# Random Permutations and Queues 

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## 1. Introduction

The rencontre problem is one of the oldest problems of probability theory. If $n$ people exchange their hats at random, the probability that no one receives their own hat is about $e^{-1}$. This classic result relates to the probability that the permutation of $[n]:=\{1, \ldots, n\}$ has no fixed points (singleton cycles). Such a permutation is called a derangement. For permutations of increasing degree one can consider a dynamic version of the rencontre problem. Given a stochastic process that sequentially constructs permutations on $n=1,2, \ldots$ elements, how often a derangement will be observed? What are other path properties of the number of fixed points seen as a process? Questions of this sort can be asked also about other functionals related to the cycle structure of the permutation.

A growth rule which received considerable attention in the literature is the following preferential attachment algorithm known as the Chinese Restaurant Process (CRP) [3, 9, 26]. Given a permutation of [ $n$ ] which has been constructed at step $n$, element $n+1$ is either appended to the permutation as a new singleton cycle with probability proportional to $\theta$, or inserted in random position within any existing cycle of size $m$ with probability proportional to $m$. The permutation obtained at step $n$ has the Ewens distribution, which is the uniform distribution on $n$ ! permutations in the case $\theta=1$. Properties of the Ewens distribution of fixed degree have been thoroughly studied. Still, there does not seem to be much work done on the dynamic properties connecting permutations with variable $n$.

We will show that for the discrete-time CRP the proportion of time when the permutation has no fixed points does not converge. An intuitive explanation for this is that the process counting fixed points slows down as $n$ increases, spending more and more steps at the same level. To achieve convergence, we employ a known embedding of the CRP in a continuous-time birth process with immigration [16]. In this realisation the process counting fixed points becomes identical with a $M / M / \infty$ queue, whose features have been intensely studied. Going deeper in this vein, the full process of cycle counts behaves as a series of such queues arranged in a tandem. The principal point of the present note is that this analogy opens the way to translate many results from the queueing theory in terms of the evolutionary properties of permutations. There is vast literature on infinite-server queues, so we do not attempt to survey the field exhaustively. Much more our strategy is to collect and complement results allowing for transparent combinatorial interpretation, with the primary focus on the small cycle counts of permutation.

## 2. Background

Let $\Pi^{(n)}$ be the random permutation of $[n]$ at the $n$th step of the CRP. The distribution of $\Pi^{(n)}$ is invariant under conjugations. For different degrees the permutations are consistent, in the sense that $\Pi^{(m)}$, for $m<n$, can be derived from $\Pi^{(n)}$ by removing elements $m+1, \ldots, n$ from their cycles and deleting empty cycles if necessary.

Let $C_{i}^{(n)}$ be the number of cycles of size $i$ in $\Pi^{(n)}$. Thus $C_{1}^{(n)}$ counts singletons (fixed points), $C_{2}^{(n)}$ doubletons, and so on. The vector of counts $C^{(n)}:=\left(C_{1}^{(n)}, C_{2}^{(n)}, \ldots\right)$ is a random integer partition representing the cycle structure of $\Pi^{(n)}$. The generic value of $\boldsymbol{C}^{(n)}$ is a vector $\boldsymbol{c}=\left(c_{1}, c_{2}, \ldots\right) \in \mathbb{Z}_{+}^{\infty}$ satisfying $\sum_{i=1}^{\infty} i c_{i}=n$ (hence $c_{i}=0$ for $i>n$ ). The law of $\Pi^{(n)}$ is Ewens' distribution, which assigns probability $\theta^{c} /(\theta)_{n}$ to permutation with $c=\sum_{i=1}^{n} c_{i}$ cycles (where $(\theta)_{n}=\theta(\theta+1) \cdots(\theta+n-1)$ ), so permutations of [ $n$ ] with the same number of cycles are equally likely.

The process $\left(\boldsymbol{C}^{(n)}, n \geq 1\right.$ ) (we shall also use shorthand notation $\boldsymbol{C}^{(\cdot)}$ ) is a nonhomogeneous Markov chain starting from the identity permutation for $n=1$ and evolving in $\mathbb{Z}_{+}^{\infty}$ with transition probabilities

$$
\begin{align*}
\mathbb{P}\left[\boldsymbol{C}^{(n+1)}=\left(c_{1}+1, c_{2}, \ldots\right) \mid \boldsymbol{C}^{(n)}=\boldsymbol{c}\right] & =\frac{\theta}{\theta+n},  \tag{1}\\
\mathbb{P}\left[\boldsymbol{C}^{(n+1)}=\left(c_{1}, \ldots, c_{i}-1, c_{i+1}+1, \ldots\right) \mid \boldsymbol{C}^{(n)}=\boldsymbol{c}\right] & =\frac{i c_{i}}{\theta+n}, \quad c_{i}>0 . \tag{2}
\end{align*}
$$

Transition of the first type occurs when $n+1$ starts a new cycle, and of the second when the element is inserted in an existing cycle of $\Pi^{(n)}$. The state distribution is widely known as the Ewens sampling formula

$$
\begin{equation*}
\mathbb{P}\left[\boldsymbol{C}^{(n)}=\boldsymbol{c}\right]=\frac{n!}{(\theta)_{n}} \prod_{i=1}^{n}\left(\frac{\theta}{i}\right)^{c_{i}} \frac{1}{c_{i}!}, \quad \sum_{i=1}^{n} i c_{i}=n \tag{3}
\end{equation*}
$$

The number of cycles in $\Pi^{(n)}$, denoted by

$$
K^{(n)}:=\sum_{i=1}^{n} C_{i}^{(n)}
$$

has probability generating function (p.g.f.)

$$
\begin{equation*}
\mathbb{E} z^{K_{n}}=\frac{(\theta z)_{n}}{(\theta)_{n}} \tag{4}
\end{equation*}
$$

which corresponds to the distribution of the sum of $n$ independent Bernoulli variables with success probabilities $i /(i+\theta-1), i=1,2, \ldots, n$. As $n \rightarrow \infty$,

$$
\begin{equation*}
K^{(n)} \sim \theta \log n \quad \text { a.s., } \quad \frac{K^{(n)}-\theta \log n}{\sqrt{\theta \log n}} \xrightarrow{d} \mathrm{~N}(0,1) . \tag{5}
\end{equation*}
$$

The counts $C_{k}^{(n)}$ for $k=1,2, \ldots, n$ are not independent because of the constraint in (3). Nevertheless, the small cycle counts of large permutation are almost independent:

$$
\begin{equation*}
\boldsymbol{C}^{(n)} \xrightarrow{d}\left(Z_{1}, Z_{2}, \ldots\right), \quad n \rightarrow \infty, \tag{6}
\end{equation*}
$$

where the random variables $Z_{k}$ are independent with Poisson distribution $Z_{k} \stackrel{d}{=} \operatorname{Poiss}(\theta / k)$. The convergence (6) holds with all moments. See [3] for detailed discussion including estimates of the convergence rate.

## 3. Time averages in the CRP

For $k \geq 1$ we denote $\boldsymbol{C}_{k}^{(n)}:=\left(C_{1}^{(n)}, \ldots, C_{k}^{(n)}\right)$ the truncated vector of the first $k$ cycle counts. Easily from (1) and (2), $\boldsymbol{C}_{k}^{(\cdot)}$ itself evolves as a Markov chain. Each transition of type (1) or (2) with $i \leq k$ triggers a jump of the truncated chain, while transitions of type (2) with $i>k$ result in a loop, by which we mean same value $\boldsymbol{C}_{k}^{(n+1)}=\boldsymbol{C}_{k}^{(n)}$. We assert that the average time spent in any given state $\left(c_{1}, \ldots, c_{k}\right)$, that is $\#\left\{m \leq n: \boldsymbol{C}_{k}^{(m)}=\left(c_{1}, \ldots, c_{k}\right)\right\} / n$, does not have a limit as $n \rightarrow \infty$. For the sake of simplicity of exposition we only consider the singleton count, the general case being completely analogous.

The process $\left(C_{1}^{(n)}, n \geq 1\right)$ is a nonhomogeneous Markov chain on $\mathbb{Z}_{+}$which in state $c$ has transition probabilities

$$
\mathbb{P}\left[C_{1}^{(n+1)}=j \mid C_{1}^{(n)}=c\right]= \begin{cases}\frac{\theta}{\theta+n}, & \text { for } j=c+1,0 \leq c \leq n,  \tag{7}\\ \frac{c}{\theta+n}, & \text { for } j=c-1,1 \leq c \leq n, \\ \frac{n-c}{\theta+n}, & \text { for } j=c, 0 \leq c \leq n\end{cases}
$$

Loops occur in the event $C_{1}^{(n+1)}=C_{1}^{(n)}$. By time $n$ there will be about $K^{(n)} \sim \theta \log n$ upward moves caused by starters, and about the same number of downward moves caused by upgrading of singletons to doubletons.

We introduce here the time spent in state $c$,

$$
T_{n}(c):=\#\left\{m \leq n: C_{1}^{(m)}=c\right\},
$$

the time spent in state $c$, which equals the number of permutations among $\Pi^{(1)}, \ldots, \Pi^{(n)}$ with exactly $c$ fixed points. We will look at the long-run behaviour of the proportion $n^{-1} T_{n}(c)$ for fixed $c \geq 0$. Specialising (6) to singletons, we have

$$
\begin{equation*}
\mathbb{P}\left[C_{1}^{(n)}=i\right] \rightarrow \frac{e^{-\theta} \theta^{c}}{c!}, \quad n \rightarrow \infty . \tag{8}
\end{equation*}
$$

Taking the Césaro average in (8) results in the limit for the mean $\mathbb{E}\left[n^{-1} T_{n}(c)\right] \rightarrow e^{-\theta} \theta^{c} / c$ !, which suggests that the proportion itself obeys the law of large numbers. But this intuition is wrong.

Theorem 1. It holds that

$$
\liminf _{n \rightarrow \infty}^{\lim } n_{n}(c)=0 \text { a.s. and } \limsup _{n \rightarrow \infty} n^{-1} T_{n}(c)=1 \text { a.s. }
$$

Proof. Let $\nu_{1}<\nu_{2}<\ldots$ be the consequitive times when sojourns at $c$ start, that is $C_{1}^{\left(\nu_{i}-1\right)} \neq c$, and $C_{1}^{\left(\nu_{i}\right)}=c$. Choose an arbitrary integer $\gamma>1$, and consider the event $A_{k}=\left\{\nu_{k+1} / \nu_{k}>\gamma\right\}$. If $A_{k}$ occurs, the proportion $n^{-1} T_{n}(c)$ exceeds $1-1 / \gamma$ for $n=\nu_{k+1}$.

Given $C_{1}^{(n)}=c$ and $m>n$, the probability that $C_{1}^{(j)}=c$ for all $j=n, \ldots, m$ is

$$
\begin{align*}
\prod_{j=n}^{m-1}\left(1-\frac{c+\theta}{j+\theta}\right) & =\exp \left(\sum_{j=n}^{m-1} \log \left(1-\frac{c+\theta}{j+\theta}\right)\right) \\
& =\exp \left(-(c+\theta) \sum_{j=n}^{m-1} \frac{1}{j+\theta}+O\left(\frac{1}{n}\right)\right) \\
& =\exp \left(-(c+\theta) \log \left(\frac{m}{n}\right)+O\left(\frac{1}{n}\right)\right) \tag{9}
\end{align*}
$$

Taking $m=\gamma n$, we obtain

$$
\prod_{j=n}^{\gamma n-1}\left(1-\frac{c+\theta}{j+\theta}\right) \rightarrow \gamma^{-(c+\theta)} \quad \text { as } n \rightarrow \infty
$$

Using the strong Markov property to replace fixed $m$ by the random stopping time $\nu_{k}$, we conclude that $\mathbb{P}\left[A_{k} \mid \Pi_{\nu_{k}}\right]$ is bounded away from 0 as $k \rightarrow \infty$. It follows that

$$
\sum_{k=1}^{\infty} \mathbb{P}\left[A_{k} \mid \Pi_{\nu_{k}}\right]=\infty \quad \text { a.s. }
$$

Noting that $A_{k}$ is $\sigma\left(\Pi_{\nu_{k+1}}\right)$-measurable, Lévy's conditional Borel-Cantelli lemma ([17], p. 108) applies to the sequence $\left(A_{k}, \sigma\left(\Pi_{\nu_{k+1}}\right)\right)$ and ensures that these events coincide:

$$
\left\{\sum_{k=1}^{\infty} \mathbb{P}\left[A_{k} \mid \Pi_{\nu_{k}}\right]=\infty\right\}=\left\{A_{k} \text { i.o. }\right\}
$$

It follows that $\mathbb{P}\left[A_{k}\right.$ i.o. $]=1$.
Letting $\gamma \rightarrow \infty$ we arrive at $\lim \sup _{n \rightarrow \infty} n^{-1} T_{n}(c)=1$ a.s. for every $c$. But then since $n^{-1} T_{n}(c)+$ $n^{-1} T_{n}(d) \leq 1$ for $d \neq c$ we also have $\liminf _{n \rightarrow \infty} n^{-1} T_{n}(c)=0$ a.s.

We leave to the reader checking that the variance of $n^{-1} T_{n}(c)$ does not vanish asymptotically, hence convergence in probability also fails. The source of irregularity is the infinite expectation of sojourn times. This phenomenon is akin to the null recurrence of time-homogeneous Markov chains like the symmetric random walk.

## 4. The embedded random walk

The convergence of time averages can be achieved by discarding the loops, that is letting the clock tick only at times of nontrivial moves.

To explore this thread, let $\left(J_{i}, i \geq 0\right)$ be a nearest-neighbour random walk on $\mathbb{Z}_{+}=\{0,1,2, \ldots\}$ which moves $\pm 1$ with time-independent probabilities

$$
\begin{equation*}
p_{c}=\frac{\rho}{c+\rho}, \quad q_{c}=\frac{c}{c+\rho} \tag{10}
\end{equation*}
$$

respectively, where $\rho>0$ is a parameter. The random walk is reversible and, checking the detailed balance equations, it is seen that it has the unique stationary distribution (cf. [11], Section 6.11, Exercise 2)

$$
\begin{equation*}
\alpha_{c}=\frac{e^{-\rho}(\rho+c) \rho^{c-1}}{2 c!}, \quad c \in \mathbb{Z}_{+} \tag{11}
\end{equation*}
$$

which can be decomposed as equi-weighted mixture of $\operatorname{Poiss}(\rho)$ distribution on $\{0,1, \ldots\}$ and the shifted Poiss $(\rho)$ distribution on $\{1,2, \ldots\}$. Appealing to the ergodic theorem, we conclude that the proportion of time that $J$. spends at 0 converges to $\alpha_{0}=e^{-\rho} / 2$. Therefore the mean time between two consequitive visits to 0 is $1 / \alpha_{0}=2 e^{\rho}$, and the mean number of upward moves between the visits is $e^{\rho}$.

Define an excursion of $J$. above level $c$ to be a segment of the path that starts from $c+1$ and terminates by hitting $c$. Assuming $J_{0}=c+1$, the first passage time

$$
\begin{equation*}
\kappa_{c}:=\inf \left\{k \geq 0: J_{k}=c\right\} \tag{12}
\end{equation*}
$$

is the length, and

$$
H_{c}:=\sup \left\{J_{k}: k \leq \kappa_{c}\right\}-c
$$

is the height of excursion above $c$.
The random walk starts anew by each visit at $c$, hence the elementary renewal theorem ensures that the number of excursions above $c$ completed within $k$ steps is asymptotic to

$$
\alpha_{c} \frac{\rho}{\rho+c} k, \quad k \rightarrow \infty
$$

which entails that the limit proportion of time spent above $c$ is

$$
\alpha_{c} \frac{\rho}{\rho+c} \mathbb{E}\left[\kappa_{c}\right],
$$

which must also be equal to $\sum_{j=c+1}^{\infty} \alpha_{j}$ by ergodicity. Recalling (11), an easy calculation gives

$$
\mathbb{E}\left[\kappa_{c}\right]=1+\frac{2 c!}{\rho^{c}} \sum_{j=c+1}^{\infty} \frac{\rho^{j}}{j!}
$$

In particular,

$$
\mathbb{E}\left[\kappa_{0}\right]=2 e^{\rho}-1
$$

To determine the variance of excursion length we adopt a formula of Harris ([15], Equation (5.9)). To that end, express the product of odds involved in the cited result in terms of Poiss $(\rho)$ probabilities as

$$
\pi_{r}:=e^{-\rho} \frac{\rho^{r}}{r!}, \quad \text { so } \quad \prod_{j=1}^{r-1} \frac{q_{j}}{p_{j}}=\left(e^{\rho} \pi_{r-1}\right)^{-1}
$$

to obtain

$$
\begin{equation*}
\operatorname{Var}\left[\kappa_{0}\right]=4 e^{\rho}\left(2 \rho-e^{\rho}+1\right)+8 e^{\rho} \sum_{r=0}^{\infty} \frac{1}{\pi_{r}}\left(\sum_{j=r+1}^{\infty} \pi_{j}\right)^{2} \tag{13}
\end{equation*}
$$

Harris ([15], Theorem 2b) also solved the generalised gambler's ruin problem: for fixed integers $0 \leq \ell<$ $s \leq u$, if the random walk starts at $s$ it will reach $u$ before visiting $\ell$ with probability

$$
\begin{equation*}
\mathbb{P}\left[J . \text { reaches } u \text { before } \ell \mid J_{0}=s\right]=\frac{\frac{1}{\pi_{\ell}}+\frac{1}{\pi_{\ell+1}}+\cdots+\frac{1}{\pi_{s-1}}}{\frac{1}{\pi_{\ell}}+\frac{1}{\pi_{\ell+1}}+\cdots+\frac{1}{\pi_{u-1}}} \tag{14}
\end{equation*}
$$

Choosing $\ell=c, s=c+1, u=h+1$ this gives the distribution of $H_{c}$, most conveniently expressed in the form of the upper tail probabilities

$$
\begin{equation*}
\mathbb{P}\left[H_{c} \geq h+1\right]=\frac{1}{\sum_{r=0}^{h} \frac{(c+1)_{r}}{\rho^{r}}}, \quad h \geq 0 \tag{15}
\end{equation*}
$$

The tails are lighter than geometric,

$$
\lim _{h \rightarrow \infty} \frac{\mathbb{P}\left[H_{0} \geq h+1\right]}{\mathbb{P}\left[H_{0} \geq h\right]}=0
$$

which suggests that the maximum of random walk satisfies a law of large numbers for discrete random variables [1]. This was indeed shown by Park et al [25]. To state their exceptionally precise result, let $\beta_{m}$ be the time of the $m$ th visit of the random walk to 0 . Then

$$
\begin{equation*}
\lim _{m \rightarrow \infty} \mathbb{P}\left[\max _{0 \leq i \leq \beta_{m}} J_{i} \in\left\{I_{m}, I_{m}+1\right\}\right]=1 \tag{16}
\end{equation*}
$$

where

$$
\begin{equation*}
I_{m}:=\left\lfloor\frac{\log m-\frac{1}{2} \log \log m-\frac{1}{2} \log 2 \pi}{\log \log m-1-\log \rho}+\frac{1}{2}\right\rfloor \tag{17}
\end{equation*}
$$

Using $\rho=1$ in (15), we obtain numerical values for the excursion height moments

$$
\begin{array}{r}
\mathbb{E}\left[H_{0}\right]=\sum_{h=0}^{\infty} \frac{1}{0!+1!+2!+\cdots+h!}=1.887 \ldots \\
\operatorname{Var}\left[H_{0}\right]=\sum_{h=0}^{\infty} \frac{2 h+1}{0!+1!+2!+\cdots+h!}-\left(\mathbb{E}\left[H_{0}\right]\right)^{2}=1.242 \ldots
\end{array}
$$

and the number of upward moves of an excursion has expectation and variance

$$
\mathbb{E}\left[\frac{\kappa_{0}-1}{2}\right]=1.718 \ldots, \quad \operatorname{Var}\left[\frac{\kappa_{0}-1}{2}\right]=7.930 \ldots
$$

Translating the results above to the CRP and taking our parameter $\rho=\theta$, the random walk $J$. is the embedded jump chain for $C_{1}^{(\cdot)}$, hence we can make conclusions about the number of fixed points in the CRP. Excursions above $c=0$ correspond to the fluctuation in the number of fixed points in the period between two consequitive derangements. The asymptotic proportion of derangements within the number of nontrivial
moves of $C_{1}^{(\cdot)}$ is $e^{-\theta} / 2$. This does not match with the value $e^{-\theta}$ that could be anticipated from (8). Starting from $C_{1}^{(n)}=c+1, c \geq 0$, the variable $H_{c}+c$ is the maximum number of singletons observed until their number falls to $c$, and $\left(\kappa_{c}-1\right) / 2$ is the number of new cycles produced by the CRP within this period. In particular, applying the above findings to the case $\theta=1$ of uniformly chosen permutations, we see that a permutation with initially one singleton (for instance, $\Pi^{(1)}$ ) will have on the average 1.778 cycles at the first time when it becomes derangement, and the expected maximum number of singletons observed by this time is about 1.887 .

For the first $n$ permutations $\Pi^{(1)}, \ldots, \Pi^{(n)}$, the number of fixed points $C_{1}^{(\cdot)}$ will change the value about $2 \theta \log n$ times, when singletons are formed and when they progress to doubletons. This implies a strong law of large numbers on the log scale

$$
\#\left\{j \leq n: \Pi^{(j)} \text { is a derangement and } \Pi^{(j-1)} \text { is not }\right\} \sim(2 \theta \log n) \alpha_{0}=e^{-\theta} \theta \log n \quad \text { a.s. }
$$

Thus $e^{-\theta}$ appears to be the asymptotic proportion of singletons that enter the permutation when it is a derangement, relative to the number of all cycles. For the maximum number of fixed points we have from (5) and (16)

$$
\lim _{n \rightarrow \infty} \mathbb{P}\left[\max _{1 \leq j \leq n} C_{1}^{(j)} \in\left\{I_{m}, I_{m}+1\right\}\right]=1
$$

where $I_{m}$ is given by (17) with a possible adjustment of $\pm 1$, with $\rho=\theta$ for $m=\lfloor\theta \log n\rfloor$. The fact that approximating the true value of $m$ by $\left\lfloor\theta e^{-\theta} \log n\right\rfloor$ only changes the expression inside $\lfloor\cdot\rfloor$ in (17) by o(1) accounts for the possible adjustment to $I_{m}$.

It is tempting to similarly link $C_{k}^{(\cdot)}$ to the random walk $J$. with parameter $\rho=\theta / k$. For $k>1$, this comparison does not work literally, because the count of $k$-cycles is not a Markov chain. The connection becomes valid asymptotically, in the sense that for $n_{0} \rightarrow \infty$ the loop-free path of ( $C_{k}^{(n)}, n \geq n_{0}$ ) conditioned on $C_{k}^{\left(n_{0}\right)}=c$ converges in distribution to $J$. with the initial state $J_{0}=c$. This will follow from the embedding in the next section. To gain some intuition, observe that the transition of $\boldsymbol{C}_{k-1}^{(\cdot)}$ causing an upward move of $C_{k}^{(\cdot)}$ has probability

$$
\mathbb{P}\left[\boldsymbol{C}_{k-1}^{(n+1)}-\boldsymbol{C}_{k-1}^{(n)}=(0, \ldots, 0,-1) \mid \Pi^{(n)}\right]=\frac{(k-1) C_{k-1}^{(n)}}{n+\theta}
$$

For large $n$ the distribution of $C_{k-1}^{(n)}$ is approximately Poisson with mean $\theta /(k-1)$, hence the unconditional probability of the said upward move is about $\theta /(n+\theta)$, to be compared with probability $k c /(\theta+n)$ of the downward move (which does not depend on the first $k-1$ counts); thus for large times given $C_{k}^{(\cdot)}$ has a move it is +1 with probability about $(\theta / k) /(c+\theta / k)$, in agreement with (10).

## 5. Embedding in continuous time

To ensure the convergence of time averages of occupation times, the temporal scale of the CRP should be changed so that the degree of permutation grows about exponentially, and hence the number of cycles grows about linearly. An elegant way to do this is to embed the permutation-valued process in continuous time. The embedding idea originated in $[4,16,38]$ and is nicely presented in [9].

Consider a permutation-valued process $(\Pi(t), t \geq 0)$ which starts with the empty permutation (of degree $0)$ and then evolves according to this rule: given permutation $\Pi(t)$ of $[n]$, element $n+1$ starts a new cycle with probability rate $\theta$ and is inserted in random position of any existing cycle of size $m$ at rate $m$. It is obvious from this description that the associated discrete-time jump chain is the CRP $\left(\Pi^{(n)}, n \geq 0\right)$.

Let $\boldsymbol{C}(t)=\left(C_{1}(t), C_{2}(t), \ldots\right)$ denote the vector counting singletons, doubletons, and so on. Clearly, $(\boldsymbol{C}(t), t \geq 0)$ is a Markov process on $\left\{\boldsymbol{c} \in \mathbb{Z}_{+}^{\infty}: \sum_{i} c_{i}<\infty\right\}$ with time-independent transition rates

$$
\begin{aligned}
\theta \text { for }\left(c_{1}, c_{2}, \ldots\right) & \rightarrow\left(c_{1}+1, c_{2}, \ldots\right), \\
i c_{i} \text { for }\left(c_{1}, c_{2}, \ldots\right) & \rightarrow\left(c_{1}, \ldots, c_{i}-1, c_{i+1}+1, \ldots\right), i \geq 2,
\end{aligned}
$$

and (by definition) the initial value $\boldsymbol{C}(0)=(0,0, \ldots)$.
Define $K(t)$ and $N(t)$ to be the number of cycles and the degree of permutation $\Pi(t)$, respectively. Thus,

$$
K(t):=\sum_{i=1}^{\infty} C_{i}(t) \quad \text { and } \quad N(t):=\sum_{i=1}^{\infty} i C_{i}(t) .
$$

The number of cycles evolves according to a Poisson process of rate $\theta$. The degree of $\Pi(t)$ follows a linear birth process with immigration, where the immigration rate is constant $\theta$ and the birth rate per capita is 1 (so that given $N(t)=n$, the birth rate is $n$ ). Sometimes the term Pascal process is used for such a process, because the conditional distribution of the increment $N(t)-N(s)$ given $N(s)=n$ is negative binomial (i.e. Pascal) $\mathrm{NB}\left(\theta+n, 1-e^{-(t-s)}\right)$, for $t>s \geq 0$. See the texts [5, 9, 29] for properties of the birth-death processes.

We let $\boldsymbol{C}_{k}(t):=\left(C_{1}(t), \ldots, C_{k}(t)\right)$ denote a truncated vector of cycle counts. Similarly to the discretetime CRP, each $\boldsymbol{C}_{k}(\cdot)$ is itself a time-homogeneous Markov process.

The next product-form result is known in much larger generality in the theory of networks and population processes [18]. The textbook proofs for the transient (pre-limit) state all employ Kolmogorov's equation. The following elegant elementary proof for the special case in focus was outlined in [4], Exercise 10.7.

Theorem 2. The random variables $C_{1}(t), C_{2}(t), \ldots$ are independent, with distribution

$$
\begin{equation*}
C_{k}(t) \stackrel{d}{=} \operatorname{Poisson}\left(\frac{\theta\left(1-e^{-t}\right)^{k}}{k}\right) . \tag{18}
\end{equation*}
$$

Therefore, as $t \rightarrow \infty$

$$
\begin{equation*}
\boldsymbol{C}(t) \xrightarrow{d}\left(Z_{1}, Z_{2}, \ldots\right), \tag{19}
\end{equation*}
$$

where $Z_{i} \stackrel{d}{=} \operatorname{Poisson}(\theta / i)$ are independent.
Proof. A singleton needs time $\xi_{1}$ to become a doubleton, then time $\xi_{2} / 2$ to become a tripleton, etc., where $\xi_{1}, \xi_{2}, \ldots$ are independent unit exponential random variables. The jump times for different cycles are independent. Hence by the theorem on marked Poisson processes (see [19], Ch. 5), the $C_{k}(t)$ 's for $k=1,2, \ldots$ are independent and Poisson distributed. It remains to compute the means.

A singleton grows to a cycle of size at least $k+1$ within time $s$ with probability

$$
\begin{equation*}
\mathbb{P}\left[\sum_{i=1}^{k} \xi_{i} / i<s\right]=\mathbb{P}\left[\max \left(\xi_{1}, \ldots, \xi_{k}\right)<s\right]=\left(1-e^{-s}\right)^{k} \tag{20}
\end{equation*}
$$

where the first identity is Rényi's representation of the exponential order statistics. Hence, the probability that this is a $k$-cycle is

$$
\left(1-e^{-s}\right)^{k-1}-\left(1-e^{-s}\right)^{k}=e^{-s}\left(1-e^{-s}\right)^{k-1} .
$$

Thus, the mean number of $k$-cycles at time $t$ is

$$
\mathbb{E}\left[C_{k}(t)\right]=\int_{0}^{t} e^{-s}\left(1-e^{-s}\right)^{k-1} \theta \mathrm{~d} s=\theta k^{-1}\left(1-e^{-t}\right)^{k}
$$

as wanted.
The convergence of $\boldsymbol{C}(t)$ in distribution follows from the first part of the statement. Note that this convergence is understood relative to the discrete product topology on $\mathbb{Z}_{+}^{\infty}$ and amounts to the weak convergence of truncated processes $\boldsymbol{C}_{k}(t)$.

The Markov chain $\left(\boldsymbol{C}_{k}(t), t \geq 0\right)$ is positive recurrent, hence application of the ergodic theorem ensures existence of the time averages. Let

$$
h_{k}:=\sum_{i=1}^{k} \frac{1}{i}
$$

denote the $k$ th harmonic number.
Corollary 1. The proportion of time spent by the process $\left(\boldsymbol{C}_{k}(t), t \geq 0\right)$ in state $\left(c_{1}, \ldots, c_{k}\right)$ converges to

$$
\mathbb{P}\left[Z_{1}=c_{1}, \ldots, Z_{k}=c_{k}\right]=e^{-\theta h_{k}} \prod_{i=1}^{k} \frac{\theta^{c_{i}}}{i^{c_{i}} c_{i}!}
$$

In particular, the average time when $\Pi(\cdot)$ is a derangement approaches $e^{-\theta}$.
We may tag a cycle by its minimal element. The growth of a cycle can be thought of as passing through phases $S_{1}, S_{2}, \ldots$ of being a singleton, doubleton, etc. The sojourn periods across different cycles and phases are independent. The input flow into $S_{1}$ is a Poisson process of rate $\theta$, and the time spent in $S_{k}$ has exponential distribution with parameter $k$. By Theorem 2 the flow from $S_{k}$ to $S_{k+1}$ is nonhomogeneous Poisson with rate $\theta\left(1-e^{-t}\right)^{k}$, hence converging to homogeneous flow with rate $\theta$.

Such a process, with general sojourn rates $\mu_{1}, \mu_{2}, \ldots$, models a network of $\mathrm{M} / \mathrm{M} / \infty$, infinite-server queues connected in a tandem $[7,12,20,28]$. In the literature the tandem is often considered as open network with finitely many phases $S_{1}, \ldots, S_{k}$, where the task departs upon passing through $S_{k}$.

The process $\left(\boldsymbol{C}_{k}(t), t \geq 0\right)$ is stationary if it starts with the product Poisson distribution $\boldsymbol{C}_{k}(0) \stackrel{d}{=}$ $\left(Z_{1}, \ldots, Z_{k}\right)$ as in (19). Following the established terminology we shall call the stationary process steady state, as opposed to the transient regime with the pre-limit state distribution (18). In the steady state the flow from $S_{i}$ to $S_{i+1}$ is Poisson with rate $\theta$, so each $C_{i}(\cdot)$ behaves like a single stationary $\mathrm{M} / \mathrm{M} / \infty$ queue; this is an instance of the seminal Burke's theorem. We stress that the steady state does not describe permutations of finite degree, but rather captures asymptotic features of small cycle counts of $\Pi(\cdot)$ at large times.

## 6. Pascalisation and big cycles

The discrete- and continuous time models are related via

$$
(\Pi(t), t \geq 0) \stackrel{d}{=}\left(\Pi^{(N(t))}, t \geq 0\right)
$$

where the permutation in the right-hand side is constructed from two independent ingredients: CRP $\left(\Pi^{(n)}, n \geq 0\right)$ and a Pascal process $(N(t), t \geq 0)$. For this kind of randomisation we propose the term
pascalisation, by analogy with the established concept of poissonisation (sampling $n$ from the Poisson distribution). These methods are most useful in the situations where they produce exact independence instead of the asymptotic independence in fixed- $n$ combinatorial models. In the context of cycle structure the pascalisation was used already in [36] for the case $\theta=1$ (where the mixing distribution is geometric), in particular to prove the convergence (6). See [24] (Section 4 and references) for the general case $\theta>0$, and [6] for pascalisation of another interesting distribution on integer partitions.

To illustrate, let $g_{n}(z)$ be the p.g.f. of $K^{(n)}$. Connecting $g_{n}(z)$ to the Poisson p.g.f. of $K(t)$ produces

$$
e^{\theta t(z-1)}=\sum_{n=0}^{\infty} \mathbb{P}[N(t)=n] g_{n}(z)=\sum_{n=0}^{\infty} \frac{(\theta)_{n}}{n!} e^{-\theta t}\left(1-e^{-t}\right)^{n} g_{n}(z)
$$

Expanding the left-hand side in powers of $\left(1-e^{-t}\right)$ and equating the coefficients yields $g_{n}(z)=(\theta z)_{n} /(\theta)_{n}$, which gives altervative proof of (4). For both discrete and continuous models, the number of cycles is about normally distributed for large times.

A more complex functional is the maximal size of a cycle. For $\Pi^{(n)}$ this has a sophisticated limit distribution ([9], Theorem 2.5), but for $\Pi(t)$ the things are rather straightforward. Let $M(t):=\max \{i$ : $\left.C_{i}(t)>0\right\}$ be the maximal size of a cycle present in $\Pi(t)$. From (18),

$$
\mathbb{P}[M(t) \leq m]=\exp \left(-\sum_{k=m+1}^{\infty} \frac{\theta\left(1-e^{-t}\right)^{k}}{k}\right)
$$

To find the limit law of $M(t)$ consider a Poisson random measure (PRM) $\mathcal{P}_{t}$ which charges point $e^{-t} k$ with mass $C_{k}(t), k=1,2, \ldots$ ans let $\mathcal{P}$ be another PRM on $(0, \infty)$ with mean measure $\lambda(\mathrm{d} x)=\theta e^{-x} \mathrm{~d} x / x$. We have $\lambda(x, \infty)=\theta E_{1}(x)$, where

$$
E_{1}(x)=\int_{x}^{\infty} \frac{e^{-s}}{s} \mathrm{~d} s
$$

is the exponential integral function.
Theorem 3. For $t \rightarrow \infty$, the PRM $\mathcal{P}_{t}$ converges weakly to $\mathcal{P}$.
Proof. It is sufficient to show that the mean measure $\lambda_{t}$ of $\mathcal{P}_{t}$ satisfies

$$
\lim _{t \rightarrow \infty} \lambda_{t}(x, \infty)=\lambda(x, \infty)
$$

for each $x>0$. We have

$$
\begin{aligned}
& \lambda_{t}\left(m e^{-t}, \infty\right)=\theta \sum_{k=m+1}^{\infty} \frac{\left(1-e^{-t}\right)^{k}}{k}=\theta \sum_{k=m+1}^{\infty} \int_{e^{-t}}^{1}(1-y)^{k-1} \mathrm{~d} y= \\
& \theta \int_{e^{-t}}^{1} \frac{(1-y)^{m}}{y} \mathrm{~d} y= \\
&=\theta \int_{1}^{e^{t}} \frac{\left(1-s e^{-t}\right)^{m}}{s} \mathrm{~d} s
\end{aligned}
$$

Setting $m=\left\lfloor x e^{t}\right\rfloor$ we obtain, by the monotone convergence,

$$
\lambda_{t}(x, \infty)=\theta \int_{1}^{e^{t}} \frac{\left(1-s e^{-t}\right)^{x e^{t}}}{s} \mathrm{~d} s+o(1) \rightarrow \theta \int_{1}^{\infty} \frac{e^{-x s}}{s} \mathrm{~d} s=\theta E_{1}(x)
$$

and the conclusion follows.

The result implies that $e^{-t} M(t)$ converges in distribution to the largest point of $\mathcal{P}$, whence

$$
\lim _{t \rightarrow \infty} \mathbb{P}\left[e^{-t} M(t) \leq x\right]=\exp \left(-\theta E_{1}(x)\right)
$$

Moreover, the whole scaled decreasing sequence of the cycle lengths of $\Pi(t)$ converges in distribution to the infinite sequence of points of $\mathcal{P}$ listed in decreasing order. The analogous limit for $\Pi^{(n)}$, the Poisson-Dirichlet distribution, can be obtained by normalising the points of $\mathcal{P}$ by their sum ([9], Theorem 2.2).

Better tractable limit laws for large cycles in $\Pi^{(n)}$ or $\Pi(t)$ appear if the cycles are listed in the age order, that is by increase of the starters. For instance, the size of the oldest cycle of $\Pi(t)$ (containing element 1 ) is asymptotic to $e^{t-\eta / \theta} \xi$ with independent unit exponential $\xi$ and $\eta$. See $[2,3,9]$ for various representations of the multivariate limit.

## 7. Excursions of a cycle count process

The $\mathrm{M} / \mathrm{M} / \infty$ queue occupancy process is a Markov chain $(X(t), t \geq 0)$ on $\mathbb{Z}_{+}$which from state $c$ jumps by $\pm 1$ with rates $\theta$ and $\mu c$, respectively. This is sometimes called a linear immigration-death process [5]. We shall follow the intuitive terminology of queueing theory, calling $\theta$ the arrival rate of the input Poisson process $K(\cdot), \mu$ the service rate (the departure rate per task), and $\rho:=\theta / \mu$ the average workload. The transient state distribution is Poisson with parameter depending on $t$, that is

$$
\mathbb{P}[X(t)=c \mid X(0)=0]=\pi_{c}(t), \quad \text { where } \quad \pi_{c}(t)=\exp \left\{-\rho\left(1-e^{-t / \mu}\right\} \frac{\left[\left(\rho\left(1-e^{-t / \mu}\right)\right]^{c}\right.}{c!}, \quad c \geq 0\right.
$$

and in the steady state the distribution is $\operatorname{Poiss}(\rho)$. The embedded jump chain for $X(\cdot)$ is the random walk $J$. with parameter $\rho$ as in Section 4.

In terms of the permutation-valued process, $X(\cdot)$ could be $C_{1}(\cdot)$, or $C_{k}(\cdot)$ for $k>1$ with Poisson inflow resulting from the output of $C_{k-1}(\cdot)\left(\right.$ or $\left.\boldsymbol{C}_{k-1}(\cdot)\right)$ in the steady state. Then the parameters are $\theta, \mu=k, \rho=$ $\theta / k$.

For $c \geq 0$. we define excursion above $c$ to be a segment of the path that starts at $c+1$ and terminates by the first passage of level $c$. The case $c=0$ is referred to as the busy period, and for the general $c$ the excursion is called the congestion period above the level. Excursions below $c \geq 1$ are defined analogously but will not be touched here (see [30] or [31] on intercongestion periods).

A visit to $c$ is followed by an excursion above $c$ if the next state is $c+1$. In the long run, the mean rate of the point process of jumps $c \rightarrow c+1$ is about $\pi_{c} \lambda$, thus by renewal theory the number of excursions above $c$ completed by time $t$ is asymptotic to $t /\left(\pi_{c} \lambda\right)=t e^{\rho} c!/\left(\lambda \rho^{c}\right)$ as $t \rightarrow \infty$.

The functionals characterising the excursion include

$$
\begin{aligned}
& \text { the duration } \quad D_{c}=\inf \{t: X(t)=c\}, \\
& \text { the height above } c \quad H_{c}=\sup \left\{X(t)-c: 0 \leq t \leq D_{c}\right\} \text {, } \\
& \text { the overflow } \quad A_{c}=\int_{0}^{D_{c}}(X(t)-c) \mathrm{d} t, \\
& \text { the number of new arrivals } \Delta_{c}=\left(\kappa_{c}-1\right) / 2,
\end{aligned}
$$

where we write definitions as if the excursion started at time 0 with $X(0)=c+1$. The variables $H_{c}$ and $\kappa_{c}$ are functionals of the embedded random walk $J$. and have the same meaning as in Section 4. In the rest of this section we put together and complement properties of these variables found in the literature.

Some relations among the moments follow by the optional sampling theorem applied to $\Delta_{c}=K\left(D_{c}\right)$ and the martingale $K(\cdot)-\theta \cdot$ where $K(\cdot)$ is the Poisson arrival process:

$$
\begin{equation*}
\mathbb{E}\left[\Delta_{c}-\theta D_{c}\right]=0, \quad \mathbb{E}\left[\Delta_{c}-\theta D_{c}\right]^{2}=\mathbb{E}\left[\Delta_{c}\right] \tag{21}
\end{equation*}
$$

By arguments from the renewal theory,

$$
\begin{equation*}
\mathbb{E}\left[D_{c}\right]=\frac{1}{\theta \pi_{c}} \sum_{j=c+1}^{\infty} \pi_{j}, \quad \mathbb{E}\left[A_{c}\right]=\frac{1}{\theta \pi_{c}} \sum_{j=c+1}^{\infty} \pi_{j}(j-c) \tag{22}
\end{equation*}
$$

and from (21)

$$
\begin{equation*}
\mathbb{E}\left[\Delta_{c}\right]=\frac{1}{\pi_{c}} \sum_{j=c+1}^{\infty} \pi_{j} \tag{23}
\end{equation*}
$$

where $\pi_{j}$ are the Poisson $(\rho)$ probabilities. The variance of $\Delta_{c}$ for $c=0$ is obvious from the connection with $\kappa_{c}$ and (13), and for $c>0$ can be also derived from (14); from (21) one can compute then the covariance between $D_{c}$ and $\Delta_{c}$.

Formulas for the Laplace transforms of these statistics have been obtained in terms of the integrals

$$
I_{c}(\alpha, \beta)=\int_{0}^{1} u^{c}(1-u)^{\alpha-1} e^{-\beta u} \mathrm{~d} u
$$

which in turn can be expressed through Kummer's confluent hypergeometric function

$$
M(a, b, z):=\sum_{i=0}^{\infty} \frac{(a)_{i}}{(b)_{i}} \frac{z^{i}}{i!}
$$

as

$$
I_{c}(\alpha, \beta)=e^{-\beta} \frac{\Gamma(c+1) \Gamma(\alpha)}{\Gamma(c+\alpha+1)} M(\alpha, \alpha+c+1, \beta)
$$

The appearance of these functions here is quite natural, since the Laplace transform of the transient state probability $\pi_{c}(t)$ is

$$
\int_{0}^{\infty} \pi_{c}(t) e^{-z t} \mathrm{~d} t=\frac{\rho^{c} \mu}{c!} I_{c}(\mu z, \rho)
$$

as one can easily calculate by

$$
\begin{aligned}
\int_{0}^{\infty} \pi_{c}(t) e^{-z t} \mathrm{~d} t & =\int_{0}^{\infty} \exp \left(-\rho\left(1-e^{-t / \mu}\right) \frac{\left[\rho\left(1-e^{-t / \mu}\right)\right]^{c}}{c!} e^{-z t} d t\right. \\
& =\int_{0}^{1} e^{-\rho u} \frac{[\rho u]^{c}}{c!}(1-u)^{\mu z} \mu(1-u)^{-1} d u \\
& =\frac{\mu \rho^{c}}{c!} \int_{0}^{1} e^{-\rho u} u^{c}(1-u)^{\mu z-1} d u \\
& =\frac{\mu \rho^{c}}{c!} I_{c}(\mu z, \rho)
\end{aligned}
$$

where $u=1-e^{-t / \mu}, d u=\frac{1}{\mu} e^{-t / \mu} d t \Rightarrow d t=\mu(1-u)^{-1} d u$ and $e^{-z t}=(1-u)^{\mu z}$. Concretely, Guillemin and Simonian [12] showed that

$$
\begin{equation*}
\mathbb{E}\left[\exp \left(-z D_{c}\right)\right]=\frac{I_{c+1}(z / \mu, \rho)}{I_{c}(z / \mu, \rho)} \tag{24}
\end{equation*}
$$

Preater [28] derived the joint Laplace transform

$$
\begin{equation*}
\mathbb{E}\left[\exp \left(-x D_{0}-y \Delta_{0}-z A_{0}\right)\right]=\frac{\mu}{z+\mu} \frac{I_{c+1}(a-b, b)}{I_{c}(a-b, b)} \tag{25}
\end{equation*}
$$

where

$$
a=\frac{x+\theta}{z+\mu}, \quad b=\frac{\theta \mu e^{-y}}{(z+\mu)^{2}}
$$

(this result is cited as Equation (20) in [31]), and in [27] obtained a continued fraction formula for the joint Laplace transform of $D_{c}$ and $A_{c}$. Preater's approach [27, 28] to continued fractions expansions relies on the fact that an excursion above $c$ decomposes in a sojourn at $c+1$ of $\operatorname{Exp}(\theta+(c+1) \mu)$-length and some Geom $((c+1) /(\rho+(c+1)))$ number of path segments, each comprised of excursion above $c+1$ and sojourn at $c$, with all ingredients being independent.

Roijers et al [31] notice that obtaining higher moments by differentiating the Laplace transforms is not straightforward due to the implicit nature of functions involved. They derived recursions in $c$ using the said decomposition of the excursion above $c$, thus eventually reducing to the case $c=0$. For the second moment of the duration they obtain a series expansion ([31], Equation (24))

$$
\begin{equation*}
\mathbb{E}\left[D_{0}^{2}\right]=\frac{2 e^{2 \rho}}{\theta \mu} \sum_{j=1}^{\infty} \frac{\pi_{j}}{j} \tag{26}
\end{equation*}
$$

which can be written as

$$
\mathbb{E}\left[D_{0}^{2}\right]=\frac{2 e^{\rho}}{\theta \mu} \int_{0}^{\rho} \frac{e^{s}-1}{s} \mathrm{~d} s
$$

Lizgin and Rudenko [22] employed a similar recursion for the moments of the first passage time from level $c$
to 0 , which led them to another derivation of (26), the third moment formula

$$
\mathbb{E}\left[D_{0}^{3}\right]=\frac{6 e^{\rho}}{\theta \mu^{2}}\left[e^{2 \rho}\left(\sum_{j=1}^{\infty} \frac{\pi_{j}}{j}\right)^{2}+e^{\rho} \sum_{j=1}^{\infty} \frac{\pi_{j}}{j^{2}}\right]
$$

and a similar more complex formula for the fourth moment.
Knessl and Young [20] (p. 217) give a representation of the density of $D_{c}$ as a series $\sum_{i=1}^{\infty} c_{i} \exp \left(-z_{i} t / \mu\right)$ where $z_{i}$ 's are the (positive) roots of $M(-z, c+1-z, \rho)=0$. For example, for $c=0, \theta=\mu=1$ this gives the leading exponential term of the order $\exp \left(-z_{1} t\right)$ with $z_{1}=0.450 \ldots$, as compared with $\exp (-t)$ tail of the service time.

For $X(0)=c+1$ the first passage time to 0 can be represented as $\sum_{j=0}^{c} D_{j}$ with independent $D_{j}$. From (22) and tail asymptotics of the Poisson distribution (cf [10], Corollary 1 (ii)) for large $c$ we have

$$
\mathbb{E}\left[D_{c}\right] \sim \frac{\pi_{c+1}}{\theta \pi_{c}}=\frac{1}{\mu(c+1)}
$$

which gives

$$
\mathbb{E}\left[\sum_{j=0}^{c} D_{j}\right] \sim \frac{\log c}{\mu}, \quad c \rightarrow \infty
$$

Robert [30] (Proposition 6.8) employs the Laplace transform to show that this asymptotics also holds in probability.

See $[20,30]$ and references therein for asymptotic results in the heavy traffic limit $\rho \rightarrow \infty$.

## 8. Multivariate excursions from the zero state

A path segment of $\boldsymbol{C}_{k}(\cdot)$ that starts with $(1,0, \ldots, 0)$ and terminates upon reaching the zero state $(0, \ldots, 0)$ is analogous to a busy period of a tandem of $\mathrm{M} / \mathrm{M} / \infty$ queues with $k$ phases $S_{1}, \ldots, S_{k}$. To study the basic characteristics of such multivariate excursion it is enough to follow the total

$$
Y(t):=C_{1}(t)+\cdots+C_{k}(t)
$$

which itself is the occupancy process of a single-phase $M / G / \infty$ queue with Poisson arrival rate $\theta$, and the generic service time $\sigma$ having distribution function

$$
\begin{equation*}
\mathbb{P}[\sigma \leq t]=\left(1-e^{-t}\right)^{k} \tag{27}
\end{equation*}
$$

Indeed, $\sigma$ is distributed like a sum of exponential variables $\xi_{1} / 1+\cdots+\xi_{k} / k$, as in $(20)$, which is the time that a cycle needs to pass through $S_{1}, \ldots, S_{k}$.

The definition of the busy period (excursion above 0 ) for $\mathrm{M} / \mathrm{G} / \infty$ requires some care, because the process is not Markovian and the periods spent by present tasks in service cannot be ignored hence must be included in description of the state [33]. With this in mind, the excursion starting at time $t_{0}$ is defined under the assumption that $Y\left(t_{0}-\right)=0$ and $Y\left(t_{0}\right)=1$. In this section we denote by $D_{0}, H_{0}, A_{0}, \Delta_{0}$ the duration, height, overflow and the number of new arrivals during the busy period of $Y(\cdot)$.

Let $\rho:=\theta \mathbb{E}[\sigma]=\theta h_{k}$. The steady state distribution is $\operatorname{Poisson}(\rho)$, and formulas (21), (22) and (23) with $c=0$ apply without change. In particular,

$$
\mathbb{E}\left[D_{0}\right]=\frac{e^{\rho}-1}{\theta}
$$

Recall a concept from the renewal theory. For a nonnegative integrable random variable $\eta$, representing the generic inter-arrival time, the variable $\eta^{*}$ with the integrated tail distribution

$$
\mathbb{P}\left[\eta^{*} \leq t\right]=\frac{1}{\mathbb{E}[\eta]} \int_{0}^{t} \mathbb{P}[\eta>x] \mathrm{d} x
$$

appears as the stationary residual lifetime. In terms of their Laplace transforms, the relationship between $\eta$ and $\eta^{*}$ is

$$
\begin{equation*}
\mathbb{E}\left[\exp \left(-z \eta^{*}\right)\right]=\frac{1-\mathbb{E}[\exp (-z \eta)]}{z \mathbb{E}[\eta]} \tag{28}
\end{equation*}
$$

We shall use this connection of $\sigma$ and $D_{0}$ to their associated variables $\sigma^{*}$ and $D_{0}^{*}$.
The transient state distribution $\pi_{c}(t):=\mathbb{P}[Y(t)=c \mid Y(0)=0]$ is Poisson with mean $\rho \mathbb{P}\left[\sigma^{*} \leq t\right]$, hence in particular

$$
\pi_{0}(t)=\exp \left\{-\rho \mathbb{P}\left[\sigma^{*} \leq t\right]\right\}
$$

From (27) one finds readily that $\sigma$ and $\sigma^{*}$ both have exponential tails: as $t \rightarrow \infty$

$$
\begin{equation*}
\mathbb{P}[\sigma>t] \sim k e^{-t}, \quad \mathbb{P}\left[\sigma^{*}>t\right] \sim \frac{k}{h_{k}} e^{-t} \tag{29}
\end{equation*}
$$

which implies that

$$
\pi_{0}(t)-\pi_{0}=\exp \left\{-\rho \mathbb{P}\left[\sigma^{*} \leq t\right]\right\}-e^{-\rho}=e^{-\rho}\left(\exp \left\{\rho \mathbb{P}\left[\sigma^{*}>t\right]\right\}-1\right) \sim \theta k e^{-\rho} e^{-t}
$$

where $\pi_{0}=\lim _{t \rightarrow \infty} \pi_{0}(t)=e^{-\rho}$.
The Laplace transform of the duration is given by

$$
\begin{equation*}
\mathbb{E}\left[\exp \left(-z D_{0}\right)\right]=1+\frac{z}{\theta}-\frac{z}{\theta L(z)} \tag{30}
\end{equation*}
$$

where

$$
\begin{equation*}
L(z)=1+\int_{0}^{\infty} e^{-z t} \pi_{0}^{\prime}(t) \mathrm{d} t \tag{31}
\end{equation*}
$$

Equation (30) is a version of the Takàcs formula ([37], Equation (2) on p. 210) for the Laplace transform of the time between beginnings of two successive busy periods. In [37] and subsequent work (e.g. Equation (5) in [32], Equation (4.6) in [21]) the authors use $L(z) / z$, which is the Laplace transform of $\pi_{0}(t)$. The form (31) is better suitable for our purpose since $L(z)$ is holomorphic in a larger halfplane $\Re z>-1$, as dictated by the asymptotics $\left|\pi_{0}^{\prime}(t)\right|=O\left(e^{-t}\right)$ for $t \rightarrow \infty$.

The second moment of the duration was derived from (31) in Liu and Shi [21] (Equation (4.13)) as

$$
\mathbb{E}\left[D_{0}^{2}\right]=\frac{2}{\theta \pi_{0}^{2}} \int_{0}^{\infty}\left(\pi_{0}(t)-\pi_{0}\right) \mathrm{d} t
$$

For $k=1$ this has a series representation (26) but for $k>1$ there does not seem to exist a simple analogue. To compare the numerics, for $\theta=1$ we get $\operatorname{Var}\left[D_{0}\right]$ about 12.7921 for $k=2$ and about 4.2123 for $k=1$. The joint Laplace transform of $D_{0}$ and $\Delta_{0}$ is found in Shanbhag [35] (Theorem 2).

We turn next to the counterpart of (29) for the duration of excursion above zero. To that end, designate

$$
F(t):=\mathbb{P}\left[D_{0} \leq t\right], \quad f^{*}(t):=\mathbb{P}\left[D_{0}^{*} \in \mathrm{~d} t\right] / \mathrm{d} t
$$

which are the distribution function of $D_{0}$ and the density function of $D_{0}^{*}$, respectively. These are related via

$$
\begin{equation*}
f^{*}(t)=\frac{1-F(t)}{\mathbb{E}\left[D_{0}\right]} \tag{32}
\end{equation*}
$$

The function $L(z)$ increases from $-\infty$ to $e^{-\rho}$ as $z$ runs from -1 to 0 , therefore there exists a unique $\beta \in(0,1)$ satisfying $L(-\beta)=0$ For instance, $\beta=0.2734 \ldots$ if $\theta=1, k=2$.

Theorem 4. As $t \rightarrow \infty$, it holds that

$$
\begin{equation*}
1-F(t) \sim \alpha e^{-\beta} \tag{33}
\end{equation*}
$$

where

$$
\alpha:=-\left(\theta \int_{0}^{\infty} e^{\beta t} t \pi_{0}^{\prime}(t) \mathrm{d} t\right)^{-1}
$$

Proof. We shall apply a result from the renewal theory. Following Makowski [23], the Takàcs formula (30) amounts to the representation of $D_{0}^{*}$ as a geometric sum

$$
D_{0}^{*} \stackrel{d}{=} \sum_{j=1}^{Q} U_{j}
$$

where all variables involved are independent, $Q$ has the geometric distribution

$$
\mathbb{P}[Q=j]=\pi_{0}\left(1-\pi_{0}\right)^{j-1}, \quad j=1,2, \ldots
$$

and the $U_{j}$ 's are i.i.d. with density

$$
u(t):=\frac{-\pi_{0}^{\prime}(t)}{1-\pi_{0}}
$$

Conditioning on $U_{1}$ we arrive at the improper renewal equation

$$
f^{*}(t)=\pi_{0} u(t)+\left(1-\pi_{0}\right) \int_{0}^{t} f^{*}(t-s) u(t) \mathrm{d} t
$$

with substochastic density $\left(1-\pi_{0}\right) u(t)$. Adopting a formula from Resnick [29] (page 258, bottom equation
where $z(\infty)=Z(\infty)=0$ should be set due to $\lim _{t \rightarrow \infty} \pi_{0}^{\prime}(t)=0$ ) we have

$$
f^{*}(t) \sim \alpha^{*} e^{-\beta t}
$$

with $\beta \in(0,1)$ as above solving $L(-\beta)=0(c f[29]$, Proposition 3.11.1) and

$$
\begin{aligned}
\alpha^{*} & =\frac{\pi_{0} \int_{0}^{\infty} e^{\beta x} u(x) \mathrm{d} x}{\left(1-\pi_{0}\right) \int_{0}^{\infty} x e^{\beta x} u(t) \mathrm{d} t} \\
& =\frac{\alpha \theta}{e^{\rho}-1}
\end{aligned}
$$

The assertion now follows by the virtue of (32).

An alternative approach is the following. Using (28) we have

$$
\begin{equation*}
\mathbb{E}\left[\exp \left(-z D_{0}^{*}\right)\right]=\int_{0}^{\infty} e^{-z t} f^{*}(t) \mathrm{d} t=\left(e^{\rho}-1\right)^{-1}\left(\frac{1}{L(z)}-1\right) \tag{34}
\end{equation*}
$$

From this the exponential tail asymptotics can be concluded by singularity analysis of the Laplace transform. Indeed, with $\Re z$ fixed, $|L(z)-1|$ is maximised for $\Im z=0$, hence and by monotonicity $L(z) \neq 0$ if $\Re z>-\beta$. On the other hand, by a property of the Laplace transform $|L(z)-1| \rightarrow 0$ as $|z| \rightarrow \infty$ uniformly in $\Re z>-\beta-\varepsilon(c f[8]$, Theorem 23.6). Thus for $\varepsilon>0$ sufficiently small, $L(z)$ has no zeros in this halfplane other than $-\beta$, hence the only singularity of (34) in the halfplane is a simple pole at $-\beta$, with residue readily identified with $\alpha^{*}$. From this (33) follows by writing $f^{*}(t)$ in the form of the inverse Laplace transform of (34), then moving the contour of integration to $\Re z=-\beta-\varepsilon$, see [8] (Section 35) for this classic technique.

Note that $\alpha$ is the residue of (30) at pole $-\beta$, but using (30) directly to justify the tail asymptotics (of the density of $D_{0}$ ) looks more difficult due to the factor $z$.

## 9. The embedded tagged cycle process

Suppose element $n$ starts a new cycle of the CRP permutation, with some number $L_{1}^{(n)}$ of singletons already present in $\Pi^{(n-1)}$, that is $L_{1}^{(n)}=C_{1}^{(n-1)}$. Let $L_{2}^{(n)}$ be the number of doubletons present immediately before this cycle moves to $S_{2}$, etc. Intuitively, $L_{k}^{(n)}$ is what an observer moving with the tagged cycle spots in $S_{k}$ when entering the phase. As $n \rightarrow \infty$ the distribution of $L_{1}^{(n)}, L_{2}^{(n)}, \ldots$ converges to a limit which has Poisson marginals as in (6) but they are not independent. It seems hard to capture features of the limit multivariate distribution without turning to the embedding of CRP in continuous time. Fortunately, a major work has been done by the queueing theorists.

To set a general scene, consider a tandem of $\mathrm{M} / \mathrm{M} / \infty$ queues with arrival rate $\theta$ and sojourn parameter $\mu_{k}$ for phase $S_{k}$, and let $\rho_{k}:=\theta / \mu_{k}$. Assuming the system in steady state and that there is a tagged arrival at time 0 , let $L_{k}$ be the occupancy of $S_{k}$ immediately before the tagged item enters $S_{k}$. The following result was obtained by Vainstein and Kreinin [39] and extended by Boxma [7] to tandems of M/G/ $\infty$ queues with arbitrary sojourn times. As above, $\xi_{1}, \xi_{2}, \ldots$ denote independent unit exponential random variables.

Theorem 5. The joint p.g.f. of $L_{j}$ and $L_{k}$ for $1 \leq j<k$ is

$$
\begin{equation*}
\mathbb{E}\left[x^{L_{j}} y^{L_{k}}\right]=\exp \left\{\rho_{j}(x-1)+\rho_{k}(y-1)\right\} \int_{0}^{\infty} \exp \left\{\rho_{j}(x-1)(y-1) \varphi_{j k}(t)\right\} \mathrm{d} \psi_{j k}(t) \tag{35}
\end{equation*}
$$

where

$$
\varphi_{j k}(t):=\mathbb{P}\left[\frac{\xi_{j}}{\mu_{j}}+\cdots+\frac{\xi_{k-1}}{\mu_{k-1}}<t<\frac{\xi_{j}}{\mu_{j}}+\cdots+\frac{\xi_{k}}{\mu_{k}}\right], \quad \psi_{j k}(t):=\mathbb{P}\left[\frac{\xi_{j}}{\mu_{j}}+\cdots+\frac{\xi_{k-1}}{\mu_{k-1}}<t\right] .
$$

Proof. By the steady-state assumption, the flow from $S_{j-1}$ to $S_{j}$ is Poisson, hence we will not lose generality by considering the case $j=1$ only. We adapt the more general argument from [7] (Theorems 2.2 and 3.1) to the $\mathrm{M} / \mathrm{M} / \infty$ tandem. We have $L_{1} \stackrel{d}{=} \operatorname{Poiss}\left(\rho_{1}\right)$, and the time, say $T$, that the tagged arrival to $S_{1}$ needs to reach $S_{k}$ has distribution function $\mathbb{P}[T \leq t]=\psi_{1 k}(t)$. For $1 \leq i \leq k$, let $N_{i}$ be the number of items in $S_{k}$ at time $T$ that were in $S_{i}$ at time 0 , and let $N_{0}$ be the number of items in $S_{k}$ at time $T$ that were not yet present in the system at time 0 . Clearly, $L_{k}=N_{0}+N_{1}+\cdots+N_{k}$. Given the tagged item finds $L_{1}=\ell_{1}$ and needs time $T=t$, the variables $N_{0}, \ldots, N_{k}$ are conditionally independent,

$$
N_{0} \stackrel{d}{=} \operatorname{Poiss}\left(\theta p_{0}\right), \quad N_{1} \stackrel{d}{=} \operatorname{Bin}\left(\ell_{1}, p_{1}\right) \quad \text { and } \quad N_{i} \stackrel{d}{=} \operatorname{Poiss}\left(\rho_{i} p_{i}\right), 2 \leq i \leq k
$$

Here,

$$
p_{0}=\int_{0}^{t} \varphi_{1 k}(x) \mathrm{d} x
$$

and $p_{i}$ for $1 \leq i \leq k$ is the probability that the generic item from $S_{i}$ is located in $S_{k}$ over time $t$, i.e.

$$
p_{i}=\mathbb{P}\left[\sum_{m=i}^{k-1} \frac{\xi_{m}}{\mu_{m}}<t<\sum_{m=i}^{k} \frac{\xi_{m}}{\mu_{m}}\right] .
$$

Note that $p_{1}=\varphi_{1 k}(t)$. The steady-state balance equation for the mean content of $S_{k}$ is

$$
\theta p_{0}+\rho_{1} p_{1}+\cdots+\rho_{k} p_{k}=\rho_{k}
$$

which allows us to write $\theta p_{0}+\rho_{2} p_{2}+\cdots+\rho_{k} p_{k}=\rho_{k}-\rho_{1} p_{1}$, and together with the above conclude that the conditional distribution of $L_{k}$ is the convolution

$$
\operatorname{Bin}\left(\ell_{1}, p_{1}\right) * \operatorname{Poiss}\left(\rho_{k}-\rho_{1} p_{1}\right)
$$

whence

$$
\mathbb{E}\left[x^{L_{1}} y^{L_{k}} \mid L_{1}=\ell_{1}, T=t\right]=x^{\ell_{1}}\{1-\varphi(t)+\varphi(t) y\}^{\ell_{1}} \exp \left\{\left(\rho_{k}-\rho_{1} \varphi(t)\right)(y-1)\right\}
$$

The result now follows by integrating out $\ell_{1}$ and $t$.

From (35) follows that (or see [7], Equation (3.4))

$$
\begin{equation*}
\operatorname{cov}\left(L_{j}, L_{k}\right)=\rho_{j} \int_{0}^{\infty} \varphi_{j k}(t) \mathrm{d} \psi_{j k}(t), \quad \operatorname{corr}\left(L_{j}, L_{k}\right)=\sqrt{\frac{\mu_{k}}{\mu_{j}}} \int_{0}^{\infty} \varphi_{j k}(t) \mathrm{d} \psi_{j k}(t) \tag{36}
\end{equation*}
$$

Vainshtein and Kreinin [40] observed that

$$
\begin{equation*}
\operatorname{corr}\left(L_{j}, L_{k}\right)=\frac{1}{2 \sqrt{\mu_{j} \mu_{k}}} \mathcal{L}(0) \tag{37}
\end{equation*}
$$

where $\mathcal{L}(\cdot)$ is the Lagrange polynomial interpolating the square root function from the data set

$$
\left(\mu_{j}^{2}, \mu_{j}\right), \ldots,\left(\mu_{k}^{2}, \mu_{k}\right)
$$

Remarkably, the correlation coefficient does not depend on $\theta$.
The case relevant to permutations

$$
\begin{equation*}
\mu_{k}=k \tag{38}
\end{equation*}
$$

will be worked out in the rest of this section. Using (37) Vainstein and Kreinin ([40], Equation (15)) evaluated (36) as

$$
\operatorname{corr}\left(L_{1}, L_{k}\right)=\frac{\sqrt{k}}{2(2 k-1)}
$$

We take a different approach which works smoothly for all $j$ but is limited to (38) (or constant multiples of (38)). Let $\xi_{j: k}, 1 \leq j \leq k$, denote the $j$ th maximal order statistic among the first $k$ exponential variables. Using Renyi's representation we have the identities

$$
\begin{aligned}
\psi_{j k}(t) & =\mathbb{P}\left[\xi_{j: k-1}<t\right] \\
\varphi_{j k}(t) & =\mathbb{P}\left[\xi_{j: k-1}<t\right]-\mathbb{P}\left[\xi_{j: k}<t\right]=\mathbb{P}\left[\xi_{j: k-1}<t<\xi_{j-1: k-1}\right] \mathbb{P}\left[\xi_{k}>t\right]
\end{aligned}
$$

where the last equality follows from the events coincidence

$$
\left\{\xi_{j: k-1}<t, \xi_{j: k} \geq t\right\}=\left\{\xi_{j: k-1}<t \leq \xi_{j-1: k-1}, \xi_{k} \geq t\right\}
$$

To express (36) via a beta integral we pass to the uniform order statistics, thus obtaining

$$
\begin{aligned}
\varphi_{j k}(-\log (1-x)) & =\binom{k-1}{j-1} x^{k-j}(1-x)^{j} \\
\mathrm{~d} \psi_{j k}(-\log (1-x)) & =(k-1)\binom{k-2}{j-1} x^{k-j-1}(1-x)^{j-1} \mathrm{~d} x
\end{aligned}
$$

whence (36) for rates (38) becomes

$$
\begin{aligned}
\operatorname{cov}\left(L_{j}, L_{k}\right) & =\theta\binom{k-1}{j}\binom{k-1}{j-1} \frac{(2 k-2 j-1)!(2 j-1)!}{(2 k-1)!} \\
\operatorname{corr}\left(L_{j}, L_{k}\right) & =\sqrt{j k}\binom{k-1}{j}\binom{k-1}{j-1} \frac{(2 k-2 j-1)!(2 j-1)!}{(2 k-1)!}
\end{aligned}
$$

Interestingly, the covariance has some symmetry, $\operatorname{cov}\left(L_{j}, L_{k}\right)=\operatorname{cov}\left(L_{k-j}, L_{k}\right)$.

Since $\varphi_{j k}(t)=\psi_{j, k}(t)-\psi_{j, k+1}(t),(36)$ implies an estimate

$$
\operatorname{corr}\left(L_{j}, L_{k}\right)<\frac{1}{2} \sqrt{\frac{j}{k}}
$$

which gives the correct decay order $k^{-1 / 2}$ of the correlation as $k \rightarrow \infty$ and $j$ is fixed.

## 10. A functional limit for the small cycle counts

Finally, we argue that $\left(\boldsymbol{C}_{k}(t), t \geq 0\right)$ appears as a weak limit of $\left(\boldsymbol{C}_{k}^{(n)}, n \geq 0\right)$ by the virtue of a nonlinear time-scale change. To that end, we interpolate the discrete time Markov chain to a piecewise constant jump process with real time parameter.

Theorem 6. Let $\boldsymbol{C}_{k}(\cdot)$ start at time 0 in some random state $\boldsymbol{C}_{k}(0)$, and let $\boldsymbol{C}_{k}^{(\cdot)}$ start at time $\nu$ in some random state $\boldsymbol{C}_{k}^{(\nu)}$. If $\boldsymbol{C}_{k}^{(\nu)}$ converges in distribution to $\boldsymbol{C}_{k}(0)$ as $\nu \rightarrow \infty$ then also

$$
\left(\boldsymbol{C}_{k}^{\left(\nu e^{t}\right)}, t \geq 0\right) \Rightarrow\left(\boldsymbol{C}_{k}(t), t \geq 0\right)
$$

where $\Rightarrow$ means weak convergence in the Skorohod space $D[0, \infty)$.
To ease notation let $X_{\nu}(t)=C_{k}^{\left(\nu e^{t}\right)}$. Since the state space is discrete, the assertion can be reduced to the case when the initial states are fixed and identical, that is $\boldsymbol{C}_{k}(0)=X_{\nu}(0)=\boldsymbol{c}(0)$. The embedded jump chains have the same transition probabilities, hence it is possible to couple the processes in such a way that they pass the same random sequence of states. Appealing to [41] (Lemma 2.12) shows that is suffices to verify that the sequence of consecutive sojourn times of $X_{\nu}(t)$, seen as a random element of $\mathbb{R}_{+}^{\infty}$, converges in distribution to the sequence of sojourn times of $\boldsymbol{C}_{k}(t)$. Given a path $\boldsymbol{c}(0), \boldsymbol{c}(1), \ldots$ of the jump chain, the sojourn times of $\boldsymbol{C}_{k}(\cdot)$ are independent exponential variables, with rates

$$
\begin{equation*}
r=\theta+\sum_{i=1}^{k} i c_{i} \tag{39}
\end{equation*}
$$

depending on $\boldsymbol{c} \in \mathbb{Z}_{+}^{k}$. The next lemma finds the limiting distribution of the sojourn time of $X_{\nu}(t)$ at an arbitrary state $\boldsymbol{c} \in \mathbb{Z}_{+}^{k}$.

Lemma 1. Given $X_{\nu}(t)=\boldsymbol{c}$ the residual sojourn time in this state converges in distribution to $\operatorname{Exp}(r)$, as $\nu \rightarrow \infty$.

Proof. Using (9), as $n \rightarrow \infty$ we obtain

$$
\mathbb{P}\left[\boldsymbol{C}_{k}^{(i)}=\boldsymbol{c}, n \leq i \leq m \mid \boldsymbol{C}_{k}^{(n)}=\boldsymbol{c}\right]=\prod_{j=n}^{m-1}\left(1-\frac{r}{j+\theta}\right)=\left(\frac{n}{m}\right)^{r}\left(1+O\left(\frac{1}{n}\right)\right)
$$

uniformly in $m \geq n$. Setting $n=\nu e^{t}, m=\nu e^{t+\delta}, \delta>0$, we conclude that $X_{\nu}(\cdot)$ spends in $\boldsymbol{c}$ some time exceding $\delta$ with probability $e^{-r \delta}+O\left(\nu^{-1}\right)$, hence the limit distribution is exponential as stated.

Let $r(0), r(1), \ldots$ be the rates for $\boldsymbol{c}(0), \boldsymbol{c}(1), \ldots$ defined by (39), and let $V_{\nu}(0), V_{\nu}(1), \ldots$ be the sojourn times that $X_{\nu}(\cdot)$ spends in these states. By the lemma, $V_{\nu}(0)$ converges in distribution to $\operatorname{Exp}(r(0))$. By the strong Markov property and because the estimate $O\left(\nu^{-1}\right)$ in the proof of the lemma is uniform in $t$, the conditional distribution of $V_{\nu}(1)$ given $V_{\nu}(0)$ converges to $\operatorname{Exp}(r(1))$. But then we also have the joint convergence of $\left(V_{\nu}(0), V_{\nu}(1)\right)$, as follows from [34] (Theorem 2). Continuing by induction, we obtain the joint convergence of the sojourn times $V_{\nu}(0), V_{\nu}(1), \ldots$ to the counterpart sequence of sojourn times of $\boldsymbol{C}_{k}(\cdot)$ and the weak convergence of $X_{\nu}(\cdot)$ follows.

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