Y_i, $i \in I_r$ preserves stabilisability and therefore that (Porter and Power (1970), Power and Porter (1970))

$$\text{rank } G^{(i)} = m \ , \tag{7.8}$$

where the sub-plant transfer-function matrix

$$G^{(i)}(s) = C^{(i)}(sI-A)^{-1}B \ , \tag{7.9}$$

and the sub-plant steady-state transfer-function matrix

$$G^{(i)} = G^{(i)}(0) = -C^{(i)}A^{-1}B \in R^{m \times m} \tag{7.10}$$

is obtained from equation (7.4).

In the case of digital self-selecting control systems with lowest-wins strategies, it is convenient to consider the behaviour of such plants on the discrete-time set $T_T = \{0,T,2T,\ldots,kT,\ldots\}$. This behaviour is governed by state and output equations of the respective forms (Kwakernaak and Sivan (1972))

$$x_{k+1} = \Phi x_k + \Psi u_k \tag{7.11}$$

and

$$y_k^{(i)} = \Gamma^{(i)} x_k \ , \tag{7.12}$$

where $x_k = x(kT) \in R^n$, $u_k = u(kT) \in R^m$, $y_k^{(i)} = y^{(i)}(kT) \in R^m$, $i \in I_r$,

$$\Phi = \exp(AT) \ , \tag{7.13}$$

$$\Psi = \int_0^T \exp(At)B \, dt \, , \tag{7.14}$$

$$\Gamma^{(i)} = C^{(i)} \, , \quad i \in I_r \, , \tag{7.15}$$

and $T \in R^+$ is the sampling period.

Furthermore, it is assumed that the overall set-point vector is

$$v = [v_1, \ldots, v_p]^T \in R^p \, , \tag{7.16}$$

the overall plant output vector is

$$y(kT) = [y_1(kT), \ldots, y_p(kT)]^T \in R^p \, , \tag{7.17}$$

and that the overall error vector is

$$e(kT) = [e_1(kT), \ldots, e_p(kT)]^T$$

$$= v - y(kT) \in R^p. \tag{7.18}$$

Then, the sub-error vector is

$$e^{(i)}(kT) = v^{(i)} - y^{(i)}(kT) \in R^m \, , \tag{7.19}$$

where the sub-plant set-point vector $v^{(i)} \in R^m$.

Here, it is noted in view of equations (7.6) and (7.7) that

$$v^{(i)} = E^{(i)}v = \begin{bmatrix} v_{s_1} \\ \cdot \\ \cdot \\ \cdot \\ v_{s_{m-1}} \\ v_{t_i} \end{bmatrix}, \qquad (7.20)$$

$$y^{(i)}(kT) = E^{(i)}y_k = \begin{bmatrix} y_{s_1}(kT) \\ \cdot \\ \cdot \\ \cdot \\ y_{s_{m-1}}(kT) \\ y_{t_i}(kT) \end{bmatrix}, \qquad (7.21)$$

and

$$e^{(i)}(kT) = E^{(i)}e_k = \begin{bmatrix} e_{s_1}(kT) \\ \cdot \\ \cdot \\ \cdot \\ e_{s_{m-1}}(kT) \\ e_{t_i}(kT) \end{bmatrix}, \qquad (7.22)$$

where $E^{(i)} \in R^{m \times p}$ consists of s_1th,...,s_{m-1}th and t_ith rows of a unit matrix I_p.

It is evident from equation (7.22) that all the sub-error vectors $e^{(i)}(kT)$ $(i=1,2,...,r)$ have $m-1$ common elements $e_{s_1}(kT),...,e_{s_{m-1}}(kT)$. Therefore, the lowest-wins strategies need to compare only the remaining r scalar signals which are not common in $e^{(1)}(kT),...,e^{(r)}(kT)$ to select the control loop. Furthermore, it follows that the index set $J(kT)$ of lowest errors and the loop index $\ell(kT)$ of the actually selected loop are defined by the respective forms

$$J(kT) = \{j : e_{t_j}(kT) = \min_{i \in I_r} e_{t_i}(kT)\} \qquad (7.23)$$

and

$$\ell(kT) \in J(kT) \subset I_r . \qquad (7.24)$$

The self-selecting controller is governed on the discrete-time set T_T by equations of the form

$$u(kT) = K_P^{(\ell(kT))} e^{(\ell(kT))}(kT) + K_I^{(\ell(kT))} z(kT) \qquad (7.25)$$

and

$$z((k+1)T) = z(kT) + Te^{(\ell(kT))} , \qquad (7.26)$$

where the controller state vector $z(kT) \in R^m$, and the controller matrices $K_P^{(\ell(kT))} \in R^{m \times m}$ and $K_I^{(\ell(kT))} \in R^{m \times m}$ are chosen from the sets $\{K_P^{(1)}, \ldots, K_P^{(r)}\}$ and $\{K_I^{(1)}, \ldots, K_I^{(r)}\}$, respectively. It is assumed that each separate closed-loop system is asymptotically stable, where there clearly exist r sepatate closed loops when $\ell(kT) = \text{const} \in I_r$. This assumption is justified by the functional controllability of each separate output, as indicated in the conditions (7.8), so that the controller design methodology described in Chapter 6 is applicable.

Since equations (7.23) and (7.24) decide which controller should be used at each sampling instant, controller switching may occur. In controller switching from loop index ℓ_1 to ℓ_2 at

time kT, the following three types of switching logic are considered:

(i) *Without bumpless transfer*

$$z(kT) = z((k-1)T) \qquad (7.27)$$

(ii) *With bumpless transfer*

$$u(kT) = u((k-1)T) \qquad (7.28)$$

and

$$z(kT) = K_I^{(\ell_2)-1} \{ u((k-1)T) - K_P^{(\ell_2)} e^{(\ell_2)}(kT) \} \qquad (7.29)$$

(iii) *Instantaneous perturbation*

$$z(kT) = z((k-1)T) + \Delta z , \qquad (7.30)$$

where Δz is bounded, ie there exists M such that

$$\|\Delta z\| < M < \infty , \qquad (7.31)$$

where $\|\cdot\|$ is the euclidean norm of $\cdot$.

It is clear from equations (7.27) to (7.31) that the switching logic (iii) includes (i) and (ii) as special cases. Therefore, the analysis is carried out only for the switching logic (iii).

The equations (7.11), (7.12), (7.16), (7.25), and (7.26) that govern the behaviour of the self-selecting control system can be written in the forms

$$\dot{x}((k+1)T) = A_{\ell(kT)}(T)x(kT) + B_{\ell(kT)}(T)v^{(\ell(kT))} \qquad (7.32)$$

where

$$x(kT) = \begin{bmatrix} x(kT) \\ z(kT) \end{bmatrix} \in R^{n+m} , \qquad (7.33a)$$

$$A_{\ell(kT)}(T) = \begin{bmatrix} \Phi - K_P^{(\ell(kT))}\Psi\Gamma^{(\ell(kT))} & , & K_I^{(\ell(kT))}\Psi \\ -T\Gamma^{(\ell(kT))} & , & I_m \end{bmatrix}$$
$$\in R^{(n+m)x(n+m)} , \qquad (7.33b)$$

$$B_{\ell(kT)}(T) = \begin{bmatrix} K_P^{(\ell(kT))}\Psi \\ TI_m \end{bmatrix} \in R^{n+m} , \qquad (7.33c)$$

and

$$v^{(\ell(kT))} = v^{(\ell(kT))} \in R^m . \qquad (7.33d)$$

It is clear that the complete closed-loop digital self-selecting control system is governed by equations (7.32) with the lowest-wins control equations (7.23), (7.24), (7.30), and (7.31). Therefore, let a solution of the governing equations of the self-selecting control system be denoted by

$$\chi(kT;x_0;v) = \begin{bmatrix} x(kT;x_0;v) \\ z(kT;x_0;v) \end{bmatrix} \quad , \quad \chi(0;x_0;v) \equiv x_0$$

where $\chi(kT;x_0;v)$ is the motion of the controlled plant and $z(kT;x_0;v)$ is the corresponding motion of the self-selecting controller on the discrete-time set T_T. The following results can then be obtained.

Definition 7.1

1 Equilibrium state

A state $x_e(T) \in R^{n+m}$ is an equilibrium state of the self-selecting control system if and only if, for each separate closed-loop system,

$$\chi(kT;x_e(T);v) = x_e(T) \quad , \quad \forall kT \in T_T \quad ;$$

2 Steady state

A state $x_s(T) \in R^{n+m}$ is a steady state of the self-selecting control system if and only if

$$\chi(kT;x_s(T);v) = x_s(T) \quad , \quad \forall kT \in T_T \quad .$$

Definition 7.2: Index sets of correct and incorrect loops

In a steady state, the index set $I_c(v)$ such that

$$I_c(v) = \{i \in I_r : y_{t_i} = v_{t_i} \, , \quad t_i \in I \backslash I^*\}$$

is the set of correct loops and the set $I_r\backslash I_c(v)$ the set of incorrect loops.

The existence of nonempty $I_c(v)$ is guaranteed by Theorem 5.1.

Proposition 7.1

In a steady state, if $i \in I_r\backslash I_c(v)$ then

$$y_{t_i} < v_{t_i} \quad .$$

Proposition 7.2

The self-selecting control system has $\#(I_c(v))$ steady states for every v, including multiplicity, where $\#(\cdot)$ means the number of elements in the set $\cdot$.

It is desirable in such self-selecting control systems that the entire set of plant outputs exibits the dynamical counterpart of limit-tracking (Definition 5.1). Therefore, a rigorous analysis is carried out for dynamical limit tracking in the following (Grujić and Porter (1980)).

Definition 7.3: Target set

The set

$$S(v) = \{x : c_{s_1}^T x = v_{s_1}, \ldots, c_{s_{m-1}}^T x = v_{s_{m-1}},$$
$$c_{t_i}^T x = v_{t_i}, \ i \in I_c(v),$$
$$c_{t_j}^T x < v_{t_j}, \ j \in I_r\backslash I_c(v), \ z \in R^m\} \subset R^{n+m}$$

or

$$\mathfrak{S}(v) = \{x : C^{(i)}x = v^{(i)}, \quad i \in I_c(v),$$

$$c_{t_j}^T x < v_{t_j}, \quad j \in I_r \backslash I_c(v), \quad z \in R^m\} \subset R^{n+m}$$

is the target set of the self-selecting control system.

In the sequel, the distance of a point $x \in R^{n+m}$ from a set $W \subset R^{n+m}$ is denoted by $\rho[x , W] = \inf [\|x - x^*\| : x^* \in W]$.

Definition 7.4

The self-selecting control system ehibits

(i) stable tracking on T_T if and only if for every $(x_0,v) \in R^{n+m}xR^p$ and for every $\epsilon > 0$, there exist $\mathcal{L} \subseteq \mathfrak{S}(v)$, $\mathcal{L} = \mathcal{L}(v)$ and $\delta = \delta(\epsilon,x_0,v,\mathcal{L}) > 0$, such that $\rho[x_0 , \mathcal{L}(v)] \leq \delta$ implies that

$$\rho[\chi(kT;x_0;v) , \mathcal{L}(v)] \leq \epsilon$$

for all $kT \in T_T$;

(ii) globally asymptotically stable tracking if and only if both it exhibits stable tracking and for every $(x_0,v) \in R^{n+m}xR^p$

$$\lim_{k\to\infty} \rho[\chi(kT;x_0;v) , \mathcal{L}(v)] = 0 .$$

More precisely, there exists some $\gamma > 0$ such that if

$\rho[\chi(k_1T;x_0;v)$, $\mathcal{L}(v)] \leq \gamma$, then for every $\bar{\epsilon} > 0$, there exists a positive $\Delta k^* = \Delta k^*(\bar{\epsilon},\gamma,x_0,v)$ such that

$$\rho[\chi(kT;x_0;v) \; , \; \mathcal{L}(v)] \leq \bar{\epsilon}$$

for all $kT \geq (k_1 + \Delta k^*)T$;

(iii) state-bounded tracking if and only if the solution $\chi(kT;x_0;v)$ is bounded for every $(x_0,v) \in R^{n+m}xR^p$;

(iv) perfect dynamical limit tracking or state-bounded globally asymptotically stable tracking if and only if it exhibits both state-bounded tracking and globally asymptotically stable tracking.

It is noted that, in Definition 7.4, (ii) implies (i), (iv) implies (i), (ii), and (iii), and that all characteristics are uniform. In the following, the practical version of perfect dynamical limit tracking is defined.

Definition 7.5: Nearly perfect dynamical limit tracking

The self-selecting control system exhibits nearly perfect dynamical limit tracking if and only if both it exhibits state-bounded tracking and for every $(x_0,v) \in R^{n+m}xR^p$ and for every $e_{th} > 0$, there exists $k^* = k^*(e_{th},x_0,v)$ such that for $kT \geq k^*T$,

$$e(kT) \geq -e_\epsilon$$

and

$$\|e^{(\ell(kT))}(kT)\| \leq e_{th} \, ,$$

where $e_\epsilon = [e_{th}, \ldots, e_{th}]^T \in R^p$.

Proposition 7.3

Definition 7.4(iv) implies Definition 7.5 in the sense that perfect dynamical limit tracking implies nearly perfect dynamical limit tracking.

7.3 Synthesis

The block diagram of the digital supervisory self-selecting controller is shown in Fig 7.1. In order to obtain not only enhanced stability but also both dynamic and static limit tracking, the controller is equipped with two special operational modes (ie Loop-excluded self-selecting control mode and Loop-fixed control mode) in addition to the normal self-selecting control mode (Section 7.2). Then, these three control modes are called, respectively, 'Normal mode', 'Loop-excluded mode', and 'Loop-fixed mode'. The transition from one mode to another is decided in two assessment blocks (ie 'Tracking assessment block' and 'Correct/Incorrect loop assessment block'). Such operation of the controller is initialised whenever the set-point command vector changes. In the following, the hierarchical structure of the controller is defined.

Definition 7.6

1 Level of self-selecting control

The level of self-selecting control is the number of excluded loops and called η.

2 Index set of level η excluded loops and

   Index set of level η candidate loops

The index set of level η excluded loops is $I_{r,ex}^{\eta}$, whilst the index set of level η candidate loops is $I_r^{\eta} = I_r \backslash I_{r,ex}^{\eta}$.

3 Index set of level η lowest errors

The index set of level η lowest errors is

$$J^{\eta}(kT) = \{j : e_{t_j}(kT) = \min_{i \in I_r^{\eta}} e_{t_i}(kT)\} \ .$$

Proposition 7.4

(i) $I_{r,ex}^0 = \emptyset$

(ii) $I_r^0 = I_r$

(iii) $\#(I_{r,ex}^{\eta}) = \eta$

(iv) $\#(I_r^{\eta}) = r - \eta$

(v) $J^0(kT) = J(kT)$

Definition 7.7

1 Normal mode

The self-selecting control system is said to be under Normal mode if and only if the level of such self-selecting control is 0.

2 Loop-excluded mode

The self-selecting control system is said to be under Loop-excluded mode if and only if the level of such self-selecting control is $\eta \geq 1$.

3 Loop-fixed mode

The self-selecting control system is said to be under Loop-fixed mode if and only if

$$\ell(kT) = \ell_f = const \in I_r$$

irrespective of $J^\eta(kT)$.

Definition 7.8: Background computation of Normal mode and

Loop-excluded mode (Fig 7.2)

$$
\left.\begin{aligned}
t_0 &= (\text{Initialised time}) + T_s \\[6pt]
t_1 &= t_0 + T_o \\[6pt]
t_2 &= t_1 + T_o \\[6pt]
&\quad . \\
&\quad . \\
&\quad . \\
t_a &= t_{a-1} + T_o \\[6pt]
&\quad . \\
&\quad . \\
&\quad .
\end{aligned}\right\} \qquad (7.34)
$$

where T_s, the initial settling time, and T_o, the observation time, are chosen by the designer.

The following variables are defined on the discrete-time set $T_{T_o}(t_2) = \{t_2, t_3, \ldots, t_a, \ldots\}$ as long as the same operational mode continues. Such discrete-time set is re-initialised whenever either Normal or Loop-excluded mode operation begins either by the set-point change or the transition from the loop-fixed mode.

$$
\max(e_s)_a = \max_{\substack{kT \in [t_{a-1}, t_a) \\ i \in [1, m-1]}} e_{s_i}(kT) \qquad (7.35a)
$$

$$
\min(e_s)_a = \min_{\substack{kT \in [t_{a-1}, t_a) \\ i \in [1, m-1]}} e_{s_i}(kT) \qquad (7.35b)
$$

$$\text{mean}(e_s)_a = [\sum_{i=1}^{m-1} \{(\sum_{kT=t_{a-1}}^{t_a} Te_{s_i}(kT))/T_o\}]/(m-1) \qquad (7.35c)$$

$$\Delta_{max}(e_s)_a = \text{max}(e_s)_a - \text{mean}(e_s)_a \qquad (7.35d)$$

$$\Delta_{min}(e_s)_a = \text{mean}(e_s)_a - \text{min}(e_s)_a \qquad (7.35e)$$

$$\text{max}(e_t)_a = \text{max } e_{t_{\ell(kT)}}(kT) \qquad (7.36a)$$
$$kT \in [t_{a-1}, t_a)$$
$$\ell(kT) \in J^{\eta}(kT)$$

$$\text{min}(e_t)_a = \text{min } (e_{t_{\ell(kT)}}(kT) , e_{t_j}(kT))$$
$$kT \in [t_{a-1}, t_a)$$
$$\ell(kT) \in J^{\eta}(kT)$$
$$j \in I^{\eta}_{r,ex}$$

$$= \text{min } e_{t_i}(kT) \qquad (7.36b)$$
$$kT \in [t_{a-1}, t_a)$$
$$i \in I_r$$

$$\text{mean}(e_t)_a = (\sum_{kT=t_{a-1}}^{t_a} Te_{t_{\ell(kT)}}(kT))/T_o \qquad (7.36c)$$

$$\Delta_{max}(e_t)_a = \text{max}(e_t)_a - \text{mean}(e_t)_a \qquad (7.36d)$$

$$\Delta_{min}(e_t)_a = \text{mean}(e_t)_a - \text{mean}(e_t)_a \qquad (7.36e)$$

Definition 7.9: Tracking assessment

The tracking of Normal and Loop-excluded modes is assessed on the discrete-time set $T_{T_o}(t_3) = \{t_3, t_4, \ldots, t_a, \ldots\}$ to decide whether such modes should continue or should be transfered to Loop-fixed mode.

If

$$\max(e_s)_a \leq e_{th} \qquad (7.37a)$$

$$\min(e_s)_a \geq -e_{th} \qquad (7.37b)$$

$$\max(e_t)_a \leq e_{th} \qquad (7.37c)$$

$$\min(e_t)_a \geq -e_{th} \qquad (7.37d)$$

or if

$$\Delta_{max}(e_s)_a \leq \alpha\Delta_{max}(e_s)_{a-1} \qquad (7.38a)$$

$$\Delta_{min}(e_s)_a \leq \alpha\Delta_{min}(e_s)_{a-1} \qquad (7.38b)$$

$$\left|_{mean}(e_s)_a\right| \leq \alpha\left|_{mean}(e_s)_{a-1}\right| \qquad (7.38c)$$

$$\Delta_{max}(e_t)_a \leq \alpha\Delta_{max}(e_t)_{a-1} \qquad (7.38d)$$

$$\Delta_{min}(e_t)_a \leq \alpha\Delta_{min}(e_t)_{a-1} \qquad (7.38e)$$

$$\left|_{mean}(e_t)_a\right| \leq \alpha\left|_{mean}(e_t)_{a-1}\right| \qquad (7.38f)$$

where the error threshold e_{th} and the admissible convergence rate α $(0 < \alpha < 1)$ are to be chosen by the designer,

then the assessment is 'Convergent' and the controller continues to operate in the same mode during the next interval $kT \in [t_a, t_{a+1})$;

else the assessment is 'Non convergent', the mode change occurs, and the controller begins to operate in Loop-fixed mode.

It is clear from equation (7.34) and Definition 7.9 that such assessment is carried out for the first time at t_3 $(=T_s+2T_0)$ after the set-point change.

Definition 7.10: Background computation of Loop-fixed mode

(Fig 7.3)

$$\left. \begin{array}{l} \tilde{t}_0 = t_a + T_s \\ \\ \tilde{t}_1 = \tilde{t}_0 + T_0 \end{array} \right\} \tag{7.39}$$

where Loop-fixed mode begins at time t_a, T_s is the initial settling time, and T_0 is the observation time.

When the operational mode changes from Normal/Loop-excluded mode to Loop-fixed mode at time t_a, the actually selected loop is fixed by the form

$$\ell(kT) = \ell_f = \ell((k-1)T) \qquad \tilde{t}_0 \le kT \le \tilde{t}_1 . \tag{7.40}$$

However, during Loop-fixed mode operation, the lowest-wins strategy continues to operate. Therefore, for $i \in I_r$ and for $\tilde{t}_0 \leq kT \leq \tilde{t}_1$,

$$\Delta T_{int}(i\ ,\ kT) = \begin{cases} 1 & i \in J(kT) \\ 0 & i \notin J(kT) \end{cases} \qquad (7.41)$$

and

$$T_{int}(i) = \sum_{kT=\tilde{t}_0}^{\tilde{t}_1} T\Delta T_{int}(i\ ,\ kT)\ . \qquad (7.42)$$

It is clear from equation (7.42) that $T_{int}(i)$ is the time interval during which the index i belongs to $J(kT)$ $(\tilde{t}_0 \leq kT \leq \tilde{t}_1)$.

Definition 7.11: Correct/Incorrect loop assessment at time $\tilde{t}_1$

If $\exists i_1 \in I_r$, $i_1 \neq \ell_f$,

$$T_{int}(\ell_f) < \beta T_{int}(i_1)\ , \qquad \beta > 0\ , \qquad (7.43)$$

then the assessment is 'Incorrect loop';

else the assessment is 'Correct loop' and Loop-fixed mode continues to operate with the horizon-reinitialisation such that

$$\tilde{t}'_0 = \tilde{t}_1 \qquad\qquad (7.44a)$$

$$\tilde{t}'_1 = \tilde{t}_0 + T_o \qquad\qquad (7.44b)$$

$$\forall i \in I_r , \quad T'_{int}(i) = 0 . \qquad\qquad (7.44c)$$

It is clear from equation (7.39) that such assessment is carried out for the first time at $\tilde{t}_1$ ($=T_s + T_o$) after Loop-fixed mode begins.

Definition 7.12: Re-initialisation in case of 'Incorrect loop'
assessment in Loop-fixed mode (Fig 7.4)

The number of excluded loops increases by one. Therefore,

$$I^\eta_{r,ex} = I^{\eta-1}_{r,ex} + \{\ell_f\} \qquad\qquad (7.45a)$$

$$\#(I^\eta_r) = \#(I^{\eta-1}_r) - 1 \qquad\qquad (7.45b)$$

$$\#(I^\eta_{r,ex}) = \#(I^{\eta-1}_{r,ex}) + 1 \qquad\qquad (7.45c)$$

If $\eta \leq r-2$ ie $\#(I^\eta_r) = r-\eta \geq 2$ then Loop-excluded mode begins.

If $\eta = r-1$ ie $\#(I^\eta_r) = 1$ then only one candidate loop remains and Loop-fixed mode begins under such loop.

If $\eta = r$ ie $\#(I^\eta_r) = 0$ then there remains no candidate loop. In such case, Normal mode begins.

The case in which the final assessment $\eta = r$ happens at time T_f in the lowest stage shown in Fig 7.4, where

$$T_f = (r - 1)\{(T_s + 2T_o) + (T_s + T_o)\} + (T_s + T_o)$$
$$= (r - 1)(2T_s + 3T_o) + (T_s + T_o) . \qquad (7.46)$$

7.4 Performance

Firstly, the controller performance of Loop-fixed mode is discussed. Exponential stability and Correct/Incorrect loop assessment are verified.

Proposition 7.5: Exponential stability - Convergence property of Loop-fixed mode

In the system under Loop-fixed mode with the fixed loop index ℓ_f, for a given $\epsilon > 0$, there exists some k_1^* such that

$$\chi(kT;x_0;v)_{\ell(kT)=\ell_f} \in x_s^{\ell}f(T) + \epsilon B \qquad kT \geq k_1^*T$$

ie

$$\rho[\chi(kT;x_0;v) , x_s^{\ell}f(T)] \leq \epsilon \qquad kT \geq k_1^*T ,$$

where

$$x_s^{\ell}f(T) = \lim_{k \to \infty} \chi(kT;x_0;v)_{\ell(kT)=\ell_f}$$

and B is the Euclidean unit ball in R^{n+m}.

Proposition 7.6: Correct loop continuity in Loop-fixed mode

If $\ell_f \in I_c(v)$ then there exists k_1^* such that

$$\forall i \in I_r \backslash I_c(v), \quad T_{int}(\ell_f) \nless \beta T_{int}(i), \quad \beta > 0, \quad \tilde{t}_0 \geq k_1^* T.$$

Proposition 7.7: Incorrect loop detection in Loop fixed mode

If $\ell_f \in I_r \backslash I_c(v)$ then there exists k_1^* such that

$$\exists i \neq \ell_f, \quad T_{int}(\ell_f) < \beta T_{int}(i), \quad \beta > 0, \quad \tilde{t}_0 \geq k_1^* T.$$

Although $\beta > 0$ is enough to prove above propositions, too large β (for example, $\beta \gg 1$) might degrade the performance of assessment blocks in view of the inequalities in Propositions 7.6 and 7.7. Therefore, β of the order of 1 is recommended. Finally, a practically important theorem can be obtained.

Theorem 7.1

Suppose that the plant input is made to be bounded. Then, the supervisory self-selecting controller can attain nearly perfect dynamical limit tracking or perfect dynamical limit tracking for every (x_0, v).

It is noted that, as indicated in equation (7.46), the supervisory self-selecting controller can attain such tracking for every (x_0,v) within time T_f as long as the set-point command remains constant and plant variation does not occur.

7.5 Illustrative example

In order to illustrate the performance characteristics of digital supervisory self-selecting controllers, it is convenient to design a controller for a simple one-input/two-output plant and to analyse the resulting closed-loop characteristics by the phase-plane method. In fact, the plant is governed by state and output equations of the respective forms

$$\left.\begin{array}{l} \dot{x}(t) = -x(t) + u(t) \\ y_1(t) = 2x(t) \\ y_2(t) = 4x(t) \end{array}\right\} . \qquad (7.47)$$

For this plant, the self-selecting control system ehibited stable responses (with or without sliding motion) or limit-cycle oscillations depending upon the controller gains and the controller switching logic (Appendix 6).

In order to demonstrate the enhanced dynamical stability of the proposed supervisory self-selecting controller, the controller gains and the switching logic are chosen as for the case of limit-cycle oscillations. Thus, the controller gains are

therefore that perfect dynamical limit tracking has been achieved.

7.6 Conclusion

Rigorous theoretical foundations for the analysis of the dynamical properties of digital self-selecting control systems have been established. Based upon these foundations, a new technique for supervising the operation of self-selecting controllers has been developed. The resulting digital supervisory self-selecting controller has three operational modes (ie Normal mode, Loop-excluded mode, and Loop-fixed mode). According to the judgements of assessment blocks (ie Tracking assessment and Correct/Incorrect loop assessment), which observe error vectors, controller switchings, and lowest-wins strategies, the controller changes the operational mode so that perfect or nearly perfect dynamical limit tracking can be achieved.

An illustrative example has shown that the digital supervisory self-selecting controller possesses enhanced stability and that perfect dynamical limit tracking can be achieved even for the case in which limit-cycle oscillations occurred under the action of a non-supervisory self-selecting controller.
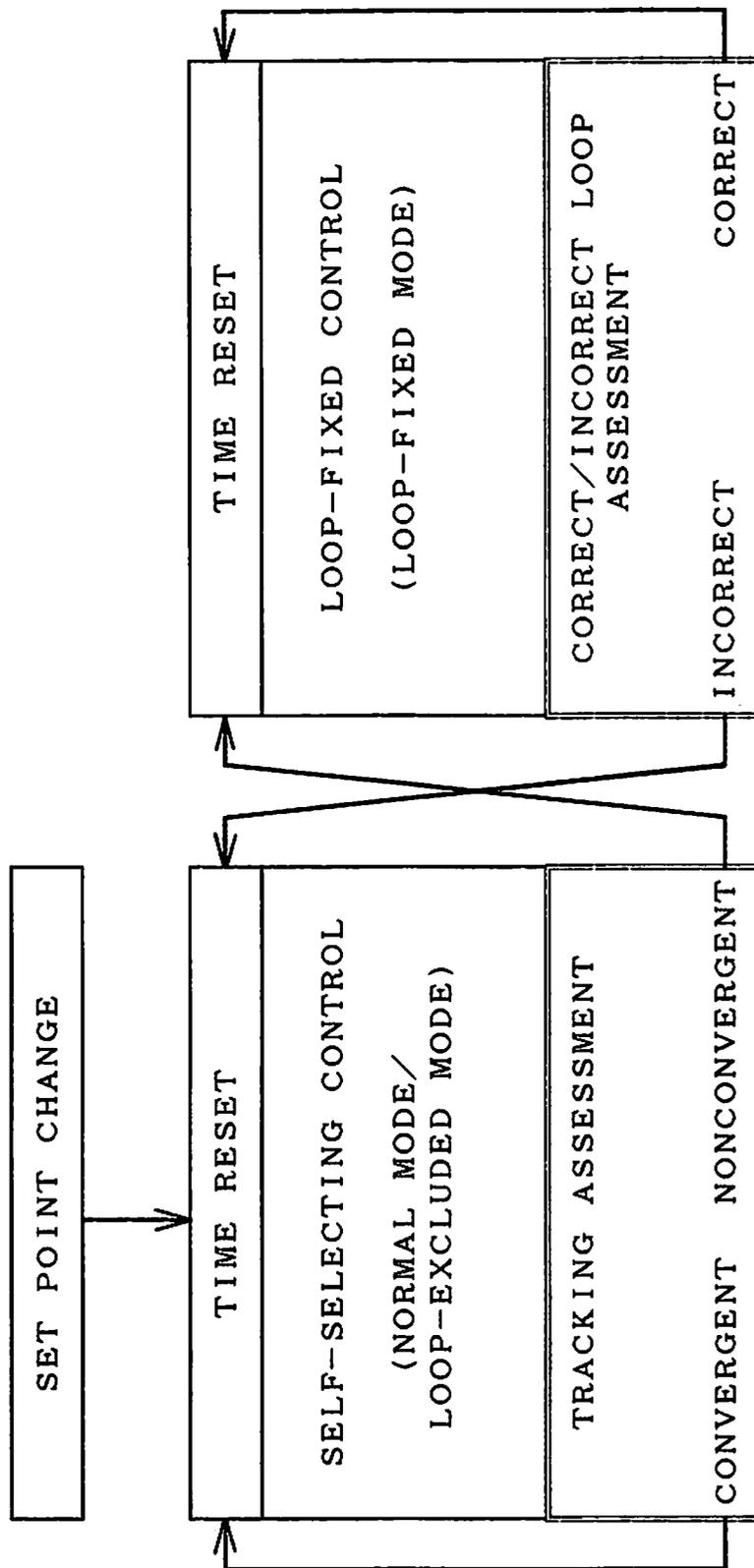
Fig 7.1 The block diagram of supervisory self-selecting controller

$e_{t_{2\ (kT)}}(kT)$

0

1 e_{max}

2 e_{max}

a-1 e_{max}

a e_{max}

1 e_{min}

2 e_{min}

a-1 e_{min}

a e_{min}

(Initialised time) t_0 t_1 t_2 t_{a-2} t_{a-1} t_a
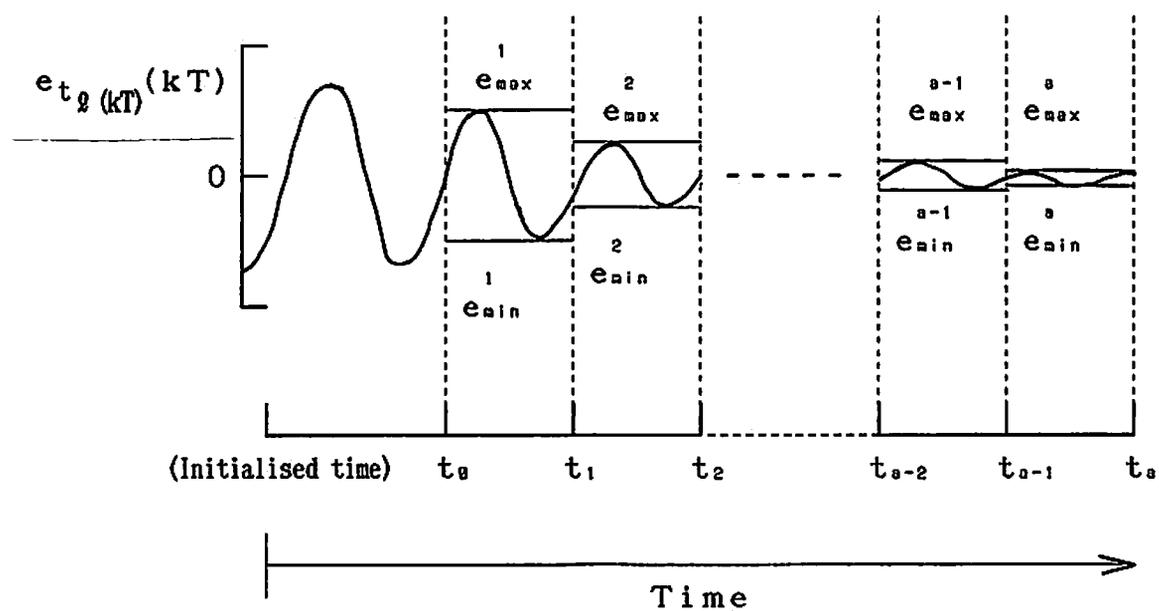
Time

Fig 7. 2 Discrete-time set of
Normal and Loop-excluded mode

t_0 $\widetilde{t}_0$ $\widetilde{t}_1$
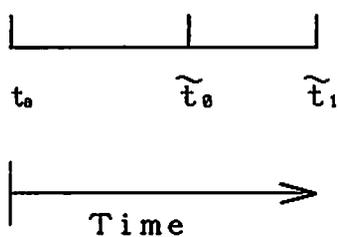
Time

Fig 7. 3 Discrete-time set
of Loop-fixed mode

Set-point change

Normal mode
Level 0   (In case of plant variation during mode transfer)
'Con'   'No Con'

Loop-fixed mode
'Corr'   'In Cor'

Loop-excluded mode
Level 1
'Con'   'No Con'

Loop-fixed mode
'Corr'   'In Cor'

.
.
.

Loop-excluded mode
Level r-2
'Con'   'No Con'

Loop-fixed mode
'Corr'   'In Cor'

Loop-fixed mode
(Loop-excluded mode)
Level r-1
'Corr'   'In Cor'

Abbreviation

'Con'	:	'Convergent'
'No Con'	:	'Non convergent'
'Corr'	:	'Correct loop'
'In Cor'	:	'Incorrect loop'

Fig 7.4   Tree diagram of Control mode
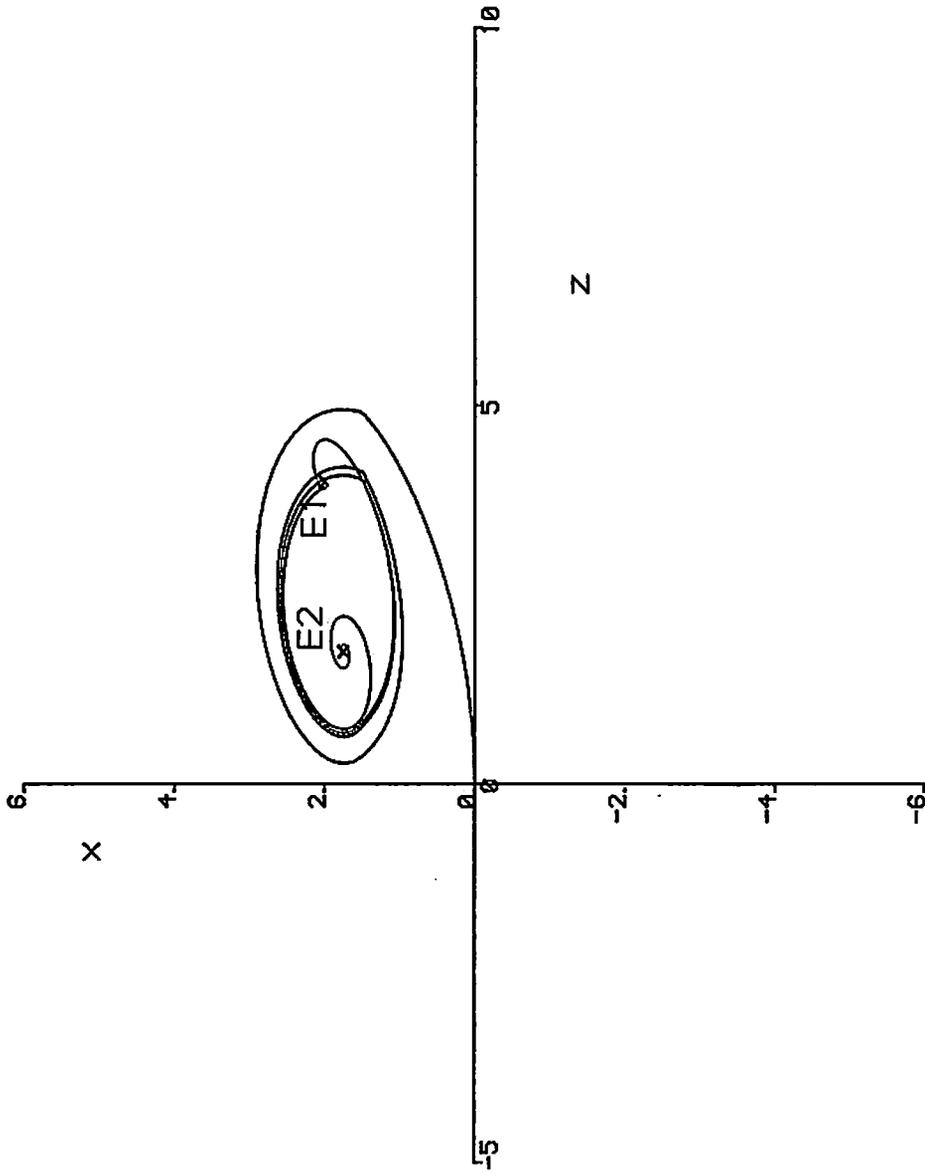
Fig 7.5   Phase trajectory under Supervisory self-selecting control.
Controller switching without bumpless transfer
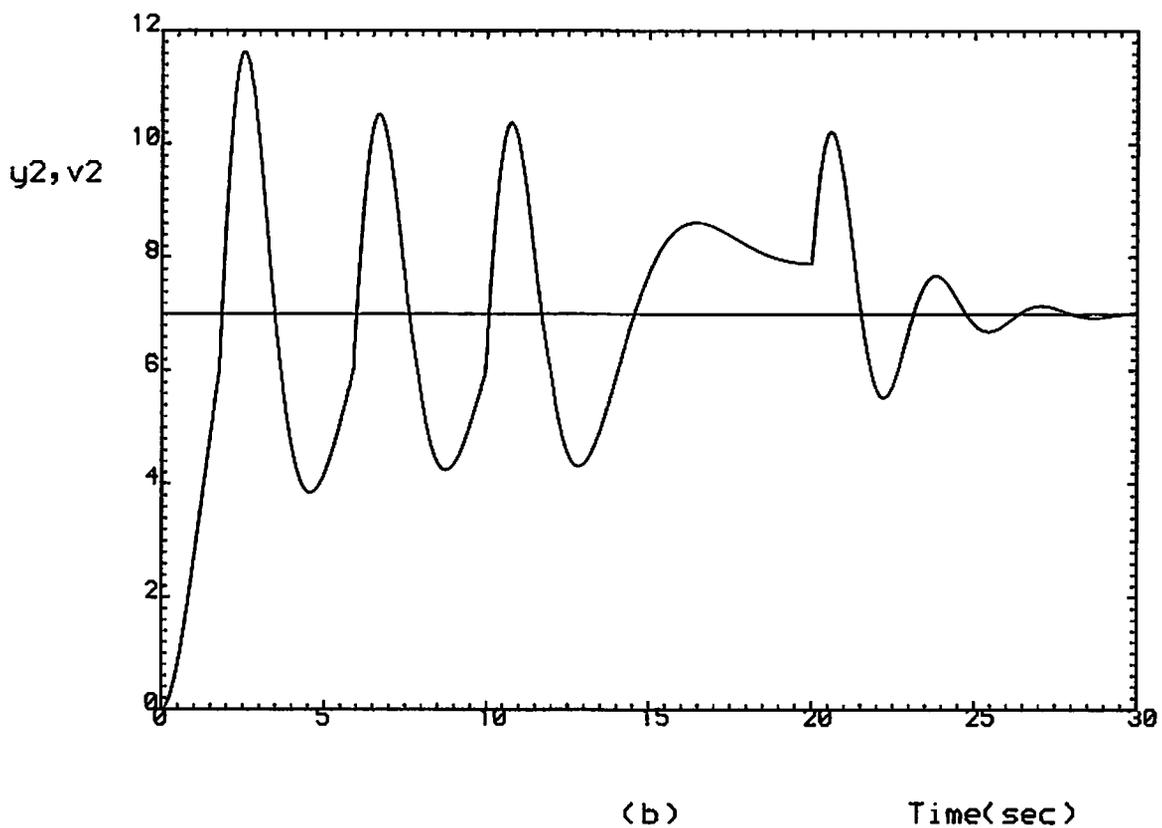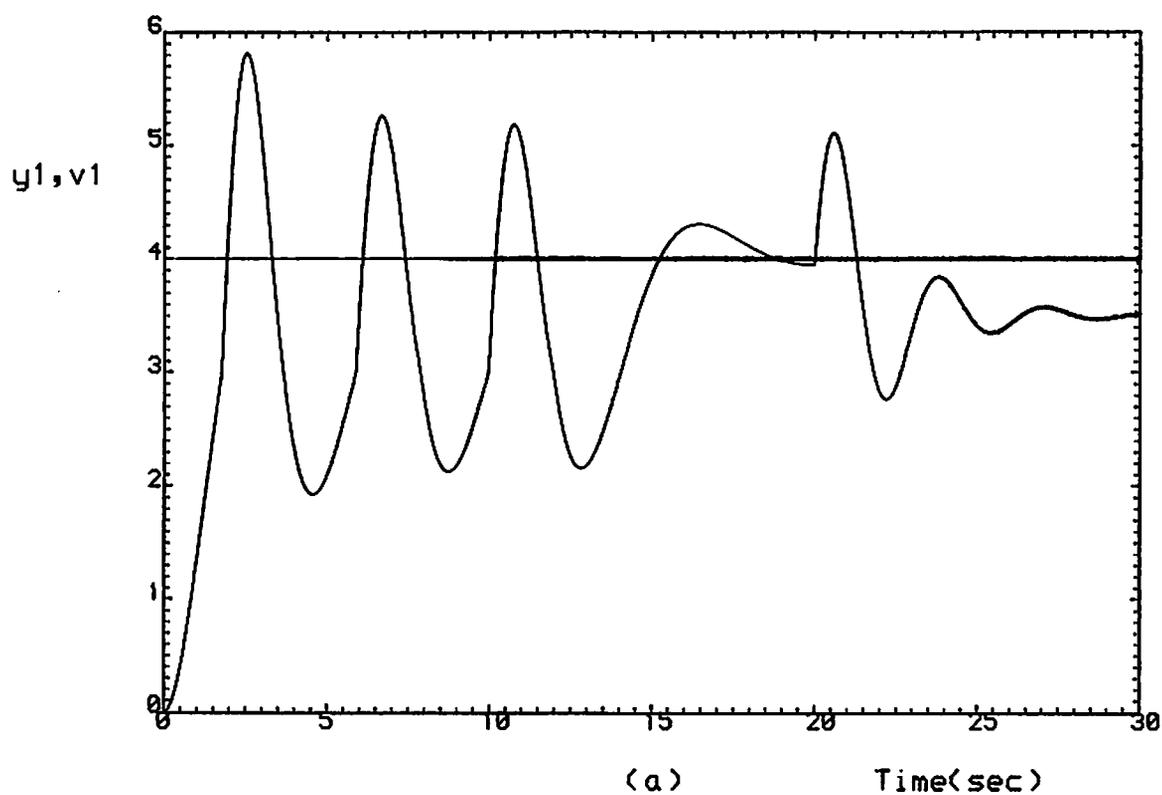
Fig 7.6   Closed-loop responses of the plant under
           digital supervisory self-selecting control
           Controller switching without bumpless transfer

INPUT

(a)   Time(sec)

LOOP

(b)   Time(sec)
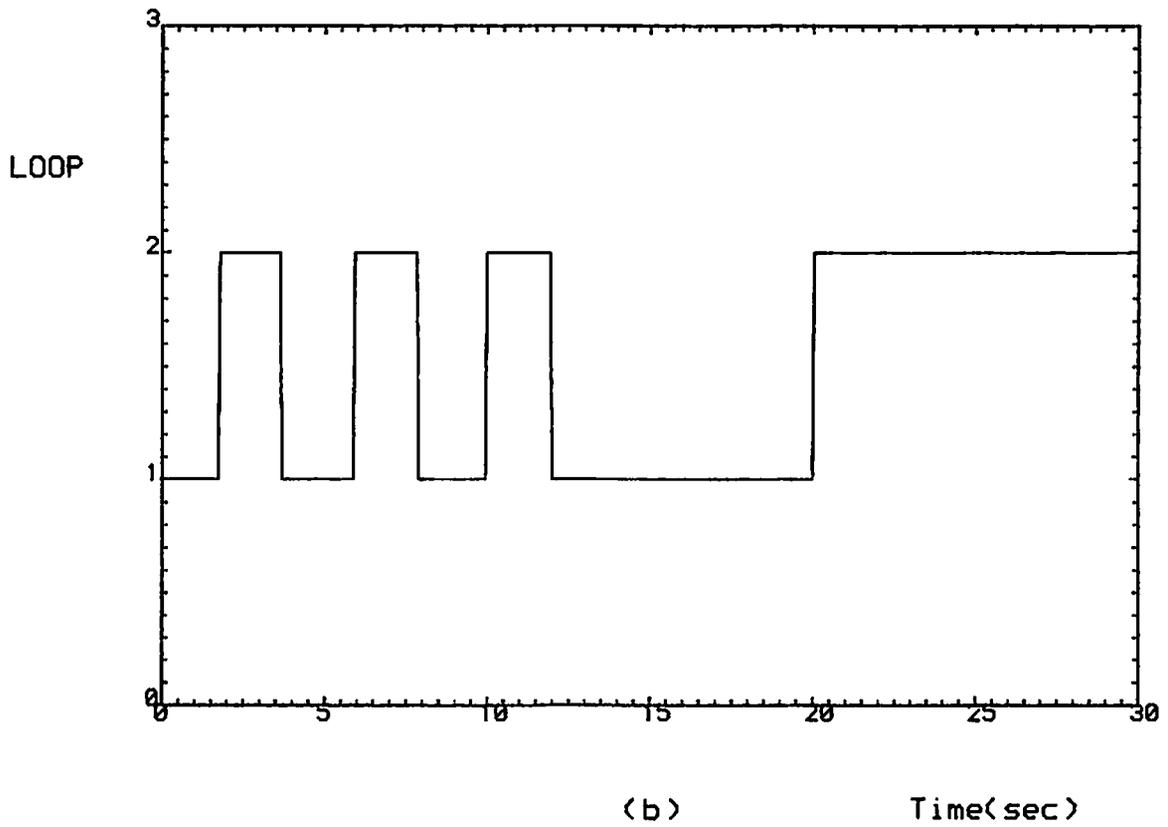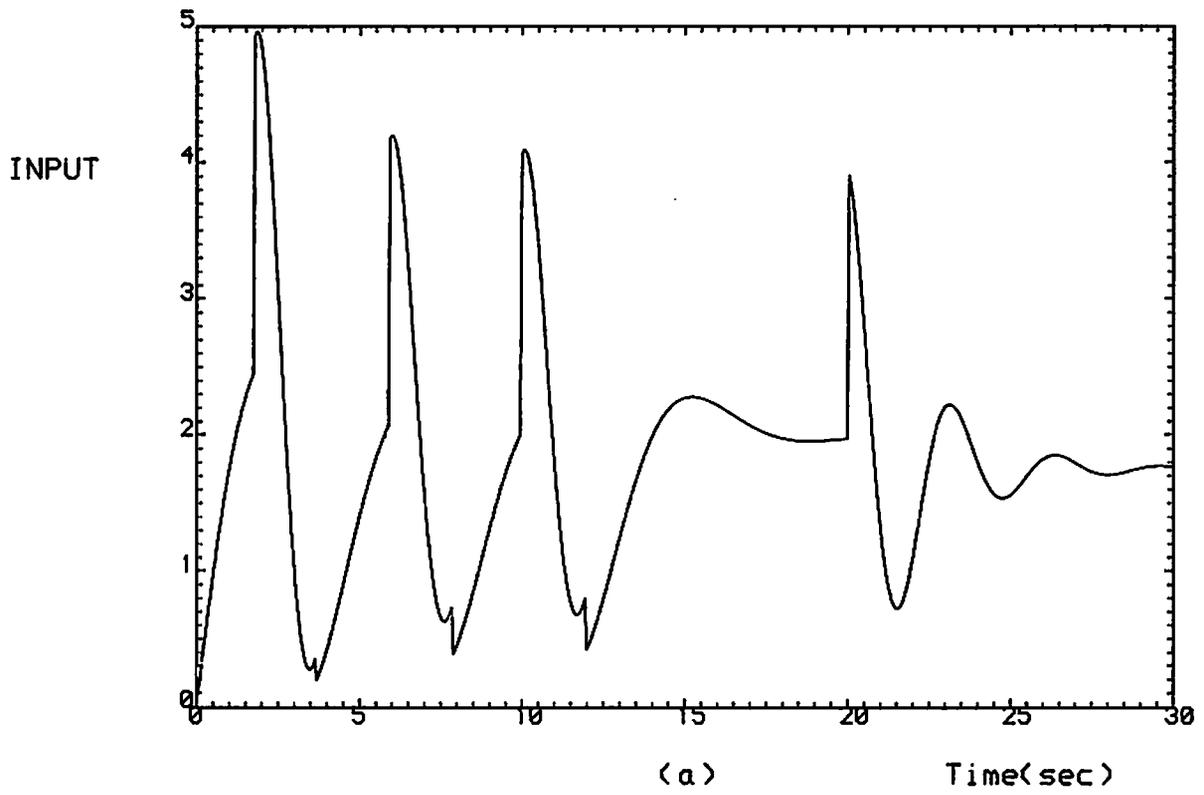
Fig 7.7   Input and loop index under
           digital supervisory self-selecting control
           Controller switching without bumpless transfer

PART IV

ROBUSTNESS OF TRACKING SYSTEMS

CHAPTER 8

ROBUSTNESS OF SET-POINT TRACKING SYSTEMS

8.1 Introduction

Controllers are *robust* when they function (ie operate with acceptable performance) in the presence of significant plant uncertainties such as unknown disturbances and plant variations. Since such uncertainties may exist in practical applications, it is desirable that controllers are robust. Therefore, a robustness investigation is carried out in this chapter to specify the uncertainties with which the controllers described in Part II can cope.

In the following sections, the robustness to unknown disturbances of tunable digital set-point tracking controllers is assessed at first. Then, the robustness to plant variations of such controllers is assessed. In this assessment, a very important Theorem 1 (Porter and Khaki-Sedigh (1989) (Appendix 7)) is utilised to characterise the admissible plant perturbations that can be tolerated by digital set-point tracking PID controllers.

8.2 Robustness of tunable digital set-point tracking PID controllers

8.2.1 Robustness in the face of unknown disturbances

The robustness to unknown constant disturbances of tunable

digital set-point tracking PID controllers (Chapter 2) can be readily investigated. Thus, the state and output equations (2.1) and (2.2) are modified to incorporate constant disturbances on the plant. Hence,

$$\dot{x}(t) = Ax(t) + Bu(t) + d \qquad (8.1)$$

and

$$y(t) = Cx(t) , \qquad (8.2)$$

where the vectors $x(t)$, $u(t)$, and $y(t)$ and the matrices A, B, and C are defined as before. It is assumed that the constant disturbance vector $d \in R^n$ is unknown. The behaviour of such plants on the discrete-time set $T_T = \{0,T,2T,\ldots,kT,\ldots\}$ is governed by state and output equations of the form (Kwakernaak and Sivan (1972))

$$x_{k+1} = \Phi x_k + \Psi u_k + \Theta d \qquad (8.3)$$

and

$$y_k = \Gamma x_k , \qquad (8.4)$$

where the vectors x_k, u_k, and y_k are defined as before, the matrices Φ, Ψ, and Γ are defined in equations (2.14) to (2.16), and

$$\Theta = \int_0^T \exp(At)\, dt \ . \tag{8.5}$$

The state and output equations of such plants under the action of error-actuated digital PID controllers governed on the discrete-time set T_T by control-law equations of the form (2.17) assume the forms

$$\begin{bmatrix} x_{k+1} \\ z_{k+1} \\ f_{k+1} \end{bmatrix} = \begin{bmatrix} \Phi - T\Psi K_1 \Gamma - \Psi K_3 \Gamma & , & T\Psi K_2 & , & -\Psi K_3 \\ -T\Gamma & , & I_m & , & 0 \\ -\Gamma & , & 0 & , & 0 \end{bmatrix} \begin{bmatrix} x_k \\ z_k \\ f_k \end{bmatrix} + \begin{bmatrix} T\Psi K_1 + \Psi K_3 \\ T I_m \\ I_m \end{bmatrix} v$$

$$+ \begin{bmatrix} \Theta \\ 0 \\ 0 \end{bmatrix} d \tag{8.6}$$

and

$$y_k = [\ \Gamma\ ,\ 0\ ,\ 0\ ] \begin{bmatrix} x_k \\ z_k \\ f_k \end{bmatrix} . \tag{8.7}$$

Therefore, provided only that T, K_1, K_2, and K_3 are such that all the eigenvalues of the closed-loop plant matrix in equation (8.6) lie in the open unit disc D^-,

$$\lim_{k \to \infty} \Delta z_k = \lim_{k \to \infty} \{z_{k+1} - z_k\} = 0 \tag{8.8}$$

and therefore

$$\lim_{k \to \infty} e_k = 0 \tag{8.9}$$

so that set-point tracking occurs simultaneously with disturbance rejection.

Such disturbance rejection properties are illustrated by the simulation results shown in Figs 8.1 and 8.2. In this simulation, the plant is the five-input/five-output linear F100 engine model at Intermediate power condition (Appendix 3), the digital PID controller is designed and tuned as before (Example in Chapter 2), and the set-point vector for the outputs is v = $[126, 93.4, 14.5, 1.78, 1.97]^T$ so that the thrust change is 500 lb. Furthermore, the constant disturbance vector is described by

$$\left. \begin{array}{l} d(i) = 0 \quad , \quad i \in [1,33] \ , \quad i \neq 2 \\ d(2) = -100 \end{array} \right\} , \qquad (8.10)$$

where $d(i)$ is the ith element of $d \in R^{33}$. This choice is made to simulate the horsepower extraction. These results indicate the excellent disturbance rejection and set-point tracking behaviour of the plant under the action of unknown constant disturbances.

8.2.2 Robustness in the face of plant variations

The robustness to plant variations of tunable digital set-point tracking PID controllers can now be assessed. In this study, the five-input/five-output linear F100 engine models (Appendix 3) are used as the nominal and actual plants.

Firstly, the controller is designed for a model obtained at Intermediate power condition (Power lever angle = 83 deg) - ie the nominal plant. The excellent set-point tracking behaviour of the F100 engine under the action of an error-actuated digital PID controller tuned such that T = 0.05 sec, $\Lambda(0.05)\Pi$ = diag{0.05, 0.2, 0.1, 0.1, 0.1}, Σ = 50.0I_5, and $\Lambda(0.05)\Delta$ = 0.001I_5 is shown in Figs 8.3 and 8.4. In this case, the set-point vector for the outputs is v = [126, 93.4, 14.5, 1.78, 1.97]^T so that the thrust change is 500 lb.

Next, in order to examine the robustness of this controller, the controller is now applied to another linear F100 engine model obtained at the different power condition (Power lever angle = 67 deg) - ie the actual plant. The steady-state transfer function matrices of the nominal plant and the actual plant are given in the forms

$$
G_n(0) = \begin{bmatrix}
0.37904 & 1238.8 & -28.508 \\
0.30777 & 660.79 & -2.8675 \\
0.20602\text{E}-01 & -39.863 & 0.25947 \\
0.15944\text{E}-02 & -12.168 & 0.38479\text{E}-01 \\
0.90309\text{E}-01 & 210.94 & -1.7403
\end{bmatrix}
$$

$$
\begin{bmatrix}
-9.2619 & -57.405 \\
-25.646 & -46.221 \\
-0.76283 & -6.8275 \\
-0.33542\text{E}-01 & -0.44527 \\
2.2101 & 12.248
\end{bmatrix} \tag{8.11}
$$

and

$$
G_a(0) = \begin{bmatrix}
0.40780 & 1220.2 & -30.646 \\
0.32555 & 197.87 & -1.9798 \\
0.27484E\text{-}01 & -6.8768 & 0.83052E\text{-}01 \\
0.24994E\text{-}02 & -7.8017 & 0.24802E\text{-}01 \\
0.95167E\text{-}01 & 72.565 & -1.2269
\end{bmatrix}
$$

$$
\begin{bmatrix}
-10.843 & -64.279 \\
-19.076 & -37.008 \\
-0.61430 & -5.5979 \\
-0.15957E\text{-}01 & -0.29314 \\
1.8282 & 10.459
\end{bmatrix} . \quad (8.12)
$$

By Theorem 1 (Porter and Khaki-Sedigh (1989) (Appendix 7)), the spectrum of the perturbation matrix $M = G_a(0)G_n^{-1}(0)$ is

$$
\{\mu_1,\mu_2,\ldots,\mu_5\} = \{1.25870,\ 1.08577,\ 0.93707,\ 0.71708,\ 0.64485\}
$$

$$
(8.13)
$$

and therefore satisfies the robustness theorem since $\mu_j \in C^+$ (j = 1,2,...,5). The set-point tracking behaviour of the F100 engine (the actual plant) under the action of the error-actuated digital PID controller designed for the nominal plant and tuned as before is shown in Figs 8.5 and 8.6 for the same set-point vector as before. The tunable digital PID controller is robust in the face of plant variations, as predicted, since this behaviour exhibits only minimal performance degradation.

8.3 Robustness of tunable digital set-point tracking PID/Pre-filter controllers

8.3.1 Robustness in the face of unknown disturbances

The robustness to unknown constant disturbances of tunable digital set-point tracking PID/Pre-filter controllers (Chapter 3) can be readily investigated. Thus, the state, output, and measurement equations (3.1) to (3.3) are modified to incorporate constant disturbances on the plant. Hence,

$$\dot{x}(t) = Ax(t) + Bu(t) + d \ , \tag{8.14}$$

$$w(t) = Ex(t) \ , \tag{8.15}$$

and

$$y(t) = Cx(t) \ , \tag{8.16}$$

where the vectors $x(t)$, $u(t)$, $w(t)$, and $y(t)$, and the matrices A, B, E, and C are defined as before. It is assumed that the constant disturbance vector $d \in R^n$ is unknown.

In the presence of such disturbances, the Laplace transforms of unmeasurable and measurable outputs are

$$w(s) = E(sI - A)^{-1}\{Bu(s) + d/s\} \tag{8.17}$$

and

$$y(s) = C(sI - A)^{-1}\{Bu(s) + d/s\} \ . \tag{8.18}$$

Therefore, in case u(t) = u = const

$$\lim_{t \to \infty} w(t) = -EA^{-1}(Bu + d)$$

$$= G_w u - EA^{-1}d \tag{8.19}$$

and

$$\lim_{t \to \infty} y(t) = -CA^{-1}(Bu + d)$$

$$= G_y u - CA^{-1}d , \tag{8.20}$$

where the steady-state transfer function matrices G_w and G_y are defined in equations (3.5) and (3.6), respectively.

The behaviour of such plants on the discrete-time set $T_T = \{0,T,2T,...,kT,...\}$ is governed by state, output, and measurement equations of the form (Kwakernaak and Sivan (1972))

$$x_{k+1} = \Phi x_k + \Psi u_k + \Theta d , \tag{8.21}$$

$$w_k = \Xi x_k , \tag{8.22}$$

and

$$y_k = \Gamma x_k , \tag{8.23}$$

where the vectors x_k, u_k, w_k, and y_k are defined as before, the matrices Φ, Ψ, Ξ, and Γ are defined in equations (3.19) to (3.22), and

$$\Theta = \int_0^T \exp(At) \, dt \ .$$ (8.24)

The state and output equations of such plants under the action of error-actuated digital PID/Pre-filter controllers governed on the dicrete-time set T_T by control-law equations of the form (3.29) assume the forms

$$
\begin{bmatrix} x_{k+1} \\ z_{k+1} \\ f_{k+1} \end{bmatrix}
=
\begin{bmatrix}
\Phi - T\Psi K_1 \Gamma - \Psi K_3 \Gamma & , & T\Psi K_2 & , & -\Psi K_3 \\
-T\Gamma & , & I_m & , & 0 \\
-\Gamma & , & 0 & , & 0
\end{bmatrix}
\begin{bmatrix} x_k \\ z_k \\ f_k \end{bmatrix}
+
\begin{bmatrix} T\Psi K_1 + \Psi K_3 \\ T I_m \\ I_m \end{bmatrix} v
$$

$$
+ \begin{bmatrix} \Theta \\ 0 \\ 0 \end{bmatrix} d
$$ (8.25)

and

$$
w_k = [\ \Xi \ , \ 0 \ , \ 0 \ ] \begin{bmatrix} x_k \\ z_k \\ f_k \end{bmatrix} \ .
$$ (8.26)

Therefore, provided only that T, K_1, K_2, and K_3 are such that all the eigenvalues of the closed-loop plant matrix in equation (8.25) lie in the open unit disc D^-,

$$
\lim_{t \to \infty} \Delta z_k = \lim_{t \to \infty} (z_{k+1} - z_k) = 0
$$ (8.27)

and therefore

$$
\lim_{t \to \infty} e_k = 0
$$ (8.28)

so that set-point tracking occurs for the measurable outputs in the sense that

$$\lim_{k \to \infty} (v - y_k) = 0 \ . \tag{8.29}$$

Here, the set-point vector for measurable outputs

$$v = Jr \in R^m \ , \tag{8.30}$$

where the set-point vector for unmeasurable outputs $r \in R^m$ and the pre-filter matrix

$$J = G_y G_w^{-1} \in R^{m \times m} \ , \tag{8.31}$$

However, in general

$$\lim_{k \to \infty} (Jr - Jw_k) \neq 0 \tag{8.32}$$

in view of equations (8.19) and (8.20). Therefore,

$$\lim_{k \to \infty} (r - w_k) \neq 0 \ . \tag{8.33}$$

This indicates that, although set-point tracking of the measurable outputs can be achieved, the unmeasurable output vector cannot be caused to track its set-point vector in the steady state. It therefore follows that, in general, tunable

digital set-point tracking PID/Pre-filter controllers cannot reject unknown disturbances in this sense.

Such properties are illustrated by the simulation results shown in Fig 8.7 to 8.9. In this simulation, the plant has five manipulated variables, five unmeasurable outputs, and five measurable outputs (Appendix 3). The digital PID/Pre-filter controller is designed and tuned as before (Example in Chapter 3) and the set-point vector for the unmeasurable outputs is $r = [500, 0, 0, 0, 0]^T$ whilst the corresponding set-point vector for the measurable outputs is $v = Jr = [126, 93.4, 14.5, 1.78, 1.97]^T$. Furthermore, the constant disturbance vector is described by

$$\left. \begin{array}{l} d(i) = 0 \quad , \quad i \in [1,33] \ , \quad i \neq 2 \\ d(2) = -100 \end{array} \right\} , \qquad (8.34)$$

where $d(i)$ is the ith element of $d \in R^{33}$. This choice is made to simulate the horsepower extraction. These results indicate that although set-point tracking together with disturbance rejection for measurable outputs can be achieved, neither set-point tracking nor disturbance rejection for unmeasurable outputs can be achieved and that such performance degradation might occur in the face of unknown disturbances.

8.3.2 Robustness in the face of plant variations

The robustness to plant variations of tunable digital set-point tracking PID/Pre-filter controllers can now be investigated.

The design equations (3.68), (3.70), and (3.72) for the proportional, integral, and derivative controller matrices K_1, K_2, and K_3 are accordingly re-expressed in the forms

$$K_1 = H_{w,n}^{-1}(T) \Lambda_{w,n}(T) \Pi J^{-1} , \qquad (8.35)$$

$$K_2 = G_{y,n}^{-1}(0) \Sigma , \qquad (8.36)$$

and

$$K_3 = H_{w,n}^{-1}(T) \Lambda_{w,n}(T) \Delta J^{-1} . \qquad (8.37)$$

Here, $H_{w,n}(T)$ and $\Lambda_{w,n}(T)$ are, respectively, the step-response and decoupling-index matrices of the nominal plant for unmeasurable outputs and $G_{y,n}(0)$ is the steady-state transfer-function matrix of the nominal plant for measurable outputs.

It is then evident from equations (3.64), (3.65), and (3.66) that $Z_c = \hat{Z}_1 \cup \hat{Z}_2 \cup \hat{Z}_3$ is now the set of closed-loop characteristic roots, where

$$\hat{Z}_1 = \{ z \in C : | zI_n - I_n - TA + 0(T^2) | = 0 \} , \qquad (8.38)$$

$$\hat{Z}_2 = \{ z \in C : | zI_m - I_m + T^2 G_{y,a}(0) G_{y,n}^{-1}(0) \Sigma + 0(T^3) | \\ = 0 \} , \qquad (8.39)$$

$$\hat{Z}_3 = \{ z \in C : | zI_m + 0(T) | = 0 \} , \qquad (8.40)$$

and $G_{y,a}(0)$ is the steady-state transfer function matrix of the actual plant for measurable outputs. It is clear that the closed-loop tracking system will remain asymptotically stable, and that set-point tracking will consequently still occur in the sense that

$$\lim_{k \to +\infty} (v - y_k) = 0 , \qquad\qquad (8.41)$$

provided that $Z_c \subset D^-$. However, in view of equations (3.23), (3.26), and (3.27),

$$\lim_{k \to +\infty} (r - w_k) \neq 0 , \qquad\qquad (8.42)$$

unless

$$G_{y,n} G_{w,n}^{-1} = G_{y,a} G_{w,n}^{-1} . \qquad\qquad (8.43)$$

Therefore, it is noted that the unmeasurable output vector may not track its set-point vector in the steady state and that performance degradation of the controller may occur in this sense.

In case the controller is designed for a linear F100 engine model obtained at Intermediate power condition (Power lever angle = 83 deg) — ie the nominal plant, the excellent set-point tracking behaviour of the F100 engine under the action of a digital PID/Pre-filter controller tuned such that T = 0.05 sec, $\Lambda_y(0.05)\Pi = 1.0I_5$, $\Sigma = 50.0I_5$, and $\Lambda_y(0.05)\Delta = 0.01I_5$ is shown

in Figs 8.10 to 8.12. In this case, the required thrust change is 500 lb so that the set-point vector for the unmeasurable outputs is $r = [500, 0, 0, 0, 0]^T$ whilst the corresponding set-point vector for the measurable outputs is $v = Jr = [126, 93.4, 14.5, 1.78, 1.97]^T$.

In order to examine the robustness of this controller, the controller is now applied to another linear F100 engine model obtained at the different power condition (Power lever angle = 67 deg) — ie the actual plant. The steady-state transfer-function matrices of the nominal plant and the actual plant are given in the forms

$$
G_n(0) = G_{y,n}(0) = \begin{bmatrix}
0.37904 & 1238.8 & -28.508 & -9.2619 & -57.405 \\
0.30777 & 660.79 & -2.8675 & -25.646 & -46.221 \\
0.20602E-01 & -39.863 & 0.25947 & -0.76283 & -6.8275 \\
0.15944E-02 & -12.168 & 0.38479E-01 & -0.33542E-01 & -0.44527 \\
0.90309E-01 & 210.94 & -1.7403 & 2.2101 & 12.248
\end{bmatrix} \quad (8.44)
$$

and

$$
G_a(0) = G_{y,a}(0) = \begin{bmatrix}
0.40780 & 1220.2 & -30.646 & -10.843 & -64.279 \\
0.32555 & 197.87 & -1.9798 & -19.076 & -37.008 \\
0.27484E-01 & -6.8768 & 0.83052E-01 & -0.61430 & -5.5979 \\
0.24994E-02 & -7.8017 & 0.24802E-01 & -0.15957E-01 & -0.29314 \\
0.95167E-01 & 72.565 & -1.2269 & 1.8282 & 10.459
\end{bmatrix} . \quad (8.45)
$$

By Theorem 1 (Porter and Khaki-Sedigh (1989) (Appendix 7)), the spectrum of the perturbation matrix $M = G_a(0)G_n^{-1}(0)$ is

$$\{\mu_1, \mu_2, \ldots, \mu_5\} = \{1.25870, \ 1.08577, \ 0.93707, \ 0.71708, \ 0.64485\}$$

$$(8.46)$$

and therefore satisfies the robustness theorem since $\mu_j \in C^+$ ($j = 1, 2, \ldots, 5$). The set-point tracking behaviour of the F100 engine (the actual plant) under the action of a digital PID/Pre-filter controller designed for the nominal plant and tuned as before is shown in Figs 8.13 to 8.15 for the same set-point vector for the unmeasurable outputs as before. The tunable digital PID/Pre-filter controller is robust in the face of plant variations in the sense that the closed-loop system remains asymptotically stable. However, performance degradation has occured in the sense that the unmeasurable output vector no longer tracks its set-point vector in the steady state.

8.4 Conclusion

In this chapter, the robustness properties of set-point tracking systems incorporating tunable digital PID or PID/Pre-filter controllers has been assessed. By considering the stability of discrete-time closed-loop tracking systems, the effect of unknown constant disturbances has been investigated for both types of controller. The robustness assessment for plant variations has been carried out using the robustness theorem – Theorem 1 (Porter and Khaki-Sedigh (1989)

(Appendix 7)). This assessment has been effected by characterising, in terms of the steady-state transfer function matrices of nominal and actual plants, the admissible plant perturbations that can be tolerated by such tunable set-point tracking controllers. In order to verify the results and predictions of this analysis, time-domain simulation results for a gas-turbine engine have been presented.

It has been shown in the analysis and the simulation results that the tunable digital set-point tracking PID controllers can achieve set-point tracking with disturbance rejection and that such controllers are robust since only minimal performance degradation has occurred in the face of plant variation.

In the case of the tunable digital set-point tracking PID/Pre-filter controllers, it has been shown that such controllers are robust in the sense that the closed-loop digital control systems remain asymptotically stable in the face of unknown constant disturbances or admissible plant variations. However, it has been shown in this case that although set-point tracking together with disturbance rejection for measurable outputs can be achieved, neither set-point tracking nor disturbance rejection can be achieved for unmeasurable outputs and that performance degradation might therefore occur in this sense in the face of unknown disturbances or plant variations. These results have been verified by the presentation of time-domain simulation results.

Fig 8.1   Measurable outputs of F100 engine
———   under digital PID control
with constant disturbance

Fig 8.2 Manipulated variables of F100 engine
under digital PID control
with constant disturbance

Fig 8.3 Measurable outputs of F100 engine
—— under digital PID control
Nominal plant, Actual plant: Intermediate (PLA 83 deg)

Fig 8.4 Manipulated variables of F100 engine
under digital PID control
Nominal plant, Actual plant: Intermediate (PLA 83 deg)

Fig 8.5   Measurable outputs of F100 engine
———   under digital PID control

Nominal plant: Intermediate (Power lever angle = 83 deg)
Actual plant: Power lever angle = 67 deg

Fig 8.6   Manipulated variables of F100 engine
 ——— under digital PID control

Nominal plant: Intermediate (Power lever angle = 83 deg)
Actual plant: Power lever angle = 67 deg
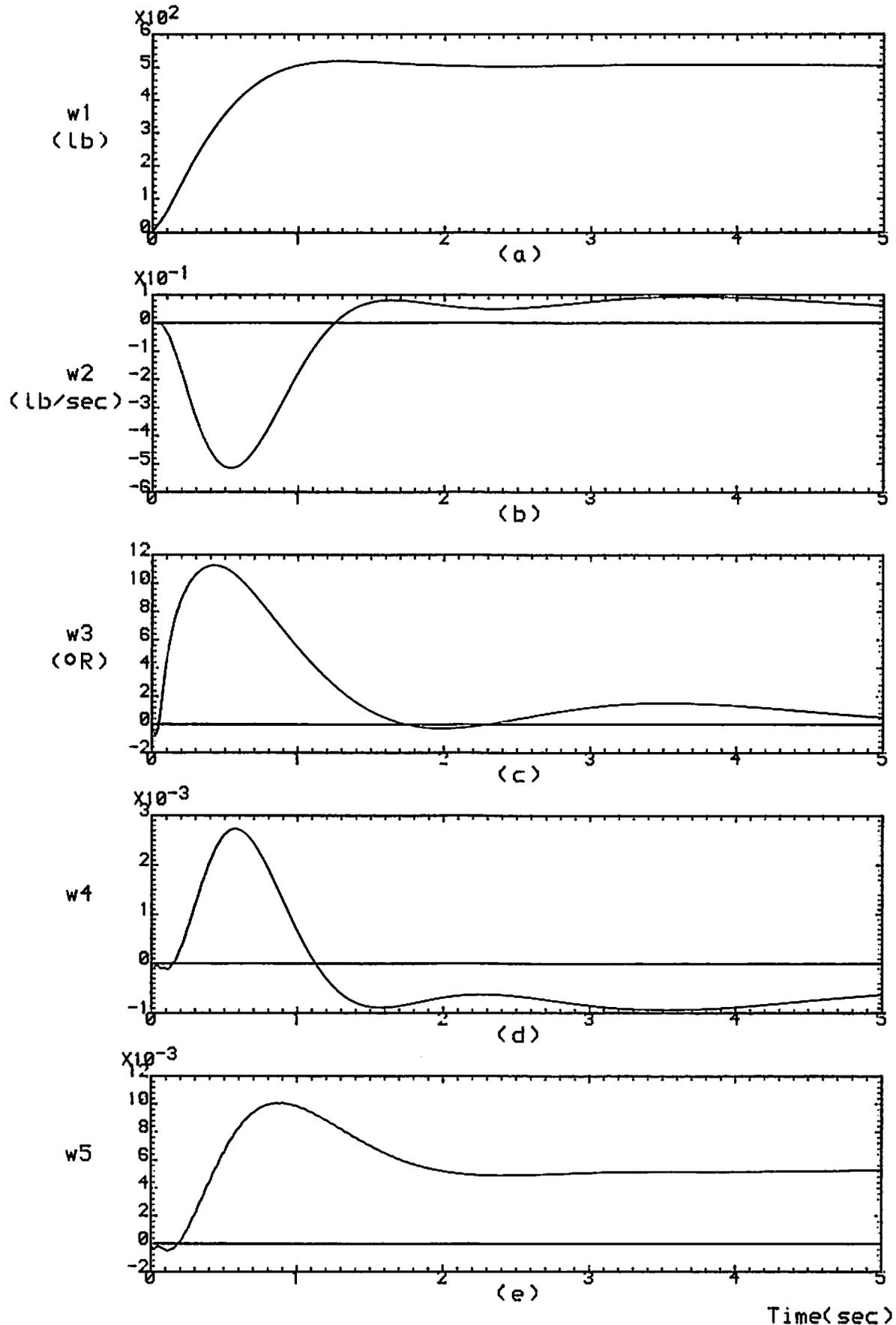
Fig 8.7 Unmeasurable outputs of F100 engine
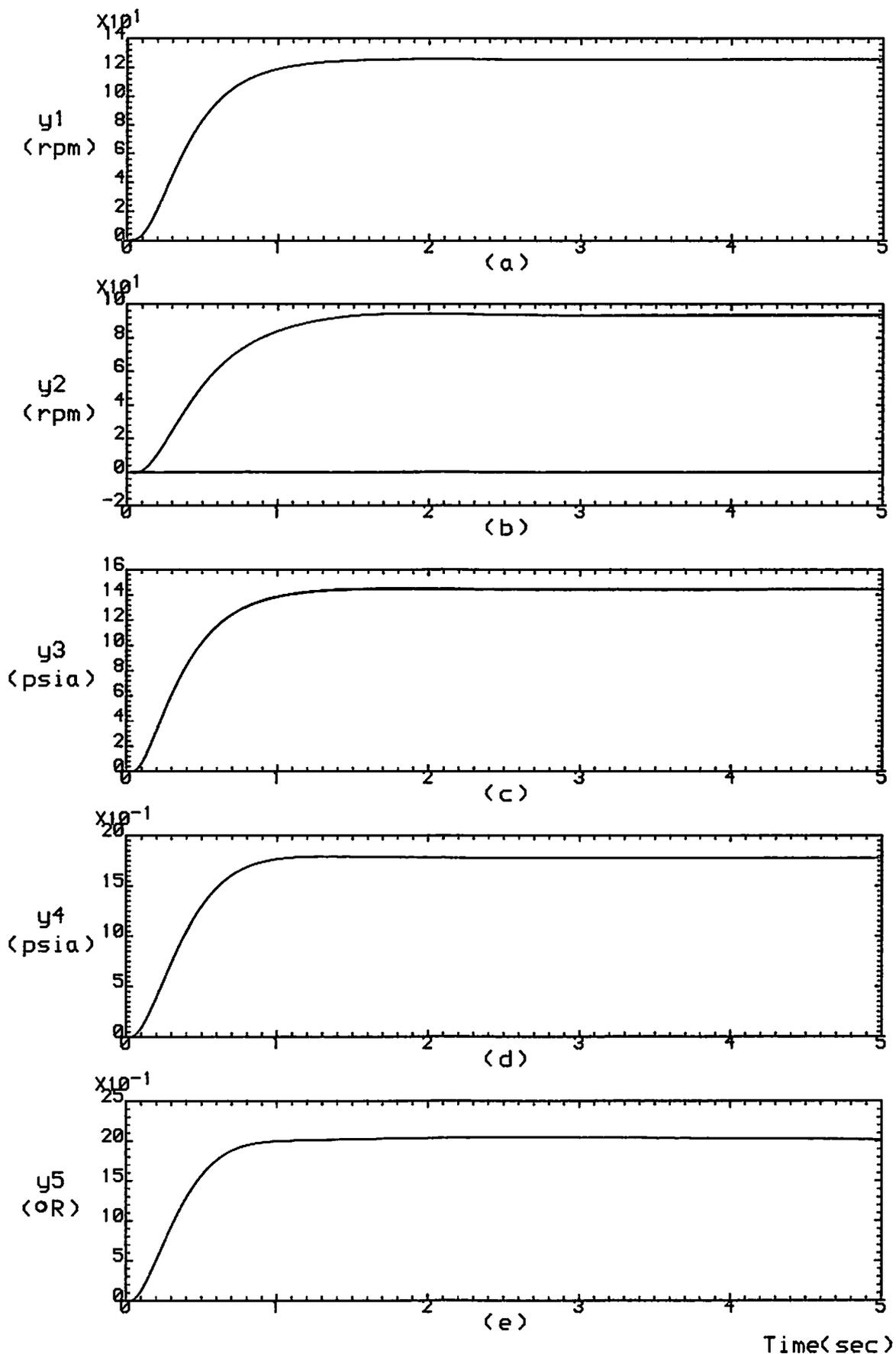under digital PID/Pre-Filter control
with constant disturbance
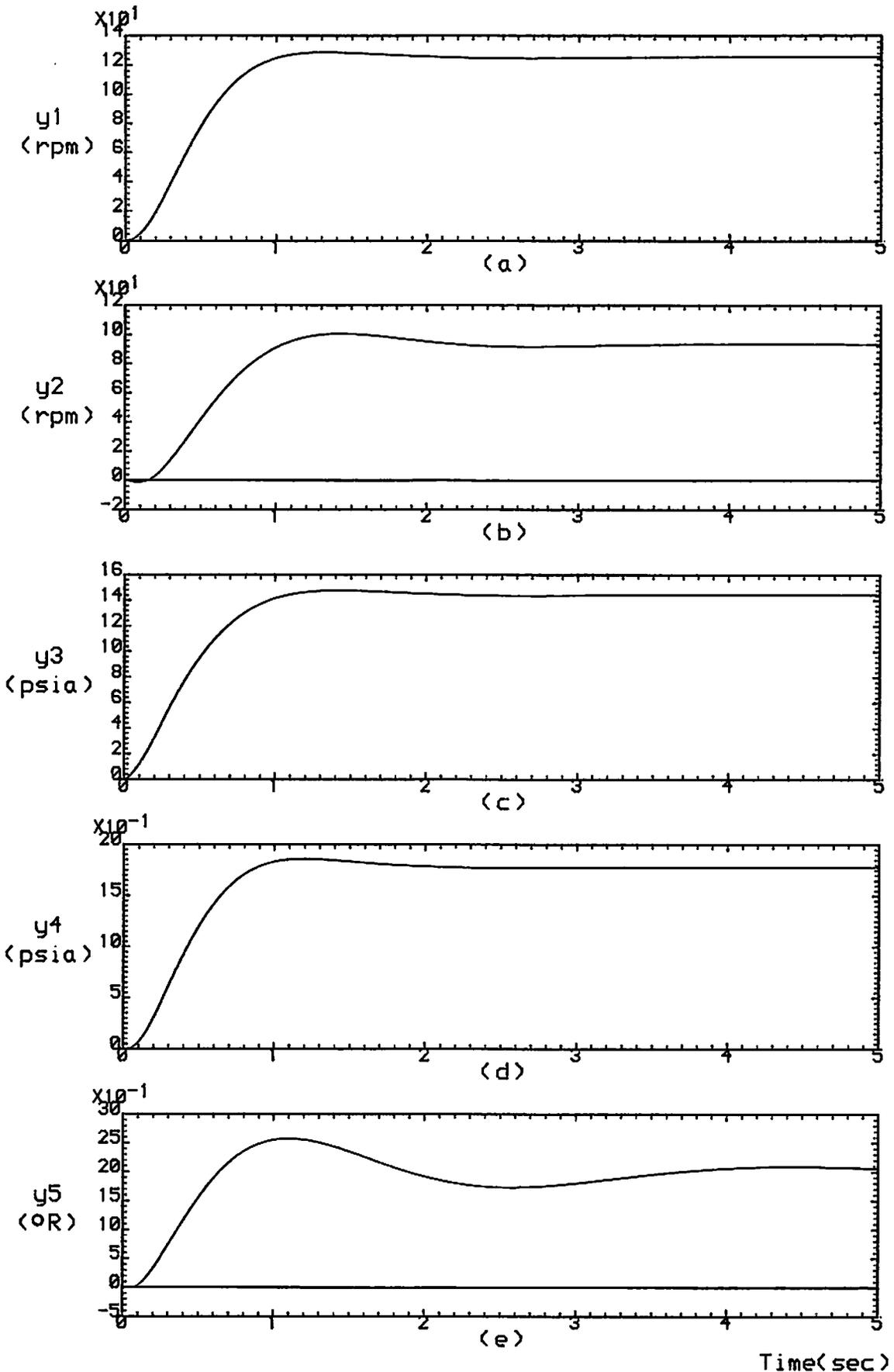
Fig 8.8 Measurable outputs of F100 engine
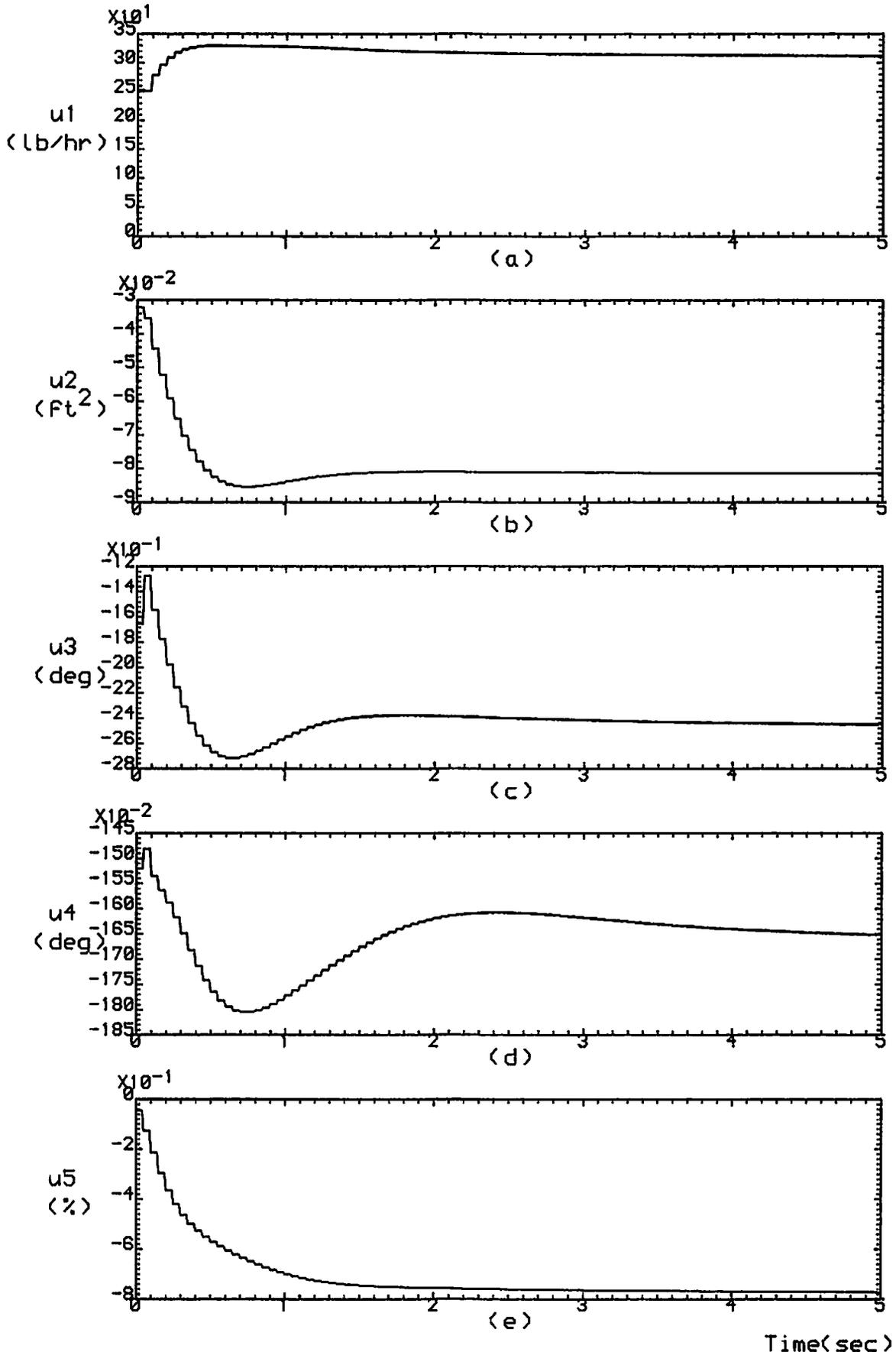— — under digital PID/Pre-filter control
with constant disturbance
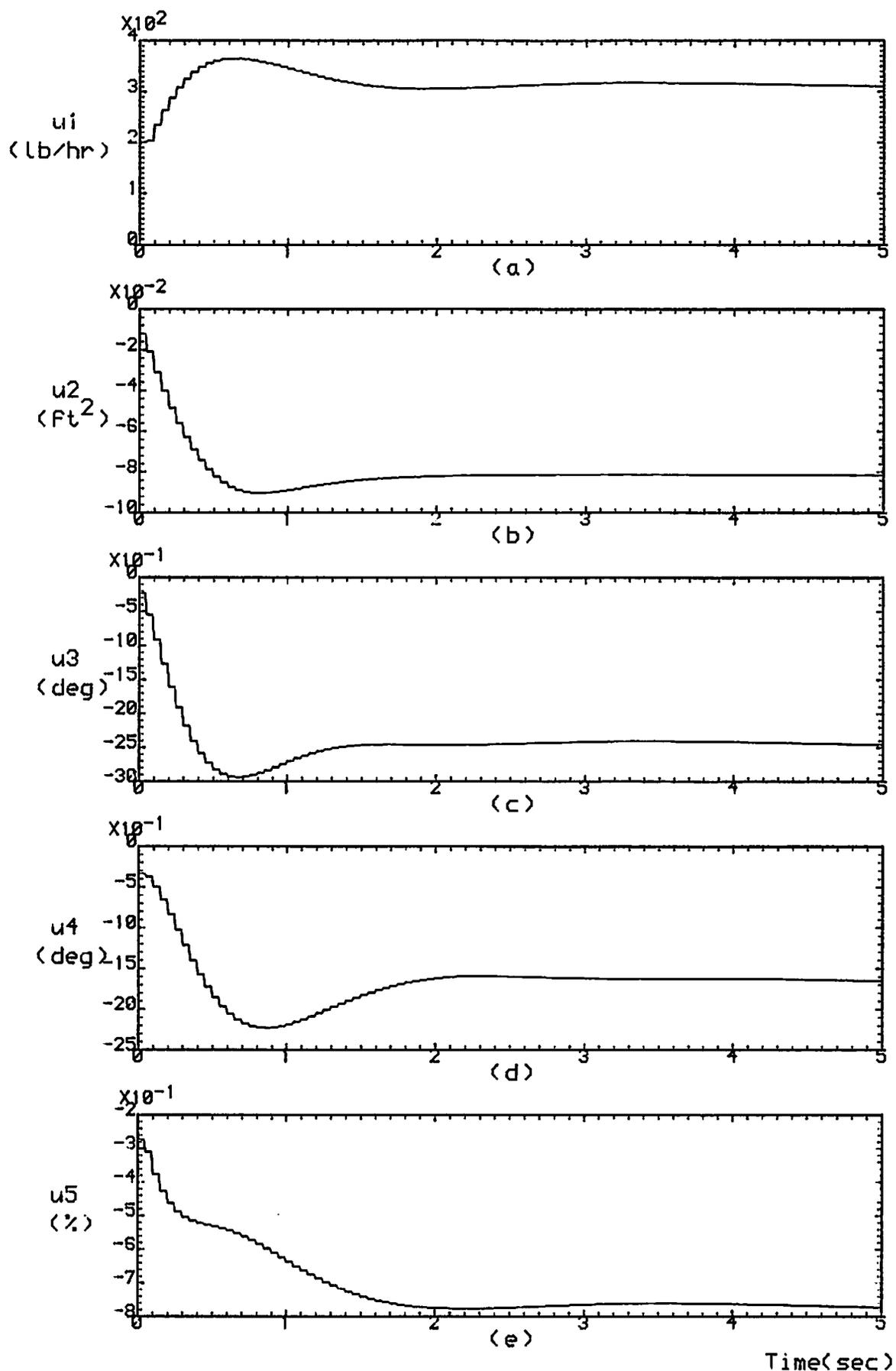
Fig 8.9   Manipulated variables of F100 engine
——   under digital PID/Pre-Filter control
with constant disturbance
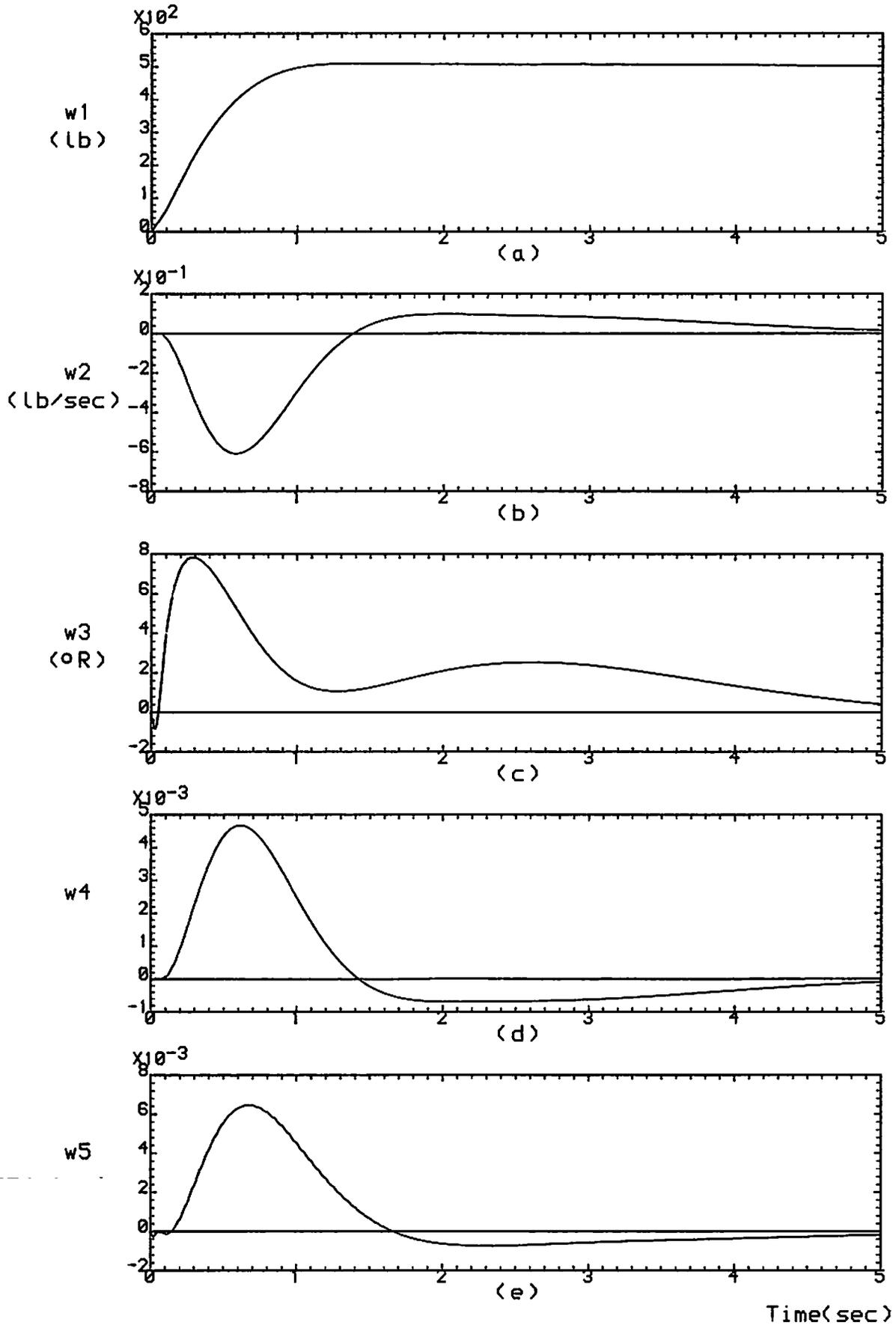
Fig 8.10 Unmeasurable outputs of F100 engine
——— under digital PID/Pre-Filter control
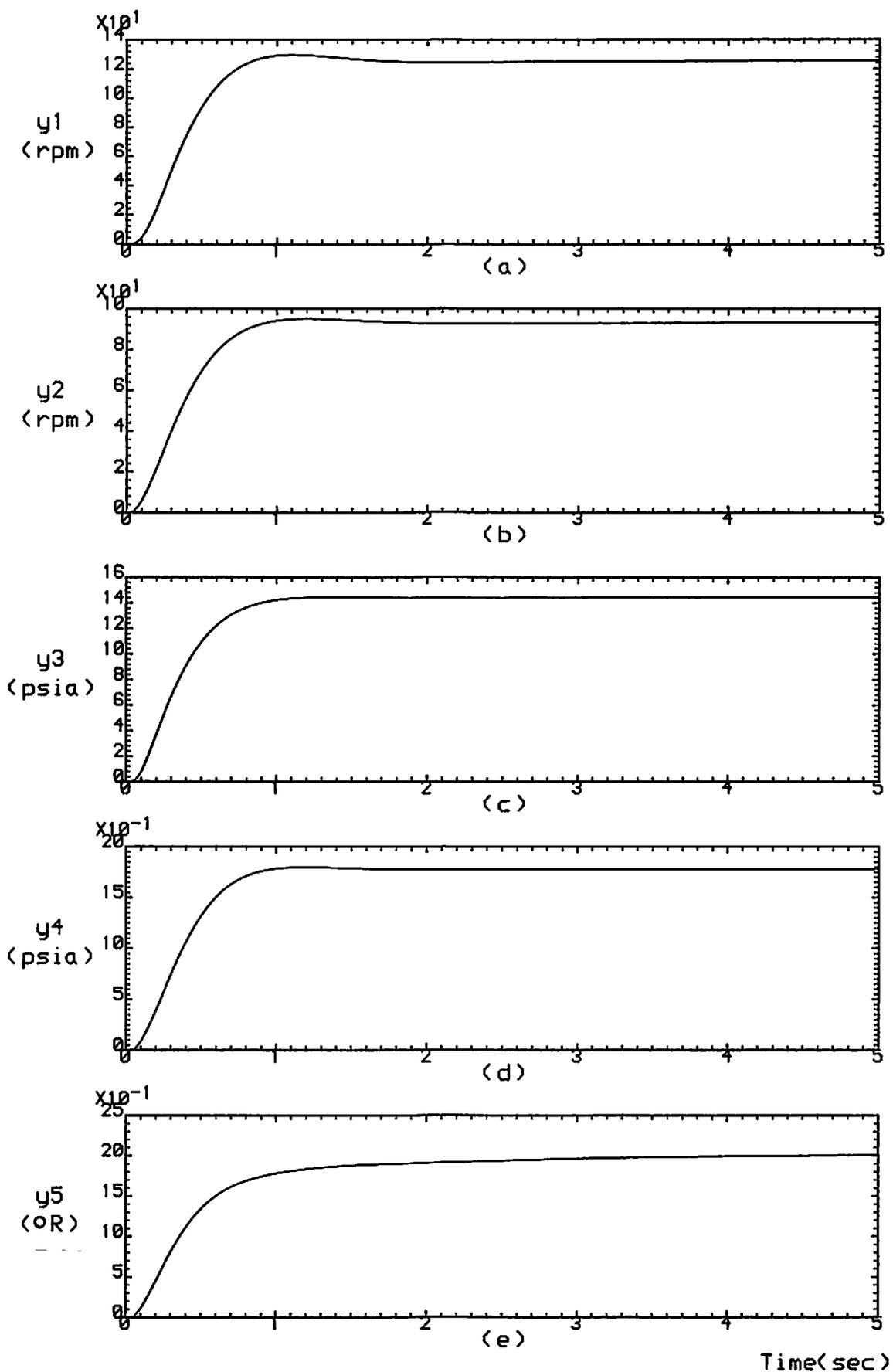Nominal plant, Actual plant: Intermediate (PLA 83 deg)
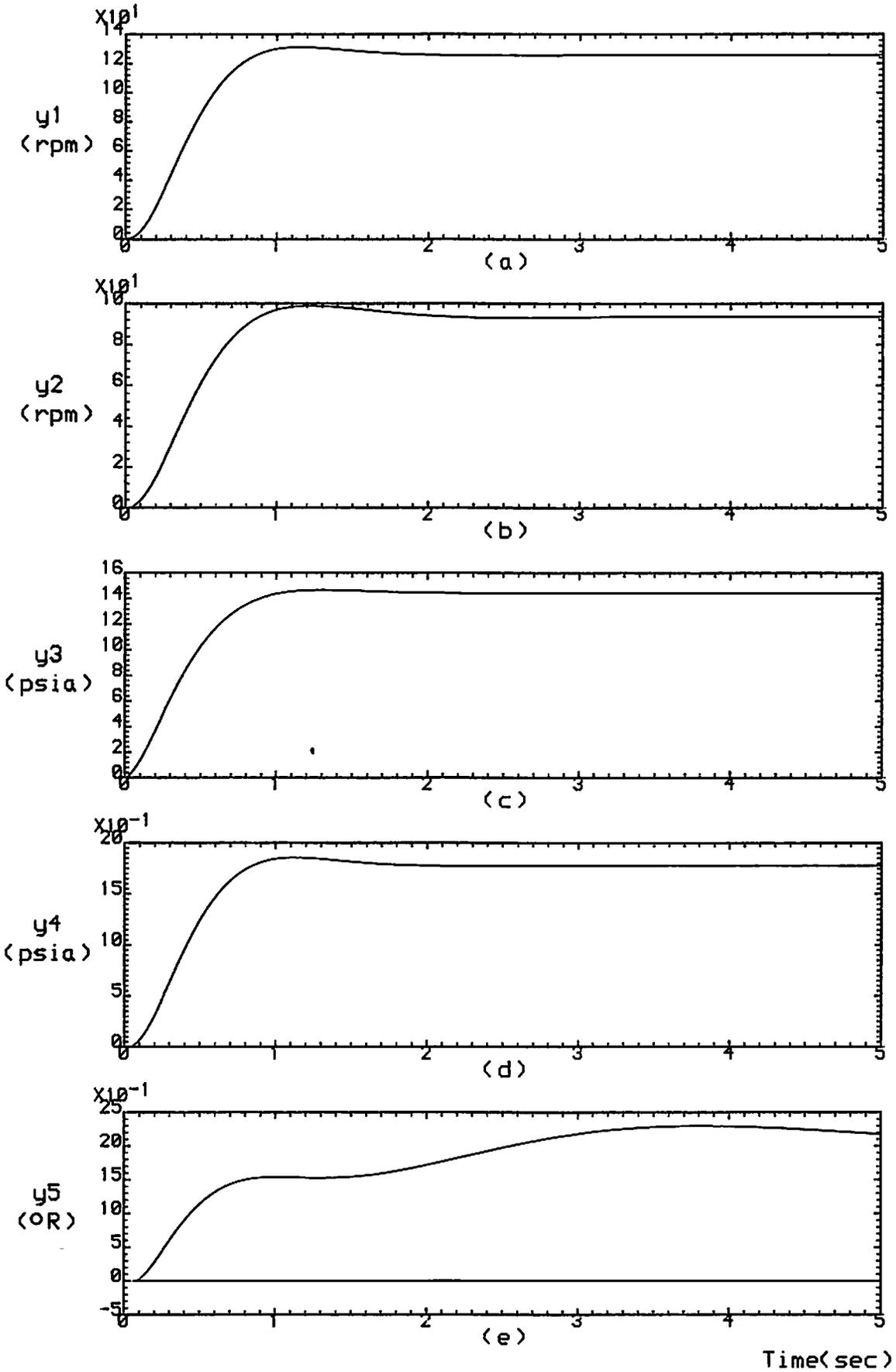
Fig 8.11 Measurable outputs of F100 engine
under digital PID/Pre-Filter control
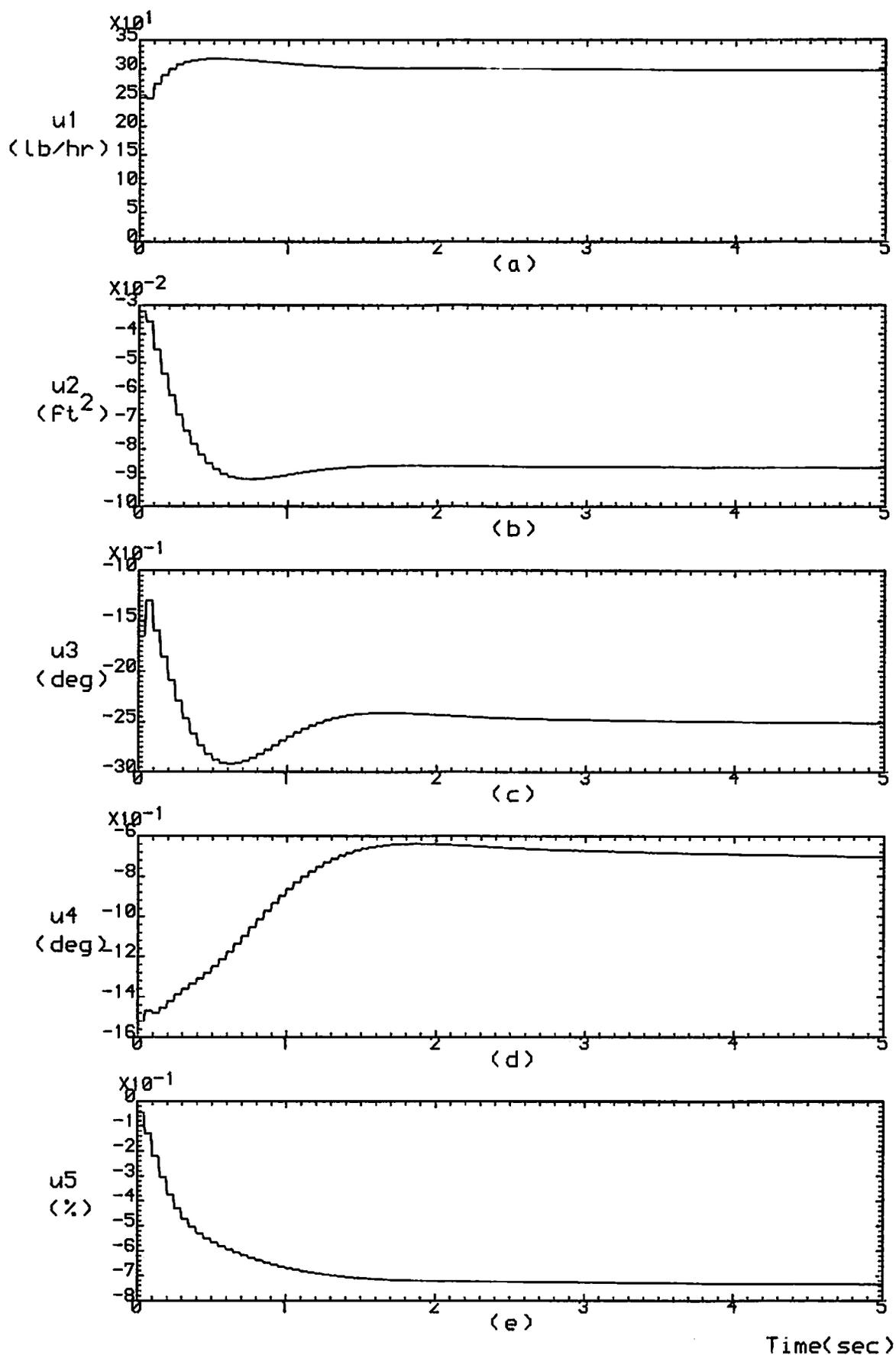Nominal plant, Actual plant: Intermediate (PLA 83 deg)
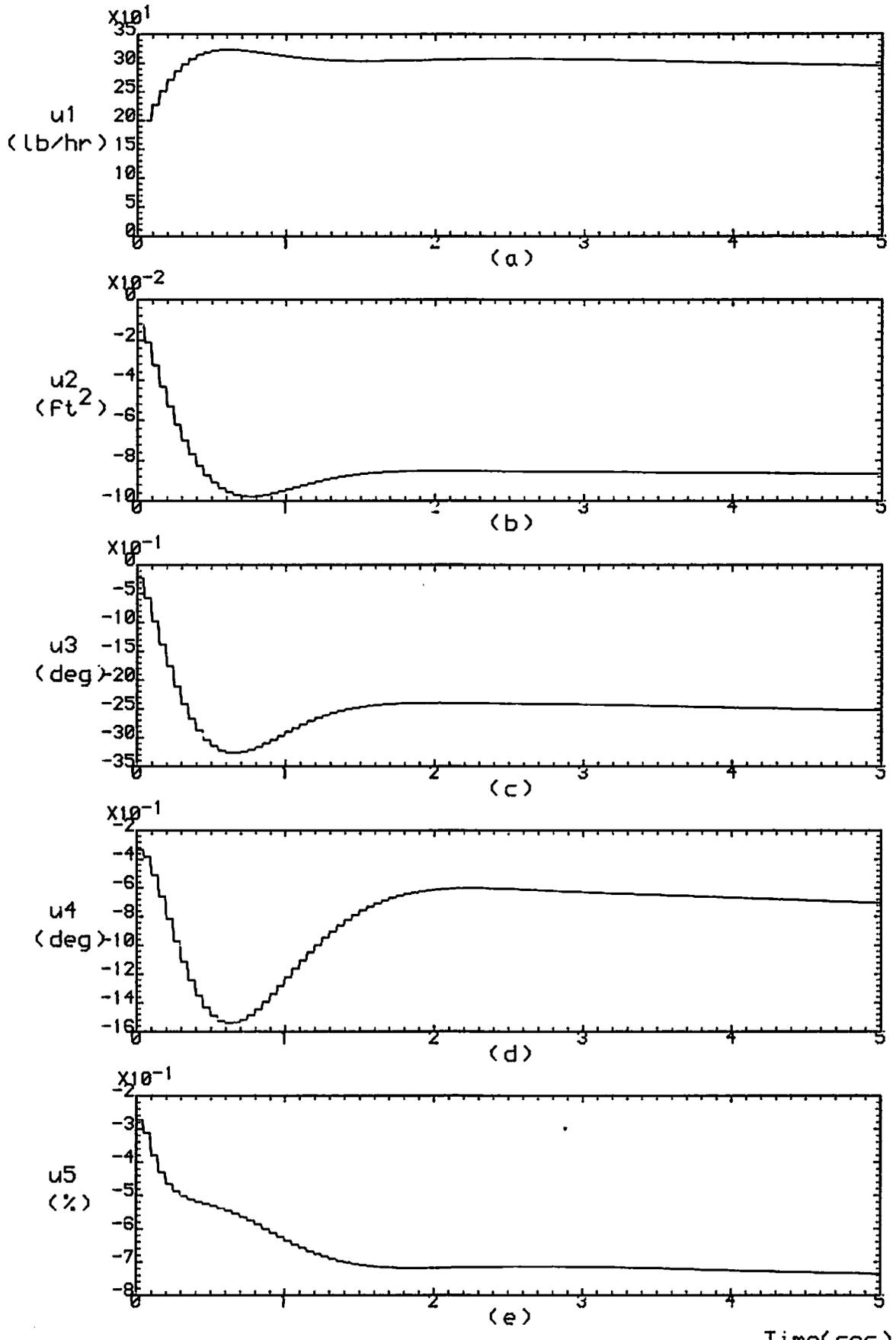
Fig 8.12 Manipulated variables of F100 engine
under digital PID/Pre-Filter control
Nominal plant, Actual plant: Intermediate (PLA 83 deg)
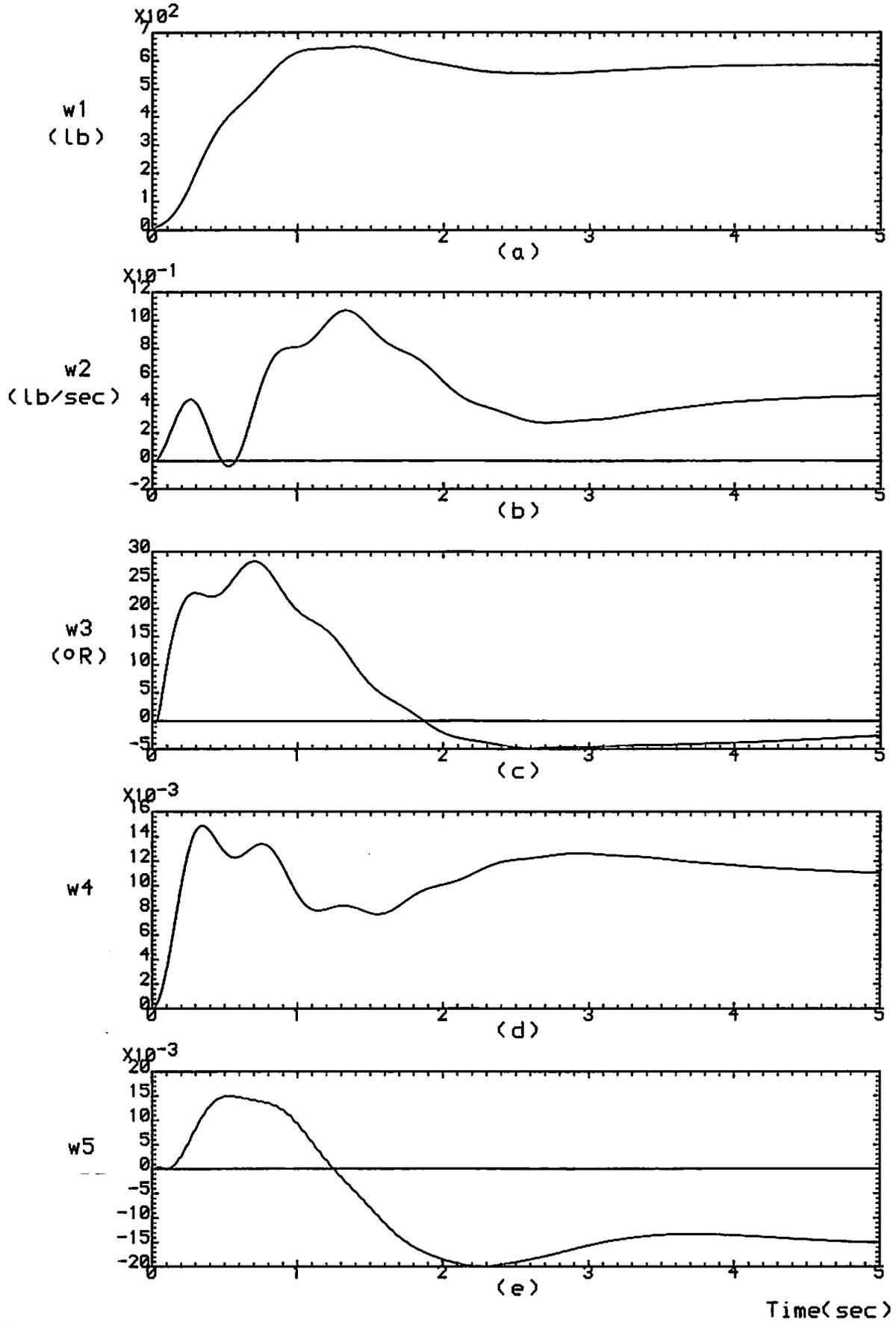
Fig 8.13 Unmeasurable outputs of F100 engine
under digital PID/Pre-Filter control

Nominal plant: Intermediate (Power lever angle = 83 deg)
Actual plant: Power lever angle = 67 deg
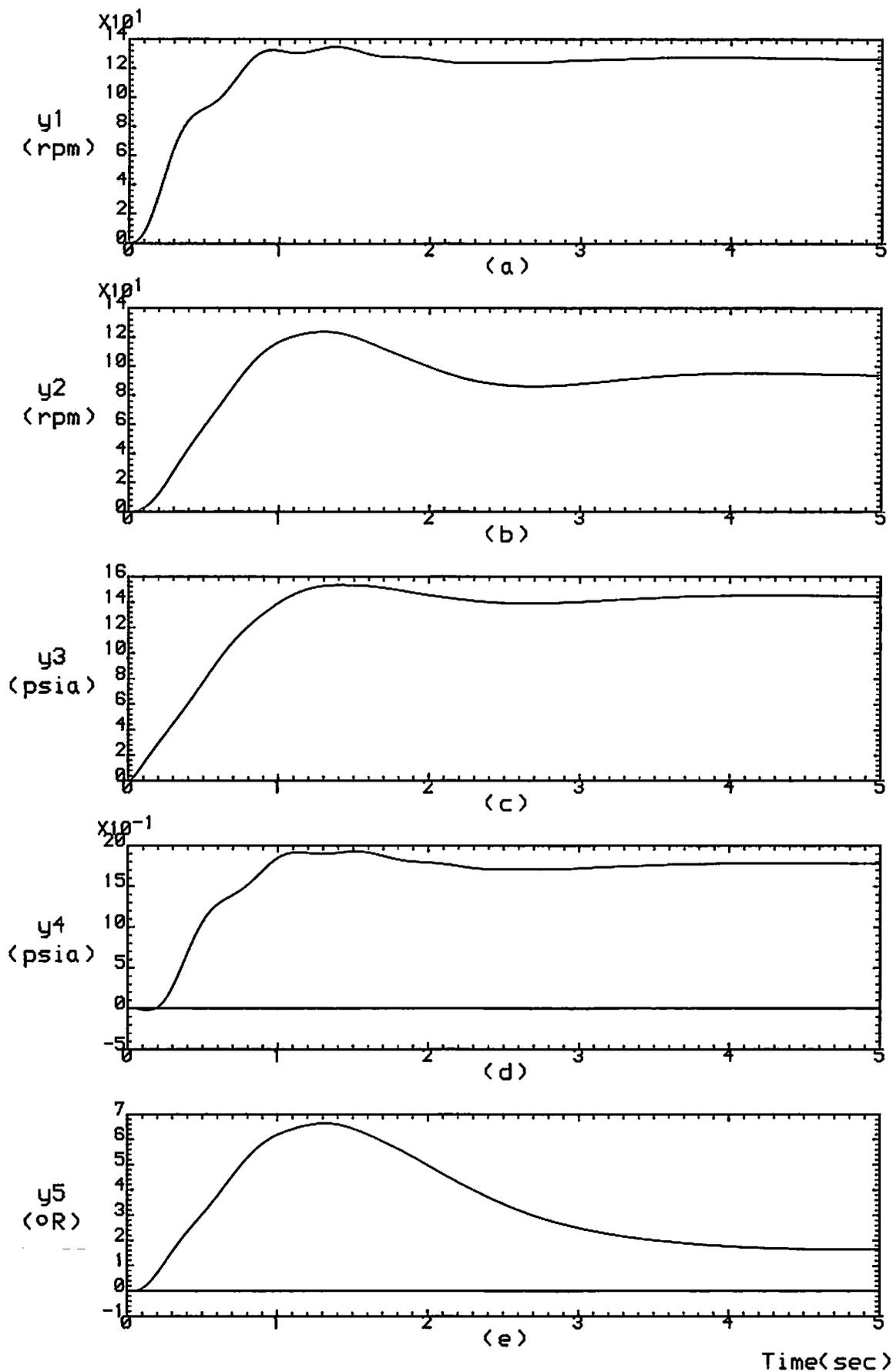
Fig 8.14 Measurable outputs of F100 engine
under digital PID/Pre-Filter control
Nominal plant: Intermediate (Power lever angle = 83 deg)
Actual plant: Power lever angle = 67 deg
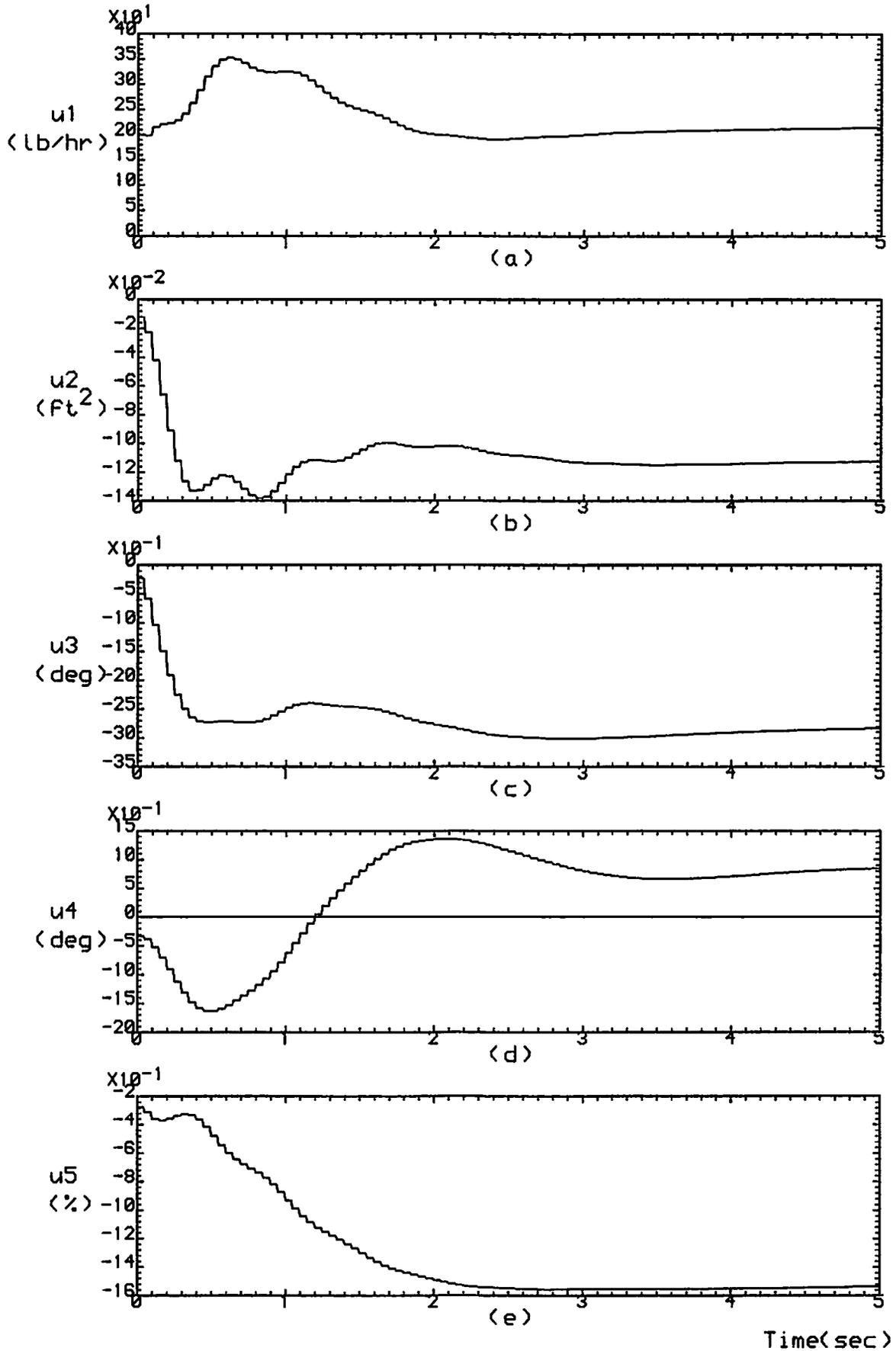
Fig 8.15 Manipulated variables of F100 engine
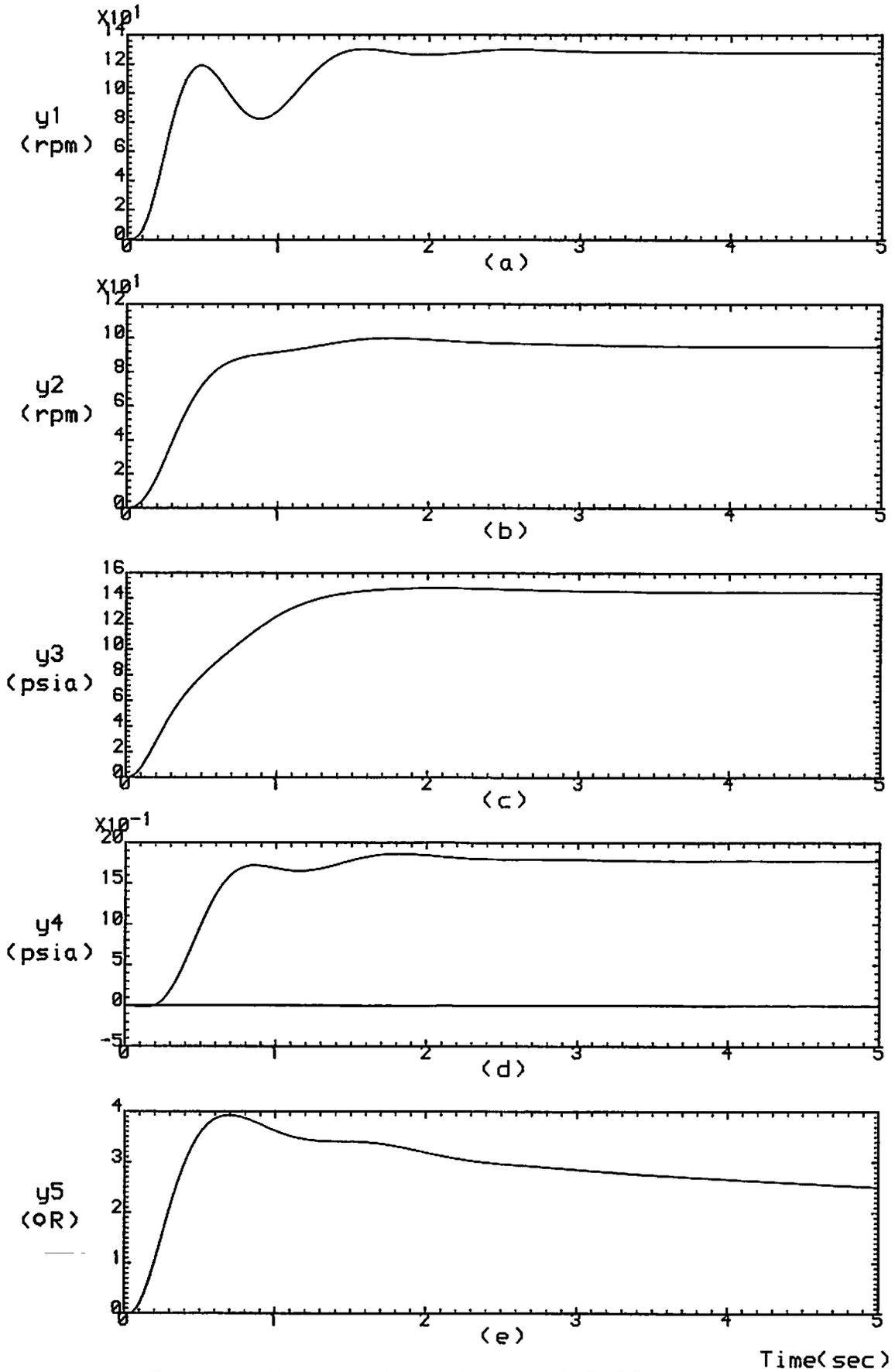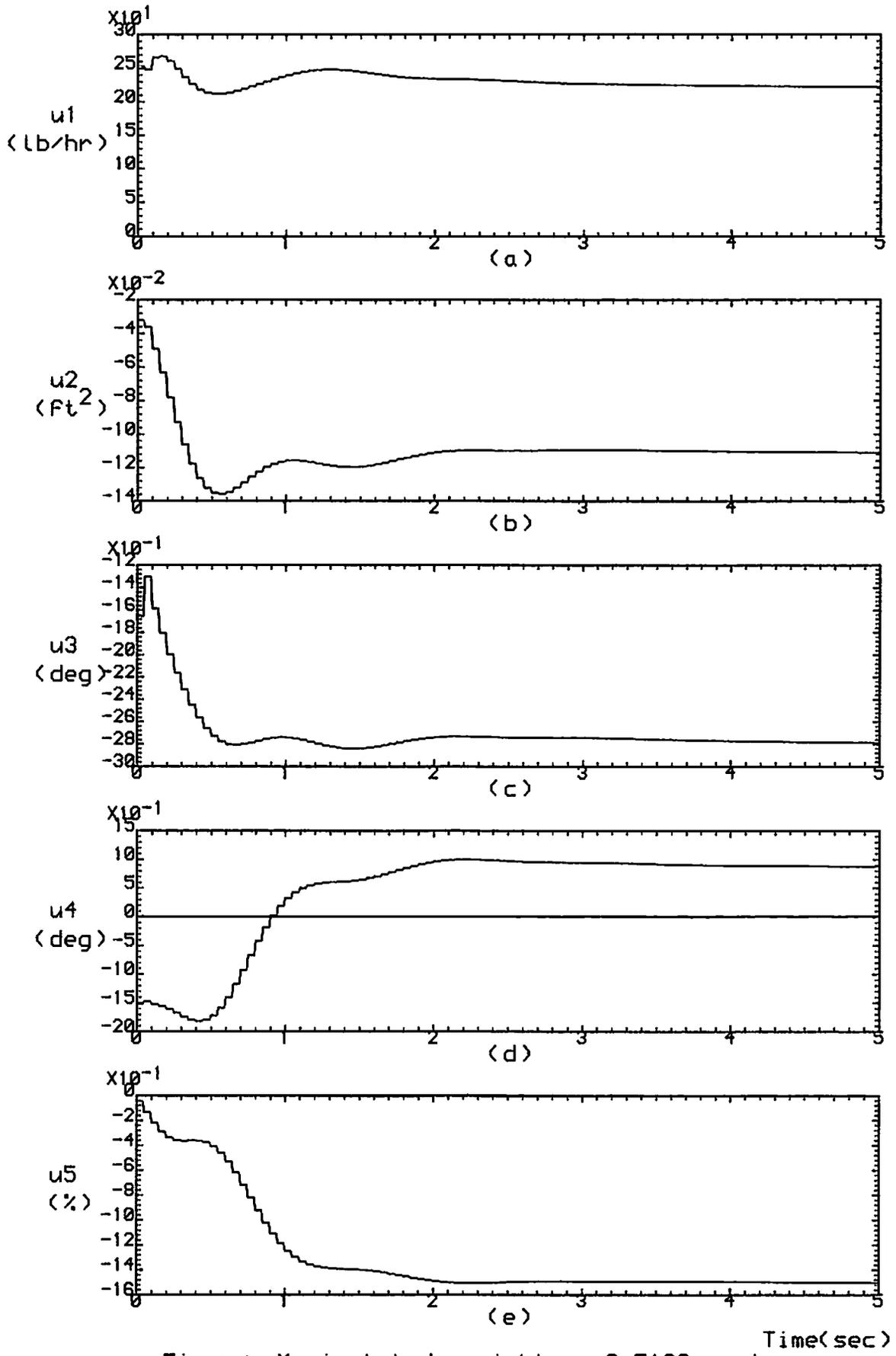under digital PID/Pre-Filter control
Nominal plant: Intermediate (Power Lever angle = 83 deg)
Actual plant: Power Lever angle = 67 deg

CHAPTER 9

ROBUSTNESS OF LIMIT-TRACKING SYSTEMS

9.1 Introduction

Since it has been shown in Chapter 4 that undertracking (Definition 4.4) is always possible in the face of unknown constant disturbances, it is expected that self-selecting controllers (Chapter 6) have good disturbance-rejection properties. Furthermore, since such controllers are extensions of tunable digital set-point tracking controllers whose robustness to plant variations has been shown in Chapter 8, self-selecting controllers are also expected to be robust in the face of plant variations.

Therefore, in this chapter, an investigation is carried out to assess the robustness of self-selecting controllers in the face of unknown constant disturbances and plant variations. Furthermore, for supervisory self-selecting controllers, the robustness of the supervisory part is investigated based on its control-mode structure.

9.2 Robustness of digital self-selecting PID controllers

9.2.1 Robustness in the face of unknown disturbances

The robustness to unknown constant disturbances of digital self-selecting PID controllers (Chapter 6) can be readily investigated. Thus, the state and output equations (6.1) and

(6.2) of linear multivariable Class I plants are modified to incorporate constant disturbances on the plants. Hence,

$$\dot{x}(t) = Ax(t) + Bu(t) + d \tag{9.1}$$

and

$$y(t) = Cx(t) , \tag{9.2}$$

where the vectors $x(t)$, $u(t)$, and $y(t)$, and the matrices A, B, and C are defined as before. It is assumed that the constant disturbance vector $d \in R^n$ is unknown. Since the plants are asymptotically stable, it follows from equations (9.1) and (9.2) that the output vector of the initially quiescent plant for a constant input vector $u(t) = u = \text{const}$ is

$$y(t) = CA^{-1}(e^{At} - I_n)(Bu + d) \tag{9.3}$$

and therefore that, in the steady state,

$$\lim_{t \to \infty} y(t) = -CA^{-1}(Bu + d)$$

$$= Gu - CA^{-1}d , \tag{9.4}$$

where $G \in R^{p \times m}$ is the steady-state transfer-function matrix defined in equation (6.4). Then, by Theorem 4.4, there always exists an input u such that

$$G u \leq v + CA^{-1}d , \tag{9.5}$$

where $v \in R^p$ is the set-point vector. This means that

$$\lim_{t \to \infty} y(t) \leq v \qquad (9.6)$$

in view of equation (9.4) and that, by considering $v + CA^{-1}d$ as a new set-point vector in Theorem 5.1, there always exists at least one limit-tracking input which satisfies equation (9.6). Therefore, the steady-state condition of limit-tracking is satisfied in the face of unknown constant disturbances.

Next, by applying the order-reduction technique (Appendix 5.1) to the plants, the sub-output vectors corresponding to subsets Y_i ($i = 1,2,\ldots,r$ $r = p-m+1$) are

$$y^{(i)}(t) = C^{(i)}x(t) \qquad (i = 1,2,\ldots,r) , \qquad (9.7)$$

where the vectors $y^{(i)}(t)$ ($i = 1,2,\ldots,r$) and the matrices $C^{(i)}$ ($i = 1,2,\ldots,r$) are defined as before. The behaviour of such sub-plants on the discrete-time set $T_T = \{0,T,2T,\ldots,kT,\ldots\}$ is governed by state and output equations of the form (Kwakernaak and Sivan (1972))

$$x_{k+1} = \Phi x_k + \Psi u_k + \Theta d \qquad (9.8)$$

and

$$y_k^{(i)} = \Gamma^{(i)} x_k , \qquad i \in I_r , \qquad (9.9)$$

where the vectors x_k, u_k, and $y_k^{(i)}$ are defined as before, the matrices Φ, Ψ, and $\Gamma^{(i)}$ are defined in equations (6.19) to (6.21), the index set I_r is defined as before, and

$$\Theta = \int_0^T \exp(At) \, dt \, . \qquad (9.10)$$

The state and output equations of such sub-plants under the action of individual error-actuated digital PID controllers governed on the discrete-time set T_T by control-law equations of the form (6.22) assume the forms

$$\begin{bmatrix} x_{k+1} \\ z_{k+1} \\ f_{k+1} \end{bmatrix} = \begin{bmatrix} \Phi - T\Psi K_1^{(i)}\Gamma^{(i)} - \Psi K_3^{(i)}\Gamma^{(i)} & , & T\Psi K_2^{(i)} & , & -\Psi K_3^{(i)} \\ -T\Gamma^{(i)} & , & I_m & , & 0 \\ -\Gamma^{(i)} & , & 0 & , & 0 \end{bmatrix} \begin{bmatrix} x_k \\ z_k \\ f_k \end{bmatrix}$$

$$+ \begin{bmatrix} T\Psi K_1^{(i)} + \Psi K_3^{(i)} \\ TI_m \\ I_m \end{bmatrix} v^{(i)} + \begin{bmatrix} \Theta \\ 0 \\ 0 \end{bmatrix} d \qquad (9.11)$$

and

$$y_k^{(i)} = [\Gamma^{(i)} , 0 , 0] \begin{bmatrix} x_k \\ z_k \\ f_k \end{bmatrix} . \qquad (9.12)$$

Therefore, provided only that T, $K_1^{(i)}$, $K_2^{(i)}$, and $K_3^{(i)}$, where $i \in I_r$, are such that all the eigenvalues of the closed-loop

sub-plant matrix in equation (9.11) lie in the open unit disc D^-,

$$\lim_{k \to \infty} \Delta z_k = \lim_{k \to \infty} \{z_{k+1} - z_k\} = 0 \qquad (9.13)$$

and therefore

$$\lim_{k \to \infty} e_k^{(i)} = 0 \qquad (9.14)$$

so that set-point tracking for the subset Y_i occurs simultaneously with disturbance rejection.

Such disturbance-rejection properties are illustrated by the simulation results shown in Figs 9.1 and 9.2 for Structure 1 (Example in Chapter 6) and in Figs 9.3 and 9.4 for Structure 2 (Example in Chapter 6). In these simulations, the plant is the two-input/three-output linear F100 engine model obtained at Intermediate power condition and the digital self-selecting controllers are designed for both structures and tuned as before. The disturbance vector is described by

$$d(j) = 0 \quad , \qquad j \in [1,25] \ , \quad j \neq 2 \ , \qquad (9.15)$$

where $d(j)$ is the jth element of $d \in R^{25}$ and $d(2)$ is shown in Figs 9.1 and 9.3. This choice is made to simulate the horsepower extraction. These results indicate the excellent disturbance-rejection and limit-tracking behaviour of the plant under the action of unknown disturbances.

9.2.2 Robustness in the face of plant variations

The robustness to plant variations of digital self—selecting PID controllers can now be assessed. Such self—selecting controllers are synthesised using Algorithm 5.1. It is recalled that a key concept of this algorithm is to discover in the set of feasible inputs $U_F(0)$ an extreme ray uniquely represented by m—1 (m is the number of inputs) hyperplanes and to limit the region of limit—tracking inputs on a corresponding line (Chapter 5). Then, even though plant variations occur, there exists a limit—tracking input on this line for any set—point command as long as the intersection of these m—1 hyperplanes continues to be an extreme ray. Therefore, for each separate controller, the design equations (6.67), (6.68), and (6.69) for the proportional, integral, and derivative controller matrices $K_1^{(i)}$, $K_2^{(i)}$, and $K_3^{(i)}$ are accordingly re-expressed in the forms

$$K_1^{(i)} = H_n^{(i)}(T)^{-1}\Lambda_n^{(i)}(T)\Pi^{(i)} ,$$ (9.16)

$$K_2^{(i)} = G_n^{(i)}(0)^{-1}\Sigma^{(i)} ,$$ (9.17)

and

$$K_3^{(i)} = H_n^{(i)}(T)^{-1}\Lambda_n^{(i)}(T)\Delta^{(i)} .$$ (9.18)

Here, $H_n^{(i)}(T)$, $G_n^{(i)}(0)$, and $\Lambda_n^{(i)}(T)$ are, respectively, the step—response, steady—state transfer—function, and decoupling matrices of the nominal sub—plant. It is then evident from

equations (6.67), (6.68), and (6.69) that $Z_c^{(i)} = \hat{Z}_1 \cup \hat{Z}_2 \cup \hat{Z}_3$ is now the set of closed-loop characteristic roots, where

$$\hat{Z}_1^{(i)} = \{z \in C : \mid zI_n - I_n -TA + 0(T^2) \mid = 0\} , \qquad (9.19)$$

$$\hat{Z}_2^{(i)} = \{z \in C : \mid zI_m - I_m + T^2 G_a^{(i)}(0)G_n^{(i)}(0)^{-1}\Sigma^{(i)}$$
$$+ 0(T^3) \mid = 0 \} , \qquad (9.20)$$

$$\hat{Z}_3^{(i)} = \{z \in C : \mid zI_m + 0(T) \mid = 0 \} , \qquad (9.21)$$

and $G_a^{(i)}(0)$ is the steady-state transfer-function matrix of the actual sub-plant. It is clear that the separate closed-loop tracking system will remain asymptotically stable, and that set-point tracking for the subset Y_i will consequently occur in the sense of equation (6.33), provided that $Z_c^{(i)} \subset D^-$. Therefore, this indicates that the robustness theorem (Theorem 1: Porter and Khaki-Sedigh (1989) (Appendix 7)) can be utilised to assess the robustness of each separate closed-loop system.

Based on these results, the following theorem is obtained for the robustness properties of the separate set-point tracking PID controllers which are incorporated in a self-selecting controller. In this robustness theorem, it is necessary to distinguish between the plant for which a controller is designed - ie the nominal plant (denoted by subscript $_n$) - and the plant to which a controller is applied - ie the actual plant (denoted by subscript $_a$).

Theorem 9.1

Consider any digital self-selecting PI/PID controller with controlled subsets Y_i, $i \in I_r$ of plant outputs in the form

$$
\left.
\begin{aligned}
Y_1 &= \{y_{s_1}, \ldots, y_{s_{m-1}}, y_{t_1}\} \\
&\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
&\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
&\quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
Y_r &= \{y_{s_1}, \ldots, y_{s_{m-1}}, y_{t_r}\}
\end{aligned}
\right\}
\tag{9.22}
$$

where $r = p-m+1$, the index set of all the control loops is $I_r = \{1,2,\ldots,r\}$, the index set $I = \{1,2,\ldots,p\}$, $I^* = \{s_1,\ldots,s_{m-1}\}$, and $I\backslash I^* = \{t_1,\ldots,t_r\}$.

Assume that the steady-state transfer-function matrix, G_a, of the actual plant is such that

(i)   the m-1 row vectors of G_a (= $G_a(0)$) corresponding to the index set I^* represent an extreme ray in the set $U_F(0)$ of feasible inputs of the actual plant,

and

(ii) every one of the r row vectors of G_a corresponding to the index set $I\backslash I^*$ is linearly independent of all the row vectors corresponding to I^*.

Then for every separate set-point tracking PI/PID controller (which controls Y_i, $i \in I_r$ and is incorporated in the self-selecting controller) with integral post-multiplier of the form

$$\Sigma^{(i)} = \sigma^{(i)} I_m \qquad (\sigma^{(i)} \in R^+) \qquad (9.23)$$

and any plant perturbation such that

$$\mu_j^{(i)} \in C^+ \qquad (j = 1,2,\ldots,m) \qquad (9.24)$$

where $\{\mu_1^{(i)}, \mu_2^{(i)}, \ldots, \mu_m^{(i)}\}$ is the spectrum of the perturbation matrix

$$M^{(i)} = G_a^{(i)}(0)G_n^{(i)}(0)^{-1} \in R^{m \times m} , \qquad (9.25)$$

there exists a sampling period $\hat{T}_i^* \in R^+$ such that set-point tracking for the subset Y_i occurs for all $T \in (0,\hat{T}_i^*]$.

Next, Theorem 9.1 is used in an illustrative example. In case the nominal and actual plant are the linear F100 engine model obtained at Intermediate power condition (Power lever angle = 83 deg), the set of feasible inputs $U_F(0)$ is shown in Fig 6.2. For the obtained Structure 1 (Example in Chapter 6), the excellent limit-tracking and switching behaviour of the F100 engine under the action of an error-actuated controller tuned such that $T = 0.05$ sec, $\Lambda^{(1)}(0.05)\Pi^{(1)} = \text{diag}\{0.04, 0.1\}$, $\Lambda^{(2)}(0.05)\Pi^{(2)} = \text{diag}\{0.04, 0.02\}$, $\Sigma^{(1)} = \Sigma^{(2)} = 50.0I_2$, and $\Lambda^{(1)} = \Lambda^{(2)} = 0.0005I_2$ is shown in Figs 6.6 and 6.7, where the loops show that $P_7(y_2)$ and FTIT(y_3) are controlled in turn, whilst $N_1(y_1)$ is permanently controlled. For the Structure 2 (Example in Chapter 6), the excellent limit-tracking and

switching behaviour of an error-actuated controller tuned such that $T = 0.05$ sec, $\Lambda^{(1)}(0.05)\Pi^{(1)} = \text{diag}\{0.1, 0.04\}$, $\Lambda^{(2)}(0.05)\Pi^{(2)} = \text{diag}\{0.1, 0.02\}$, $\Sigma^{(1)} = \Sigma^{(2)} = 50.0I_2$, and $\Lambda^{(1)} = \Lambda^{(2)} = 0.0005I_2$ is shown in Figs 6.8 and 6.9, where the loops show that $N_1(y_1)$ and $FTIT(y_3)$ are controlled in turn, whilst $P_7(y_2)$ is permanently controlled.

In order to examine the robustness of these controllers, they are now applied to another linear F100 engine model obtained at the different power condition (Power lever angle = 67 deg). The steady-state transfer-function matrices of the nominal plant and the actual plant are given in the forms

$$G_n = G_n(0) = \begin{bmatrix} 0.37904 & 1238.8 \\ 0.15944e{-}2 & -12.168 \\ 0.90309e{-}1 & 210.94 \end{bmatrix} \tag{9.26}$$

and

$$G_a = G_a(0) = \begin{bmatrix} 0.40780 & 1220.2 \\ 0.24994e{-}2 & -7.8017 \\ 0.95167e{-}1 & 72.565 \end{bmatrix}. \tag{9.27}$$

The input space for the actual plant is shown in Fig 9.5. It is clear from Fig 6.2 for the nominal plant and Fig 9.5 that the conditions (i) and (ii) of Theorem 9.1 are satisfied and therefore that the robustness assessment using perturbation matrices for the separate set-point tracking controllers of either Structure 1 or Structure 2 is effective. For Structure 1, the spectra of the perturbation matrices are

$$\{\mu_1^{(1)}, \ \mu_2^{(1)}\} = \{1.1317, \ 0.8359\} \tag{9.28}$$

$$\{\mu_1^{(2)}, \ \mu_2^{(2)}\} = \{1.0808, \ 2.5082\} \tag{9.29}$$

and therefore the condition (9.24) of Theorem 9.1 is satisfied. For Structure 2, the spectra of the perturbation matrices are

$$\{\mu_1^{(1)}, \ \mu_2^{(1)}\} = \{2.5082, \ 1.0808\} \tag{9.30}$$

$$\{\mu_1^{(2)}, \ \mu_2^{(2)}\} = \{1.2167, \ 0.5291\} \tag{9.31}$$

and therefore the condition (9.24) of Theorem 9.1 is satisfied. Thus, by Theorem 9.1, the separate set-point tracking PID controllers incorporated in the digital self-selecting controller based upon either Structure 1 or Structure 2 can cope with such plant variations.

Now, the robustness of complete closed-loop systems can be demonstrated in time-domain simulation. The limit-tracking and switching behaviour of the F100 engine (the actual plant) under the action of such digital self-selecting controllers designed for the nominal plant and tuned as before is shown in Figs 9.6 and 9.7 for Structure 1 and in Figs 9.8 and 9.9 for Structure 2. The digital self-selecting PID controller is robust in the face of plant variations in the sense that the separate set-point tracking controllers and the integrated self-selecting controller remain asymptotically stable and that only minimal performance degradation has occurred.

9.3 Robustness of digital supervisory self-selecing controllers

The robustness of digital supervisory self-selecting controllers (Chapter 7) can now be investigated. The controller operations of Normal and Loop-excluded modes are in principle the same since the lowest-wins strategies have authority to decide the controlled subsets at each time instant. Therefore, in case the controller matrices of separate set-point tracking controllers are designed by the methodology described in Chapter 6, the robustness assessment of both control modes can be effected by Theorem 9.1. Furthermore, since the controller operation of Loop-fixed mode is the same as that of the tunable digital set-point tracking controller, the robustness theorem (Theorem 1: Porter and Khaki-Sedigh (1989) (Appendix 7)) is applicable.

Next, the choice of controller parameters of the supervisory part is discussed in the context of robustness. It is considered that the initial settling time T_s (Definition 7.8) does not affect crucially the stability of the complete closed-loop system although it affects the timing at which Tracking or Correct/Incorrect loop assessment begins. Furthermore, it is considered that a too short observation time T_o (Definition 7.8) might give an incorrect assessment in the assessment blocks. Therefore the robustness to the choice of T_o needs to be studied. In addition, the effects of the choice of α ($0 < \alpha < 1$) and β ($\beta > 0$) need to be studied. In the following simulation studies, the controller gains, the controller switching logic, and the initial settling time T_s

are chosen the same as in the example in Chapter 7. The effect of T_o is shown in Figs 9.10 and 9.11, where T_o = 0.5 sec, α = 0.5, and β = 0.5. The first tracking assessment at 3.5 sec (T_s+2T_o) in Normal mode is 'Non convergent', Loop-fixed mode $(\ell_f=2)$ begins, and Correct/Incorrect loop assessment at 6.5 sec $(T_s+T_o+3.5)$ is 'Correct loop'. After that, the same assessment continues at 7.0 sec, 7.5 sec, ... Therefore, perfect dynamical limit tracking has been achieved.

The effect of α is shown in Figs 9.12 and 9.13, where T_o = 5.0 sec, α = 0.9, and β = 0.5. In Normal mode, the first Tracking assessment at 12.5 sec (T_s+T_o) is generously 'Convergent' because of a large α. However, the second Tracking assessment at 17.5 sec $(T_o+12.5)$ is 'Non convergent', and Loop-fixed mode $(\ell_f=1)$ begins. Since Correct/Incorrect loop assessment at 25 sec $(T_s+T_o+17.5)$ is 'Incorrect loop', Loop-fixed mode $(\ell_f=2)$ begins. And finally, Correct/Incorrect loop assessment at 30 sec (T_o+25) is 'Correct loop', so that perfect dynamical limit tracking has been achieved.

The effect of β is shown in Figs 9.14 and 9.15 for β = 0.1, and Figs 9.16 and 9.17 for β = 0.9. It is evident from these figures that these responses are the same as the responses of the example in Chapter 7 and therefore that perfect dynamical limit tracking has been achieved independently of the value of β.

9.4 Conclusion

In this chapter, the robustness properties of limit-tracking systems incorporating digital self-selecting controllers or digital supervisory self-selecting controllers have been assessed. It has been shown that there always exists at least one limit-tracking input (in the steady state) in the presence of unknown disturbances, that set-point tracking for separate subsets of plant outputs occurs with disturbance rejection, and therefore that limit tracking for the complete plant occurs with disturbance rejection. Time-domain simulation results have demonstrated such properties.

Next, by considering the conditions under which limit-tracking is possible in the face of plant variations, a robustness theorem (Theorem 9.1) has been constructed so as to assess the robustness properties of the separate set-point tracking controllers which are incorporated in the self-selecting controllers. Illustrative examples together with time-domain simulation results have demonstrated the effectiveness of the theorem and the robustness of self-selecting controllers.

Then, the robustness properties of supervisory self-selecting controllers have been studied. It has been shown that Theorem 9.1 is applicable to Normal and Loop-excluded modes, and that Theorem 1 (Porter and Khaki-Sedigh (1989) (Appendix 7)) is applicable to Loop-fixed mode. In order to investigate the effects of the controller parameters of the supervisory part, time-domain simulation results have been presented. It has been shown that the supervisory self-selecting controllers are

robust in the sense that dynamical limit tracking can be achieved in the presence of variations of controller parameters.

It is noted that, although the stability of complete self-selecting control systems cannot be assessed by Theorem 9.1 in the case of non-supervisory self-selecting controllers, the present analysis and simulation results together with application examples (Jones et al (1988), (1990)) show the implicit robustness of such control systems. In case such implicit robustness is not enough to guarantee the stability of complete self-selecting control systems, supervisory self-selecting controllers can be applied.
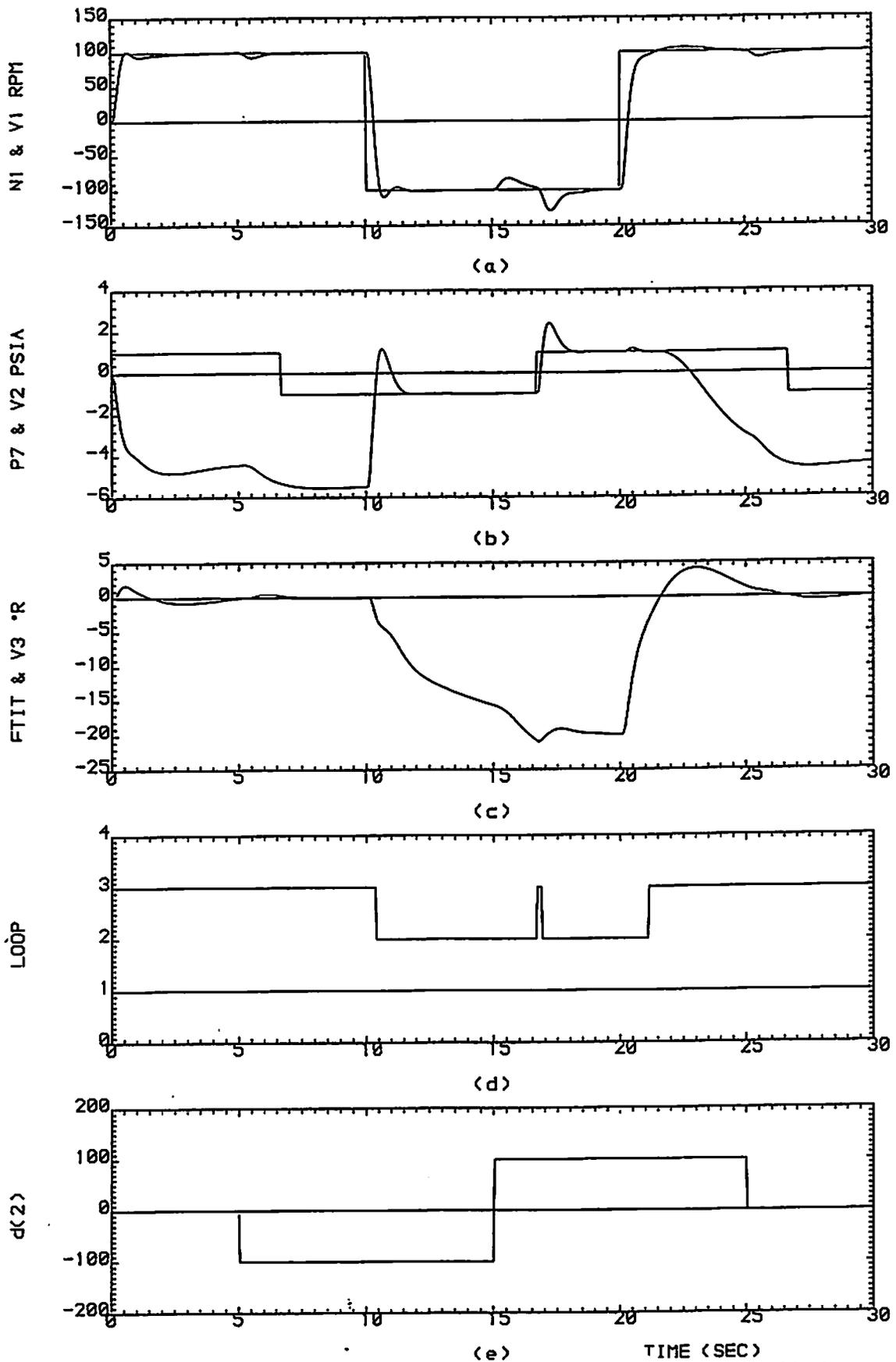
Fig 9.1   Responses of F100 engine under digital
self-selecting PID control with disturbance
Structure 1: [N1,P7] & [N1,FTIT]
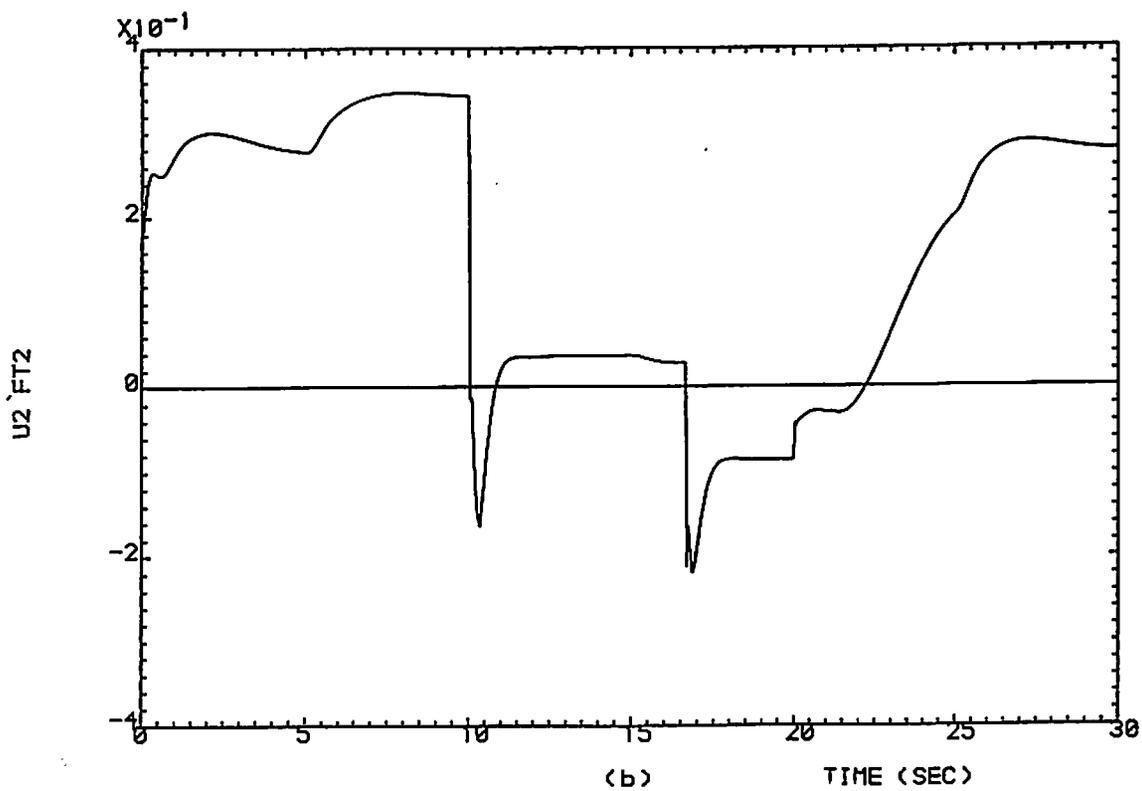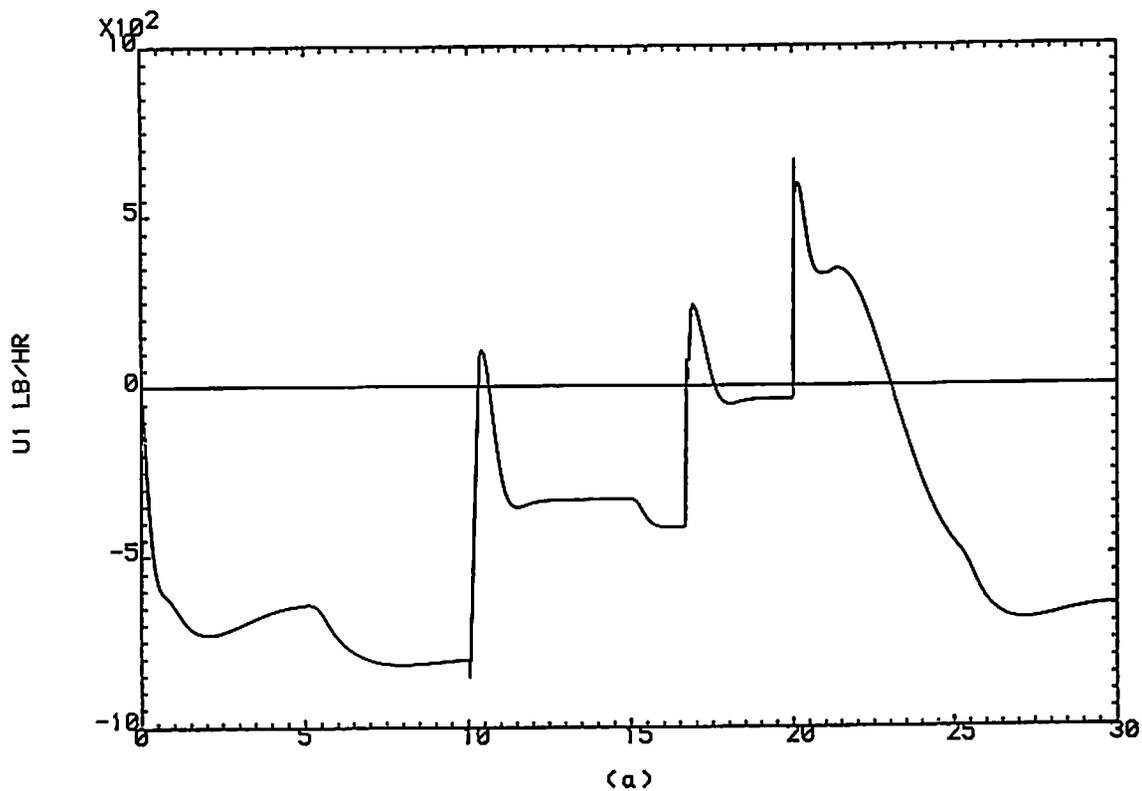
Fig 9.2   Manipulated variables of F100 engine
under digital self-selecting PID control
with disturbance
Structure 1: [N1,P7] & [N1,FTIT]
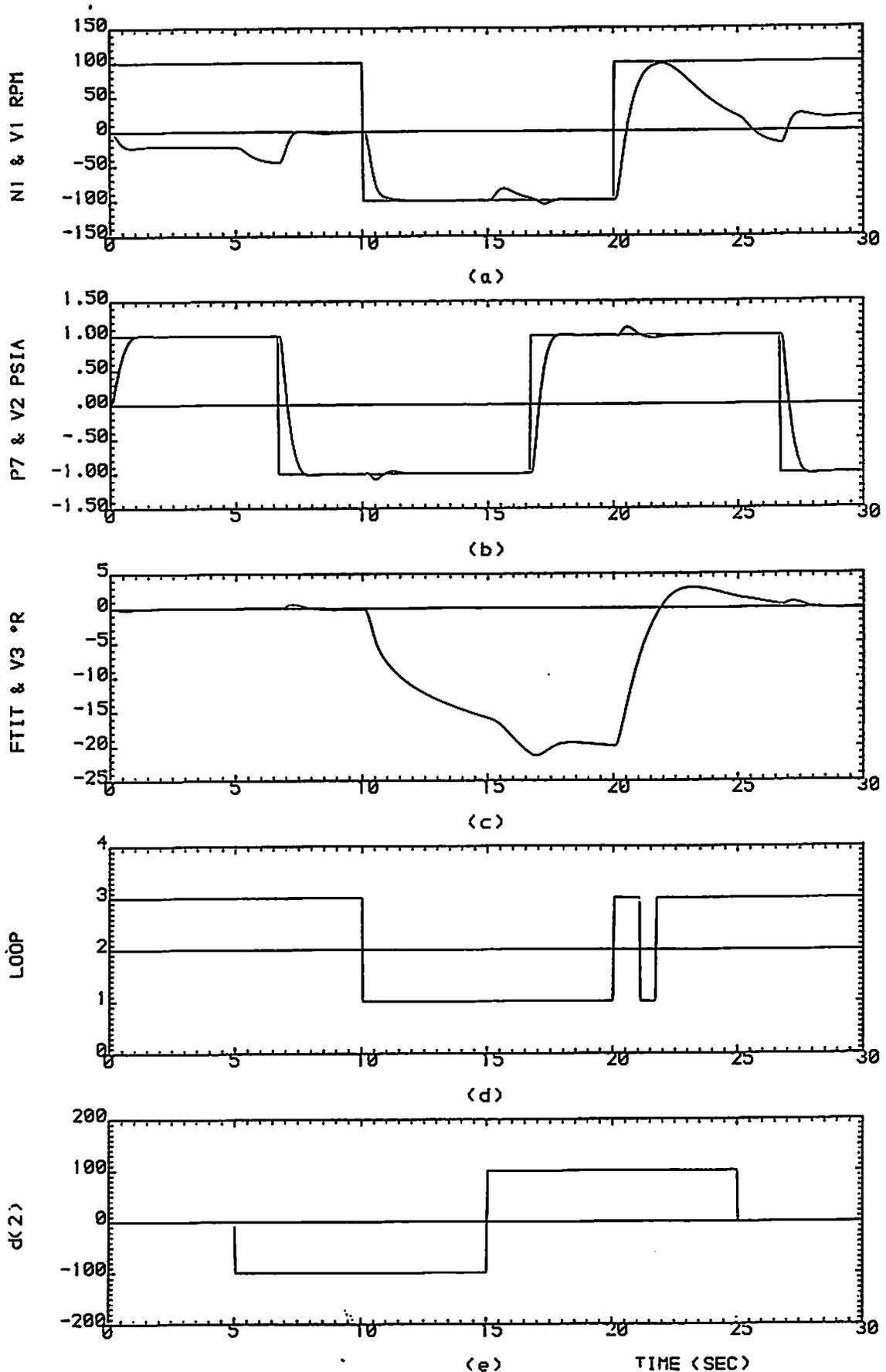
Fig 9.3 Responses of F100 engine under digital
self-selecting PID control with disturbance
Structure 2: [P7,N1] & [P7,FTIT]
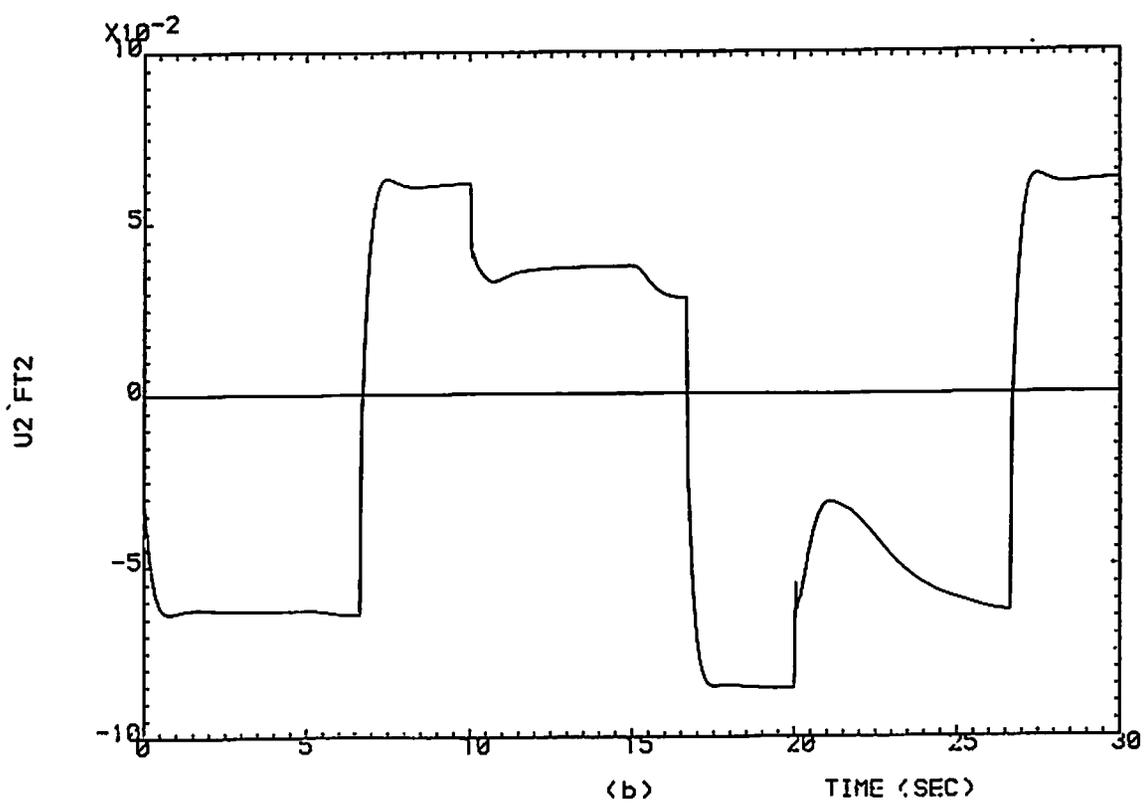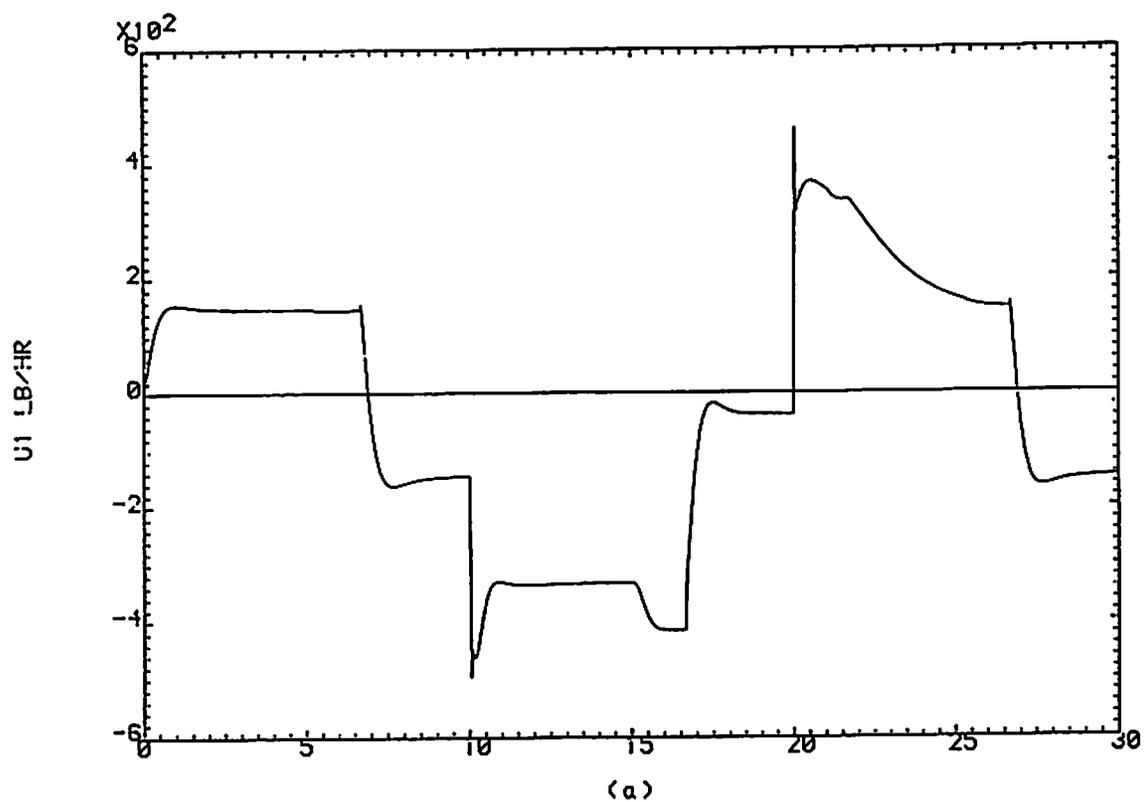
Fig 9.4   Manipulated variables of F100 engine
under digital self-selecting PID control
with disturbance
Structure 2: [P7,N1] & [P7,FTIT]
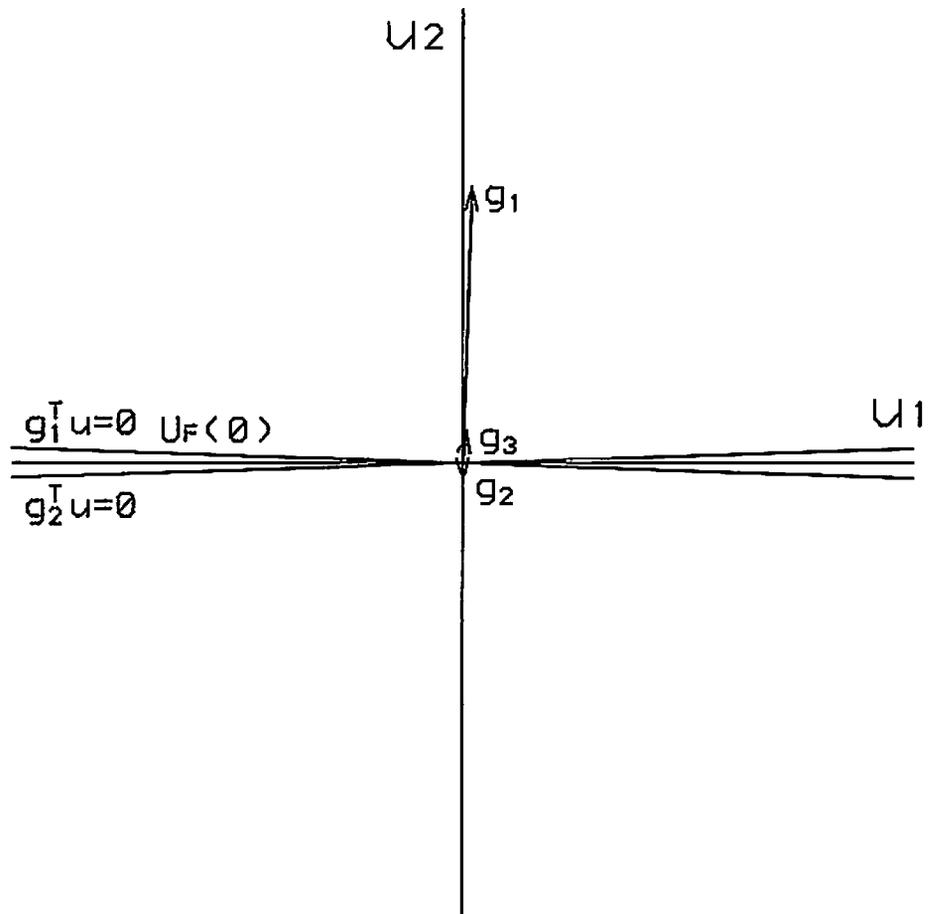
Fig 9.5 U-space

Plant: Linear F100 engine model at PLA 67 deg
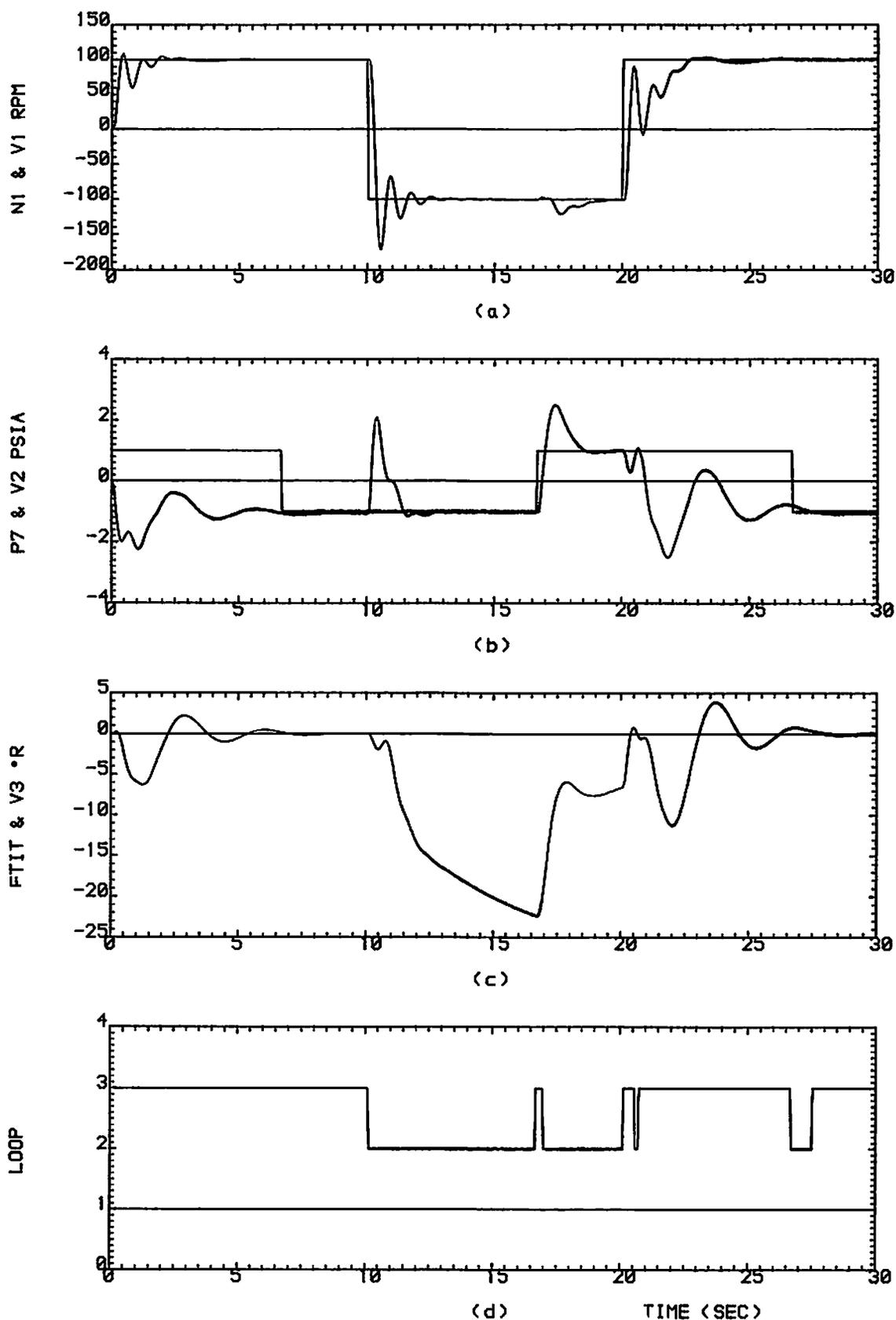
Fig 9.6   Responses of F100 engine under self-selecting control
Nominal plant: Intermediate (PLA 83 deg)
Actual plant: PLA 67 deg
Structure 1: [N1,P7] & [N1,FTIT]
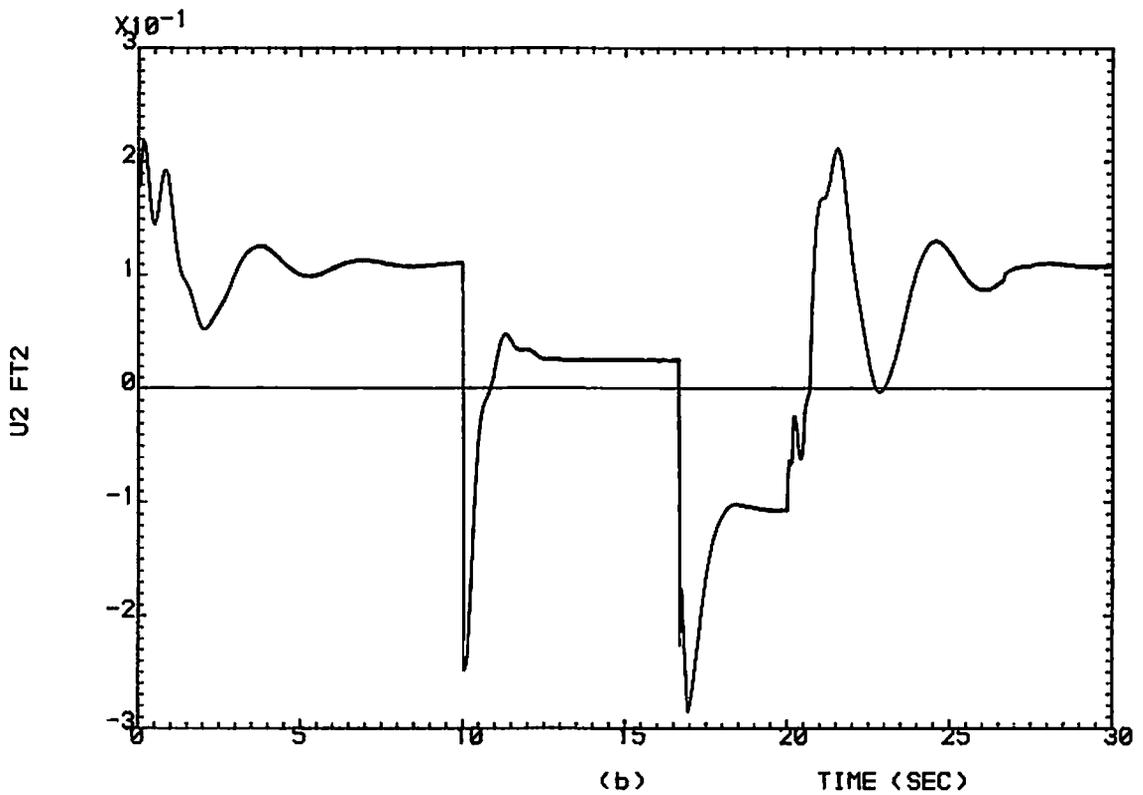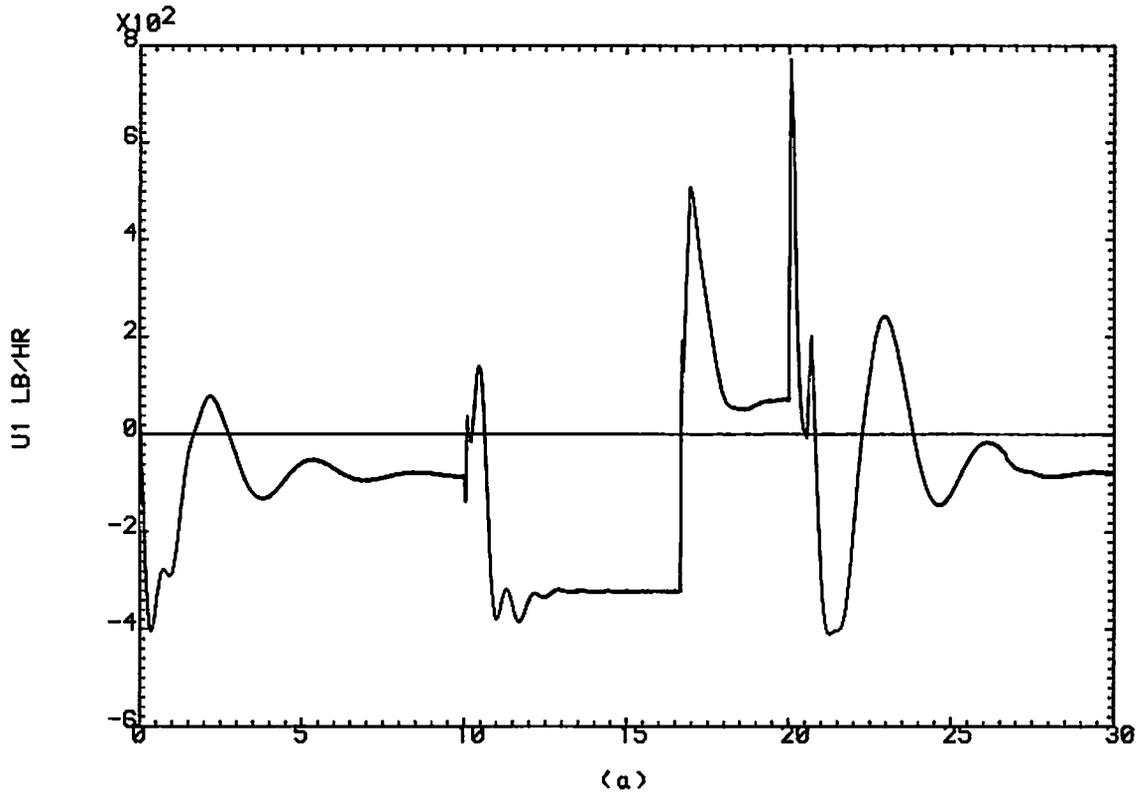
Fig 9.7   Manipulated variables of F100 engine
under self-selecting control
Nominal plant: Intermediate (PLA 83 deg)
Actual plant: PLA 67 deg
Structure 1: [N1,P7] & [N1,FTIT]
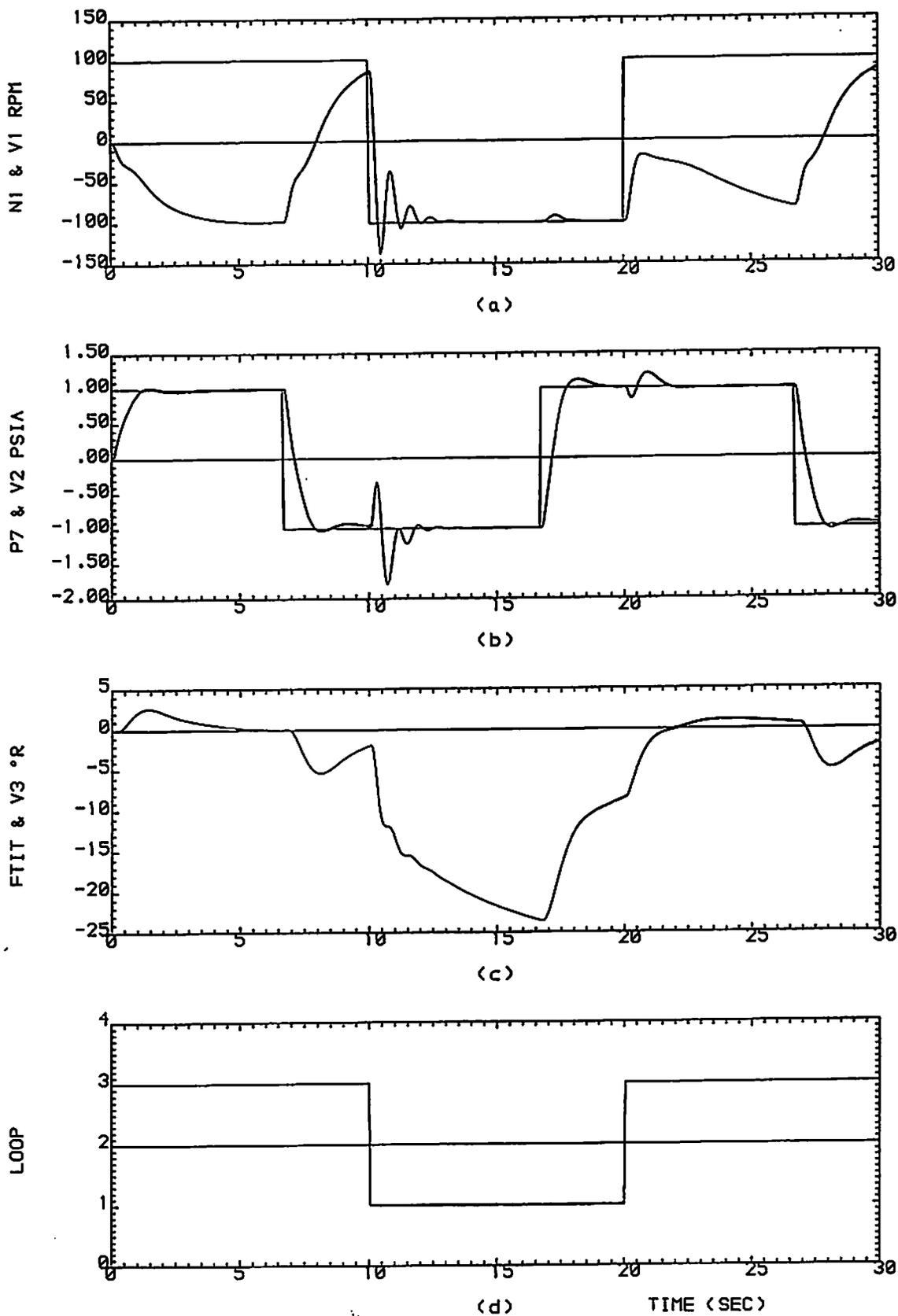
Fig 9.8   Responses of F100 engine under self-selecting control
Nominal plant: Intermediate (PLA 83 deg)
Actual plant: PLA 67 deg
Structure 2: [P7,N1] & [P7,FTIT]
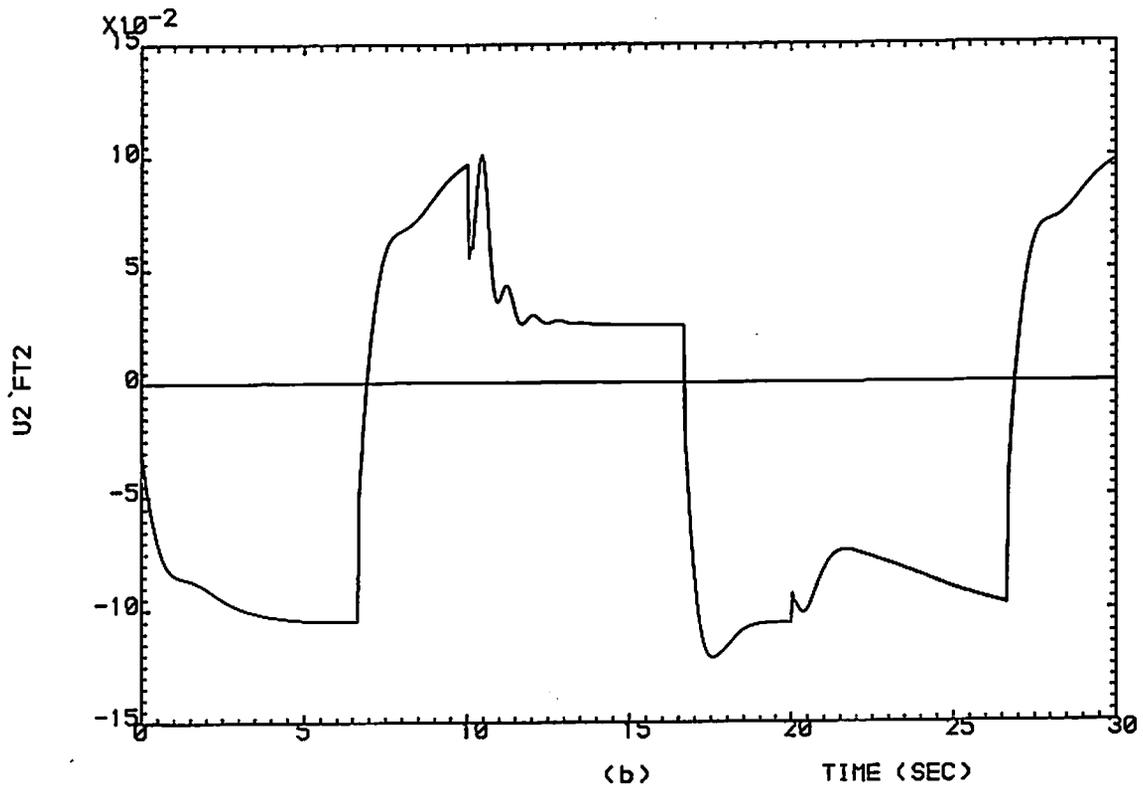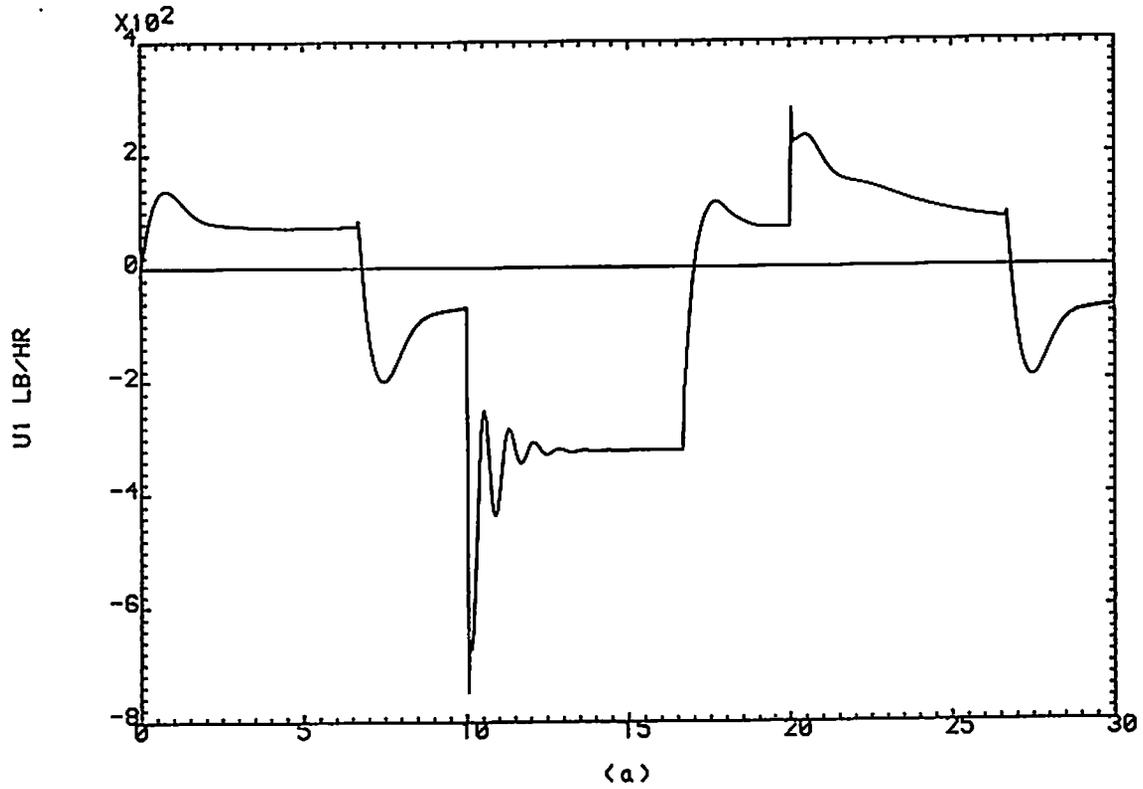
Fig 9.9   Manipulated variables of F100 engine
under self-selecting control
Nominal plant: Intermediate (PLA 83 deg)
Actual plant: PLA 67 deg
Structure 2: [P7,N1] & [P7,FTIT]
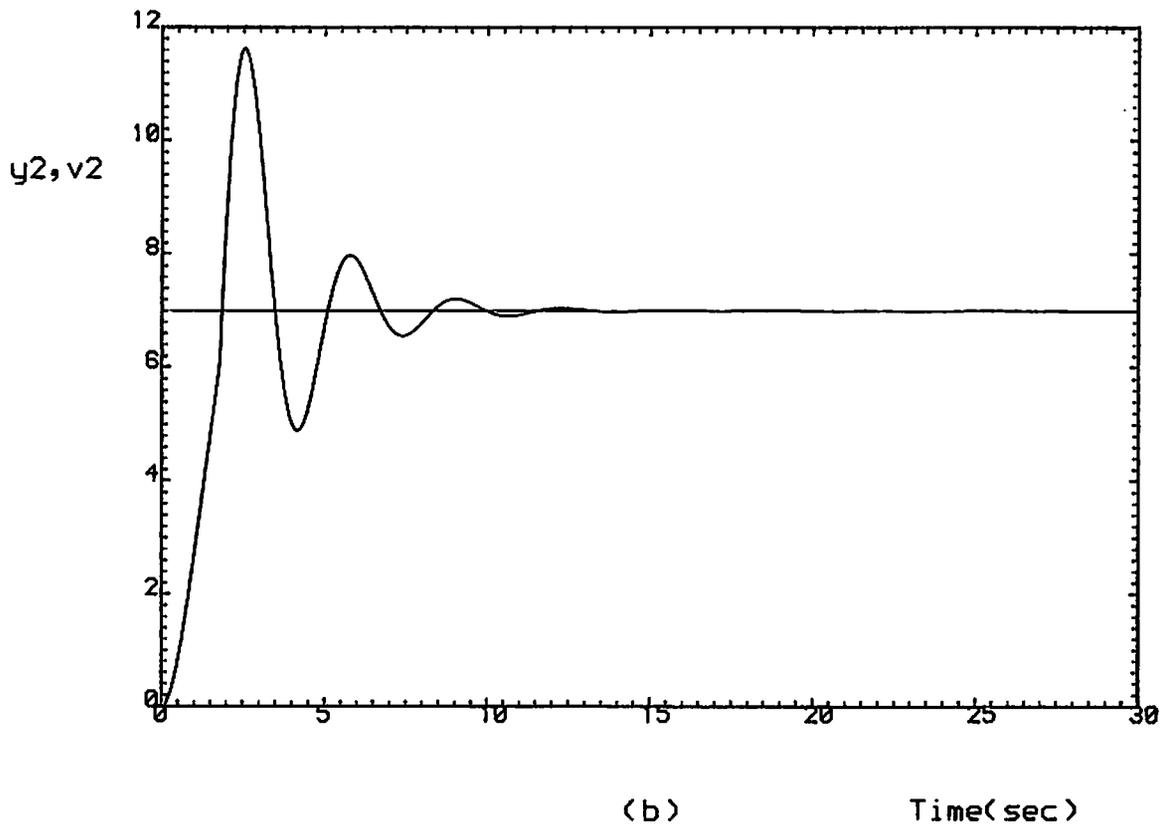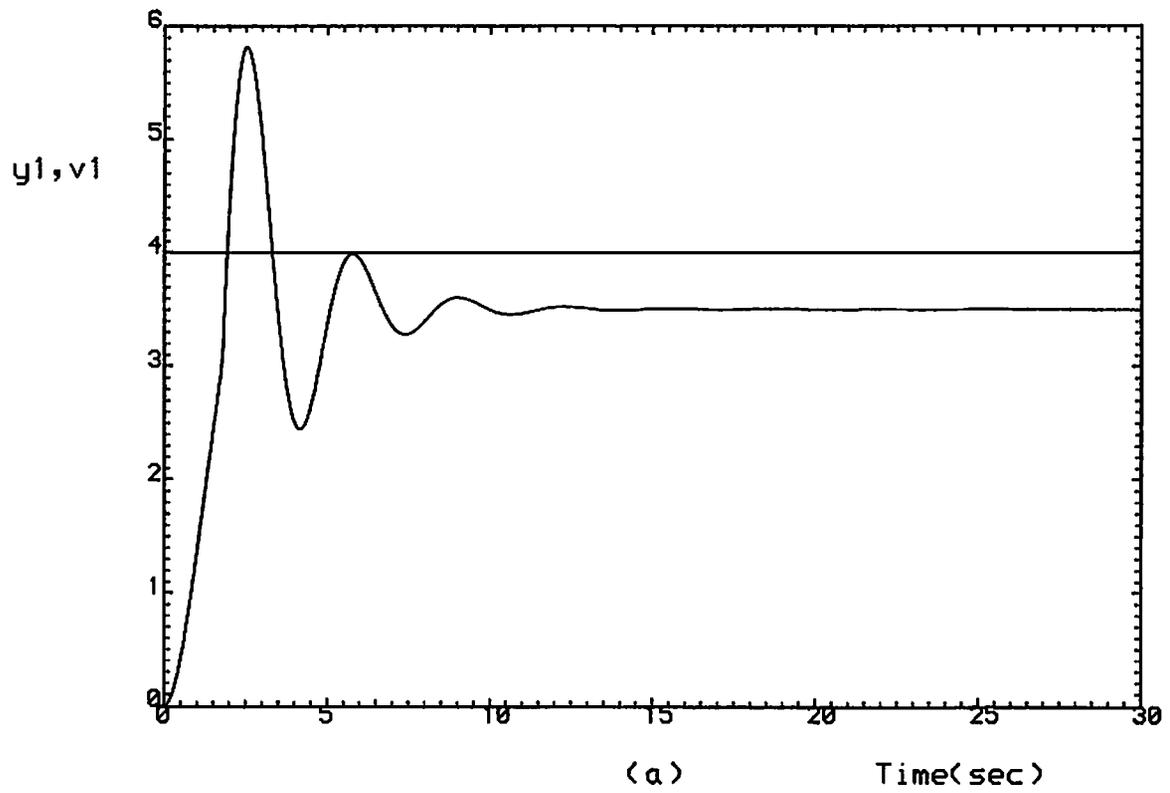
Fig 9.10 Closed-loop responses of the plant under
          digital supervisory self-selecting control
          Controller switching without bumpless transfer
Controller parameter: To=0.5sec, Alpha=0.5, Beta=0.5

INPUT

5

4

3

2

1

0

0   5   10   15   20   25   30

(a)   Time(sec)

LOOP

3

2

1

0

0   5   10   15   20   25   30

(b)   Time(sec)
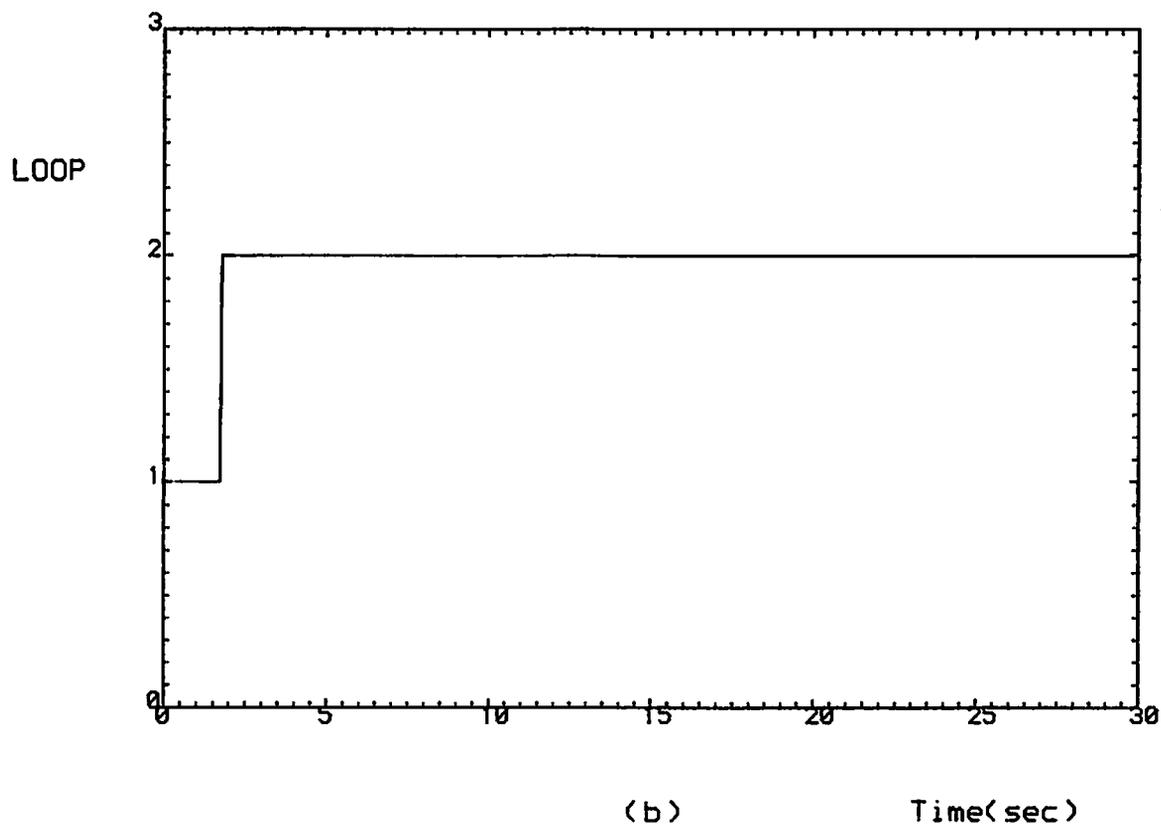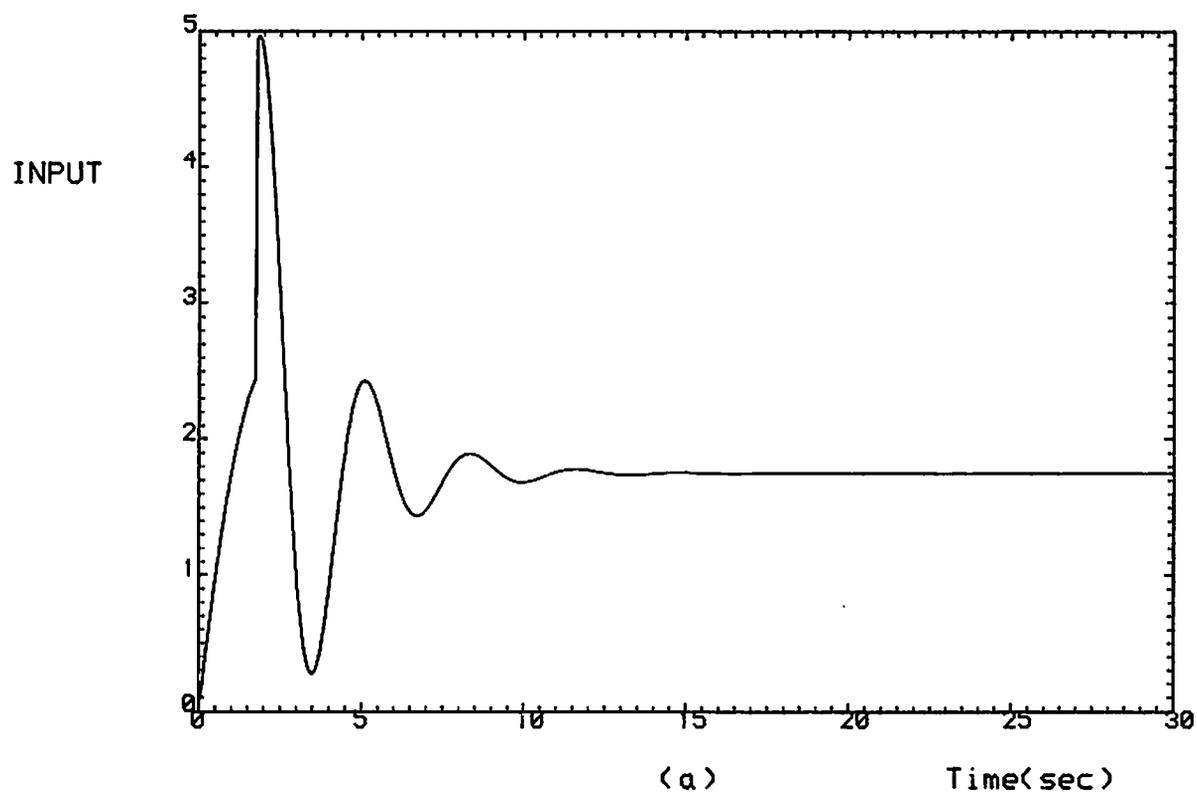
Fig 9.11   Input and loop index under
            digital supervisory self-selecting control
            Controller switching without bumpless transfer
Controller parameter: To=0.5sec, Alpha=0.5, Beta=0.5
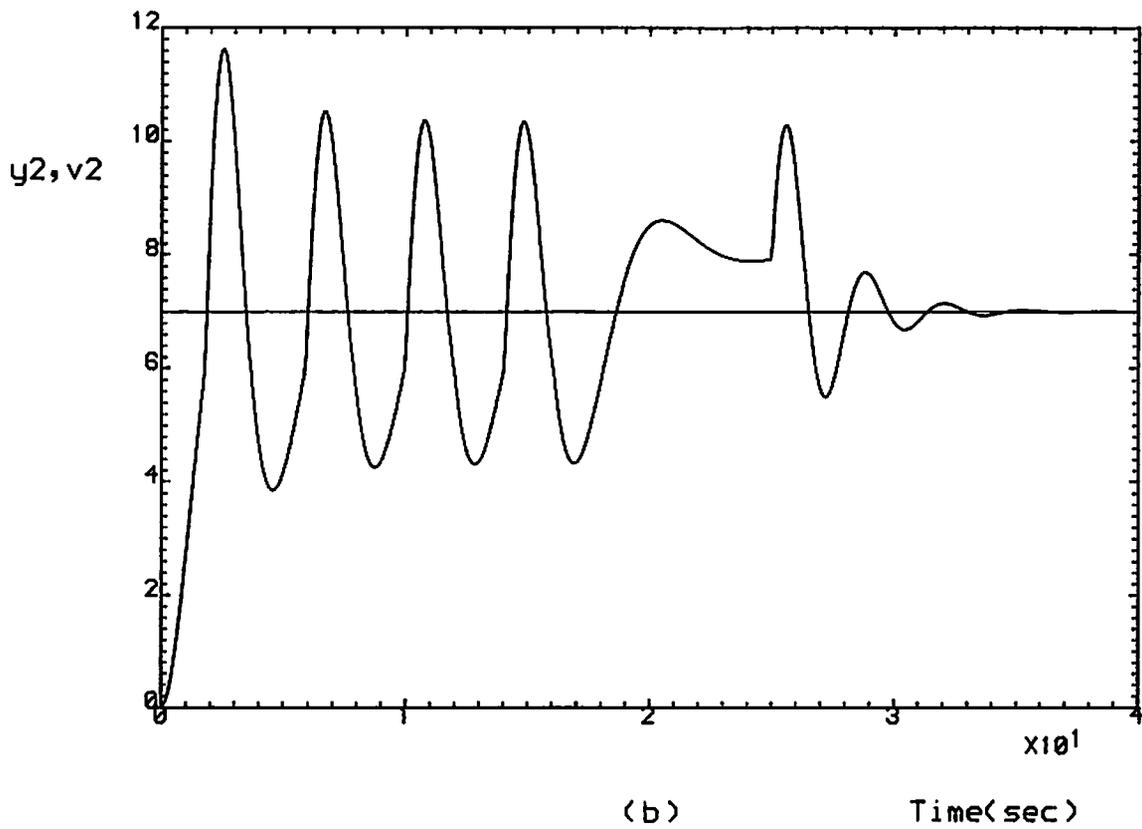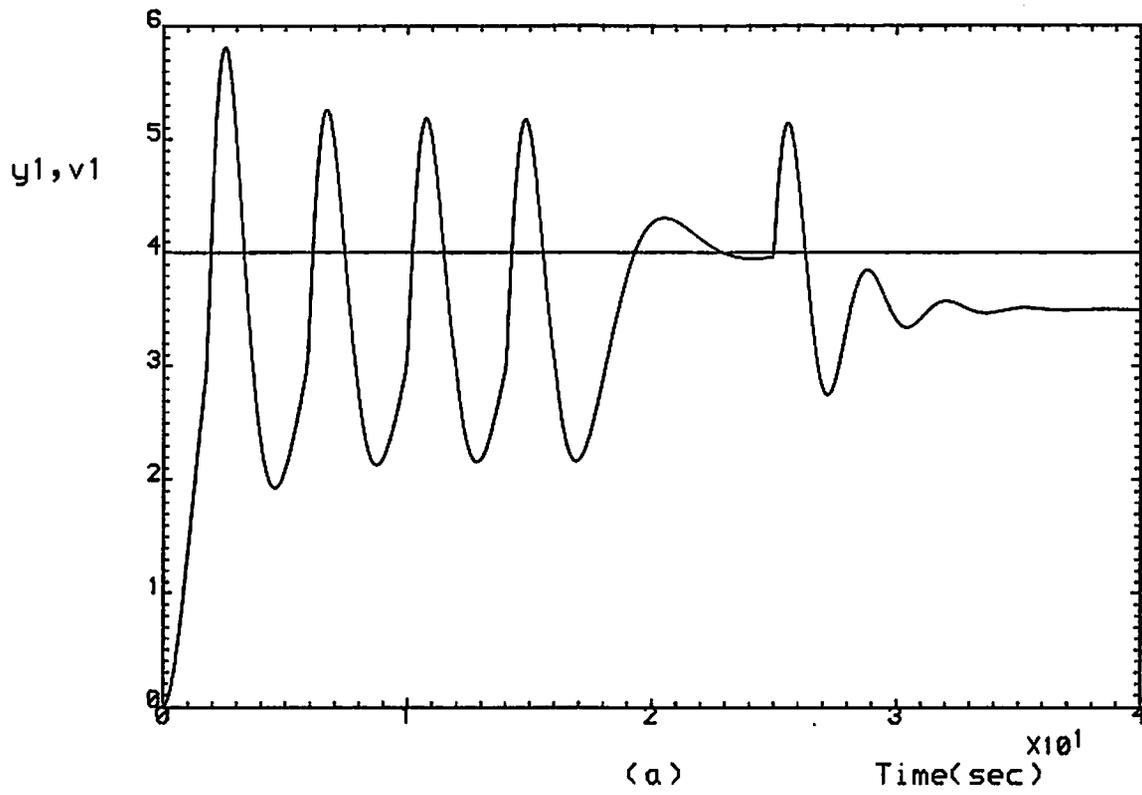
Fig 9.12   Closed-loop responses of the plant under
           digital supervisory self-selecting control
           Controller switching without bumpless transfer
Controller parameter: To=5.0sec, Alpha=0.9, Beta=0.5

INPUT

(a)   Time(sec)

LOOP

(b)   Time(sec)
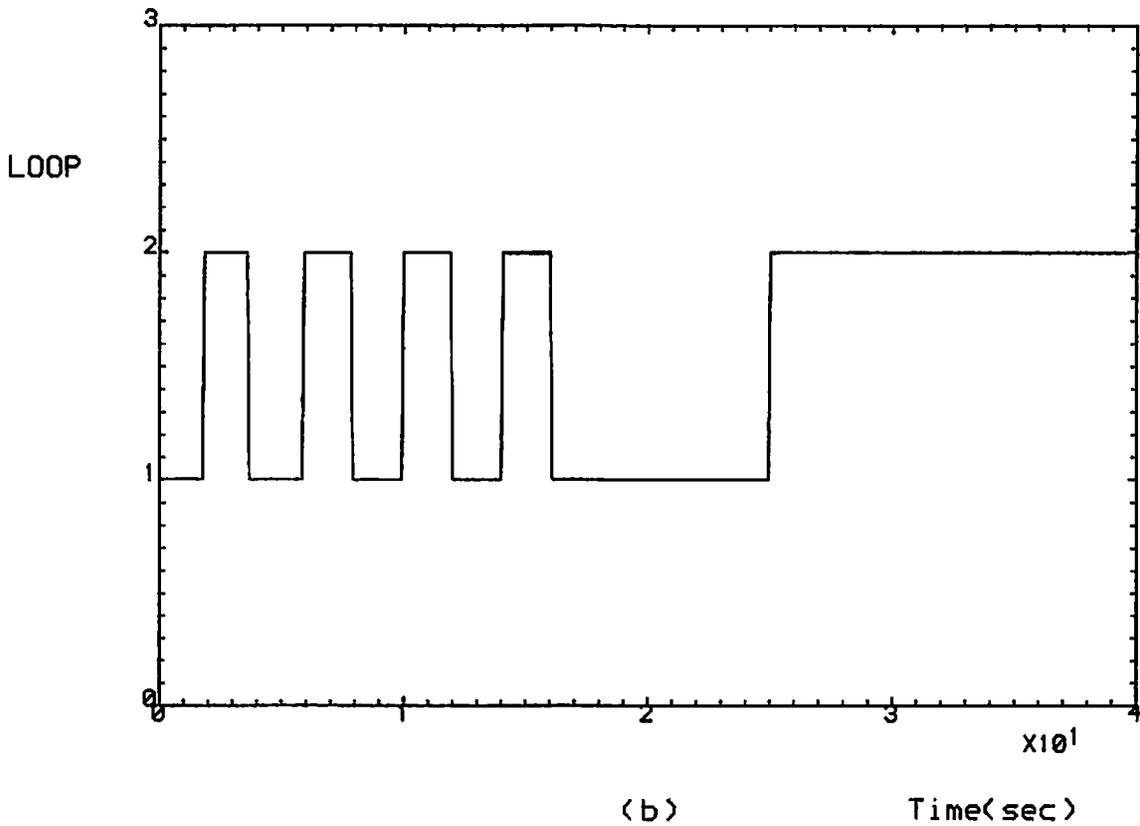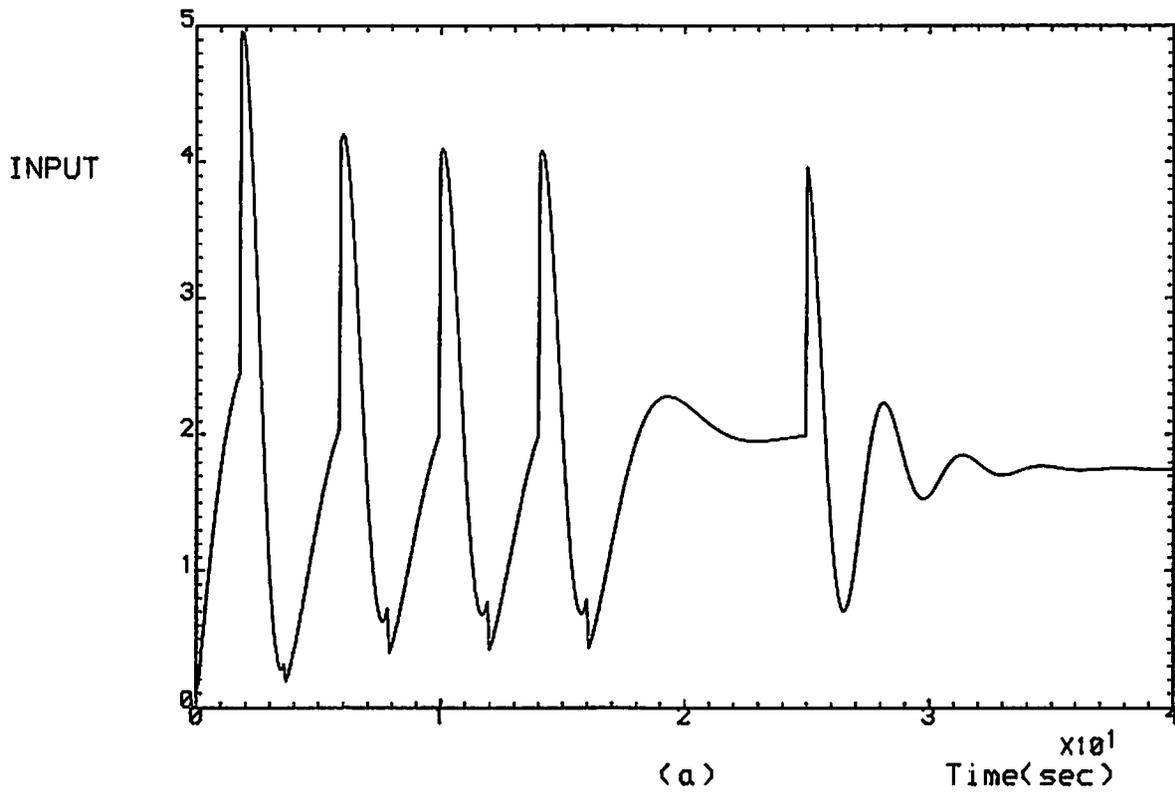
Fig 9.13   Input and loop index under
            digital supervisory self-selecting control
            Controller switching without bumpless transfer
Controller parameter: To=5.0sec, Alpha=0.9, Beta=0.5
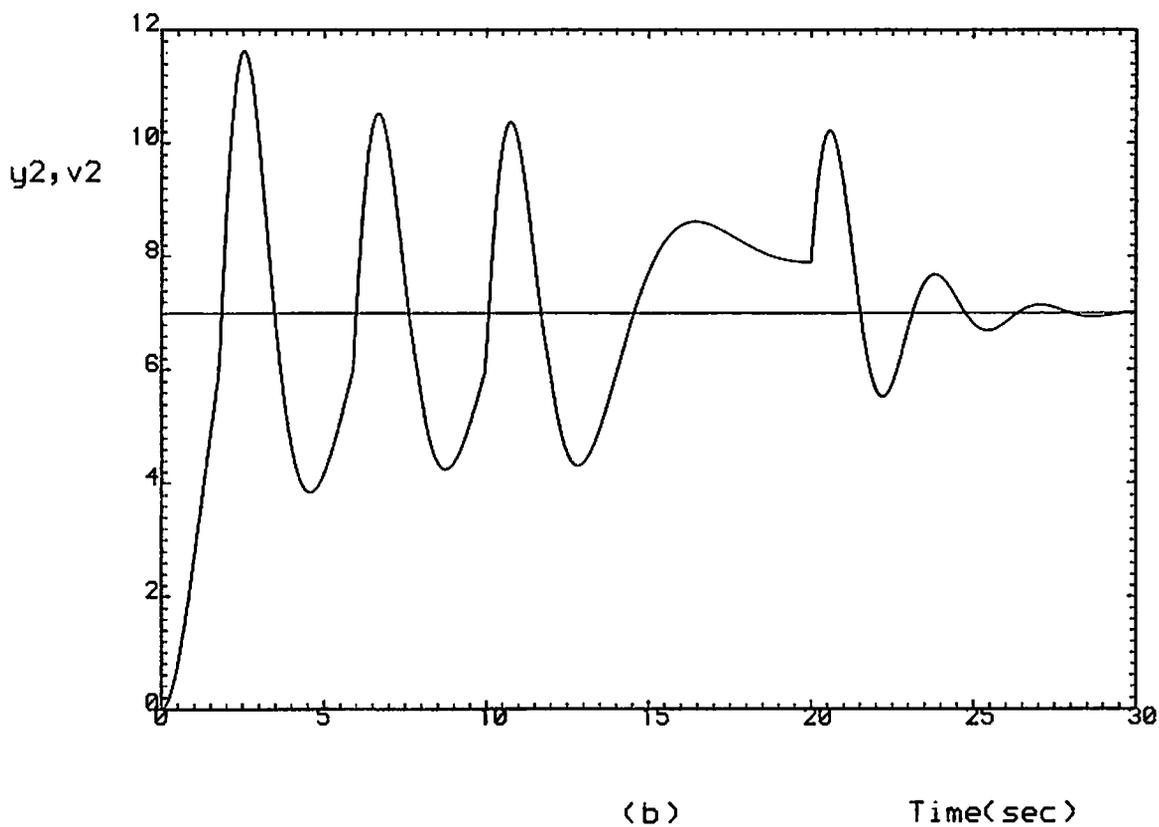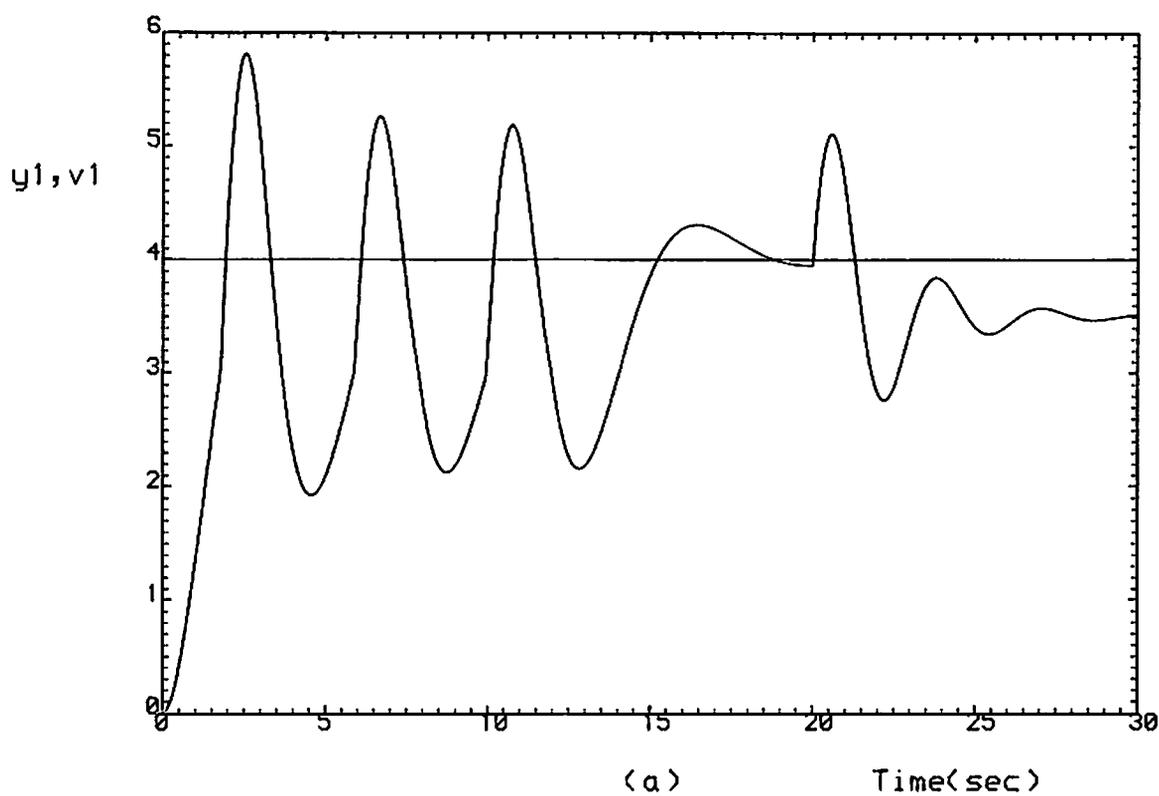
Fig 9.14   Closed-loop responses of the plant under
           digital supervisory self-selecting control
           Controller switching without bumpless transfer
Controller parameter: To=5.0sec, Alpha=0.5, Beta=0.1

INPUT

(a)   Time(sec)

LOOP

(b)   Time(sec)
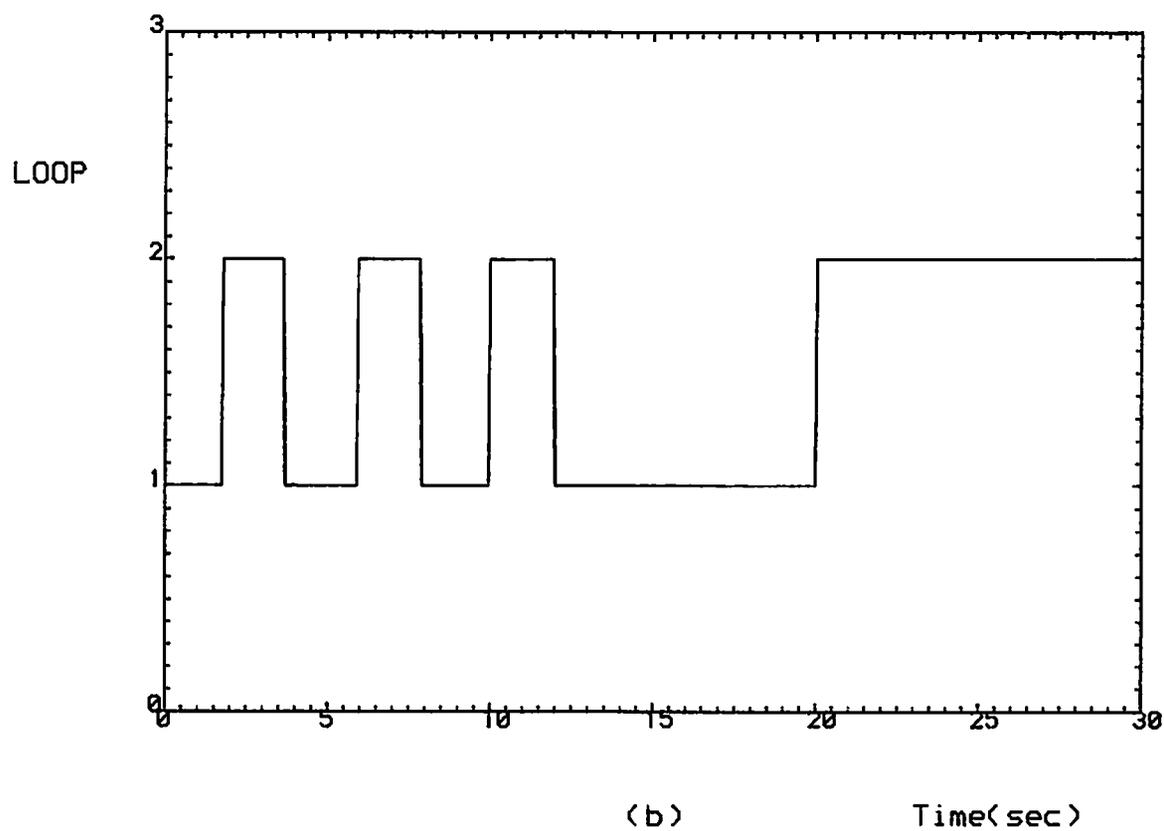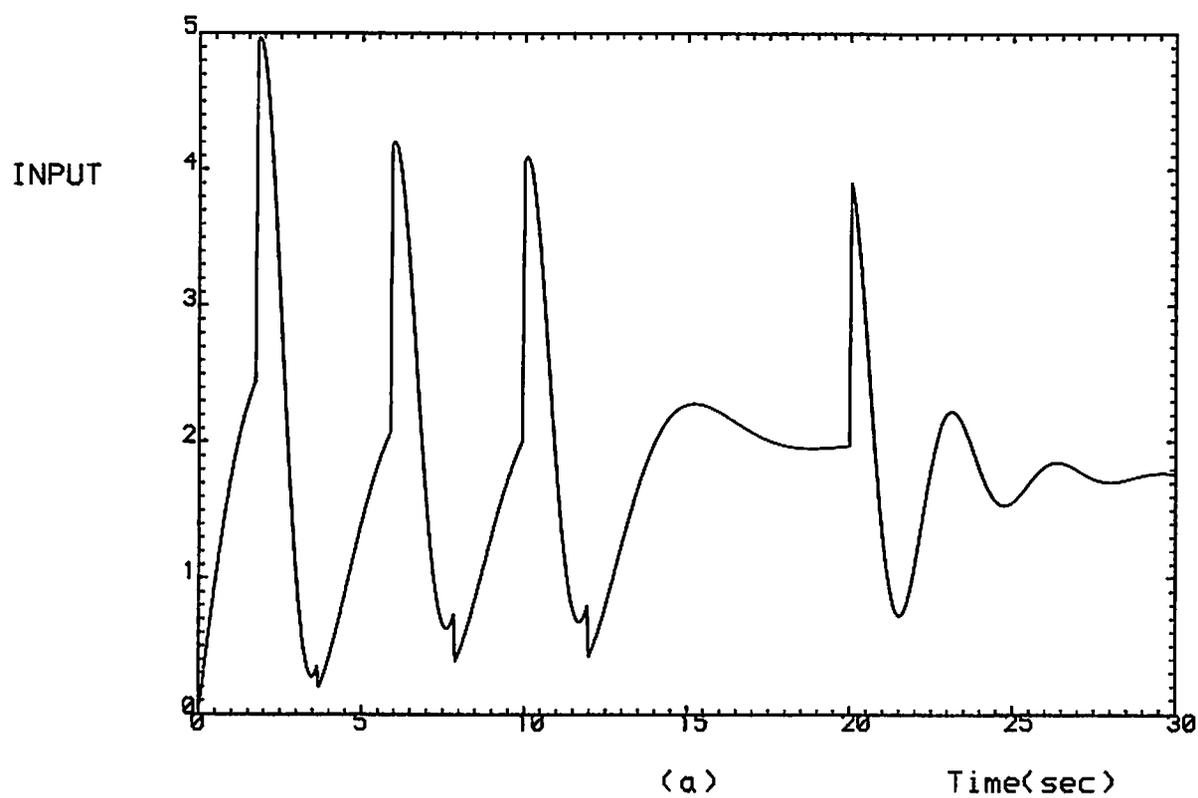
Fig 9.15   Input and loop index under
            digital supervisory self-selecting control
            Controller switching without bumpless transfer
Controller parameter: To=5.0sec, Alpha=0.5, Beta=0.1

Fig 9.16   Closed-loop responses of the plant under
            digital supervisory self-selecting control
            Controller switching without bumpless transfer
Controller parameter: To=5.0sec, Alpha=0.5, Beta=0.9

INPUT

(a)   Time(sec)

LOOP

(b)   Time(sec)
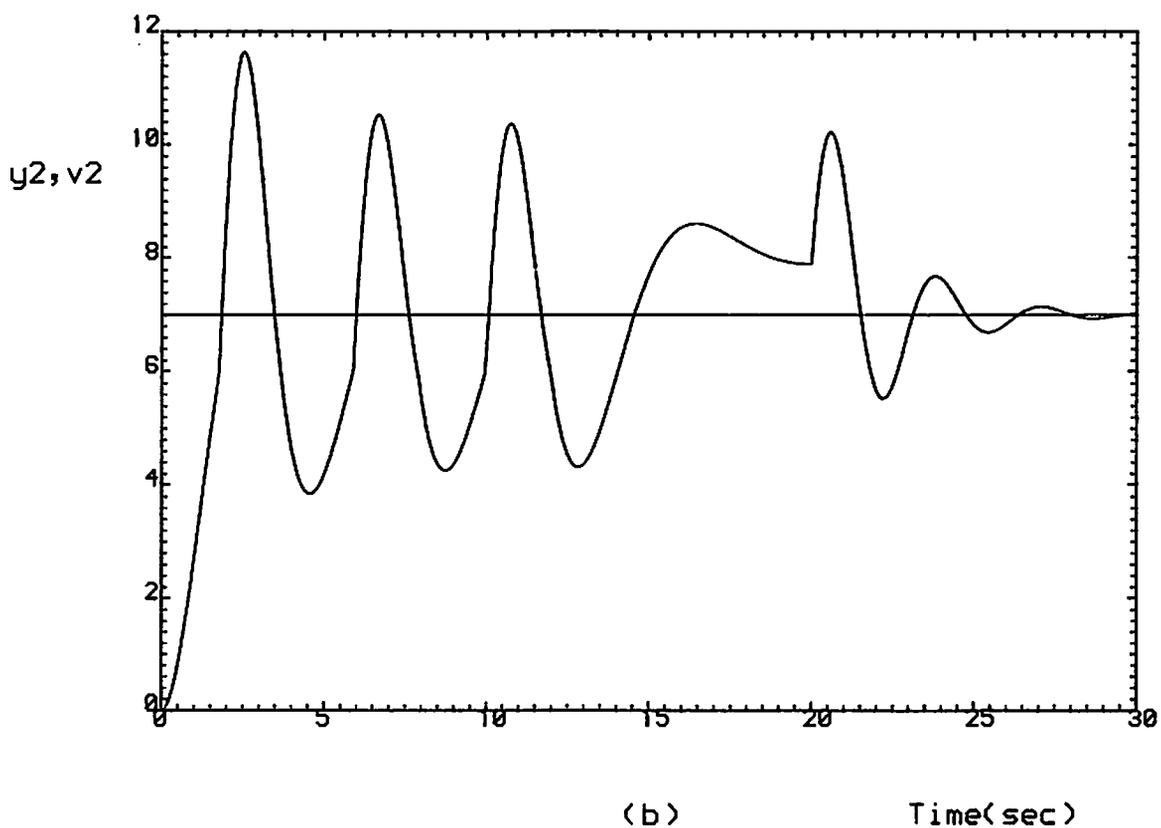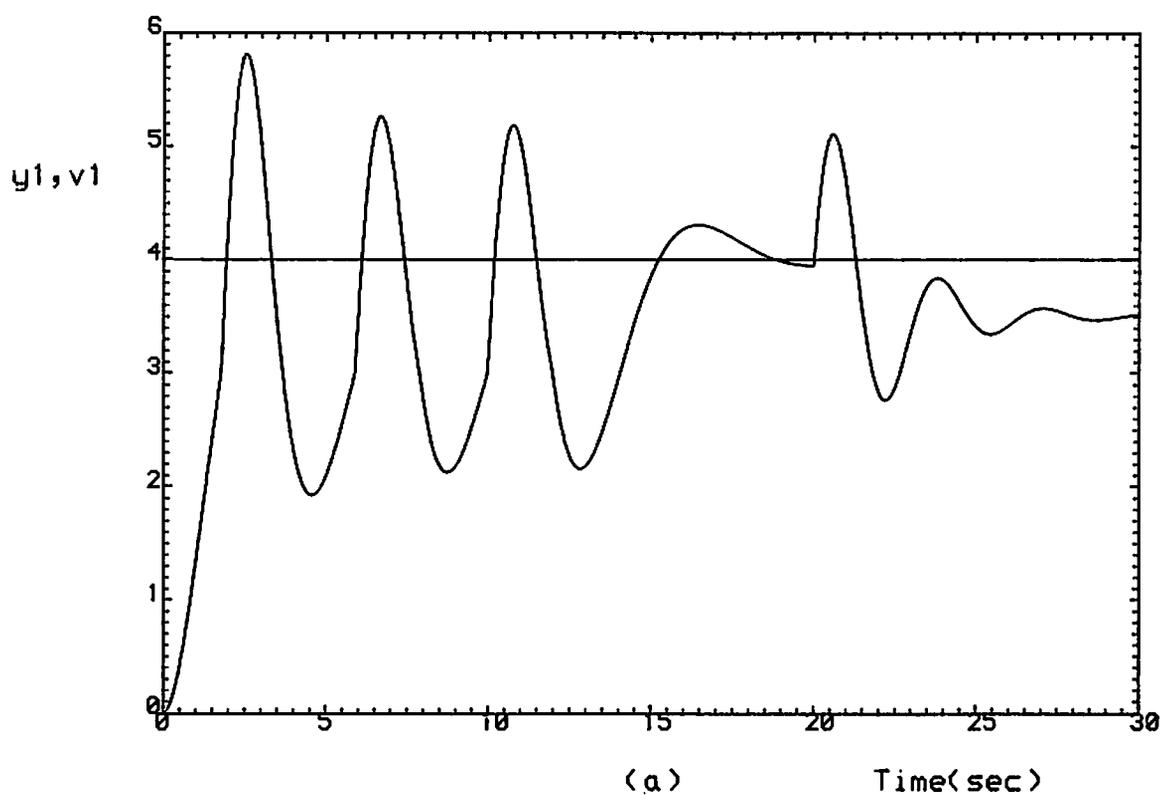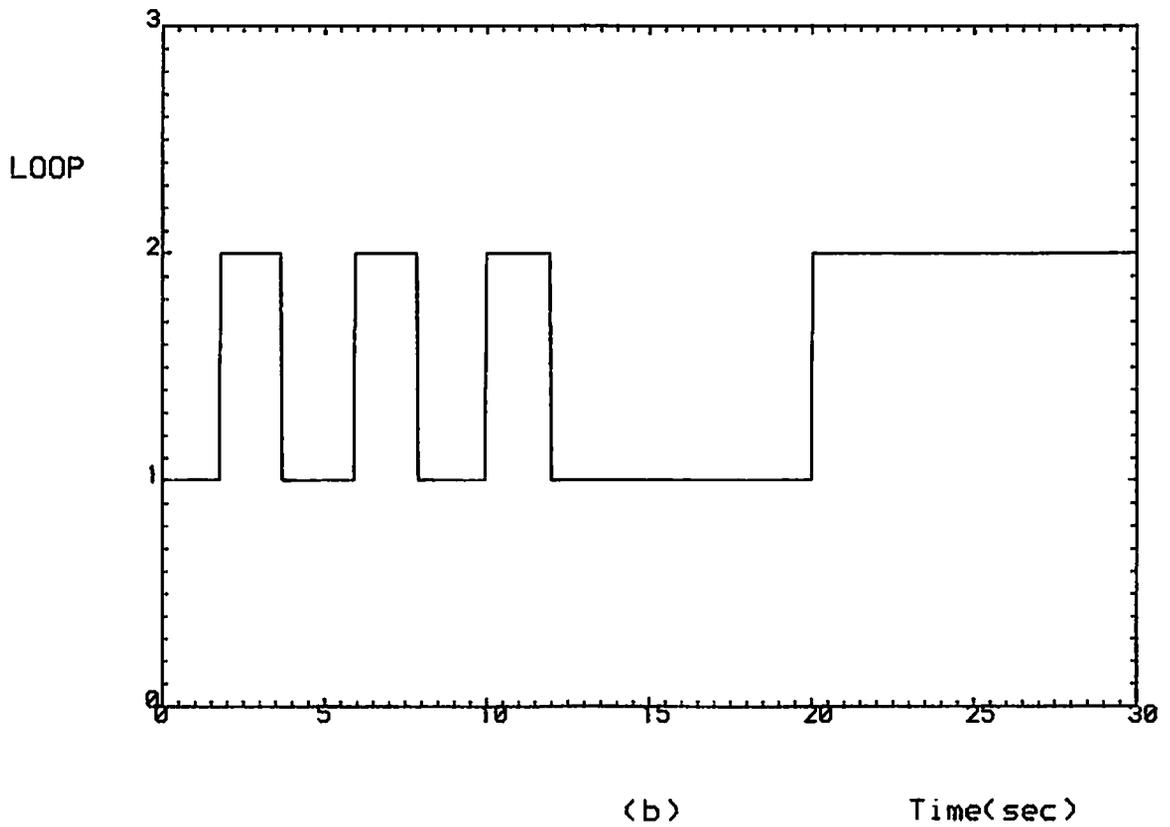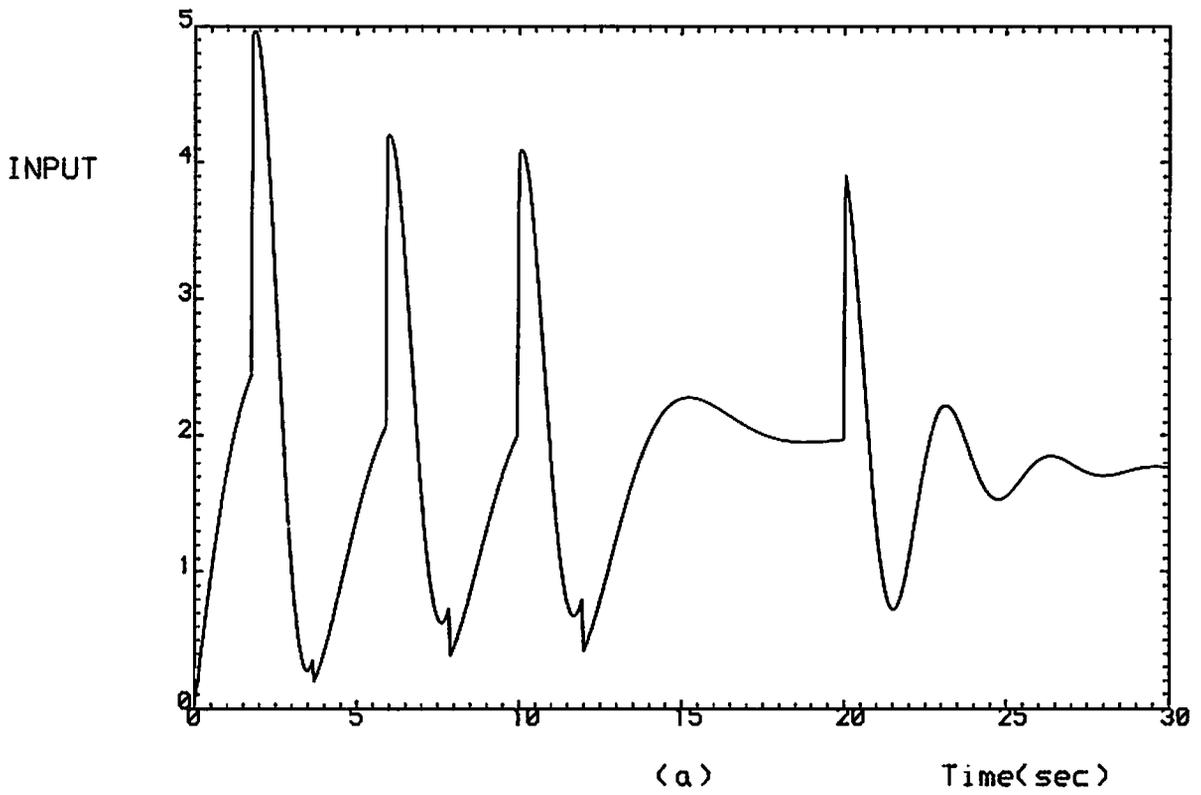
Fig 9.17   Input and loop index under
           digital supervisory self-selecting control
           Controller switching without bumpless transfer
Controller parameter: To=5.0sec, Alpha=0.5, Beta=0.9

PART V

DESIGN EXAMPLE

CHAPTER 10

DESIGN OF DIGITAL SELF-SELECTING CONTROLLERS

FOR F100 GAS-TURBINE ENGINE

10.1 Introduction

The controller design methodologies discussed in Parts II to IV are concerned with linear multivariable plants. However, most physical plants have more or less nonlinear characteristics. It is accordingly necessary to verify that these controllers can function for complex nonlinear plants. Therefore, in this chapter, a digital self-selecting controller (Chapter 6) is designed for a nonlinear F100 engine model (Appendix 4) and the adaptability of the controller to nonlinear complex multivariable plants is demonstrated.

10.2 Controller design

In order to compare the obtained results with those of the example in Chapter 6, it is convenient to choose a design point at Sea Level Static (SLS)/Intermediate power condition. The two manipulated inputs chosen are u_1: main burner fuel flow (lb/hr) and u_2: nozzle jet area (ft^2). The five controlled outputs chosen are $y_1(N_1)$: fan speed (rpm), $y_2(N_2)$: compressor speed (rpm), $y_3(P_3)$: compressor discharge pressure (psia), $y_4(P_7)$: augmentor pressure (psia), and $y_5(FTIT)$: fan-turbine inlet temperature ($^\circ$R). The obtained open-loop step responses of the F100 engine at SLS/Intermediate are shown

in Figs 10.1 and 10.2. The steady-state transfer-function
matrix is

$$
G = G(0) = \begin{bmatrix}
2.52760e-1 & , & 1451.03 \\
1.35074e-1 & , & -7.04544 \\
2.22808e-2 & , & -4.56090 \\
2.08151e-3 & , & -7.70092 \\
1.46573e-2 & , & 2.96426
\end{bmatrix} \in R^{5 \times 2}, \quad (10.1)
$$

so that it is clear from Theorem 4.5 that $G \in$ Class I.
Therefore, in order to apply Algorithm 5.1, the set of feasible
inputs $U_F(0)$ (Definition 4.2) is shown in Fig 4.5, where
$g_1^T, \ldots, g_5^T$ are the row vectors of G. Then, it is evident from
Fig 4.5 that both N_1 and P_7 represent extreme rays of $U_F(0)$,
that both extreme rays have unique representations, and
therefore that self-selecting controllers can be synthesised
based on either N_1 or P_7 as the permanently controlled output.
However, since the structure based on P_7 provided the better
overshooting characteristics in the examples in Chapter 6, this
structure is chosen. Therefore, the controlled subsets of
plant outputs are

$$
\left.
\begin{aligned}
Y_1 &= \{P_7 , N_1\} \\
Y_2 &= \{P_7 , N_2\} \\
Y_3 &= \{P_7 , P_3\} \\
Y_4 &= \{P_7 , FTIT\}
\end{aligned}
\right\} . \qquad (10.2)
$$

It is assumed that the sub-plant step-response matrices $H^{(i)}(t)$
$\in R^{2 \times 2}$ $(i = 1,2,3,4)$ and the sub-plant steady-state
transfer-function matrices $G^{(i)}(0) \in R^{2 \times 2}$ $(i = 1,2,3,4)$

correspond to these subsets Y_i (i = 1,2,3,4). The corresponding minimum singular values of the sub-plant step-response matrices $(\sigma_{min}[H^{(i)}(t)]$ (i = 1,2,3,4)) are obtained from Figs 10.1 and 10.2 and their plots are shown in Figs 10.3 to 10.6. It is considered from these figures that none of the $\sigma_{min}[H^{(i)}(t)]$ (i = 1,2,3,4) vanishes and therefore that all sub-plants corresponding to Y_i (i = 1,2,3,4) have minimum-phase characteristics (Porter and Jones (1985c)). Furthermore, these figures indicate that the $G^{(i)}(0)$ (i = 1,2,3,4) are well-conditioned since $\sigma_{min}[H^{(i)}(+\infty)]$ (i = 1,2,3,4) are not small.

The sampling period is chosen as 0.05 seconds which reflects both the required speed of response and the fact that the $\sigma_{min}[H^{(i)}(0.05)]$ (i = 1,2,3,4) are not small so that the $H^{(i)}(0.05)$ are well-conditioned, where the $H^{(i)}(0.05)$ are sub-matrices of H(0.05) obtained from Figs 10.1 to 10.5 and

$$H(0.05) = \begin{bmatrix} 0.428846e{-}3 & , & 2.86270 \\ 0.406083e{-}3 & , & 0.524400e{-}1 \\ 0.151410e{-}3 & , & -0.148817e{-}2 \\ 0.400079e{-}5 & , & -0.598165 \\ 0.169012e{-}4 & , & -0.163060e{-}2 \end{bmatrix}. \tag{10.3}$$

The self-selecting controller is governed on the discrete-time set $T_T = \{0,T,2T,\ldots,kT,\ldots\}$ by the equations

$$J(kT) = \{j : e_{t_j}(kT) = \min_{i \in I_r} e_{t_i}(kT)\}, \tag{10.4}$$

$$\ell_k = \ell(kT) \in J(kT) \subset I_r, \tag{10.5}$$

and

$$u_k = u_{k-1} + TK_1^{(\ell_k)}(e_k - e_{k-1}) + T^2 K_2^{(\ell_k)} e_{k-1}$$
$$+ K_3^{(\ell_k)}(e_k - 2e_{k-1} + e_{k-2}) \qquad . \qquad (10.6)$$

Here, the index set of all the control loops is $I_r = \{1,2,3,4\}$, the index set of lowest-errors is $J(kT)$, the loop index of the actually selected loop is ℓ_k, the sub-error vector is $e_k = e_k^{(\ell_k)}$, and the input vector is $u_k = u(kT) \in R^m$. It is noted that the elements of the sub-error vectors which are used in the lowest-wins strategy equation (10.4) and in the control-law equation (10.6) have been scaled so that the steady-state gains of the open-loop plant for the fuel flow are equal (Chapter 6).

10.3 Nonlinear simulation

The excellent limit-tracking behaviour of the F100 engine at SLS/Intermediate power condition under the action of the resulting digital self-selecting controller tuned such that $A^{(1)}(0.05)\Pi^{(1)} = \text{diag}\{0.2, 0.01\}$, $A^{(2)}(0.05)\Pi^{(2)} = \text{diag}\{0.2, 0.05\}$, $A^{(3)}(0.05)\Pi^{(3)} = \text{diag}\{0.2, 0.1\}$, $A^{(4)}(0.05)\Pi^{(4)} = \text{diag}\{0.2, 0.05\}$, $\Sigma^{(1)} = \ldots = \Sigma^{(4)} = 50.0 I_2$, and $A^{(1)}(0.05)A^{(1)} = \ldots = A^{(4)}(0.05)A^{(4)} = 0.0 I_2$ is shown in Figs 10.7 and 10.8, where the loops show that N_1, N_2, P_3, and FTIT are controlled in turn whilst P_7 is permanently controlled.

Next, in order to verify the effectiveness of this controller in the face of a large thrust change, a fast acceleration from 80% N_2 to Intermediate is carried out. The resulting

closed-loop responses are shown in Figs 10.9 and 10.10, where the engine initially accelerates on the open-loop accelerating schedule and the self-selecting control begins operation at 1.3 seconds. It is clear from these figures that, although the nozzle area hits the minimum position at 1.3 seconds, the limit-tracking performance at Intermediate is satisfactory.

Finally, in order to examine the robustness of the controller (which is designed for Intermediate power condition - ie the nominal plant), the controller is applied to the different power condition (80% N_2, which corresponds to Power lever angle $\simeq$ 40 deg) - ie the actual plant. The steady-state transfer-function matrix of the actual plant is given in the form

$$
G_a = G_a(0) = \begin{bmatrix} 1.43313 & , & 811.014 \\ 6.81174e-1 & , & 142.496 \\ 3.74045e-2 & , & 4.90223 \\ 2.28222e-3 & , & -2.29121 \\ 4.45136e-2 & , & -10.3136 \end{bmatrix} \in R^{5 \times 2}. \quad (10.7)
$$

The input space for the actual plant is shown in Fig 10.11. It is clear from Fig 4.5 for the nominal plant and Fig 10.11 that the conditions (i) and (ii) of Theorem 9.1 are satisfied and therefore that the robustness assessment using perturbation matrices for the separate set-point tracking controllers is effective. The spectra of the perturbation matrices are

$$
\{\mu_1^{(1)}, \mu_2^{(1)}\} = \{0.3445, 3.0007\} \quad (10.8)
$$

$$
\{\mu_1^{(2)}, \mu_2^{(2)}\} = \{0.3440, 5.3464\} \quad (10.9)
$$

$$\{\mu_1^{(3)}, \ \mu_2^{(3)}\} \ = \ \{0.3418, \ 1.7490\} \hspace{3cm} (10.10)$$

$$\{\mu_1^{(4)}, \ \mu_2^{(4)}\} \ = \ \{0.2351, \ 2.8030\} \hspace{3cm} (10.11)$$

and therefore the condition (9.23) of Theorem 9.1 is satisfied. Thus, by Theorem 9.1, the separate set-point tracking PID controllers incorporated in the digital self-selecting controller can cope with such plant variations. The robustness of the complete closed-loop system can be verified in time-domain simulation. Therefore, a fast deceleration from Intermediate to 80% N_2 is carried out, where the controller is tuned as before. The resulting closed-loop responses are shown in Figs 10.12 and 10.13, where the open-loop decelerating schedule decides the fuel flow until 1.2 seconds. It is clear from these figures that, although the nozzle area hits the minimum position until 1.5 seconds, the digital self-selecting controller is robust in the face of plant variations in the sense that the separate set-point tracking controllers and the integrated self-selecting controller remain asymptotically stable and that only minimal performance degradation has occurred.

10.4 Conclusions

In this design example, a self-selecting controller has been designed for a nonlinear model of the F100 gas-turbine engine and simulation studies have been performed. The demonstrated excellent closed-loop performance and robustness property of

the designed limit-tracking system indicates that the self-selecting controllers can readily be designed for complex multivariable plants and that the adaptability of set-point tracking controllers - which underlie the self-selecting controllers - is very high.

In the case of highly nonlinear plants, a few controllers are designed for corresponding separate operating points. Then, it is important to check the convex structure of $U_F(0)$ (Definition 4.2) at such operating points before designing self-selecting controllers. If a particular set of plant outputs represents an extreme ray at all the operating points, then the synthesis of self-selecting controllers can be based upon such a set and therefore the effort in controller design and implementation can be reduced.
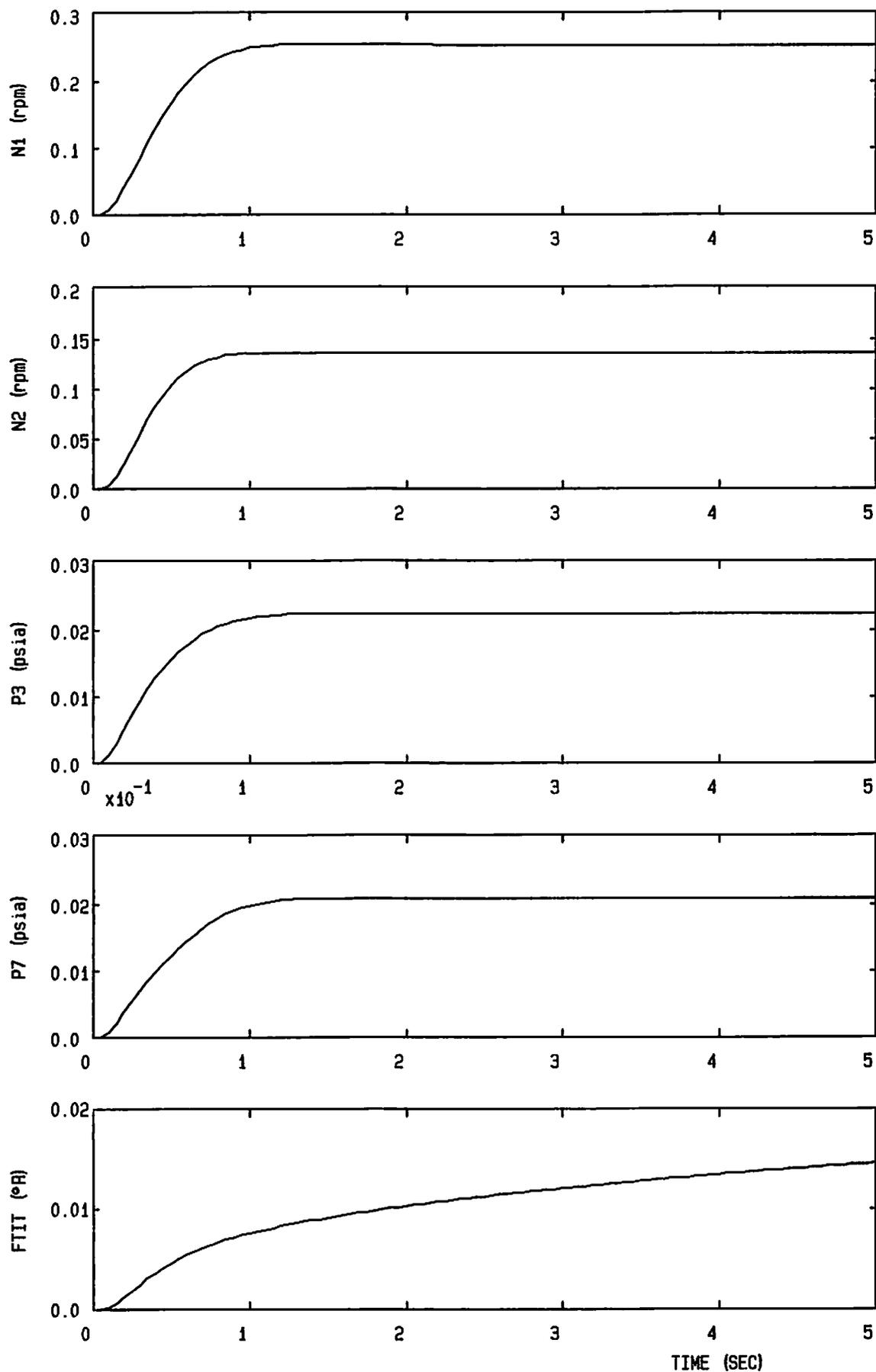
Fig 10.1 Open-loop step-responses of measurable outputs
F100 engine nonlinear model u=[1 0 0 0 0]
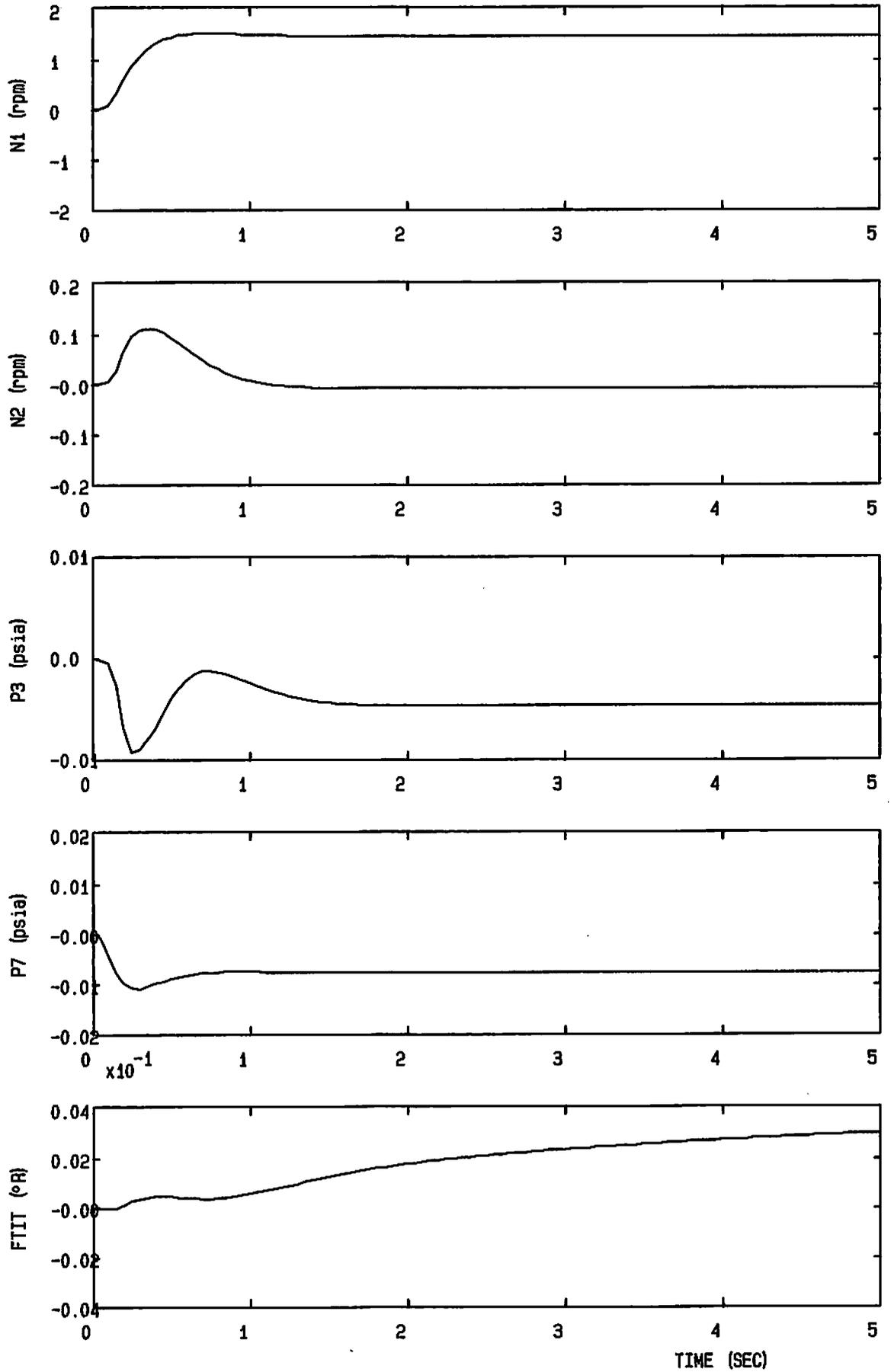
Fig 10.2 Open-loop step-responses of measurable outputs
F100 engine nonlinear model u=[0 0.001 0 0 0]
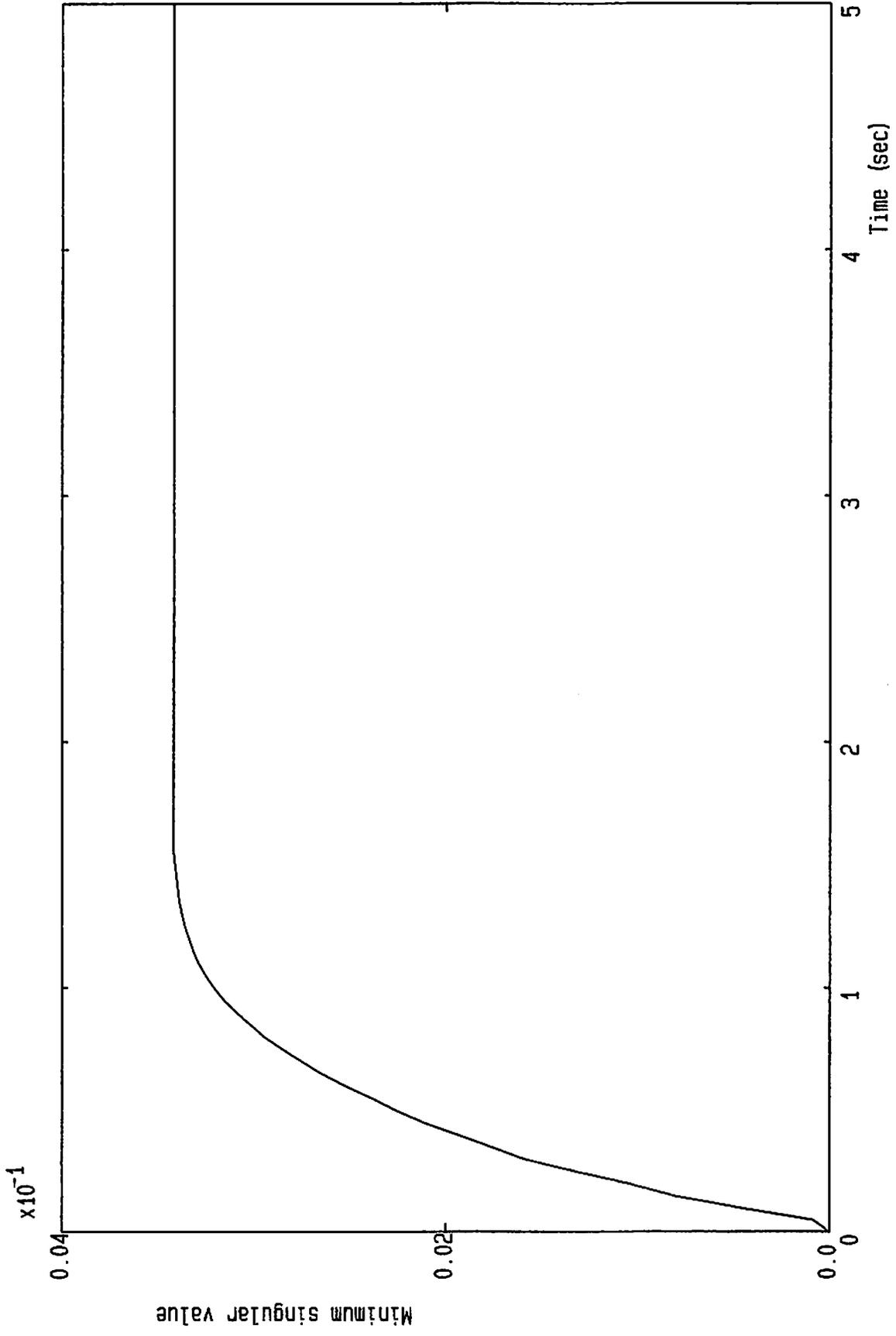
Fig 10.3 Minimum singular value plot of the plant step-response matrix
F100 engine nonlinear model Output [N1 P7] Input [WFMB AJ]
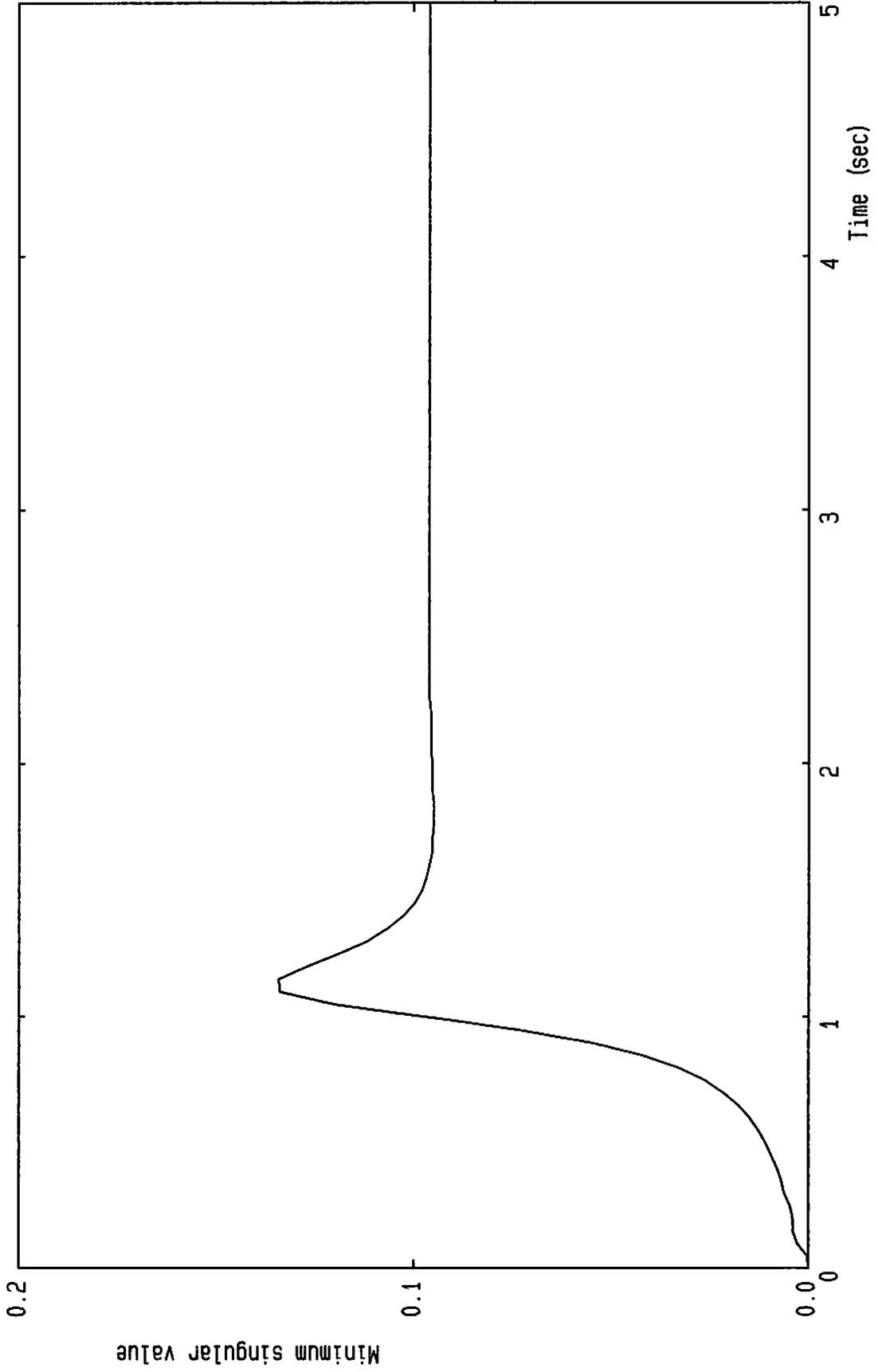
Fig 10.4   Minimum singular value plot of the plant step-response matrix
F100 engine nonlinear model   Output [N2 P7]   Input [WFMB AJ]
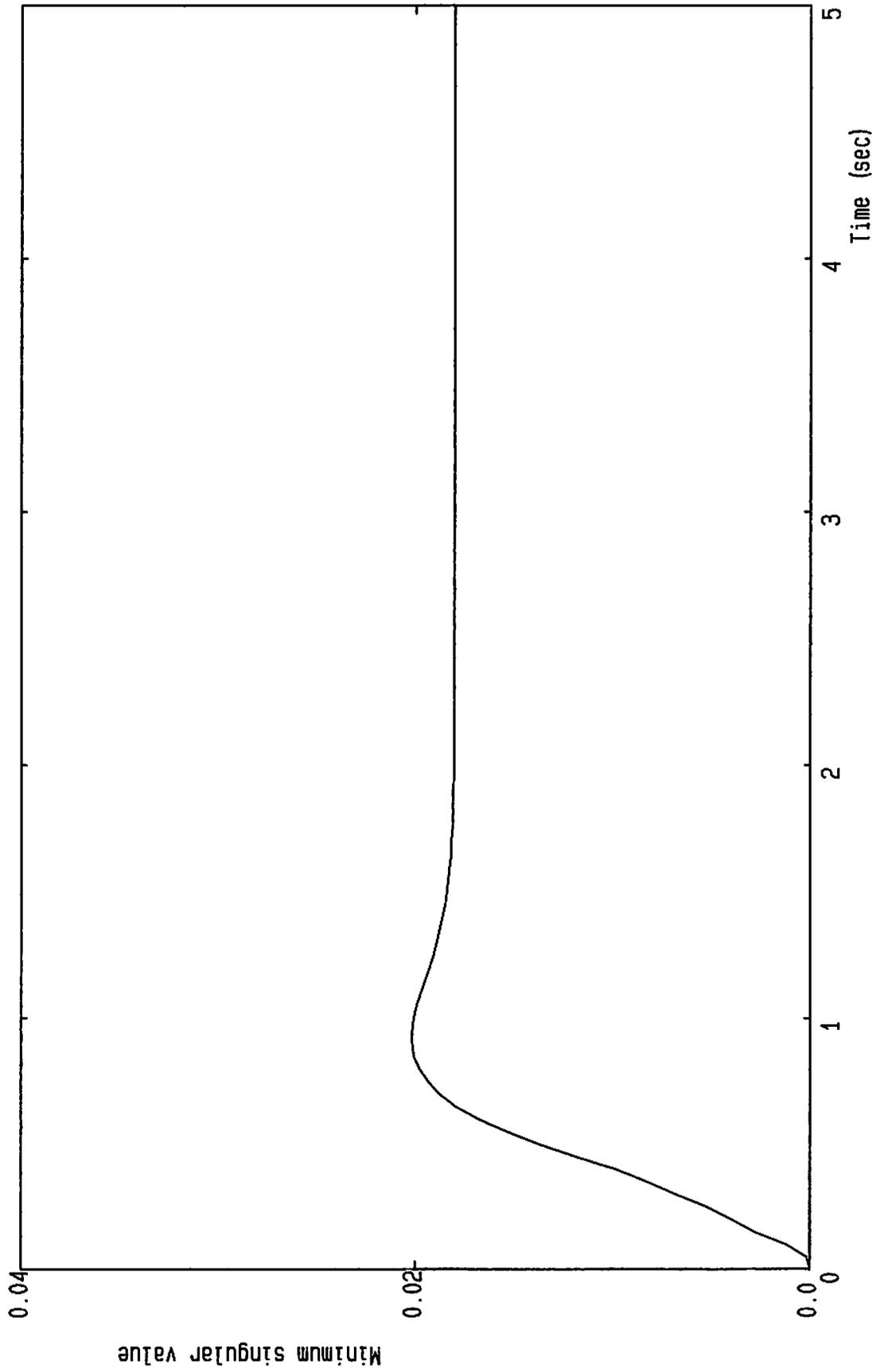
Fig 10.5 Minimum singular value plot of the plant step-response matrix
F100 engine nonlinear model Output [P3 P7] Input [WFMB AJ]
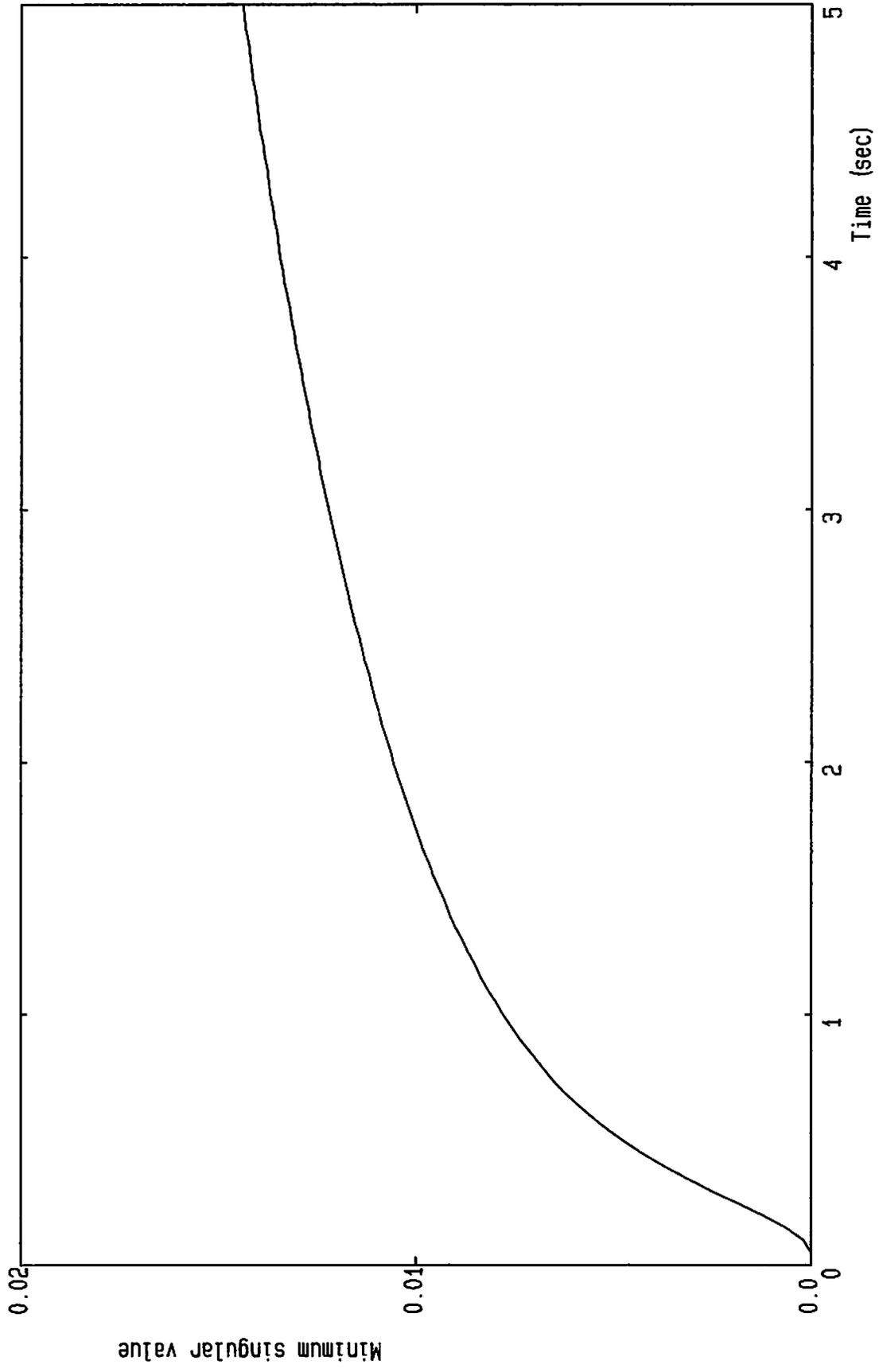
Fig 10.6 Minimum singular value plot of the plant step-response matrix
F100 engine nonlinear model Output [FTIT P7] Input [WFMB AJ]
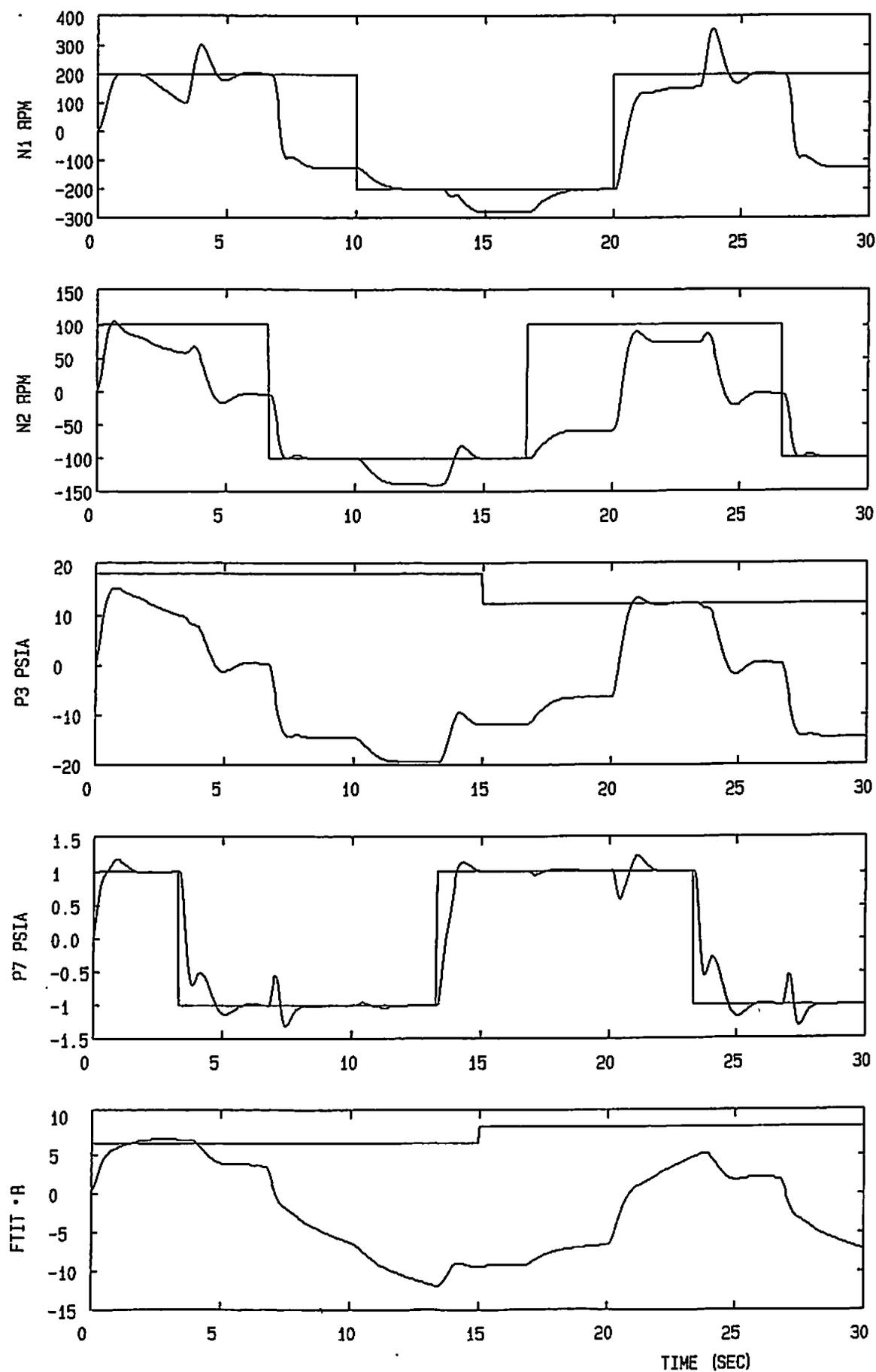
Fig 10.7 RESPONSES OF F100 ENGINE UNDER SELF-SELECTING CONTROL
RATING: INTERMEDIATE
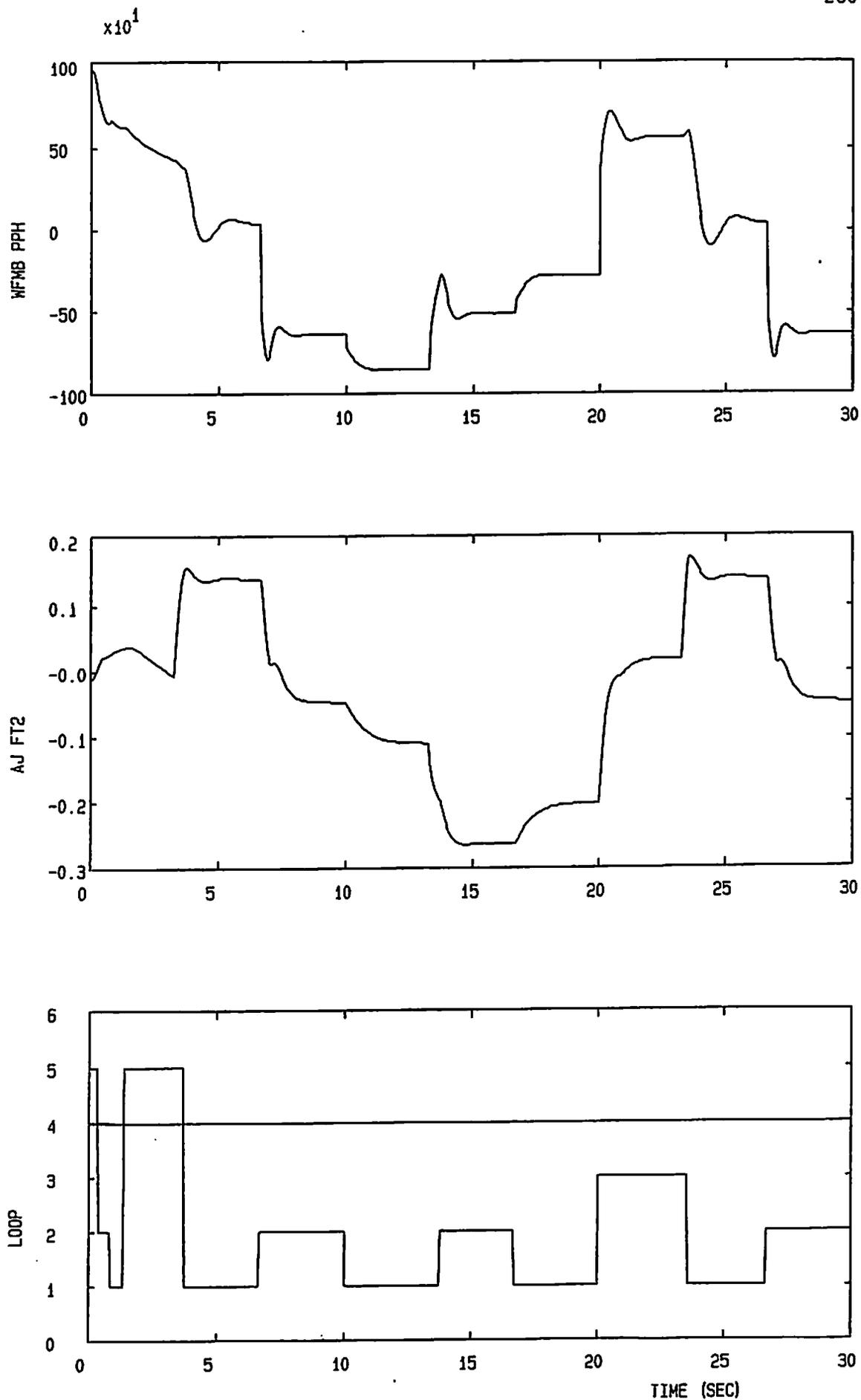
Fig 10.8 RESPONSES OF F100 ENGINE UNDER SELF-SELECTING CONTROL
RATING: INTERMEDIATE
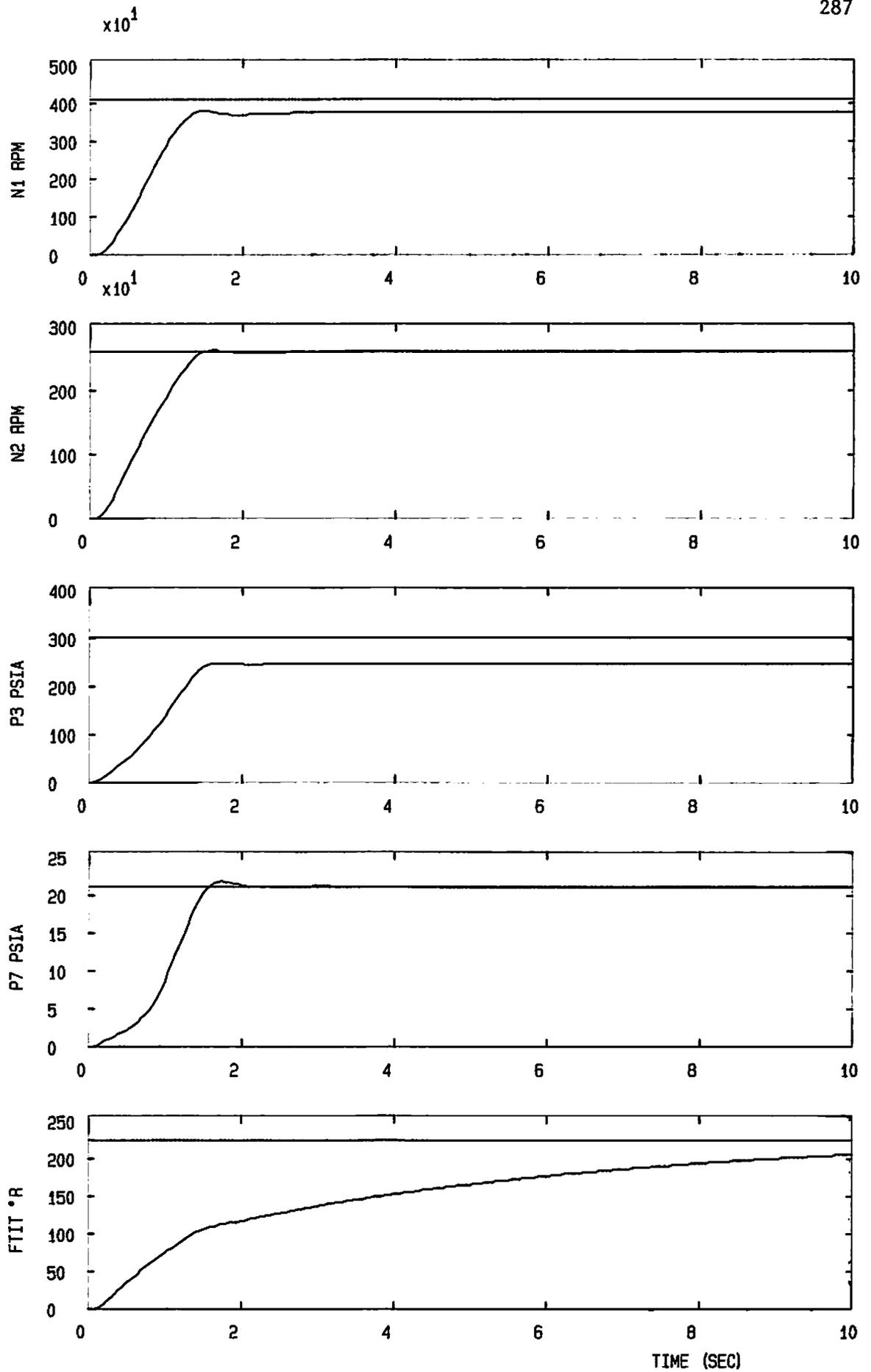
Fig 10.9 FAST ACCELERATION OF F100 ENGINE
80% N2 --> INTERMEDIATE
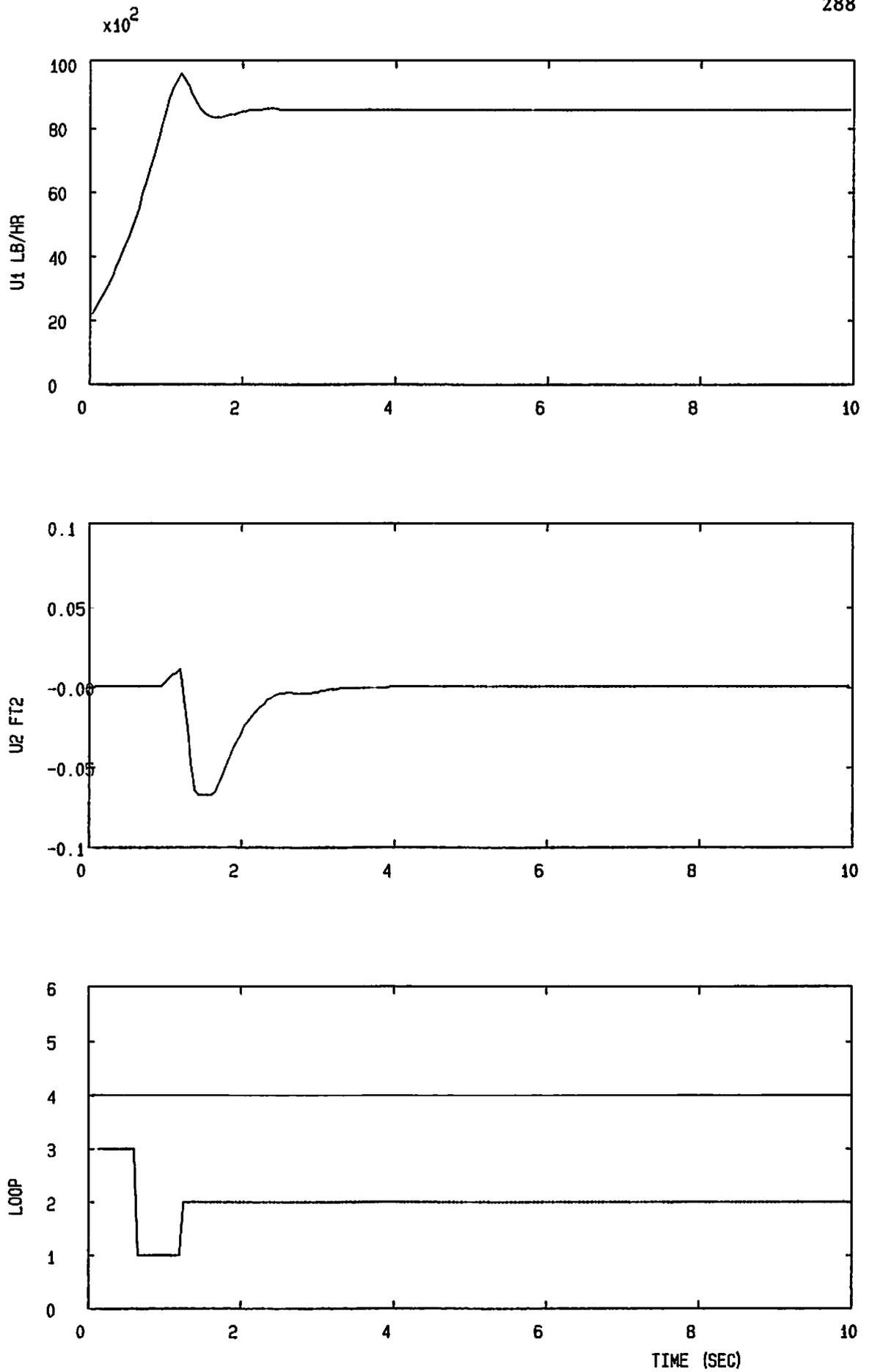
Fig 10.10   FAST ACCELERATION OF F100 ENGINE
80% N2 --> INTERMEDIATE
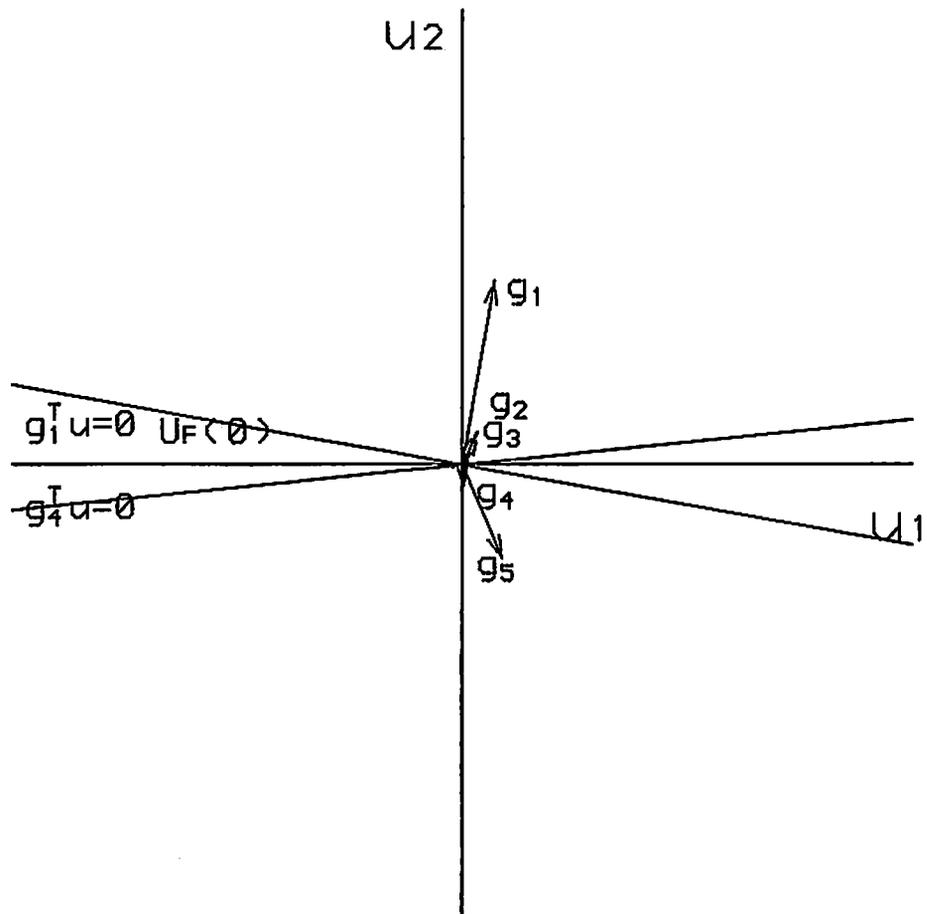
Fig 10.11   U-space

Plant: Nonlinear F100 engine model at 80% N2
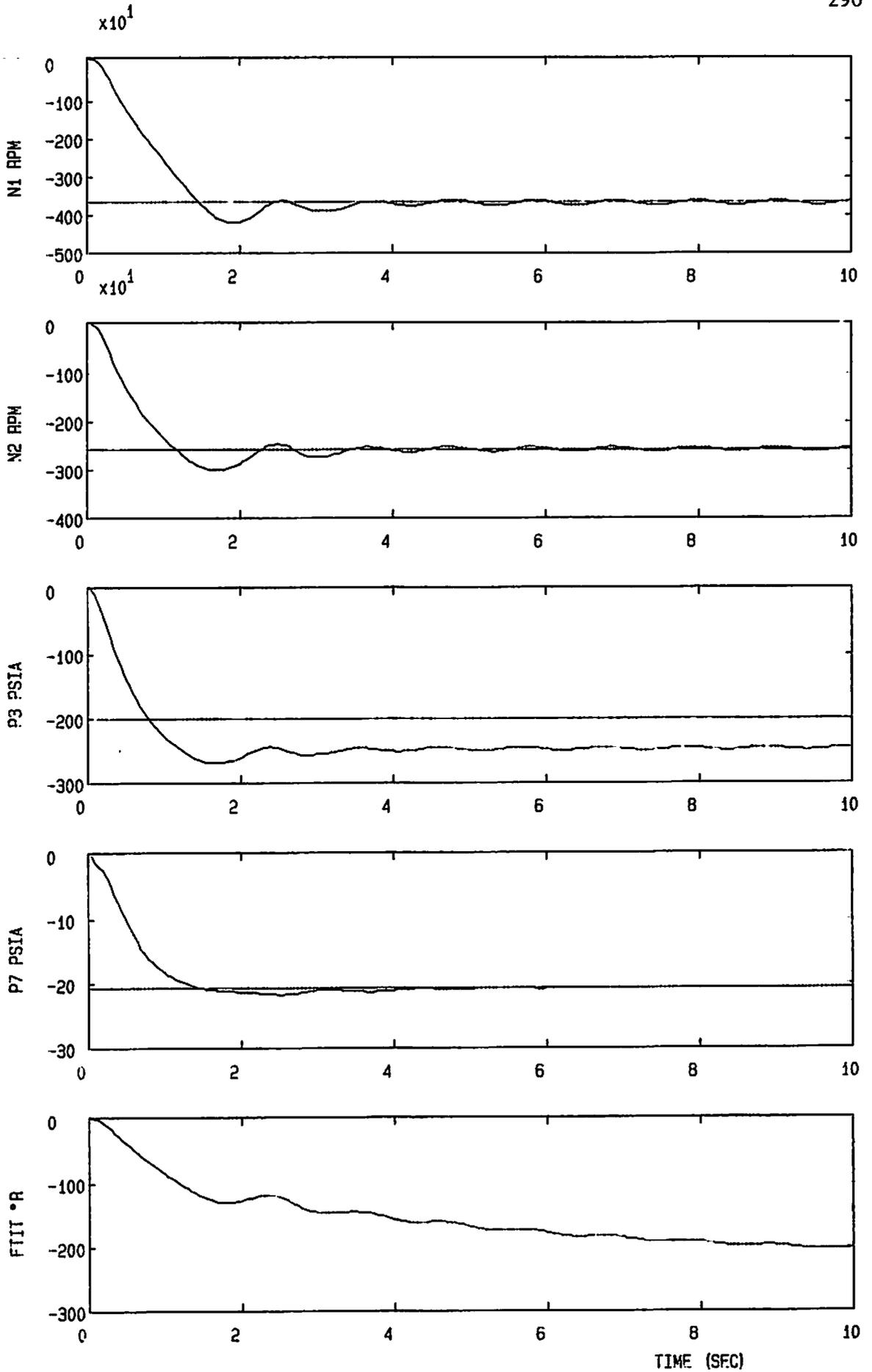
Fig 10.12 FAST DECELERATION OF F100 ENGINE
INTERMEDIATE --> 80% N2
Nominal plant: Intermediate Actual plant: 80% N2
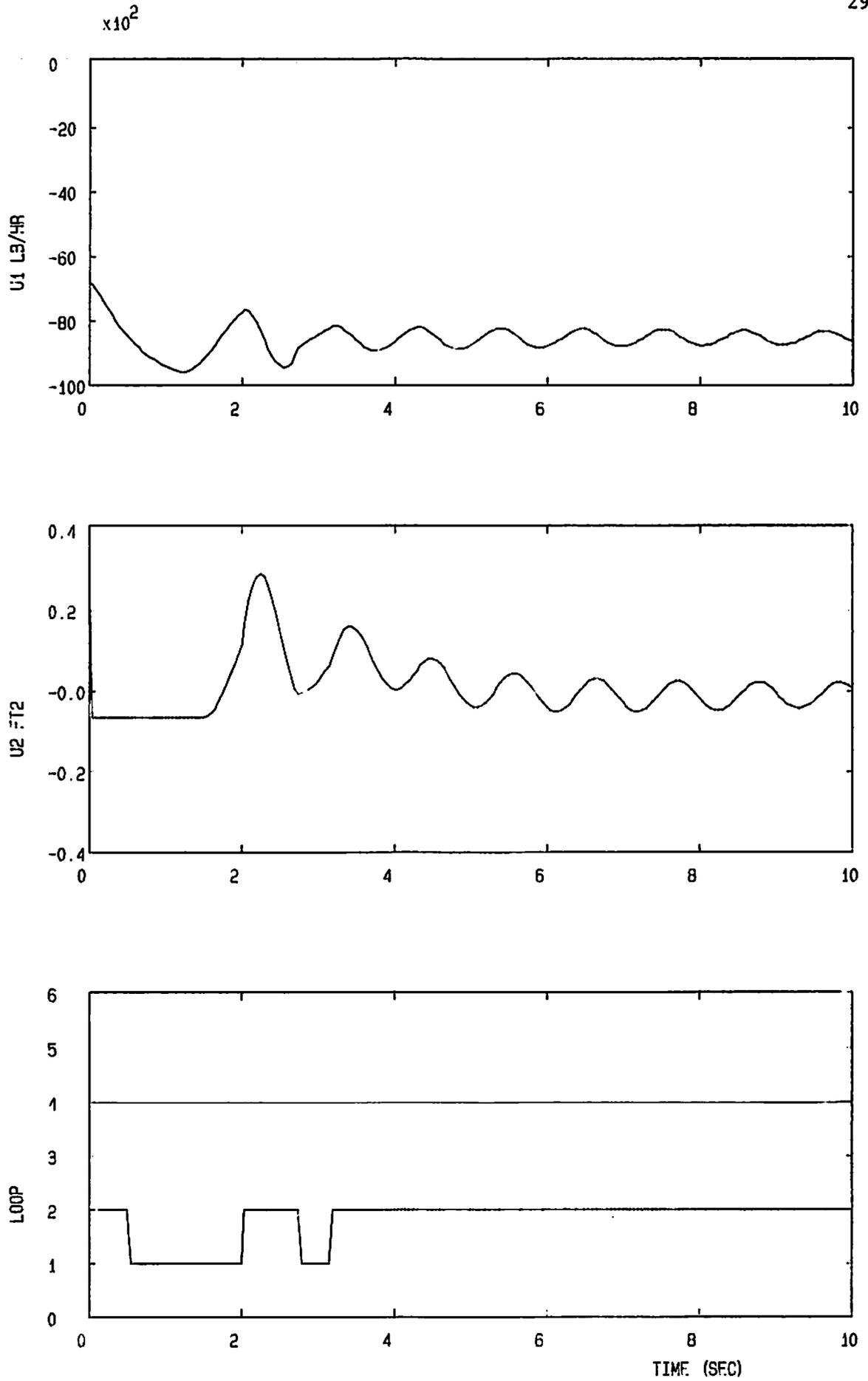
Fig 10.13 FAST DECELERATION OF F100 ENGINE
INTERMEDIATE --> 80% N2
Nominal plant: Intermediate Actual plant: 80% N2

PART VI

CONCLUSIONS AND RECOMMENDATIONS

CHAPTER 11

CONCLUSIONS AND RECOMMENDATIONS

11.1 Conclusions

Industrial plants are becoming more complicated than before so as to satisfy many consistent or inconsistent requirements such as performance, versatility, safety, environmental friendliness, etc. Most of such plants are inevitably MIMO multivariable, high order, possessing some uncertainties, and therefore their detailed mathematical models in either state-space or transfer-function matrix form are difficult to obtain. Furthermore, plants might have unmeasurable controlled outputs or have more controlled outputs than manipulated inputs. Therefore, the need for a broad range of methodologies − that are free from a heavy reliance upon accurate plant models − is clearly felt for the design of tracking systems incorporating various classes of multivariable plants.

Set-point tracking systems − which incorporate error-actuated so-called 'low-gain' controllers and multivariable plants with measurable controlled outputs whose numbers do not exceed the numbers of manipulated inputs − were developed by Porter and co-workers (Porter and Jones (1984a), (1985a)). Such tracking systems inherit the structure of SISO classical robust proportional-integral-derivative control systems, achieve with initial non-interaction the practical decoupling of MIMO plants and excellent transient performance in the available range of

tuning, and exhibit robustness in the face of uncertainties such as plant variations, disturbances, etc (Khaki-Sedigh (1988), Porter and Khaki-Sedigh (1989)). Therefore, the evolving methodologies for the design of tracking systems have been further developed in this thesis so as to incorporate various classes of plants with measurable/unmeasurable outputs or with more outputs than inputs whilst keeping these desirable properties.

For plants with measurable outputs, the methodology is applicable provided that the asymptotically stable plants satisfy the fundamental condition of functional controllability for the preservation of stabilisability in the presence of integral action (Porter and Power (1970), Power and Porter (1970)), and that input-output decoupling is achievable (Falb and Wolovich (1967)). The designed tracking systems incorporate error-actuated digital PID controllers in which the controller matrices can be directly obtained from open-loop tests performed on the plants (Appendix 1). It has been shown that the resulting tracking systems exhibit both set-point tracking and minimal interaction.

For plants with unmeasurable outputs, the developed methodology is also applicable provided that the asymptotically stable plants satisfy the fundamental condition of functional controllability for the preservation of stabilisability in the presence of integral action (Porter and Power (1970), Power and Porter (1970)), and that input-output decoupling is achievable (Falb and Wolovich (1967)). The designed tracking systems

incorporate error-actuated digital PID controllers and associated pre-filters in which both the controller and pre-filter matrices can be directly obtained from open-loop tests performed on the plants (Appendix 1). However, since the proportional and derivative controller matrices involve the inverse of the step-response matrix for unmeasurable outputs, such unmeasurable outputs have to be measurable in the "off-line" controller design stages – this assumption is by no means impractical (Chapter 3). Under these assumptions, it has been shown that the resulting tracking systems exhibit both set-point tracking and initial non-interaction for unmeasurable outputs together with minimal transient interaction among unmeasurable/measurable outputs.

For plants with more outputs than inputs, rigorous theoretical foundations have been constructed for the design of tracking systems incorporating such plants, self-selecting controllers (which themselves consist of a number of set-point tracking controllers), and lowest-wins and/or highest-wins strategies. Such foundations include a characterisation of general tracking systems (ie undertracking and overtracking which are expressed by sets of inequalities), a classification of linear multivariable plants into Class I and Class II plants, the concept of limit tracking, a feasibility-assessment procedure for the design of limit-tracking systems, and an order-reduction technique which decides the minimum numbers of different subsets of plant outputs to be controlled by corresponding set-point tracking controllers. It has been shown in the case of m-input/p-output plants and lowest-wins

strategies that, provided the plant belongs to Class I, not only undertracking but also limit tracking is possible for any set-point command, that p-m+1 subsets are the minimum, and that only steady-state transfer-function matrices of the plants are required in the feasibility assessment and the order reduction. Next, the methodology for the design of tracking systems has been presented based upon this order-reduction technique and lowest-wins strategies. This methodology is applicable to the asymptotically stable Class I plants provided that the set $U_F(0)$ of feasible inputs has at least one extreme ray uniquely represented by m-1 hyperplanes, that all the determined sub-plants satisfy the fundamental condition of functional controllability for the preservation of stabilisability in the presence of integral action (Porter and Power (1970), Power and Porter (1970)), and that input-output decoupling is achievable for all the sub-plants (Falb and Wolovich (1967)). The designed tracking systems incorporate error-actuated digital self-selecting PID controllers whose controller matrices can be directly obtained from open-loop tests performed on the plants (Appendix 1). It has been shown that the resulting tracking systems exhibit excellent limit-tracking and controller switching behaviour. However, it has been shown that the stability of separate closed-loop systems is not enough to guarantee the stability of complete closed-loop systems incorporating self-selecting controllers and that peculiarities such as limit-cycle oscillations might occur (Appendix 6). Therefore, theoretical foundations for the dynamical analysis of self-selecting control systems have been built and the

methodology for the design of supervisory self-selecting controllers has been developed in order to enhance closed-loop stability. It has been shown that enhanced stability can be achieved using such supervisory controllers for the case in which the non-supervisory controller causes limit-cycle oscillations.

In practice, the controllers are often exposed to uncertainties such as unknown disturbances, plant variations, etc. Therefore, the robustness of tracking systems has been assessed. It has been shown under the action of unknown constant disturbances that the tunable digital set-point tracking PID or PID/Pre-filter controllers can reject such disturbances and achieve set-point tracking for measurable outputs, that the tunable digital set-point tracking PID/Pre-filter controllers can neither reject such disturbances nor achieve set-point tracking for unmeasurable outputs, and that the digital self-selecting PID controllers can reject such disturbances and achieve limit tracking. The admissible plant perturbations that can be tolerated by digital controllers have been characterised in terms of the steady-state transfer-function matrices of the nominal and actual plants using the robustness theorems - Theorem 1: Porter and Khaki-Sedigh (1989) (Appendix 7) and Theorem 9.1. It has been shown in the face of plant admissible variations that closed-loop digital tracking systems can remain stable and achieve set-point or limit tracking for measurable outputs but that set-point tracking for unmeasurable outputs is no longer possible in the case of set-point tracking PID/Pre-filter

controllers. Furthermore, the effect of the controller parameters of supervisory self-selecting controllers on tracking performance has also been studied. It has been shown that the supervisory self-selecting controllers can achieve dynamical limit tracking for a wide range of choice of controller parameters.

The adaptability and effectiveness of tracking systems for complex nonlinear plants have been shown by designing a digital self-selecting controller for a nonlinear model of a gas-turbine engine.

It can accordingly be concluded that the requirements for the design of tracking systems outlined in Section 1.3 have been achieved and that the design of tracking systems incorporating multivariable plants with measurable/unmeasurable outputs or with more outputs than inputs has been successfully completed. Illustrative examples have demonstrated how to apply these design methodologies and verified the effectiveness of the methodologies.

11.2 Recommendations

The controller matrices of digital set-point tracking controllers are determined by directly measurable input-output data of plants - ie step-response matrices and steady-state transfer-function matrices. Therefore, tunable digital set-point tracking controllers have been rendered adaptive using on-line recursive least square identifiers (for example, Jones and Porter (1987)). It has been reported that the

controller matrices of the resulting adaptive digital set-point tracking controllers are not dependent on the individual elements of the identified ARMA models of the plants but are obtained from a unique relationship involving the elements of the ARMA models and therefore that such adaptive controllers are very robust in the face of gross underparameterisation (Porter and Khaki-Sedigh (1988)). Since self-selecting controllers consist of numbers of set-point tracking controllers, it is possible and important to render the self-selecting controllers adaptive using recursive identifiers.

In the implementation of both set-point tracking and self-selecting controllers for multivariable plants, it is assumed that the positive diagonal "tuning" matrices in the control-law design equations of such controllers are chosen by the designer. In fact, experienced control engineers have seldom met with difficulties in tuning these controllers. However, some intelligent technique such as real-time expert systems could be introduced in order to enhance tuning capabilities (Porter (1988)).

In the analysis of tracking systems and the design of self-selecting controllers, only the lowest-wins strategies have been extensively investigated. However, it is also important to consider self-selecting controllers based upon highest-wins strategies or the combination of lowest- and highest-wins strategies. Furthermore, it is noted that the controller switching itself is a subject of research and

therefore that other approaches can be used in the controller switching of self-selecting controllers (for example, Hanus et al (1987)).

Finally, in the case of actuator failures in set-point tracking systems, the numbers of live inputs become less than the numbers of controlled outputs. Therefore, the methodology for the design of limit-tracking systems might be applied to the failure accommodation of set-point tracking systems in the face of such actuator failures.

APPENDICES

APPENDIX 1

MULTIVARIABLE REACTION CURVE

The Multivariable Reaction Curve technique is an "off—line" open—loop test procedure in which a known step change is made in each of the manipulated input variables of an asymptotically stable plant, separately, in turn.

The use of the Multivariable Reaction Curve technique is outlined by the following procedure:

Step 1: The plant must be in a steady—state condition, when a known step change, u_i, ($i \in \{1,2,\ldots,m\}$) is made in one of the manipulated input variables and allowed to act for a chosen period of time. The plant must then again be brought back to a steady—state condition. During test, traces of the output are taken in consonance with the time—domain solution of vector differential equation of the form (2.1) and (2.2) (Chapter 2) such that

$$y_i(t) = \int_0^t Ce^{A(t-\tau)}b_i u_i d\tau$$

$$= CA^{-1}(e^{At} - I_n)b_i u_i \,, \qquad\qquad (A1.1)$$

where

$$B = [b_1, b_2, \ldots, b_m] \; ; \quad b_i \in R^{n \times 1} \;\; (i = 1, 2, \ldots, m) \; .$$

Step 2: The procedure of step 1 is repeated with a known step-change now made in another input variable.

Step 3: → Step m: Step 2 is repeated until response curves for all m input variables have been determined.

Then, after tests 1,2,...,m have been carried out, the step-response matrix H(t) can be determined according to the formula

$$H(t) = [y_1(t), y_2(t), \ldots, y_m(t)][\text{diag } \{u_1, u_2, \ldots, u_m\}]^{-1}. \quad (A1.2)$$

In the case of the set-point tracking PID controllers of Part II and the self-selecting controllers of Part III, this formula is used in determining H(T) and H(∞) where H(∞) is obtained using the steady-state conditions.

APPENDIX 2

PROOFS OF THEOREMS AND PROPOSITIONS

Throughout the proofs, [1] means Rockafellar (1970).

(Proof of Proposition 4.1)

$$U_F(v) = \{u : \begin{bmatrix} g_1^T \\ \cdot\cdot \\ \cdot\cdot \\ \cdot\cdot \\ g_p^T \end{bmatrix} u \leq \begin{bmatrix} v_1 \\ \cdot\cdot \\ \cdot\cdot \\ \cdot\cdot \\ v_p \end{bmatrix}\}$$

Since $g_i^T u \leq v_i$ $(i=1,\ldots,p)$ represents a closed halfspace in U, $U_F(v)$ is the intersection of p halfspaces. This means that $U_F(v)$ is a polyhedral set and closed [1].

Let $u_1 \in U_F(v)$, $u_2 \in U_F(v)$. Then $Gu_1 \leq v$ and $Gu_2 \leq v$. Therefore, for $\lambda \in [0,1]$,

$$G\{\lambda u_1 + (1-\lambda)u_2\} = G\lambda u_1 + G(1-\lambda)u_2$$
$$= \lambda Gu_1 + (1-\lambda)Gu_2$$
$$\leq \lambda v + (1-\lambda)v$$
$$= v$$

This means that

$$\lambda u_1 + (1-\lambda)u_2 \in U_F(v).$$

So, $U_F(v)$ is convex.

Since $y_i \leq v_i$ $(i=1,\ldots,p)$ or $e_i^T y \leq v_i$ $(i=1,\ldots,p)$, where

$$e_i = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \quad (i$$

represents a closed halfspace in Y, $Y_A(v)$ is the intersection of p halfspaces. This means that $Y_A(v)$ is a polyhedral set and closed.

Let $y_1 \in Y_A(v)$, $y_2 \in Y_A(v)$.
Then, for $\lambda \in [0,1]$,

$$\lambda y_1 + (1-\lambda)y_2 \leq \lambda v + (1-\lambda)v = v.$$

This means that

$$\lambda y_1 + (1-\lambda)y_2 \in Y_A(v).$$

So, $Y_A(v)$ is convex.

For $u_1 \in U$, $u_2 \in U$, let $y_1 = Gu_1$, $y_2 = Gu_2$.
Then, for $\lambda \in R$,

$$(1-\lambda)y_1 + \lambda y_2 = (1-\lambda)Gu_1 + \lambda Gu_2$$
$$= G\{(1-\lambda)u_1 + \lambda u_2\} .$$

Since $(1-\lambda)u_1 + \lambda u_2 \in U$, it follows that $(1-\lambda)y_1 + y_2 \in Y_R$. This means that Y_R is an affine set. So, Y_R is closed and convex. Furthermore, since $0 = G0 \in Y_R$, ie Y_R contains the origin, Y_R is a subspace (Theorem 1.1[1]).

Now

$$G\,(U_F(v)) = \{y \in Y : y = G\,u\,,\quad u \in U_F(v)\}$$

$$= Y_R \cap Y_A(v)$$

$$= Y_F(v).$$

Since G is a linear transformation from R^m to R^p and $U_F(v)$ is a closed polyhedral convex set in $U = R^m$, so also is $G\,(U_F(v))$ (Theorem 19.3[1]).   QED

(Proof of Proposition 4.2)

1(i) implies 1(ii):

Suppose that $U_F(v) = \phi$. If $Y_F(v) \neq \phi$, there exists $y \in Y_F(v)$ and $u \in U_F(v)$ such that $y = Gu \leq v$. This means that $U_F(v) \neq \phi$ and contradicts the assumption. So, $Y_F(v) = \phi$.

1(ii) implies 1(i):

Suppose that $Y_F(v) = \phi$. If $U_F(v) \neq \phi$, there exists $y \in Y_F(v)$ and $u \in U_F(v)$ such that $y = Gu \leq v$. This means that $Y_F(v) \neq \phi$ and contradicts the assumption. So, $U_F(v) = \phi$.

2:

2 is clear from 1.   QED

(Proof of Proposition 4.3)

Since $G0 = 0 \leq v$ for $v \geq 0$, $0 \in U_F(v)$ and $0 \in Y_F(v)$. This
proves the statement.   QED

(Proof of Theorem 4.1)

1(i) implies 1(ii):

Suppose that 1(ii) does not hold, ie there exists a hyperplane
separating $Y_A(0)$ and Y_R properly. By Theorem 11.3[1], such a
hyperplane exists if and only if ri $Y_A(0)$ and ri Y_R have no
point in common, ie

$$\text{ri } Y_A(0) \cap \text{ri } Y_R = \emptyset,$$

where ri $\cdot$ is the relative interior of the set $\cdot$. Since Y_R is
an affine set, ri $Y_R = Y_R$. And $Y_A(v) \subset$ ri $Y_A(0)$ for $v < 0$.
So, $Y_F(v) = Y_A(v) \cap Y_R = \emptyset$ for $v < 0$.

1(ii) implies 1(i):

There exists $y \in$ ri $Y_A(0) \cap$ ri Y_R such that $y = Gu$. Clearly,
$y \notin$ rb $Y_A(0)$ and $y < 0$, where rb $\cdot$ is the relative boundary of
the set $\cdot$.

For $v < 0$, there exists $\lambda > 0$ such that $\lambda y < v$. Then,
$\lambda y \in Y_A(v) \cap Y_R = Y_F(v)$. So, $Y_F(v) \neq \emptyset$ for $v < 0$.

2(i) implies 2(ii):

As a negative statement of 1, the statement that $\exists v < 0$, $Y_F(v) = \emptyset$ is equivalent to the one that there exists a hyperplane separating $Y_A(0)$ and Y_R properly. So, 2(i) is sufficient for 2(ii).

2(ii) implies 2(i):

$$\text{ri } Y_A(0) \cap \text{ri } Y_R = \text{ri } Y_A(0) \cap Y_R = \emptyset$$

Since

$$Y_A(v) \subset \text{ri } Y_A(0) \text{ for } v < 0,$$

$$Y_F(v) = Y_A(v) \cap Y_R = \emptyset \text{ for } v < 0.$$

QED

(Proof of Proposition 4.4)

If there exists a hyperplane separating $Y_A(0)$ and Y_R properly,

$$\text{ri } Y_A(0) \cap \text{ri } Y_R = \text{ri } Y_A(0) \cap Y_R = \emptyset.$$

Since $0 \in Y_A(0) \cap Y_R$, $Y_A(0) \cap Y_R = Y_A(0) \cap \text{ri } Y_R \neq \emptyset$.

Both $Y_A(0)$ and Y_R are polyhedral convex sets. By Theorem 20.2[1], such a hyperplane contains Y_R and does not contain $Y_A(0)$.

QED

(Proof of Theorem 4.2)

1(i) implies 1(ii):

By Definition 4.3.1, $U_F(v) \neq \emptyset$, $Y_F(v) \neq \emptyset$ for $v < 0$ and by Proposition 4.3, $U_F(v) \neq \emptyset$, $Y_F(v) \neq \emptyset$ for $v \geq 0$. For $v \not< 0$ and $v \not\geq 0$, there exist $v' < 0$ and $v'' \geq 0$ such that $v = v' + v''$. Then for $u \in U_F(v') \neq \emptyset$, $Gu \leq v' \leq v' + v'' = v$. This means that $U_F(v) \neq \emptyset$.

2(i) implies 2(ii):

Contraposition of 1.

3(i) implies 3(ii):

This is clear from Definition 4.3.2.

4(i) implies 4(ii):

Contraposition of 3.

QED

(Proof of Proposition 4.5)

1:

Let $y = [y_1, \ldots, y_p]^T$, $v = [v_1, \ldots, v_p]^T$.

$$U_F(v) = \bigcap_{i=1}^{p} \{u : g_i^T u \leq v_i\}$$

If $g_i = 0$, $g_i^T u = 0$ and $\{u : g_i^T u \leq v_i , v_i < 0\} = \emptyset$.

This means that $\exists v$, $v_i < 0$, $U_F(v) = \emptyset$.

So, $G \in$ Class II.

2:

Contraposition of 1.

QED

(Proof of Theorem 4.3)

Clearly by Corollary 2.5.1[1], $U_F(0)$ is a convex cone.

(i) implies (ii):

Suppose that $G \in$ Class I. By Proposition 4.5, $g_i \neq 0$, $\forall i \in [1,p]$. And

$$U_F(v) = \{u : Gu \leq v\} \neq \emptyset \text{ for } v < 0.$$

This means that $\{u : Gu < 0\} \neq \emptyset$. By Theorem 22.2[1], there do not exist non-negative real numbers $\lambda_1, \ldots, \lambda_p$, such that at least one of them is not zero, and

$$\sum_{i=1}^{p} \lambda_i g_i = 0$$

ie for the system

$$\sum_{i=1}^{p} \lambda_i g_i = 0 \quad , \quad \lambda_i \geq 0 \quad , \quad i \in [1,p],$$

the only solution is $\lambda_i = 0$, $i \in [1,p]$. Let U_F stand for $U_F(0) = \{u : Gu \leq 0\}$. The polar U_F^o of U_F is given in the following [1]

$$U_F^o = \{u : u = \sum_{i=1}^{p} \bar{\lambda}_i g_i \ , \quad \bar{\lambda}_i \geq 0\} \ .$$

By Corollary 14.6.1[1],

$$\dim U_F = \dim U - \dim \text{Li } U_F^o$$
$$= m - \dim \text{Li } U_F^o$$

where the linearity space $\text{Li } U_F^o$ of U_F^o is defined by using the recession cone $0^+ U_F^o$ of U_F^o in the form

$$0^+ U_F^o = \bigcap_{\epsilon > 0} \epsilon U_F^o$$

$$\text{Li } U_F^o = (-0^+ U_F^o) \cap 0^+ U_F^o.$$

If $\exists u \neq 0$, $u \in \text{Li } U_F^o$, then

$$u \in (-0^+ U_F^o) \cap 0^+ U_F^o.$$

This implies that

$$u = \sum_{i=1}^{p} (-\epsilon) \tilde{\lambda}_i g_i = \sum_{i=1}^{p} \epsilon \bar{\lambda}_i g_i \ , \quad \tilde{\lambda}_i, \bar{\lambda}_i \geq 0 \ , \quad \epsilon > 0 \ ,$$

so that

$$\sum_{i=1}^{p} \epsilon (\bar{\lambda}_i + \tilde{\lambda}_i) g_i = 0.$$

This happens only if $\bar{\lambda}_i = \tilde{\lambda}_i = 0$, so that

$$\text{Li } U_F^o = \{ \ 0 \ \}$$

$$\dim \text{Li } U_F^o = 0$$

$$\dim U_F = m - \dim \text{Li } U_F^o = m.$$

(ii) implies (i):

Suppose that $G \in$ Class II and that $g_i \neq 0$, $\forall i \in [1,p]$.

$$U_F(v) = \{u : Gu \leq v\} = \emptyset \quad \text{for} \quad v < 0$$

ie

$$\{u : Gu < 0\} = \emptyset$$

By Theorem 22.2[1], there exist non-negative real numbers $\lambda_1,\ldots,\lambda_p$, such that at least one of them is not zero and

$$\sum_{i=1}^{p} \lambda_i g_i = 0.$$

Let λ_{i_0} be one of the non-zero λ_i. Then

$$\lambda_{i_0} g_{i_0} = - \sum_{\substack{i=1\\i\neq i_0}}^{p} \lambda_i g_i \quad , \quad \lambda_i \geq 0 \quad , \quad \lambda_{i_0} > 0 \quad ,$$

thus.

$$g_{i_0} = - \sum_{\substack{i=1\\i\neq i_0}}^{p} \frac{\lambda_i}{\lambda_{i_0}} g_i \quad , \quad \lambda_i \geq 0 \quad , \quad \lambda_{i_0} > 0 .$$

Since $g_{i_0} \neq 0$, there still exists non-zero λ_{i_1}, $i_1 \in [1,p]$, $i_1 \neq i_0$. And by the definition of $0^+U_F^o$ and $(-0^+U_F^o)$,

$$0^+U_F^o = \bigcap_{\epsilon>0} \{u : u = \sum_{i=1}^{p} \epsilon \bar{\lambda}_i g_i \quad , \quad \bar{\lambda}_i \geq 0\}$$

$$= \bigcap_{\epsilon>0} \{u : u = \sum_{\substack{i=1\\i\neq i_0}}^{p} \epsilon \bar{\lambda}_i g_i - \sum_{\substack{i=1\\i\neq i_0}}^{p} \epsilon \bar{\lambda}_{i_0} \frac{\lambda_i}{\lambda_{i_0}} g_i, \bar{\lambda}_i \geq 0, \lambda_i \geq 0, \lambda_{i_0} > 0\}$$

$$= \bigcap_{\epsilon>0} \{u : u = \sum_{\substack{i=1\\i\neq i_0}}^{p} \epsilon (\bar{\lambda}_i - \bar{\lambda}_{i_0} \frac{\lambda_i}{\lambda_{i_0}}) g_i, \bar{\lambda}_i \geq 0, \lambda_i \geq 0, \lambda_{i_0} > 0\}.$$

Since $\bar{\lambda}_i$, $i \in [1,p]$ are arbitrary, for an index i_1 of non-zero

λ_{i_1} it follows that

$$g = \{u : u = \lambda g_{i_1} , \quad \lambda \in R\} \in 0^+ U_F^o.$$

Similarly, $g \in (-0^+ U_F^o)$.

This means that

$$g \in \text{Li } U_F^o$$

$$\dim \text{Li } U_F^o \geq 1$$

$$\dim U_F = m - \dim \text{Li } U_F^o$$

$$\leq m - 1 .$$

<div align="right">QED</div>

(Proof of Proposition 4.6)

Suppose that $G \in$ Class I. Clearly, the recession cone $0^+ U_F(v)$ of $U_F(v)$ is $U_F(0)$. By Theorem 4.3, $U_F(0)$ is an m-dimensional convex cone and does not consist of the zero vector alone. Thus, by Theorem 8.4[1], $U_F(v)$ is unbounded. Furthermore,

$$\dim U_F(0) = \dim \text{aff } U_F(0) = m$$

where aff • is the affine hull of the set •.

Since $0 \in U_F(0)$, to express $y \in U_F(0)$ as the linear combination of the vectors in the form

$$y = \lambda_1 u_1 + \ldots + \lambda_m u_m,$$

the vectors $u_1, \ldots, u_m$ must be linearly independent. Then

$$\text{aff } U_F(0) = \{0, u_1, \ldots, u_m\}, \quad u_i \in U_F(0), \quad i \in [1,m].$$

Let $\dim U_F(v) = k$. Then

$$\text{aff } U_F(v) = \{a_0, a_1, \ldots, a_k\}, \quad a_i \in U_F(v), \quad i \in [0,k]$$

$$= \{\lambda_0 a_0 + \ldots + \lambda_k a_k : a_i \in U_F(v), \ i \in [0,k], \ \lambda_0 + \ldots + \lambda_k = 1\}$$

$$= \{\lambda_1(a_1 - a_0) + \ldots + \lambda_k(a_k - a_0) + a_0 : a_i \in U_F(v), \ i \in [0,k]\}$$

where $a_0, a_1, \ldots, a_k$ are affinely independent, ie vectors $a_1 - a_0, \ldots, a_k - a_0$ are linearly independent.

For $x \in U_F(v) \subset \text{aff } U_F(v)$

$$x = \lambda_1(a_1 - a_0) + \ldots + \lambda_k(a_k - a_0) + a_0.$$

Since $G(x+y) = Gx + Gy \leq Gx \leq v$, for $y \in U_F(0)$,

$$x + y \in U_F(v)$$

$$x + y = \bar{\lambda}_1(a_1 - a_0) + \ldots + \bar{\lambda}_k(a_k - a_0) + a_0$$

$$y = x + y - x$$

$$= (\bar{\lambda}_1 - \lambda_1)(a_1 - a_0) + \ldots + (\bar{\lambda}_k - \lambda_k)(a_k - a_0).$$

This means that y must be expressed as the linear combination of $a_1 - a_0, \ldots, a_k - a_0$ and that $k \geq m$.

But dim U = m means that k = m.   QED

(Proof of Theorem 4.4)

If $G \in$ Class I, $\{u : Gu \leq v - d\} \neq \phi$ for $\forall v$, $\forall d$.

If $G \in$ Class II, $\{u : Gu \leq v - d\} = \phi$ for $v - d < 0$.

                                                        QED

(Proof of Theorem 4.5)

Theorem 22.1[1] states that one and only one of the following alternatives holds:

(a) There exists u such that

   $Gu \leq v$.

(b) There exists w such that

   $w \geq 0$,

   $G^T w = 0$,

$$v^T w < 0 .$$

Statement (a) exactly concerns Class I. The two systems in (a) and (b) are dual to each other.

If $\exists i \in [1,m]$, $g_{c_i} > 0$, then there exists no solution w for $\forall v$ in

$$w \geq 0 ,$$

$$g_{c_i}^T w = 0 ,$$

$$v^T w < 0 .$$

This means that statement (a) holds for $\forall v$, ie $G \in$ Class I.

QED

(Proof of Theorem 5.1)

Part 1: rank $G = m$

1 $\forall v$, ext $U_F(v) \neq \emptyset$:

Suppose that $U_F(v)$ contains an entire line.
Then, there exists $d \neq 0$ such that

$$\{u + \lambda d : \lambda \in R, u \in U_F(v)\} \subset U_F(v) .$$

This means that

$$G(u + \lambda d) = Gu + \lambda Gd \leq v \quad \text{for } u \in U_F(v), \quad \lambda \in R.$$

Therefore, it is required that $Gd = 0$. However, rank $G = m = \dim U$ implies $d = 0$. This contradicts the assumption. So, $U_F(v)$ contains no lines. Furthermore, since $G \in$ Class I, $\forall v$, $U_F(v) \neq \emptyset$. It follows by Corollary 18.5.3[1] that $\forall v$, exit $U_F(v) \neq \emptyset$.

2(i) implies 2(ii):

It is shown that if $u \in U_F(v)$ fails to satisfy the definition of limit-tracking input, $u \notin$ exit $U_F(v)$.

(1) $k = 0$: Suppose that $u \in U_F(v)$ satisfies

$$g_i^T u < v_i \, , \quad i \in [1,p].$$

Then

$$u \in \bigcap_{i=1}^{p} \text{ri } H_i$$

where

$$H_i = \{u : g_i^T u \leq v_i\} \, .$$

Since $\bigcap_{i=1}^{p} \text{ri } H_i \neq \emptyset$, by Theorem 6.5[1],

$$\bigcap_{i=1}^{p} \text{ri } H_i = \text{ri} \bigcap_{i=1}^{p} H_i = \text{ri } U_F(v) \, .$$

So, $u \notin rb \ U_F(v)$. This implies that $u \notin ext \ U_F(v)$.

(2) $k \geq 1$: Suppose that $u \in U_F(v)$ satisfies

$$g_{s_i}^T u = v_{s_i} \quad i \in [1,k]$$

$$g_{t_j}^T u < v_{t_j} \quad i \in [1,p-k]$$

$$rank \ G_s < m \ ,$$

where

$$1 \leq s_i \ , \quad t_j \leq p$$

$$G_s = \begin{bmatrix} g_{s_i}^T \\ \cdots \\ \cdots \\ \cdots \\ g_{s_k}^T \end{bmatrix} .$$

This means that

$$u \in (\overset{k}{\underset{i=1}{\cap}} rb \ H_{s_i}) \cap (\overset{p-k}{\underset{j=1}{\cap}} ri \ H_{t_j}) \ .$$

Since $\overset{p-k}{\underset{j=1}{\cap}} ri \ H_{t_j} \neq \emptyset$, by Theorem 6.5[1],

$$u \in (\bigcap_{i=1}^{k} \text{rb } H_{s_i}) \cap (\text{ri} \bigcap_{j=1}^{p-k} H_{t_j}) .$$

There exists at least one non-zero vector $x \in U$ such that $G_s x = 0$.

By Proposition 4.6, $G \in$ Class I implies dim $U_F(v) = m$.

Then by Corollary 6.4.1[1],

$$\text{for } u \in \text{ri} \bigcap_{j=1}^{p-k} H_{t_j} = \text{int} \bigcap_{j=1}^{p-k} H_{t_j} \text{ and for } x \in U ,$$

where int $\cdot$ is the interior of the set $\cdot$,

there exists some $\epsilon_1 > 0$ such that

$$u + \epsilon_1 x \in \bigcap_{j=1}^{p-k} H_{t_j} .$$

Similarly, there also exists some $\epsilon_2 > 0$ such that

$$u + \epsilon_2 (-x) \in \bigcap_{j=1}^{p-k} H_{t_j} .$$

Let $\epsilon = \min (\epsilon_1, \epsilon_2) > 0$, then for $\epsilon > 0$

$$g_{s_i}^T (u + \epsilon x) = g_{s_i}^T u + \epsilon g_{s_i}^T x , \quad i \in [1,k]$$

$$= v_{s_i} , \quad i \in [1,k] .$$

So, $u + \epsilon x \in \bigcap_{i=1}^{k} \text{rb } H_{s_i}$, $i \in [1,k]$.

Similarly, $u - \epsilon x \in \bigcap_{i=1}^{k} \text{rb } H_{s_i}$, $i \in [1,k]$.

It follows that

$$u + \epsilon x \in (\bigcap_{i=1}^{k} \text{rb } H_{s_i}) \cap (\bigcap_{j=1}^{p-k} H_{t_j}) \subset U_F(v)$$

$$u - \epsilon x \in (\bigcap_{i=1}^{k} \text{rb } H_{s_i}) \cap (\bigcap_{j=1}^{p-k} H_{t_j}) \subset U_F(v)$$

$$u + \epsilon x \neq u - \epsilon x$$

$$u = \frac{1}{2}(u + \epsilon x) + \frac{1}{2}(u - \epsilon x) .$$

This implies that $u \notin \text{ext } U_F(v)$.

It follows by contraposition that if $u \in \text{ext } U_F(v)$, then

$$\text{rank } G_s = m , \quad 1 \leq s_i \leq p , \quad i \in [1,k]$$

and

$$k \geq m .$$

Therefore, u is limit-tracking input.

2(ii) implies 2(i):

Suppose that $u \notin \text{ext } U_F(v)$.

There exist $u_1, u_2 \in U_F(v)$, $u_1 \neq u_2$, $0 < \lambda < 1$ such that

$$u = (1-\lambda)u_1 + \lambda u_2$$

$$g^T_{s_i} = v_{s_i} \; , \quad i \in [1,k]$$

$$g^T_{t_j} < v_{t_j} \; , \quad j \in [1,p-k]$$

where $1 \leq s_i$, $t_j \leq p$.

Let

$$G_s = \begin{bmatrix} g^T_{s_1} \\ \cdots \\ \cdots \\ g^T_{s_k} \end{bmatrix} .$$

For every s_i, $i \in [1,k]$, since

$$g^T_{s_i} u_1 \leq v_{s_i} \; ,$$

$$g^T_{s_i} u_2 \leq v_{s_i} \; ,$$

there exist ϵ_{1_i}, $\epsilon_{2_i} \geq 0$ such that

$$g_{s_i}^T u_1 + \epsilon_{1_i} = v_{s_i} \ ,$$

$$i \in [1,k] \ .$$

$$g_{s_i}^T u_2 + \epsilon_{2_i} = v_{s_i} \ ,$$

Since $g_{s_i}^T u = v_{s_i}$,

$$g_{s_i}^T \{(1 - \lambda)u_1 + \lambda u_2\} = v_{s_i} \ ,$$

$$(1 - \lambda)(v_{s_i} - \epsilon_{1_i}) + \lambda(v_{s_i} - \epsilon_{2_i}) = v_{s_i} \ ,$$

$$(1 - \lambda)(-\epsilon_{1_i}) + \lambda(-\epsilon_{2_i}) = 0 \ , \quad i \in [1,k] \ .$$

This means that $\epsilon_{1_i} = \epsilon_{2_i} = 0$, $i \in [1,k]$ and

$$g_{s_i}^T u_1 = v_{s_i} \ ,$$

$$i \in [1,k] \ ,$$

$$g_{s_i}^T u_2 = v_{s_i} \ ,$$

$$u_1 \neq u_2 \ .$$

So,

$$G_s(u_1 - u_2) = 0$$

$$u_1 - u_2 \neq 0 \ .$$

This implies that rank $G_s < m$ and that u is not a limit-tracking input.   Part 1 QED

Part 2: rank $G < m$

Firstly, the proof of Proposition 5.1 is given. Then, the proof of the Theorem follows for the case rank $G < m$.

(Proof of Proposition 5.1)

(i) implies (ii):

Suppose that $\bar{G} \in$ Class II and $\forall i \in [1,p]$, $\bar{g}_i \neq 0$. Then

$$\bar{U}_F(v) = \{\bar{u} : \bar{G}\,\bar{u} \leq v\} = \emptyset \quad \text{for } v < 0 ,$$

ie $\quad \{\bar{u} : \bar{G}\,\bar{u} < 0\} = \emptyset$, where $\bar{u} \in \bar{U} = R^q$.

By Theorem 22.2[1], there exist non-negative real numbers $\lambda_1,\ldots,\lambda_p$, such that at least one of them is not zero, and

$$\sum_{i=1}^{p} \lambda_i \bar{g}_i = 0 .$$

By applying a similar argument to that used in the latter part of the proof of Theorem 4.3,

$$\dim \text{Li } \bar{U}_F^\circ \geq 1 ,$$

$$\dim \bar{U}_F = \dim \bar{U} - \dim \text{Li } \bar{U}_F^\circ ,$$

$$\leq q - 1 .$$

Let $\tilde{G} \in R^{p \times (m-q)}$ be the remaining columns of G when $\bar{G} \in R^{p \times q}$ is removed. Then,

$$G u = \bar{G} \bar{u} + \tilde{G} \tilde{u}$$

$$= [\ \bar{G}\ ,\ \tilde{G}\ ] \begin{bmatrix} \bar{u} \\ \tilde{u} \end{bmatrix}$$

where $\tilde{u} \in \tilde{U} = R^{m-q}$. Then,

$$\dim U_F = \dim \bar{U} + \dim \tilde{U} \leq q - 1 + m - q = m - 1\ .$$

This means that $G \in$ Class II.

(ii) implies (i):

$$\forall v,\ \bar{U}_F(v) = \{\ \bar{u}\ :\ \bar{G}\ \bar{u} \leq v\ \} \neq \emptyset\ .$$

Let $\tilde{G}$ be defined in the same way as in the former part.

Since $\tilde{G}\ \tilde{u} = 0$ for $\tilde{u} = 0$, where $\tilde{u} \in \tilde{U} = R^{m-q}$,

$$G u = \bar{G}\ \bar{u} + \tilde{G}\ \tilde{u}$$

$$= \bar{G}\ \bar{u} \qquad \text{for } \forall \bar{u} \in \bar{U} \text{ and } \tilde{u} = 0.$$

If the columns of G and the elements of u and v are re-arranged, it is possible to obtain

$$u = \begin{bmatrix} \bar{u} \\ 0 \end{bmatrix} \quad \text{for } \tilde{u} = 0$$

so that $u \in U_F(v)$. This means that $G \in$ Class I.

<div align="right">Proposition 5.1 QED</div>

(Proof of Theorem 5.1 for the case rank $G < m$)

Suppose that $G \in$ Class I and that rank $G = q < m$.

By applying Proposition 5.1, $\bar{G} \in$ Class I ie

$$\forall v, \quad \bar{U}_F(v) = \{ \bar{u} : \bar{G}\bar{u} \leq v \} \neq \emptyset .$$

By similar arguments to those used in the first part of the proof of Theorem 5.1, $\bar{U}_F(v)$ contains no lines. It follows by Corollary 18.5.3[1] that ext $\bar{U}_F(v) \neq \emptyset$. Furthermore, by applying Theorem 5.1 to $\bar{G}$, there exists $\bar{u} \in$ ext $\bar{U}_F(v)$ such that

$$\bar{G}_s \bar{u} = \begin{bmatrix} \bar{g}^T_{s_1} \\ \cdots \\ \cdots \\ \bar{g}^T_{s_k} \end{bmatrix} \bar{u} = \begin{bmatrix} v_{s_1} \\ \cdots \\ \cdots \\ v_{s_k} \end{bmatrix} \quad \underline{k \geq q} ,$$

$$\bar{g}^T_{t_j} \bar{u} < v_{t_j} , \qquad j \in [1, p-k] ,$$

where

$$1 \leq s_i, t_j \leq p ,$$

rank $\bar{G}_s = q$.

Let $\tilde{G} \in R^{p \times (m-q)}$ be the remaining columns of G which was removed from $\bar{G}$, then

rank $\bar{G}$ = rank G ,

rank $[\ \bar{G}\ ,\ \tilde{G}\ ]$ = rank $\bar{G}$.

There exists $w \in R^{q \times (m-q)}$ such that

$$\bar{G}\ w = \tilde{G}\ .$$

If the columns of G and the elements of u and v are re-arranged, it is possible to obtain that

$$\bar{G}\ \bar{u} = \bar{G}\ \bar{u} - \tilde{G}\ \tilde{u} + \tilde{G}\ \tilde{u}$$

$$= \bar{G}\ \bar{u} - \bar{G}\ w\ \tilde{u} + \tilde{G}\ \tilde{u}$$

$$= [\ \bar{G}\ ,\ \tilde{G}\ ] \begin{bmatrix} \bar{u} - w\ \tilde{u} \\ \tilde{u} \end{bmatrix}$$

$$= G \begin{bmatrix} \bar{u} - w\ \tilde{u} \\ \tilde{u} \end{bmatrix} \qquad \text{for } \tilde{u} \in \tilde{U}\ ,$$

where $G = [\ \bar{G}\ ,\ \tilde{G}\ ]$.

So, for $u = \begin{bmatrix} \bar{u} - w\,\tilde{u} \\ \tilde{u} \end{bmatrix}$, $\bar{u} \in \text{ext } \bar{U}_F(v)$, $\tilde{u} \in \tilde{U}$,

$$G\,u = \bar{G}\,\bar{u}$$

so that

$$g_{s_i}^T\,u = \bar{g}_{s_i}^T\,\bar{u} = v_{s_i} \qquad\qquad i \in [1,k]$$

$$g_{t_j}^T\,u = \bar{g}_{t_j}^T\,\bar{u} < v_{t_j} \qquad\qquad j \in [1,p-k]$$

where

$$1 \le s_i \ , \quad t_j \le p$$

$$G_s = \begin{bmatrix} g_{s_i}^T \\ \vdots \\ \cdots \\ \vdots \\ g_{s_k}^T \end{bmatrix}$$

$$\text{rank } G_s = \text{rank } \bar{G}_s = q = \text{rank } G$$

and

$$k \ge \text{rank } G \ .$$

This means that u is a limit-tracking input in the sense of Defintion 5.1.   Part 2 QED

(Proof of Proposition 5.2)

1 and 2:

Let

$$\bar{G} = \begin{bmatrix} g_{s_1}^T \\ \vdots \\ \cdots \\ \vdots \\ g_{s_{m-1}}^T \end{bmatrix} \quad , \quad \bar{v} = \begin{bmatrix} v_{s_1} \\ \vdots \\ \cdots \\ \cdots \\ v_{s_{m-1}} \end{bmatrix} \quad ,$$

$$S = \{u : \bar{G}u = \bar{v}\} \quad ,$$

$$h_i(u) = g_{s_i}^T u \quad , \qquad\qquad i \in [1,m-1] \quad ,$$

$$L_i = \{u : h_i(u) = v_{s_i}\} \cap U_F(v) \quad , \qquad i \in [1,m-1] \quad ,$$

$$L = \bigcap_{i=1}^{m-1} L_i \quad .$$

By Theorem 5.1, there exists at least one set of points in $U_F(v)$ which satisfies equations (5.10) or (5.11). Therefore,

$$L_i \neq \emptyset \quad , \qquad i \in [1,m-1] \quad ,$$

and

$$L \neq \emptyset \quad .$$

Clearly, S is the intersection of m-1 hyperplanes $g_{s_i}^T u = v_{s_i}$,

$i \in [1, m-1]$. Since L_i is the set of points where a linear function h_i achieves its maximum over $U_F(v)$, L_i is a face of $U_F(v)$.

Since $h_i(v) = v_{s_i} =$ constant on a line segment of L_i, L_i is an exposed face. Furthermore,

$$
\begin{aligned}
\dim L &= \dim \text{ aff } (L) \\
&= \dim S \\
&= m - \text{rank } \bar{G} \\
&= 1 \ ,
\end{aligned}
$$

a half line $\{u + \lambda d : \lambda \geq 0, \ \bar{G}u = \bar{v}, \ \bar{G}d = 0\} \subset U_F(v)$, and by the proof of Theorem 5.1, $U_F(v)$ contains no lines. This means that L is a half line. Therefore, L is an exposed half-line face.

In the case $v_{s_i} = 0$, $i \in [1, m-1]$, L is an extreme ray.

3:

Let

$$
\bar{G} = \begin{bmatrix} g_{s_1}^T \\ \cdots \\ \cdots \\ g_{s_{m-1}}^T \end{bmatrix} \quad , \quad \bar{v} = \begin{bmatrix} v_{s_1} \\ \cdots \\ \cdots \\ v_{s_{m-1}} \end{bmatrix} \quad ,
$$

$$\tilde{G} = \begin{bmatrix} g^T_{t_1} \\ \vdots \\ \cdots \\ \cdots \\ g^T_{t_{p-m+1}} \end{bmatrix} \quad , \quad \tilde{v} = \begin{bmatrix} v_{t_1} \\ \vdots \\ \cdots \\ \cdots \\ v_{t_{p-m+1}} \end{bmatrix} \quad .$$

An extreme ray corresponding to equation (5.10) is expressed in the form

$$L_0 = \{u : \overline{G}u = 0 \quad , \quad \tilde{G}u \leq 0\}$$
$$= \{u : \overline{G}u = 0\} \cap U_F(0)$$
$$= \{\lambda d_0 : \lambda \geq 0\} \quad , \quad d_0 \neq 0 \ .$$

An exposed half-line face corresponding to equation (5.11) is expressed in the form

$$L = \{u : \overline{G}u = \overline{v} \quad , \quad \tilde{G}u \leq \tilde{v}\}$$
$$= \{u : \overline{G}u = \overline{v}\} \cap U_F(v) \ .$$

Then, for $\lambda d_0 \in L_0$ and $u \in L$,

$$\overline{G}(u + \lambda d_0) = \overline{G}u$$
$$= \overline{v}$$

and

$$\tilde{G}(u + \lambda d_0) = \tilde{G}u + \lambda \tilde{G}d_0$$
$$\leq \tilde{v} \ .$$

This means that

$$u + \lambda d_0 \in L \ .$$

Therefore, L_0 and L have the same direction called the extreme direction. It is now evident that a line corresponding to an exposed half-line face L is parallel to a line corresponding to such an extreme ray L_0.

4:

Suppose that an exposed half-line face of $U_F(v)$ emanates from a point u_0 and is expressed in the form

$$L = \{u : \bar{G}u = \bar{v} \quad , \quad \tilde{G}u \leq \tilde{v}\}$$
$$= \{u_0 + \lambda d : \lambda \geq 0\} \quad , \quad d \neq 0 \quad , \quad \bar{G}d = 0 \ ,$$

where $\bar{G}$, $\tilde{G}$, $\bar{v}$, and $\tilde{v}$ are defined in the former part of the proof.

Clearly, a point u_0 is the unique vertex of L. And for $\lambda < 0$, $u_0 + \lambda d \notin L$. Therefore, for $\lambda < 0$,

$$\bar{G}(u_0 + \lambda d) \neq \bar{v}$$

or

$$\tilde{G}(u_0 + \lambda d) \nleq \tilde{v} \ .$$

However,

$$\bar{G}(u_0 + \lambda d) = \bar{G}u_0 + \lambda\bar{G}d$$
$$= \bar{G}u_0$$
$$= \bar{v} \ .$$

This implies that

$$\tilde{G}\ (u_0\ +\ \lambda d)\ \nleq\ \tilde{v}\ .$$

Therefore, there exits at least one index s_m among $t_1, t_2, \ldots, t_{p-m+1}$ such that for $\lambda < 0$,

$$g_{s_m}^T (u_0\ +\ \lambda d)\ \nleq\ v_{s_m}$$

ie

$$g_{s_m}^T (u_0\ +\ \lambda d)\ >\ v_{s_m}\ .$$

For such index s_m, if $g_{s_m}^T u_0 < v_{s_m}$ then $u_0 \in \text{Int } H_{s_m}$, where $H_{s_m} = \{u : g_{s_m}^T u \leq v_{s_m}\}$. It follows by Corollary 6.4.1[1] that for $-d$, there exists some $\epsilon > 0$ such that $u_0 + \epsilon(-d) \in H_{s_m}$. Therefore,

$$g_{s_m}^T (u_0\ -\ \epsilon d)\ \leq\ v_{s_m} \qquad \text{for } \epsilon > 0$$

and this is a contradiction. So,

$$g_{s_m}^T u_0\ \nless\ v_{s_m}$$

ie

$$g_{s_m}^T u_0\ \geq\ v_{s_m}\ .$$

However, since $u_0 \in L \subset U_F(v)$, $g_{s_m}^T u_0 = v_{s_m}$.

Next, if g_{s_m} is dependent of $g_{s_1}, \ldots, g_{s_{m-1}}$, g_{s_m} is the linear combination of $g_{s_1}, \ldots, g_{s_{m-1}}$ and such that

$$g_{s_m} = \lambda_1 g_{s_1} + \ldots + \lambda_{m-1} g_{s_{m-1}} \cdot$$

Then,

$$
\begin{aligned}
g_{s_m}^T d &= (\lambda_1 g_{s_1}^T + \ldots + \lambda_{m-1} g_{s_{m-1}}^T) d \\
&= \lambda_1 g_{s_1}^T d + \ldots + \lambda_{m-1} g_{s_{m-1}}^T d \\
&= 0 \cdot
\end{aligned}
$$

Therefore, for $\lambda < 0$,

$$
\begin{aligned}
g_{s_m}^T (u_0 + \lambda d) &= g_{s_m}^T u_0 + \lambda g_{s_m}^T d \\
&= v_{s_m} \cdot
\end{aligned}
$$

Since $g_{s_m}^T$ is one of rows of $\widetilde{G}$, this is a contradiction. So, $g_{s_1}, \ldots, g_{s_m}$ are independent and u_0 satisfies

$$g_{s_i}^T u_0 = v_{s_i} \quad , \quad i \in [1,m]$$

$$g_{t_j}^T u_0 \leq v_{t_j} \quad , \quad j \in [1,p-m] \cdot$$

This means that u_0 is 0-dimensional face ie an extreme point.

QED

(Proof of Proposition 7.1)

Suppose that

$$\chi(0;x_s(T);v) = x_s(T)$$

and that there exists $i \in I_r \backslash I_c(v)$ such that $y_{t_i}(0) \not< v_{t_i}$. Since $i \notin I_c(v)$, this means that $y_{t_i}(0) > v_{t_i}$. Then,

$$\forall j \in J(0), \quad e_{t_j}(0) \leq e_{t_i}(0) = v_{t_i} - y_{t_i}(0) < 0 .$$

Since $\ell(0) \in J(0)$, $e_{t_{\ell(0)}} < 0$. However, since $x_s(T)$ is a steady state,

$$A_{\ell(0)}(T)x_s(T) + B_{\ell(0)}(T)v^{(\ell(0))} = 0 .$$

Therefore,

$$e^{(\ell(0))} = 0 .$$

This is a contradiction and implies that $y_{t_i}(0) < v_{t_i}$ for $i \in I_r \backslash I_c(v)$.

<div align="right">QED</div>

(Proof of Proposition 7.2)

The asymptotic stability of each closed-loop system

corresponding to loop index i ensures that all the eigenvalues of A_i lie in the open unit disc D^-. This means that all the solutions of equation

$$\det (\lambda I_{n+m} - A_i) = 0$$

satisfy $|\lambda| < 1$, that for $\lambda = 1$ (clearly, $|\lambda| = 1$),

$$\det (\lambda I_{n+m} - A_i) \neq 0 ,$$

and therefore that

$$\text{rank} (I_{n+m} - A_i) = n + m .$$

Since $\text{rank} [I_{n+m} - A_i , B_i] \geq \text{rank} [I_{n+m} - A_i]$, $\text{rank} [I_{n+m} - A_i , B_i] = n + m$. Hence,

$$\text{rank} [I_{n+m} - A_i , B_i] = \text{rank} [I_{n+m} - A_i]$$

and a steady state is determined as the unique solution x_i of the equation

$$(I_{n+m} - A_i)x_i = B_i v^{(i)} \qquad i \in I_c(v) .$$

For $i \in I_r \backslash I_c(v)$, if x_i satisfies

$$(I_{n+m} - A_i)x_i = B_i v^{(i)}$$

then $y^{(i)} = v^{(i)}$, so that $y_{t_i} = v_{t_i}$. This means that $i \in I_c(v)$ and contradicts the fact that $i \in I_r \backslash I_c(v)$. So, only x_i, $i \in I_c(v)$ are steady states of the system.   QED

(Proof of Proposition 7.3)

Let $\rho_0 = \rho[x_0 , \mathcal{L}(v)]$. Though ρ_0 is unknown, there exists $\gamma = \gamma(x_0, \mathcal{L}(v))$ such that if $\rho_0 \leq \gamma$, then for every $\bar{\epsilon} > 0$, there exists a positive $k_1^* = k_1^*(\bar{\epsilon}, \gamma)$ such that

$$\rho[\chi(kT; x_0; v) , \mathcal{L}(v)] \leq \bar{\epsilon} \qquad \text{for } kT \geq k_1^* T .$$

This means that for $kT \geq k_1^* T$,

$$\inf \left[\left\| \begin{bmatrix} x \\ z \end{bmatrix} - \begin{bmatrix} x^* \\ z^* \end{bmatrix} \right\| : C^{(i)} x^* = v^{(i)}, \ i \in I_c(v), \right.$$

$$c_{t_j}^T x^* < v_{t_j}, \ j \in I_r \backslash I_c(v),$$

$$z^* \in R^m,$$

$$\left. \begin{bmatrix} x^* \\ z^* \end{bmatrix} \in \mathcal{L}(v) \right] \leq \bar{\epsilon} .$$

By Schwarz's inequality, for $\forall i \in I_c(v)$ and for $\forall x^* \in \mathcal{S}(v)$

$$\|e^{(i)}\| = \|C^{(i)} x - v^{(i)}\|$$

$$= \|[C^{(i)} , 0](x - x^*)\|$$

$$\leq \|[C^{(i)} , 0]\| \|x - x^*\| .$$

Since $\mathcal{L}(v) \subseteq \mathfrak{Z}(v)$, for $\forall i \in I_c(v)$

$$\|e^{(i)}\| \leq \|[C^{(i)}, 0]\inf[\|x - x^*\| : x^* \in \mathcal{L}(v)]$$

$$\leq \|[C^{(i)}, 0]\|\bar{\epsilon}$$

$$\leq \max_{i \in I_c(v)} \|[C^{(i)}, 0]\|\bar{\epsilon} \qquad \text{for } kT \geq k_1^*T .$$

For $\forall j \in I_r \backslash I_c(v)$ and for $\forall x^* \in \mathfrak{Z}(v)$, let $e_{t_j}^* = v_{t_j} - c_{t_j}^T x^*$. Then,

$$|e_{t_j} - e_{t_j}^*| = |[c_{t_j}^T, 0](x - x^*)|$$

$$\leq \|[c_{t_j}^T, 0]\|\|x - x^*\| .$$

Since $\mathcal{L}(v) \subseteq \mathfrak{Z}(v)$, for $\forall j \in I_r \backslash I_c(v)$

$$|e_{t_j} - e_{t_j}^*| \leq \|[c_{t_j}^T, 0]\|\inf[\|x - x^*\| : x^* \in \mathcal{L}(v)]$$

$$\leq \|[c_{t_j}^T, 0]\|\bar{\epsilon}$$

$$\leq \max_{j \in I_r \backslash I_c(v)} \|[c_{t_j}^T, 0]\|\bar{\epsilon} \qquad \text{for } kT \geq k_1^*T .$$

Let $\bar{\epsilon} = e_{th}/\max_{\substack{i \in I_c(v) \\ j \in I_r \backslash I_c(v)}} (\|[C^{(i)}, 0]\|, \|[c_{t_j}^T, 0]\|)b$

then for $kT \geq k_1^*T$

$$\forall i \in I_c(v), \quad \|e^{(i)}(kT)\| \leq e_{th}$$

and since $e_{t_j}^* > 0$, $-e_{th} < e_{t_j}$.

Furthermore, for $i \in I_c(v)$ let

$$H_i = \{x : e_{s_1} = \ldots = e_{s_{m-1}} = 0, \ e_{t_i} < e_{t_j}, \ j \in I_r \backslash I_c(v)\}$$

and in case $I_r \backslash I_c(v) = \emptyset$,

$$H_i = \{x : e_{s_1} = \ldots = e_{s_{m-1}} = 0\} \ .$$

If $x \in \mathcal{L}(v)$, then $\forall i \in I_c(v)$, $e^{(i)} = 0$,

ie $e_{s_1} = \ldots = e_{s_{m-1}} = 0$, $e_{t_i} = 0$ and $\forall j \in I_r \backslash I_c(v)$, $e_{t_j} > 0$.

So, $x \in \bigcap_{i \in I_c(v)} H_i$.

Therefore, there exists $\tilde{\epsilon} = \tilde{\epsilon}(v) > 0$ such that

$$\mathcal{L}(v) + \tilde{\epsilon}B \subset \bigcap_{i \in I_c(v)} H_i \ ,$$

where B is the Euclidean unit ball in R^{n+m}.

Similarly to the former part of the proof, for this $\tilde{\epsilon}$ there exists a positive $k_2^* = k_2^*(\tilde{\epsilon}, \gamma)$ such that

$$\rho[\chi(kT; x_0; v) \ , \ \mathcal{L}(v)] \leq \tilde{\epsilon} \quad \text{for all } kT \geq k_2^* T.$$

Then

$$J(kT) \subset I_c(v) \qquad\qquad kT \geq k_2^*T$$

and

$$\|e^{(\ell(kT))}(kT)\| \leq \max_{i \in I_c(v)} \|[C^{(i)}, 0]\|\tilde{\epsilon} \qquad kT \geq k^*_2 T.$$

So, if k^* is chosen such that

$$k^* = \max(k_1^*, k_2^*)$$

then for $kT \geq k^*T$

$$\ell(kT) \in J(kT) \subset I_c(v)$$

$$\|e^{(\ell(kT))}(kT)\| \leq e_{th}$$

$$\forall i \in I_c(v), \|e^{(i)}(kT)\| \leq e_{th}$$

$$\forall j \in I_r \backslash I_c(v), \quad e_{t_j}(kT) > -e_{th}.$$

This implies that there exists $k^* = k^*(e_{th}, x_0, v)$ such that for $kT \geq k^*T$

$$e(kT) \geq -e_\epsilon$$

and

$$\| e^{(\ell(kT))}(kT) \| \leq e_{th},$$

where $e_\epsilon = [e_{th}, \ldots, e_{th}]^T \in R^p$.

<div align="right">QED</div>

(Proof of Proposition 7.4)

This is clear from Definition 7.6.

<div align="right">QED</div>

(Proof of Proposition 7.5)

The system under Loop-fixed mode is linear time-invariant and $x_s^\ell f(T)$ is an asymptotically and exponentially stable steady state of such system of equation (7.32) with $\ell(kT) = \ell_f$. Therefore, there exist m, $\alpha > 0$ such that the solution $\chi = \chi(kT;x_0;v)$ satisfies the condition

$$\rho[\chi \ , \ x_s^\ell f(T)] \leq me^{-\alpha kT} \rho[x_0 \ , \ x_s^\ell f(T)] \ .$$

For a given $\epsilon > 0$, let $k_1^* = k_1^*(m,\alpha,\epsilon,\rho[x_0 \ , \ x_s^\ell f(T)])$ be such that

$$me^{-\alpha k_1^* T} \rho[x_0 \ , \ x_s^\ell f(T)] \leq \epsilon \ .$$

It follows that

$$e^{-\alpha k_1^* T} \leq \frac{\epsilon}{m\rho[x_0 \ , \ x_s^\ell f(T)]}$$

$$-\alpha k_1^* T \leq \ln \frac{\epsilon}{m\rho[x_0 \ , \ x_s^{\ell} f(T)]}$$

$$k_1^* T \geq \frac{1}{\alpha} \ln \frac{m\rho[x_0 \ , \ x_s^{\ell} f(T)]}{\epsilon} \qquad \cdot$$

Therefore, it is evident that

$$\rho[\chi \ , \ x_s^{\ell} f(T)] \leq \epsilon \qquad k \geq k_1^* T \ .$$

QED

(Proof of Proposition 7.6)

Since each separate closed-loop is asymptotically stable,

$$_s e^{(\ell_f)} = \lim_{k \to \infty} e^{(\ell_f)}(kT) = 0 \ .$$

In the sequel, $_s \cdot$ means $\lim_{k \to \infty} \cdot$.

Since $\ell_f \in I_c(v)$,

$$_s e_{s_1} = \cdots = {}_s e_{s_{m-1}} = 0$$

$$\forall i \in I_c(v), \quad _s e_{t_i} = 0$$

and

$$\forall j \in I_r \backslash I_c(v), \quad {}_se_{t_j} > 0 \quad \text{(Proposition 7.1)} \ .$$

For $i \in I_c(v)$ let

$$H_i = \{x : e_{s_1} = \ldots = e_{s_{m-1}} = 0, \ e_{t_i} < e_{t_j}, \ j \in I_r \backslash I_c(v)\}$$

and in case $I_r \backslash I_c(v) = \emptyset$,

$$H_i = \{x : e_{s_1} = \ldots = e_{s_{m-1}} = 0\} \ .$$

Let

$$x_s^\ell f(T) = \begin{bmatrix} x_s^\ell f(T) \\ z_s^\ell f(T) \end{bmatrix} = \lim_{k \to \infty} \chi(kT; x_0; v)_{\ell(kT)=\ell_f} \ .$$

Clearly, $x_s^\ell f(T) \in \bigcap_{i \in I_c(v)} H_i$.

Since H_i is open, int $H_i = H_i$ and there exists some $\epsilon > 0$ such that

$$x_s^\ell f(T) + \epsilon B \subset \bigcap_{i \in I_c(v)} H_i \ .$$

Therefore, using Proposition 7.5,

$$\chi(kT; x_0; v)_{\ell(kT)=\ell_f} \in \bigcap_{i \in I_c(v)} H_i \qquad k \geq k_1^*(\epsilon, x_0, v) \ .$$

This means that

$$\sum_{i \in I_c(v)} T_{int}(i) = T_o \ ,$$

where $T_o = \tilde{t}_1 - \tilde{t}_0$, $\tilde{t}_0 \geq k_1^* T$, and that

$$\forall \ j \in I_r \backslash I_c(v), \quad T_{int}(j) = 0, \quad \tilde{t}_0 \geq k_1^* T \ .$$

So,

$$\forall \ j \in I_r \backslash I_c(v), \quad T_{int}(\ell_f) \nleq \beta T_{int}(j), \quad \beta > 0, \quad \tilde{t}_0 \geq k_1^* T \ .$$

$$\text{QED}$$

(Proof of Proposition 7.7)

In the sequel, $_s\cdot$ means $\lim_{k \to \infty} \cdot$. By the assumption, $_s e_{t_{\ell_f}} = 0$.

Furthermore, if

$$\forall i \in I_c(v), \quad _s e_{t_i} \geq 0$$

and

$$\forall j \in I_r \backslash I_c(v), \ j \neq \ell_f, \quad _s e_{t_j} \geq 0$$

then

$$_se_{t_{\ell_f}} = \min (_se_{t_i}, \ _se_{t_j}), \ i \in I_c(v), \ j \in I_r \backslash I_c(v)$$

$$= \min \ _se_{t_i}, \quad i \in I_r \ .$$

This implies that $\ell_f \in I_c(v)$ and contradicts the assumption. So, there exists at least one index $i \neq \ell_f$, $i \in I_r$ such that

$$_se_{t_{\ell_f}} > \ _se_{t_i}$$

ie

$$\exists i \in I_r, \quad x_s^{\ell}f(T) \in \{x : e_{t_{\ell_f}} > e_{t_i}\}$$

and there exists some $\epsilon > 0$ such that

$$\exists i \in I_r, \quad x_s^{\ell}f(T) + \epsilon B \subset \{x : e_{t_{\ell_f}} > e_{t_i}\} \ .$$

This means that for some ϵ and for $x \in x_s^{\ell}f(T) + \epsilon B$

$$\ell_f \notin J^0(kT)$$

and that, by Proposition 7.5, there exists some k_1^* such that

$$\chi(kT; x_0; v) \in x_s^{\ell}f(T) + \epsilon B \qquad kT \geq k_1^*T \ .$$

So,

$$\ell_f \notin J^0(kT) \qquad\qquad\qquad kT \geq k_1^*T$$

$$T_{int}(\ell_f) = 0 \qquad \tilde{t}_0 \geq k_1^* T \;.$$

However, this means that

$$\sum_{\substack{i \in I_r \\ i \neq \ell_f}} T_{int}(i) = T_0 \qquad \tilde{t}_0 \geq k_1^* T \;.$$

Therefore, there exists k_1^* such that

$$\exists i \neq \ell_f, \quad T_{int}(\ell_f) < \beta T_{int}(i), \quad \beta > 0, \quad \tilde{t}_0 \geq k_1^* T \;.$$

QED

(Proof of Theorem 7.1)

Since the plant is asymptotically stable and the plant input is bounded, the closed-loop system exhibits state-bounded tracking. Next, the following cases are considered:

1: In Normal or Loop-excluded mode, the tracking assessment is continuously 'Convergent'.

2: In Normal or Loop-excluded mode (Level 1 to r-1), the tracking assessment is 'Non convergent'.

Case 1:

In the case of 'Convergent' assessment in Normal or Loop-excluded mode,

(i)

$$\max(e_s)_a \leq e_{th} \tag{A2.1a}$$

$$\min(e_s)_a \geq -e_{th} \tag{A2.1b}$$

$$\max(e_t)_a \leq e_{th} \tag{A2.1c}$$

$$\min(e_t)_a \geq -e_{th} \tag{A2.1d}$$

or

(ii)

$$\Delta_{\max}(e_s)_a \leq \alpha\Delta_{\max}(e_s)_{a-1} \tag{A2.2a}$$

$$\Delta_{\min}(e_s)_a \leq \alpha\Delta_{\min}(e_s)_{a-1} \tag{A2.2b}$$

$$|_{\text{mean}}(e_s)_a| \leq \alpha|_{\text{mean}}(e_s)_{a-1}| \tag{A2.2c}$$

$$\Delta_{\max}(e_t)_a \leq \alpha\Delta_{\max}(e_t)_{a-1} \tag{A2.2d}$$

$$\Delta_{\min}(e_t)_a \leq \alpha\Delta_{\min}(e_t)_{a-1} \tag{A2.2e}$$

$$|_{\text{mean}}(e_t)_a| \leq \alpha|_{\text{mean}}(e_t)_{a-1}| \tag{A2.2f}$$

are obtained for $t \geq t_a$. Therefore, in case of (i), from

equations (A2.1a) and (A2.1b),

$$\forall i \in [1, m-1] \quad -e_{th} \leq e_{s_i}(kT) \leq e_{th} \qquad kT \geq t_{a-1} \ . \qquad (A2.3)$$

From equation (A2.1c),

$$e_{t_{\ell}(kT)}(kT) \leq e_{th} \qquad kT \geq t_{a-1} \ . \qquad (A2.4)$$

From equation (A2.1d),

$$\min_{\substack{kT \in [t_{a-1}, t_a) \\ i \in I_r}} e_{t_i}(kT) \geq -e_{th} \ , \qquad (A2.5)$$

so that

$$\forall i \in I_r, \quad e_{t_i}(kT) \geq -e_{th} \qquad kT \geq t_{a-1} \ . \qquad (A2.6)$$

It follows from equations (A2.3) and (A2.6) that for $kT \geq t_{a-1}$,

$$\forall i \in [1, m-1], \quad e_{s_i}(kT) \geq -e_{th} \qquad (A2.7a)$$

$$\forall j \in I_r, \quad e_{t_j}(kT) \geq -e_{th} \qquad (A2.7b)$$

and therefore that for $kT \geq t_{a-1}$,

$$e(kT) \geq -e_{\epsilon} \ , \qquad (A2.8)$$

where $e_\epsilon = [e_{th}, \ldots, e_{th}]^T \in R^p$.

It follows from equations (A2.3), (A2.4), and (A2.5) that for $kT \geq t_{a-1}$,

$$\| e^{(\ell(kT))}(kT) \| \leq e_{th} . \qquad (A2.9)$$

In case of (ii), it follows from equation (A2.2a) that

$$\Delta_{max}(e_s)_a \leq \alpha^{a-1} \Delta_{max}(e_s)_1 . \qquad (A2.10)$$

Therefore, there exists a_{11}^* such that

$$\Delta_{max}(e_s)_a \leq e_{th}/2 \qquad \text{for } a \geq a_{11}^* \quad (t_a \geq t_{a_{11}^*}) , \qquad (A2.11)$$

where

$$
\begin{aligned}
a_{11}^* &= a_{11}^*(\alpha, e_{th}, \Delta_{max}(e_s)_1) \\
&= INT[\log_\alpha(e_{th}/(2\Delta_{max}(e_s)_1) + 2)] \qquad (A2.12)
\end{aligned}
$$

and $INT[\cdot]$ is the integer function.

Similarly, it frollows from equation (A2.2b) that there exists a_{12}^* such that

$$\Delta_{min}(e_s)_a \leq e_{th}/2 \qquad \text{for } a \geq a_{12}^* \quad (t_a \geq t_{a_{12}^*}) , \qquad (A2.13)$$

where

$$a_{12}^* = a_{12}^*(\alpha, e_{th}, \Delta_{min}(e_s)_1)$$

$$= INT[\log_\alpha(e_{th}/(2\Delta_{min}(e_s)_1) + 2)] \ . \qquad (A2.14)$$

Furthermore, it follows from equation (A2.2c) that there exists a_{13}^* such that

$$|_{mean}(e_s)_a| \le e_{th}/2 \qquad \text{for } a \ge a_{13}^* \quad (t_a \ge t_{a_{13}^*}) \ , \quad (A2.15)$$

where

$$a_{13}^* = a_{13}^*(\alpha, e_{th}, |_{mean}(e_s)_1|)$$

$$= INT[\log_\alpha(e_{th}/2|_{mean}(e_s)_1|) + 2)] \qquad (A2.15)$$

Let $a_1^* = \max(a_{11}^*, a_{12}^*, a_{13}^*)$. Then, by Definition 7.8,

$$_{max}(e_s)_a - _{mean}(e_s)_a \le e_{th}/2 \qquad (A2.16a)$$

$$_{mean}(e_s)_a - _{min}(e_s)_a \le e_{th}/2 \qquad (A2.16b)$$

$$-e_{th}/2 \le _{mean}(e_s)_a \le e_{th}/2 \ . \qquad (A2.16c)$$

Therefore, for $a \ge a_1^*$ $(t_a \ge t_{a_1^*})$

$$\forall i \in [1, m-1], \quad -e_{th} \le e_{s_i}(kT) \le e_{th} \ . \qquad (A2.17)$$

Now, it follows from equations (A2.2d) to (A2.2f) that there exists a_2^* such that for $a \ge a_2^*$,

$$\max(e_t)_a - \operatorname{mean}(e_t)_a \leq e_{th}/2 \tag{A2.18a}$$

$$\operatorname{mean}(e_t)_a - \min(e_t)_a \leq e_{th}/2 \tag{A2.18b}$$

$$-e_{th}/2 \leq \operatorname{mean}(e_t)_a \leq e_{th}/2 \ . \tag{A2.18c}$$

Therefore, for $a \geq a_2^*$,

$$e_{t_{\ell(kT)}}(kT) \leq e_{th} \tag{A2.19}$$

and

$$\min_{\substack{kT \in [t_{a-1}, t_a) \\ i \in I_r}} e_{t_i}(kT) = \min(e_t)_a \geq -e_{th} \ . \tag{A2.20}$$

Thus,

$$-e_{th} \leq e_{t_{\ell(kT)}}(kT) \leq e_{th} \ . \tag{A2.21}$$

It follows from equations (A2.17) and (A2.20) that for $a \geq a_2^*$ $(kT \geq t_{a_2^*})$,

$$e(kT) \geq -e_\epsilon \ , \tag{A2.22}$$

where $e_\epsilon = [e_{th}, \ldots, e_{th}]^T \in R^p$.

It follows from equations (A2.17) and (A2.21) that for $a \geq a_2^*$ ($kT \geq t_{a_2}^*$),

$$\|e^{(\ell(kT))}(kT)\| \leq e_{th} . \qquad (A2.23)$$

Therefore, it follows from equations (A2.8), (A2.9), (A2.22), and (A2.23) that if the tracking assessment is continuously 'Convergent', nearly perfect dynamical limit tracking is attained for $kT \geq t_{a-1}$ or for $kT \geq t_{a_2}^*$.

Case 2:

In the case of 'Non convergent' assessment in Normal or Loop-excluded mode, Loop-fixed mode begins by Definition 7.9. Then, if $\ell_f \in I_c(v)$ then by Definitions 7.11 and 7.12, and Proposition 7.6, such mode continues to operate. Therefore,

$$\mathcal{L}(v) = \lim_{k \to \infty} \chi(kT;x_0;v)_{\ell(kT)=\ell_f} = x_s^{\ell}f(T) \in \mathfrak{I}(v)$$

and

$$\lim_{k \to \infty} \rho[\chi(kT;x_0;v)_{\ell(kT)=\ell_f} , \mathcal{L}(v)] = 0 .$$

Furthermore, since $x_s^{\ell}f(T)$ corresponds to a steady state of the original self-selecting control system,

$$\forall i \in I_c(v), \quad \lim_{k \to \infty} e^{(i)}(kT) = 0$$

and by Proposition 7.1,

$$\forall j \in I_r \backslash I_c(v), \quad \lim_{k \to \infty} e_{t_j}(kT) > 0 .$$

Hence, there exists $k^* = k^*(e_{th}, x_0, v)$ such that for $kT \geq k^*T$,

$$e(kT) \geq -e_\epsilon$$

and

$$\| e^{(\ell(kT))}(kT) \| = \| e^{(\ell_f)}(kT) \| \leq e_{th} ,$$

where $e_\epsilon = [e_{th}, \ldots, e_{th}]^T \in R^p$.

Therefore, nearly perfect dynamical limit tracking is achieved for $kT \geq k^*T$.

If $\ell_f \notin I_c(v)$, ie $\ell_f \in I_r \backslash I_c(v)$, then by Definition 7.11 and Proposition 7.7, 'Incorrect loop' assessment is obtained and such loop ℓ_f is excluded. Furthermore, if $\#(I_r^\eta) \geq 2$, by Definition 7.12, Loop-excluded mode begins and the analysis of Case 1 can be applied. If $\#(I_r^\eta) = 1$, by Definition 7.12, the control loop is fixed to the remaining loop. From the previous discussion, such a remaining loop must be a correct loop unless a plant variation has occured or a design parameter such as T_0, α, β is inappropriate (as indicated in the lowest stage, Fig 7.4). **QED**

APPENDIX 3

LINEAR F100 ENGINE MODEL

The linearised state-space model of the F100 turbofan engine (Figs A3.1 to A3.3) is governed on the continuous-time set $T = [0, \infty)$ by state, unmeasurable output, and measurable output equations of the respective forms (Miller and Hackney (1976))

$$\dot{x}_p = A_p x_p + B_p u_p \qquad (A3.1)$$

$$w = C_p^u x_p + D_p u_p \qquad (A3.2)$$

and

$$y_p = C_p^m x_p \qquad . \qquad (A3.3)$$

Here, the plant state vector $x_p \in R^{16}$, the plant input vector $u_p \in R^m$, the unmeasurable plant output vector $w \in R^5$, the measurable plant output vector $y_p \in R^p$, the plant state matrix $A_p \in R^{16 \times 16}$, the plant input matrix $B_p \in R^{16 \times m}$, the plant output matrix for unmeasurable outputs $C_p^u \in R^{5 \times 16}$, the plant direct coupled matrix for unmeasurable outputs $D_p \in R^{5 \times 5}$, and the plant output matrix for measurable outputs $C_p^m \in R^{p \times m}$. The control actuators are governed on T by state and output equations of the respective forms

$$\dot{x}_a = A_a x_a + B_a u \qquad (A3.4)$$

and

$$u_p = C_a x_a \qquad . \qquad (A3.5)$$

Here, the actuator state vector $x_a \in R^{n_a}$, the actuator input vector $u \in R^m$, the actuator output vector is the plant input vector $u_p \in R^m$, the actuator state matrix $A_a \in R^{n_a \times n_a}$, the actuator input matrix $B_a \in R^{n_a \times m}$, and the actuator output matrix $C_a \in R^{m \times n_a}$. Furthermore, the measurement sensors are governed on T by state and output equations of the respective forms

$$\dot{x}_s = A_s x_s + B_s y_p \qquad (A3.6)$$

$$y = C_s x_s \qquad . \qquad (A3.7)$$

Here, the sensor state vector $x_s \in R^{n_s}$, the sensor input vector is the measurable plant output vector $y_p \in R^p$, the sensor output vector $y \in R^p$, the sensor state matrix $A_s \in R^{n_s \times n_s}$, the sensor input matrix $B_s \in R^{n_s \times p}$, and the sensor outut matrix $C_s \in R^{p \times n_s}$.

It follows from equations (A3.1) to (A3.7) that the behaviour of systems consisting of such a plant, actuators, and sensors is governed on T by state, unmeasurable output, and measurable output equations of the respective forms

$$\dot{x} = Ax + Bu \qquad (A3.8)$$

$$w = Ex \tag{A3.9}$$

and

$$y = Cx \tag{A3.10}$$

where

$$x = \begin{bmatrix} x_p \\ x_a \\ x_s \end{bmatrix} \in R^{16+n_a+n_s} \tag{A3.11}$$

$$A = \begin{bmatrix} A_p & B_p C_a & 0 \\ 0 & A_a & 0 \\ B_s C_p^m & 0 & A_s \end{bmatrix} \tag{A3.12}$$

$$B = \begin{bmatrix} 0 \\ B_a \\ 0 \end{bmatrix} \tag{A3.13}$$

$$E = [C_p^u, \ D_p C_a, \ 0] \tag{A3.14}$$

and

$$C = [0, \ 0, \ C_s] \tag{A3.15}$$

The steady-state transfer-function matrices are given in the

forms for unmeasurable outputs

$$G_w = -EA^{-1}B$$

$$= (-C_p^u A_p^{-1} B_p + D_p)(-C_a A_a^{-1} B_a) \in R^{5 \times m} \qquad (A3.16)$$

and for measurable outputs

$$G_y = -CA^{-1}B$$

$$= (-C_s A_s^{-1} B_s)(-C_p^m A_p^{-1} B_p)(-C_a A_a^{-1} B_a) \in R^{p \times m}. \qquad (A3.17)$$

In the case of $m = 5$, $p = 5$, $n_a = 11$, and $n_s = 6$, the five manipulated variables are

u_1: main burner fuel flow (lb/hr)

u_2: nozzle jet area (ft^2)

u_3: inlet guide vane position (deg)

u_4: variable stator position (deg)

u_5: compressor bleed flow (%)

the five unmeasurable output variables are

w_1 (Fn)   : engine net thrust (lb)

w_2 (WFAN): total engine airflow (lb/s)

w_3 (TT4) : turbine inlet temperature (°R)

w_4 (SMAF): fan stall margin

w_5 (SMHC): compressor stall margin

and the five measurable output variables are

y_1 (N_1) : fan speed (rpm)

y_2 (N_2) : compressor speed (rpm)

y_3 (P_3) : compressor discharge pressure (psia)

y_4 (P_7) : augmentor pressure (psia)

y_5 (FTIT): fan-turbine inlet temperature ($^{\circ}$R).

In this case, the matrices A_p, B_p, C_p^u, D_p, and C_p^m are given in Tables A3.2(a),(b),(c) and A3.3(a),(b),(c) for two operating conditions [ie Sea Level Static (SLS)/Intermediate (Power Lever Angle (PLA) 83 deg) and Sea Level Static (SLS)/Power Lever Angle (PLA) 67 deg], where the data format is shown in Table A3.1. The matrices A_a, B_a, and C_a are given in Table A3.4. The matrices A_s, B_s, and C_s are given in Table A3.5.

In the case of different numbers of inputs or outputs, the corresponding parts of the input/output matrices of the plant and the corresponding parts of the state/input/output matrices of the actuators and the sensors are used.

Compressor Bleed Flow— ⌐Main Burner Fuel Flow

Inlet Guide Vane
Position

Variable Stator
Position

Nozzle Jet Area
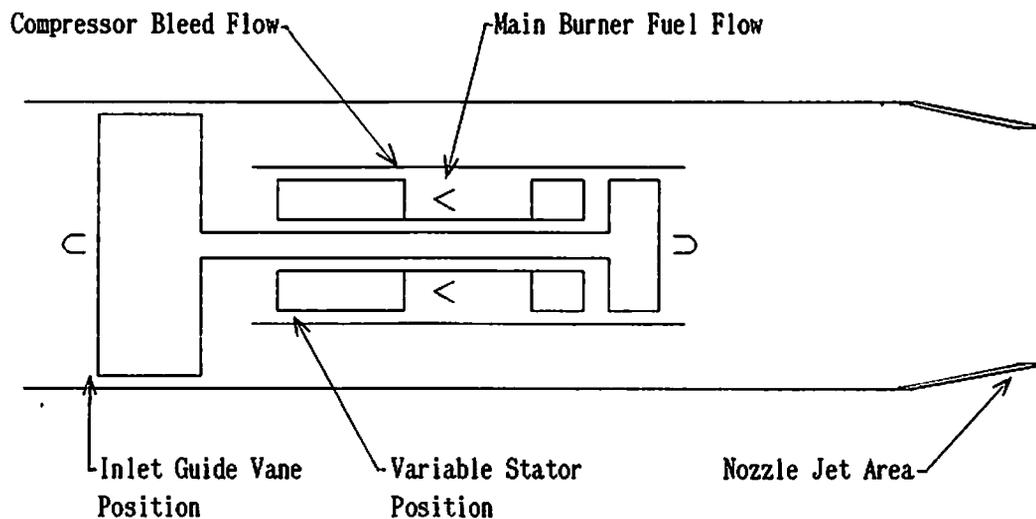
Fig A3.1 Manipulated variables of F100 engine

⌐SMAF   ⌐SMHC   ⌐TT4

Fg

WFAN   Fn=Fg-Drag
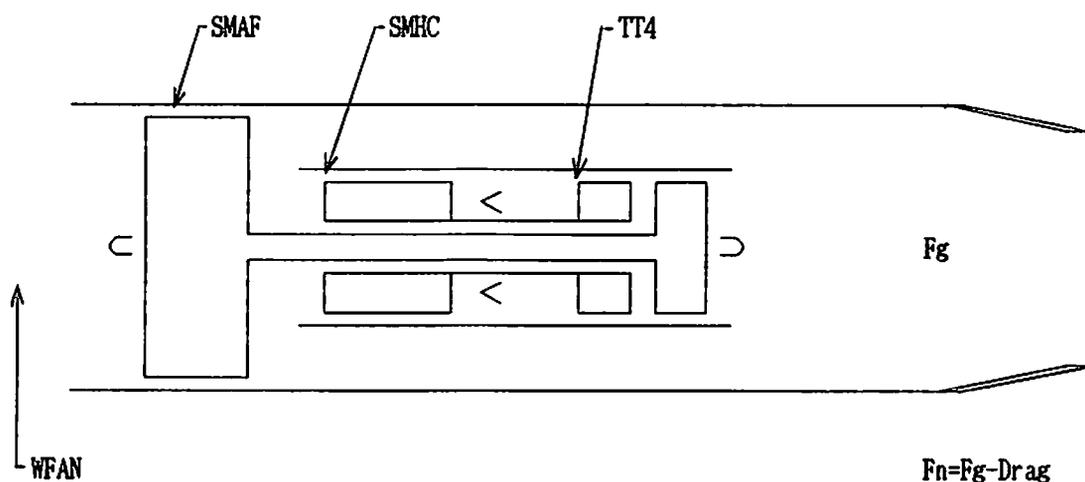
Fig A3.2 Unmeasurable output variables of F100 engine

⌐N1   ⌐N2 ⌐P3   ⌐FTIT   ⌐P7

Fig A3.3 Measurable output variables of F100 engine

Third page of each table
The C^m_p matrix

$$
\begin{matrix}
C^m_{P_{11}} & C^m_{P_{12}} & \cdots\cdots\cdots & C^m_{P_{1,16}} \\
C^m_{P_{21}} & & & \\
\vdots & & \ddots & \\
C^m_{P_{51}} & & \cdots\cdots\cdots & C^m_{P_{5,16}}
\end{matrix}
$$

Second page of each table
The B_p, C^u_p, and D_p matrices

$$
\begin{matrix}
B_{P_{11}} & B_{P_{12}} & \cdots & B_{P_{15}} \\
B_{P_{21}} & & & \\
\vdots & & & \\
B_{P_{16,1}} & \cdots\cdots & B_{P_{16,5}}
\end{matrix}
$$

$$
\begin{matrix}
C^u_{P_{11}} & C^u_{P_{12}} & \cdots\cdots\cdots & C^u_{P_{18}} \\
C^u_{P_{29}} & & & \\
\vdots & & & \\
C^u_{P_{51}} & & \cdots\cdots\cdots & C^u_{P_{5,8}}
\end{matrix}
$$

$$
\begin{matrix}
C^u_{P_{1,9}} & C^u_{P_{1,10}} & \cdots\cdots\cdots & C^u_{P_{1,16}} \\
C^u_{P_{29}} & & & \\
\vdots & & & \\
C^u_{P_{59}} & & \cdots\cdots\cdots & C^u_{P_{5,16}}
\end{matrix}
$$

$$
\begin{matrix}
D_{P_{11}} & D_{P_{12}} & \cdots & D_{P_{15}} \\
D_{P_{21}} & & & \\
\vdots & & & \\
D_{P_{51}} & & \cdots\cdots & D_{P_{55}}
\end{matrix}
$$
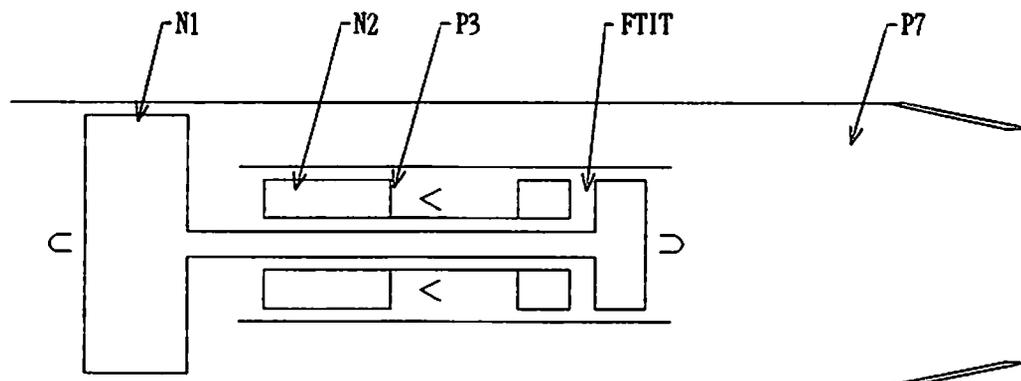
First page of each table
The A_p matrix

$$
\begin{matrix}
A_{P_{11}} & A_{P_{12}} & \cdots\cdots\cdots & A_{P_{18}} \\
A_{P_{21}} & & & \\
\vdots & & & \\
A_{P_{16,1}} & & \cdots\cdots\cdots & A_{P_{16,8}}
\end{matrix}
$$

$$
\begin{matrix}
A_{P_{1,9}} & \cdots\cdots\cdots & A_{P_{1,16}} \\
A_{P_{29}} & & \\
\vdots & & \\
A_{P_{16,9}} & \cdots\cdots\cdots & A_{P_{16,16}}
\end{matrix}
$$

Table A3.1   Linear F100 engine model data format

The A_p matrix

-4.328	.1714	5.376	401.6	-724.6	-1.933	1.020	-.9820
-.4402	-5.643	127.5	-233.5	-434.3	26.59	2.040	-2.592
1.038	6.073	-165.0	-4.483	1049.	-82.45	-5.314	5.097
.5304	-.1086	131.3	-578.3	102.0	-9.240	-1.146	-2.408
.8476E-2	-.1563E-1	.5602E-1	1.573	-10.05	.1952	-.8804E-2	-.2110E-1
.8350	-.1249E-1	-.3567E-1	-.6074	37.65	-19.79	-.1813	-.2962E-1
.6768	-.1264E-1	-.9683E-1	-.3567	80.24	-.8239E-1	-20.47	-.3928E-1
-.9696E-1	.8666	16.87	1.051	-102.3	29.66	.5943	-19.97
-.8785E-2	-.1636E-1	.1847	.2169	-8.420	.7003	.5666E-1	6.623
-.1298E-3	-.2430E-3	.2718E-2	.3214E-2	-.1246	-.1039E-1	.8395E-3	.9812E-1
-1.207	-6.717	26.26	12.49	-1269.	103.0	7.480	36.84
-.2730E-1	-.4539	-52.72	198.8	-28.09	2.243	.1794	9.750
-.1206E-2	-.2017E-1	-2.343	8.835	-1.248	.9975E-1	.8059E-2	.4333
-.1613	-.2469	-24.05	23.38	146.3	1.638	.1385	4.486
-.1244E-1	.3020E-1	-.1198	-.4821E-1	5.675	-.4525	19.81	.1249
-1.653	1.831	-3.822	113.4	341.4	-27.34	-2.040	-.6166

.9990	1.521	-4.062	9.567	10.08	-.6017	-.1312	.9602E-1
11.32	10.90	-4.071	-.5739E-1	-.6063	-.7488E-1	-.5936	-.9602E-1
-.9389E-2	.1352	5.638	-.2246E-1	.1797	.2407E-1	1.100	.2743E-1
-3.081	-4.529	5.707	-.2346	-2.111	-.2460	-.4686	-.3223
-.2090E-2	-.5256E-1	-.4077E-1	-.9182E-2	-.8178E-1	.3428E-1	.4995E-2	-.1256E-1
-.1953E-1	-.1622	-.6439E-2	-.2346E-1	-.2201	-.2514E-1	-.3749E-2	-.3361E-1
-.1878E-1	-.2129	-.9337E-2	-.3144E-1	-.2919	-.3370E-1	.8873E-1	-.4458E-1
-.2253E-1	.1791	.8371E-2	.2645E-1	.2560	.2835E-1	-.3749E-1	.3635E-1
-49.99	.6760E-1	39.46	.4991E-2	.8983E-1	.5349E-2	0	.1372E-1
-.6666	-.6657	.5847	.6654E-4	.1347E-2	.7131E-4	0	.2057E-3
.2854	2.332	-47.65	.3406	3.065	.3624	-.4343	.4681
-9.627	-9.557	38.48	-50.01	.1011	.1203E-1	-.4686E-1	.1715E-1
-.4278	-.4245	1.710	-2.000	-1.996	.5349E-3	-.1999E-2	.7544E-3
-4.414	-4.354	17.66	-3.113	-3.018	-19.77	-.4999E-1	.1509E-1
-.1127E-2	-.6760E-2	.1835E-1	-.9981E-3	-.1347E-1	-.1070E-2	-20.00	-.2057E-2
.5004	-.1437	-2.416	-.1073	-1.078	30.63	19.89	-50.16

Table A3.2(a) Matrix of linear F100 engine model at SLS/Intermediate

The B$_p$ matrix

-.4570E-1	-451.6	-105.8	-1.506	851.5
.1114	-546.1	-6.575	-107.8	3526.
.2153	1362.	13.46	20.14	-.6777E+5
.3262	208.0	-2.888	-1.653	-269.1
.99948E-2	-98.39	.5069	-.1940	-94.70
.2728E-1	71.62	9.608	-.3160	-184.1
.1716E-1	71.71	8.571	.7989	-515.2
-.7741E-1	-141.2	-.8215	39.74	1376.
.3855E-1	-7.710	-.4371E-1	-.1024	-6684.
.5707E-3	-.1144	-.6359E-3	-.1432E-2	-99.02
5.727	-1745.	-8.940	-17.96	.8898E+5
.1392	-24.30	-.2736	-.3403	-6931.
.6172E-2	-1.082	-.1183E-1	-.1452E-1	-307.7
.6777E-1	16.60	.3980	.2311E-1	-2588.
.1880E-2	9.147	-.8241	.8984E-1	-32.31
.1677	435.8	-89.94	4.900	-295.5

The C$_p^u$ matrix

.4866	-.6741	5.392	95.42	24.03	10.52	.8190	-.4492
.1383E-1	.2789E-5	.0	.0	-.1081E-1	-.5545E-4	.4722E-4	.0
.0	.0	.0	.0	.0	.0	.0	.0
.7418E-4	.5496E-5	.4790E-5	.1478E-3	-.1504E-1	-.6503E-4	.8820E-4	.4999E-5
.1538E-4	.1201E-3	-.2579E-2	-.1609E-3	.1618E-1	-.1071E-2	-.9561E-4	-.5503E-5
.5195	.8437	-1.863	.5709E-1	.4815	3.428	2.161	.7681E-1
.0	.0	.0	.0	.0	.0	.0	.0
.0	.0	1.000	.0	.0	.0	.0	.0
.3434E-5	.2727E-4	.1128E-5	.4002E-5	.3673E-4	.4290E-5	-.4958E-5	.5609E-5
-.3732E-5	-.2996E-4	-.1234E-5	-.4380E-5	-.4024E-4	-.4721E-5	.5324E-5	-.6103E-5

The D$_p$ matrix

-.6777E-1	-420.5	32.97	-1.824	1245.
.1282E-3	.3353	.6804	-.5605E-4	-.1199E-1
.0	.0	.0	.0	.0
.1030E-5	-.1193E-1	-.5806E-2	.6015E-4	.4463E-1
.8109E-5	.2328E-1	.1178E-3	-.5538E-2	-.1039

Table A3.2(b) Matrices of linear F100 engine model at SLS/Intermediate

The C_p^m matrix

$$
\begin{array}{cccccccccccccc}
1.0 & .0 & .0 & .0 & .0 & .0 & .0 & .0 & .0 & .0 & .0 & .0 & .0 & .0 \\
.0 & 1.0 & .0 & .0 & .0 & .0 & .0 & .0 & .0 & .0 & .0 & .0 & .0 & .0 \\
.0 & .0 & 1.0 & .0 & .0 & .0 & .0 & .0 & .0 & .0 & .0 & .0 & .0 & .0 \\
.0 & .0 & .0 & 1.0 & .0 & .0 & .0 & .0 & .0 & .0 & .0 & .0 & .0 & .0 \\
.0 & .0 & .0 & .0 & .0 & .0 & .0 & .0 & .0 & 1.0 & .0 & .0 & .0 & .0 \\
.0 & .0 & .0 & .0 & .0 & .0 & .0 & .0 & .0 & .0 & 1.0 & .0 & .0 & .0 \\
\end{array}
$$

Table A3.2(c)   Matrix of linear F100 engine model at SLS/Intermediate

The A_p matrix

-4.132	.6039	5.880	428.3	-726.1	-6.646	-.7558	-.3874	1.166	3.629	-3.188	8.502	11.64	-.2480	.6694	.5527
-1.243	-5.204	124.8	-223.7	-227.1	25.91	-.9276	-1.628	10.20	12.78	-3.237	.4379	3.889	-.4309	.6967	.6096
3.002	5.331	-160.5	-20.21	552.1	-68.58	2.735	3.013	-.6178	-5.501	4.052	-.8206	-7.281	-.8308	-1.134	-1.142
.5780	-.6019E-1	121.1	-566.5	87.71	-7.600	-.5359	-2.131	-2.766	-5.638	4.596	-.4234	-3.862	-.3906	-.9358	-.6055
.1820E-1	-.9528E-2	.2287	2.232	-7.382	.2233	.4044E-1	.1992E-1	.2430E-1	.1603	-.1514E-1	-.2273E-1	.2041	.6591E-1	.8521E-1	.3168E-1
.7680	-.8363E-1	.1685	-.6931	39.35	-18.90	.2652	-.2941E-1	-.1907E-1	-.1717	-.1304E-1	-.2552E-1	-.2140	-.2480E-1	-.4098E-2	-.3333E-1
.3466	-.1697	.3134	-1.553	78.53	2.222	-19.48	-.7102E-1	-.4248E-1	-.3863	-.3129E-1	-.5741E-1	-.5167	-.5394E-1	-.2596E-1	-.8047E-1
-.4025	.8823	22.62	1.476	-75.12	29.02	-.4810	-19.93	.4205E-1	.3590	.3017E-1	.5335E-1	.4802	.4774E-1	.2186E-1	.7478E-1
-.2535E-1	-.1324E-1	.1853	-.3209E-1	-4.445	.5493	-.2061E-1	6.377	-49.99	.3902E-1	39.52	.2900E-2	.2610E-1	-.1550E-2	-.6831E-2	.1626E-1
-.3760E-3	-.1980E-3	.2696E-2	-.8557E-3	-.6618E-1	.8102E-2	-.3206E-3	.9448E-1	-.6666	-.6661	.5855	.3866E-4	.3480E-3	-.4133E-4	-.1366E-3	.2167E-3
-4.506	-7.261	31.62	15.63	-841.3	99.80	-5.504	38.69	-.4335	3.794	-47.51	.5640	4.972	.5440	.1673	.7803
-.5226E-1	-.3814	-62.97	234.7	-9.260	1.232	-.5153E-1	9.541	-9.864	-9.735	38.81	-49.99	.1305	.2790E-1	.3415E-2	.2845E-1
-.2328E-2	-.1695E-1	-2.799	10.43	-.4119	.5479E-1	-.2336E-2	.4240	-.4384	-.4327	1.725	-1.999	-1.994	.1240E-2	.1366E-3	.1219E-2
-.1569	-.2174	-28.66	21.56	188.2	1.219	-.3298E-1	4.405	-4.517	-4.358	17.84	-3.091	-2.892	-19.73	.2869E-1	.3739E-1
-.8900E-2	-.2400E-1	-.9773E-1	-.5134E-1	2.963	-.3131	19.89	.1011	-.1734E-2	-.7803E-2	.1527E-1	-.2320E-2	-.2088E-1	-.1240E-2	-20.00	-.3251E-2
-2.109	1.654	-2.013	135.8	207.9	-21.22	1.841	-.3928	.6481	1.005	-2.291	.5220E-1	.5872	28.38	22.80	-49.91

Table A3.3(a) Matrix of linear F100 engine model at SLS/PLA 67 deg

The B_p matrix

-.4711	-238.0	-106.3	3.619	2016.
-.4616	-137.0	-23.33	-77.02	6038.
1.355	408.6	56.19	13.52	-.5727E+5
.6243	206.3	.5061	-2.668	-2131.
-.2991E-1	-82.45	1.023	.1759	222.3
.2201E-1	55.74	8.338	.1616	271.5
.5017E-1	122.8	8.619	-.2908	206.3
-.1779	-75.42	-7.548	32.68	2466.
.4516E-1	1.794	-.5636	-.2229	-5634.
.6831E-3	.2378E-1	-.8480E-2	-.3354E-2	-83.55
7.200	-653.8	-90.61	-28.70	.7812E+5
.1219	-.8765	-.9822	-.1042	-6134.
.5432E-2	-.4241E-1	-.4386E-1	-.4874E-2	-273.0
.5094E-1	-21.26	.4157	.3855E-1	-2439.
-.2444E-2	3.554	-.5694	.8113E-1	-32.60
-.8299E-2	351.6	-89.02	8.981	1128.

The C_p^u matrix

.6649	-.5442	5.667	89.00	120.0	8.090	-.2001	-.1681
.2601E-1	.3519E-3	-.6213E-3	.3048E-2	-.1659	-.4643E-2	-.1074E-2	.1367E-3
.0	.0	.0	.0	.0	.0	.0	.0
.1234E-3	.2592E-4	-.4714E-4	.2244E-3	-.1222E-1	-.3417E-3	-.7966E-4	.1002E-4
.6857E-4	.3171E-3	-.3571E-2	-.2378E-3	.1280E-1	-.2798E-2	.8381E-4	-.1047E-4

.5140	1.614	-1.337	.1726	1.562	2.762	2.355	.2490
.8806E-4	.7925E-3	.6169E-4	.1178E-3	.9950E-3	.1085E-3	.4269E-4	.1549E-3
.0	.0	1.000	.0	.0	.0	.0	.0
.6313E-5	.5572E-4	.4538E-5	.8347E-5	.7289E-4	.7931E-5	.2635E-5	.1144E-4
-.6657E-5	-.5830E-4	-.4729E-5	-.8736E-5	-.7620E-4	-.8227E-5	-.2635E-5	-.1201E-4

The D_p matrix

-.2484	-355.2	33.28	-.7662	2130.
-.7176E-4	-1.581	.9528	-.1128E-2	-1.128
.0	.0	.0	.0	.0
-.8166E-5	-.2304E-1	-.5410E-2	.4397E-6	-.5882E-1
.2165E-4	.1297E-1	.1330E-2	-.8858E-2	-.2342

Table A3.3(b)   Matrices of linear F100 engine model at SLS/PLA 67 deg

The C_p^m matrix

```
1.0  .0  .0  .0  .0  .0  .0  .0  .0  .0  .0  .0  .0  .0
 .0 1.0  .0  .0  .0  .0  .0  .0  .0  .0  .0  .0  .0  .0
 .0  .0 1.0  .0  .0  .0  .0  .0  .0  .0 1.0  .0  .0  .0
 .0  .0  .0 1.0  .0  .0  .0  .0  .0 1.0  .0  .0  .0  .0
```

Table A3.3(c) Matrix of linear F100 engine model at SLS/PLA 67 deg

$$
A_a =
\begin{bmatrix}
-50.0 & .0 & .0 & .0 & .0 & .0 & .0 & .0 & .0 \\
10.0 & -10.0 & .0 & .0 & .0 & .0 & .0 & .0 & .0 \\
.0 & .0 & -100.0 & .0 & 1.0 & .0 & .0 & .0 & .0 \\
.0 & .0 & 1421. & -1421. & -42.22 & .0 & .0 & .0 & .0 \\
.0 & .0 & .0 & .0 & -50.0 & -50.0 & .0 & .0 & .0 \\
.0 & .0 & .0 & .0 & 1200. & 1200. & -100.0 & -1200. & .0 \\
.0 & .0 & .0 & .0 & .0 & .0 & 1.0 & -100.0 & -4000. \\
.0 & .0 & .0 & .0 & .0 & .0 & .0 & 1.0 & .0 \\
.0 & .0 & .0 & .0 & .0 & .0 & .0 & .0 & -50.0
\end{bmatrix}
$$

$$
B_a =
\begin{bmatrix}
50.0 & .0 & .0 & .0 & .0 \\
.0 & 100.0 & .0 & .0 & .0 \\
.0 & .0 & 50.0 & .0 & .0 \\
.0 & .0 & .0 & .0 & .0 \\
.0 & .0 & .0 & .0 & .0 \\
.0 & .0 & .0 & 4000. & .0 \\
.0 & .0 & .0 & .0 & .0 \\
.0 & .0 & .0 & .0 & 50.0
\end{bmatrix}
$$

$$
C_a =
\begin{bmatrix}
.0 & 1.0 & .0 & .0 & .0 & .0 & .0 & .0 & .0 \\
.0 & .0 & 1.0 & .0 & .0 & .0 & .0 & .0 & .0 \\
.0 & .0 & .0 & .0 & 1.0 & .0 & .0 & 1.0 & .0 \\
.0 & .0 & .0 & .0 & .0 & .0 & .0 & .0 & .01
\end{bmatrix}
$$

Table A3.4 F100 engine actuator matrices

$$A_s = \begin{bmatrix} -33.0 & .0 & .0 & .0 & .0 \\ .0 & -20.0 & .0 & .0 & .0 \\ .0 & .0 & -20.0 & .0 & .0 \\ .0 & .0 & .0 & -1.6807 & .0 \\ .0 & .0 & .0 & .0 & -.1821 \end{bmatrix}$$

$$B_s = \begin{bmatrix} 33.0 & .0 & .0 & .0 & .0 \\ .0 & 20.0 & .0 & .0 & .0 \\ .0 & .0 & 20.0 & .0 & .0 \\ .0 & .0 & .0 & .5193 & .0 \\ .0 & .0 & .0 & .1259 & .0 \end{bmatrix}$$

$$C_s = \begin{bmatrix} 1.0 & .0 & .0 & .0 & .0 \\ .0 & 1.0 & .0 & .0 & .0 \\ .0 & .0 & 1.0 & .0 & .0 \\ .0 & .0 & .0 & 1.0 & .0 \\ .0 & .0 & .0 & .0 & 1.0 \end{bmatrix}$$

Table A3.5 F100 engine sensor matrices

APPENDIX 4

NONLINEAR F100 ENGINE MODEL

The F100 engine (Fig A4.1) is an axial, mixed-flow, augmented, twin-spool, low-bypass-ratio turbofan. The digital computer simulation of the F100 engine was implemented on a mini computer. The nonlinear mathematical model of the simulation is based on the hybrid computer simulation developed by Szuch and Seldner (1975), where the model utilises wide-range, overall performance maps of the engine's components so as to provide wide-range, steady-state accuracy. Factors such as rotor inertias, fluid compressibility, fluid momentum, and energy storage are also included in the model so as to provide transient capability. Although it was reported by Yamane and Kagiyama (1988) and Yamane and Takahara (1988) that factors such as heat capacity of combustor and ignition time lag of fuel also affect the dynamical characteristics of the engine, such factors are neglected in this simulation.

The computational flow diagram and the simplified dynamical representation of the F100 engine simulation are shown in Figs A4.2 and A4.3, respectively. The rotor moments of inertia are the most significant factors in determining the transient behaviour of a turbofan engine. Rotor speeds are computed from the dynamical form of the angular momentum equation.

Intercomponent volumes are assumed at engine locations where either (1) gas dynamics are considered to be important or (2) gas dynamics are required to avoid the need for iterative solution of equations. In these volumes, the storage of mass and energy occurs. The dynamical forms of the continuity, energy, and state equations are solved for the stored mass, temperature, and pressure in each volume. When mixing of gases is not involved, a simple first-order lag form of the energy equation is used.

The effects of fluid momentum on the transient behaviour of the F100 engine are considered only in the fan duct and augmentor tailpipe. The contribution of flow dynamics in the compressor, main combustor, and turbines is assumed to be primarily high frequency ($>$ 10 Hz) in nature and is consequently ignored.

It is assumed that the control actuators and the measurement sensors are the same as those of the linear F100 engine model (Appendix 3) and therefore that the actuators and the sensors are governed on T by equations (A3.4) to (A3.7).

Finally, the five manipulated variables are

u_1: main burner fuel flow (lb/hr)

u_2: nozzle jet area (ft^2)

u_3: inlet guide vane position (deg)

u_4: variable stator position (deg)

u_5: compressor bleed flow (%)

the five unmeasurable output variables are

w_1: engine net thrust (lb)

w_2: total engine airflow (lb/s)

w_3: turbine inlet temperature (°R)

w_4: fan stall margin

w_5: compressor stall margin

and the five measurable output variables are

y_1 (N_1) : fan speed (rpm)

y_2 (N_2) : compressor speed (rpm)

y_3 (P_3) : compressor discharge pressure (psia)

y_4 (P_7) : augmentor pressure (psia)

y_5 (FTIT): fan-turbine inlet temperature (°R).

In case the inputs are u_1 and u_2 only, a steady-state transfer-function matrix at Sea Level Static/Idle condition is

$$ G = \begin{bmatrix} 2.68035 & , & 102.516 \\ 4.50972 & , & 135.000 \\ 1.02400e\text{-}1 & , & 2.76397 \\ 1.34031e\text{-}3 & , & -2.43161e\text{-}1 \\ -1.21223e\text{-}1 & , & -9.60357 \end{bmatrix} \qquad (A4.1) $$

and a steady-state transfer-function matrix at Sea Level Static/Intermediate condition is

$$ G = \begin{bmatrix} 2.52760e\text{-}1 & , & 1451.03 \\ 1.35074e\text{-}1 & , & -7.04544 \\ 2.22808e\text{-}2 & , & -4.56090 \\ 2.08151e\text{-}3 & , & -7.70092 \\ 1.46573e\text{-}2 & , & 2.96426 \end{bmatrix} . \qquad (A4.2) $$

Furthermore, the open-loop step-responses of measurable outputs at SLS/Intermediate condition are shown in Figs A4.4 to A4.6 for u_3, u_4, and u_5, where such responses for u_1 and u_2 are shown in Figs 10.1 and Fig 10.2 (Chapter 10).
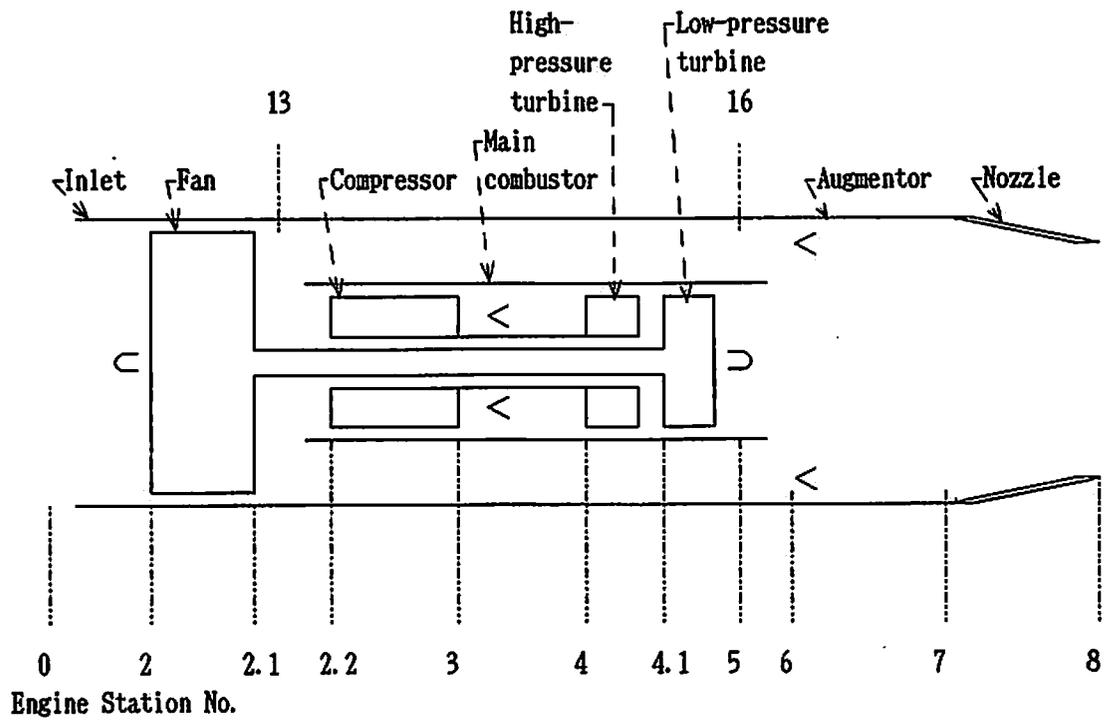
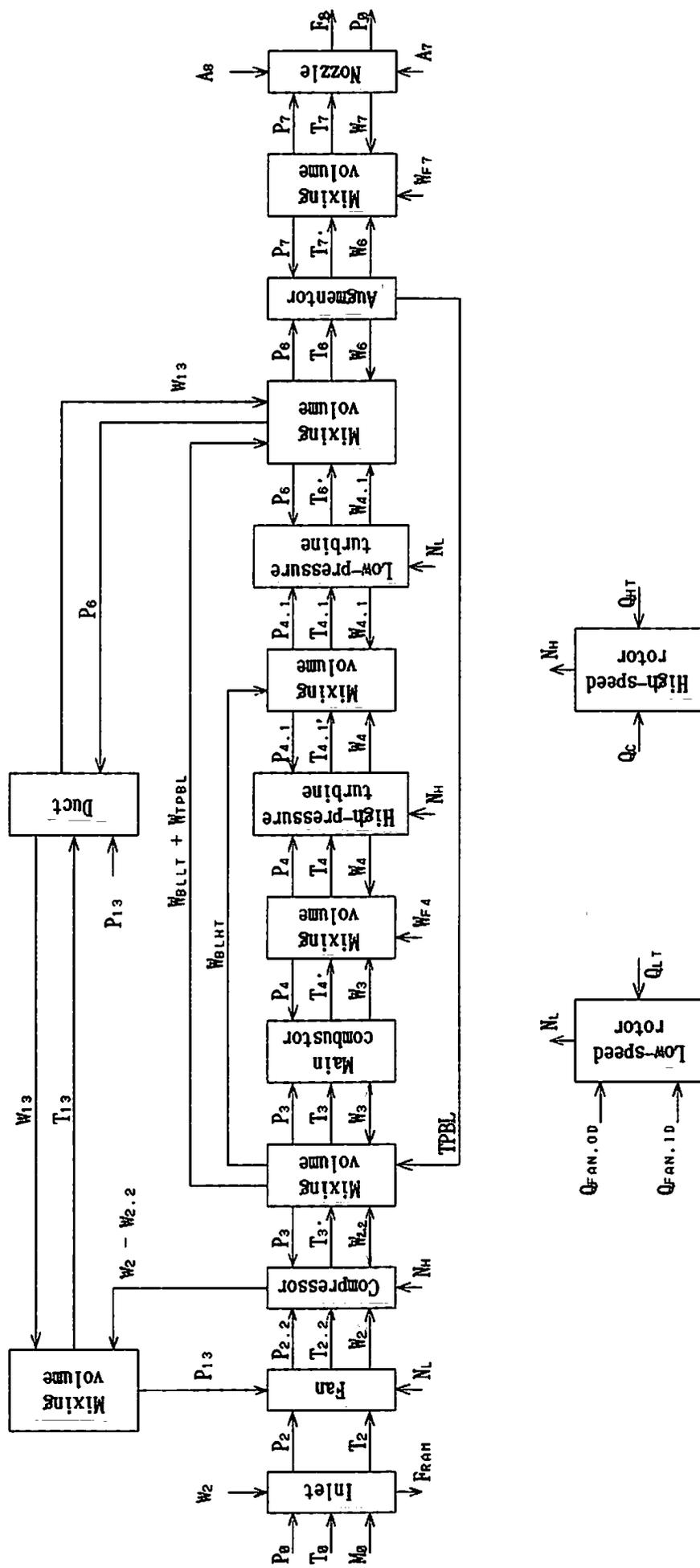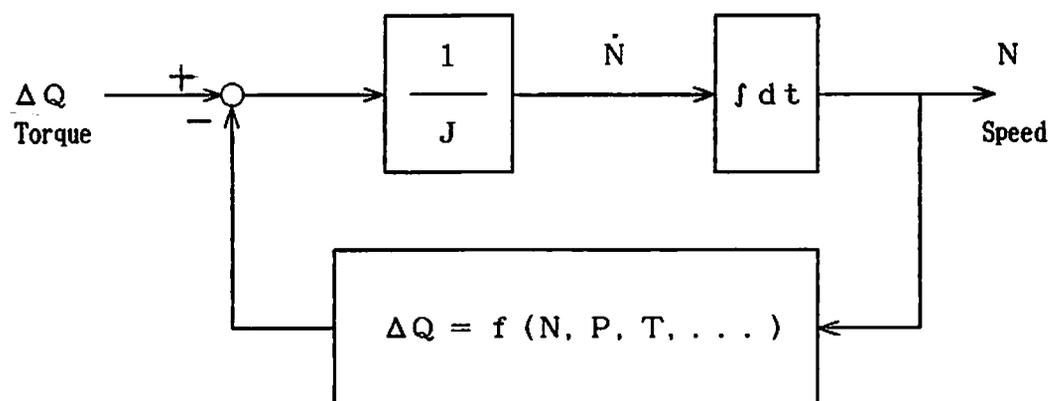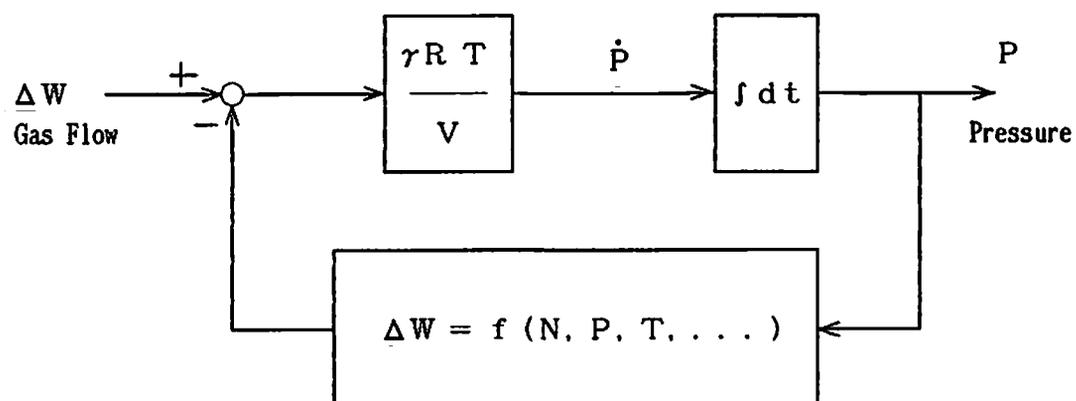Fig A4.1   Schematic representation of F100 turbofan engine

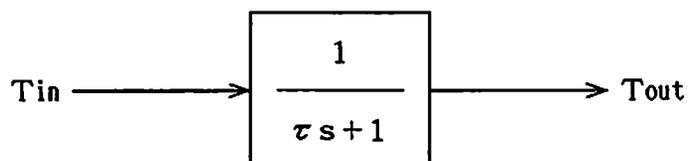Fig A4.2 Computational flow diagram of F100 engine simulation

(a) Rotor Dynamics (Inertia)

$$\Delta Q \text{ Torque} \xrightarrow{+} \bigcirc \xrightarrow{} \boxed{\dfrac{1}{J}} \xrightarrow{\dot{N}} \boxed{\int d\,t} \xrightarrow{} N \text{ Speed}$$

$$\boxed{\Delta Q = f\,(N,\ P,\ T,\ \ldots\ )}$$

(b) Pressure Dyanamics (Gas Volume)

$$\Delta W \text{ Gas Flow} \xrightarrow{+} \bigcirc \xrightarrow{} \boxed{\dfrac{\gamma R\ T}{V}} \xrightarrow{\dot{P}} \boxed{\int d\,t} \xrightarrow{} P \text{ Pressure}$$

$$\boxed{\Delta W = f\,(N,\ P,\ T,\ \ldots\ )}$$

(c) Temperature Dynamics (Thermal Capacitance)

$$T_{in} \xrightarrow{} \boxed{\dfrac{1}{\tau s + 1}} \xrightarrow{} T_{out}$$

Where $\gamma = f\,(m,\ \gamma,\ h,\ A,\ \ldots\ )$

Fig A4.3 Simplified dynamical representation
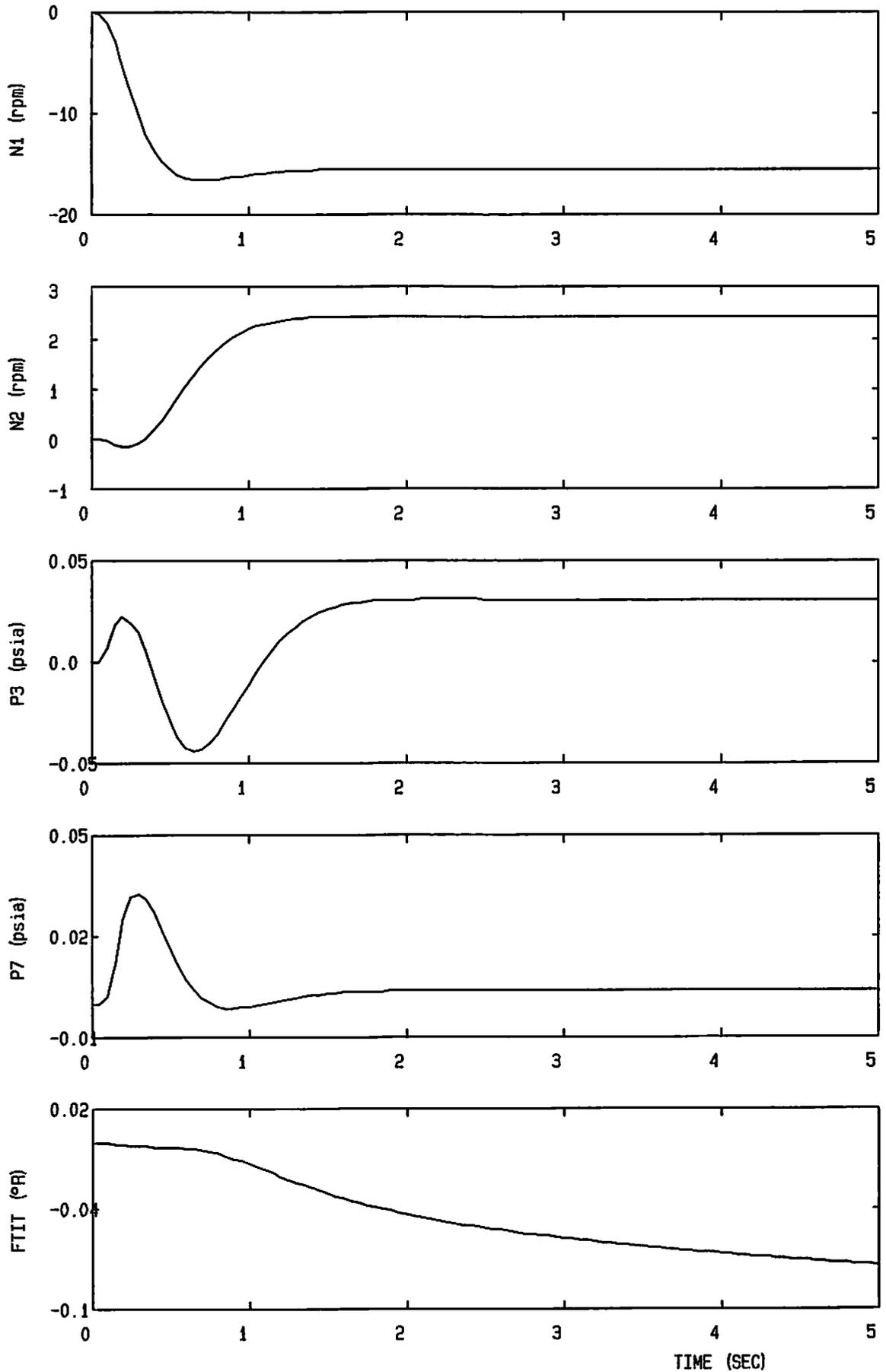
Fig A4.4   Open-loop step-responses of measurable outputs
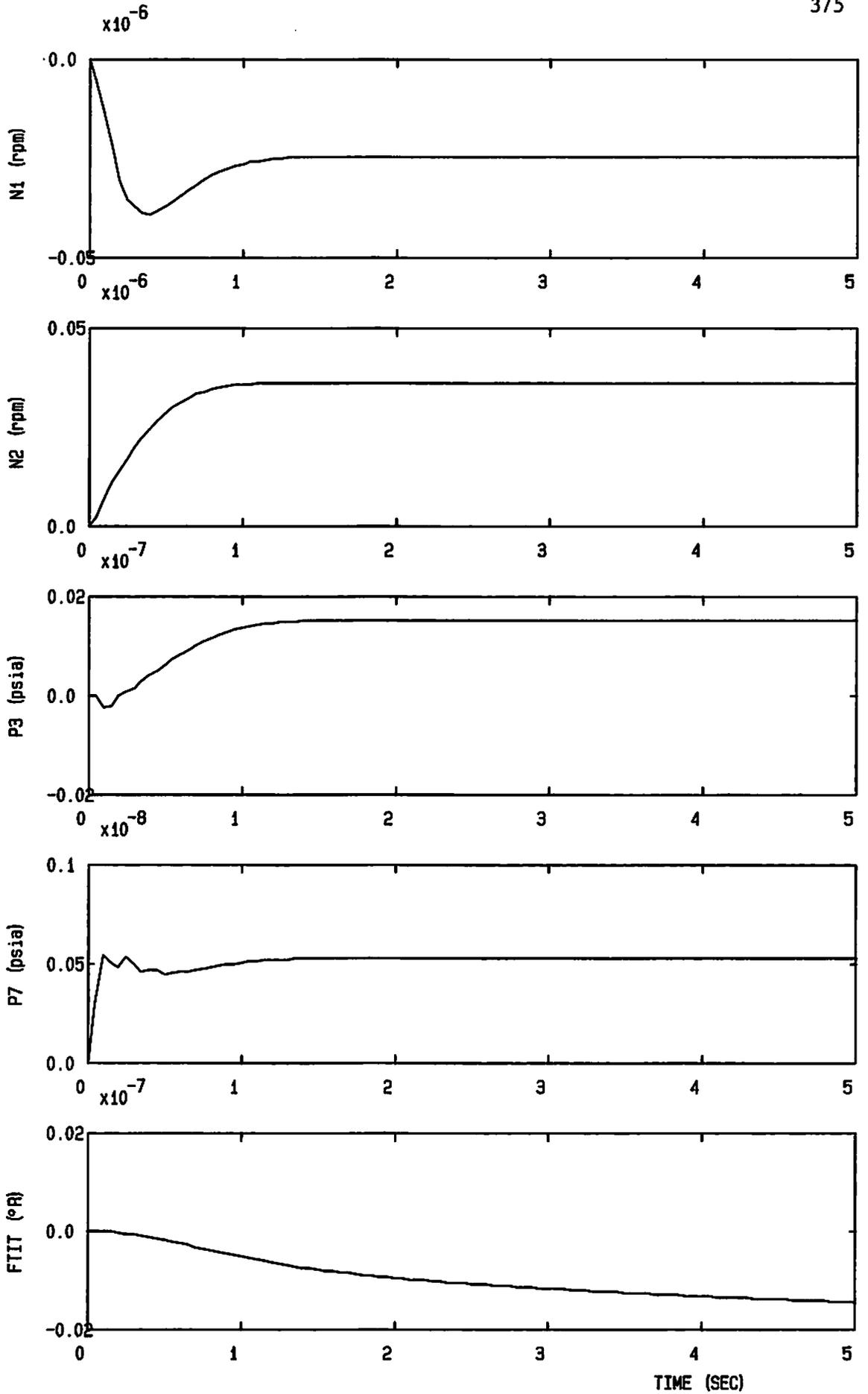F100 engine nonlinear model u=[0 0 1 0 0]

Fig A4.5   Open-loop step-responses of measurable outputs
F100 engine nonlinear model u=[0 0 0 1 0]
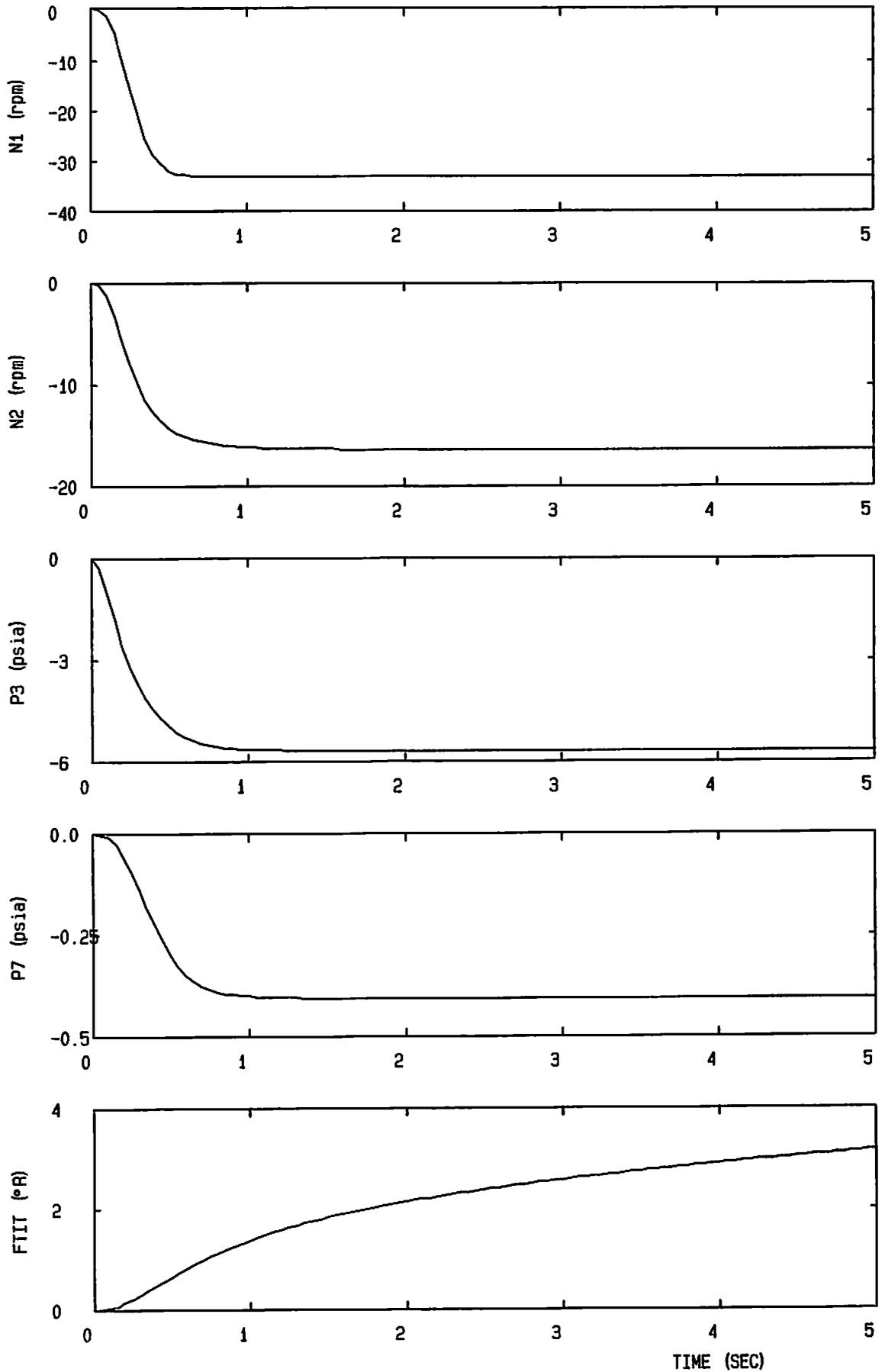
Fig A4.6 Open-loop step-responses of measurable outputs
F100 engine nonlinear model u=[0 0 0 0 1]

APPENDIX 5

CONNECTION BETWEEN CONVEX ANALYSIS OF TRACKING SYSTEMS

AND LINEAR PROGRAMMING PROBLEMS

There is a close connection between the convex analysis of limit-tracking systems and linear programming problems, although no objective function is specified. Indeed, the set $U_F(v)$ of feasible inputs has been defined in Chapter 4 in the form

$$U_F(v) = \{u \in U : G u \leq v\} \qquad (A5.1)$$

where $G \in R^{p \times m}$, the input vector $u \in U = R^m$, and the set-point command vector $v \in Y = R^p$. Therefore, if the vector u is replaced by $u' - u''$, where u', $u'' \geq 0$, and u', $u'' \in R^m$, and if the vector of slack variables $u_s \geq 0$, $u_s \in R^p$ is introduced (Bazaraa and Jarvis (1977)), $U_F(v)$ can be transformed into the feasible region $X(v)$ of linear equation with nonnegativity constraint of the form

$$X(v) = \{x : A x = v , \quad x \geq 0\} \qquad (A5.2)$$

where

$$A = [G , -G , I_p] \in R^{p \times (2m+p)} \qquad (A5.3)$$

$$x = \begin{bmatrix} u' \\ u'' \\ u_s \end{bmatrix} \in R^{2m+p} . \qquad (A5.4)$$

It is clear from equations (A5.1) and (A5.2) that the existence of nonempty $U_F(v)$ is equivalent to the existence of such $X(v)$. Therefore, the following results are obtained.

Definition A5.1: Classification (Alternative of Definition 4.3)

1 Class I plant

$$\text{Class I} = \{G : X(v) \neq \emptyset \text{ for } v < 0\} \tag{A5.5}$$

2 Class II plant

$$\text{Class II} = \{G : X(v) = \emptyset \text{ for } v < 0\} \tag{A5.6}$$

Theorem A5.1

1 (i) If $G \in$ Class I, then (ii) $\forall v$, $X(v) \neq \emptyset$.

2 (i) If $\exists v$, $X(v) = \emptyset$, then (ii) $G \in$ Class II.

3 (i) If $G \in$ Class II, then (ii) $\forall v < 0$, $X(v) = \emptyset$.

4 (i) If $\exists v < 0$, $X(v) \neq \emptyset$, then (ii) $G \in$ Class I.

(Proof)

$U_F(v)$ is not empty if and only if $X(v)$ is not empty. Therefore, the result is evident.   QED

Thus, the classification of linear multivariable plants has been related to linear programming problems. It should be

noted that, although linear programming is applicable to the classification of plants, the results obtained by convex analysis in Chapter 4 are geometrically simple and more easily applicable to the classification of two- or three-input multivariable plants.

Next, limit tracking and the limit-tracking input are discussed. In linear programming, the set of basic feasible solutions corresponds to the set of extreme points of X(v) and both are nonempty, provided that the feasible region is not empty (Theorem 1 (Bazaraa and Jarvis (1977)). Therefore, in case $G \in$ Class I, $\forall v$, X(v) is not empty and at least one basic feasible solution exists. However, in the case of limit-tracking systems, the objective function is unspecified, the set-point vector v might be unknown, and furthermore, an unknown disturbance vector $d \in R^p$ might exist. Hence, neither linear programming nor the simplex method provide the detailed features of such solutions (ie limit-tracking input). It is noted in this sense that Definition 5.1 and Theorem 5.1 have provided such detailed features of limit tracking and have guaranteed the existence of such a special form of the basic feasible solution without solving any linear programming problem. Indeed, the following result is obtained.

Proposition A5.1

The limit-tracking input of Definition 5.4 (in case rank $G = m$) is equivalent to a special basic feasible solution of the form

$$x = B^{-1}v \hspace{4cm} (A5.7)$$

where the basic matrix $B \in R^{p \times p}$ consists of m column basis vectors of $[G , -G]$ and of p-m columns $I_{t_1}, \ldots, I_{t_{p-m}}$ of I_p, and m slack variables $u_{s_{s_1}}, \ldots, u_{s_{s_m}}$ are zero.

Thus, the limit-tracking input corresponds to a basic feasible solution with special form. It is noted that such forms change depending upon the set-point v and the unknown disturbance d, and therefore that it is difficult to use the linear programming technique to specify the form of solution without knowledge of v and d.

Furthermore, it is noted that the order-reduction technique proposed in Chapter 5 uniquely exploits the facial structure of $U_F(v)$, ie the internal structure of the matrix A in equation (A5.3).

Therefore, to summarise the discussion, the results that have been obtained with novelty in Chapters 4 and 5 are:

1: The characterisation of tracking for systems incorporating self-selecting controllers and multivariable plants,

2: The classification of linear multivariable plants in terms of simple geometrical features ie the m-dimensional convex cone in U-space and the separating hyperplane in Y-space,

(Remark)

The classification of plants in terms of the feasible region of a linear equation with nonnegativity constraint, which is common in linear programming problems, is possible and has

been shown. Such a linear programming technique might be used to classify a given plant by using Proposition A5.1, although the geometrical interpretation of this classification is not so simple as those obtained in U-space and Y-space.

3: The creation of rigorous theoretical foundations for the design of limit-tracking systems such as Definition 5.1 and Theorem 5.1 (Existence of limit-tracking for Class I plants),

(Remark)

The interpretation of the limit-tracking input in terms of the basic feasible solution of the linear equation with nonnegativity constraint has been given (Proposition A5.1) in which a limit-tracking input corresponds to a special basic feasible solution of such linear equation. Theorems of linear programming (such as Theorem 1 (Bazaraa and Jarvis (1977)) guarantee the existence of basic feasible solutions for Class I plants. However, they neither specify the form of solution nor guarantee the existence of such a special solution as a limit-tracking input.

4: The formulation of a synthesis technique for limit-tracking systems that exploits the facial structure of the polyhedral convex set $U_F(v)$.

APPENDIX 6

DYNAMICAL PECULIARITIES OF SELF-SELECTING CONTROL SYSTEMS

A6.1 Introduction

In designing the digital self-selecting controllers proposed in Chapter 6, it was assumed that the complete closed-loop system can be made asymptotically stable. However, due to the selection of different controllers, systems incorporating self-selecting controllers (ie self-selecting control systems or limit-tracking systems) change their structures discontinuously, ie, they are variable-structure systems. Therefore, even though each control loop produces asymptotically stable behaviour when considered separately, the stability of the complete closed-loop system is not guaranteed and limit-cycle oscillations may occur.

In previous studies, Foss (1981a) analysed the stability of single-input self-selecting control systems. In this analysis, discontinuous systems were transformed into continuous systems with nonlinear elements, and describing-function criteria or passivity criteria were used to assess the stability of the complete systems. These criteria were also used by Glattfelder and co-workers to analyse the stability of control systems with nonlinearity such as saturation and antireset-windup circuits (Glattfelder and Schaufelberger (1983), Glattfelder et al (1988)). However, this approach is not in general effective for the analysis of self-selecting control systems which are

untransformable.

Much effort has been devoted to studies of variable-structure systems, which are discontinuous dynamical systems described by differential equations with discontinuous right-hand sides. The existence of sliding modes is recognised as one the typical characteristics of such systems. Filippov (1964) gave a definition of the solution of the equations of motion of such systems and studied the properties of these solutions. If various non-idealities such as hysteresis, delay, and dynamic non-idealities (which are present in a real sliding mode) are made to tend to zero, this limiting process leads to the same equations that result from Filippov's method. Filippov's trajectories can therefore be considered as the ideal representation of the trajectories obtained in real systems, thus indicating one of the reasons for the wide use of Filippov's method in studies of variable-structure systems (Utkin (1978)).

However, it is shown in this appendix that a more general solution concept than Filippov's is necessary to describe the behaviour of self-selecting control systems and that even simple self-selecting control systems exhibit dynamical peculiarities such as sliding motion and limit-cycle oscillation. Such peculiarities have never previously been investigated systematically. It is noted that the whole analysis is carried out on the continuous-time set in order to simplify the discussion.

A6.2 System description

The linear multivariable Class I plants under consideration are assumed to be governed on the continuous-time set $T = [0,+\infty)$ by state and output equations of the respective forms

$$\dot{x}(t) = Ax(t) + bu(t) \qquad (A6.1)$$

and

$$\left.\begin{array}{l} y_1(t) = c_1^T x(t) \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ y_p(t) = c_p^T x(t) \end{array}\right\} \qquad (A6.2)$$

where the state vector $x(t) \in R^n$, the input $u(t) \in R$, and the outputs $y_i(t) \in R$ $(i=1,2,\ldots,p)$ are to be controlled by the self-selecting controller. The plant matrix $A \in R^{n \times n}$, whose eigenvalues all lie in the open left-half plane C^-, the input vector $b \in R^n$, and the output vectors are $c_i \in R^n$ $(i=1,2,\ldots,p)$. It is assumed that the introduction of integral action preserves stabilisability, ie, $g_i(s)$ $(i=1,2,\ldots,p)$ represents a functionally controllable plant and therefore that (Porter and Power (1970), Power and Porter (1970))

$$g_i \neq 0 \qquad (i=1,2,\ldots,p) \qquad (A6.3)$$

where the plant transfer function matrix

$$G(s) = \begin{bmatrix} c_1^T \\ \vdots \\ \vdots \\ \vdots \\ c_p^T \end{bmatrix} (sI_n - A)^{-1}b = \begin{bmatrix} g_1(s) \\ \cdots \\ \cdots \\ \cdots \\ g_p(s) \end{bmatrix} \qquad (A6.4)$$

and

$$G = \begin{bmatrix} g_1 \\ \vdots \\ \vdots \\ \vdots \\ g_p \end{bmatrix} = -\begin{bmatrix} c_1^T \\ \vdots \\ \vdots \\ \vdots \\ c_p^T \end{bmatrix} A^{-1}b \ . \qquad (A6.5)$$

In the case of self-selecting control systems with lowest-wins strategies, the index set of all the control loops is $I = \{1,2,\ldots,p\}$ and the error $e_i(t) \in R$, $i \in I$, is

$$e_i(t) = v_i - y_i(t) \qquad (A6.6)$$

where the set-point vector $v = [v_1,\ldots,v_p]^T \in R^p$. Furthermore, the index set $J(t)$ of lowest errors and the loop index $\ell(t)$ of the actually selected loop are defined by the respective forms

$$J(t) = \{j : e_j(t) = \min_{i \in I} e_i(t)\} \qquad (A6.7)$$

and

$$\ell(t) \in J(t) \subset I \ . \qquad (A6.8)$$

The self-selecting controller is governed on the

continuous-time set $T = [0, +\infty)$ by equations of the form

$$\dot{z}(t) = e_{\ell(t)}(t) \tag{A6.9}$$

and

$$u(t) = k_P^{(\ell(t))} e_{\ell(t)}(t) + k_I^{(\ell(t))} z(t) \tag{A6.10}$$

where the controller state $z(t) \in R$, and the controller gains $k_P^{(\ell(t))} \in R$ and $k_I^{(\ell(t))} \in R$ are chosen from the sets $\{k_P^{(1)}, \ldots, k_P^{(p)}\}$ and $\{k_I^{(1)}, \ldots, k_I^{(p)}\}$, respectively. It is assumed that each separate closed-loop system is asymptotically stable, where there clearly exist p separate closed-loops when $\ell(t) = \text{const} \in I$. This assumption is justified by the functional controllability of each separate output, as indicated in the conditions (A6.3).

Since equations (A6.7) and (A6.8) decide which controller should be used at each instant, controller switching may occur. In controller switching from loop index ℓ_1 to ℓ_2 at time t, the following two types of switching logic are considered:

(i) *Without bumpless transfer*

$$z(t) = \lim_{\Delta t \to +0} z(t - \Delta t) \tag{A6.11}$$

(ii) *With bumpless transfer*

$$u(t) = \lim_{\Delta t \to +0} u(t-\Delta t) \qquad (A6.12)$$

and

$$z(t) = \frac{1}{k_I^{(\ell_2)}} \{ \lim_{\Delta t \to +0} u(t-\Delta t) - k_P^{(\ell_2)} e_{\ell_2}(t) \}. \qquad (A6.13)$$

A6.3 Analysis

The equations (A6.1), (A6.2), (A6.6), (A6.9), and (A6.10) that govern the behaviour of the self-selecting control system can be written in the form

$$\dot{x}(t) = A_{\ell(t)} x(t) + b_{\ell(t)} v_{\ell(t)} \qquad (A6.14)$$

where

$$x(t) = \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} \in R^{n+1} , \qquad (A6.15a)$$

$$A_{\ell(t)} = \begin{bmatrix} A - k_P^{(\ell(t))} bc_{\ell(t)}^T & , & k_I^{(\ell(t))} b \\ -c_{\ell(t)}^T & , & 0 \end{bmatrix} \in R^{(n+1) \times (n+1)} , \qquad (A6.15b)$$

$$b_{\ell(t)} = \begin{bmatrix} k_P^{(\ell(t))} b \\ 1 \end{bmatrix} \in R^{n+1} , \qquad (A6.15c)$$

and

$$v_{\ell(t)} = v_{\ell(t)} \in R .$$ (A6.15d)

It is clear that the system equations (A6.14) with the lowest-wins control equations (A6.7), (A6.8), and (A6.11) (or (A6.A6), (A6.8), (A6.12), and (A6.13)) have discontinuous right-hand sides. Therefore, let a solution of the governing equations of the self-selecting control system be denoted by

$$\chi(t;x_0;v) = \begin{bmatrix} x(t;x_0;v) \\ z(t;x_0;v) \end{bmatrix} , \quad \chi(0;x_0;v) \equiv x_0$$

where $\chi(t;x_0;v)$ is the motion of the controlled plant and $z(t;x_0;v)$ is the corresponding motion of the self-selecting controller. Since the absolute continuity of $\chi(t;x_0;v)$ is lost at the controller switching instants in the case of the controller switching equations (A6.12) and (A6.13), Filippov's definition of solutions of differential equations with discontinuous right-hand sides is not enough. So, piecewise continuous $\chi(t;x_0;v)$ are admitted as solutions.

Many fundamental properties of closed-loop systems embodying multivariable plants and digital self-selecting controllers are established in Chapter 7. The following definitions and propositions are the anologue version of such properties and summarise those concepts needed to understand the results presented in the next section:

Definition A6.1

(i) Equilibrium state

A state $x_e \in R^{n+1}$ is an equilibrium state of the self-selecting control system if and only if, for each separate closed-loop system,

$$\chi(t;x_e;v) = x_e \quad , \quad \forall t \in R^+ \quad ;$$

(ii) Steady state

A state $x_s \in R^{n+1}$ is a steady state of the self-selecting control system if and only if

$$\chi(t;x_s;v) = x_s \quad , \quad \forall t \in R^+ \quad .$$

Definition A6.2: Index sets of correct and incorrect loops

In a steady state, the index set $I_c(v)$ such that

$$I_c(v) = \{i \in I : y_i = v_i\}$$

is the set of correct loops and the set $I \backslash I_c(v)$ the set of incorrect loops.

The existence of nonempty $I_c(v)$ is guaranteed by Theorem 5.1.

Proposition A6.1

In a steady state, if $i \in I \backslash I_c(v)$ then

$$y_i < v_i \quad .$$

(Proof of Proposition A6.1)

Suppose that

$$\chi(0; x_s; v) = x_s$$

and that there exists $i \in I \backslash I_c(v)$ such that $y_i(0) \nless v_i$. Since $i \notin I_c(v)$, this means that $y_i(0) > v_i$. Then,

$$\forall j \in J(0), \quad e_j(0) \leq e_i(0) = v_i - y_i(0) < 0 \quad .$$

Since $\ell(0) \in J(0)$, $e_{\ell(0)} < 0$. However, since x_s is a steady state,

$$A_{\ell(0)} x_s + b_{\ell(0)} v_{\ell(0)} = 0 \quad .$$

Therefore,

$$\dot{z}(0) = e_{\ell(0)} = 0 \quad .$$

This is a contradiction and implies that $y_i(0) < v_i$ for $i \in I \backslash I_c(v)$.

QED

Proposition A6.2

The self-selecting control system has $\#(I_c(v))$ steady states for every v, including multiplicity, where $\#(\cdot)$ means the number of elements in the set $\cdot$.

(Proof of Proposition A6.2)

The asymptotic stability of each closed-loop system corresponding to loop index i ensures that all the eigenvalues of A_i lie in the open left-half plane C^- and that rank A_i = n+1.

Since rank $[A_i , b_i] \geq$ rank A_i, rank $[A_i , b_i]$ = n+1. Hence, rank $[A_i , b_i]$ = rank A_i and a steady state is determined as the unique solution x_i of the equation

$$0 = A_i x_i + b_i v_i \qquad i \in I_c(v) .$$

For $i \in I \backslash I_c(v)$, if x_i satisfies

$$0 = A_i x_i + b_i v_i$$

then $y_i = v_i$. This means that $i \in I_c(v)$ and contradicts the fact that $i \in I \backslash I_c(v)$. So, only x_i, $i \in I_c(v)$ are steady states of the system.

QED

A6.4 Illustrative example

In order to illustrate these concepts, it is convenient to design self-selecting controllers for a simple one-input/two-output plant and to analyse the resulting closed-loop characteristics by the phase-plane method. In fact, the plant is governed by state and output equations of the respective forms

$$
\left.\begin{array}{l}
\dot{x}(t) = -x(t) + u(t) \\[2ex]
y_1(t) = 2x(t) \\[2ex]
y_2(t) = 4x(t)
\end{array}\right\} . \qquad (A6.16)
$$

The responses of this self-selecting control system in the case of controller switching without bumpless transfer are shown in Figs A6.1 to A6.3 when the controller parameters are

$$
\left.\begin{array}{ll}
k_P^{(1)} = 0.1 & k_I^{(1)} = 0.5 \\[2ex]
k_P^{(2)} = 0.2 & k_I^{(2)} = 0.25
\end{array}\right\} . \qquad (A6.17)
$$

Indeed, these figures show the phase trajectories, the set-point commands and outputs, and the plant input and loop index, respectively. In this case, stable responses with sliding modes are observed. However, when the same controller parameters are used in the case of controller switching with bumpless transfer, the responses of the system are shown in Figs A6.4 to A6.6. In this case, stable responses without

sliding motion are observed but it is important to note that the discontinuity when x = 1.5 in Fig A6.4 arises from bumpless transfer. Finally, the responses of this system in the case of controller switching without bumpless transfer are shown in Figs A6.7 to A6.9 when the controller parameters are

$$
\left.
\begin{array}{ll}
k_P^{(1)} = 0 & k_I^{(1)} = 0.5 \\
\\
k_P^{(2)} = 0 & k_I^{(2)} = 1.0
\end{array}
\right\} .
\tag{A6.18}
$$

In this case, despite the fact that each control loop is separately asymptotically stable, limit–cycle oscillations are observed. In each of these cases, the sampling period of digital simulation is 0.01 sec, E1, E2 are the equilibrium states of the corresponding separate closed loops,

$$
v = \begin{bmatrix} 4 \\ 7 \end{bmatrix} ,
\tag{A6.19}
$$

and

$$
x_0 = \begin{bmatrix} 4 \\ 0 \end{bmatrix} .
\tag{A6.20}
$$

A6.5 Conclusion

It has been shown that self–selecting control systems with lowest–wins strategies are discontinuous dynamical systems. Equilibrium states, steady states, index sets of correct and

incorrect loops have been defined and characterised for every set-point vector. Furthermore, it has been shown that dynamical peculiarities can occur even in a very simple first-order plant with one input and two outputs under self-selecting control. In this case, it has been demonstrated that the complete system exhibits stable responses (with or without sliding motion) or limit-cycle oscillations depending upon the controller gains and the controller switching logic. These peculiarities indicate both the richness of the possible responses of higher-order multivariable self-selecting control systems and the difficulty of analysing such systems. They thus stimulate further research into powerful design methods for self-selecting control systems which guarantee the well-regulated behaviour of complex engineering systems.
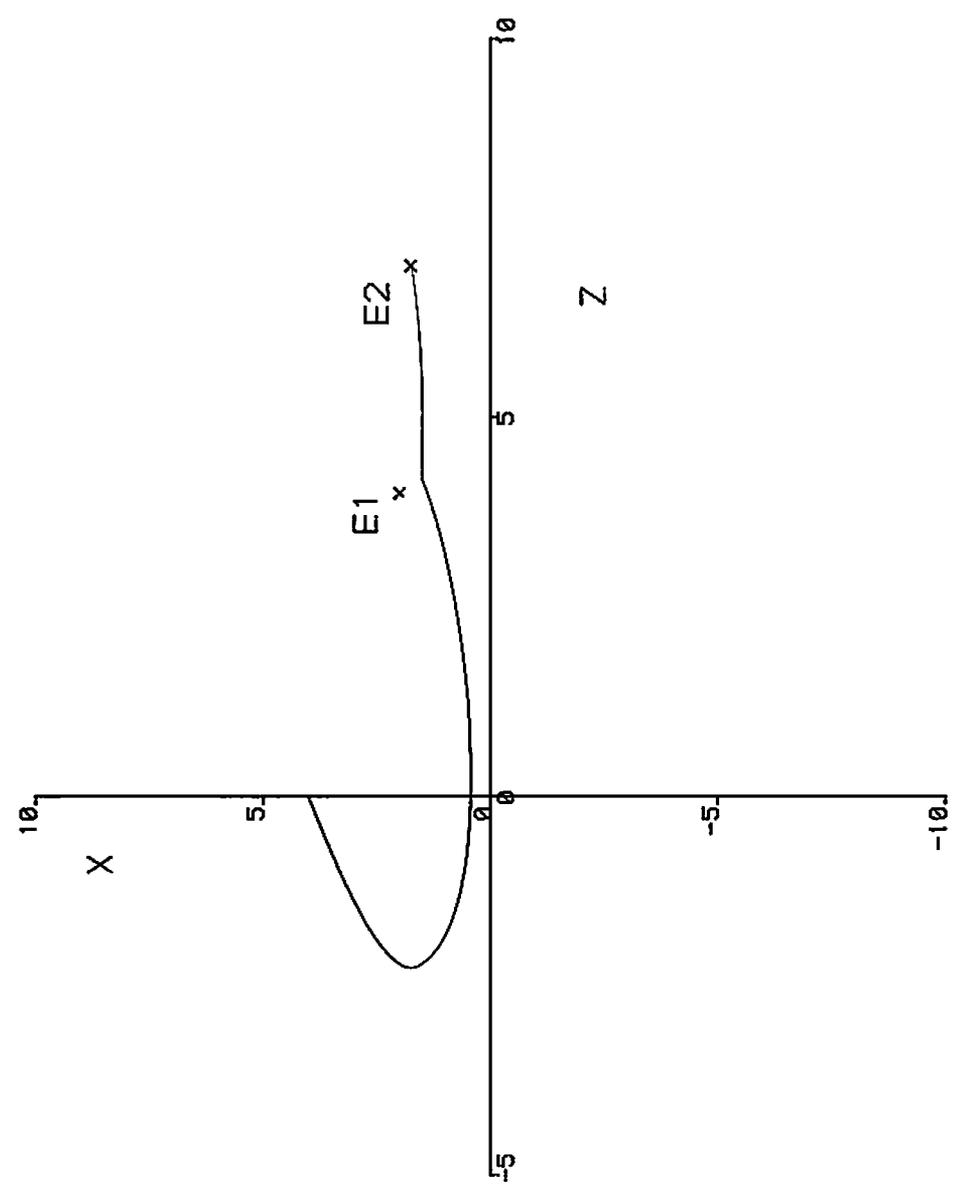
Fig.A6.1 Phase trajectory with sliding motion
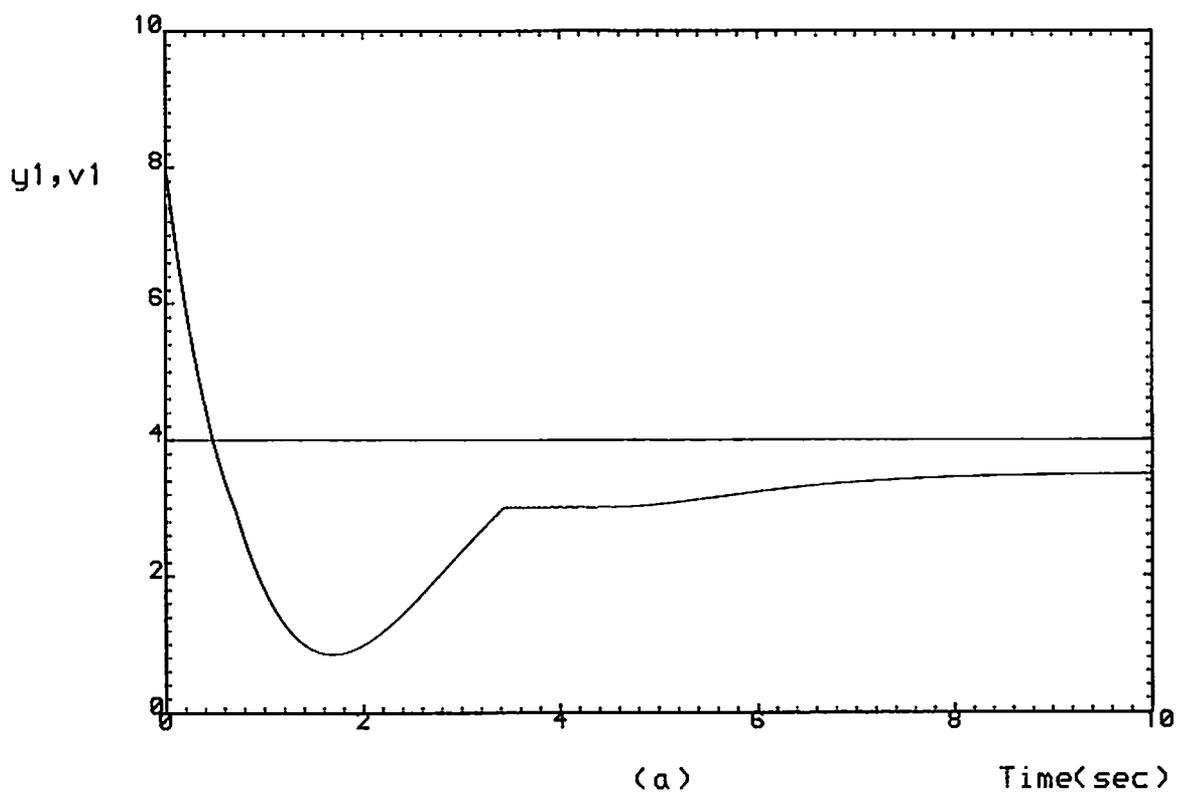Controller switching without bumpless transfer

(a)   Time(sec)

(b)   Time(sec)
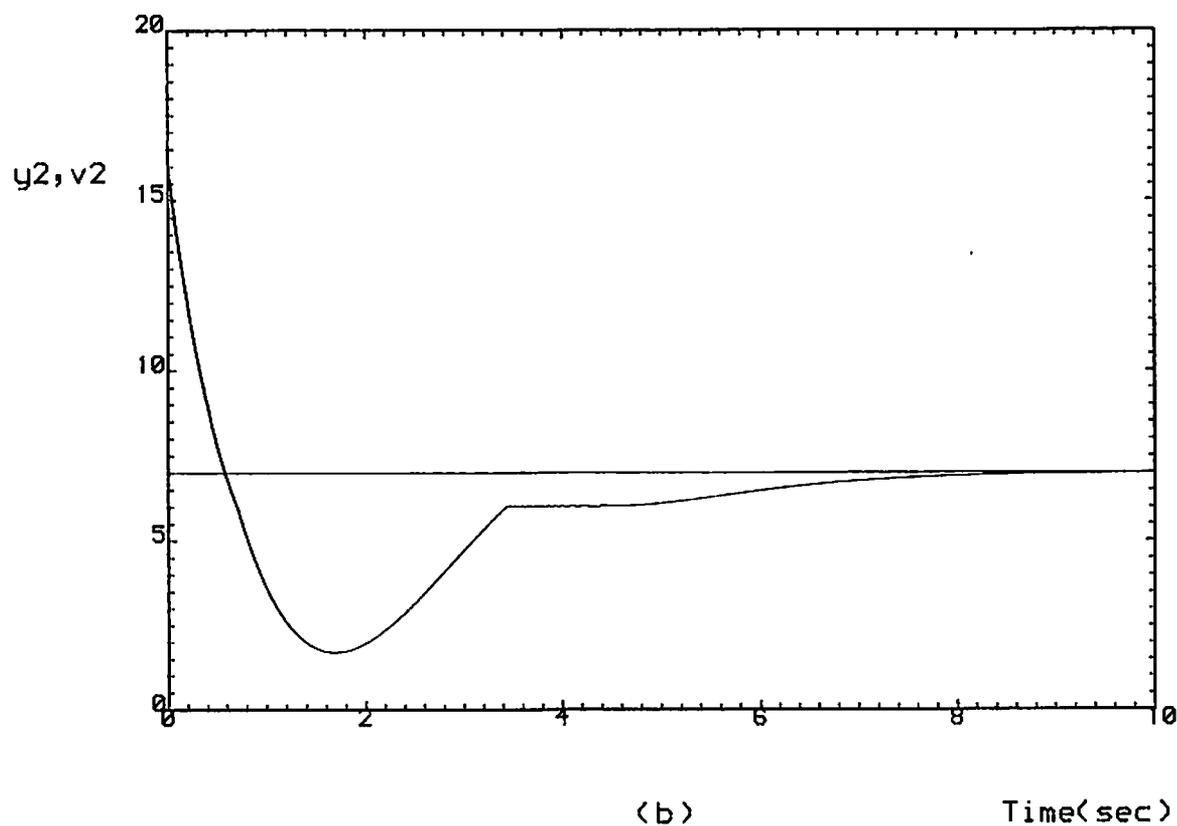
Fig A6.2 Stable responses with sliding motion
——— Controller switching without bumpless transfer
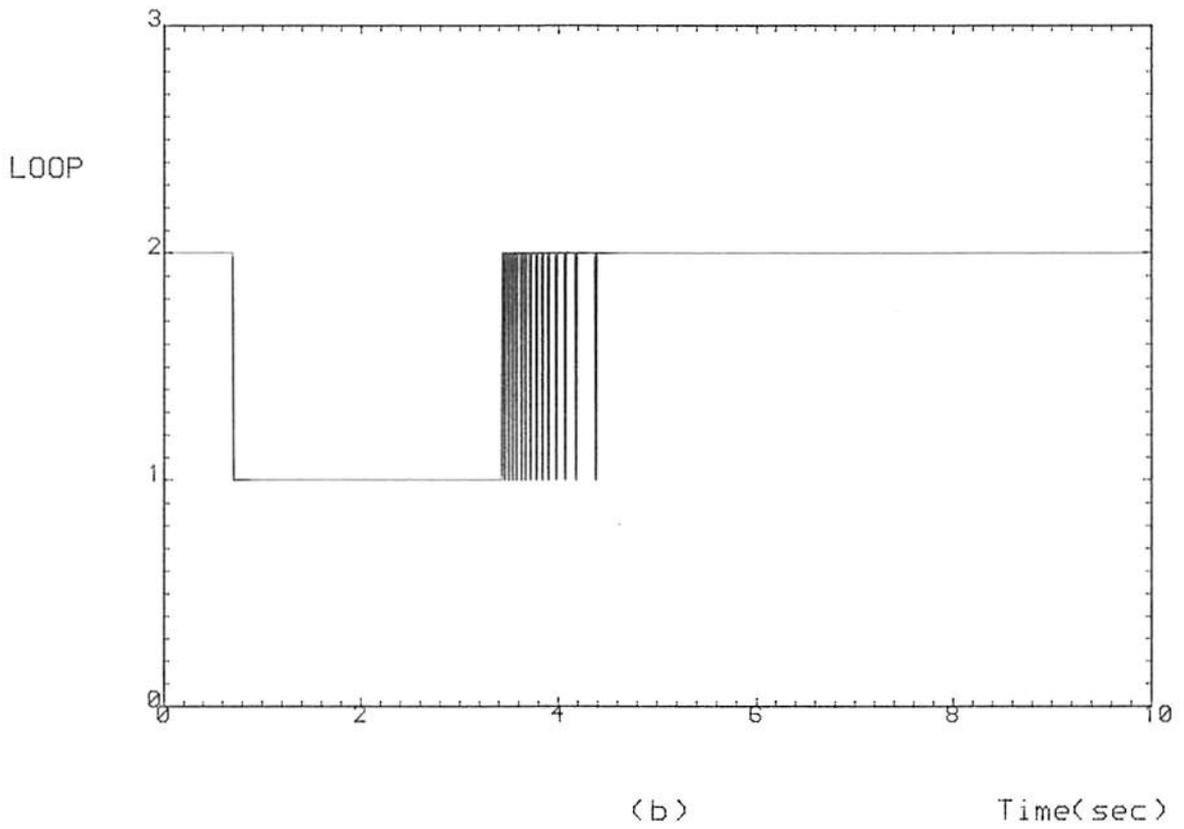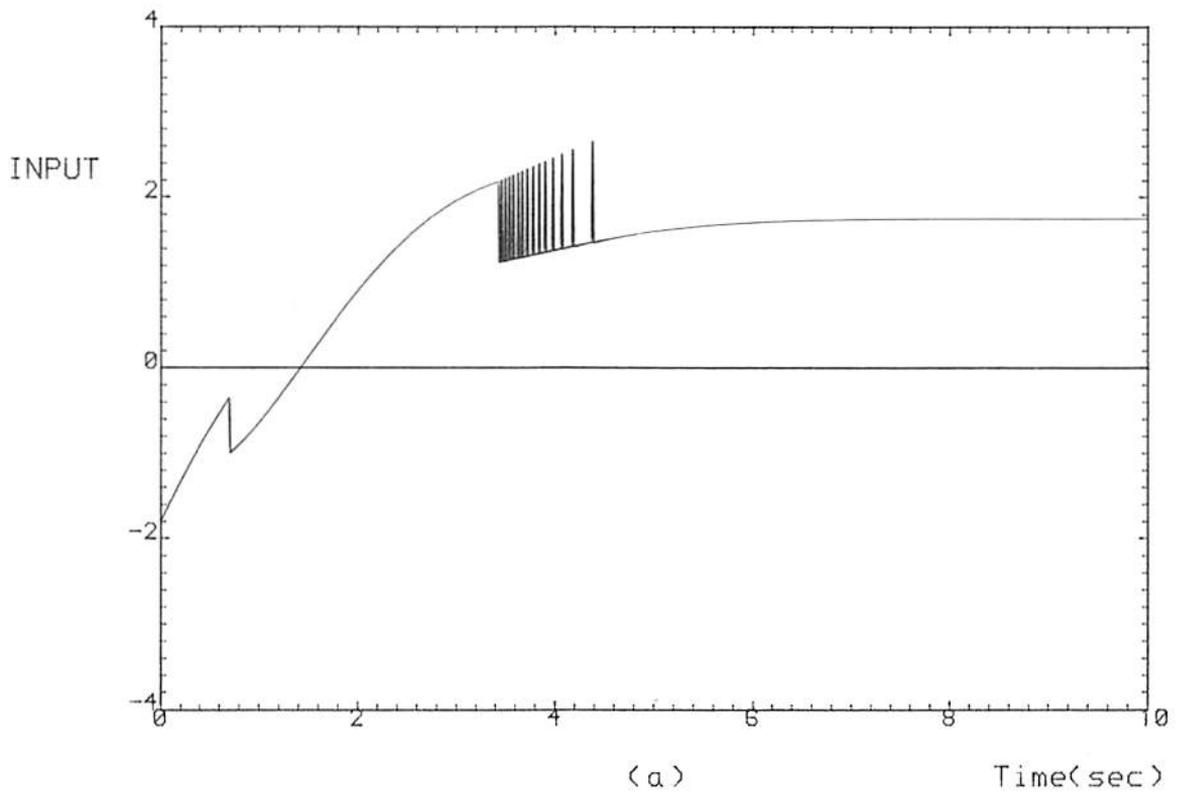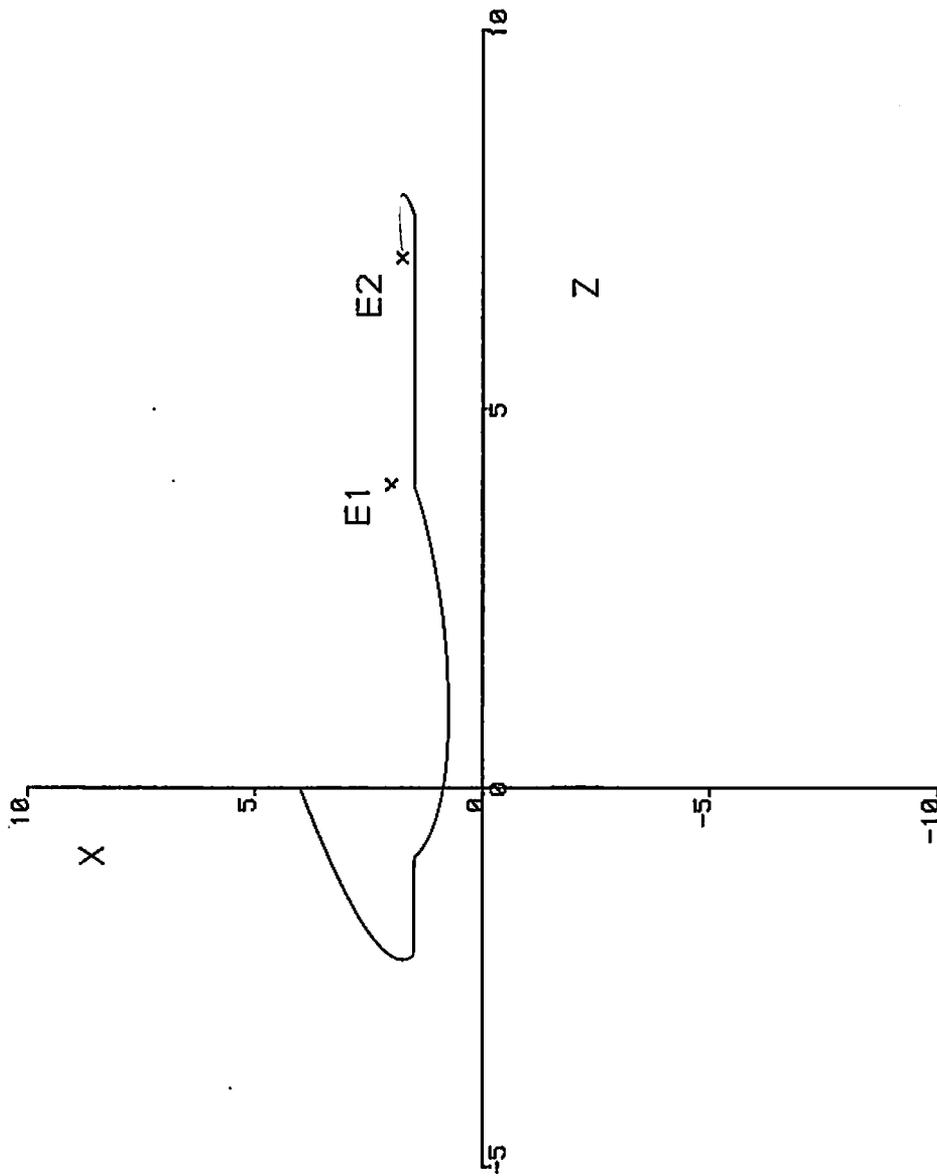
(a)   Time(sec)

(b)   Time(sec)

Fig A6.3   Input and loop index
           Controller switching without bumpless transfer

Fig A6.4 Phase trajectory without sliding motion
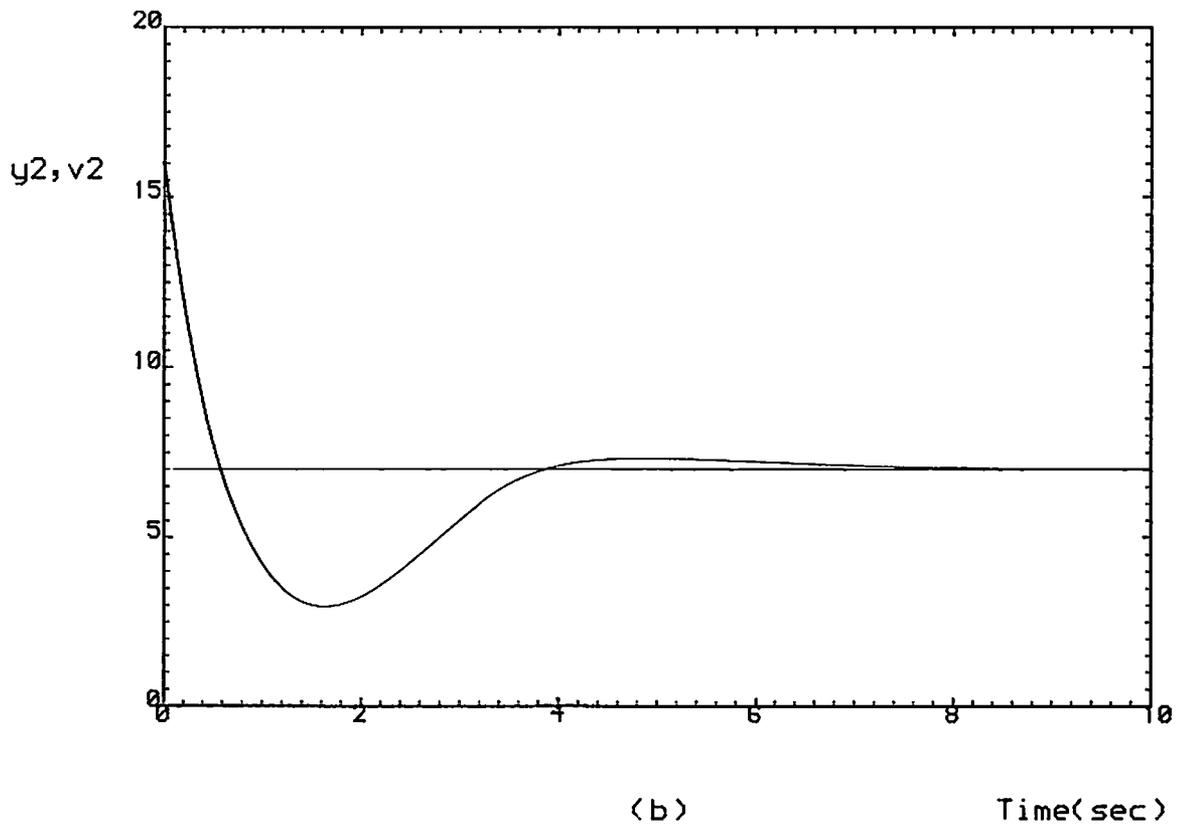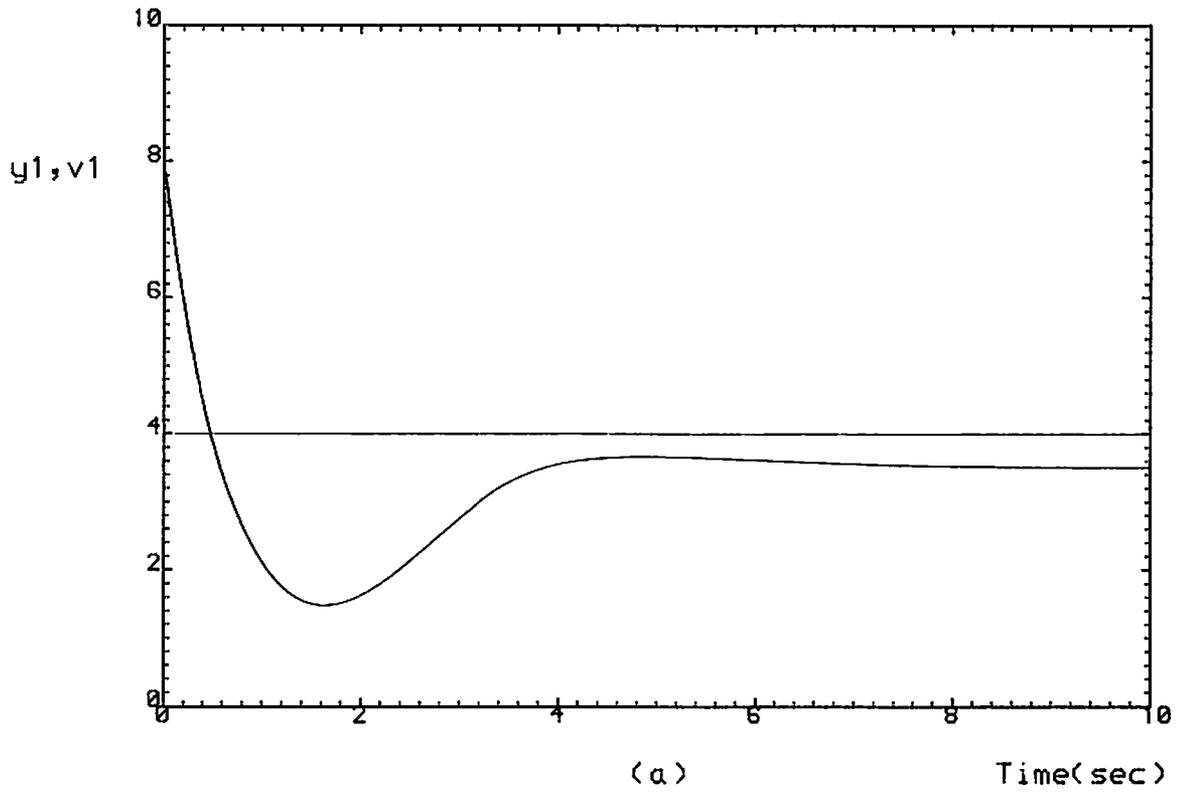Controller switching with bumpless transfer

(a)   Time(sec)

(b)   Time(sec)

Fig A6.5 Stable responses without sliding motion
          Controller switching with bumpless transfer

(a)   Time(sec)
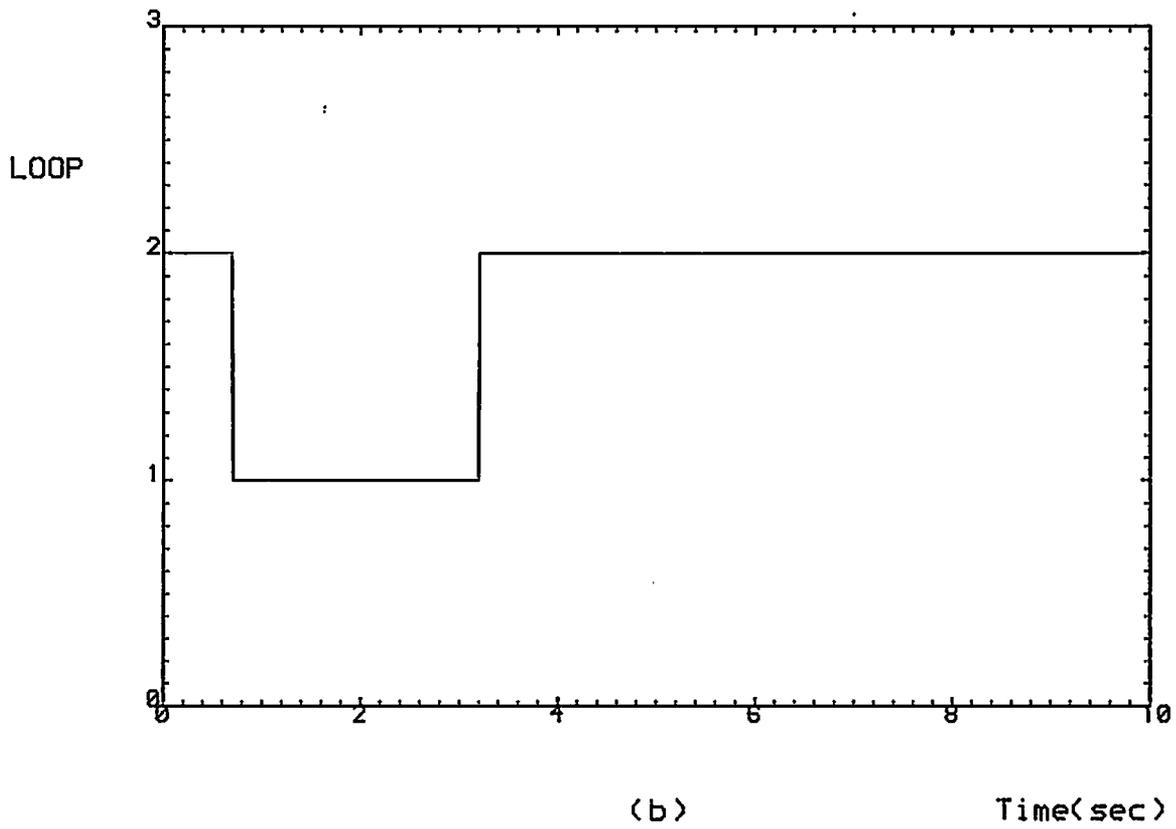
(b)   Time(sec)
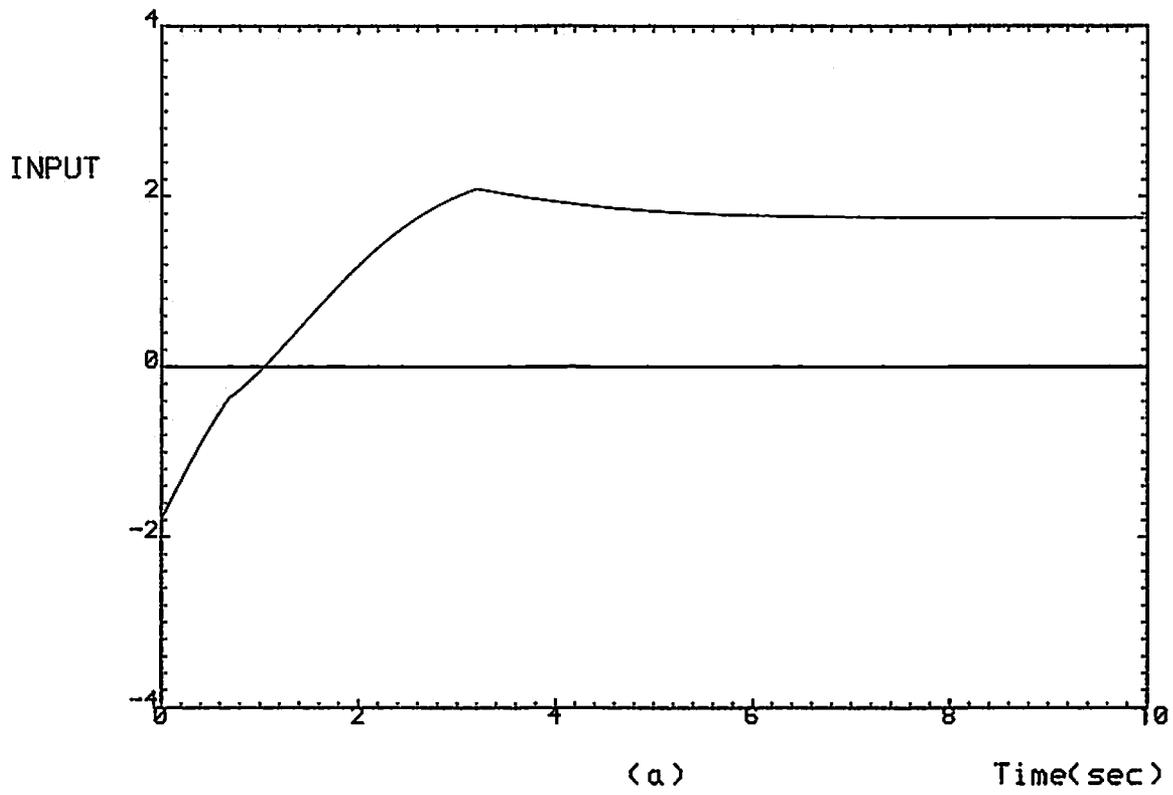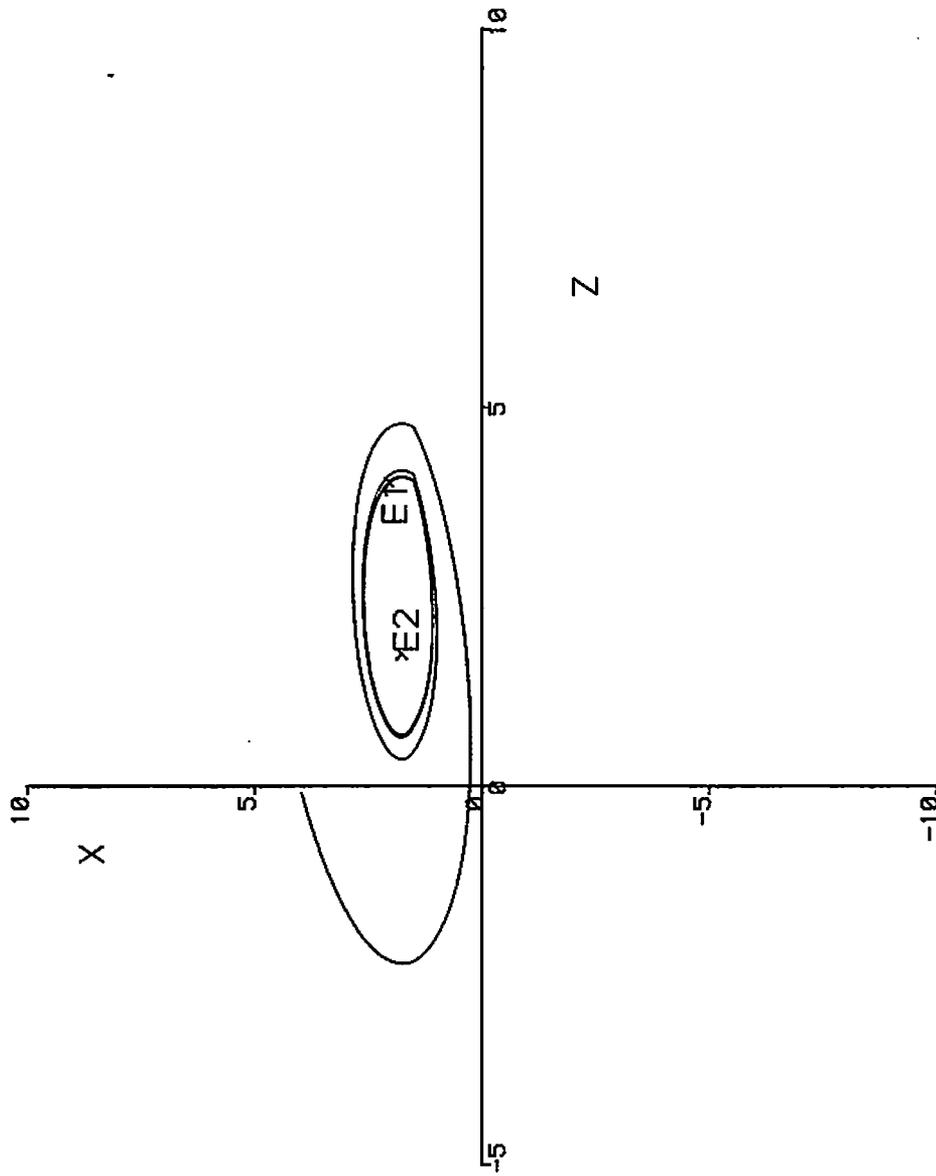
Fig A6.6   Input and loop index
           Controller switching with bumpless transfer

Fig A6.7 Phase trajectory with limit-cycle
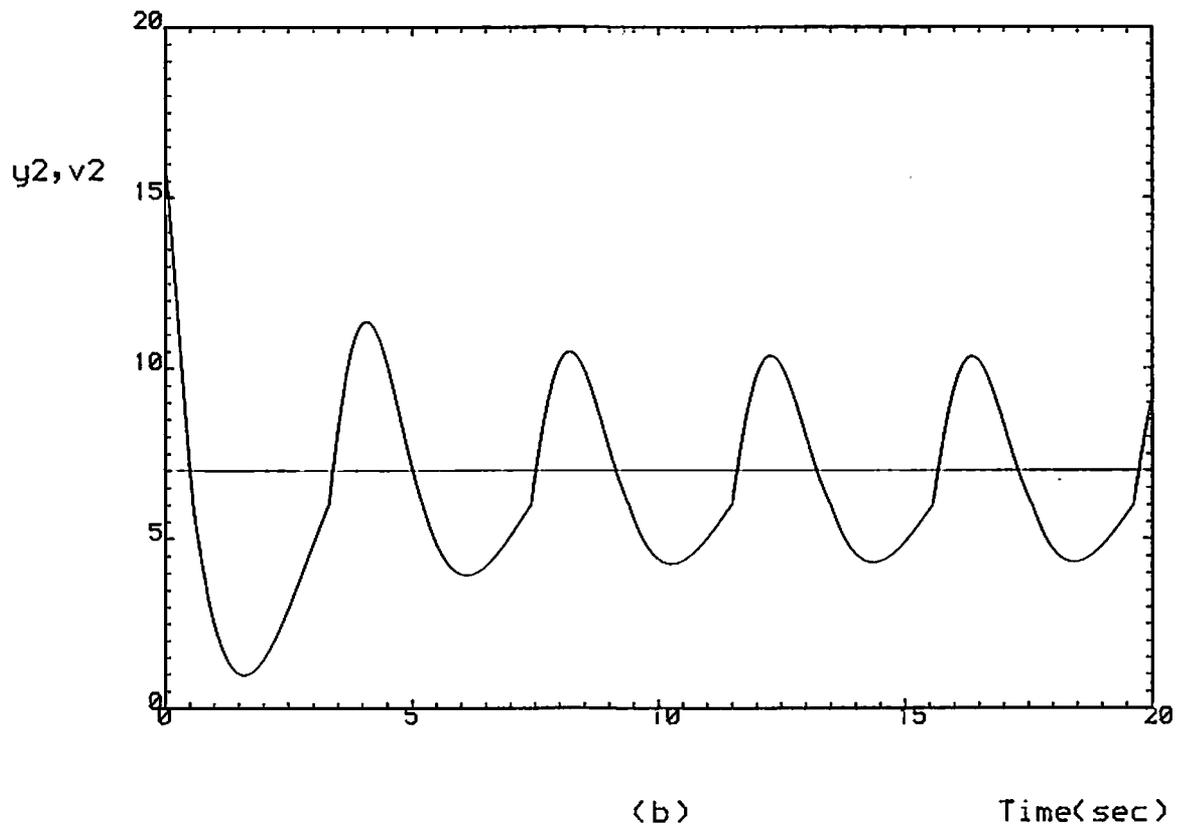Controller switching without bumpless transfer

(a)   Time(sec)

(b)   Time(sec)
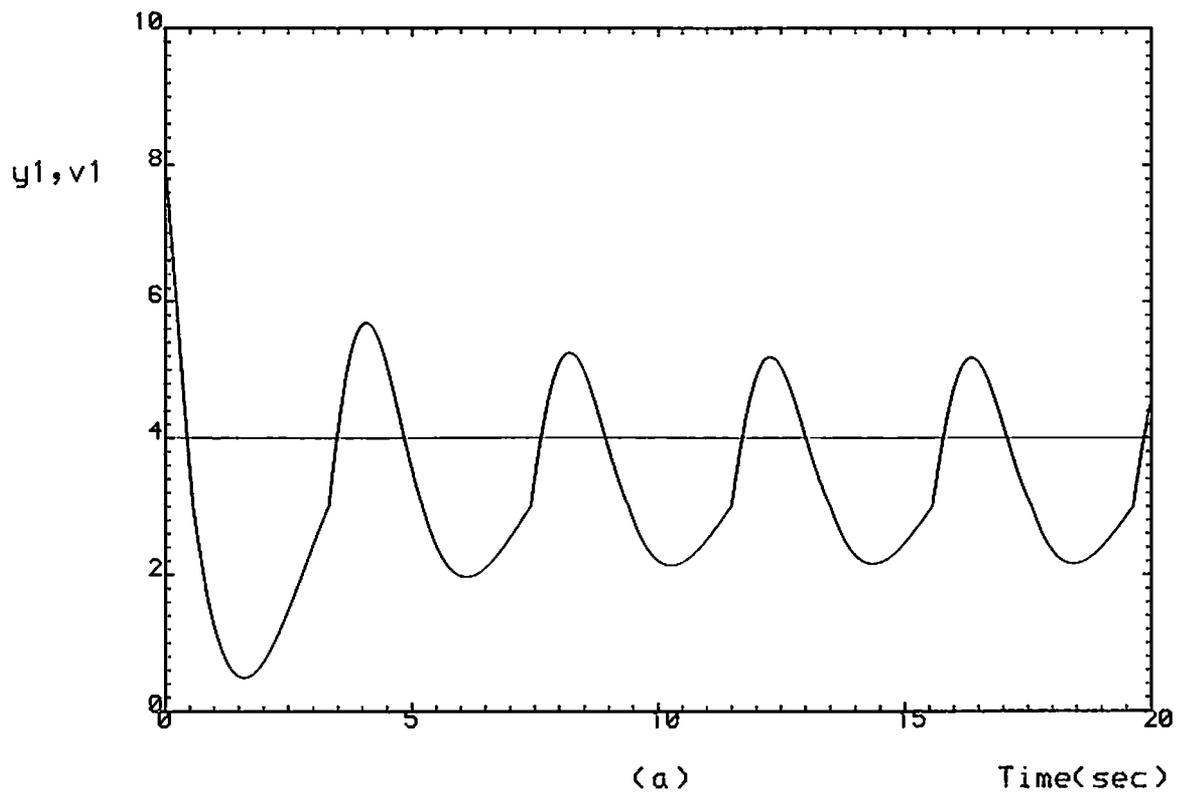
Fig A6.8 Limit-cycle oscillations
———— Controller switching without bumpless transfer

INPUT

(a)   Time(sec)

LOOP
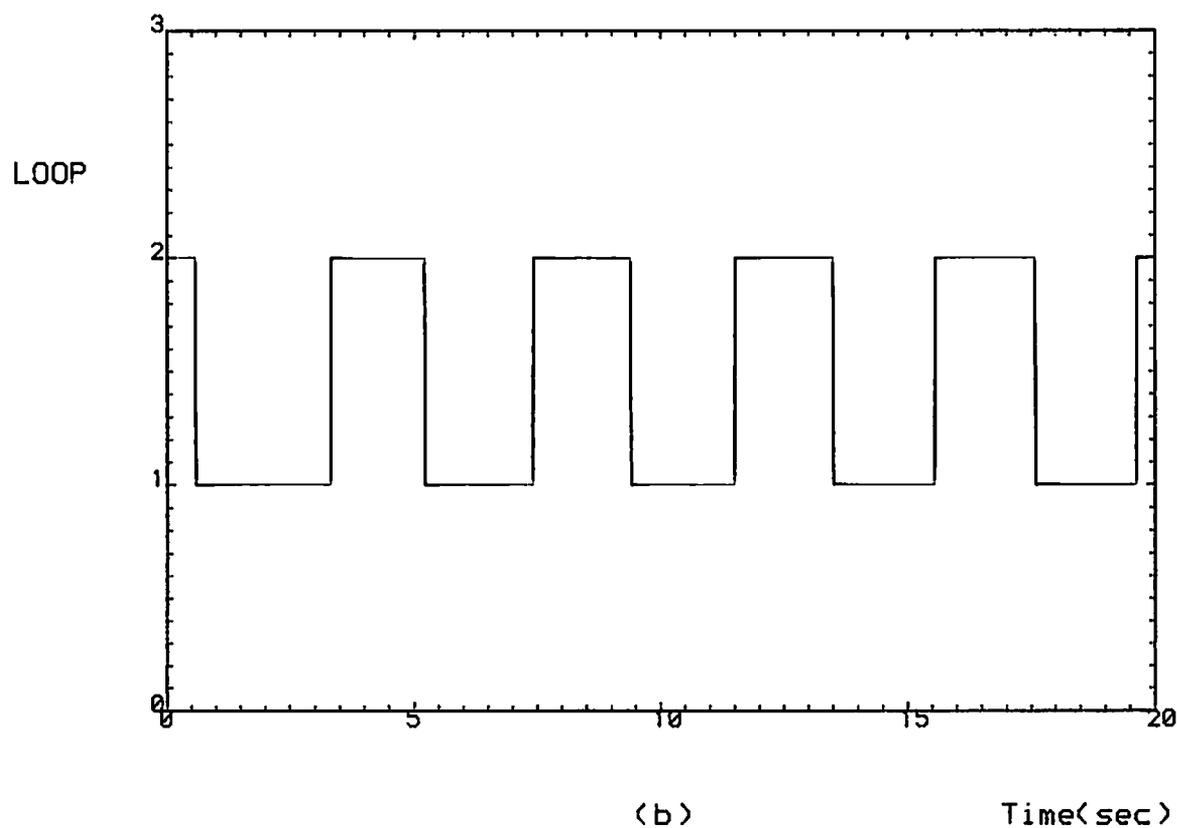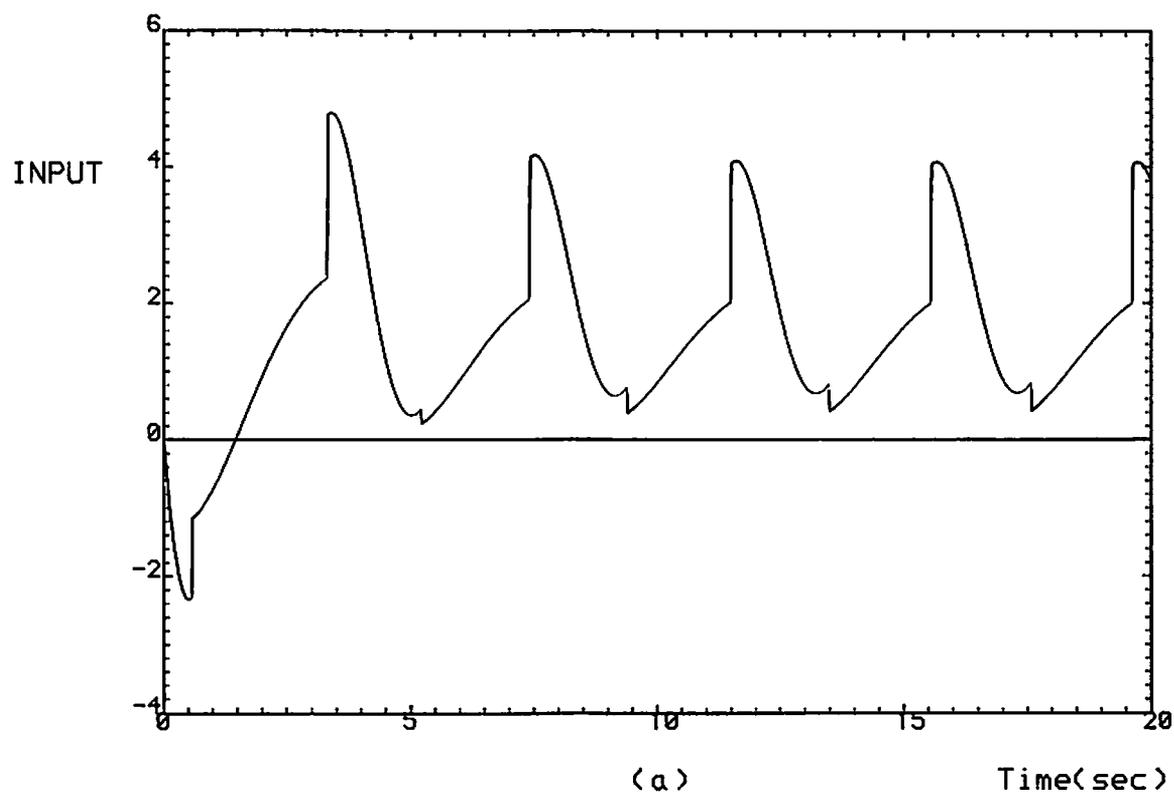
(b)   Time(sec)

Fig A6.9   Input and loop index
           Controller switching without bumpless transfer

APPENDIX 7

ROBUSTNESS THEOREM

In the following robustness theorem, it is necessary to distinguish between the plant for which a controller is designed - ie the *nominal* plant (denoted by subscript $_n$) - and the plant to which a controller is applied - ie the *actual* plant (denoted by subscript $_a$).

Theorem 1 (Porter and Khaki-Sedigh (1989))

In the case of any tunable digital PID controller with integral post-multiplier of the form

$$\Sigma = \sigma \, I_m \qquad (\sigma \in R^+)$$

and any plant perturbation such that

$$\mu_j \in C^+ \qquad (j = 1,2,\ldots,m)$$

where $\{\mu_1, \mu_2, \ldots, \mu_m\}$ is the spectrum of the perturbation matrix

$$M = G_a(0)G_n^{-1}(0) \in R^{m \times m} \, , \qquad\qquad (A7.1)$$

there exists a sampling period $\hat{T}^* \in R^+$ such that set-point tracking occurs for all $T \in (0, \hat{T}^*]$.

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