Highly symmetric embeddings of graphs on surfaces

Thesis

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Highly symmetric embeddings of graphs on surfaces

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Abstract

This thesis considers highly symmetric maps, that is embeddings of graphs in surfaces such that the automorphism group is "large". This may be when the automorphism group of the map acts regularly on the flag-set of the map, as for the fully regular maps studied in Part I. In contrast, Part II focusses on a class of maps where the automorphism group has (up to) two orbits on the flag-set and may not be edge-transitive.

Part I is dedicated to advancing the understanding of fully regular maps with external symmetries. Chapter 2 proves that for arbitrary valency greater than three, a fully regular map with Trinity symmetry exists, extending the previously-known existence of such a map for every even valency. Chapter 3 addresses a group of operators which acts on fully regular maps whose automorphism group is isomorphic to $\mathrm{SL}(2, 2^{\alpha})$. The group of operators, which depends on the value of α and is defined more precisely in Chapter 3, includes the dual and Petrie operators as well as the allowable hole operators. One approach is by exploring the orbits of this group as it acts on the space of all maps with automorphism group isomorphic to $\mathrm{SL}(2, 2^{\alpha})$ for the given α. A detailed investigation is presented for the group of operators acting on the set consisting of all maps with automorphism group $\mathrm{A}_5 \cong \mathrm{SL}(2, 4)$.

In Part II, the focus is on edge-biregular maps. These maps can be identified with group presentations which have a particular form, namely they are generated by four involutions which partition into two distinct sets each consisting of a pair of commuting involutions. Edge-biregular maps correspond to the most symmetric examples of maps with bipartite medial graph. By the definition, each edge-biregular map inherits a two-colouring on the edges, and so long as the map is not degenerate in some way, both the valency and the face length are even. In Chapter 4 these maps are introduced, foundations are laid and degeneracies are addressed. Chapter 5 is a partial classification covering edge-biregular maps whose colour-preserving automorphism group is dihedral, and/or whose surface has Euler characteristic which is either non-negative or negative and prime. The context for Chapter 6 is edge-biregular maps whose underlying group is symmetric or alternating. A *genuinely* edge-biregular map is an edge-biregular map which (when disregarding the colouring of edges) is not a fully regular map. The chapter includes a proof that, with the exception of some small cases, a genuinely edge-biregular map of every feasible type exists such that the colour preserving automorphism group is symmetric or alternating.

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Dedication

In loving memory of Elizabeth Dowman

to whom I am forever grateful

for sponsoring me through my time

as a Mathematics undergraduate

at Oriel College, Oxford.

AND

To all those who have ever inspired and encouraged me:

... family, friends, mentors...

Thank you!!!

Contents

Chapter 1

Introduction

1.1 Definitions and assumptions

This thesis concerns highly symmetric maps, and here we present definitions which are common in the field, and state the underlying assumptions which we make for the majority of the following work. We assume a certain degree of familiarity with common terminology in algebraic and topological graph theory, abstract algebra and basic topology. Important definitions will be given in the text, and for ease of reference Appendix B is a glossary while there is a list of notation in Appendix C.

Unless stated otherwise, we make the following assumptions:

> Graphs are connected and contain no semi-edges.
>
> Surfaces are compact, connected 2-manifolds without boundary.
>
> Maps and their automorphism groups are finite.

A *map*, $\mathcal{M}$, is a cellular embedding of a connected graph, Γ, on a surface, $\mathcal{S}$. The embedding is such that removal of the image of Γ splits up the surface $\mathcal{S}$ into disjoint regions, each of which is homeomorphic to an open disc. Each of the resulting regions corresponds to the interior of a face, and thus the map consists of vertices, edges and faces. Incidence of a pair of these objects (vertices, edges and faces) is defined by non-empty intersection, and the surface $\mathcal{S}$ is called the *supporting surface*.

Each *flag* is a vertex-edge-face triple, where the vertex, edge and face are pairwise incident, and can be thought of as a triangle drawn on the surface, with its corners at the corresponding vertex, the midpoint of the edge and the centre of the face. There are *degeneracies* when a flag cannot be determined by such a triple. Each flag corresponds to, and may be defined as, a face of the barycentric subdivision of the map. We say that two flags are *adjacent* when they share an edge in that barycentric subdivision of the map, so each flag is adjacent to at most three other flags. See Figure 1.1 which shows part of a map whose edges are drawn in bold, and the surrounding flags, one of which is highlighted. The dots in the diagram indicate the possible presence of further vertices, edges and faces. Note that any given edge is

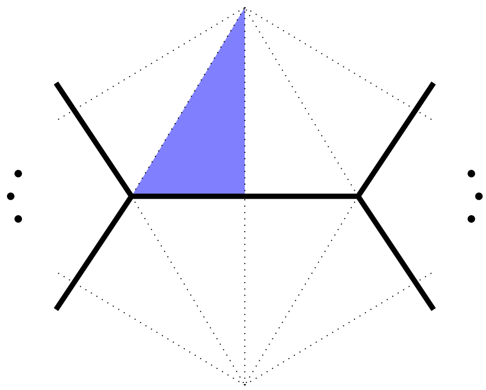

Figure 1.1: Part of a map showing one edge and four corners with the corresponding flags, one of which is shaded.

incident to exactly four flags, and the set of flags covers the surface.

An *automorphism* of a map is a permutation of the set of flags, preserving all the types of adjacencies between flags, and so retaining the structure of the map. If two flags in a map are adjacent "along an edge", "across an edge" or "across a corner", then their images under a given automorphism will also have the same type of adjacency, that is the images will be adjacent respectively along an edge, across an edge or across a corner. An automorphism can be thought of as a symmetry of the map and the set of automorphisms form a group, G, under composition in the natural way.

A map is called a *fully regular map* when the automorphism group G acts both semi-regularly and also transitively on the flag set. A consequence of this is that, in a fully regular map, all the vertices must have the same degree, k, and all faces have boundary walks of the same length, ℓ. The map is then said to have *type* (k, ℓ), and the automorphism group G is referred to as the *underlying group*.

A fully regular map of type (k, ℓ) is spherical, Euclidean, or hyperbolic according to whether $\frac{1}{k} + \frac{1}{\ell}$ is greater than, equal to or less than a half. This relates to the Euler-Poincaré formula, and corresponds to the supporting surface for the map having respectively positive, zero, or negative Euler characteristic, and the regular tessellation of the corresponding type is supported by, respectively, the sphere, the Euclidean plane or the hyperbolic plane, see [21].

The Platonic solids are examples of regular maps where the supporting surface is the sphere. This same surface also supports embeddings of the n-cycle and the n-regular dipole to give two further (infinite) families of spherical regular maps. For completeness of the general theory one needs to consider maps in which semi-edges appear. Then the embedding of an *n-semi-star*, that is a graph consisting of one vertex and n semi-edges, also yields a regular map on the sphere. It is well-known that including the resulting family completes the list of regular maps on the sphere.

1.2 History and context

As already mentioned, Platonic solids are examples of regular maps, and as such highly symmetric maps have been appreciated for thousands of years. Non-spherical examples of regular maps occur in the work of Kepler in the 17th century, where they are presented as stellated polyhedra. The study of maps gained attention in the late 19th century with Heawood's conjecture which ignited interest in the field now known as topological graph theory. When finally proved much later, Heawood's conjecture became the famous Map Colour Theorem. More recently, in the middle half of the 20th century, symmetric maps were studied by Brahana [6, 7], Coxeter and Moser [21], Threlfall [61], Sah [56], and others. In early references to "regular" maps, often the surface is orientable and the group considered is generated by rotations and so consists only of orientation-preserving automorphisms. In this work the phrase "fully regular map" is sometimes abbreviated to "regular map" and *always* refers to a map in which the automorphism group acts transitively on the flag set.

The modern interest and study of symmetric maps is thanks in large part to Jones and Singerman, who in [42], presented the algebraic and topological theory of maps on orientable surfaces. Shortly afterwards this theory was extended by Bryant and Singerman to address non-orientability and surfaces with boundary in [10]. They demonstrate a correspondence between a symmetric map and its underlying group, thus allowing study of the properties of regular maps by using group theory. Section 1.3 presents an overview of the topic without referencing individual items. For further details about the theory of symmetric maps see [7, 10, 42, 46, 56, 57].

Embeddings of graphs on surfaces can be studied from different standpoints. The most common three approaches focus on one of the following: the underlying group; the supporting surface; the graph. For each case one may consider whether and when it is possible to build maps which have certain properties. For example, if a group can be generated by three involutions, two of which commute, then this necessarily yields a fully regular map, and so studying such group presentations is one way of studying fully regular maps. In this work I will focus on the first two approaches.

Huge advances in the field came with computer-based searches, notably by Conder and Dobcsányi [14] who created a list of all fully regular maps of orientable genus 2 to 15, and non-orientable genus 4 to 30. This work has been extensively expanded up to genus 101 (or 202 depending on orientability) and beyond by Conder and is listed online at [11].

In this thesis I take what is to me an intuitive, often naive and elementary "by hand" approach, only occasionally referring to a computer to check things using GAP [27]. There are also places where I cannot resist including a record of my own personal motivations, hopes, reflections, disappointment or frustration, and indeed delight at some of the surprises.

1.3 Fully regular maps and their automorphism groups

The group G of automorphisms of a regular map $\mathcal{M}$ is generated by three involutions, two of which commute. Each of the three involutions can be thought of as a local reflection in one of the boundary components of a given distinguished flag, and so preserves all the adjacency relationships between flags. As shown in Figure 1.2 the involutions act locally on the given (shaded) flag as follows: r_0 as a reflection in the edge bisector; r_2 as a reflection across the edge; r_1 as a reflection in the angle bisector at the vertex. The dots on the diagram indicate where there may be further vertices, edges and faces while the dashed lines outline each of the flags of this part of the map.

The subscripts in the notation are natural in the sense that, in Figure 1.2, the involutions r_i act locally as reflections in the sides of a given flag, where r_i changes the i-dimensional component of the flag's vertex-edge-face triple, leaving the other two components of the triple unchanged. This introduction is all with reference to a particular distinguished flag, and as such we say that the map is *rooted*.

The study of regular maps is thus equivalent to the study of group presentations of

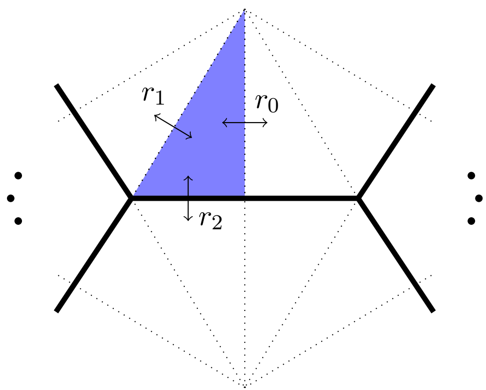

Figure 1.2: The action of automorphisms r_0, r_1 and r_2 on the shaded flag

the form

$$G = Aut(\mathcal{M}) = \langle\, r_0, r_1, r_2 \mid r_0^2, r_1^2, r_2^2, (r_0r_2)^2, (r_1r_2)^k, (r_0r_1)^\ell, \dots \rangle \tag{1.1}$$

where k is the vertex degree, ℓ the face length, and the dots indicate the potential for further relators equivalent to the identity which are not listed. We assume also that the orders shown are indeed the true orders of those elements in the group. As such, the group $G = Aut(\mathcal{M})$ is isomorphic to a smooth quotient of $T_{k,\ell}$, the *full triangle group of type* (k, ℓ) with presentation
$T_{k,\ell} = \langle\, R_0, R_2, R_1 \mid R_0^2, R_2^2, R_1^2, (R_0R_2)^2, (R_1R_2)^k, (R_0R_1)^\ell \,\rangle$, by some torsion-free normal subgroup. Occasional reference will be made to the "parent" map which is associated with the corresponding full triangle group. For the most part, the parent map is not finite and is supported by a non-compact, simply connected surface, namely the Euclidean or hyperbolic plane.

The *maps* $\mathcal{M}$ *and* $\mathcal{M}'$ *are isomorphic* to each other if and only if they have respective automorphism group presentations

$$G = Aut(\mathcal{M}) = \langle\, r_0, r_1, r_2 \mid r_0^2, r_1^2, r_2^2, (r_0r_2)^2, (r_1r_2)^k, (r_0r_1)^\ell \dots \rangle \text{ and}$$

$$G' = Aut(\mathcal{M}') = \langle\, r_0', r_1', r_2' \mid {r_0'}^2, {r_1'}^2, {r_2'}^2, (r_0'r_2')^2, (r_1'r_2')^{k'}, (r_0'r_1')^{\ell'}, \dots \rangle,$$

and there is a group isomorphism from G to G' which sends $r_i \to r_i'$ for each
$i \in \{0, 1, 2\}$.

For ease of reading, typing and reference, we immediately define a new (slightly less intuitive, but nevertheless more practical) notation according to the following dictionary: $x := r_0$, $y := r_2$, $z := r_1$. Using this, a fully regular map will be denoted by $\mathcal{M} = (G; x, y, z)$ where G has group presentation of the following form:
$G \cong \langle\, x, y, z, \mid x^2, y^2, z^2, (yz)^k, (zx)^\ell, (xy)^2, \dots \rangle$.

An alternative approach to looking at a map by means of studying its underlying group(s) is to consider the monodromy group which acts on the flag set as follows. The three generators are $\mathcal{R}_0$, $\mathcal{R}_1$ and $\mathcal{R}_2$, each of which is an involutory permutation of the flag set such that two flags are transposed by $\mathcal{R}_i$ if and only if the two flags are adjacent and differ only in the i-dimensional part of the vertex-edge-face triple. The monodromy group may then be used to construct a map from a set of flags and three involutory permutations of the flag set as described. Since they encode the adjacencies between the flags, the involutions $\mathcal{R}_0$, $\mathcal{R}_1$ and $\mathcal{R}_2$ provide a set of "gluing instructions" with which to build the map from the flag set. The orbits of the flag set under $\langle\mathcal{R}_0, \mathcal{R}_2\rangle$, $\langle\mathcal{R}_1, \mathcal{R}_2\rangle$, and $\langle\mathcal{R}_0, \mathcal{R}_1\rangle$, correspond respectively to (the flags associated with) each edge, vertex and face, and incidence of a pair of these objects is defined by non-empty intersection of the relevant orbits. In the case of fully regular maps, the monodromy group is isomorphic to the automorphism group.

1.4 Operators acting on regular maps, and external symmetries

Given a regular map $\mathcal{M}$, where $\mathcal{M}$ has underlying graph Γ, and supporting surface $\mathcal{S}$, there are certain operations one can do which will generate other regular maps. We list some such operators in what follows.

Every regular map has an associated *dual map* which is also a regular map. Informally, the dual map is created by forming a vertex at the centre of each original face and considering each of the original vertices as the centre of a face. Each edge of the dual map is thereby formed by linking a pair of neighbouring (new) vertices across one of the original edges. The concept of the (geometric) dual of an embedded graph is well-known [29, 50, 63], and essentially interchanges the roles of the vertices and faces of the map $\mathcal{M}$. By construction the resulting map $\mathcal{M}\mathbf{D}$ is embedded in $\mathcal{S}$, the same surface as $\mathcal{M}$, but $\mathcal{M}\mathbf{D}$ may have underlying graph different from Γ. An example of a dual pair of maps is the cube and the octahedron, both of which are embedded in the sphere. The cube has eight vertices and six faces, while the octahedron has eight faces and six vertices.

Given a graph embedded on a surface one can also find its Petrie map. Again informally, the Petrie walk for an orientable map is as follows: trace along edges of the map, turning left at the first vertex, and right at the next, repeating this left, right "zigzag" pattern for the length of the walk. See Figure 1.3 for an example of a Petrie polygon, that is a closed Petrie walk, on the dodecahedron.

The *Petrie map* of a map $\mathcal{M}$, can be formed by embedding Γ into a surface such that the Petrie polygons resulting from the original map now form the face boundaries of the new map $\mathcal{M}\mathbf{P}$. Petrie polygons are described in [21], while a comprehensive example of the action of the Petrie operator on a map appears in [65]. While the Petrie map $\mathcal{M}\mathbf{P}$ has the same edges and vertices as the original map $\mathcal{M}$, the faces are different. That is, the underlying graph Γ is the same, but the embedding is different, and hence the supporting surface for the Petrie map may also be different from $\mathcal{S}$.

In a fully regular map $\mathcal{M}$, the automorphism group acts regularly on the set of flags, and in some sense the object is as symmetric as possible. In fact there may be further, more subtle, symmetries, namely external symmetries, relating to operations

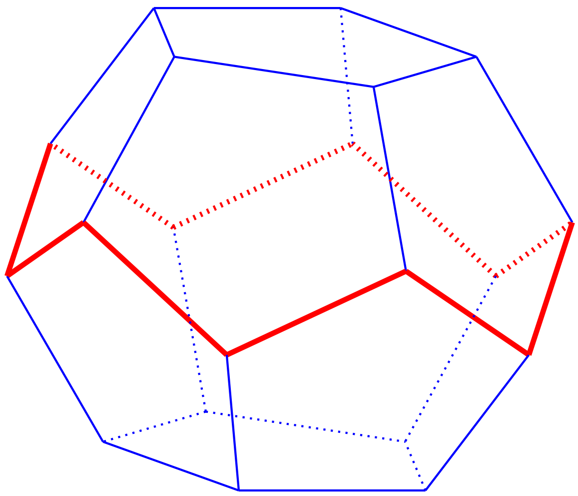

Figure 1.3: A Petrie polygon shown in red on a dodecahedron

which themselves preserve both the edge stabiliser $\langle x, y\rangle$ and the automorphism group G of the fully regular map. Note that an operator does not necessarily induce an isomorphism of the map. The clearest example is a map of type (k, ℓ) whose dual is of type (ℓ, k): when $k \neq \ell$ the dual operator yields a fully regular map which is certainly not isomorphic to the original. However, when one of the above-described operators $\mathbf{T}$ yields a map which is isomorphic to the original fully regular map, that is when $\mathcal{M}\mathbf{T} \cong \mathcal{M}$, we say the map $\mathcal{M}$ has an *external symmetry*. The *external symmetry group Ext*$(\mathcal{M})$ contains the operators which are map isomorphisms for the given map $\mathcal{M}$, and the group operation is composition.

When the associated dual or Petrie map is isomorphic to the original map $\mathcal{M}$, we call the map $\mathcal{M}$ *self-dual* or *self-Petrie* respectively. The work in Chapter 2 explores necessary and sufficient conditions for a regular map to have each of these properties when the automorphism group $G \cong \mathrm{PSL}(2, q)$ or $G \cong \mathrm{PGL}(2, q)$, for odd prime power q. If a map is both self-dual and also self-Petrie, it is referred to as having *Trinity symmetry*. This phrase surely refers to the equivalences between the three involutions within the edge stabiliser $\langle r_0, r_2\rangle = \langle x, y\rangle$, and is so called by strong preference of the late Dan Archdeacon [58]. I have adopted this terminology because I like it. The existence of such maps for odd valency is established in Chapter 2.

The best known example of external symmetry in a map is the tetrahedron, a Platonic solid which is self-dual.

Example 1.1. The tetrahedron $\mathcal{T}$: Being a Platonic solid, it is well known that this is an example of a regular map. With the usual notation, the automorphism group is given by $G = \langle x, y, z \mid x^2, y^2, z^2, (xy)^2, (yz)^3, (zx)^3\rangle$.

Inspection of the tetrahedron will easily yield that the Petrie polygons are quadrangles and so we may include an extra relator $(xyz)^4$ in the above presentation.

The dual operator, $\mathbf{D} : \mathcal{T} \to \mathcal{T}'$, is defined by sending $x \to y'$, $z \to z'$ and $y \to x'$, so we may form the dual of the tetrahedron T', which will have presentation as follows: $G' = \langle x', y', z' \mid (y')^2, (x')^2, (z')^2, (y'x')^2, (x'z')^3, (z'y')^3 \rangle$.

The tetrahedron is self-dual, and this is clear from the symmetry in the presentation of G: swapping the role of x and y respects all relators.

On the other hand, the Petrie operator $\mathbf{P} : \mathcal{T} \to \mathcal{T}''$, is defined by sending $x \to x''y''$, $y \to y''$ and $z \to z''$, so we may form the Petrie-dual of the tetrahedron T'', which will have presentation as follows:
$G'' = \langle x'', y'', z'' \mid (x''y'')^2, (y'')^2, (z'')^2, (x'')^2, (y''z'')^3, (z''x''y'')^3 \rangle$.

It is already clear that the tetrahedron is not self-Petrie and this is borne out in the presentation of the automorphism group for its Petrie map. Applying the same logic, the extra relator $(xyz)^4$ would become $(x''z'')^4$, and this demonstrates the difference in valency between the faces of the tetrahedron and those of its Petrie map. It is then obvious that there is no group isomorphism between G and G'' sending $x \to x''$, $y \to y''$ and $z \to z''$. •

A further operator, the *jth hole operator*, was introduced by Coxeter [20] and later studied by Wilson [65]. The operators in $\langle \mathbf{D}, \mathbf{P} \rangle$ respect both the automorphism group of the object map and at the same time either maintain the underlying graph, or replace it with the corresponding dual graph, and as such each of them essentially changes the adjacency relationships between the four flags around each edge. We can study further operations on regular maps by considering the adjacencies of flags at vertices in the following way. When j is coprime to the vertex degree of a regular map on an orientable surface, the jth hole operator $\mathbf{H}_j$ replaces the local cyclic permutation of edges at every vertex (each defined consistently with the orientation of the surface) with its jth power. Hole operators thereby also maintain the underlying graph of the object map, and there is a natural extension to this concept which defines hole operators on non-orientable maps. More recently the operation $\mathbf{H}_j$ has been studied by Nedela and Škoviera and when the resulting map is isomorphic to the original, j is called an *exponent* of the map [48]. When all possible values of j are exponents, then the map is called *kaleidoscopic*.

External symmetries relate to outer automorphisms of the group G and maps admitting them have been investigated by Archdeacon, Conder, Erskine, Fraser, Hriňáková, Jeans, Jones, Kwon, Poulton, and Širáň, among others, see [3, 16, 25, 26, 41].

In 2012 Archdeacon, Conder and Širáň proved the existence of a kaleidoscopic regular map with Trinity symmetry for any given even degree [3]. Their proof relies on the existence of a base map, in particular the two-cycle embedded in the sphere, which has the required external symmetries, that is it is self-dual, self-Petrie and kaleidoscopic. They then use a voltage assignment, see [4], to lift this map in order to construct an example of a regular map which still retains the external symmetries, but has a valency (and hence face-length and Petrie-walk length) of $2n$ for any given positive integer n.

Although their existence was known for all even valencies, until recently it was unknown whether a regular map with Trinity symmetry exists for any given odd valency. The existence of such maps was proved by Fraser, Jeans (aka Reade) and Širáň in 2019, see [26], and is the subject of Chapter 2 of this thesis. However, the resulting maps are known not to be kaleidoscopic, and there is no obvious way to doctor the resulting maps to demand this property. So, the question of the existence of a kaleidoscopic regular map with Trinity symmetry for arbitrary odd degree remains open, although it is strongly suspected by Conder that they do not exist for odd prime valency [12]. Looking for small examples is the motivation for Chapter 3, and the question remains open.

1.5 Groups of operators

In [65], Wilson demonstrated how given a regular map $\mathcal{M}$, the actions of the two dual operators, namely D and P, generate one, two, three or six maps (up to isomorphism). When applied to a fully regular map, the operators D and P respectively yield the dual map and the Petrie map, while their compositions are described in terms of redrawing the underlying graph and/or surface surgery. Also both operators maintain the same adjacencies of flags across corners as in the original map $\mathcal{M}$, and so the role of what we have called $\mathcal{R}_1$ remains unchanged.

The dual, Petrie and hole operators have also been described informally in section 1.4, and it is natural to see how they work by considering flag adjacencies, that is with reference to the monodromy group. For a fully regular map the monodromy group is isomorphic to the automorphism group, these being respectively the regular right and left action on the flag set of the map. For a more detailed analogue example describing the relationship between monodromy and automorphisms, see Section 4.2.4. It is this isomorphic correspondence that allows us to consider the action of the operators $\mathbf{D}$, $\mathbf{P}$ and $\mathbf{H}_j$ as defined on the generators of the automorphism group G, and this will be the perspective throughout this thesis. This is consistent with the

	$\mathcal{M}$	$\mathcal{M}$**Id**	$\mathcal{M}$**D**	$\mathcal{M}$**P**	$\mathcal{M}$**PD**	$\mathcal{M}$**DP**	$\mathcal{M}$**DPD**
r_0'	r_0	r_0	r_2	r_0r_2	r_2	r_0r_2	r_0
r_2'	r_2	r_2	r_0	r_2	r_0r_2	r_0	r_0r_2
$r_0'r_2'$	r_0r_2	r_0r_2	r_0r_2	r_0	r_0	r_2	r_2
r_1'	r_1	r_1	r_1	r_1	r_1	r_1	r_1

Table 1.1: The map $\mathcal{M} = (G; r_0, r_2, r_1)$ and its image under the operators **D**, **P** and their compositions.

choice to study highly symmetric maps by focussing on their automorphism groups. For the purposes of the formal definitions in Table 1.1, we use the original and more intuitive notation to highlight the natural relationship with monodromy.

In the automorphism group $G = Aut(\mathcal{M})$, the two elements r_0 and r_2 generate the stabiliser of the distinguished edge, which is isomorphic to the Klein 4-group V_4. The elements of this group $\{e, r_0, r_2, r_0r_2\}$ correspond to the four flags surrounding that edge, with the identity element being the marked shaded flag. We define the action of **D** and **P** (and so any composition of these operators) by mapping $r_i \to r_i'$ as shown in Table 1.1.

The automorphism group of V_4 is S_3, so we know that there can be no more than these six maps generated by the action of **D** and **P**. The table shows one map corresponding to each of the six permutations of the three non-identity elements of V_4. The operator **Id** fixes the map. Indeed the group generated by these two dual operators is $\langle \mathbf{D}, \mathbf{P} \rangle \cong \mathrm{S}_3$, and is well-known in the field.

It is clear from the table that the operation $\mathbf{DPD} = \mathbf{PDP}$ has order two and so is another duality. The existence of this duality is implicit in the work of Coxeter and Moser [21], and Wilson calls the image under this operation the *opposite* map [65], while Lins [45] calls it the *phial*. Self-opposite maps are rarely mentioned in the literature. This may be because any self-opposite map $\mathcal{M}$ is the dual of a self-Petrie map and at the same time is the Petrie map of a self-dual map. That is, a self-opposite map $\mathcal{M}$ is such that, by definition, $\mathcal{M} \cong \mathcal{M}\mathbf{PDP} \cong \mathcal{M}\mathbf{DPD}$ and so immediately we have $\mathcal{M}\mathbf{D} \cong \mathcal{M}\mathbf{DP}$ and $\mathcal{M}\mathbf{P} \cong \mathcal{M}\mathbf{PD}$.

The table also makes it clear that the operation **DP** has order three. In "Regular maps admitting trialities but not dualities" [41], Jones and Poulton constructed infinite families of maps which are invariant under **DP** and so also **PD**, but which are not invariant under the action of the dual operators.

Very little is known in general about the external symmetry group for a *super-symmetric map*, that is a kaleidoscopic map with Trinity symmetry. For a given super-symmetric map $\mathcal{M}$, with valency k, the external symmetry group is generated by the dual operator, the Petrie operator and the jth hole operators for all j coprime to k, and denoted by $Ext(\mathcal{M}) = \langle \mathbf{D}, \mathbf{P}, \mathbf{H}_j \mid (j,k) = 1 \rangle$. It has been proved by Conder, Kwon and Širáň [16] that, even for a family of maps with constant degree 8, the external symmetry group can be arbitrarily large, and otherwise not much appears in the literature. The motivation for the work in Chapter 3 is to further understanding of this mysterious beast.

There are, however, a few things which are relatively well-known and very easy to prove, and this gives some understanding of subgroups of the external symmetry group. For example the Petrie operator and hole operators commute, while the group of hole operators is isomorphic to the group of units modulo k. However mixing duality and hole operators generally creates, informally, a mess! In Chapter 3 I have found further subgroups of the external symmetry group. As part of this I have also investigated the orbit structure of groups of operators acting on the space of maps which share the same underlying group. These are very small steps towards the goal of understanding external symmetry groups.

1.6 Classification results for symmetric maps

Classifications of regular maps are usually done by genus, automorphism group or underlying graph. The state of the art in classification of regular maps is presented in the survey article [57] by Širáň. Here we mention just a few results which are directly relevant to this research project.

Conder, building on work with Dobcsányi [14], has used a computer to generate a classification of all regular maps on surfaces with Euler characteristic $\chi \geq -600$ and there is a comprehensive list of these maps and their properties available online [11]. More generally, classifications now exist for regular maps on non-orientable surfaces with Euler characteristic $\chi = -p$, $\chi = -p^2$ and $\chi = -3p$ for prime p, see [9, 18, 17] for details. The classification of fully regular maps by supporting surface is clearly incomplete and has come across a pause in terms of recent progress. In Chapter 5 the classification of symmetric maps on surfaces of negative prime Euler characteristic is extended to include edge-biregular maps on those surfaces.

The groups $\mathrm{PSL}(2,q)$ and $\mathrm{PGL}(2,q)$ are a source of many regular maps. These were studied by Sah [56], and those of type (k,ℓ) on $\mathrm{PSL}(2,q)$ have been enumerated in [1] by Adrianov. Meanwhile explicit forms of generating matrices for $\mathrm{PSL}(2,q)$ and $\mathrm{PGL}(2,q)$ regular maps are presented by Conder, Potočnik and Širáň in [19]. For odd prime powers, this forms the basis for the work in Chapter 2. Meanwhile the case when q is even is the foundation for the investigations found in Chapter 3.

1.7 Biregular maps

Maps which are not necessarily fully regular but still display a lot of symmetry are also a topic of much research. A map on an orientable surface for which the group of orientation-preserving automorphisms (which is generated by the rotations about a vertex and an incident face) acts regularly on the set of darts is called *rotary* or *orientably regular*. Such a map is called *reflexible* if and only if there is an orientation-reversing automorphism of the map. A rotary map which is not reflexible is called *chiral* and, as such, is an example of a symmetric map which is not fully regular. Chiral maps are very well-studied in their own right. In a similar vein, and looking at maps which are in some sense *almost* regular, recently Breda d'Azevedo, Catalano and Širáň [8] introduced and classified the bi-rotary maps for surfaces of negative prime characteristic.

Depending on the values of k and ℓ, there are up to seven index two subgroups of the full triangle group of type (k, ℓ). The tessellations corresponding to these index two subgroups may be thought of as the parent maps for *biregular* maps. In a biregular map, the relevant automorphism group may be identified with a smooth quotient of an index two subgroup of the full triangle group. Both chiral maps and also bi-rotary maps are examples of maps which are biregular. Part II of this thesis is dedicated to one of these classes of highly symmetric biregular maps, which until now have not enjoyed much attention. We lay the foundations and make progress towards classifying this distinct class of biregular maps: the unique, much-neglected and non-edge-transitive class. We call these maps edge-biregular, and begin to catch up with the classification results which are already known for their more popular cousins.

Part I

Regular maps

Chapter 2

Regular maps with Trinity symmetry

A regular map which is both self-dual and self-Petrie is described as having *Trinity symmetry*. This chapter addresses the question:

Given a positive integer k, is there a fully regular map of type (k, k) which has Trinity symmetry?

Archdeacon, Conder and Širáň have answered the above question in the affirmative for all even valencies, see [3]. The work relies on a lifting construction from the map of type $(2, 2)$ embedded on a sphere. The lifting preserves the properties (external symmetries) of self-duality and self-Petriality, and so using the aforementioned map as a base map, the existence of fully regular maps with Trinity symmetry is proved for all even values of k.

In this chapter, a similar method is applied to answer the question in the affirmative also for maps of odd valency $k \geq 5$. A sketch of the proof structure is as follows:

1. Section 2.2: Look in well-known groups, namely $\mathrm{PSL}(2, q)$ where q is a power of the odd prime p, for necessary and sufficient conditions for a given map to be self-dual and/or self-Petrie.
2. Corollary 2.10: Choose a pleasant sufficient condition for the existence of a regular map with Trinity symmetry of type (π, π) where π is a prime.
3. Section 2.3: Prove that this condition can be satisfied for all primes $\pi \geq 5$.
4. Section 2.4: Invoke the lifting trick from [3], for which we must prove it extends to the non-orientable case, to extend the claim to all types $(b\pi, b\pi)$ for multiples of π where $\pi \geq 5$.
5. Section 2.5: Mind the gap! Use Example 2.11 and Proposition 2.21 to address any occasions for which the previous item fails, and draw conclusion.

The work in this chapter is based on what appears in two joint papers. Sections 2.1 and 2.2 summarise respectively the background and the particulars from the work with Erskine and Hriňáková [25] while Sections 2.3, 2.4 and 2.5 reflect what appears in the paper with Fraser and Širáň, [26].

2.1 Preliminaries

2.1.1 Context

It had been an open problem for some time as to whether there exists a self-dual and self-Petrie regular map for any given valency k. In this chapter we prove:

Theorem 2.1 (**Fraser, Jeans (aka Reade), Širáň**). *For every odd $k \geq 5$, there exists a fully regular map of type (k, k) which is both self-dual and self-Petrie.*

The motivation and the inspiration came from Theorem 2.2 of [3] in which Archdeacon, Conder and Širáň proved the existence of such a map for any even valency:

Theorem 2.2 (**Archdeacon, Conder, Širáň**). *For every integer $n \geq 1$, there is a map of degree $2n$ which is fully regular, kaleidoscopic, and has Trinity symmetry.*

First, we note the following Lemma, which addresses the type $(3, 3)$.

Lemma 2.3. *There is no fully regular map with Trinity symmetry of type $(3, 3)$.*

Proof. A map of type $(3, 3)$ must have the tetrahedral graph as its underlying graph. It is well-known that the embedding of this graph in the sphere is self-dual. However the length of a closed Petrie walk is 4, not 3, so this regular map can not be self-Petrie. Since the tetrahedron is the only option for a (necessarily fully regular) map of this type, we conclude that there is no fully regular map of type $(3, 3)$ which is both self-dual and self-Petrie. □

We continue by stating the background material and results which we will use. This chapter is founded on and applies to work done by Conder, Potočnik and Širáň in [19] which provides a detailed analysis of reflexible regular hypermaps for triples (k, ℓ, m) on projective two-dimensional linear groups including explicit generating sets for the associated groups. In particular we are concerned only with maps, not hypermaps, nor maps with semi-edges, and so, without loss of generality, we let $m = 2$.

By [19] we have explicit generating involutions (X, Y, Z) for each (isomorphism class of) fully regular map whose automorphism group is isomorphic to $\mathrm{PSL}(2, q)$. As such this seemed a natural place to start looking for necessary and sufficient conditions for such a map to be self-dual and self-Petrie.

2.1.2 Regular maps on PSL$(2, q)$ or PGL$(2, q)$

The surface on which a regular map is embedded could be orientable or non-orientable. If the regular map is on an orientable surface then G has a subgroup of index two which corresponds to the orientation preserving automorphisms. Instead of the group generated by the three involutions x, y and z, we can consider the group of orientation-preserving automorphisms which is generated by the two rotations $R = yz$ and $S = zx$. On a non-orientable surface these two elements will still generate the full automorphism group and we can say that studying these maps is equivalent to studying groups which have presentations of the form $G = \langle R, S \mid R^k, S^\ell, (RS)^2, ...\rangle$.

We focus on regular maps of type (k, ℓ) where the associated group $G = \langle x, y, z\rangle$ is isomorphic to PSL$(2, q)$ or PGL$(2, q)$ where q is a power of a given odd prime p. By [19], each of k and ℓ are either equal to p or divide one of $q - 1$ or $q + 1$. In particular, for the case of PSL$(2, q)$ where q is odd, each of k and ℓ are either equal to p or divide one of $\frac{q-1}{2}$ or $\frac{q+1}{2}$. Following the convention and notation of [19], in the latter two cases we let ξ_κ and ξ_λ be primitive $2k$th and 2ℓth roots of 1 in the field as follows: if k or ℓ divides $q - 1$ then the corresponding primitive root is in the field GF(q); otherwise it is in the unique quadratic extension GF(q^2). We also define $\omega_i = \xi_i + \xi_i^{-1}$ for $i \in \{\kappa, \lambda\}$. Note that ω_i is thus in the field GF(q). We too assume that (k, ℓ) is a hyperbolic pair, that is $1/k + 1/\ell < 1/2$. This implies that $k \geq 3$ and $\ell \geq 3$. The conditions in this paragraph are what we refer to as *the usual setup*.

Recall that we can consider the dual and the Petrie dual, as defined in Table 1.1, as operators which may be applied to a map $\mathcal{M} = (G; x, y, z)$, where G is the automorphism group associated with the regular map. In particular, with reference to the edge associated with the distinguished flag, the involutions x and y are the reflections (r_0 and r_2) respectively in the edge-bisector and in the edge itself. Meanwhile z is the reflection (r_1) across the angle bisector at the vertex associated with the distinguished flag. The dual of a map is obtained by swapping the vertices for faces and vice versa, and in so doing, each edge is replaced by a new edge which is perpendicular to the original one. Hence, in terms of the involutions x, y, z, the dual operator interchanges x and y, while fixing z. Naturally, any dual operator is an involution, and self-duality is therefore equivalent to the existence of an involutory automorphism of the group G which fixes z and interchanges x with y. By reference to Table 1.1 it may be seen that the Petrie operator replaces x with xy and fixes both y and z. Hence a map is self-Petrie if and only if there is an involutory automorphism of the group $G = \langle x, y, z\rangle$ which sends x to xy and fixes both y and z.

The first half of this chapter is devoted in large part to finding necessary and sufficient conditions for the existence of involutory automorphisms which imply self-duality and/or that the map is self-Petrie. Later in the chapter we will use a selection of these results and apply them in proving the main result.

If, as we have assumed, $G \cong \mathrm{PSL}(2, q)$ or $G \cong \mathrm{PGL}(2, q)$, then the automorphism group for G is $\mathrm{P\Gamma L}(2, q)$, the semidirect product $\mathrm{PGL}(2, q) \rtimes \mathrm{C}_e$ where $q = p^e$. Elements $(A, j) \in \mathrm{P\Gamma L}(2, q)$ act as follows: $(A, j)(T) = A\phi_j(T)A^{-1}$ where ϕ_j is the repeated Frobenius field automorphism of the finite field, $\phi_j : x \to x^r$ with $r = p^j$. The function ϕ_j acts element-wise on a representative matrix and we use the general rule for composition in $\mathrm{P\Gamma L}(2, q)$ which is $(B, j)(A, i) = (B\phi_j(A), i + j)$. Observe that, when $e = 1$, this group is essentially $\mathrm{PGL}(2, p)$ and so, in the case where $G \cong \mathrm{PGL}(2, p)$, all automorphisms are inner automorphisms.

When $(A, j) \in \mathrm{P\Gamma L}(2, p^e)$ is an involution, it must be such that $(A, j)(A, j) = (A\phi_j(A), 2j)$ is the identity, so $2j \equiv 0 \pmod{e}$. One case is when there is no field automorphism involved, that is $j = 0$ and $A^2 = I$. Alternatively $e = 2j$ is even, and then we need $\phi_j(A) = A^{-1}$. This is summarised in Lemma 2.4.

Lemma 2.4. *The non-identity element $(A, j) \in P\Gamma L(2, p^e)$ is an involution if and only if one of the following conditions holds:*

1. *$j = 0$ and $A^2 = I$*
2. *$2j = e$ and $\phi_j(A) = A^{-1}$.*

Explicit generating sets are known for regular maps with automorphism group G isomorphic to $\mathrm{PSL}(2, q)$ or $\mathrm{PGL}(2, q)$. For details we refer the interested reader to [19, 46] while noting that, for our purposes, it does not matter whether G is isomorphic to $\mathrm{PSL}(2, q)$ or $\mathrm{PGL}(2, q)$. We present the results for maps of each type as required.

We will need to consider performing operations on the elements x, y and z of G. As such we denote elements of the group $\mathrm{PSL}(2, q)$ or $\mathrm{PGL}(2, q)$, by a representative matrix with square brackets. This allows us to perform the necessary calculations. We can then determine whether or not two resulting matrices are equivalent within G, that is whether or not they correspond to the same element of the group G. A pair of matrices are in the same equivalence class, that is they represent the same *element* of G if one is a scalar multiple of the other. We use curved brackets for X, Y and Z, the matrix representatives for x, y, and z.

Lemma 2.4 can then be used to find conditions for the elements of the matrix part A of an involutory automorphism (A, j) as follows.

Lemma 2.5. *Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. *The automorphism denoted* $(A, j) \in P\Gamma L(2, p^e)$ *is an involution, if and only if* a, b, c, d *satisfy the following equations, with* $r = p^j$ *for* $j = 0$ *or* $2j = e$.

1. $a^{r+1} = d^{r+1}$,
2. $bc^r = cb^r$,
3. $ab^r + bd^r = 0$,
4. $ca^r + dc^r = 0$.

Proof. By Lemma 2.4, and letting $r = p^j$ we have

$$A\phi_j(A) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a^r & b^r \\ c^r & d^r \end{bmatrix} = \begin{bmatrix} a^{r+1} + bc^r & ab^r + bd^r \\ ca^r + dc^r & cb^r + d^{r+1} \end{bmatrix} = I.$$

By comparing the leading diagonal entries we see that

$$a^{r+1} + bc^r = cb^r + d^{r+1}.$$

Applying the field automorphism ϕ_j yields $a^{r^2+r} + b^r c^{r^2} = c^r b^{r^2} + d^{r^2+r}$. We notice this yields

$$a^{1+r} + b^r c = c^r b + d^{1+r}.$$

Subtracting these two equations, and remembering that q is odd, we get the first two equations, while looking at the off-diagonal immediately gives rise to the final two equations. □

2.2 In search of a sufficient condition for Trinity symmetry

The scene has now been set with the background information about regular maps on $\mathrm{PSL}(2, q)$ or $\mathrm{PGL}(2, q)$ groups, and so we begin by investigating regular maps of type (p, p), (k, p) and (p, ℓ) when k and ℓ are coprime to p. As it turns out, type (p, p) is self-dual but it is not self-Petrie, and it is clear that the other types cannot yield a self-dual map. Next we address the necessary and sufficient conditions for a map of type (k, ℓ) to be self-dual and self-Petrie respectively. Then we highlight a special case, namely maps of type $(5, 5)$ whose orientation-preserving automorphism groups turn out to be isomorphic to A_5.

Much of the work in this section appears in a joint paper with Hriňáková and Erskine

[25]. Among other things, the published paper presents necessary and sufficient conditions for the properties of self-duality and self-Petriality. As it happened, I derived some conditions which are equivalent to the results in Hriňáková's thesis [32], although at that time I hadn't had sight of her work. I have included here my own version of these proofs as well as the presentation of the equivalent and sometimes identical results, while acknowledging her priority for Propositions 2.8 and 2.9. I am also very grateful to Dr Grahame Erskine for his support with checking and verifying numerous claims by computer.

2.2.1 First false starts and non-starters: Regular maps on linear fractional groups of type (p, p), (k, p) and (p, ℓ) where p is an odd prime

For odd prime p, by Proposition 3.1 in [19], maps of the type (p, p) have the following representatives $X = X_1$, $Y = Y_1$ and $Z = Z_1$, where $\alpha^2 = -1$:

$$X_1 = -\alpha \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}, \; Y_1 = -\alpha \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \text{ and } Z_1 = \alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Proposition 2.6. *With the usual setup, a map of type (p, p) is self-dual.*

Proof. For self-duality we need $G \cong \langle x, y, z \rangle$ to admit an automorphism such that x and y are interchanged, and z is fixed. So the question is: can we find an automorphism $(A, j) \in \mathrm{P\Gamma L}(2, q)$ such that $A\phi_j(X)A^{-1} = Y$, $A\phi_j(Y)A^{-1} = X$, and $A\phi_j(Z)A^{-1} = Z$. It is easy to verify that $(A, 0)$, where A has the form $A = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ satisfies these conditions, so this type of map is self-dual. □

Guaranteed self-duality seems to be a promising start in our hunt for regular maps with Trinity symmetry, but, as in the case of the tetrahedron, such naive hopes are soon dashed:

Proposition 2.7. *With the usual setup, a map of type (p, p) is not self-Petrie.*

Proof. In order to be self-Petrie, the group G needs to admit an involutory automorphism (B, j) which fixes z and y, and exchanges x with xy.

First notice that $\phi_j(Z) = Z$ and $\phi_j(Y) = Y$ so if B exists, it must be of a form which commutes with both Z and Y. To commute with Z, the necessarily non-identity element B must be either $B_1 = \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$ or $B_2 = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$. Note that $0 \notin \{a, b, c, d\}$ and $a \neq d$. As shown below, neither of these commute with Y.

$$B_1 Y = -\alpha \begin{bmatrix} 0 & -b \\ c & -c \end{bmatrix} \neq Y B_1 = -\alpha \begin{bmatrix} -c & b \\ -c & 0 \end{bmatrix}$$

$$B_2 Y = -\alpha \begin{bmatrix} a & -a \\ 0 & -d \end{bmatrix} \neq Y B_2 = -\alpha \begin{bmatrix} a & -d \\ 0 & -d \end{bmatrix}$$

Hence this type of map is not self-Petrie. □

Maps of type (k, p) and (p, ℓ) where k and ℓ are coprime to p clearly cannot be self-dual since the vertex degree and face lengths differ. In the interests of streamlining this chapter-long proof, and given we are looking for maps with Trinity symmetry, not maps which are self-Petrie and not self-dual, we omit the necessary and sufficient conditions for such maps and refer the interested reader to the paper [25].

2.2.2 More promising hunting grounds: Regular maps on linear fractional groups of type (k, ℓ) where both k and ℓ are coprime to p

Perhaps unsurprisingly, this turns out to be more fruitful in the quest to find promising conditions for regular maps with Trinity symmetry.

In this case we have different generating triples for the group G. As per Proposition 3.2 in [19], the triple (X, Y, Z) has representatives as defined below where $D = \omega_\kappa^2 + \omega_\lambda^2 - 4$, $\beta = -1/\sqrt{-D}$ and $\eta = (\xi_\kappa - \xi_\kappa^{-1})^{-1}$.

$$X_4 = \eta\beta \begin{pmatrix} D & D\omega_\lambda\xi_\kappa \\ -\omega_\lambda\xi_\kappa^{-1} & -D \end{pmatrix}, \; Y_4 = \beta \begin{pmatrix} 0 & \xi_\kappa D \\ {\xi_\kappa}^{-1} & 0 \end{pmatrix}, \; Z_4 = \beta \begin{pmatrix} 0 & D \\ 1 & 0 \end{pmatrix} \tag{2.1}$$

We will also consider the pair of matrices which represent R and S, the rotations around a vertex and a face respectively, which by Proposition 2.2 in [19] are:

$$R_4 = \begin{pmatrix} \xi_\kappa & 0 \\ 0 & \xi_\kappa^{-1} \end{pmatrix} \text{ and } S_4 = \eta \begin{pmatrix} -\omega_\lambda \xi_\kappa^{-1} & -D \\ 1 & \omega_\lambda \xi_\kappa \end{pmatrix}. \tag{2.2}$$

At this point we note that there is an exception for maps of type $(5,5)$, which is addressed in subsection 2.2.3.

Proposition 2.8. *Under the usual setup, a regular map of type (k,k) is self-dual if and only if $\omega_\lambda = \pm\omega_\kappa^r$ where $r = p^j$, and $j = 0$ or $2j = e$.*

Proof. Suppose the map is self-dual.

There is an involutory automorphism of G which fixes Z and interchanges X and Y. This is equivalent to interchanging the rotations $R^{-1} = (YZ)^{-1} = ZY$ and $S = ZX$ around a vertex and a face respectively. That is there is an automorphism (A,j) which interchanges $\pm R^{-1}$ with S. Here the $\pm$ takes into account both representative elements for R. So $A(\pm\phi_j(R^{-1}))A^{-1} = S$. Remembering that conjugation preserves traces this implies $\pm\phi_j \mathrm{tr}(R^{-1}) = \mathrm{tr}(S)$ which immediately yields the condition $\pm\omega_\kappa^r = \omega_\lambda$.

Conversely suppose $\pm\omega_\kappa^r = \omega_\lambda$.

We note that $\omega_\kappa^{2r} = \omega_\lambda^2 \iff \omega_\kappa^2 = \omega_\lambda^{2r}$ and so
$D^r = (\omega_\kappa^2 + \omega_\lambda^2 - 4)^r = \omega_\kappa^{2r} + \omega_\lambda^{2r} - 4 = \omega_\lambda^2 + \omega_\kappa^2 - 4 = D.$

We aim to find an involutory automorphism (A,j) which demonstrates this map is self-dual. Consider $A = \begin{bmatrix} a & D \\ -1 & -a \end{bmatrix}$ which, by Lemma 2.5, so long as $a^r = a$, satisfies all the equations necessary for the element (A,j) to be involutory. Notice that (A,j) also fixes Z. We also need X and Y to be interchanged by the automorphism in which case the following matrices are equivalent.

$$A\phi_j(X) = \eta^r\beta^r \begin{bmatrix} D(a - \omega_\lambda^r \xi_\kappa^{-r}) & D(a\omega_\lambda^r \xi_\kappa^r - D) \\ \omega_\lambda^r \xi_\kappa^{-r} a - D & D(a - \omega_\lambda^r \xi_\kappa^r) \end{bmatrix} \text{ and } YA = \beta \begin{bmatrix} -\xi_\kappa D & -a\xi_\kappa D \\ a\xi_\kappa^{-1} & D\xi_\kappa^{-1} \end{bmatrix}$$

Ratio of elements in the leading diagonal: $-\xi_\kappa^2 = (a - \omega_\lambda^r \xi_\kappa^{-r})/(a - \omega_\lambda^r \xi_\kappa^r)$

Ratio of elements in the left column: $-D\xi_\kappa^2/a = D(a - \omega_\lambda^r \xi_\kappa^{-r})/(\omega_\lambda^r \xi_\kappa^{-r} a - D)$

Ratio of elements in the top row: $1/a = (a - \omega_\lambda^r \xi_\kappa^{-r})/(a\omega_\lambda^r \xi_\kappa^r - D)$

The last ratio listed yields the following quadratic in a: $0 = a^2 - a\omega_\lambda^r(\xi_\kappa^{-r} + \xi_\kappa^r) + D$, that is $0 = a^2 - a\omega_\lambda^r\omega_\kappa^r + D$. This is consistent with all the necessary ratios. All that remains is to check that a value of a satisfying this quadratic is invariant under the repeated Frobenius field automorphism. The discriminant $\Delta = \omega_\kappa^2\omega_\kappa^{2r} - 4(\omega_\kappa^2 + (\omega_\kappa^r)^2 - 4) = ((\omega_\kappa^r)^2 - 4)(\omega_\kappa^2 - 4)$. Furthermore the expression for $a = (\omega_\lambda^r\omega_\kappa^r \pm \sqrt{\Delta})/2$ is invariant under the transformation $x \to x^r$ as required. Hence the map is self-dual. $\square$

Proposition 2.9. *With the usual setup, where k, ℓ are coprime to p, a map of type (k, ℓ) is self-Petrie if and only if one of the following conditions is fulfilled:*

1. $\omega_\lambda^2 = -D$

2. $q = r^2 = p^{2j}$, $\omega_\lambda^{2r} = -D$ *and* $k \mid r \pm 1$.

Proof. First suppose the map is self-Petrie. So there exists $(B, j) \in \mathrm{P\Gamma L}(2, q)$ such that $B\phi_j(X)B^{-1} = XY$, $B\phi_j(Y)B^{-1} = Y$, and $B\phi_j(Z)B^{-1} = Z$. By comparing the traces of $\phi_j(ZX)$ and ZXY we get the necessary condition: $\omega_\lambda^{2r} = -D$ where $r = p^j$. In the case where the field automorphism is non-trivial, that is when $j \neq 0$, comparing the traces of ZY and $\phi_j(ZY)$ we see that $\omega_\kappa = \omega_\kappa^r$ is invariant so $\omega_\kappa \in \mathrm{GF}(p^j)$ which in turn implies $k \mid r \pm 1$.

For the rest of the proof we split the situation into two cases: the first when $j = 0$ and we do not consider any field automorphism, and the second case where a field automorphism is included.

Case 1: $j = 0$.

Suppose $\omega_\lambda^2 = -D$. Notice that $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ fixes both Y and Z. The map is self-Petrie if $BX = \eta\beta \begin{bmatrix} D & D\omega_\lambda\xi_\kappa \\ \omega_\lambda\xi_\kappa^{-1} & D \end{bmatrix}$ and $XYB = \eta\beta^2 \begin{bmatrix} D\omega_\lambda & -D^2\xi_\kappa \\ -D\xi_\kappa^{-1} & D\omega_\lambda \end{bmatrix}$ are also equivalent. Comparing these and applying our assumption that $\omega_\lambda^2 = -D$ we conclude this map is self-Petrie.

Case 2: $2j = e$. By Lemma 2.4 we include the repeated Frobenius automorphism.

Suppose $\omega_\lambda^{2r} = -D$ and $k \mid r \pm 1$.

The map is self-Petrie if there is an involutory automorphism which not only fixes Z but also fixes Y and interchanges XY with X. We hope to find $(B, j) = (\begin{bmatrix} a & b \\ c & d \end{bmatrix}, j)$, the associated element of PΓL. In addition to the conditions for a, b, c, d established in Lemma 2.5, we require $B\phi_j(Z)B^{-1} = Z$ and $B\phi_j(Y)B^{-1} = Y$. In order to fix Z we must have $bd = acD^r$ and $D(d^2 - c^2D^r) = a^2D^r - b^2$. Fixing Y yields two further equations: $bd = ac\xi_\kappa^{2r}D^r$ and $\xi_\kappa^2 D(d^2\xi_\kappa^{-r} - c^2\xi_\kappa^r D^r) = a^2\xi_\kappa^r D^r - b^2\xi_\kappa^{-r}$. Since $\xi_\kappa^{2r} \neq 1$ and $D \neq 0$, notice that $bd = ac\xi_\kappa^{2r}D^r = acD^r$ tells us that either $a = 0$ or $c = 0$.

If $a = 0$, we immediately see $d = 0$ too and so we can assume $b = 1$ without loss of generality. The equations for a, b, c, d tell us that to fix Z we have $c^2 = \frac{1}{D^{r+1}}$ and to fix Y we have $c^2\xi^{2r+2} = \frac{1}{D^{r+1}}$. So this automorphism exists only if $\xi_\kappa^{2r+2} = 1$. By definition ξ_κ is a primitive $2k$th root of unity and
$\xi_\kappa^{2r+2} = 1 \iff 2k \mid 2r + 2 \iff k \mid r + 1$, which is the case by our assumption.

$B\phi_j(XY) = \eta^r\beta^{2r}\begin{bmatrix} -D^r\xi^{-r} & \mp D^r\sqrt{-D} \\ \pm cD^r\sqrt{-D} & cD^{2r}\xi_\kappa^r \end{bmatrix}$ and $XB = \eta\beta\begin{bmatrix} cD\omega_\lambda\xi_\kappa & D \\ -cD & -\omega_\lambda\xi_\kappa^{-1} \end{bmatrix}$ are also equivalent if the map is self-Petrie so we compare the ratios of the elements in turn. This yields $\pm c = \frac{1}{\omega_\lambda\xi_\kappa^{r+1}\sqrt{-D}} = \frac{\omega_\lambda\sqrt{-D}}{D^{r+1}\xi_\kappa^{r+1}}$ which is true only if $D^r = -\omega_\lambda^2$, which is again the case by our assumption. These conditions are consistent with our other requirements for the value of c, (namely that $c^r = c$) so we have an automorphism demonstrating that this map is self-Petrie.

If on the other hand $c = 0$ then we have $b = 0$ and we assume $a = 1$ without loss of generality. Fixing Z yields $d^2D = D^r$. Fixing Y yields $d^2D = \xi_\kappa^{2r-2}D^r$. So the map is self-Petrie only if $\xi_\kappa^{2r-2} = 1$, which is the case by our assumption.

Now $B\phi_j(XY) = \eta^r\beta^{2r}D^r\begin{bmatrix} \omega_\lambda^r & D^r\xi_\kappa^r \\ -d\xi_\kappa^{-r} & -d\omega_\lambda^r \end{bmatrix}$ and $XB = \eta\beta\begin{bmatrix} D & dD\omega_\lambda\xi_\kappa \\ -\omega_\lambda\xi_\kappa^{-1} & -dD \end{bmatrix}$.

Again, the map is self-Petrie if these two elements are equivalent, that is if both $D^r\xi_\kappa^{r-1} = d\omega_\lambda^{r+1}$ and $\omega_\lambda^{r+1}\xi_\kappa^{r-1} = dD$. Applying our assumption $\omega_\kappa^{2r} = -D$, we conclude the map is self-Petrie. □

Using the fact that when $q = p$ the conditions are often much simpler to state, the preceding two results, Theorem 2.8 and Theorem 2.9, indicate a *sufficient* condition for a regular map of type (k, k) to be both self-dual and self-Petrie, namely $\omega_\kappa^2 = \omega_\lambda^2 = -D$. Corollary 2.10 shows this becomes a tractable sufficient condition for a map to be both self-dual and self-Petrie. This is the result on which our proof of Theorem 2.1 will rely.

Corollary 2.10. *Let (k, ℓ) be a hyperbolic pair such that $k = \ell$. Let ξ_κ and ξ_λ be primitive $2k$th roots of unity in some finite field $GF(q^2)$, and define* $\omega_i := \xi_i + \xi_i^{-1} \in GF(q)$ *for* $i \in \{\kappa, \lambda\}$.

If $\omega = \omega_\kappa = \omega_\lambda$ and $3\omega^2 = 4$ then the associated fully regular map of type (k, k), determined by the representative matrices given in Equation (2.1)*, is both self-dual and self-Petrie.*

As stated this corollary follows immediately from Propositions 2.8 and 2.9. Later in the chapter we will refine this further for our purposes, specifically to consider the case where k is a prime, but for now we illustrate the idea with an example.

Example 2.11. Let $k = \ell = 9$. We seek a finite field which has 18th roots of unity such that the equations in Corollary 2.10 are satisfied.

Let ζ be a primitive element of GF(73), and hence $\zeta^{72} = 1$. Notice that ζ^4 is therefore an 18th root of unity in the finite field containing 73 elements.

In particular, working modulo 73, the element $-4 = \xi_\kappa$ is an 18th root of unity, and its multiplicative inverse is $18 = \xi_\kappa^{-1}$. Their sum, $\omega_\kappa := \xi_\kappa + \xi_\kappa^{-1} = -4 + 18 = 14$, is such that $3\omega_\kappa^2 = 3 \times 14^2 = 3 \times 196 = 3 \times 50 = 150 = 4$ modulo 73. Now let $\xi_\lambda = 18$, which is also an 18th root of unity in GF(73), so that $\omega_\kappa = \omega_\lambda = 14$.

By Corollary 2.10, the fully regular map determined by the following representative matrices is both self-dual and self-Petrie, and thus has Trinity symmetry:

The triple (X, Y, Z) has representatives as defined below where $D = \omega_\kappa^2 + \omega_\lambda^2 - 4 = 23$, $\beta = -1/\sqrt{50}$ and $\eta = (\xi_\kappa - \xi_\kappa^{-1})^{-1} = 14^{-1} = -26$.

$$X_4 = \eta\beta \begin{pmatrix} D & D\omega_\lambda\xi_\kappa \\ -\omega_\lambda\xi_\kappa^{-1} & -D \end{pmatrix} = \eta\beta \begin{pmatrix} 23 & 26 \\ 40 & -23 \end{pmatrix}$$

$$Y_4 = \beta \begin{pmatrix} 0 & \xi_\kappa D \\ {\xi_\kappa}^{-1} & 0 \end{pmatrix} = \beta \begin{pmatrix} 0 & -19 \\ 18 & 0 \end{pmatrix}$$

$$Z_4 = \beta \begin{pmatrix} 0 & D \\ 1 & 0 \end{pmatrix} = \beta \begin{pmatrix} 0 & 23 \\ 1 & 0 \end{pmatrix}$$

Since the valency, 9, is odd and the map is self-Petrie, the supporting surface for the map must be non-orientable and therefore the orientation-preserving automorphisms R and S also generate the full automorphism group G. As such we may use the matrix representatives from Equation (2.2), where the entries are in GF(73):

$$R_4 = \begin{pmatrix} \xi_\kappa & 0 \\ 0 & \xi_\kappa^{-1} \end{pmatrix} = \begin{pmatrix} -4 & 0 \\ 0 & 18 \end{pmatrix}$$

$$S_4 = \eta \begin{pmatrix} -\omega_\lambda \xi_\kappa^{-1} & -D \\ 1 & \omega_\lambda \xi_\kappa \end{pmatrix} = -26 \begin{pmatrix} 40 & -23 \\ 1 & 17 \end{pmatrix} = \begin{pmatrix} -18 & 14 \\ -26 & -4 \end{pmatrix}.$$

Finally, by Proposition 6.4 in [19], the automorphism group generated is isomorphic to PSL(2, 73). •

2.2.3 Regular maps of type (5, 5) whose orientation-preserving automorphism group $\langle R, S\rangle$ is isomorphic to $\mathbf{A}_5$.

Adrianov's [1] enumeration of regular hypermaps on PSL$(2, q)$ includes a constant which deals with the special case which occurs for maps of type (5, 5).

For us to be considering a map of type (k, ℓ) we must have $2k \mid q \pm 1$ and $2\ell \mid q \pm 1$, and it is known, see [44], that PSL$(2, q)$ has subgroup A_5 when $q \equiv \pm 1 \pmod{10}$. The constant in Adrianov's enumeration, which is 2 for maps of type (5, 5) and zero otherwise, is subtracted to account for the cases when the group $\langle R, S\rangle$ collapses into the subgroup $\mathrm{A}_5 \leq \mathrm{PSL}(2, q)$.

The following result, with the usual definitions for ω_κ and ω_λ, indicates when the orientation-preserving automorphism group of a type (5, 5) map is not the linear fractional group that we might expect. As such it addresses an occasion which could be (but for our purposes turns out not to be) an exception for the case of type (5, 5).

Proposition 2.12. *The group $\langle R, S\rangle$ of a regular map of type (5,5), generated by the representative matrices R_4 and S_4, is isomorphic to A_5 if and only if $\omega_\lambda \neq \omega_\kappa$.*

Proof. From [54] we know a presentation of the group A_5 is:
$\langle a, b \mid a^5, b^5, (ab)^2, (a^4b)^3\rangle$.

Considering the group $\langle R, S \mid R^5, S^5, (RS)^2, \ldots\rangle$, it is clear that this will be isomorphic to A_5 if and only if the condition $(R^4S)^3 = I$ is also satified. This is the case if and only if $R^{-1}S$ has order 3.

$$R^{-1}S = \eta \begin{bmatrix} \xi_\kappa^{-1} & 0 \\ 0 & \xi_\kappa \end{bmatrix} \begin{bmatrix} -\omega_\lambda \xi_\kappa^{-1} & -D \\ 1 & \omega_\lambda \xi_\kappa \end{bmatrix} = \eta \begin{bmatrix} -\omega_\lambda \xi_\kappa^{-2} & -\xi_\kappa^{-1} D \\ \xi_\kappa & \omega_\lambda \xi_\kappa^2 \end{bmatrix}$$

$$(R^{-1}S)^3 = \eta^3 \begin{bmatrix} \omega_\lambda(2D\xi_\kappa^{-2} - \omega_\lambda^2\xi_\kappa^{-6} - D\xi_\kappa^2) & D\xi_\kappa^{-2}(D\xi_\kappa + \omega_\lambda^2(\xi_\kappa - \xi_\kappa^5 - \xi_\kappa^{-3})) \\ -(D\xi_\kappa + \omega_\lambda^2(\xi_\kappa - \xi_\kappa^5 - \xi_\kappa^{-3})) & \omega_\lambda(D\xi_\kappa^{-2} + \omega_\lambda^2\xi_\kappa^6 - 2D\xi_\kappa^2) \end{bmatrix}$$

The off diagonal elements are both zero if and only if $D\xi_\kappa + \omega_\lambda^2(\xi_\kappa - \xi_\kappa^5 - \xi_\kappa^{-3}) = 0$. This condition is equivalent to $(\omega_\kappa + 2)(\omega_\kappa - 2)(1 + \omega_\kappa\omega_\lambda)(1 - \omega_\kappa\omega_\lambda) = 0$ and we know that $\omega_\kappa \neq \pm 2$ so long as $k \neq p$.

The leading diagonal entries are equal if and only if $D(\xi_\kappa^{-2} + \xi_\kappa^2) = \omega_\lambda^2(\xi_\kappa^6 + \xi_\kappa^{-6})$. Applying $\xi_\kappa^{10} = 1$ and eliminating D shows this is equivalent to

$$(\omega_\kappa^2 - 4)(\omega_\kappa^2 - \omega_\kappa^2\omega_\lambda^2 + \omega_\lambda^2 - 2) = 0.$$

Assume $\omega_\kappa^2\omega_\lambda^2 = 1$. The off-diagonals are clearly zero, and the leading diagonal entries are equal since $(\omega_\kappa^2 - \omega_\kappa^2\omega_\lambda^2 + \omega_\lambda^2 - 2) = \omega_\kappa^{-2}(\omega_\kappa^4 - 3\omega_\kappa^2 + 1) = 0$ is always the case since the expression inside the bracket is the sum of powers of ξ_κ^2, a 5th root of unity. Then $R^{-1}S$ has order 3.

Conversely, assume $(R^{-1}S)^3 = I$. Then we instantly have $\omega_\kappa^2\omega_\lambda^2 = 1$ since $p \neq 5$.

We conclude that $(R^{-1}S)^3 = I$ if and only if $\omega_\kappa\omega_\lambda = \pm 1$. By considering the two possible values for ω_κ and ω_λ we see that this will happen if and only if $\omega_\kappa \neq \omega_\lambda$. □

We can see that this does not cause problems for us, since in our selective condition for Corollary 2.10 we force $\omega_\kappa = \omega_\lambda$. Indeed the structure of our intended proof only requires the existence of a regular map with Trinity symmetry of type (π, π) for each prime $\pi \geq 5$. However, we are yet to present the proof of when such conditions can be satisfied, and so for those with a nervous disposition, we now provide a concrete example of a reflexible regular map of type $(5, 5)$ with Trinity symmetry.

Example 2.13. The finite field consisting of eleven elements may be considered as the congruence classes of the integers modulo 11. Now, let $\xi_\kappa = -4$, which is a 10th root of unity, and then $\xi_\kappa^{-1} = -3$ and so $\omega_\kappa = 4$. Now let $\xi_\lambda = -3$ so that $\omega_\lambda = \omega_\kappa = 4$. Thus the condition for Proposition 2.8 is satisfied.

Now, $D := \omega_\kappa^2 + \omega_\lambda^2 - 4 = 4^2 + 4^2 - 4 = 5 + 5 - 4 = 6$, and $\eta = (\xi_\kappa - \xi_\kappa^{-1})^{-1} = (-4 + 3)^{-1} = -1$. Notice also that $\omega_\lambda^2 = 5 = -D = -6 \pmod{11}$, and so the condition for Proposition 2.9 is also satisfied.

Hence the fully regular map determined by the following generators is both self-dual and self-Petrie: $R_4 = \begin{pmatrix} \xi_\kappa & 0 \\ 0 & \xi_\kappa^{-1} \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$ and

$$S_4 = \eta \begin{pmatrix} -\omega_\lambda \xi_\kappa^{-1} & -D \\ 1 & \omega_\lambda \xi_\kappa \end{pmatrix} = -1. \begin{pmatrix} -4 \times -3 & -6 \\ 1 & 4 \times -4 \end{pmatrix} = \begin{pmatrix} -1 & 6 \\ -1 & 5 \end{pmatrix}.$$

This example corresponds to the map N35.3 in Conder's census [11]. •

Remark 2.14. The above work has only addressed regular maps on linear fractional groups where the associated finite field has odd characteristic. Many of the calculations would look quite different if we were to consider the case when $p = 2$. Regular maps whose underlying group is $\mathrm{PSL}(2, 2^\alpha) \cong \mathrm{SL}(2, 2^\alpha)$ for some α are investigated in Chapter 3. ⋆

2.2.4 Summary

At this point it is useful to summarise the section and condense it into one Corollary which will motivate the work in the next section.

Corollary 2.15. *Let $k \geq 5$ be odd. Assume that there is a prime $p \geq 5$ such that $p \equiv \pm 1 \pmod{2k}$ and also $p \equiv \pm 1 \pmod{12}$, and one of the finite fields $GF(p)$ or $GF(p^2)$ has a primitive kth root of unity ζ such that $3(\zeta + \zeta^{-1}) + 2 = 0$. Then there exists a (necessarily non-orientable) self-dual and self-Petrie regular map of valency k with automorphism group $G \cong PSL(2, p)$.*

Proof (sketch). This corollary is essentially a rewording of Corollary 2.10, where we assume $k \geq 5$, let $\xi = \xi_\kappa = \xi_\lambda$, define $\zeta := \xi^2$, and rearrange the condition $3\omega^2 = 4$ to the equivalent $3(\zeta + \zeta^{-1}) + 2 = 0$.

The exceptional type where $k = 5$ is not an issue, since by the choice of ξ we have $\omega = \omega_\kappa = \omega_\lambda$, and by Proposition 2.12 the resulting automorphism group G will not collapse to A_5.

Assuming $p \equiv \pm 1 \pmod{2k}$ ensures there is a primitive $2k$th root of unity in the field $\mathrm{GF}(p)$ or $\mathrm{GF}(p^2)$, namely ξ which implies that ζ is a primitive kth root of unity as described. Notice that $\omega = \xi + \xi^{-1}$ can be compared with the quadratic equation $x^2 + \omega x + 1 = 0$ which has roots ξ and ξ^{-1} in, at worst $\mathrm{GF}(p^2)$, and so $\omega \in \mathrm{GF}(p)$, thus ensuring, by the work in [19] that the resulting automorphism group is $\mathrm{PSL}(2, p)$.

Meanwhile, the condition that $p \equiv \pm 1 \pmod{12}$ simply ensures that the equation $3\omega^2 = 4$ has a solution for ω in $\mathrm{GF}(p)$, which is necessary for the above, and this will be the case if and only if 3 is a square in the field. □

2.3 A regular map with Trinity symmetry, for given prime valency $k = \ell = \pi \geq 5$

In this section we take a detour into cyclotomic polynomials and algebraic number theory in order to prove that the conditions for Corollary 2.15 can indeed be satisfied for every prime greater than or equal to five. The inspiration for this approach is the similar successful application used by Jones, Mačaj and Širáň to complete the proof of the existence of a non-orientable regular map for all hyperbolic types, see [40]. The remaining sections of this chapter present joint work with Jay Fraser and Jozef Širáň which is published in [26].

Given a prime $k = \pi \geq 5$, the aim is to build a finite field which contains a primitive kth root of unity $\overline{\alpha}$ such that $3(\overline{\alpha} + \overline{\alpha}^{-1}) + 2 = 0$, thereby satisfying the conditions for Corollary 2.15. In this section we prove:

Proposition 2.16. *Let $k \geq 5$ be an odd prime. Then there exists a prime p such that $p \equiv \pm 1 \pmod{2k}$ and $\pmod{12}$, and the finite field $GF(p)$ or $GF(p^2)$ contains an element ζ, a primitive kth root of unity such that $3(\zeta + \zeta^{-1}) + 2 = 0$.*

The proof of the Proposition 2.16 can be seen to rely on the following set-up and Lemmas 2.17, 2.18 and Proposition 2.19. Proposition 2.16 is then an easy corollary.

Let $k := \pi \geq 5$ be an odd prime. Let α be a primitive complex k-th root of unity. The minimal polynomial for α is the k-th cyclotomic polynomial. Define $h := \alpha + \alpha^{-1}$ and let $K = \mathbb{Q}(h)$ be the field obtained by adjoining h to the rationals. It is known [62, Proposition 2.16] that the ring $\mathcal{O}$ of algebraic integers of K is $\mathbb{Z}(h)$. We will focus on the algebraic integer $g = 3h + 2 \in \mathcal{O}$. Note that the form of g resembles the condition in Corollary 2.15, thereby giving a hint of the aim and structure of this section: to find a ring $\mathcal{O}'$ with some maximal ideal J, which contains g and is of finite index in $\mathcal{O}'$, such that $\mathcal{O}'/J$ is a finite field. Observe also that $g \neq 0$, for otherwise α would be a root of a quadratic polynomial over $\mathbb{Z}$, contrary to $k \geq 5$.

The first useful Lemma is as follows:

Lemma 2.17. *If $k \geq 5$ is a prime, then $N(g) \neq \pm 1$; in particular, the non-zero element $g \in \mathcal{O}$ is not a unit of the ring $\mathcal{O}$. Moreover, for every prime factor p of $N(g)$ one has $p \geq 5$.*

The proof of this first lemma relies on a pleasing coincidence between our calculations and modular arithmetic, as we will see in due course. For now we continue with the overview of the proof:

Consider now the field $K' = \mathbb{Q}(\alpha)$, an extension of K of degree two. Let $\mathcal{O}'$ be the ring of algebraic integers of K'; it is well-known [62, Theorem 2.6] that $\mathcal{O}' = \mathbb{Z}(\alpha)$, and, of course, $[\mathcal{O}' : \mathcal{O}] = 2$. The norm $N'(z)$ of any $z \in \mathcal{O}'$ is now the product $\prod_t \sigma_t(z)$ taken over all injective homomorphism $\sigma_t : \mathcal{O}' \to \mathbb{C}$ given by $\sigma_t(\alpha) = \alpha^t$ for t between 1 and $k-1$ coprime to k, and again one has $N'(z) \in \mathbb{Z}$. The two norms, N on $\mathcal{O}$ and N' on $\mathcal{O}'$, are related by $N'(y) = (N(y))^2$ for each $y \in \mathcal{O}$.

We will keep assuming that $k \geq 5$ is an odd prime, and we let $p \geq 5$ be an arbitrary prime divisor of $N(g)$, which exists by Lemma 2.17. We continue by considering the ideal $\langle g, p\rangle$ of $\mathcal{O}' = \mathbb{Z}(\alpha)$ generated by the elements g and p.

Lemma 2.18. *If $k \geq 5$ is a prime and if $p \geq 5$ is a prime divisor of $N(g)$, the ideal $\langle g, p\rangle$ is proper in the ring $\mathcal{O}'$.*

To maintain the flow of this section we defer the proof and continue as follows:

By Lemma 2.18, the ideal $\langle g, p\rangle$ is contained in some maximal ideal $J = J_p$ of the ring $\mathcal{O}'$. Since $\mathcal{O}'$ is a Dedekind domain, the ideal J has finite index in $\mathcal{O}'$ and so $\mathcal{O}'/J$ is a finite field F of characteristic p, that is, $F \cong \mathrm{GF}(p^m)$ for some $m \geq 1$. Recalling our assumption of primality of k we show that the (multiplicative) order of the element $\overline{\alpha} = \alpha + J$ in the field $F = \mathcal{O}'/J$ is equal to k. Indeed, suppose this is not the case. Then, because of primality of k, the order of $\overline{\alpha}$ in F would have to be one, meaning that $\overline{\alpha} = 1$ in F. But then, since the element $\overline{g} = g + J$ is equal to zero in F, we would have $0 = \overline{g} = 3(\overline{\alpha} + \overline{\alpha}^{\,-1}) + 2 = 8$ in F, a contradiction as p is odd. Observe also that $k \neq p$ since no element in F has multiplicative order p.

This way we have constructed a finite field F of characteristic p containing a primitive k-th root $\overline{\alpha}$ of unity such that $3(\overline{\alpha} + \overline{\alpha}^{\,-1}) + 2 = 0$. We now invoke the analysis concerning Corollary 2.15, which fully applies to our situation. As the result we conclude that F is the prime field F_p if and only if $\overline{\alpha} \in F_p$ for $p \equiv 1 \pmod{2k}$; otherwise F is a quadratic extension of F_p for $p \equiv -1 \pmod{2k}$. In both cases we have $p \equiv \pm 1 \pmod{12}$ because 3 has to be a square in F_p. Summing up, we will have proved:

Proposition 2.19. *Let $k \geq 5$ be an odd prime and let α be a primitive complex k-th root of unity. Further, let $g = 3(\alpha + \alpha^{-1}) + 2$ and let $N(g)$ be the norm of g in the ring $\mathbb{Z}(\alpha)$. Then, $N(g) \notin \{0, \pm 1\}$, every prime divisor p of $N(g)$ satisfies $p \geq 5$, and $p \equiv \pm 1 \pmod{2k}$ and $\pmod{12}$, and for every such p there is a finite field F of order p or p^2 containing a primitive k-th root $\overline{\alpha}$ of 1 such that $\overline{g} = 3(\overline{\alpha} + \overline{\alpha}^{\,-1}) + 2 = 0$ in F.*

It remains to fill in the fine detail by proving Lemmas 2.17 and 2.18.

Proof of Lemma 2.17. Recall that the norm $N(y)$ of an element $y \in \mathcal{O}$ is defined as the product $\prod_t \sigma_t(y)$, where σ_t denotes the injective homomorphism $\mathcal{O} \to \mathbb{C}$ into the field of complex numbers, uniquely determined by $\sigma_t(\alpha) = \alpha^t$, and t ranges over all integers between 1 and $(k-1)/2$ that are relatively prime to k. It is well-known that $N(y)$ is an integer for any $y \in \mathcal{O}$, which is a consequence of the invariance of $N(y)$ under the endomorphisms σ_t.

For the norm of our element $g \in \mathcal{O}$ we thus have $N(g) = \prod_t (3\sigma_t(h) + 2)$, the product being taken over all t between 1 and $(k-1)/2$, coprime to k. The $\varphi(k)/2$ images $\sigma_t(h)$ appearing in this product are precisely the roots of the minimal polynomial $\Psi(x)$ of degree $\varphi(k)/2$ for $h = \alpha + \alpha^{-1}$, see for example [40]. So, if $\Psi(x) = \prod_t (x - \sigma_t(h)) = \sum_j a_j x^j$ where j ranges from 0 to $\varphi(k)/2$, then the integral coefficients a_j will also appear in the expansion of the above product. More precisely, letting $r = \varphi(k)/2$ and $u = -2/3$, one has

$$N(g) = \prod_t (3\sigma_t(h) + 2) = (-3)^r \prod_t (u - \sigma_t(h)) = (-3)^r \sum_{j=0}^{r} a_j u^j = \sum_{j=0}^{r} (-3)^{r-j} 2^j a_j \ . \tag{2.3}$$

Let us consider what happens when we look at (2.3) modulo 9. Up to the last two terms all the remaining ones are a multiple of 9 and so, noting that $a_r = 1$, we have

$$N(g) \equiv 2^r - 3{\cdot}2^{r-1} a_{r-1} \pmod 9 \ . \tag{2.4}$$

We will show that if $k \geq 5$ and k is a *prime*, then the norm $N(g)$ is not equal to ± 1, which means that g is then *not* a unit of the ring $\mathcal{O}$. Indeed, let $k \geq 5$ be a prime, so that $r = \varphi(k)/2 = (k-1)/2$. By [60] we then also have $a_{r-1} = 1$, and the congruence (2.4) becomes

$$N(g) \equiv 2^{(k-1)/2} - 3{\cdot}2^{(k-3)/2} \equiv -2^{(k-3)/2} \pmod 9 \ .$$

It is easy to check that $2^j \equiv \pm 1 \pmod 9$ for a positive integer j if and only if j is a multiple of 3. This means that if k is prime, the norm $N(g)$ can be congruent to ± 1 modulo 9 only if $(k-3)/2$ is a multiple of 3, giving a contradiction if $k \geq 5$. Further, from (2.3) with the help of $a_r = 1$ and $a_0 = \pm 1$ [60] it follows that if $k \geq 5$ is a prime, then $N(g)$ is divisible neither by 2 nor by 3. □

For interest, here we include, courtesy of Širáň, Table 2.1 showing the first few values of $N(g)$ for odd k between 5 and 29, with $\Phi(n)$ standing for the prime factorisation of

k	$N(g)$	$\Phi(\lvert N(g)\rvert)$
5	-11	prime
7	-13	prime
9	-73	prime
11	$+263$	prime
13	-131	prime
15	-239	prime
17	-4079	prime
19	$+15503$	37×419
21	$+5209$	prime
23	-4093	prime
25	$+56149$	prime
27	-16417	prime
29	$+3161869$	59×53591

Table 2.1: Values of $N(g)$ for odd k between 5 and 29, and the prime divisors.

n. The same table appears in [26].

Finally we fill the last gap in proving Proposition 2.19.

Proof of Lemma 2.18: Suppose that $\langle g, p\rangle = \mathcal{O}'$, which means that $1 = Ag + Bp$ for some $A, B \in \mathcal{O}'$. Clearly $A \neq 0$, for otherwise $1 = N'(B)N'(p) = N'(B)p^{k-1}$ and so $N'(B)$ would not be an integer. Now, $1 = N'(1) = N'(Ag + Bp) = \prod_\sigma \sigma(Ag + Bp)$, where the product is being taken over all the $\varphi(k) = k - 1$ embeddings $\sigma :\ \mathcal{O}' \to \mathbb{C}$. Expansion of this product gives $N'(Ag + Bp) = N'(A)N'(g) + cp$ for some $c \in \mathcal{O}'$. Thus, $cp \in \mathbb{Z}$ and so either $c \in \mathbb{Z}$ or $c = \pm 1/p$. As p is a divisor of $N'(g) = (N(g))^2$ and $N'(A)$ is a non-zero integer, in either case it follows that $N'(Ag + Bp) \neq 1$, a contradiction. □

This completes the proof of Proposition 2.16, and this along with Corollary 2.15, yields the following existence result for regular maps with Trinity symmetry and prime valency $k \geq 5$.

Theorem 2.20. *For every odd prime $k \geq 5$ there exists a prime $p \equiv \pm 1 \pmod{2k}$ and $\pmod{12}$ such that $PSL(2, p)$ is the automorphism group of a (non-orientable) regular, self-dual and self-Petrie-dual map of valency k.*

To motivate the next section we will have the above maps, each of which has prime valency, in mind for use as base maps to prove the general result, Theorem 2.1.

2.4 Magic trick - lifting construction

Theorem 2.20 in the previous section proves that there is a self-dual self-Petrie map for every prime valency $k = \pi \geq 5$. The following Theorem is based on the method used by Archdeacon, Conder and Širáň in [3] to lift their base map, a self-dual, self-Petrie (indeed kaleidoscopic) map of type $(2, 2)$, and thereby prove the existence of such maps for all even valencies.

As alluded to earlier, the proof of the main result will be done with the help of coverings, and more specifically using a non-orientable analogue of Theorem 2.1 of [3]. We state it here in a restricted version sufficient for our purpose.

Proposition 2.21. ***Non-orientable Analogue of Theorem by Archdeacon, Conder, Širáň.*** *If there is a finite non-orientable regular map of odd valency $d \geq 5$ with Trinity symmetry and with automorphism group G, then for any odd integer $n \geq 3$ there is a non-orientable regular map of degree nd with Trinity symmetry and automorphism group isomorphic to $(\mathbb{Z}_n)^{1+|G|/4} \rtimes G$.*

Proof (sketch). As indicated, this result was proved in [3, Theorem 2.1] for orientable maps (and, in this category, in a much more general setting that included also external symmetries induced by hole operators). The parts of the proof in [3] that refer to regularity, self-duality and self-Petrie-duality apply almost word-by-word to the non-orientable case and we thus give only a sketch of the arguments here. We will assume familiarity with the theory of lifts of maps by corner voltage assignments as explained e.g. in [4, 5, 3]; a *corner* of a regular map $M = (G; x, y, z)$ is any 2-subset of the form $\{g, gz\}$ for $g \in G$.

Now let $\mathcal{M} = (G; x, y, z)$ be a regular map as in the statement. For odd $n \geq 3$ let $H = \mathbb{Z}_n^{|G|/2}$ be the space of all $|G|/2$-tuples with entries from $\mathbb{Z}_n$ and let $\mathcal{E}$ be the set of unit vectors (those with exactly one non-zero coordinate, equal to 1) in H. Define a corner voltage assignment σ on flags of $\mathcal{M}$ – that is, on the elements of G – in the group H by assigning the $|G|/2$ two-element subsets $\{\varepsilon, -\varepsilon\}$ for $\varepsilon \in \mathcal{E}$ to the $|G|/2$ corners $\{g, gz\}$ for $g \in G$ in an arbitrary one-to-one fashion. By arguments in the proof of Theorem 2.1 in [3] that do not depend on orientability, the lift of the map $\mathcal{M}$ of type (d, d) by the voltage assignment σ has $n^{-1+|G|/4}$ components, each isomorphic to a regular map $\mathcal{M}^\sigma(G^\sigma; x^\sigma, y^\sigma, z^\sigma)$ of type (nd, nd) for the group $G^\sigma = (\mathbb{Z}_n)^{1+|G|/4} \rtimes G$ and suitable involutory generators $x^\sigma, y^\sigma, z^\sigma$ of G^σ. Moreover, by the reasoning in the same proof (again applying also to non-orientable maps), Trinity symmetry of $\mathcal{M}$ implies Trinity symmetry of $\mathcal{M}^\sigma$. Note that both $\mathcal{M}$ and $\mathcal{M}^\sigma$ are non-orientable as their Petrie walks (of length d and nd) have odd length. □

2.5 Conclusion, with proof of Theorem 2.1

Drawing together the work from earlier in the chapter, we are now in a position to prove Theorem 2.1.

Proof. Theorem 2.20 proves that there exists a fully regular map with Trinity symmetry and type (π, π) for all primes $\pi \geq 5$. Then Proposition 2.21 may be used to extend this claim to maps of type $(b\pi, b\pi)$, for all multiples of the prime $\pi \geq 5$.

Mind the gap! The above fails to address the cases where $k = 3^a$ for $a \geq 2$. However, Example 2.11 gives a fully regular map with Trinity symmetry and type $(9, 9)$. Applying the same method (Proposition 2.21) covers these remaining cases, and so we may conclude that the claim holds for all odd $k \geq 5$. □

Combining this proof with the previously known construction for even valencies, we may conclude:

Corollary 2.22. *Given an arbitrary degree $k \geq 4$, there exists a reflexible regular map with Trinity symmetry and type (k, k).*

Chapter 3

Groups of operators on regular maps

A map is said to be *super-symmetric* if it is kaleidoscopic and also has Trinity symmetry. The motivation for this chapter is to consider the existence of such maps, and in particular small examples of such maps. The previously mentioned result of Archdeacon, Conder and Širáň [3] settles their existence for even valencies, while it is implicit in the work of Jones [37] that we can build such maps for any valency $k = 2^{2\alpha} - 1$. Meanwhile Conder [12] strongly suspects that there are no super-symmetric maps with odd prime valency. In Chapter 2 we have proved that fully regular maps with Trinity symmetry and odd valency exist, however there is no obvious way to extend our previous result if we demand the map also to be kaleidoscopic. Hence the question of existence for super-symmetric maps of odd valency remains largely open.

By their nature, super-symmetric maps possess many external symmetries. The work in this chapter focusses on understanding the group of operators generated by the dual operator, the Petrie operator and the appropriate hole operators for a given fully regular map, with a view to applying these results to a map which is super-symmetric. This super-symmetric map is built by amalgamating the maps in a single orbit in such a way that each operator simply permutes the maps within the so-called parallel product. For this reason it is also of interest to study the orbit structure of the action of these operators on the set of fully regular maps with a given underlying group. Much of the work presented in this chapter is subject to further research.

When considering operators being applied to a fully regular map we make the same assumption as previously: the edge stabiliser of the map is isomorphic to V_4, or equivalently that the map contains edges but no semi-edges, that is it is not a semistar. Thus we may safely assume that $x \neq y \neq xy \neq x$.

In this chapter we will use the notation (k, ℓ, m) to refer to a fully regular map of type (k, ℓ) whose Petrie faces have boundary walks of length m. We call this the extended type of the map, but where confusion will not occur we may omit the word "extended". It is in our interests to ensure that $2 \notin \{k, \ell, m\}$, and for $\mathcal{M}(G; x, y, z)$ this is automatically given when the automorphism group G supports no spherical regular maps.

3.1 Operators acting on sets of fully regular maps

An operator is a function which maps a given fully regular map to a fully regular map with the same underlying group. In particular, for the cases which we consider, an operator $\mathbf{T}$ maps $\mathcal{M}(G; x, y, z)$ to $\mathcal{M}\mathbf{T} = \mathcal{M}'(G; x', y', z')$ where $\langle x, y\rangle = \langle x', y'\rangle \cong \mathrm{V}_4$. It is important to note at this stage that some operators may not respect the above condition when applied to every fully regular map. Indeed for an operator to be valid for our purposes it must be invertible and so there may be certain conditions on, for example, the valency of the map.

Let G be a given group, and let Ω_G be the set which consists of all fully regular maps $\mathcal{M}$ defined by $\mathcal{M}(G; x, y, z)$ for some $x, y, z \in G$. There are always some operators which *do* satisfy the above condition when applied to all the maps in Ω_G and such an operator $\mathbf{T}$ may be regarded as a permutation $\mathbf{T} : \Omega_G \to \Omega_G$. In this context it is clear that $\mathbf{T}$ will generate a permutation group, and we may then also include other such operators as generators to yield further groups of operators. In this way we will consider groups of operators acting on the set Ω_G of regular maps with a given underlying group G.

This approach will be useful later, but for now we focus on groups of operators which are always clearly defined and invertible. In order to ensure this, we must be certain that each generating operator may be applied to every fully regular map, yielding a fully regular map, and so the composition of operators is well-defined, and also that each generating operator has a unique inverse operator. In that case, the set of operators, along with composition of functions, forms a group. The identity element $\mathbf{Id}$ is the operator which sends every $\mathcal{M}(G; x, y, z)$ to itself.

Operators are often defined in terms of monodromy, which makes a lot of sense when visualising how to build the resulting image map $\mathcal{M}\mathbf{T}$ from $\mathcal{M}$. In this work we take advantage of the full regularity of the maps with which we work, by defining the operators with respect to their action on the automorphism group G of the fully regular map $\mathcal{M}(G; x, y, z)$. It is the isomorphism between the monodromy group for a fully regular map and its automorphism group which allows us to take this liberty.

3.1.1 Well-known dualities and trialities: $\langle\, \mathbf{D}, \mathbf{P}\,\rangle$

The duality operators $\mathbf{D}$ and $\mathbf{P}$ are well-known and, for ease of reference, in Table 3.1, we recreate the information presented in the introduction, using the current working notation, and including lines to indicate z and the extended type of the

	$\mathcal{M}$	$\mathcal{M}\mathbf{D}$	$\mathcal{M}\mathbf{P}$	$\mathcal{M}\mathbf{DP}$	$\mathcal{M}\mathbf{PD}$	$\mathcal{M}\mathbf{DPD}$
x'	x	y	xy	xy	y	x
y'	y	x	y	x	xy	xy
$x'y'$	xy	xy	x	y	x	y
z'	z	z	z	z	z	z
Type	(k, ℓ, m)	(ℓ, k, m)	(k, m, ℓ)	(ℓ, m, k)	(m, k, ℓ)	(m, ℓ, k)

Table 3.1: The action of the operators $\mathbf{D}$ and $\mathbf{P}$ on $\mathcal{M}(G; x, y, z)$

resulting maps. In each case z is fixed and the image $\mathcal{M}\mathbf{T}$ of the map $\mathcal{M}(G; x, y, z)$ under the operator $\mathbf{T}$ is $\mathcal{M}'(G; x', y', z')$.

The group of operators generated by these two dual operators is well-known, and the following fact has been widely used for decades, [21, 65].

Lemma 3.1 (Wilson)**.** *The group* $\langle \boldsymbol{D}, \boldsymbol{P} \rangle$ *is isomorphic to* S_3*.*

Proof. Remember that in general we assume that a graph (and hence a fully regular map) has no semi-edges, and as a result we know that $x \neq y \neq xy \neq x$. The elements of the group $\langle \mathbf{D}, \mathbf{P} \rangle$ permute the three involutions in the edge-stabiliser $\langle x, y \rangle$, and fix the generator z. It is easy to demonstrate that all such permutations are generated and so the group is isomorphic to S_3. Note that this group corresponds to the outer automorphism group of the edge-stabiliser $\langle x, y \rangle \cong \mathrm{V}_4$. □

There are further operators, namely hole operators, which may be applied to a given regular map of valency k, and these are as described in the following subsection, along with the caveat that more work is necessary to ensure they are fit to be defined as group elements and so be composed with the operators in this section.

Recall that the operators in $\langle \mathbf{D}, \mathbf{P} \rangle$ respect both the automorphism group of the object map and at the same time either maintain the underlying graph, or replace it with the coresponding dual graph. As discussed in [57], the hole operators are the only further operators which maintain both the underlying graph of the object map and also the automorphism group, while fixing (pointwise) the elements in the edge stabiliser $\langle x, y \rangle$. Taken together these dualities and hole operators, along with their compositions, are thus the only operations which have a natural geometric foundation and motivation with respect to the original object fully regular map. Thus investigation of operators on regular maps has a natural goal: to fully understand how hole operators interact with the operators in $\langle \mathbf{D}, \mathbf{P} \rangle$. In the following sections we make a start on this challenging quest.

	$\mathcal{M}$	$\mathcal{M}\mathbf{H}_{k-1}$	$\mathcal{M}\mathbf{H}_j$	$\mathcal{M}\mathbf{H}_j\mathbf{H}_i = \mathcal{M}\mathbf{H}_{ij}$
x'	x	$x \quad (= x^y)$	x	x
y'	y	$y \quad (= y^y)$	y	y
$x'y'$	xy	xy	xy	xy
z'	z	$(zy)^{-1}y = yzy = z^y$	$(zy)^j y = (zy)^{j-1}z$	$((zy)^j y.y)^i y = (zy)^{ji} y$
Type	(k, ℓ, m)	(k, ℓ, m)	$(k, ?, ?)$	$(k, ?, ?)$

Table 3.2: Hole operators applied to the map $\mathcal{M}(G; x, y, z)$

3.1.2 Hole operators modulo k : $\langle\, \mathbf{H}_j \mid (j, k) = 1 \,\rangle$

Let $\mathcal{M}_k(G; x, y, z)$ be a fully regular map of valency k. The rotational power $\mathbf{H}_j$, so denoted as some authors call this the jth hole operator, can be applied to the fully regular map $\mathcal{M}_k$ whenever k and j are coprime, and when this is the case the hole operator is said to be *valid*. The operator $\mathbf{H}_j$ is then defined as follows: $\mathbf{H}_j$ sends $\mathcal{M} = (G; x, y, z)$ to $\mathcal{M}\mathbf{H}_j = (G; x, y, (zy)^j y)$. This operator fixes x and y and so, informally and thinking in terms of the equivalent definition in terms of monodromy, it doesn't interfere with edges, but only the cyclic ordering of edges in the embedding at each vertex. As such the image map has the same underlying graph as the original map, thereby preserving the valency k, and so ensuring that composition of the hole operators may be properly defined. It is of course possible that the image of a map $\mathcal{M}\mathbf{H}_j$ is isomorphic to the original fully regular map $\mathcal{M}$, in which case j is said to be an *exponent of the map* $\mathcal{M}$.

Table 3.2 presents how the $(k-1)$th and the jth rotational powers are defined when applied to the map $\mathcal{M}_k(G; x, y, z)$, the latter operator having an inverse only when $(j, k) = 1$.

Let us now restrict ourselves to the set $\mathcal{H}_k = \{\mathbf{H}_j\}$ where $(j, k) = 1$ and the index j is such that $-1 \le j \le k-2$, working modulo k, where $k \ge 3$. Clearly $\mathbf{H}_1 \in \mathcal{H}_k$ is the identity operator. Meanwhile the coprimality of j and k ensures that the hole operator $\mathbf{H}_j$ has a unique and well-defined inverse $\mathbf{H}_i$ where $-1 \le i \le k-2$ and $ij \equiv 1 \pmod{k}$. Indeed, under composition, the set of hole operators $\mathcal{H}_k$ is closed and forms the group $\langle\, \mathbf{H}_j \mid (j, k) = 1 \,\rangle$, where the index j is always assumed to be taken modulo k for some given k.

By the commutativity of multiplication, along with the final column of Table 3.2, it is clear that the hole operators $\mathbf{H}_i$ and $\mathbf{H}_j$ commute, so the group $\langle\, \mathbf{H}_j \mid (j, k) = 1 \,\rangle$ is Abelian. In fact this group is isomorphic, in a very natural way, to U_k, the well-known multiplicative group of units in the ring of integers modulo k.

Lemma 3.2. *The group of hole operators modulo k is Abelian and*

$\langle\, \boldsymbol{H}_j \mid (j,k)=1 \,\rangle \cong U_k$. □

The notation in Table 3.2 leads us naturally to consider conjugating a given map $\mathcal{M}(G;x,y,z)$ by an element of the group $g \in G$.

Lemma 3.3. *Let $\mathcal{M}(G;x,y,z)$ be a fully regular map and $g \in G$. Then $\mathcal{M}^g$ is defined as $\mathcal{M}^g(G;x^g,y^g,z^g)$ and this is also a fully regular map, isomorphic to $\mathcal{M}$.*

Proof. Conjugation by an element of G is always an automorphism of the group G, and so by the definition we have the map isomorphism sending $\mathcal{M}$ to $\mathcal{M}^g$, namely conjugation of each generating element by g. □

Remark 3.4. The above lemma should be no surprise. $\mathcal{M}^g$ may be thought of as the same object as the original map $\mathcal{M}$, the only difference being that it is rooted at a different distinguished flag. Indeed many authors would not distinguish between these two objects, preferring instead to identify a map with the conjugacy class of isomorphic maps, or equivalently with the (necessarily normal) *map subgroup*, that is the kernel of the natural epimorphism which maps the full triangle group $T_{k,\ell}$ to the automorphism group G. ⋆

Since $\langle x,y \rangle$ is Abelian, an immediate corollary is as follows.

Corollary 3.5. *Conjugation by an element in $\langle x,y \rangle$ of any map $\mathcal{M}(G;x,y,z)$ yields a map $\mathcal{M}'(G;x,y,z')$ in the same isomorphism class as $\mathcal{M}$.* □

That $\mathcal{M}\mathbf{H}_{k-1} = \mathcal{M}^y$ immediately yields the well-known fact:

Lemma 3.6. $-1 \equiv k-1$ *is an exponent of any given fully regular map of valency k.* □

Since $k-1$ is always coprime to k, the operator $\mathbf{H}_{-1}$ is always invertible. Furthermore -1 is an exponent of any reflexible map, so it makes sense to have an efficient notation for it. We denote $\mathbf{S} := \mathbf{H}_{-1}$, and so $\mathbf{S}$ transforms $\mathcal{M} = (G;x,y,z)$ to $\mathcal{M}\mathbf{S} = (G;x,y,yzy)$.

3.1.3 Mixing operators: $\langle\, \mathbf{P}, \mathbf{H}_j \mid (j,k)=1 \,\rangle$, $\langle\, \mathbf{DP}, \mathbf{S} \,\rangle$ and $\langle\, \mathbf{D}, \mathbf{P}, \mathbf{S} \,\rangle$

The task in hand is to determine more about the structure of the group of operators which, for given k, is generated by the hole operators $\{\mathbf{H}_j \mid (j,k)=1\}$ and the dual operators $\{\mathbf{D}, \mathbf{P}\}$. We must be careful: the dualities and their compositions do not always leave the valency of a map invariant, and yet the hole operator $\mathbf{H}_j$ (when applied to a given fully regular map) only has a unique inverse operator if j is coprime to the valency k. For the moment we continue to avoid having to consider this technicality, mostly by limiting ourselves to groups of operators generated by the involutory hole operator $\mathbf{S}$ which is self-inverse and always defined on any fully regular map (regardless of valency), (indeed it is a map isomorphism as we have just seen), along with the dualities $\mathbf{D}$, $\mathbf{P}$, which are also always well-defined, self-inverse operators. Furthermore we are dealing with the general case and so we must make no assumptions about the group structure G. Such assumptions may be made later, when studying the action of a group of operators on the set of maps with a given group, as in section 3.2.2.

Another thing to note is that, when applied to any map $\mathcal{M}(G; x, y, z)$, the operator $\mathbf{P}$ only changes x (with reference to y) and hence leaves the valency invariant. Meanwhile $\mathbf{H}_j$ only alters z, also with reference just to y. So, since the Petrie operator and hole operators modulo a given k thus may be composed, we have the following easy lemma:

Lemma 3.7. *The operator* $\boldsymbol{P}$ *commutes with every hole operator* $\boldsymbol{H}_j$. *Hence* $\langle\, \boldsymbol{P}, \boldsymbol{H}_j \mid (j,k) = 1 \,\rangle = \langle\, \boldsymbol{P} \,\rangle \times \langle\, \boldsymbol{H}_j \mid (j,k) = 1 \,\rangle \cong C_2 \times U_k$. □

Being careful to ensure that we are still dealing with a group of operators, and in particular that composition of operators is well-defined, this allows us, on occasion, to focus initially on understanding, for a given k, the group $\langle\, \mathbf{DP}, \mathbf{H}_j \mid (j,k) = 1 \,\rangle$ since it is clear that:

Lemma 3.8. *Where it is properly defined, the group* $\langle\, \boldsymbol{D}, \boldsymbol{P}, \boldsymbol{H}_j \mid (j,k) = 1 \,\rangle$ *is isomorphic to* $\langle\, \boldsymbol{DP}, \boldsymbol{H}_j \mid (j,k) = 1 \,\rangle \rtimes \langle\, \boldsymbol{P} \,\rangle$.

Proof. The operator $\mathbf{P}$ commutes with $\mathbf{H}_j$ for all j coprime with k and conjugation by $\mathbf{P}$ inverts $\mathbf{DP}$. Note that the subgroup $\langle\, \mathbf{DP}, \mathbf{H}_j \mid (j,k) = 1 \,\rangle$ does not contain the involution $\mathbf{P}$, and it is normal, with index 2, in $\langle\, \mathbf{D}, \mathbf{P}, \mathbf{H}_j \mid (j,k) = 1 \,\rangle$. □

Not quite so easy to see are the following relationships, which must surely be known, but I haven't seen them mentioned anywhere:

Lemma 3.9. *The group* $\langle\, \boldsymbol{DP}, \boldsymbol{S} \mid \boldsymbol{S}^2, (\boldsymbol{DP})^3, (\boldsymbol{DPS})^3 \,\rangle$ *is isomorphic to* A_4.

Proof. $\mathbf{DP}$ is well-known to have order three and, as shown in the following table, $\mathbf{SPD}$ has order three while their product $\mathbf{S}$ has order two. Thus the group

$\langle\, \mathbf{DP}, \mathbf{SPD}\,\rangle = \langle\, \mathbf{DP}, \mathbf{S}\,\rangle$ corresponds to the orientation-preserving group of automorphisms of a regular map of type $(3,3)$. There is only one such isomorphism class of map, that of the tetrahedron, and it is known that $Aut^+(\mathcal{T}) \cong \mathrm{A}_4$.

	$\mathcal{M}$	$\mathcal{M}\mathbf{PD}$	$\mathcal{M}\mathbf{DP}$	$\mathcal{M}\mathbf{S}$	$\mathcal{M}\mathbf{SPD}$	$\mathcal{M}(\mathbf{SPD})^2$	$\mathcal{M}(\mathbf{SPD})^3$
x'	x	y	xy	x	y	xy	x
y'	y	xy	x	y	xy	x	y
z'	z	z	z	$yzy = z^y$	yzy	xzx	$(xzx)^x = z$

It can easily be checked that the group $\langle\, \mathbf{DP}, \mathbf{S}\,\rangle$ is no smaller than A_4 by considering its behaviour when applied to the map $\mathcal{M}(G; x, y, z)$, some of which is shown in the table. It is easy to see that there are four possible values for z', that is z, z^x, z^y and z^{xy}, which are in general distinct, and each of which has, by the application of $\mathbf{DP}$, three possible combinations for the ordered pair (x', y'). $\square$

Incorporating the Petrie operator, we establish the following.

Lemma 3.10. *The group $\langle\, \boldsymbol{D}, \boldsymbol{P}, \boldsymbol{S}\,\rangle$ is isomorphic to the symmetric group of degree four and has presentation $\langle\, \boldsymbol{D}, \boldsymbol{P}, \boldsymbol{S} \mid \boldsymbol{D}^2, \boldsymbol{P}^2, \boldsymbol{S}^2, (\boldsymbol{PS})^2, (\boldsymbol{DP})^3, (\boldsymbol{SDP})^3, (\boldsymbol{SD})^4\,\rangle \cong S_4$. Also $\langle\, \boldsymbol{D}, \boldsymbol{P}, \boldsymbol{S}\rangle = \langle\, \boldsymbol{DP}, \boldsymbol{DS}\rangle$.*

Proof. The key to proving the first claim is noting that, in general, the element $\mathbf{SD}$ has order 4. This is demonstrated in the table below. From the previous lemma, we have $\mathrm{A}_4 \cong \langle\, \mathbf{DP}, \mathbf{S}\,\rangle \leq \langle\, \mathbf{D}, \mathbf{P}, \mathbf{S}\,\rangle$ while $\langle\, \mathbf{D}, \mathbf{P}, \mathbf{S}\,\rangle = \langle\, \mathbf{DP}, \mathbf{S}\,\rangle \rtimes \langle\, \mathbf{P}\,\rangle$ since conjugation by $\mathbf{P}$ inverts $\mathbf{DP}$ and fixes $\mathbf{S}$. Thus the cardinality of the group is 24. It is certain that the group is not also isomorphic to a direct product of A_4 and C_2 since we know it contains an element of order 4, and so the only remaining possibility is $\langle\, \mathbf{S}, \mathbf{P}, \mathbf{D}\,\rangle \cong \mathrm{S}_4$.

	$\mathcal{M}$	$\mathcal{M}\mathbf{D}$	$\mathcal{M}\mathbf{S}$	$\mathcal{M}\mathbf{SD}$	$\mathcal{M}(\mathbf{SD})^2$	$\mathcal{M}(\mathbf{SD})^3$	$\mathcal{M}(\mathbf{SD})^4$
x'	x	y	x	y	x	y	x
y'	y	x	y	x	y	x	y
z'	z	z	yzy	yzy	$(yzy)^x = xyzyx$	$(xyzyx)^y = xzx$	$(xzx)^x = z$

An alternative way to prove the first claim is as follows. We again note (as above) that the element $\mathbf{DS}$ has order 4, and that the group must have cardinality 24. Now $\mathbf{PD}$ has order 3, $\mathbf{DS}$ has order 4, while their product is $\mathbf{PS}$ which has order 2. The involutions $\mathbf{S}$ and $\mathbf{P}$ commute so we have a group, $\langle\, \mathbf{D}, \mathbf{P}, \mathbf{S}\,\rangle$, generated by three involutions, two of which commute. Thus the group must be isomorphic to the (full) automorphism group for a fully regular map of type $(4,3)$ (or its dual of type $(3,4)$). There are only a very small number of such maps. It is natural to first consider the octahedron (or its dual the cube) both of which are orientable and are known to have orientation-preserving automorphism group isomorphic to S_4. In these cases the full automorphism group has cardinality 48 which is too large. Now consider the only alternative: regular maps of these types supported by the projective plane. Each of these has full automorphism group with cardinality 24 since the non-orientability guarantees that the orientation-preserving automorphism group $Aut^+ = \langle \mathbf{PD}, \mathbf{DS} \rangle$ is equal to the full automorphism group $Aut = \langle \mathbf{D}, \mathbf{P}, \mathbf{S} \rangle$, and again, it is known that the automorphism group is isomorphic to S_4.

This second approach leads us naturally to conclude that the second claim is true, and this may be verified another way by realising that $\mathbf{DP.PS.PD.PS.DP} = \mathbf{S}$, which easily can be seen by remembering that **DPS** has order three. □

At this stage we make a few general remarks.

Remark 3.11. Since $\mathcal{M}(\mathbf{SD})^2 = \mathcal{M}^{xy}$, it is clear that $(\mathbf{SD})^2$ is an external symmetry for any given fully regular map. ⋆

Remark 3.12. In the above work, we have made no assumption regarding the structure of the group G. In particular we do not know whether xy commutes with z. In the case of the map $\mathcal{M}(G; x, y, z)$ where xy does commute with z in G, the element $(xyz)^2 = 1$ and so the Petrie walk of $\mathcal{M}$ would have length 2. In such a case we find that $(\mathbf{SD})^2$ fixes $\mathcal{M}(G; x, y, z)$ and the group G would have to be the automorphism group for a fully regular map of spherical type. ⋆

Remark 3.13. By observing $\mathcal{M}(\mathbf{SD})^2 = \mathcal{M}^{xy} = ((\mathcal{M}\mathbf{PD})\mathbf{S})\mathbf{DP}$ it is clear that $\mathbf{SDSD} = \mathbf{PDSDP}$ and so it is a general truth that the involution $\mathbf{S}$ is equal to $(\mathbf{PDSD})^2$. This allows us to write down an alternative presentation for the group: $\langle \mathbf{D}, \mathbf{P}, \mathbf{S} \rangle = \langle \mathbf{PD}, \mathbf{SD} \mid (\mathbf{PD})^3, (\mathbf{SD})^4, (\mathbf{PDSD})^4, ((\mathbf{SD})^{-1}(\mathbf{PDSD})^2)^2 \rangle \cong S_4$ ⋆

Lemma 3.14. *The operator $(\boldsymbol{DS})^2 = (\boldsymbol{SD})^2$ commutes with all hole operators.*

Proof. We have already seen that **DS** has order four and both **D** and **S** are involutions so clearly $(\mathbf{DS})^2 = (\mathbf{SD})^2$. Let $\mathbf{H}_j$ be a hole operator for any allowable j. Remembering that $\mathcal{M}(\mathbf{DS})^2 = \mathcal{M}^{xy}$ where $\mathcal{M} = (G; x, y, z)$, the operator $\mathcal{M}(\mathbf{DS})^2\mathbf{H}_j = (G : x, y, (z^{xy}y)^j y)$. Meanwhile $\mathcal{M}\mathbf{H}_j(\mathbf{DS})^2 = (G : x, y, [(zy)^j y]^{xy})$. Now $(z^{xy}y)^j y = (xyzx)^j y = (x(yz)^j xy) = x(y(zy)^j y)xy = [(zy)^j y]^{xy}$, demonstrating that the two operators are equivalent, that is that $(\mathbf{DS})^2$ commutes with the hole operator $\mathbf{H}_j$. □

Remark 3.15. It is important to reiterate that at this stage we have simply considered the group of operators $\langle \mathbf{D}, \mathbf{P}, \mathbf{S} \rangle$ as just that, a group of operators under composition, without much regard for the individual objects they may be acting on. This is justified in the context of fully regular maps since each of the operators is always defined on a fully regular map, and in each case the underlying group is respected, meaning that the operators may be composed. The operators $\mathbf{D}$ and $\mathbf{P}$ are always defined, on any given fully regular map. Meanwhile, $j = -1$ is the only (non-trivial) value for which the operator $\mathbf{H}_j$ is necessarily defined on any given fully regular map of unknown valency $k \geq 3$, since it must be the case that $(j, k) = 1$. This is why the hole operator $\mathbf{S} = \mathbf{H}_{-1}$ has been given special treatment here. Beyond this, if we hope to study a larger group of operators then it is clearly crucial that composition of operator functions is well-defined, and, naturally, each element having a uniquely defined inverse is also critical. ⋆

There are examples where we know that a certain hole operator acts on all maps with a given underlying group. For example, in the case of a group $G \cong \mathrm{SL}(2, 2^\alpha)$ such that $\alpha \geq 2$, every fully regular map $\mathcal{M}(G; x, y, z)$ has odd valency. Thus the operator $\mathbf{H}_2$ is defined on all such maps (thereby ensuring that composition of this hole operator with the operators in $\langle\, \mathbf{D}, \mathbf{P}\,\rangle$ is well-defined) and, when composed with $\mathbf{D}$, behaves as per the following example.

Example 3.16. Let G be a group such that every non-involution has odd order, and $\mathcal{M}(G; x, y, z)$ be a given fully regular map. The condition stated ensures that the valency of any such map is odd, including after the application of $\mathbf{D}$, and so $\mathbf{H}_2$ is defined on all such maps. Repeated applications of $\mathbf{H}_2\mathbf{D}$ yields

$\mathcal{M}\mathbf{H}_2\mathbf{D} = (G; y, x, zyz)$,

$\mathcal{M}(\mathbf{H}_2\mathbf{D})^2 = (G; x, y, zyzxzyz)$,

$\mathcal{M}(\mathbf{H}_2\mathbf{D})^3 = (G; y, x, (zyzxzyz)y(zyzxzyz))$,

$\mathcal{M}(\mathbf{H}_2\mathbf{D})^4 = (G; x, y, (zyzxzyz)y(zyzxzyz)x(zyzxzyz)y(zyzxzyz))$, and then

$\mathcal{M}(\mathbf{H}_2\mathbf{D})^5 = (G; y, x,$
$(zyzxzyz)y(zyzxzyz)x(zyzxzyz)y(zyzxzyz)y(zyzxzyz)y(zyzxzyz)x(zyzxzyz)y(zyzxzyz))$
and so forth.

This illustrates how complicated composing $\mathbf{D}$ alternately with $\mathbf{H}_2$ can be, even after just a few iterations. Although there is an obvious structure to the increasingly long word representing the third involution, if, when and how it may simplify to z depends entirely on the group G which contains the involutions x, y and z. However in general, we may make no extra assumptions about the structure of G and so we cannot deduce the order of the operator $\mathbf{H}_2\mathbf{D}$. •

3.2 Introducing three related groups

The end of the previous section illustrates how the context in which group elements are defined matters, and how this context makes a difference to the structure of the resulting group. For this reason, we now introduce three closely-related groups, the first being an abstract universal group which will yield quotients groups which are isomorphic to operator groups.

3.2.1 The universal parent group $\mathfrak{Op}_k$ for operators for each given k

As mentioned previously, care is required when considering “groups” of operators, which may include hole operators, when defined on regular maps since the image under a particular operation does not necessarily have the same valency as the pre-image, and the hole operator $\mathbf{H}_j$ is only defined when the valency is coprime to j, meaning that the composition of operators is not always a given.

In contrast, some operators are always defined, and there are ways in which operators interact which are true in general, regardless of the objects (fully regular maps) to which they are applied, including some compositions of operators which necessarily have finite order. Determining this overarching structure is the purpose of the concept introduced in this subsection. The abstract group which we define here should *not* be thought of as a group of operators, for the same reason as outlined above. However it will give rise to finite quotient groups which are isomorphic to groups of operators, and for which the group operation is composition of functions. The overarching structure will provide the foundation, including the general truths about interactions between the elements for the more specific quotient groups, each of which will have more relators according to the set of maps on which the quotient is defined.

For a given valency k, each universal parent group $\mathfrak{Op}_k$ will be defined such that it gives rise to quotient groups which are groups of operators defined to act on maps of valency k. In order to motivate the definition of the universal parent groups, we now remind ourselves about some specifics concerning the properties of the operator groups. These general properties are inherited from the structure which will be built in to the definition for the universal parent groups.

1. The identity **Id** is the operator which, when applied to any fully regular map $\mathcal{M}(G; x, y, z)$, returns exactly the same original map $\mathcal{M}(G; x, y, z)$.
2. The operators **D** and **P** are self-inverse by definition.

3. For each j coprime to a given k there exists a unique $-1 \leq i \leq k-2$ such that $ij \equiv 1 \pmod{k}$ and hence the inverse of $\mathbf{H}_j$ is $\mathbf{H}_i$.
4. $\mathbf{S} := \mathbf{H}_{-1} \in \langle\, \mathbf{H}_j \mid (j,k)=1 \,\rangle \cong \mathrm{U}_k$, the group of units modulo k.
5. $\langle\, \mathbf{D}, \mathbf{P}, \mathbf{S} \mid \mathbf{D}^2, \mathbf{P}^2, \mathbf{S}^2, (\mathbf{DP})^3, (\mathbf{DS})^4 \,\rangle \cong \mathrm{S}_4$.
6. $\langle\, \mathbf{P}, \mathbf{H}_j \mid (j,k)=1 \,\rangle = \langle\, \mathbf{H}_j \mid (j,k)=1 \,\rangle \times \langle\, \mathbf{P} \,\rangle \cong \mathrm{U}_k \times \mathrm{C}_2$ is Abelian.
7. The two groups $\langle\, \mathbf{D}, \mathbf{P}, \mathbf{S} \,\rangle$ and $\langle\, \mathbf{P}, \mathbf{H}_j \mid (j,k)=1 \,\rangle$ have non-empty intersection containing $\langle\, \mathbf{P}, \mathbf{S} \,\rangle \cong \mathrm{V}_4$.
8. The order of, for example, $\mathbf{DH}_j$ in not clear in general, although, if finite, it may be assumed to be even.
9. The order of, for example, $\mathbf{DPH}_j$ in not clear in general, although, if finite, it may be assumed to be a multiple of three.

Lemmas from earlier in this chapter guarantee how some operators interact with each other, as summarised in the above list, and so we have some ready-made relators for the parent group $\mathfrak{Op}_k$.

However, for the order of an element $\mathfrak{T} \in \mathfrak{Op}_k$, we make no further assumptions about the fully regular map(s) to which $\mathbf{T}$ (the corresponding operator in a quotient group) may be applied, and certainly no assumptions concerning the structure of the automorphism group of said map. Thus the apparent order of $\mathbf{T}$, a (composition of) operator(s), will depend on context, and may be a divisor of the theoretical (possibly infinite) order of the corresponding element in $\mathfrak{T} \in \mathfrak{Op}_k$. To illustrate this point, refer back to Example 3.16: for odd k, the pre-image of the operator $\mathbf{H}_2\mathbf{D}$ is the element $\mathfrak{H}_2\mathfrak{D} \in \mathfrak{Op}_k$ which must be assumed to have infinite order in the general parent group. However, when applied to a fully regular map $\mathcal{M}(G; x, y, z)$ with finite G, the operator $\mathbf{H}_2\mathbf{D}$ clearly has finite order.

Drawing all this together, we now define the parent group which possesses the required characteristics.

For a given value $k \geq 3$, we define the *kth universal parent group* to be $\mathfrak{Op}_k :=$ $\langle\, \mathfrak{D}, \mathfrak{P}, \mathfrak{S}, \mathfrak{H}_j \mid \mathfrak{D}^2, \mathfrak{P}^2, \mathfrak{S}^2, (\mathfrak{D}\mathfrak{P})^3, (\mathfrak{D}\mathfrak{S})^4, \mathfrak{S}\mathfrak{H}_{-1}, [\mathfrak{H}_j, \mathfrak{S}], [\mathfrak{H}_j, \mathfrak{H}_{j'}], [\mathfrak{P}, \mathfrak{S}], [\mathfrak{P}, \mathfrak{H}_j], \mathfrak{H}_j^i \,\rangle$ where $(j,k)=1$, and $ij \equiv 1 \pmod{k}$, while $1 \leq i, j, j' \leq k-1$, with the group operation being right multiplication.

Remember that $\mathbf{S}$ is a hole operator, so the corresponding element $\mathfrak{S} = \mathfrak{H}_{-1}$. The group $\mathfrak{Op}_k$ is then the amalgamated free product of

$\langle \mathfrak{D}, \mathfrak{P}, \mathfrak{S} \mid \mathfrak{D}^2, \mathfrak{P}^2, \mathfrak{S}^2, (\mathfrak{D}\mathfrak{P})^3, (\mathfrak{D}\mathfrak{S})^4 \rangle \cong \mathrm{S}_4$ and
$\langle \mathfrak{P}, \mathfrak{H}_j \mid [\mathfrak{H}_j, \mathfrak{H}_{j'}], [\mathfrak{P}, \mathfrak{S}], [\mathfrak{P}, \mathfrak{H}_j], \mathfrak{H}_j^i \mid (j,k) = 1, ij \equiv 1 \pmod{k} \rangle \cong \mathrm{U}_k \times \mathrm{C}_2$, whose intersection $\langle \mathfrak{P}, \mathfrak{S} \rangle \cong \mathrm{V}_4$, and so $\mathfrak{Op}_k \cong \mathrm{S}_4 *_{\mathrm{V}_4} (\mathrm{U}_k \times \mathrm{C}_2)$. The above definition is therefore such that the following remark holds.

Remark 3.17. $\mathfrak{Op}_k$ is a group isomorphic to $\mathrm{S}_4 *_{\mathrm{V}_4} (\mathrm{U}_k \times \mathrm{C}_2)$. ⋆

Henceforth let us focus on concrete examples of finite groups of operators. Each of these groups of operators consists of elements which correspond to elements in $\mathfrak{Op}_k$, and which must respect the equivalent relators in $\mathfrak{Op}_k$. Each group of operators may thus be considered as a quotient group $\mathfrak{Op}_k/N$, or some subgroup of $\mathfrak{Op}_k/N$. As a foretaste of how this works, we give an example, with further details in the following sections.

Example 3.18. Let $G = \mathrm{SL}(2, 2^\alpha)$, and let $k = (2^\alpha + 1)(2^\alpha - 1)$. It is known that G contains elements of order $2^\alpha + 1$ and elements of order $2^\alpha - 1$, and furthermore that all non-involutions in G have odd order dividing either $2^\alpha + 1$ or $2^\alpha - 1$. Also G supports no fully regular map of spherical type. The integer k is then the least common multiple of all possible valencies for a fully regular map $\mathcal{M} = (G; x, y, z)$. The operators $\mathbf{D}, \mathbf{P}, \mathbf{S}$ and $\mathbf{H}_j$, where $(j,k) = 1$, are all defined to act on any such map $\mathcal{M}$, and so generate a group of operators under composition. The group of operators $\langle \mathbf{D}, \mathbf{P}, \mathbf{S}, \mathbf{H}_j \mid (j,k) = 1 \rangle$ is then isomorphic to a quotient of the group $\mathfrak{Op}_k$. •

3.2.2 The group of operators $\mathbf{Op}_k(\Omega_G)$, defined as permutations on Ω_G

So long as we may ensure that composition of the operators is valid, we may think of operators acting as permutations on Ω_G, the set consisting of all the fully regular maps with automorphism group G.

First, let us define the *non-involution exponent of a group* G, that is the least common multiple of the orders of all the non-involutory elements of the group G. In many cases, this will be the same as the least common multiple of the valencies for the set Ω_G, and so for this reason we denote the non-involutory exponent of G by $k(G)$.

Lemma 3.19. *Let G be a finite group which supports fully regular maps, but does not support any spherical fully regular maps, and let $k = k(G)$. The operators $\boldsymbol{D}, \boldsymbol{P}$, and $\boldsymbol{H}_j$ such that $-1 \le j \le k-2$ and $(j,k) = 1$ are defined to act on the set of all maps whose automorphism group is G. Moreover,*
$\boldsymbol{Op}_k(\Omega_G) := \langle \boldsymbol{D}, \boldsymbol{P}, \boldsymbol{H}_j \mid -1 \le j \le k-2,\ (j,k) = 1 \rangle$, with composition of operators, may be considered as a permutation group of the set of fully regular maps Ω_G.

Proof. First we must be convinced that each of these generating operators is well-defined and each has a unique inverse. This is clear for the well-known dualities. Since the hole operators, $\mathbf{H}_j$ are defined such that j is read modulo k and j is coprime to k, so long as $\mathbf{H}_j$ is well-defined on every map within the set Ω_G, there is a unique inverse for each hole operator. The only remaining thing under question here is whether composition of operators is valid.

Let $\mathcal{M}(G; x, y, z)$ be a fully regular map of valency $k' \geq 3$.

It is clear that the (involutory) duality operators $\mathbf{D}, \mathbf{P}$ (and hence their composite functions) are well-defined to be applied to any fully regular map $\mathcal{M}(G; x, y, z)$, and also the operators respect the underlying automorphism group G of each map,

Each of the hole operators $\mathbf{H}_j$ is defined on the map $\mathcal{M}(G; x, y, z)$ so long as j and k' are mutually prime. Now $k' \mid k = k(G)$, and so certainly $(j, k') = 1$, and hence the hole operator $\mathbf{H}_j$ is defined on $\mathcal{M}$ (and hence on every such map) while the group G (which, it may help to remember, is isomorphic to the monodromy group) is respected too.

The set Ω_G is thus closed under the action of the operations, composition of operators is well-defined, and so the group $\mathbf{Op}_k(\Omega_G)$, defined with binary operation being composition of functions, may be considered as a permutation group acting on the (finite) set Ω_G. As such $\mathbf{Op}_k(\Omega_G)$ is isomorphic to a subgroup of $\mathrm{S}_{|\Omega_G|}$. $\square$

In particular, for a given finite group G with $k = k(G)$, each operator in $\mathbf{Op}_k(\Omega_G)$ may be considered as a permutation of the set Ω_G, and so must have finite order. In contrast, the order of the corresponding element in the more general abstract parent group $\mathfrak{Op}_k$ could be infinite.

Since it is a free product, the group $\mathfrak{Op}_k$ is clearly infinite. By the definition of the non-involution exponent, along with Lemma 3.19, where $k(G) = k$, the group $\mathfrak{Op}_k$ can be considered as the parent group for $\mathbf{Op}_k(\Omega_G)$. The extra relators in the group $\mathbf{Op}_k(\Omega_G)$ arise from the structure of the group G which allows for simplification of what would otherwise be infinitely long words in x, y and z.

The group $\mathbf{Op}_k(\Omega_G)$ is then naturally defined by the epimorphism which maps the general elements in $\mathfrak{Op}_k$ to those corresponding operators which are applied to fully regular maps with underlying group G, that is $\mathfrak{D} \to \mathbf{D}$, $\mathfrak{P} \to \mathbf{P}$ and $\mathfrak{H}_j \to \mathbf{H}_j$ for each j coprime to k. Thus, $\mathbf{Op}_k(\Omega_G) = \langle\, \mathbf{D}, \mathbf{P}, \mathbf{H}_j \mid (j, k) = 1 \,\rangle \cong \mathfrak{Op}_k / \mathfrak{N}$ for some $\mathfrak{N}$, where $\mathfrak{N}$ is the kernel of the natural epimorphism from $\mathfrak{Op}_k$ to $\mathbf{Op}_k(\Omega_G)$ which maps each $\mathfrak{T}$ onto the corresponding $\mathbf{T}$.

We use similar notation for the corresponding elements in each group, and it should be clear from context whether we are considering: the parent group $\mathfrak{Op}_k = \langle \mathfrak{D}, \mathfrak{P}, \mathfrak{H}_j \mid (j,k) = 1, -1 \leq j \leq k-2 \rangle$; or one of its finite quotient groups $\mathbf{Op}_k(\Omega_G) = \langle \mathbf{D}, \mathbf{P}, \mathbf{H}_j \mid (j,k) = 1 \rangle$ whose elements are operators which act as permutations on the set of maps with automorphism group G; or indeed something isomorphic to a subgroup of one of those quotient groups.

3.2.3 The external symmetry group of a given fully regular map

Given a fully regular map $\mathcal{M}(G; x, y, z)$ of valency k, the external symmetry operators of $\mathcal{M}$ are those operators which are map isomorphisms of $\mathcal{M}$.

The external symmetry operators for a given finite fully regular map $\mathcal{M}$ of valency k form a group of operators, the *external symmetry group*, $Ext(\mathcal{M})$, which is naturally isomorphic to a subgroup of a quotient group of $\mathfrak{Op}_k$.

It is straightforward to see that the set of external symmetry operators form a group under composition. Indeed the identity, **Id**, is the operator which when applied to a fully regular map $\mathcal{M}(G; x, y, z)$, returns the exact same map $\mathcal{M}(G; x, y, z)$, and so **Id** is certainly a map isomorphism. Meanwhile the group operator is composition of functions, clearly well-defined since every $\mathbf{T} \in Ext(\mathcal{M})$ is by definition a map isomorphism (and its inverse is also a map isomorphism), and under which the set of external symmetry operators is certainly closed.

The external symmetry group for a non-degenerate and non-spherical fully regular map is never trivial. Indeed, by earlier work in this chapter, it is easy to see that some non-trivial operators, for example $\mathbf{S} := \mathbf{H}_{-1}$, are always included. Remember that $\mathbf{S}$ is certainly not the identity since the non-spherical condition implies the valency $k \geq 3$.

Corollary 3.20. *Let $\mathcal{M}$ be a fully regular map with extended type (k, ℓ, m) such that $2 \notin \{k, \ell, m\}$. The operator group $\langle \boldsymbol{S}, (\boldsymbol{SD})^2 \rangle \cong V_4$ is a subgroup of $Ext(\mathcal{M})$.*

Proof. By Remark 3.11, the operator $(\mathbf{SD})^2 \in Ext(\mathcal{M})$, and also $\mathbf{S}$ is an involutory map isomorphism. Meanwhile the condition on the extended type of the map ensures that the operator $(\mathbf{SD})^2$ does not fix $\mathcal{M}$, according to Remark 3.12. Finally it is known that the two operators are commuting involutions since $(\mathbf{SD})$ has order four by Lemma 3.10. $\square$

In general, the external symmetry group for a fully regular map is often very far from being trivial, and this is the case especially for super-symmetric maps.

Any super-symmetric map $\mathcal{M}$ of valency k, and hence type (k,k,k), will by definition, have the following generating external symmetry operators: $\mathbf{D}$ for self-duality; $\mathbf{P}$ for self-Petrie-duality; and (generators for) all the hole operators $\mathbf{H}_j$ where j is read modulo k and is such that $(j,k)=1$. As such the external symmetry group for this map $\mathcal{M}$ will be $Ext(\mathcal{M}) = \langle\, \mathbf{D}, \mathbf{P}, \mathbf{H}_j \mid (j,k)=1 \,\rangle$.

3.3 Existence of super-symmetric regular maps

Work by Jones and Thornton [43] demonstrates how, given a regular map which is not self-dual, a covering map may be built which does have this property. Meanwhile Wilson [65] describes the construction of a parallel product of groups and highlights applications in the theory of maps. Of particular relevance to us is the observation that, given any fully regular map $\mathcal{M}$ we can form a reflexible map with Trinity symmetry by forming a parallel product of rooted maps:
$\mathcal{M} \,||\, \mathcal{M}\mathbf{D} \,||\, \mathcal{M}\mathbf{P} \,||\, \mathcal{M}\mathbf{DP} \,||\, \mathcal{M}\mathbf{PD} \,||\, \mathcal{M}\mathbf{DPD}$. Since each operator then merely permutes the maps within the parallel product, the resulting fully regular map has Trinity symmetry. Meanwhile, if (for example) the original map $\mathcal{M}$ was self-dual then we would only need the first, third and fifth elements of the parallel product to yield a map with Trinity symmetry. Extending this to consider building kaleidoscopic maps with Trinity symmetry, with each being a super-symmetric map built from a parallel product of component maps, we will only need a single component map to represent each isomorphism class.

We now remind the reader of the background material on which this relies.

3.3.1 Background: parallel products of maps

The motivation is to build super-symmetric maps from a given orbit of fully regular maps under the action of $\mathbf{Op}_k(\Omega_G)$. The notation $Orb(\mathcal{M})$ is used for the orbit under $\mathbf{Op}_k(\Omega_G)$ of the fully regular map $\mathcal{M}(G;x,y,z)$.

Let $G = \langle x,y,z \rangle$ and $G' = \langle x',y',z' \rangle$. The *parallel product of the two groups* G and G' is denoted $G \,||\, G'$ and forms a subgroup of $G \times G'$ where the generators are defined to be (x,x') and (y,y') and (z,z'). Group operations are performed element-wise and a word in the group is the identity in the parallel product if and only if the corresponding word is the identity within each of the component groups G and G'. The group $G \,||\, G'$ is isomorphic to the quotient group of F, the free group on the elements x, y and z, modulo the intersection $(K \cap K')$ where $G = F/K$ and $G' = F/K'$. The theory then extends naturally to allow the (simultaneous) parallel product of more than two groups to be defined, relying on the known associativity of the intersection of groups.

Analogously, the parallel product of two maps $\mathcal{M}(G; x, y, z)$ and $\mathcal{M}'(G'; x', y', z')$ is denoted $\mathcal{M} \,||\, \mathcal{M}'$ and has automorphism group $G \,||\, G' = \langle\, (x, x'), (y, y'), (z, z')\,\rangle$. The marked flag is identified with the element (id, id'), and the group generated may not be as large as the direct product $G \times G'$.

However, in some circumstances the group is the direct product of the components. Using exactly the same reasoning as Jones in section 6 of [36], the following analogous claim may be proved.

Lemma 3.21. *Let G be a finite, non-Abelian simple group and let $\overline{\mathcal{M}}$ be the parallel product of r mutually non-isomorphic maps on G. Then $Aut(\overline{\mathcal{M}})$ is isomorphic to the direct product G^r.*

The above relies on the simplicity of the group G. Since $\mathrm{SL}(2, 2^\alpha)$, the groups we want to work with, are simple, we will use this in our later investigations. The operators $\mathbf{D}$, $\mathbf{P}$ and valid $\mathbf{H}_j$ are then defined such that an operator is applied element-wise to each map within a parallel product of maps, that is $(\mathcal{M} \,||\, \mathcal{M}')\mathbf{T} := (\mathcal{M})\mathbf{T} \,||\, (\mathcal{M}')\mathbf{T}$.

3.3.2 Orbits of maps with the same group

As we have seen, the group $\mathbf{Op}_k(\Omega_G)$ may be considered as a permutation group of Ω_G, the space of all fully regular maps with a given underlying group G, and, of course, it is possible that the action is not transitive.

In any case, it is possible to build the parallel product of all the maps in a given orbit, and thereby have a ready-made super-symmetric map. Remember that the existence of super-symmetric maps for most (odd) valencies is unknown, and for this reason it could be of interest to identify the smallest orbits, so as to minimise the size (and possibly even limit the valency) of the resulting super-symmetric map.

More investigations into the structure of orbits of equivalent fully regular maps under the action of $\mathbf{Op}_k(\Omega_G)$ where $G \cong \mathrm{SL}(2, 2^\alpha)$ is presented in section 3.5, while section 3.6 gives a specific detailed example for $G = \mathrm{A}_5 \cong \mathrm{SL}(2, 4)$.

3.3.3 Building a super-symmetric map $\mathcal{N}_G$

Consider this informal thought experiment:

Example 3.22. Let us take the set of maps consisting of the whole orbit $Orb(\mathcal{M})$ of a given map $\mathcal{M}(G; x, y, z)$ under the group $\mathbf{Op}_k(\Omega_G)$, remembering that the group is generated by the operations of $\mathbf{D}, \mathbf{P}$ and $\mathbf{H}_j$s where $k = k(G)$ and $(j, k) = 1$. If we assume that the group G is finite, we know that there is a finite number of regular maps on this group, and so the set $Orb(\mathcal{M})$ is finite.

Let k' be the least common multiple of all the valencies of maps occurring in this orbit $Orb(\mathcal{M})$. Now form the parallel product of all the maps in the aforementioned orbit $Orb(\mathcal{M})$, to get a fully regular map $\mathcal{N}$ of valency k', which might be a proper divisor of $k(G)$.

Clearly the group of units modulo k' gives precisely the values for j such that $\mathbf{H}_j$ is defined on a map of valency k'. We may work element-wise with the maps within the parallel product, and so every such $\mathbf{H}_j$ is a valid hole operator for each map within the orbit (many of which may have valency a divisor of k'). Moreover, each such operator simply permutes the elements (maps) in the parallel product. Similarly the duality operators $\mathbf{D}$ and $\mathbf{P}$ shuffle the maps within the parallel product. Thereby any operator in $\mathbf{T} \in \mathbf{Op}_{k'}(\Omega_G)$ produces $\mathcal{N}\mathbf{T}$, a map isomorphic to $\mathcal{N}$.

And so the map $\mathcal{N}$ is super-symmetric and $Ext(\mathcal{N})$ contains all the same operators as $\mathbf{Op}_{k'}(\Omega_G) = \langle\, \mathbf{D}, \mathbf{P}, \mathbf{H}_j \mid (j, k') = 1 \,\rangle$. •

Remark 3.23. The above thought experiment leaves open the question as to whether the valency k' might be a proper divisor of $k(G)$? In the case where $G = \mathrm{SL}(2, 2^\alpha)$, the work in section 3.5 leads me to conjecture that $k' = k(G)$. Other cases remain unexplored, and so are subject to further research. ⋆

By the same reasoning, given a group G which is the underlying group for at least one fully regular map, it is easy to build a *natural super-symmetric map*, henceforth denoted by $\mathcal{N}_G$, using the parallel product of a selection of maps from Ω_G, making sure there is precisely one example included for each map isomorphism class. Each operator will then permute the isomorphism classes within the parallel product, thus yielding a fully regular map which is isomorphic to $\mathcal{N}_G$.

Remark 3.24. I'm making no claim that the notion of $\mathcal{N}_G$ is uniquely defined as a rooted fully regular map - when described as above, it is certainly not - but I do claim that, for a given G, all such maps are isomorphic to each other. This is an occasion where it would be advantageous to consider maps up to isomorphism. However, since much of this chapter relies on distinguishing between distinct but isomorphic maps, which is in keeping with the perspective for much of this work where I consider specific rooted maps, I will omit defining $\mathcal{N}_G$ in terms of its map subgroup. ⋆

However, considering fully regular maps up to isomorphism yields simplifications, and we benefit from this perspective when we note the following remark.

Remark 3.25. It is worth noting again that the automorphism group for the map $\mathcal{N}_G$ is not necessarily a direct product of copies of G, however Jones [36] proves that if the group G is simple, then $Aut(\mathcal{N}_G) \cong G \times G \times G \times \cdots \times G = G^r$ where r is the number of isomorphism classes of fully regular map on G. ⋆

Again, this is relevant to our investigations where $G \cong \mathrm{SL}(2, 2^\alpha)$. The group $\mathbf{Op}_k(\Omega_G)$ then acts on a natural parallel product map $\mathcal{N}_G$ in a very natural way, permuting the isomorphism classes of map within the direct product. As such, all the operators in $\mathbf{Op}_k(\Omega_G)$ are external symmetries, and so may be thought of as elements of $Ext(\mathcal{N}_G)$.

Many of the concepts introduced and discussed in sections 3.2.2 and 3.3 are rather abstract. These ideas, their motivation and their applications will be made clear in the following section.

3.4 Operators acting on Ω_G, where $G \cong \mathbf{SL}(2, 2^\alpha)$

The family of groups $\mathrm{SL}(2, 2^\alpha)$ is a good source for regular maps with odd valency due to the fact that in such a group every element of even order is an involution. We may then use these maps as building blocks for a super-symmetric map which also has odd valency.

Henceforth in this chapter, we will consider fully regular maps whose automorphism group G is isomorphic to $\mathrm{SL}(2, 2^\alpha)$, the special linear group over a field of characteristic 2, with a view to studying fully regular maps which are formed as a parallel product of such maps for a given value of α. This allows us to make some assumptions about the automorphism group G of the maps on which a given operator is acting. In doing so we make further progress in identifying group elements of $\mathbf{Op}_k(\Omega_G)$ and how they interact, which informs further study when they are considered as operators in $Ext(\mathcal{N}_G)$ for a given natural super-symmetric map $\mathcal{N}_G$.

To further our understanding of operator groups, and orbits of maps under the action of valid operators, we will investigate a specific example in detail. We consider the action of the group of valid operators on the set of all maps with a given group $G = \mathrm{A}_5$, focussing on one single orbit. We then apply our results to $\mathrm{A}_5 \times \mathrm{A}_5 \times \mathrm{A}_5$ which supports a kaleidoscopic fully regular map with Trinity symmetry and valency 15. The group $\mathrm{A}_5 \times \mathrm{A}_5 \times \mathrm{A}_5$ is well-known for supporting a super-symmetric map of valency 15, and an example, including explicit generators, may be found in [3]. Note also that $\mathrm{A}_5 \cong \mathrm{SL}(2, 2^2)$ and that this is the smallest non-trivial example of a natural super-symmetric map $\mathcal{N}_G$ where $G = \mathrm{SL}(2, 2^\alpha)$.

With further research, this could lead to a better understanding of the structure of the mysterious external symmetry group, at least in the case of a super-symmetric map $\mathcal{N}_G$, which has valency $q^2 - 1$ and is built as a parallel product of fully regular maps in Ω_G for $G \cong \mathrm{SL}(2, q)$ where q is even.

To help with our endeavour to understand these external symmetry groups, we study the group of operators for maps on a given $\mathrm{SL}(2, 2^\alpha)$ group, using the notation as follows. For a given finite group $G \cong \mathrm{SL}(2, 2^\alpha)$, which has non-involution exponent $k(G)$, the group of operators is $\mathbf{Op}_{k(G)}(\Omega_G) := \langle \mathbf{D}, \mathbf{P}, \mathbf{H}_j \mid (j, k(G)) = 1 \rangle$. This can be considered as a permutation group acting on the space Ω_G consisting of all fully regular maps $\mathcal{M}(G; x, y, z)$ which have the same automorphism group G. We will investigate the orbit structure of the group $\mathbf{Op}_{k(G)}(\Omega_G)$ acting on Ω_G, the space of all maps with the given (finite) automorphism group $G \cong \mathrm{SL}(2, 2^\alpha)$.

3.4.1 General truths regarding this case

In this section we let $G \cong \mathrm{SL}(2, q)$, where $q = 2^\alpha \geq 4$, be the automorphism group of a fully regular map $\mathcal{M}$. Regular maps on these special linear groups are well-understood - for further information see [19]. In particular we will often rely on the following list of truths, noting that further details can be found in [22] and [44].

We now list some facts about the finite simple group $\mathrm{SL}(2, q)$ where $q = 2^\alpha$ and $\alpha \geq 2$:

- Isomorphism class of groups: $\mathrm{SL}(2, q) \cong \mathrm{PGL}(2, q) \cong \mathrm{PSL}(2, q)$.
- Automorphism group: $\mathrm{P\Gamma L}(2, q) \cong \mathrm{SL}(2, q) \rtimes \langle \mathrm{C}_\alpha \rangle$.
- Order of the group: $q(q-1)(q+1)$.
- Exponent of the group : $2(q-1)(q+1)$.
- Non-involution exponent : $k(\mathrm{SL}(2, q)) = (q-1)(q+1)$
- Orders of elements: 2, $q-1$, $q+1$ and hence divisors of these.

3.4.2 Elements and subgroups of $\mathbf{Op}_{k(G)}(\Omega_G)$

Let $\mathcal{M}(G; x, y, z)$ be a fully regular map such that $G \cong \mathrm{SL}(2, q)$ where $q = 2^\alpha \geq 4$. The valency, $k \geq 3$, of $\mathcal{M}$ must thus be such that either $k \mid q+1$ or $k \mid q-1$. In particular the valency of the map is odd. We let $\mathbf{R}$ denote $\mathbf{H}_2$ and highlight that this will always be a defined operator on any such map $\mathcal{M}$ since 2 is coprime with k. Similarly, since $\mathbf{R}^n = \mathbf{H}_{2^n}$, hole operators for other powers of two are also well-defined on such maps. In particular the hole operator $\mathbf{Q} := \mathbf{H}_q = \mathbf{R}^\alpha$ is always defined on any such map $\mathcal{M}$. It will be useful to investigate the hole operator $\mathbf{Q}$'s action on individual maps, for reasons hinted at by the following lemma.

Lemma 3.26. *Let $G \cong SL(2, q)$ where $q = 2^\alpha$ and $\alpha \geq 2$. Let $\mathcal{M}(G; x, y, z)$ be a fully regular map. The operator $\boldsymbol{Q} \in \boldsymbol{Op}_{k(G)}(\Omega_G)$ is a map isomorphism and has order 2.*

Proof. Let $\mathcal{M}(\mathrm{SL}(2, 2^\alpha); x, y, z)$ be a map with valency k such that $k \mid q \pm 1$. Hence $(zy)^k = 1$ and so, with signs read consistently with the assumption, $(zy)^{q\pm1} = 1$, that is $(zy)^q = (zy)^{\mp1}$. Thus $\mathcal{M}\mathbf{Q} = \mathcal{M}'(G; x, y, (zy)^q y) = \mathcal{M}'(G; x, y, (zy)^{\mp1} y)$, thereby proving the following: if the map $\mathcal{M}(G; x, y, z)$ has valency dividing $q + 1$, the operator $\mathbf{Q}$ acts in the same way as $\mathbf{S}$; meanwhile the operator $\mathbf{Q}$ fixes all maps which have valency dividing $q - 1$.

By Lemma 3.6 we know that $\mathbf{S}$ is a map isomorphism, so this means that the hole operator $\mathbf{H}_q := \mathbf{Q}$ fixes the isomorphism class of any map $\mathcal{M}(G; x, y, z)$ where $q = 2^\alpha$ and $G \cong \mathrm{SL}(2, q)$. Finally, it is easy to see that $\mathbf{Q} \in \mathbf{Op}_{k(G)}(\Omega_G)$ has order 2. $\square$

Remark 3.27. Notice that the order of $\mathbf{R}$ is 2α, and so depends on α whereas the order of $\mathbf{Q}$ is two. This gives hope that we may be able to find some general truths for $\mathbf{Q} \in \mathbf{Op}_{k(G)}(\Omega_G)$. $\star$

Now, $\mathbf{Q}$ is a map isomorphism and so q is an exponent of every fully regular map $\mathcal{M}(\mathrm{SL}(2, 2^\alpha); x, y, z)$. We cannot assume $\mathbf{Q}$ is a non-trivial element in $Ext(\mathcal{M})$, since its action depends on the valency of the map $\mathcal{M}$, which could be $q - 1$. However it will necessarily be a non-trivial element of $Ext(\mathcal{N}_G)$. Since hole operators commute, the following is immediate.

Lemma 3.28. *Let $\mathcal{N}_G$ be a natural super-symmetric map, that is a parallel product of maps $\mathcal{M}(G; x, y, z)$ with one map in each map isomorphism class, where $G \cong SL(2, q)$ and $q = 2^\alpha \geq 4$. Then $V_4 \cong \langle\, \boldsymbol{Q}, \boldsymbol{S} \rangle \leqslant Ext(\mathcal{N}_G)$.* $\square$

Now, for any map $\mathcal{M}(G; x, y, z)$ where $q = 2^\alpha \geq 4$ and $G \cong \mathrm{SL}(2, q)$, we know that the order of every (non-identity and non-involution) element of G divides either $q - 1$ or $q + 1$. This makes it is easier than otherwise to see what will happen when we compose operations with $\mathbf{Q} \in \mathbf{Op}_{k(G)}(\Omega_G)$. For example:

Lemma 3.29. *Let $G \cong SL(2, q)$ where $q = 2^\alpha \geq 4$. The operator $\boldsymbol{QDP} \in \boldsymbol{Op}_{k(G)}(\Omega_G)$ has order six.*

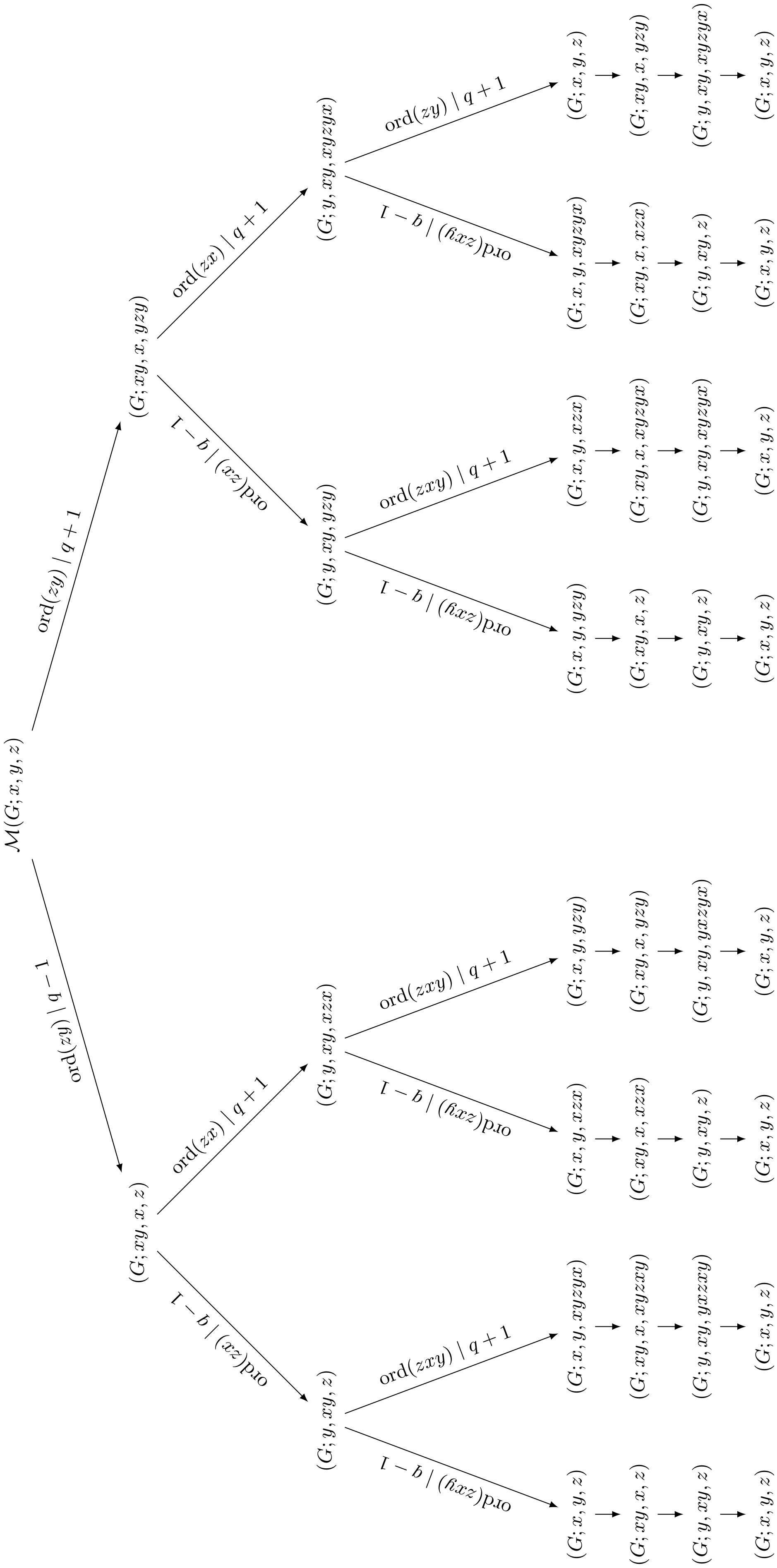

Figure 3.1: The action of **QDP** on a map where $G = \mathrm{SL}(2, q)$ for even q.

Proof. Let $\mathcal{M}(G; x, y, z)$ be a fully regular map. The operator **QDP** applied to the map $\mathcal{M}$ sends $x \to xy$, and $y \to x$ and $z \to (zy)^q y$. Figure 3.1 shows what the map $\mathcal{M}(G; x, y, z)$ is transformed into by each successive application of **QDP**. Note that which map one gets to after each step depends only on the valency, face length and Petrie-walk length of the original map $\mathcal{M}$. Now, as the diagram shows, by the time the operation **QDP** has been applied six times, the original map $\mathcal{M}$ is returned, regardless of the extended type of $\mathcal{M}$. Apart from at the root, this is the first time that every map at a level in the diagram is the original map $\mathcal{M}(G; x, y, z)$, thereby proving the claim that **QDP** has order six. □

Furthermore the information in Figure 3.1 implies that
$\mathcal{M}(\mathbf{QDP})^3 = \mathcal{M}^a = \mathcal{M}'(G; x, y, z^g)$ for some involution $g \in \langle x, y \rangle$, and so $\mathcal{M}(\mathbf{QDP})^3$, being a conjugate of $\mathcal{M}$, is isomorphic to $\mathcal{M}$. Hence $(\mathbf{QDP})^3 \in Ext(\mathcal{M})$, and it is an involution unless $\mathcal{M}(\mathbf{QDP})^3 = \mathcal{M}\mathbf{Id}$ which occurs if and only if all values in the extended type (k, ℓ, m) of $\mathcal{M}$ divide precisely one of $(q + 1)$ and $(q - 1)$.

Remark 3.30. Note that the order of other operators in $\mathbf{Op}_{k(G)}(\Omega_G)$ can be shown by constructing similar trees. This often seems to highlight powers of operators which are map isomorphisms too. For example $(\mathbf{PDQS})^3$ is a map isomorphism, while **PDQS** itself has order six. ⋆

For a short while the order of **QDP** (and hence also the order of its conjugates and its inverse **PDQ**) being six distracted me. I was thinking that I could establish the isomorphism class for the group $\langle\, \mathbf{DP}, \mathbf{Q} \,\rangle = \langle\, \mathbf{DP}, \mathbf{PDQ} \,\rangle$ since it must correspond to the orientation preserving automorphism group of a fully regular map of type $(3, 6)$. These maps are toroidal and fully classified, however, there are infinitely many of them, so initially this didn't help.

Then, I spotted the obvious analogue with Lemma 3.10. Being a hole operator, **Q** commutes with **P**, so this forms a pair of commuting involutions. We may then hope, and indeed it is the case, that including a third involution **D**, will yield a well-known regular map group. And so we may again apply our knowledge about regular map groups to identify the isomorphism class of this group.

Lemma 3.31. *Let $G \cong SL(2, q)$ where $q = 2^\alpha$. The group $\langle\, \boldsymbol{D}, \boldsymbol{P}, \boldsymbol{Q} \,\rangle$ is a subgroup of $\boldsymbol{Op}_{k(G)}(\Omega_G)$. As such,*
$\langle\, \boldsymbol{D}, \boldsymbol{P}, \boldsymbol{Q} \mid \boldsymbol{D}^2, \boldsymbol{P}^2, \boldsymbol{Q}^2, (\boldsymbol{DP})^3, (\boldsymbol{PQ})^2, (\boldsymbol{QD})^4, (\boldsymbol{DPQ})^6 \,\rangle \cong S_4 \times C_2.$

Proof. The statement that the group $\langle\, \mathbf{D}, \mathbf{P}, \mathbf{Q} \,\rangle$ is a subgroup of $\mathbf{Op}_{k(G)}(\Omega_G)$ is clear by now, and requires no proof. It is well-known that the order of **DP** is three.

$$\begin{tikzcd}
 & & & (G;x,y,z) \arrow[dll, "\mathrm{ord}(zy)\mid q-1"'] \arrow[drr, "\mathrm{ord}(zy)\mid q+1"] & & & \\
 & (G;y,x,z) \arrow[dl, "\mathrm{ord}(zx)\mid q-1"'] \arrow[dr, "\mathrm{ord}(zx)\mid q+1"] & & & & (G;y,x,yzy) \arrow[dl, "\mathrm{ord}(zx)\mid q-1"'] \arrow[dr, "\mathrm{ord}(zx)\mid q+1"] & \\
(G;x,y,z) \arrow[d] & & (G;x,y,xzx) \arrow[d] & & (G;x,y,yzy) \arrow[d] & & (G;x,y,xyzyx) \arrow[d] \\
(G;y,x,z) \arrow[d] & & (G;y,x,xzx) \arrow[d] & & (G;y,x,z) \arrow[d] & & (G;y,x,xzx) \arrow[d] \\
(G;x,y,z) & & (G;x,y,z) & & (G;x,y,z) & & (G;x,y,z)
\end{tikzcd}$$

Figure 3.2: The action of **QD** on a map where $G = \mathrm{SL}(2, q)$ for even q.

Meanwhile **QP** has order two since both **Q** and **P** are involutions and hole operators commute with the Petrie operator.

The diagram in Figure 3.2 shows that **QD** has order 4, since when $(\mathbf{QD})^4$ is applied to $\mathcal{M}' =(G; x, y, z)$, an arbitrary map with underlying group G, it yields $\mathcal{M}' = (G; x, y, z)$. As an aside, one may notice that $(\mathbf{QD})^2$ is a map isomorphism for $\mathcal{M}(G; x, y, z)$, any fully regular map such that $G \cong \mathrm{SL}(2, q)$ where $q = 2^\alpha \geq 4$.

Mimicking earlier arguments, the subgroup $\langle\, \mathbf{DP}, \mathbf{QD} \,\rangle \leqslant \langle\, \mathbf{D}, \mathbf{P}, \mathbf{Q} \,\rangle$ thus corresponds to a spherical regular map of type $(3, 4)$. This must be either the full automorphism group of the (non-orientable) map of type $(3, 4)$ on the projective plane, or the orientation preserving group of automorphisms of the cube, each of which is isomorphic to S_4. Therefore $\langle\, \mathbf{DP}, \mathbf{QD} \,\rangle \cong \mathrm{S}_4$. We may now discount the projective plane case: non-orientability would imply $Aut{+} = Aut$ which would lead us to conclude that $\langle\, \mathbf{DP}, \mathbf{QD} \,\rangle = \langle\, \mathbf{D}, \mathbf{P}, \mathbf{Q} \,\rangle \cong \mathrm{S}_4$, but we know by Lemma 3.29 that **PDQ** has order six, which gives a contradiction. Thus it must be the case that $\langle\, \mathbf{D}, \mathbf{P}, \mathbf{Q} \,\rangle = \langle\, \mathbf{DP}, \mathbf{QP} \,\rangle \rtimes \langle\, \mathbf{P} \,\rangle \cong \mathrm{S}_4 \rtimes \mathrm{C}_2$. Moreover, relying once again on knowledge of small regular maps and their automorphism groups, since it corresponds to the full automorphism group of an orientable fully regular map of type $(3, 4)$, we may conclude $\langle\, \mathbf{D}, \mathbf{P}, \mathbf{Q} \,\rangle \cong \mathrm{S}_4 \times \mathrm{C}_2$. □

Applying Lemma 3.8 to the knowledge that $|\langle\, \mathbf{D}, \mathbf{P}, \mathbf{Q} \,\rangle| = 48$, we know the order of the group $\langle\, \mathbf{DP}, \mathbf{Q} \,\rangle = \langle\, \mathbf{DP}, \mathbf{PDQ} \,\rangle$ is 24. As such it must correspond to the orientation preserving automorphism group of an orientably-regular map of type $(3, 6)$

which, by [21], has presentation of the form
$\langle R, S \mid R^3, S^6, (RS)^2, (RS^{-2})^{a+b}(R^{-1}S^2)^{-b}\rangle$ for some non-negative a and b such that $a > 0$ and $(a^2 + ab + b^2) = 4$. The only pair of values for a and b is 2 and 0, contributing to the next lemma.

Lemma 3.32. *The group* $\langle\, \boldsymbol{PD}, \boldsymbol{Q}\,\rangle$ *has order* 24 *and may be presented as follows:*

$$\langle\, \boldsymbol{PD}, \boldsymbol{Q} \mid \boldsymbol{Q}^2, (\boldsymbol{PD})^3, (\boldsymbol{DPQ})^6, (\boldsymbol{PDQPDQPD})^2\,\rangle \cong V_4 \rtimes C_6.$$

Alternatively,

$$\langle \boldsymbol{PD}, \boldsymbol{Q}\rangle = \langle \boldsymbol{PD}, (\boldsymbol{PDQ})^2 \mid (\boldsymbol{PD})^3, ((\boldsymbol{PDQ})^2)^3, ((\boldsymbol{PDQ})^2\boldsymbol{PD})^2\,\rangle \times \langle\, (\boldsymbol{PDQ})^3\,\rangle \cong A_4 \times C_2.$$

Proof. The first presentation follows from the working beforehand. Allowing manipulations which are consistent with what we know about the interactions between the individual operators **D**, **P** and **Q** in the natural supergroup $\langle\, \mathbf{D}, \mathbf{P}, \mathbf{Q}\,\rangle$ the isomorphism may be shown by first realising that $\langle\, \mathbf{PDQPDQPD}, \mathbf{PDQPDPDQ} \mid (\mathbf{PDQPDQPD})^2, (\mathbf{PDQPDPDQ})^2, (\mathbf{PDQPDQPD.PDQPDPDQ})^2\,\rangle \cong \mathrm{V}_4$.

The workings are as follows. The inverse of **PDQPDQPD** is **DPQDPQDP** which is **DPQDQPDP** and, using **QDQD** = **DQDQ**, this becomes **PDPQDQPD**, and since **P** commutes with **Q**, we conclude that the first generator, **PDQPDQPD**, is self-inverse. Also the second generator **PDQPDPDQ** which may be simplified to **PDQDPQ** and then **PDQDQP** has order two since it is a conjugate of $(\mathbf{DQ})^2$. Meanwhile their product is **PDQPDQPDPDQPDPDQ** which becomes **PDQPDQDPQDPQ** and further simplifications of **PDP(QDQD)PQDPQ** quickly yields **DPQPDQ** which, as is it equivalent to **DQDQ**, also has order two.

Furthermore $\langle\, \mathbf{DPQ} \mid (\mathbf{DPQ})^6\,\rangle \cong \mathrm{C}_6$ acts to cyclically permute the non-identity elements in the copy of V_4. The first generator is sent to the product: **QPD**.**PDQPDQPD**.**DPQ** becomes **QPDPDQPD** which is **QDQD**. And the product is thus sent to **QPD**.**QDQD**.**DPQ**, that is **QPQDQDPQ** which is the second generator **PDQDPQ**. To complete the three-cycle **QPD**.**PDQDPQ**.**DPQ** is **QDPQDPQDPQ** and, since **QDP** has order six, this is equivalent to **Q.QPDQPDQPD** which is clearly **PDQPDQPD**.

We now address the second claim. Using the final relator in the above V_4 presentation, that is $(\mathbf{PDQPDQPD})^2$, the group $\langle\, \mathbf{PD}, (\mathbf{PDQ})^2\,\rangle$ is generated by two elements of order three whose product is an involution. Thus it may be presented as the tetrahedral group $\langle\, \mathbf{PD}, (\mathbf{PDQ})^2 \mid (\mathbf{PD})^3, ((\mathbf{PDQ})^2)^3, ((\mathbf{PDQ})^2\mathbf{PD})^2\,\rangle \cong \mathrm{A}_4$. By Lemma 3.29, **PDQ** has order six and so, including $\langle(\mathbf{PDQ})^3\rangle$ as a generator, the whole group $\langle\, \mathbf{PD}, \mathbf{Q}\,\rangle$ is generated. Furthermore $(\mathbf{PDQ})^3$ certainly has order two, and then it can be proved that it is central in $\langle\, \mathbf{PD}, \mathbf{Q}\,\rangle$ by noting that it commutes with both of the generators. $\square$

We are now in a position to make a general claim about the group $\langle\, \mathbf{Q}, \mathbf{S}, \mathbf{D}, \mathbf{P}\,\rangle$ as a subgroup of $\mathbf{Op}_{k(G)}(\Omega_G)$ for $G = \mathrm{SL}(2, q)$ where $q = 2^\alpha \geq 4$.

Theorem 3.33. *Let $G \cong SL(2, q)$ where $q = 2^\alpha \geq 4$. The group $\langle\, \boldsymbol{D}, \boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{S}\,\rangle$ is equal to the abstract product of groups $\langle\, \boldsymbol{P}, \boldsymbol{D}, \boldsymbol{Q}\,\rangle.\langle\, \boldsymbol{DS}\,\rangle \cong (S_4 \times C_2).C_4$ which is a subgroup of $\boldsymbol{Op}_{k(G)}(\Omega_G)$, with order 192 and presentation as follows:*

$$\langle\, \boldsymbol{D}, \boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{S} \mid \boldsymbol{D}^2, \boldsymbol{P}^2, \boldsymbol{Q}^2, \boldsymbol{S}^2, (\boldsymbol{PD})^3, (\boldsymbol{QS})^2, (\boldsymbol{PQ})^2, (\boldsymbol{PS})^2,$$
$$(\boldsymbol{DPS})^3, (\boldsymbol{PDQ})^6, (\boldsymbol{DQ})^4, (\boldsymbol{SD})^4, (\boldsymbol{QDSD})^2\,\rangle$$

Proof. We know from Lemma 3.32 that
$\langle\, \mathbf{D}, \mathbf{P}, \mathbf{Q} \mid \mathbf{D}^2, \mathbf{P}^2, \mathbf{Q}^2, (\mathbf{DP})^3, (\mathbf{PQ})^2, (\mathbf{DQ})^4, (\mathbf{DPQ})^6\,\rangle \cong \mathrm{S}_4 \times \mathrm{C}_2$. Also, by Lemma 3.10, the operator $\mathbf{DS}$ has order four, while $\mathbf{DPS}$ has order three. Now, $\mathbf{S}$ is an involutory hole operator and as such satisfies the relators $\mathbf{S}^2$, $(\mathbf{QS})^2$, $(\mathbf{PS})^2$.

The group $\langle\, \mathbf{D}, \mathbf{P}, \mathbf{Q}\,\rangle$ is small enough that it can be checked, by hand if necessary, that it has trivial intersection with $\langle\, \mathbf{DS}\,\rangle$, by considering the action of the relevant operators on a map $\mathcal{M} = (G; x, y, z)$. In doing so, note that one may straight away discount all of the operators $\langle\, \mathbf{D}, \mathbf{P}, \mathbf{Q}\,\rangle$ which involve a single occurence of $\mathbf{P}$. This, and similar reasoning makes the task less daunting. Meanwhile the observation that the effect of the action of $\mathbf{Q}$ depends on the valency of the map, compared with $\mathbf{S}$ which is always a map isomorphism, should make the claim no surprise.

Furthermore $\mathbf{DS}.\mathbf{D} = \mathbf{SDSDS} = \mathbf{D}.(\mathbf{DS})^3$ and $\mathbf{DS}.\mathbf{P} = \mathbf{DPS} = \mathbf{DPD}.\mathbf{DS}$ and $\mathbf{DS}.\mathbf{Q} = \mathbf{DQS} = \mathbf{DQD}.\mathbf{DS}$. Also $(\mathbf{DS})^3 = (\mathbf{DS})^{-1} = \mathbf{SD}$ and we have $(\mathbf{DS})^{-1}.\mathbf{D} = \mathbf{SDD} = \mathbf{S} = \mathbf{D}.\mathbf{DS}$. Since $\mathbf{DPS}$ has order three, we have $(\mathbf{DS})^{-1}.\mathbf{P} = \mathbf{SDP} = \mathbf{PDSPDS} = \mathbf{PDPSDS} = \mathbf{PDP}(\mathbf{DD})\mathbf{SDS} = \mathbf{DP}(\mathbf{DS})^2$, and $(\mathbf{DS})^2.\mathbf{P} = \mathbf{DSDSP} = \mathbf{D}.\mathbf{SD}.\mathbf{PS} = \mathbf{D}.\mathbf{DP}(\mathbf{DS})^2.\mathbf{S} = \mathbf{DPD}.\mathbf{DSDSS} = \mathbf{PD}.(\mathbf{DS})^{-1}$.

The question then remains as to whether higher powers of $\mathbf{DS}$ may "get past" $\mathbf{Q}$, and for this we must notice something about how the operators work. By considering the action on any given map in Ω_G it may be checked that $\mathbf{QDSD}$ is an involution. This allows us to note that $(\mathbf{DS})^{-1}.\mathbf{Q} = \mathbf{SDQ} = \mathbf{D}.\mathbf{DSDQ} = \mathbf{D}.\mathbf{QDSD} = \mathbf{DQD}.(\mathbf{DS})^{-1}$. It also follows that $(\mathbf{DS})^2.\mathbf{Q} = \mathbf{DSDSQ} = \mathbf{DSDQS} = \mathbf{QDSDS} = \mathbf{Q}.(\mathbf{DS})^2$.

Hence the subgroup $\langle\, \mathbf{DS}\,\rangle$ permutes with the subgroup $\langle\, \mathbf{D}, \mathbf{P}, \mathbf{Q}\,\rangle$ and the cardinality of the group $\langle\, \mathbf{D}, \mathbf{P}, \mathbf{Q}, \mathbf{S}\,\rangle$ is thus $48 \times 4 = 192$. The group may be presented with relators showing the interaction of powers of $\mathbf{DS}$ with elements of $\langle\, \mathbf{D}, \mathbf{P}, \mathbf{Q}\,\rangle$ as below:

$$\langle\, \mathbf{D}, \mathbf{P}, \mathbf{Q}, \mathbf{DS} \mid \mathbf{D}^2, \mathbf{P}^2, \mathbf{Q}^2, (\mathbf{DP})^3, (\mathbf{PQ})^2, (\mathbf{QD})^4, (\mathbf{DPQ})^6, (\mathbf{DS})^4, (\mathbf{DS}.\mathbf{D})^2,$$

$$\mathbf{DS}.\mathbf{P}(\mathbf{DS})^{-1}\mathbf{DPD}, (\mathbf{DS})^{-1}.\mathbf{P}(\mathbf{DS})^2\mathbf{PD},\ (\mathbf{DS})^2.\mathbf{PDSDP},$$

$$\mathbf{DSQ}(\mathbf{DS})^{-1}\mathbf{DQD}, (\mathbf{DS})^{-1}.\mathbf{QDSDQD},\ ((\mathbf{DS})^2.\mathbf{Q})^2\,\rangle$$

However, we know by Lemma 3.9 that $\mathbf{DPS}$ has order three, and we also have the earlier observation that $\mathbf{QDSD}$ is an involution. Furthermore $\mathbf{S}$, which is an involution, is a hole operator and so it commutes with the Petrie operator. This all gives us the relators in the group presentation in the theorem, and a computer may be used to check that the two presentations are equivalent. □

Remark 3.34. Note that the above theorem does not depend on the value of $\alpha \geq 2$ and so is a general result within $\mathbf{Op}_{k(G)}(\Omega_G)$ for $G = \mathrm{SL}(2, 2^\alpha)$. ⋆

If we knew anything about the group $\langle\, \mathbf{R}, \mathbf{DP}\,\rangle$ then that would be great since $\mathbf{Q} \in \langle\, \mathbf{R}\,\rangle$, and $\langle\, \mathbf{R}, \mathbf{S}\,\rangle$ is a substantial part (indeed it is sometimes all) of the group of hole operators modulo $k(G)$. In general, this is a much trickier prospect, if only because the order of $\mathbf{R}$ depends on α. I have, however, investigated the intricacies of the smallest case, where $\langle \mathbf{R}, \mathbf{S}\rangle$ happens to generate all the valid hole operators. Part of that investigation has been generalised to give the above theorem. See section 3.6 for further details.

3.5 Orbits of Ω_G under the action of $\mathbf{Op}_{k(G)}(\Omega_G)$ where $G = \mathbf{SL}(2, 2^\alpha)$

In this section, we let $G \cong \mathrm{SL}(2, q)$ where $q = 2^\alpha$ and $\alpha \geq 2$.

Remember, we may consider the group $\mathbf{Op}_{k(G)}(\Omega_G) = \langle\, \mathbf{D}, \mathbf{P}, \mathbf{H}_j \mid (j, q^2 - 1) = 1\,\rangle$ as a permutation group acting on Ω_G, the set containing all the fully regular maps with underlying group G. Also, since q is a power of 2 in this case, we know that $\mathbf{R} := \mathbf{H}_2$ is defined on all the maps in Ω_G.

3.5.1 Preliminaries

Once more we refer to the work in [19] in order to study fully regular maps by focussing on the traces of elements of the group $G = \mathrm{PSL}(2, q)$ which themselves are represented by matrices with entries written in terms of elements from the finite fields $F = \mathrm{GF}(q)$ and its quadratic extension $K = \mathrm{GF}(q^2)$. As before, for $\mathcal{M}(G; x, y, z)$, a fully regular map of type (k, ℓ), the triple of generating matrices, representing the group elements x, y and z, are denoted X, Y and Z respectively.

From [19], the triple (X, Y, Z) has representatives as reproduced below where $D = \omega_\kappa^2 + \omega_\lambda^2 - 4$, $\beta = -1/\sqrt{-D}$ and $\eta = (\xi_\kappa - \xi_\kappa^{-1})^{-1}$.

$$X = \eta\beta \begin{pmatrix} D & D\omega_\lambda\xi_\kappa \\ -\omega_\lambda\xi_\kappa^{-1} & -D \end{pmatrix}, \; Y = \beta \begin{pmatrix} 0 & \xi_\kappa D \\ {\xi_\kappa}^{-1} & 0 \end{pmatrix}, \; Z = \beta \begin{pmatrix} 0 & D \\ 1 & 0 \end{pmatrix}$$

The group $G = \mathrm{SL}(2, 2^\alpha)$ is simple, so has no index two subgroup, meaning that any fully regular map $\mathcal{M}(G; x, y, z)$ is non-orientable. Working up to isomorphism of maps, we may therefore consider the group G as being generated by two rotations, which, by [19], have representative matrices R and S as per Equation (2.2), reproduced here for ease of reference.

$$R = \begin{pmatrix} \xi_\kappa & 0 \\ 0 & \xi_\kappa^{-1} \end{pmatrix} \text{ and } S = \eta \begin{pmatrix} -\omega_\lambda\xi_\kappa^{-1} & -D \\ 1 & \omega_\lambda\xi_\kappa \end{pmatrix}.$$

We restrict the investigation to the case where the field $\mathrm{GF}(q)$ has characteristic 2, and this is to our advantage for simplifying some of the associated algebraic expressions. For a start it means that $\kappa = k$ and $\lambda = \ell$, so ξ_κ and ξ_λ are respectively primitive kth and ℓth roots of unity.

We remind the reader that $R = YZ$ determines a rotation around a vertex with associated trace $\omega_k := \xi_\kappa + \xi_\kappa^{-1}$, while $S = ZX$ corresponds to a rotation around a face and has trace $\omega_\ell := \xi_\lambda + \xi_\lambda^{-1}$.

Given $G = \mathrm{SL}(2, 2^\alpha)$ where $\alpha \geq 2$, and a map $\mathcal{M}(G; x, y, z)$ we define the *trace triple* to be the ordered triple $(Tr(YZ), Tr(ZX), Tr(XYZ))$ which we denote $(\omega_k, \omega_\ell, \omega_m)$. The notation is thus consistent with that in [19] and also Chapter 2 of this work, and we sometimes denote this map by $\mathcal{M}(\omega_k, \omega_\ell, \omega_m)$.

This section investigates the orbit structure and action of $\mathbf{Op}_{q^2-1}(\Omega_G)$ on the set of maps with automorphism group $G = \mathrm{SL}(2, 2^\alpha)$, by considering the effect of each of the generating operators on the trace triple. Initially the focus will be on the size of orbits of trace triples, with no regard for the isomorphism classes of the associated maps. We will address map isomorphisms between trace triples a little later, in section 3.5.4.

Remark 3.35. In contrast to elsewhere in this thesis, where we have focussed on individual rooted maps, rather than isomorphism classes, the notation $\mathcal{M}(\omega_k, \omega_\ell, \omega_m)$

does not determine a rooted fully regular map uniquely, since any conjugate map will have the same trace triple. ⋆

Remark 3.36. The trace for a non-trivial element is zero if and only if the element in question is an involution. Since all fully regular maps with automorphism group $G = \mathrm{SL}(2, 2^\alpha)$ are non-spherical, we may assume k, ℓ and m are all greater than or equal to three, and so a trace triple will never contain zero as an element. Thus $\omega_i \in \mathrm{GF}(q)^*$ for all $i \in \{k, \ell, m\}$. ⋆

Underlying much of the work in this section are the following two facts, presented as lemmas, each of which relies on the field $\mathrm{GF}(q)$ having characteristic two. The first demonstrates how closely the three traces ω_k, ω_ℓ and ω_m are related to each other. The second fact was a great surprise, and relates to how nicely hole operators act on trace triples - it originally appeared in answer to a vague question asked by my supervisor as to how the Petrie length of a map might be affected by hole operators.

Lemma 3.37. *Let $\mathcal{M}(\omega_k, \omega_\ell, \omega_m)$ be a fully regular map with underlying group $G = SL(2, 2^\alpha)$ where $\alpha \geq 2$. The sum of the elements in the trace triple is zero.*

Proof (sketch). With some routine algebraic manipulation (which relies on the field having characteristic two) it may be checked that the trace, $Tr(XYZ) = \omega_m$, corresponding to a rotation around the face of the associated Petrie map is such that $\omega_m = \omega_k + \omega_\ell$. □

Lemma 3.38. *Let $\mathcal{M}(\omega_k, \omega_\ell, \omega_m)$ be a given fully regular map where $G \cong SL(2, q)$ and $q = 2^\alpha$ for $\alpha \geq 2$. Let j be coprime to k so that $\boldsymbol{H}_j$ is a valid hole operator acting on $\mathcal{M}$. Then $\mathcal{M}\boldsymbol{H}_j = \mathcal{M}'(\xi_k^j + \xi_k^{-j}, \frac{\omega_\ell}{\omega_k}(\xi_k^j + \xi_k^{-j}), \frac{\omega_m}{\omega_k}(\xi_k^j + \xi_k^{-j}))$.*

Proof (sketch). The action of the hole operator $\mathbf{H}_j$ fixes x and y and sends z to $(zy)^j y$. In particular $\mathbf{H}_j : R \to R^j$, and so the first element in the trace triple becomes $\xi_k^j + \xi_k^{-j}$.

Meanwhile a small amount of manipulation yields that the second element of the trace triple is $Tr((ZY)^j YX) = \frac{\omega_\ell}{\omega_k}(\xi_k^j + \xi_k^{-j})$, and then Lemma 3.37, completes the proof of the claim. □

This revelation was a joy: a completely unexpected, and very useful result, which allowed much progress as outlined below.

We will abuse the language slightly by referring to orbits of trace triples under the action of certain operators, and thereby we somewhat lose sight of the associated regular maps. This has both advantages and disadvantages, as will become clear.

We define the *triple trace ratio* for a given map $\mathcal{M}(\omega_k, \omega_\ell, \omega_m)$ to be the triple ratio $\omega_k : \omega_\ell : \omega_m$ which, being a ratio, is equivalent to $1 : \frac{\omega_\ell}{\omega_k} : \frac{\omega_m}{\omega_k}$.

This is worth defining because of the following easy corollary to the remarkable Lemma 3.38.

Corollary 3.39. *Let $\mathcal{M}(\omega_k, \omega_\ell, \omega_m)$ be a given fully regular map where $G \cong SL(2, q)$ and $q = 2^\alpha$ for $\alpha \geq 2$. The triple trace ratio of $\mathcal{M}$ is preserved by all valid hole operators.*

The above corollary presents a very useful fact when we consider the size of orbits of trace triples under the action of $\mathbf{Op}_{q^2-1}(\Omega_G)$ as it gives us an immediate and very easy upper bound.

Lemma 3.40. *Let $\mathcal{M}(\omega_k, \omega_\ell, \omega_m)$ be a given fully regular map where $G \cong SL(2, q)$ and $q = 2^\alpha$ for $\alpha \geq 2$. The orbit of $\mathcal{M}$ under the action of the group $\boldsymbol{Op}_{q^2-1}(\Omega_G) = \langle\, \boldsymbol{D}, \boldsymbol{P}, \boldsymbol{H}_j \,\rangle$ consists of a set of maps realising no more than $6(q-1)$ trace triples.*

Proof. First notice that ω_k, ω_ℓ, and ω_m are three distinct elements of $\mathrm{GF}(q)$. The operators $\mathbf{D}, \mathbf{P}$ simply permute the elements within a given trace triple giving a total of precisely six trace triples in a single orbit of $\mathcal{M}$ under $\langle\, \mathbf{D}, \mathbf{P} \,\rangle$. Similarly, this process yields at most six triple trace ratios for the associated maps. Meanwhile, for a given map in Ω_G, the triple trace ratio is fixed by all valid hole operators. Finally, realise that there are precisely $q-1$ trace triples for each given triple trace ratio. The claim then follows immediately. □

The aforementioned longest possible orbits almost always exist, as we shall now see.

3.5.2 A trace triple orbit of length $6(q-1)$

First we note the following Lemma, detailing how nicely $\mathbf{R}$ behaves, thanks largely to the Freshers' dream. This is key in the thinking behind the following reasoning and algorithm.

Lemma 3.41. *Let $\mathcal{M}(\omega_k, \omega_\ell, \omega_m)$ be a fully regular map on $SL(2, 2^\alpha)$. Then $\mathcal{M}\boldsymbol{R}$ has trace triple $(\omega_k^2, \omega_\ell\omega_k, \omega_m\omega_k)$.*

Proof. By Lemma 3.38, being a hole operator, $\mathbf{R}$ respects triple trace ratios. The field $\mathrm{GF}(2^\alpha)$ has characteristic two, so $\xi^2 + \xi^{-2} = (\xi + \xi^{-1})^2$ which confirms the first element in the trace triple for $\mathcal{M}\mathbf{R}$ is ω_k^2. Corollary 3.39 then completes the proof of the claim. □

Let β be a primitive element of the finite field F_q where $q = 2^\alpha$. Let $\mathcal{B}_\beta$ be a fully regular map on SL$(2, q)$ which has trace triple $(1, \beta, 1 + \beta)$. Since the first trace in the triple is $\omega_k := \xi_\kappa + \xi_\kappa^{-1} = 1$, this map has valency 3, and we refer to it as the *base map* for the element β. As the valency is three, any valid hole operator $\mathbf{H}_j$ applied to this map must be such that $j \equiv \pm 1 \pmod 3$, and so $\mathbf{H}_j$ will behave either like the identity $\mathbf{Id}$, or like $\mathbf{S} = \mathbf{H}_{-1}$. Thus hole operators maintain the isomorphism class of $\mathcal{B}_\beta$, and indeed fix the trace triple. However, from the dual of the base map, we may use rotational powers to build more maps. In fact, from $\mathcal{B}_\beta$, and limiting ourselves to the operators $\mathbf{R}$ and $\mathbf{D}$, we can build $2(q - 1)$ maps with different trace triples.

Let $\mathcal{M}_\beta = (\mathcal{B}_\beta)\mathbf{D}$ be the dual of the base map, and so $\mathcal{M}_\beta$ has trace triple $(\beta, 1, 1 + \beta)$. In terms of powers of the given primitive element β, the notation $[1, 0, c]$ is sometimes used to describe a map with trace triple $(\beta, 1, 1 + \beta)$.

By a combination of the operators $\mathbf{D}$ and $\mathbf{R}$, I claim it is possible to obtain, from $\mathcal{B}_\beta$, any other map $\mathcal{M}'$ such that the trace triple has the form $(f\beta, f, f(\beta + 1))$, that is $[a + 1, a, a + c]$, for any $f = \beta^a \in F_q$, where a may be assumed to be such that $0 \leq a \leq q - 2$.

The reasoning is as follows. The difference between a and $a + 1$ is one, and so precisely one of them is even. Apply $\mathbf{D}$ if necessary to ensure the first entry in the trace triple is (written as) an even power of β. The even power of β (which is now the first entry in the trace triple) may be halved by a single application of $\mathbf{R}^{-1}$. However the ratio within the trace triple remains unchanged by the application of a hole operator, and hence the difference (in the first two elements of the trace triple) between the powers of β is still one, meaning that, again, precisely one of them is even. Repeating this argument as many times as is necessary will eventually result in the base map.

In particular, if the number a is written in binary (with the leading digit naturally being 1) then the operator which transforms the map $\mathcal{M}_\beta$ into the map with trace triple $(f\beta, f, f(\beta + 1))$ can be "read off" according to this algorithm.

> Reading from left to right, for each binary digit write down $\mathbf{R}$, all the while inserting $\mathbf{D}$ every time the binary digit swaps from zero to one or vice versa. Finally write down $\mathbf{D}$ if and only if the binary number ends with a 1.

An example or two may help demonstrate how to apply this algorithm.

Example 3.42. Find an operator which maps $\mathcal{M}_\beta$, with trace triple $(\beta, 1, 1+\beta)$, to a map with trace triple $(\beta^{28}, \beta^{29}, \beta^{28} + \beta^{29})$.

First write a in binary: $a = 28 = 11100$. Marking the transition(s) between binary digits yields $111 \dagger 00$, at which point we may "write down" **RRR D RR** and check that $\mathcal{M}$**RRRDRR** is indeed a map with trace triple $(\beta^{28}, \beta^{29}, \beta^{28}(1+\beta))$. •

It is clear how to modify the algorithm to find the operator which maps the base map $\mathcal{B}_\beta$ to a map $\mathcal{M}'$ which has a consistent triple trace ratio, simply by including **D** at the beginning.

Example 3.43. Find an operator which maps $\mathcal{B}_\beta$ which has trace triple $(1, \beta, 1+\beta)$ to the map $\mathcal{M}'$ with trace triple $(\beta^{89}, \beta^{90}, \beta^{89} + \beta^{90})$.

Now $89 = 1011001$ which becomes $1 \dagger 0 \dagger 11 \dagger 00 \dagger 1\dagger$, where the final dagger reminds us that the binary number ended with a 1. Hence
$(\mathcal{M}_\beta)$**R D R D RR D RR D R D** $= \mathcal{M}'$. But $\mathcal{M}_\beta = (\mathcal{B}_\beta)$**D** so the required operator is **DRDRDRRDRRDRD**. •

Armed with this algorithm, we may now prove that, under the action of $\mathbf{Op}_{k(G)}(\Omega_G)$, these longest possible orbits consisting of $6(q-1)$ trace triples nearly always occur.

Proposition 3.44. *Let $G \cong SL(2, q)$ where $q = 2^\alpha$ and $\alpha \geq 2$. Under the action of $\boldsymbol{Op}_{q^2-1}(\Omega_G)$, trace triple orbits of cardinality $6(q-1)$ occur if and only if $\alpha \geq 3$.*

Proof. Let β be a primitive element of the finite field $\mathrm{GF}(q)$, and let $\mathcal{M}(\beta, 1, 1+\beta)$ be a fully regular map on G. The aim is to prove that, in almost all cases, the orbit of $\mathcal{M}$ contains precisely $6(q-1)$ trace triples. We remember that, by Lemma 3.40, it cannot contain more than this.

By the reasoning behind the above algorithm, under the action of $\langle\, \mathbf{D}, \mathbf{R}\,\rangle$, the single trace triple of the particular form $(\beta, 1, 1+\beta)$, yields $(q-1)$ trace triples, $(f\beta, f, f(1+\beta))$, all within the same orbit. In particular these $(q-1)$ trace triples may be assumed to be distinct since they share the same (ordered) triple trace ratio, and yet the trace triples begin with $(q-1)$ different values. The operator **P**, along with **D**, then generates all six permutations of elements within each trace triple. At a glance this seems to be the $6(q-1)$ trace triples which we are looking for, and the only thing which remains is to convince ourselves that there is no double-counting.

Double-counting in the above situation can clearly only happen when two maps have the same first value in the trace triple. This leaves the only loophole being when they have apparently different (but actually equal) triple trace ratios. However, the triple trace ratios are limited to at most six, and the list is as follows: $\beta : 1 : 1+\beta$, and $1 : \beta : 1+\beta$, and $1+\beta : 1 : \beta$, and $\beta : 1+\beta : 1$, and $1 : 1+\beta : \beta$, and $1+\beta : \beta : 1$.

Suppose there is some double-counting going on, and two of these ratios are the same. They cannot all be the same, since $\beta = 1+\beta$ has no solution. So, either the set of six splits into three non-equal ratios, or two. We take each case in turn.

Consider the first case when the equivalence is such that two elements within the triple trace ratio are swapped and the other remains the same. Without loss of generality, the only possible case is when $\beta : 1 : 1+\beta$ and $1 : \beta : 1+\beta$ are equivalent, that is $\beta = \beta^{-1}$. But then $\beta^2 = 1$ and, since β is assumed to be a primitive element, the finite field is GF(2). This field is too small for the situations we are considering, so there is no double-counting of trace triples arising in this way.

Now consider the case when the equivalence is by cyclic permutation of the elements in the ratio, that is $\beta : 1 : 1+\beta$ is equal as a triple ratio to $1 : 1+\beta : \beta$. In this case $\beta(\beta+1) = 1$, or equivalently $\beta^2 = 1+\beta$. Remembering that β is a primitive element of the field GF(q), this yields $q = 4$.

In all other cases the six triple trace ratios are distinct, and so we may now conclude that the orbit of $\mathcal{M}$ contains maps with precisely $6(q-1)$ ordered trace triples if and only if $\alpha \geq 3$. $\square$

Remark 3.45. At this stage it is worth pointing out that we have only considered fully regular maps on G, each with reference just to its trace triple. There is nothing so far to suggest that two maps having different trace triples must be non-isomorphic. However, by considering the automorphism group of $\mathrm{SL}(2,q)$, which is $\mathrm{PGL}(2,q) \rtimes \mathrm{C}_\alpha$, and remembering that conjugation preserves traces, the only automorphisms of the group which change the trace triples are the Frobenius field automorphisms. It is important to keep this in mind if we wish to study and/or enumerate the orbits of maps *up to isomorphism*, since, for a given trace triple, there may be up to α distinct trace triples all of which correspond to mutually isomorphic maps. $\star$

It so happens that the example orbit given to prove the above Proposition, nearly always consists of maps from $6(q-1)$ different map isomorphism classes. In section 3.5.4 we establish the precise conditions under which two distinct trace triples actually correspond to isomorphic maps.

For now we present some of the initial investigations and associated observations which motivated the lemmas in this section, and which were the inspiration for the later conjectures regarding orbit decomposition of Ω_G.

3.5.3 A small motivating example

The elements in the finite field $\mathrm{GF}(2^\alpha)$ are naturally partitioned into orbits by the action of the Frobenius field automorphism which maps x to x^2. To save paper and ink, I refer to these orbits as *Frorbits*.

Studying these Frorbits for small finite fields $\mathrm{GF}(q)$, in particular GF(64) and GF(256), and considering the maps supported by the corresponding groups $\mathrm{SL}(2, q)$, lead me to realise a few things about how the orbit structure of the operator group behaves with respect to the isomorphism classes of maps in $\Omega_{\mathrm{SL}(2,q)}$. These observations are recorded as Remarks, using GF(64) as an example, and formed the foundations for the lemmas in the following section.

Example 3.46 (The finite field GF(64) = GF(2^6)). Let β be a primitive element of the field $\mathrm{GF}(64) = \mathrm{GF}(2^6)$. We refer to elements of the finite field by the corresponding powers of this distinguished primitive element β. For example $[\beta^{19}]$ denotes the Frorbit containing β^{19}. The Frorbits are as follows:

$[\beta^0] = \{1\}$

$[\beta^1] = \{\beta, \beta^2, \beta^4, \beta^8, \beta^{16}, \beta^{32}\} = [\beta^{-31}]$

$[\beta^3] = \{\beta^3, \beta^6, \beta^{12}, \beta^{24}, \beta^{48}, \beta^{33}\} = [\beta^{-15}]$

$[\beta^5] = \{\beta^5, \beta^{10}, \beta^{20}, \beta^{40}, \beta^{17}, \beta^{34}\} = [\beta^{-23}]$

$[\beta^7] = \{\beta^7, \beta^{14}, \beta^{28}, \beta^{56}, \beta^{49}, \beta^{35}\} = [\beta^{-7}]$

$[\beta^9] = \{\beta^9, \beta^{18}, \beta^{36}\} = [\beta^{-27}]$

$[\beta^{11}] = \{\beta^{11}, \beta^{22}, \beta^{44}, \beta^{25}, \beta^{50}, \beta^{37}\} = [\beta^{-13}]$

$[\beta^{13}] = \{\beta^{13}, \beta^{26}, \beta^{52}, \beta^{41}, \beta^{19}, \beta^{38}\} = [\beta^{-11}]$

$[\beta^{15}] = \{\beta^{15}, \beta^{30}, \beta^{60}, \beta^{57}, \beta^{51}, \beta^{39}\} = [\beta^{-3}]$

$[\beta^{21}] = \{\beta^{21}, \beta^{42}\} = [\beta^{-21}]$

$[\beta^{23}] = \{\beta^{23}, \beta^{46}, \beta^{29}, \beta^{58}, \beta^{53}, \beta^{43}\} = [\beta^{-5}]$

$[\beta^{27}] = \{\beta^{27}, \beta^{54}, \beta^{45}\} = [\beta^{-9}]$

$[\beta^{31}] = \{\beta^{31}, \beta^{62}, \beta^{61}, \beta^{59}, \beta^{55}, \beta^{47}\} = [\beta^{-1}]$ •

Remark 3.47. The multiplicative identity $\beta^0 = 1$ is naturally in its own Frorbit

$[\beta^0]$. $\star$

Remark 3.48. Some Frorbits contain both β^a and β^{-a}. For example $[\beta^7] = [\beta^{-7}]$ $\star$

Remark 3.49. $\mathrm{GF}(2^6)$ has non-trivial subfields, namely $\mathrm{GF}(2^2)$ and $\mathrm{GF}(2^3)$ leading to Frorbits of length 2 and 3 respectively. $\star$

Remark 3.50. Choosing two distinct elements of the field as values for ω_k and ω_ℓ yields a fully regular map with trace triple $(\omega_k, \omega_\ell, \omega_k + \omega_\ell)$ whose automorphism group is $\mathrm{SL}(2, K)$ where K is the smallest subfield containing both ω_k and ω_ℓ. The two elements in question may be chosen to come from the same Frorbit, in which case it is possible, but not necessarily the case that their sum is also in the same Frorbit. $\star$

Remark 3.51. The single non-trivial Frorbit within the $\mathrm{GF}(4)$ subfield has two elements. We've seen this pair of field elements before - they appeared in the proof of Proposition 3.44. The elements of this Frorbit sum to one, $\beta^{21} + \beta^{42} = 1$, and correspond to a map on $\mathrm{SL}(2, 4)$. The corresponding triple trace ratio and the corresponding maps are particularly remarkable, as will become clear. $\star$

Remark 3.52. In reference to the previous two remarks, the Frorbit $[\beta^7]$ is such that there is a map whose trace triple values are all evenly spaced within the same Frorbit, namely $\mathcal{M}(\beta^7, \beta^{28}, \beta^{49})$. More precisely the field automorphism $x \to x^{2^2}$ maps $\beta^7 \to \beta^{28} \to \beta^{49} \to \beta^7$, and since $\beta^7 + \beta^{28} + \beta^{49} = \beta^7(1 + \beta^{21} + \beta^{42})$ we have $\beta^{49} = \beta^7 + \beta^{28}$. $\star$

Remark 3.53. Choosing two trace elements evenly spaced within the same (non-subfield) Frorbit is possible since $\alpha = 6$ is even. This process yields self-dual maps. For example the map $\mathcal{M}(\beta^5, \beta^{40}, \beta^5 + \beta^{40})$ has dual map $\mathcal{M}^*(\beta^{40}, \beta^5, \beta^5 + \beta^{40})$ which is isomorphic to $\mathcal{M}$ since the field automorphism $x \to x^{2^3}$ interchanges the two. $\star$

Lemma 3.54. *Let $\alpha = 2d$. Then self-dual maps exist on $SL(2, 2^\alpha)$.*

Proof. Let γ be an element of $\mathrm{GF}(q)$ but not in any proper subfield. The map $\mathcal{M}(\gamma, \gamma^{2^d}, \gamma + \gamma^{2^d})$ is transformed, by the involutory field automorphism $x \to x^{2^d}$, into $\mathcal{M}'(\gamma^{2^d}, \gamma, \gamma + \gamma^{2^d})$. Thus $\mathcal{M}$ is isomorphic to its dual map. $\square$

Remark 3.55. Taking, for example, γ to be the first element listed from each Frorbit in Example 3.46, the self-dual maps found in the above-described way have, respectively, triple trace ratios

(not possible)
$1 : \beta^7 : \ldots,$

$1 : \beta^{21} : \ldots,$

$1 : \beta^{35} : \ldots,$

$1 : \beta^{49} : \ldots,$

(not possible),

$1 : \beta^{14} : \ldots,$

$1 : \beta^{28} : \ldots,$

$1 : \beta^{42} : \ldots,$

$1 : \beta^{21} : \ldots,$

$1 : \beta^{35} : \ldots,$

(not possible),

$1 : \beta^{28} : \ldots.$

The second entry in each of these standardised triple trace ratios is necessarily $\gamma^{(2^d-1)}$, and this belongs to a Frorbit which contains its own inverse elements since $(\gamma^{(2^d-1)})^{2^d} = \gamma^{(1-2^d)}$. ⋆

Remark 3.56. The subfield $\mathrm{GF}(2^3)$ yields Frorbits of length three, for example $[\beta^9] = \{\beta^9, \beta^18, \beta^36\}$ while $[\beta^{27}]$ is the other non-trivial Frorbit from the $\mathrm{GF}(2^3)$ subfield. ⋆

Remark 3.57. The Frorbit [3] has a certain vibe of the subfield $\mathrm{GF}(2^3)$ which is generated by β^9. In fact $\beta^3 + \beta^{12} + \beta^{48} = \beta^3(1 + \beta^9 + \beta^{45}) = 0$ and so there is a map $\mathcal{M}(\beta^3, \beta^{12}, \beta^{48})$. This map is self-trial since the cyclic permutation of elements within the trace triple is equivalent to the field automorphism $x \to x^{2^2}$. ⋆

Remark 3.58. For a map admitting triality but not duality we need a γ not in any proper subfield of $\mathrm{GF}(2^\alpha)$, such that $3 \mid \alpha$ so $\alpha = 3d$ and $\gamma + \gamma^{2^d} + \gamma^{2^{2d}} = 0$. Jones and Poulton [41] refer to such a γ as a “useful generator” and prove such an element always exists. Furthermore [41] provides a rigorous proof that this is the only source for such maps on $\mathrm{SL}(2, 2^\alpha)$. ⋆

We now continue the same example by considering the triples of field elements including the element 1 and summing to zero. Arranging the elements within such a triple so that 1 is the first value, these triples naturally correspond to base maps and their associated triple trace ratios. Remember that the base map $\mathcal{B}_\gamma$ is a map with trace triple $(1, \gamma, 1 + \gamma)$ and associated triple trace ratio $1 : \gamma : 1 + \gamma$.

Example 3.59 (Base maps for maps on $\mathbf{SL}(2, K)$ where $K \leq \mathbf{GF}(64)$). Let β be a primitive element of the field $\mathrm{GF}(64)$, and suppose its minimal polynomial is $x^6 + x + 1$. It may be checked that, up to isomorphism within the field, the ordered triples of field elements with the fixed first element being 1 and the three elements summing to zero are as listed below.

$\mathcal{B}_{\beta}(1, \beta, \beta^6)$
$\mathcal{B}_{\beta^3}(1, \beta^3, \beta^{32})$
$\mathcal{B}_{\beta^5}(1, \beta^5, \beta^{62})$
$\mathcal{B}_{\beta^7}(1, \beta^7, \beta^{26})$
$\mathcal{B}_{\beta^9}(1, \beta^9, \beta^{45})$
$\mathcal{B}_{\beta^{11}}(1, \beta^{11}, \beta^{25})$
$\mathcal{B}_{\beta^{13}}(1, \beta^{13}, \beta^{35})$
$\mathcal{B}_{\beta^{15}}(1, \beta^{15}, \beta^{23})$
$\mathcal{B}_{\omega}(1, \beta^{21}, \beta^{42})$
$\mathcal{B}_{\beta^{23}}(1, \beta^{23}, \beta^{15})$
$\mathcal{B}_{\beta^{27}}(1, \beta^{27}, \beta^{18})$
$\mathcal{B}_{\beta^{31}}(1, \beta^{31}, \beta^{34})$ •

At this point there are further remarks and observations which inspired both proofs and conjectures in the next sections.

Remark 3.60. When $\beta^a = \delta$ is in a proper subfield, then the base map $\mathcal{B}_\delta$ has automorphism group $\mathrm{SL}(2, 2^d)$ for some $d \mid \alpha$ such that $1 \leq d < \alpha$. However even in such a case there are maps with the same triple trace ratio as $\mathcal{B}_\delta$, that is $1 : \delta : 1 + \delta$, whose automorphism group is still the whole group $\mathrm{SL}(2, 2^\alpha)$. The map with trace triple $(\beta, \beta\delta, \beta(1 + \delta))$ is an example of such a map. ⋆

Example 3.61 (Example 3.59 continuation). The following observations will allow us to arrange the base maps into sets.

- Applying $x \to x^2$ to the map $(\mathcal{B}_{\beta^3})\mathbf{P}$ yields $\mathcal{B}_\beta$.
- $\mathcal{B}_{\beta^5}(1, \beta^5, \beta^{62})$ has the same triple trace ratio as $(\mathcal{B}_\beta)\mathbf{DP}$
- $\mathcal{B}_{\beta^9}(1, \beta^9, \beta^{45})$ all entries in GF(8) so $1 : \beta^9 : \beta^{45}$ yields a shorter orbit.
- $\mathcal{B}_{\beta^{11}}(1, \beta^{11}, \beta^{25})$ is self-Petrie. This can be seen since the second and third entry of the trace triple are evenly spaced within the same Frorbit. More precisely $x \to x^{2^3}$ interchanges β^{11} and β^{25}.
- Applying $x \to x^2$ to the map $(\mathcal{B}_{\beta^{13}})\mathbf{P}$ yields $\mathcal{B}_{\beta^7}$.
- $\mathcal{B}_{\beta^{15}}(1, \beta^{15}, \beta^{23})\mathbf{DP}$ has the triple trace ratio $1 : \beta^8 : 1 + \beta^8$, the same as the image under $x \to x^{2^3}$ of $\mathcal{B}_\beta$.
- $\mathcal{B}_\omega(1, \beta^{21}, \beta^{42})$ This triple keeps turning up! It turns out to be very special and yields a very short orbit(s?). See Example 3.73.

- $\mathcal{B}_{\beta^{23}}(1, \beta^{23}, \beta^{15})$ is simply $\mathcal{B}_{\beta^{15}}\mathbf{P}$.
- $\mathcal{B}_{\beta^{27}}(1, \beta^{27}, \beta^{18})$ has all its trace elements in GF(8). Applying $x \to x^2$ to $\mathcal{B}_{\beta^9}\mathbf{P}$ yields $\mathcal{B}_{\beta^{27}}$.
- $\mathcal{B}_{\beta^{31}}(1, \beta^{31}, \beta^{34})$ transformed by $x \to x^2$ has the same *triple trace ratio* as $\mathcal{B}_\beta\mathbf{D}$.

The above observations allows us to group the base maps into sets according to the action of $\langle\, \mathbf{P}\, \rangle$ and modulo field automorphisms.

$\{\mathcal{B}_\beta(1, \beta, \beta^6),\ \mathcal{B}_{\beta^3}(1, \beta^3, \beta^{32})\}$
$\{\mathcal{B}_{\beta^5}(1, \beta^5, \beta^{62}),\ \mathcal{B}_{\beta^{31}}(1, \beta^{31}, \beta^{34})\}$
$\{\mathcal{B}_{\beta^7}(1, \beta^7, \beta^{26}),\ \mathcal{B}_{\beta^{13}}(1, \beta^{13}, \beta^{35})\}$
$\{\mathcal{B}_{\beta^9}(1, \beta^9, \beta^{45}),\ \mathcal{B}_{\beta^{27}}(1, \beta^{27}, \beta^{18})\}$
$\{\mathcal{B}_{\beta^{11}}(1, \beta^{11}, \beta^{25})\}$
$\{\mathcal{B}_{\beta^{15}}(1, \beta^{15}, \beta^{23}),\ \mathcal{B}_{\beta^{23}}(1, \beta^{23}, \beta^{15})\}$
$\{\mathcal{B}_\omega(1, \beta^{21}, \beta^{42})\}$ •

Remark 3.62. The existence of a self-dual map or a self-trial map within an orbit of trace triples ensures that the corresponding set of isomorphism classes of map contains less than $6(q-1)$ elements. ⋆

Remark 3.63. By considering how the trace elements would have to be arranged within a single Frorbit, it is clear that a map with Trinity symmetry cannot exist on $\mathrm{SL}(2, 2^\alpha)$. ⋆

Remark 3.64. Given the base map $\mathcal{B}_\beta(1, \beta, 1+\beta)$, its dual map has trace triple $(\beta, 1, 1+\beta)$ and so has triple trace ratio $1 : \beta^{-1}, 1+\beta^{-1}$. Since β is a primitive element, by Proposition 3.44, the map $\mathcal{B}_\beta$ has in its $\langle\, \mathbf{D}, \mathbf{P}, \mathbf{H}_j\, \rangle$-orbit the base maps $\mathcal{B}_\gamma$ for all $\gamma \in \{\beta, \beta^{-1}, 1+\beta, 1+\beta^{-1}, (1+\beta)^{-1}, \beta(1+\beta)^{-1}\}$. ⋆

Following the example of the above remark, and working up to field automorphisms (so only considering one base map per Frorbit) we may group together the base maps $\mathcal{B}_\epsilon$ for $\epsilon \in \{\gamma, \gamma^{-1}, 1+\gamma, 1+\gamma^{-1}, (1+\gamma)^{-1}, \gamma(1+\gamma)^{-1}\}$. Each such set of base maps, denoted $\mathfrak{B}_\gamma$, represents the (up to) six possible triple trace ratios within a given orbit under the action of $\langle\, \mathbf{D}, \mathbf{P}\, \rangle$. We reiterate that the members of $\mathfrak{B}_\gamma$ may in fact correspond to maps on $\mathrm{SL}(2, K)$ for some proper subfield $K \leq \mathrm{GF}(2^\alpha)$.

Note also that at this stage, with the exception of the case when γ is a primitive element of $\mathrm{GF}(2^\alpha)$, we have no evidence that the base maps within $\mathfrak{B}_\gamma$ are in the same orbit as each other, since we have to rely on the action of hole operators to ensure this. In particular, it remains a conjecture that, for a given γ, the set containing all maps on $\mathrm{SL}(2, 2^{\alpha})$ with the same triple trace ratio as one of those in $\mathfrak{B}_\gamma$ does not decompose into more than one orbit under the action of $\langle\, \mathbf{D}, \mathbf{P}, \mathbf{H}_j \,\rangle$. However, investigations of small examples suggest that each $\mathfrak{B}_\gamma$ does provide the set of triple trace ratios for a single true orbit of maps within $\mathrm{SL}(2, 2^{\alpha})$ for any given α.

Example 3.65 (Example 3.59 further continuation - orbits of maps). It may be checked that the base maps within each of the following sets are indeed in the same orbit, and so (modulo field automorphisms) represent all the possible triple trace ratios in the corresponding full orbit. Furthermore, since by Lemma 3.38 hole operators respect triple trace ratios, these sets of base maps are closed under the action of the group $\langle\, \mathbf{D}, \mathbf{P}, \mathbf{H}_j \,\rangle$.

- The set of mutually non-isomorphic base maps in $\mathfrak{B}_\beta$ is as follows: $\{\mathcal{B}_\beta,\, \mathcal{B}_{\beta^3},\, \mathcal{B}_{\beta^5},\, \mathcal{B}_{\beta^{31}},\, \mathcal{B}_{\beta^{15}},\, \mathcal{B}_{\beta^{23}}\}$. This orbit is the longest possible and contains $6(q-1) = 378$ mutually non-isomorphic maps.

- Meanwhile $[13] = [-11]$ while $[7] = [-7]$, and so the set of mutually non-isomorphic base maps including $\mathcal{B}_{\beta^{13}}$ is: $\{\mathcal{B}_{\beta^7},\, \mathcal{B}_{\beta^{13}},\, \mathcal{B}_{\beta^{11}}\}$. The presence of a base map $\mathcal{B}_\gamma$ from a self-inverse Frorbit $[\gamma] = [\gamma^{-1}]$ indicates double counting due to potential self-duality. In particular, since $\mathcal{B}_\gamma \cong \mathcal{B}_{\gamma^{-1}}$, there are fewer than six isomorphism classes of map in the set $\mathfrak{B}_\gamma$. This orbit contains $3(q-1) = 189$ mutually non-isomorphic maps.

- The maps whose triple trace ratios is based on the subfield $\mathrm{GF}(2^3)$ form their own set of mutually non-isomorphic maps with just two base maps: $\{\mathcal{B}_{\beta^9},\, \mathcal{B}_{\beta^{27}}\}$. Both these base maps are invariant under the involutory field automorphism $x \to x^{2^3}$. This orbit contains $64 - 8 = 56$ mutually non-isomorphic maps.

- And finally, the exceptional case $\mathcal{B}_\omega(1, \beta^{21}, \beta^{42})$ is on its own: $\{\mathcal{B}_\omega\}$. This orbit contains 20 mutually non-isomorphic maps. •

Remark 3.66. This is, of course, one of the smallest examples possible, and so some of the structure is squashed into the available space. It is also the first example where it is possible to appreciate some of the rich underlying structure resulting in different lengths of orbit. It is an interesting exercise to investigate the orbits of non-isomorphic maps on the larger group $\mathrm{SL}(2, 2^{12})$. This demonstrates further the possible different lengths of orbits, some of which have been suppressed in the example in this section because $\mathrm{SL}(2, 64)$ is too small. ⋆

These observations, and hours spent playing with the other fields of characteristic two up to and including GF(256) lead me to believe that short orbits are fundamentally related to subfields, and cannot occur without them. The more specific conjectures that I made regarding sizes of orbits in small cases were verified by computer calculations performed by Dr Grahame Erskine for $\mathrm{SL}(2, 2^\alpha)$ where $\alpha \leq 13$. This I found enormously encouraging. Some of these inklings remain conjectures, but I believe the following subsection will demonstrate how closely the shorter orbits (of isomorphism classes of map, not just trace triples) are related to proper subfields of $\mathrm{GF}(q)$.

3.5.4 Trace triples for isomorphic maps

Two maps are isomorphic to each other if and only if the trace triple of one is the image of the trace triple for the other under element-wise application of a single field automorphism.

In this way the base map $\mathcal{B}_\gamma$ with trace triple $(1, \gamma, 1+\gamma)$ belongs to an equivalence class consisting of $c \leq \alpha$ base maps $\{\mathcal{B}_\gamma, \mathcal{B}_{\gamma^2}, \mathcal{B}_{\gamma^4}, \ldots \mathcal{B}_{\gamma^{2c-1}}\}$ respectively with trace triples,
$\{(1, \gamma, 1+\gamma), (1, \gamma^2, 1+\gamma^2), (1, \gamma^4, 1+\gamma^4), (1, \gamma^8, 1+\gamma^8) \ldots, (1, \gamma^{2^{c-1}}, 1+\gamma^{2^{c-1}})\}$, all of which are in the same isomorphism class of map. This set consists of α distinct trace triples (and indeed α distinct triple trace ratios) if and only if γ does not belong to a proper subfield of $\mathrm{GF}(2^\alpha)$.

Working up to isomorphism of maps, if γ is not in a proper subfield of $\mathrm{GF}(2^\alpha)$, the orbits containing each of the (pairwise isomorphic) base maps
$\{\mathcal{B}_\gamma, \mathcal{B}_{\gamma^2}, \mathcal{B}_{\gamma^4}, \ldots \mathcal{B}_{\gamma^{2\alpha-1}}\}$, may be considered equivalent. This makes sense of our previous workings on the example where we considered base maps modulo field automorphisms.

What happens when the element γ is in a proper subfield is worth investigating in its own right, and we address this case further in section 3.5.5 - the title being a clue as to why this situation is interesting! The following lemma also gives a hint as to how such cases are important.

Lemma 3.67. *Let $\mathcal{M}(\omega_k, \omega_\ell, \omega_m)$ and $\mathcal{M}'(\omega'_k, \omega'_\ell, \omega'_m)$ be two maps each with automorphism group $G = SL(2, 2^\alpha)$, distinct trace triples, and yet the same triple trace ratio $1 : \delta : 1+\delta$.*

The maps $\mathcal{M}$ and $\mathcal{M}'$ are isomorphic to each other if and only if δ is in a proper subfield $GF(2^c) < GF(2^\alpha)$ and $\omega_k^{2^c} = \omega'_k$.

Proof. Let the two maps, $\mathcal{M}$ and $\mathcal{M}'$ be as described in the lemma, with distinct trace triples, but the same triple trace ratio $1 : \delta : 1+\delta$. Given $\mathcal{M}(\omega_k,\, \omega_k\delta,\, \omega_k(1+\delta))$ where $\mathcal{M}(G; R, S)$, then $\mathcal{M}'(G; R', S')$ must be such that $\mathcal{M}'(\omega'_k,\, \omega'_k\delta,\, \omega'_k(1+\delta))$ and $\omega_k \neq \omega'_k$. The two maps are isomorphic if and only if there is an automorphism of the group G which sends $R \to R'$ and $S \to S'$.

The only automorphisms of the group $G = \mathrm{SL}(2, 2^\alpha)$ which alter the traces are the field automorphisms. So, the two maps are isomorphic if and only if there is a c such that $1 \leq c < \alpha$ and $\omega'_i = (\omega_i)^{2^c}$ for all $i \in \{k, \ell, m\}$. Equivalently $\omega'_k = (\omega_k)^{2^c}$ and also, for the same value of c such that $1 \leq c < \alpha$ we must have $\omega'_k\delta = (\omega_k\delta)^{2^c}$, that is $\delta = \delta^{2^c}$, which happens if and only if $\delta \in \mathrm{GF}(2^c) < \mathrm{GF}(2^\alpha)$. □

This lemma has implications for the number of isomorphism classes of map which occur within a set of trace triples which share the same triple trace ratio.

In contrast, when the triple trace ratios are different, maps are isomorphic according to the following lemma.

Lemma 3.68. *Let $\mathcal{M}(\omega_k, \omega_\ell, \omega_m)$ and $\mathcal{M}'(\omega'_k, \omega'_\ell, \omega'_m)$ be two maps each with automorphism group $G = SL(2, 2^\alpha)$ and yet distinct triple trace ratios $1 : \gamma : 1+\gamma$ and $1 : \gamma' : 1+\gamma'$ respectively.*

Then $\mathcal{M}$ and $\mathcal{M}'$ are isomorphic maps if and only if there exists c such that $1 \leq c < \alpha$ and $\gamma^{2^c} = \gamma'$ and $\omega_k^{2^c} = \omega'_k$.

Proof. Suppose the two maps $\mathcal{M}$ and $\mathcal{M}'$ are isomorphic. This is the case if and only if there is a non-trivial automorphism of the field sending ω_i to ω'_i for all $i \in \{k, \ell, m\}$. That is, for some fixed $1 \leq c < \alpha$ we have $\omega_k^{2^c} = \omega'_k$ and also $\omega_\ell^{2^c} = \omega'_\ell$ which is equivalent to $\omega_k^{2^c}\gamma^{2^c} = \omega_k^{2^c}\gamma'$, that is $\gamma^{2^c} = \gamma'$.

Conversely, suppose $\gamma^{2^c} = \gamma'$ and $\omega_k^{2^c} = \omega'_k$. Then it is clear that the field automorphism $x \to x^{2^c}$ induces the required map isomorphism from $\mathcal{M}$ to $\mathcal{M}'$. □

The above lemmas, in conjunction with Proposition 3.44 yield the following theorem with regard to the longest possible orbit of isomorphism classes of maps.

Theorem 3.69. *Let $G \cong SL(2, q)$ where $q = 2^\alpha$ and $\alpha \geq 3$. Under the action of $\boldsymbol{Op}_{q^2-1}(\Omega_G)$, there is an orbit consisting of $6(q-1)$ distinct isomorphism classes of fully regular map if and only if $\alpha \geq 5$.*

Proof. Since $\alpha \geq 3$, the proof of Proposition 3.44 highlights an example of an orbit of $6(q-1)$ trace triples. The trace triples are partitioned into six sets, each of size

$(q-1)$, such that the maps within each set share the same (ordered) triple trace ratio. One of the ratios is $1 : \beta : 1+\beta$ for a primitive element β of the field $\mathrm{GF}(2^\alpha)$. By definition β does not belong to a proper subfield, and so, by Lemma 3.67 these $q-1$ distinct trace triples yield $q-1$ non-isomorphic maps.

Applying the dual and Petrie operators then permutes the three elements in each of these trace triples. Since $\alpha \geq 3$, and by the reasoning in Lemma 3.40, this yields $6(q-1)$ distinct trace triples. These trace triples are thus arranged into six distinct sets of $q-1$ maps, while each set contains $q-1$ mutually non-isomorphic maps. Each set has a naturally associated base map. In particular, the trace triples of the maps within the set with base map $\mathcal{B}_\beta(1, \beta, 1+\beta)$ have triple trace ratio $1 : \beta : 1+\beta$. The elements within these trace triples are permuted by the action of $\mathbf{D}$ and $\mathbf{P}$ giving six distinct triple trace ratios. These distinct ratios may then be standardised such that the first entry is one, and then the associated base map becomes obvious.

The question remains as to whether there are further isomorphic pairings of maps between the distinct sets of $q-1$ maps. By Lemma 3.68, this can only happen if the different triple trace ratios are related by a field automorphism. This is equivalent to the corresponding base maps being isomorphic, and indeed there may be isomorphisms between the base maps. For example it may be shown that $\mathrm{SL}(2,8)$ only has two isomorphism classes of base map, while $\mathrm{SL}(2,16)$ has just three. It is also known, see section 3.6, that $\mathrm{SL}(2,4)$ only has three isomorphism classes of map. This explains the absence of orbits consisting of $6(q-1)$ isomorphisms of maps in the cases for $\mathrm{SL}(2,4)$, $\mathrm{SL}(2,8)$ and $\mathrm{SL}(2,16)$.

To be explicit, Lemma 3.68 tells us that a pair of maps from the six distinct sets are isomorphic only if $\beta^{2^c} \in \{\beta^{-1}, 1+\beta, \beta(1+\beta)^{-1}, 1+\beta^{-1}, (1+\beta)^{-1}\}$ for some c such that $1 \leq c < \alpha$. The aim from here is to convince ourselves that, in all the other cases, that is when $\alpha \geq 5$, there is a primitive element β which does not satisfy this condition. The existence of such a β would mean the orbit based on $\mathcal{B}_\beta$ includes precisely $6(q-1)$ distinct isomorphism classes of map.

We now investigate the implications of each condition being satisfied. Let γ be a primitive element of the field $\mathrm{GF}(q) = \mathrm{GF}(2^\alpha)$.

First notice that $\gamma^{2^c} = \gamma^{-1}$ if and only if $\gamma^{2^c+1} = 1$, that is $2^c + 1 \equiv 0 \pmod{2^\alpha - 1}$. We have assumed $\alpha \geq 3$, so this is never the case. For completeness, we note that $\gamma^{2^c} = \gamma^{-1}$ and $\alpha \geq 2$ with γ being a primitive element implies that $c = 1$ and $\alpha = 2$. Indeed both primitive elements in the field $\mathrm{GF}(4)$ are in the same Frorbit and satisfy the equation $\mathrm{X}^2 = \mathrm{X}^{-1}$. We may therefore conclude that there is only one isomorphism class of base map on $\mathrm{SL}(2,4)$, and so certainly no orbit containing $6(q-1)$ different isomorphism classes of map.

Next suppose $\gamma^{2^c} = 1+\gamma$ for some c such that $1 \leq c < \alpha$. Raising everything to the power $2^{\alpha-c}$ yields $\gamma = 1+\gamma^{2^{\alpha-c}}$ which implies $\gamma^{2^c} = \gamma^{2^{\alpha-c}}$. Now raising this to the power 2^c gives $\gamma^{2^{2c}} = \gamma$. Now, γ is not in a proper subfield of $\mathrm{GF}(2^\alpha)$ and hence $2c = \alpha$, while $\mathrm{X} = \gamma$ is a solution to the equation $\mathrm{X}^{2^c} + \mathrm{X} + 1 = 0$. When α is even there are up to $2^c = \sqrt{q}$ elements γ such that $\gamma^{2^c} = 1+\gamma$, and the element β must avoid these in order that the orbit of $\mathcal{B}_\beta$ might yield $6(q-1)$ different isomorphism classes of map.

Similarly, let us suppose that $\gamma^{2^c} = \gamma(1+\gamma)^{-1}$ for some c such that $1 \leq c < \alpha$. Then $1+\gamma = \gamma^{1-2^c}$ and also $\gamma^{2^c+1} + \gamma^{2^c} + \gamma = 0$. Now $\gamma \neq 0$ so $\gamma^{2^c} + \gamma^{2^c-1} + 1 = 0$, which when raised to the power $2^{\alpha-c}$ yields $\gamma + \gamma^{1-2^{-c}} + 1 = 0$. This may be rearranged to be $1+\gamma = \gamma^{1-2^{-c}}$. Therefore $\gamma^{2^c} = \gamma^{2^{-c}}$ and so $\gamma^{2^{2c}} = \gamma$. Again, since γ generates the whole field $\mathrm{GF}(2^\alpha)$, the only way this may happen is if $\alpha = 2c$. Now there are up to $2^c = \sqrt{q}$ such elements of the field γ, which are roots of $\mathrm{X}^{2^c} + \mathrm{X}^{2^c-1} + 1 = 0$, and so satisfy $\gamma^{2^c} = \gamma(1+\gamma)^{-1}$. Notice that the polynomial $\mathrm{X}^{2^c} + \mathrm{X}^{2^c-1} + 1$ is the reciprocal of $\mathrm{X}^{2^c} + \mathrm{X} + 1$ which appears in the above paragraph, and so its roots are the inverses of those from the previous paragraph. The required element β must also avoid these values in order that the orbit based on $\mathcal{B}_\beta$ might yield $6(q-1)$ different isomorphism classes of map.

Now consider when $\gamma^{2^c} = 1+\gamma^{-1}$ for some c such that $1 \leq c < \alpha$. This is equivalent to $\mathcal{B}_\gamma \cong \mathcal{B}_\gamma\mathbf{DP}$. Raising this to the power 2^c yields $\gamma^{2^{2c}} = 1+\gamma^{-2^c}$ and rearranging gives $\gamma^{2^{2c}}\gamma^{2^c} = \gamma^{2^c} + 1$ which is equal to γ^{-1}. Hence $\gamma^{2^{2c}+2^c+1} = 1$, and since γ is a primitive element, $(2^\alpha - 1)$ divides $2^{2c} + 2^c + 1 \equiv 0$. This is severely limiting, and leaves us with only two options, $c \in \{1,2\}$ while $\alpha = 3$. Thus we may conclude that the finite field $\mathrm{GF}(2^\alpha)$ in which $\gamma^{2^c} = 1+\gamma^{-1}$ for some primitive element γ is $\mathrm{GF}(8)$.

Finally consider when $\gamma^{2^c} = (1+\gamma)^{-1}$ for some c such that $1 \leq c < \alpha$. Notice that this condition is equivalent to $\mathcal{B}_\gamma \cong \mathcal{B}_\gamma\mathbf{PD}$, which is the case if and only if $\mathcal{B}_\gamma \cong \mathcal{B}_\gamma\mathbf{DP}$. This is precisely the condition which we have just addressed, and, for primitive element γ, it is the case only when $\alpha = 3$.

If γ is a primitive element which does not satisfy any of the above equations, then the orbit of $\mathcal{B}_\gamma$ under the action of $\mathbf{Op}_{q^2-1}(\Omega_G)$ thus contains $6(q-1)$ non isomorphic maps. Henceforth we may consider the cases where $\alpha \geq 4$. Then the above workings mean we are only concerned with finding a primitive element β such that it is not the case that $\beta^{2^c} \in \{1+\beta, \beta(1+\beta)^{-1}\}$ for any $1 \leq c < \alpha$. The last thing to do is to convince ourselves that there are more primitive elements than elements which satisfy the above condition. At worst, there is a set consisting of up to $2 \times \sqrt{q}$ primitive field elements, in which β must not be contained.

First let us suppose $\alpha \geq 5$ is odd. Then by the above calculations, any primitive element may be chosen to be β, and the orbit associated with base map $\mathcal{B}_\beta$ has precisely $6(q-1)$ isomorphism classes of map.

Finally suppose $\alpha \geq 4$ is even. Then $\alpha = 2c$ and by the above workings there is a set of at most $2 \times 2^c = 2\sqrt{q}$ elements which β must avoid. There are $\varphi(q-1)$ primitive elements in the field, where φ denotes the Euler function. The smallest example in question is when $q = 16$: then $2\sqrt{16} = 8$ and $\varphi(15) = 8$ so this is an example where we cannot guarantee that there is a primitive element β which ensures the condition that $\beta^{2^c} \notin \{1+\beta, \beta(1+\beta)^{-1}\}$ for any $1 \leq c < \alpha$. This explains the aforementioned absence of an orbit of length $6(q-1)$ on $\mathrm{SL}(2, 16)$. The next example is $q = 64$: then $2\sqrt{64} = 16$ whereas $\varphi(63) = 36$ which gives us an abundance of choice for the primitive element β which, from the base map $\mathcal{B}_\beta$, yields an orbit of $6(q-1) = 6 \times 63$ non-isomorphic maps. Meanwhile for $n > 42$, it is known by folklore, building on the work of Hardy and Wright [30], that $\varphi(n) > n^{2/3}$. Conveniently, it may be shown that $(q-1)^{2/3} > 2\sqrt{q}$ when $\alpha \geq 8$. So, for $\alpha \geq 8$ we have $\varphi(q-1) > (q-1)^{2/3} > 2\sqrt{q}$, which is all we need.

Drawing this all together we may conclude that $\mathbf{Op}_{q^2-1}(\Omega_G)$ yields an orbit consisting of $6(q-1)$ distinct isomorphism classes of map on $G = \mathrm{SL}(2, 2^\alpha)$ if and only if $\alpha \geq 5$. $\square$

The orbit whose existence is proved in the above theorem may be used to build a super-symmetric map, as described in the next claim.

Corollary 3.70. *Let $G \cong SL(2, q)$ where $q = 2^\alpha$ and $\alpha \geq 5$. There is a super-symmetric map with valency $q^2 - 1$ and automorphism group $G^{6(q-1)}$.*

Proof. The maps from the orbit of $6(q-1)$ non-isomorphic maps from Theorem 3.69 may be combined as a parallel product of maps. The valency is the lowest common multiple of the valencies of the component maps. Thus the valency is $k(G) = q^2 - 1$, since for each possible first trace element ω_k, there is a component map within the orbit. The maps all have the same underlying finite non-Abelian simple group G, and so we may call upon Lemma 3.21 to deduce the isomorphism class of automorphism group of the resulting map. $\square$

3.5.5 Shorter orbits, of length less than $6(q-1)$

Let $\delta \in \mathrm{GF}(2^d) < \mathrm{GF}(2^\alpha)$ for some $2 \leq d < \alpha$ such that $d \mid \alpha$. Then $\delta^{2^d} = \delta$, and thus there are d distinct trace triples in the set $\{(1,\delta,1+\delta),(1,\delta^2,1+\delta^2),(1,\delta^4,1+\delta^4),(1,\delta^8,1+\delta^8)\ldots,(1,\delta^{2^{d-1}},1+\delta^{2^{d-1}})\}$ and the corresponding maps are all within the same isomorphism class as that with trace triple $(1,\delta,1+\delta)$. In the context of $\Omega_{\mathrm{SL}(2,2^\alpha)}$, these maps are considered to be degenerate because their automorphism group is a proper subgroup of $\mathrm{SL}(2,2^\alpha)$.

Furthermore, when δ is in $\mathrm{GF}(2^d)$, a proper subfield of $\mathrm{GF}(q) = \mathrm{GF}(2^\alpha)$, it is clear that all elements in the trace triple $(1,\delta,1+\delta)$ are members of the same subfield. The result is the existence of orbits of $\mathbf{Op}_{q^2-1}(\Omega_{\mathrm{SL}(2,q)})$ containing fewer maps than the maximal number, which is $6(q-1)$. This can be seen by realising that the base map $\mathcal{B}_\delta$ itself has automorphism group $\mathrm{SL}(2,2^d)$, not the whole group $G = \mathrm{SL}(2,2^\alpha)$, and so cannot be in an orbit of the set in which we are interested, that is $\Omega_{\mathrm{SL}(2,q)}$. With respect to the group $\mathrm{SL}(2,2^\alpha)$ the map $\mathcal{B}_\delta$ is degenerate.

Moreover there will likely be other maps with the same triple trace ratio as $\mathcal{B}_\delta$ whose trace elements are all contained in a proper subgroup of $\mathrm{GF}(2^\alpha)$, be it $\mathrm{GF}(2^d)$ or some intermediate field K such that $\mathrm{GF}(2^d) < K < \mathrm{GF}(2^\alpha)$. In every such case, it is clear that the associated map will not have $\mathrm{SL}(2,2^\alpha)$ as its automorphism group, but rather $\mathrm{SL}(2,2^d)$ or $\mathrm{SL}(2,K)$.

The lattice of subfields for $\mathrm{GF}(2^\alpha)$ may then be used to calculate the number of trace triples with the same (ordered) triple trace ratio as $\mathcal{B}_\delta$, and yet full automorphism group $\mathrm{SL}(2,2^\alpha)$. This is done by considering which (and so how many) elements in $\mathrm{GF}(2^\alpha)$ will result in the map 'collapsing' into a smaller group $\mathrm{SL}(2,K)$ for some subfield $K < \mathrm{GF}(2^\alpha)$.

This is best demonstrated with an example, and once again the focus is on a set of trace triples sharing the same ratio, without considering potential isomorphisms between the corresponding maps.

Example 3.71. Let $\alpha = 30$ and let δ be a primitive element of $\mathrm{GF}(2^5)$. Now, there are $2^{30}-1$ trace triples with triple trace ratio $1:\delta:1+\delta$, and given a specific primitive element β, each of these trace triples has the form $(\beta^a,\beta^a\delta,\beta^a(1+\delta))$ for some $1 \leq a < q-1$. However, some of these trace triples are for maps which are degenerate with respect to the group $\mathrm{SL}(2,2^{30})$. How many of the trace triples with triple trace ratio $1:\delta:1+\delta$ are non-degenerate?

The maximal proper subfields of $\mathrm{GF}(2^{30})$ are $\mathrm{GF}(2^{15})$, $\mathrm{GF}(2^{10})$ and $\mathrm{GF}(2^6)$. The last in the list has trivial intersection with $\mathrm{GF}(2^5)$, and when $\beta^a \in \mathrm{GF}(2^6)$, the trace triple $(\beta^a, \beta^a\delta, \beta^a(1+\delta))$ yields a map on $\mathrm{SL}(2, 2^{30})$ which does not collapse into a smaller group. In contrast, $\mathrm{GF}(2^5) \subseteq \mathrm{GF}(2^{15})$, so there is no $\beta^a \in \mathrm{GF}(2^{15})$, such that the trace triple $(\beta^a, \beta^a\delta, \beta^a(1+\delta))$ yields a map on $\mathrm{SL}(2, 2^{30})$ since they will all be on $\mathrm{SL}(2, 2^{15})$ or its subgroup $\mathrm{SL}(2, 2^5)$. A similar argument applies to $\beta^a \in \mathrm{GF}(2^{10})$. Then we must remember that the intersection $\mathrm{GF}(2^{15}) \cap \mathrm{GF}(2^{10}) = \mathrm{GF}(2^5)$, and these must not be double-(dis)counted.

In total there are $2^{30} - 2^{15} - 2^{10} + 2^5 = 1073708064$ many elements in the field $\mathrm{GF}(2^{30})$ which are not in some maximal proper subfield containing δ. This is clearly less than $2^{30} - 1$ and is the same as the number of trace triples for maps on $\mathrm{SL}(2, 2^{30})$ with having triple trace ratio $1 : \delta : 1 + \delta$. •

Remark 3.72. We reiterate that the focus in the above example has been on the set of trace triples which share the same triple trace ratio as $\mathcal{B}_\delta$ but contain elements not all of which are in the same maximal proper subfield of $\mathrm{GF}(2^\alpha)$, rather than the number of distinct isomorphism classes of map. Neither has any claim has been made that this set actually falls within the same orbit under the action of $\mathbf{Op}_{q^2-1}(\mathrm{SL}(2, 2^\alpha)$, although computational evidence does show that this is the case for $\alpha \leq 13$. This is subject to further research. $\star$

The above example does not consider the number of distinct isomorphism classes of map within the set of trace triples, and nor is the action of $\langle\, \mathbf{D}, \mathbf{P}\,\rangle$ taken into account. In contrast, in the next example, we do identify maps which are mutually isomorphic, as well as addressing the possibility of permutations of the trace triple elements under the action of $\langle\, \mathbf{D}, \mathbf{P}\,\rangle$. When the base triple is within a proper subfield, the upper limit for the number of pairwise non-isomorphic maps in an orbit is less than the number of distinct trace triples due to the map isomorphisms induced by the field automorphisms which fix the trace ratio. The following example demonstrates this.

Example 3.73 (The special case with triple trace ratio $1 : \omega : \omega^2$). Let $q = 2^\alpha$. When α is even, the element $\omega \in \mathrm{GF}(q)$ where $\omega^3 = 1$, and hence so are maps with triple trace ratio $(1, \omega, \omega^2)$. Up to isomorphism, the subfield $\mathrm{GF}(4)$ contains only one base map, $\mathcal{B}_\omega(1, \omega, \omega^2)$, and so the set $\mathfrak{B}_\omega$ is associated with precisely one triple trace ratio. This means there is a maximum of $q - 1$ different trace triples for pairwise non-isomorphic maps. In the same way, $q - 1$ is the maximum size of an orbit of non-isomorphic maps under $\mathbf{Op}_{q^2-1}(\Omega_{\mathrm{SL}(2,q)})$. Indeed when $\alpha = 2$, there are three distinct trace triples, corresponding to three isomorphism classes of map, all within the same orbit.

Meanwhile if $\alpha = 2d$, then within the set of non-degenerate maps with triple trace ratio $1 : \omega : \omega^2$ there are d-tuples of mutually isomorphic maps on $\mathrm{SL}(2, 2^\alpha)$ as follows. Let $f \in \mathrm{GF}(2^{2d})$, and remember that the maps with trace triples $(f, f\omega, f\omega^2)$ and $(f^{2^i}, f^{2^i}\omega^{2^i}, f^{2^i}(\omega^2)^{2^i})$ are isomorphic to one another. In particular, if f is such that f is not in a proper subfield of $\mathrm{GF}(2^{2d})$, then the triples $(f, f\omega, f\omega^2)$, $(f^{2^2}, f^{2^2}\omega, f^{2^2}\omega^2)$, $(f^{2^4}, f^{2^4}\omega, f^{2^4}\omega^2), \ldots$ and $(f^{2^{2d}}, f^{2^{2d}}\omega, f^{2^{2d}}\omega^2)$ form a set of d distinct trace triples which all have the same triple trace ratio $1 : \omega : \omega^2$, *and* correspond to the same map isomorphism class. Furthermore if f is such that f is in K, where K is a proper subfield of $\mathrm{GF}(2^\alpha)$ which contains $\mathrm{GF}(4)$, then the map with trace triple $(f, f\omega, f\omega^2)$ is degenerate since it has as its automorphism group a proper subgroup of $SL(2, 2^\alpha)$. The only remaining alternative is if f in $\mathrm{GF}(2^d)$ and d is odd. Then, just as before, the triples $(f, f\omega, f\omega^2)$, $(f^{2^2}, f^{2^2}\omega, f^{2^2}\omega^2)$, $(f^{2^4}, f^{2^4}\omega, f^{2^4}\omega^2), \ldots$ and $(f^{2^{2d}}, f^{2^{2d}}\omega, f^{2^{2d}}\omega^2)$ form a set of d distinct trace triples which all have the same triple trace ratio $1 : \omega : \omega^2$, *and* correspond to the same map isomorphism class.

Hence, when $\alpha = 2d$, within the orbit of $\mathcal{M}(f, f\omega, f\omega^2)$ under the action of $\mathbf{Op}_{q^2-1}(\Omega_{\mathrm{SL}(2,q)}) = \langle\, \mathbf{D}, \mathbf{P}, \mathbf{H}_j \mid (j, q^2 - 1) = 1 \,\rangle$, the maximum number of distinct isomorphism classes of map on $\mathrm{SL}(2, 2^{2d})$ is

$$\frac{1}{d}\left(2^{2d} - \left|\bigcup_{i|d, 1 \le i < d} \mathrm{GF}(2^{2i})\right|\right).$$

•

The above example is an extreme since $\gamma = \omega$ is the only case where, up to isomorphism, there is only one base map within the set $\mathfrak{B}_\gamma$. The result is a remarkably small set of isomorphism classes of map. This set may be an orbit, or possibly it is a union of orbits, but it is certainly closed under the action of $\mathbf{Op}_{q^2-1}(\Omega_{\mathrm{SL}(2,q})$ for $q = 2^\alpha$ where α is even.

When three divides α there is similar potential for a remarkably small orbit of mutually non-isomorphic maps by using those which have the triple trace ratio $1 : \zeta : 1 + \zeta$ where ζ is a primitive element of $\mathrm{GF}(8)$. This is because there are only two isomorphism classes of map within the set $\mathfrak{B}_\zeta$. Meanwhile, depending on the value of α, some maps with triple trace ratio $1 : \zeta : 1 + \zeta$ will generate proper subgroups of $\mathrm{SL}(2, 2^\alpha)$. By reasoning analogous to that found in Example 3.73, the number of maps which are mutually non-isomorphic and within the same orbit is therefore considerably smaller than the maximum $6(q - 1)$.

The above are examples of upper bounds on the lengths of orbits being limited by the triple trace ratio being contained within a subfield of $\mathrm{GF}(2^\alpha)$. There is also potential for shorter orbits where the base map $\mathcal{B}_\gamma$ is such that γ generates the whole field $\mathrm{GF}(2^\alpha)$. By Lemma 3.67, this eliminates the possibility of isomorphic pairs of maps sharing the same ratio, thereby yielding $a(q-1)$ different isomorphisms of map, where $a \in \{1, 2, 3, 6\}$. When $a = 6$ the result is the longest orbit, whose existence has already been proved in Theorem 3.69, while $a = 1$ is the special case in GF(4). Meanwhile $a \in \{3, 2\}$ happens when $\mathcal{B}_\gamma \cong \mathcal{B}_\epsilon$ for $\epsilon \in \{1+\gamma, \gamma(1+\gamma)^{-1}, 1+\gamma^{-1}, (1+\gamma)^{-1}\}$. As in the proof of Theorem 3.69, these conditions correspond respectively to $\mathcal{B}_\gamma$ being isomorphic to its Petrie map, its opposite map or in a triple of isomorphic maps under triality. In each case, γ satisfies one of the equations as follows:

1. $\alpha = 2c$ and $\gamma^{2^c} = 1 + \gamma$ for an orbit of at most $3(q-1)$ mutually non-isomorphic maps associated with a self-Petrie base map.
2. $\alpha = 2c$ and $\gamma^{2^c} = \gamma(1+\gamma)^{-1}$ for an orbit of at most $3(q-1)$ mutually non-isomorphic maps.
3. $3 \mid \alpha$ and $\gamma^{2^c} = 1 + \gamma^{-1}$ for some c such that $1 \le c < \alpha$ for an orbit of at most $2(q-1)$ mutually non-isomorphic maps including a self-trial map, in which case we would also have $\gamma^{2^{2c}} = (1+\gamma)^{-1}$.

It is an exciting prospect to try to prove the existence of such a γ for each case, and the computational evidence is encouraging in this regard. This would still leave to be proved that each set is indeed an orbit, and not a union of more than one orbit. With regard to this thesis, these firm hopes will be left in the form of conjecture - see section 3.5.6. We note that regular maps with $G = \mathrm{SL}(2, q)$ and $q = 2^\alpha$ have been enumerated using the Möbius function by Downs and Jones in [23].

Remark 3.74. Let $\alpha = 3d$ and "useful generator" δ from [41] be such that the map with trace triple $(\delta, \delta^{2^d}, \delta^{2^{2d}})$ demonstrates self-triality without self-duality. Then the triple trace ratio is $1 : \delta^{2^d-1} : \delta^{2^{2d}-1}$ and it should be no surprise that letting $\gamma = \delta^{2^d-1}$ satisfies the latter two equations with $c = d$ and $c = 2d$ respectively. ⋆

3.5.6 Conjectures

The work in this chapter is subject to further research. Currently, I believe the following conjectures to be true.

Conjecture 3.75. *Let $G \cong SL(2, 2^\alpha)$ be such that α is even. The set of mutually non-isomorphic maps consisting of those with triple trace ratio $1 : \omega : \omega^2$ form a single orbit under the action of $\boldsymbol{Op}_{k(G)}(\Omega_G)$.*

Frustratingly, this is still a conjecture, but I live in hope of converting it into a theorem! On a more optimistic day, and based on small computer experiments, the claim may be even more general, and extend to the following conjecture. I expect the proofs for these two conjectures, if indeed they turn out to be true, will be essentially the same.

Conjecture 3.76. *If $\mathcal{M}$ and $\mathcal{M}'$ are non-isomorphic fully regular maps with automorphism group $G \cong SL(2, 2^\alpha)$ and the same triple trace ratio, then $\mathcal{M}$ and $\mathcal{M}'$ are in the same orbit of $\boldsymbol{Op}_{k(G)}(\Omega_G)$*

This in turn, along with computational evidence from small examples, fosters the hope that the following may be true:

Conjecture 3.77. *Let $G \cong SL(2, 2^\alpha)$ where $\alpha \geq 7$. Orbits of pairwise non-isomorphic maps under the action of $\boldsymbol{Op}_{k(G)}(\Omega_G)$ occur as follows:*

- *At least one orbit of length $6(q-1)$ if and only if $\alpha \geq 5$;*
- *At least one orbit of length $3(q-1)$ if and only if $2 \mid \alpha$ and $\alpha \geq 4$;*
- *At least one orbit of length $2(q-1)$ if and only if $3 \mid \alpha$ and $\alpha \neq 6$;*
- *Further orbits, each of length less than $6(q-1)$, and each sharing the same set of triple trace ratios as an orbit of maps with automorphism group $G \cong SL(2, K)$ for proper subfield $K = GF(2^d) < GF(2^\alpha)$ where $d \geq 4$.*

A consequence of these conjectures would be the following:

Conjecture 3.78. *Each orbit under $\boldsymbol{Op}_{q^2-1}(\Omega_{SL(2,q)})$, where q is even, is a set of maps such that the least common multiple of the valencies is $(q-1)(q+1)$.*

If true, this is slightly disappointing, since the motivation for this investigation was an attempt to find an orbit of maps under $\mathbf{Op}_{2^{2\alpha}-1}(\Omega_{\mathrm{SL}(2,q)})$ such that the parallel product of those maps would have valency less than $(q-1)(q+1)$.

However, we may console ourselves with the knowledge that we can use the shortest orbit (even if the currently conjectured shortest orbit turns out itself to be a union of orbits) to build a super-symmetric map which is significantly smaller than $\mathcal{N}_G$, the so-called natural super-symmetric map for the given group $G = \mathrm{SL}(2, q)$.

3.6 A detailed example: A super-symmetric map on $\mathbf{A}_5 \times \mathbf{A}_5 \times \mathbf{A}_5$

Thinking in terms of traces lulled me into a false sense of security: this smallest example was easy! I knew everything there was to know......

“ There are only three non-zero elements in the field GF(4), and so there are only three non-zero traces. The trace values pleasingly happen to be linked by the equation $1 + \beta + \beta^2 = 0$, where β is a particular primitive element of the field. This leaves us in no doubt as to the form of the six possible trace triples for fully regular maps on $\mathrm{SL}(2, 4) \cong \mathrm{A}_5$:

$(1, \beta, \beta^2)$ $(\beta^2, 1, \beta)$ $(\beta, \beta^2, 1)$ and their images under the transformation $x \to x^2$, respectively $(1, \beta^2, \beta)$ $(\beta, 1, \beta^2)$ $(\beta^2, \beta, 1)$.

How the operators **D**, **P**, **S**, and **R** act on these triples was also easy to see. Meanwhile the Frobenius automorphism induces a map isomorphism so it is clear that there are just three isomorphism classes of maps on this group.

That's that, then. Done! There is just one orbit and it consists of six trace triples, representing the three different isomorphism classes of map, and the action of the operators on the set of trace triples is clear. What more is there to know? ”

This naivety is what drew me on to study more about the orbits of trace triples, which was an interesting diversion, the results of which have been presented in the previous section. It has some bearing on building super-symmetric maps as parallel products, and the orbit structures are still subject to further research.

So, what had I missed - what more was there more to know about this smallest example?

The reader may have the advantage of having seen the more general results from earlier in this chapter, in particular Theorem 3.33, which implies a deeper structure than is presented by this small example of my own wrong thinking. It is, of course, much more involved than looking only at what happens to the trace triples, since each trace triple represents many different (yet isomorphic) maps. This is why I then looked in detail at the smallest example. This in turn yielded the generalisations presented earlier, and so this chapter is not written in chronological order, but (hopefully!) in a logical order where the general truths are brought to the fore, and more particular results follow.

Now, if we hope to understand the group of operators $\mathbf{Op}_{k(G)}(\Omega_G)$, we need to know, not just when a composition of operators yields a map isomorphism, but when the composition leaves all fully regular maps on G invariant, that is, when it is equivalent to the identity operator $\mathbf{Id} \in \mathbf{Op}_{k(G)}(\Omega_G)$.

In order to better understand how the operators interact, and therefore what the structure of the external symmetry group was, I decided to consider the action of $\mathbf{Op}_{15}(\Omega_{\mathrm{SL}(2,4)}) = \langle \mathbf{D}, \mathbf{P}, \mathbf{R}, \mathbf{S} \rangle$ on the space consisting of all maps on $\mathrm{SL}(2,4) \cong \mathrm{A}_5$. It turns out that this group has order 2304, and I certainly didn't know everything about it!

We draw this chapter, and so this part of the thesis to a close with the detailed investigation for this smallest example.

3.6.1 Permutations of maps on $\mathbf{A}_5$

We consider the fully regular maps whose automorphism group is isomorphic to $\mathrm{SL}(2,4) \cong \mathrm{A}_5$. There are 360 such maps, and we will consider the dual and rotational power operators as permutations acting on the set of all such maps.

It is well-known that there are only three isomorphism classes of fully regular maps whose automorphism group is isomorphic to A_5. The proof is omitted since the above reasoning in terms of trace triples makes this very clear. We use the wide hat notation to indication a map isomorphism class.

Lemma 3.79. *Let $\mathcal{M}$ be a fully regular map with automorphism group $G \cong A_5$. Then $\mathcal{M}$ falls into one of the following three isomorphism classes:*

1 $\widehat{\mathcal{M}_I}$ where $\mathcal{M}_I$ has extended type $(3,5,5)$

2 $\widehat{\mathcal{M}_{II}}$ where $\mathcal{M}_{II}$ has extended type $(5,3,5)$

3 $\widehat{\mathcal{M}_{III}}$ where $\mathcal{M}_{III}$ has extended type $(5,5,3)$

Hence in the case where $G \cong \mathrm{A}_5$, the isomorphism class of a given map is uniquely determined by its type, and vice versa. This simplifies the associated calculations, and makes the following claims easy to see.

Lemma 3.80. *The operators act in the following way.*

- ***D*** *The dual operator **D** fixes $\widehat{\mathcal{M}_{III}}$ (setwise) and interchanges maps isomorphic to $\mathcal{M}_I$ with maps in $\widehat{\mathcal{M}_{II}}$.*

- ***P*** *The Petrie operator **P** fixes $\widehat{\mathcal{M}_I}$ and interchanges maps isomorphic to $\mathcal{M}_{II}$ with maps isomorphic to $\mathcal{M}_{III}$.*

S *The rotational power* $\boldsymbol{S} = \boldsymbol{H}_{-1}$ *fixes the isomorphism class of the map.*

R *The rotational power* $\boldsymbol{R} = \boldsymbol{H}_2$ *fixes the isomorphism class* $\widehat{\mathcal{M}_I}$ *and interchanges maps isomorphic to* $\mathcal{M}_{II}$ *with maps isomorphic to* $\mathcal{M}_{III}$*.*

Proof. By the previous Lemma, where the automorphism group of a fully regular map is $G \cong \mathrm{A}_5$, the type of the map uniquely determines its isomorphism class. The dual operator **D** interchanges maps of type (k, ℓ, m) with maps of type (ℓ, k, m), thus proving item **D**. The Petrie operator **P** interchanges maps of type (k, ℓ, m) with maps of type (k, m, ℓ), proving item **P**. Meanwhile, by Lemma 3.6, the rotational power **S** is an exponent and so fixes the isomorphism class of the map.

It remains to address how the operator **R** behaves when applied to a map in each of the the isomorphism classes. Any rotational power fixes the valency of a map and so **R** necessarily fixes the isomorphism class of $\mathcal{M}_I$. Now suppose $\mathcal{M}(\mathrm{A}_5; x, y, z)$ has type $(5, 3, 5)$ and so xz has order three, and yz has order 5. Then $\mathcal{M}\mathbf{R} = \mathcal{M}'(\mathrm{A}_5; x', y', z')$ where $z' = zyz$, so the face rotation $x'z' = xzyz = zxzxyz = zxzyxz$ which is conjugate to zy. This shows that the face length of the map $\mathcal{M}'$ is five. Hence $(\mathcal{M}_{II})\mathbf{R}$ is isomorphic to $\mathcal{M}_{III}$. Conversely, similar reasoning will prove that $(\mathcal{M}_{III})\mathbf{R} \cong \mathcal{M}_{II}$. □

However, each map isomorphism class contains many maps, so we now consider the action of the operators in $\mathbf{Op}_{15}(\Omega_{\mathrm{A}_5})$ with respect to the individual maps in Ω_{A_5}.

We turn our attention to the concrete example of maps with automorphism group $G = \mathrm{A}_5 \cong \mathrm{SL}(2, 4)$, considered as the group of even permutations on the set $\{1, 2, 3, 4, 5\}$, with permutations acting on the right. Note that it is easy to see that A_5 has only one conjugacy class of the subgroup V_4, since they are all Sylow 2-subgroups. Since all the operators fix the edge stabiliser $\langle x, y\rangle$, in the investigation of orbit structure of $Op_{15}(\Omega_{\mathrm{A}_5})$, we will only focus on one such copy of V_4.

We first fix an arbitrary but specific copy of the Klein four group, $\mathrm{K}_1 \cong \mathrm{V}_4$, as our edge stabiliser $\langle x, y\rangle = \{I, (23)(45), (24)(35), (25)(34)\}$. By definition, all the generating operators $\mathbf{D}, \mathbf{P}, \mathbf{R}$, and $\mathbf{S}$ (and hence their compositions) fix the edge stabiliser $\langle x, y\rangle$ setwise. The remaining twelve involutions (those which are in A_5 but not in the subgroup K_1) all permute the element labelled 1. Moreover the group generated by one of these twelve involutions and $\mathrm{K}_1 \cong \mathrm{V}_4$ is the whole group A_5. Hence these twelve involutions form the list of possible elements for z such that $\mathcal{M}(\mathrm{A}_5; x, y, z)$ is a fully regular map and $\langle x, y, z\rangle$ generates the whole group $\langle x, y, z\rangle \cong \mathrm{A}_5$. For ease of reference we name these involutions as shown in Table 3.3.

K_1	$X = (24)(35)$	$Y = (25)(34)$	$XY = (23)(45)$	K_1
K_2	$C_2 = (13)(45)$	$B_2 = (14)(35)$	$A_2 = (15)(34)$	$\mathrm{K}_1^{(12)}$
K_3	$C_3 = (12)(45)$	$A_3 = (14)(25)$	$B_3 = (15)(24)$	$\mathrm{K}_1^{(13)}$
K_4	$B_4 = (12)(35)$	$A_4 = (13)(25)$	$C_4 = (15)(23)$	$\mathrm{K}_1^{(14)}$
K_5	$A_5 = (12)(34)$	$B_5 = (13)(24)$	$C_5 = (14)(23)$	$\mathrm{K}_1^{(15)}$

Table 3.3: Conjugate copies of K_1: involtuions in A_5

Note that the element denoted 'A_5' is an involution in the alternating group G which is denoted A_5, and likewise A_4, C_3, C_4, and C_5 are group elements within $G = \mathrm{A}_5$. This notation for group elements applies only locally, within section 3.6, and should not cause confusion since discussion of other alternating groups A_n or cyclic groups C_n does not feature here.

There is a certain logic behind the labelling, and that is as follows. The subscript i simply indicates that the element in question belongs to K_i, the copy of the Klein four group within the group A_5 which fixes i. Now let us suppose $x = X := (24)(35)$ and $y = Y := (25)(34)$, and consider the map $\mathcal{M} = (\mathrm{A}_5; x, y, z)$. Then if $z \in \{A_2, A_3, A_4, A_5\}$ then the map is of type $(3, 5, 5)$ while if $z \in \{B_2, B_3, B_4, B_5\}$ then the map is of type $(5, 3, 5)$ and if $z \in \{C_2, C_3, C_4, C_5\}$ then the map is of type $(5, 5, 3)$. We formalise and extend this in the following lemma.

Lemma 3.81. *Let $x = X := (24)(35)$ and $y = Y := (25)(34)$. Then the map $\mathcal{M}(A_5; x, y, z)$ is in the following isomorphism class:*

1. $\widehat{\mathcal{M}_I} \iff z \in \{A_2, A_3, A_4, A_5\}$
2. $\widehat{\mathcal{M}_{II}} \iff z \in \{B_2, B_3, B_4, B_5\}$
3. $\widehat{\mathcal{M}_{III}} \iff z \in \{C_2, C_3, C_4, C_5\}$

Proof. By Lemma 3.79, a fully regular map $\mathcal{M} = (\mathrm{A}_5; x, y, z)$ has valency three if and only if the order of yz is three. Similarly the map has triangular faces if and only if the order of xz is three, while the map has Petrie faces with boundary-walk length being three if and only if the order of xyz is three. Notice that A_2, A_3, A_4, A_5 are precisely the involutions which contain exactly one of the transpositions (34) or (25) which occur in $y = (25)(34)$. Hence $z \in \{A_2, A_3, A_4, A_5\}$ yields yz with order three. Conversely, if the involution $z \notin \mathrm{K}_1$ does not contain either of the transpositions which occur in $y = (25)(34)$, then the product yz has order five. By analogous reasoning we have $z \in \{B_2, B_3, B_4, B_5\}$ if and only if xz has order three, while xyz has order three if and only if $z \in \{C_2, C_3, C_4, C_5\}$. □

In this small example concerning maps whose automorphism group is A_5, the type of the map uniquely determines its isomorphism class. This is not the case in general, and leads us to make the following definition. An *isoset* for a particular map $\mathcal{M} = (G; x, y, z)$ is the set $\mathcal{Z}$ of involutions such that $z' \in \mathcal{Z}$ if and only if $\mathcal{M} = (G; x, y, z')$ is isomorphic to the map $\mathcal{M} = (G; x, y, z)$.

In this way the elements in the isoset for the map $\mathcal{M} = (G; x, y, z)$ correspond to the images of z under the action of automorphisms of the group A_5 which fix x and y element-wise.

The centraliser in A_5 of the subgroup $K_1 = \langle x, y \rangle \cong V_4$ is K_1 itself and so each isoset contains four elements. In particular, the isoset for the map $(A_5; X, Y, z)$ thus consists of the orbit of z under conjugation by K_1. Specifically $D_2^I = D_2$, $D_2^X = D_4$, $D_2^Y = D_5$ and $D_2^{XY} = D_3$, for each $D \in \{A, B, C\}$ in the above notation. This generalises to give the following lemma in which we condense the notation $\mathcal{M} = (G; x, y, z)$ to $\mathcal{M}(G; x, y, z)$ for clarity.

Lemma 3.82. *Let $G = SL(2, 2^\alpha)$ and $\mathcal{M}(G; x, y, z)$ be a regular map. Then $\mathcal{M}(G; x, y, z) \cong \mathcal{M}(G; x, y, z^y) \cong \mathcal{M}(G; x, y, z^{xy}) \cong \mathcal{M}(G; x, y, z^x)$ and $\{z, z^x, z^y, z^{xy}\}$ is the isoset for $\mathcal{M}(G; x, y, z)$.*

Proof. Trivially, conjugation of each of the generators x, y, and z by a single group element is necessarily an automorphism of the group, and hence an isomorphism of the map, while conjugation of x and y by any element in the Abelian group $\langle x, y \rangle \cong V_4$ fixes the element in question. Being conjugates of each other, it is thus clear that $\mathcal{M}(G; x, y, z) \cong \mathcal{M}(G; x, y, z^y) \cong \mathcal{M}(G; x, y, z^{xy}) \cong \mathcal{M}(G; x, y, z^x)$.

It remains to prove there are no other elements in the isoset. The centraliser of V_4 in $SL(2, 2^\alpha)$ is V_4, and so there are no further automorphisms of the group $G \cong SL(2, 2^\alpha)$ which will fix both x and y. $\square$

Remark 3.83. Given a fully regular map $\mathcal{M}(G; x, y, z)$, the operators **DSD**, **S**, and **PDSDP** yield respectively the maps $\mathcal{M}(G; x, y, z^y)$, $\mathcal{M}(G; x, y, z^{xy})$, and $\mathcal{M}(G; x, y, z^x)$ which share the same isoset as $\mathcal{M}(G; x, y, z)$. Hence **DSD**, **S**, and **PDSDP** are all map isomorphisms, and so elements of $Ext(\mathcal{M})$. However, bearing in mind Remark 3.13, we know that **PDSDP** = **SDSD** is an involution and so $\langle\, \mathbf{S}, \mathbf{DSD} \,\rangle$ forms a subgroup of $Ext(\mathcal{M})$ which is isomorphic to V_4. ⋆

3.6.2 The permutation diagram

Figure 3.3 shows the action of $\langle \mathbf{D}, \mathbf{P}, \mathbf{S}, \mathbf{R} \rangle$ on the set of maps $\{\mathcal{M}(\mathrm{A}_5; x, y, z)\}$ with $\langle x, y \rangle = \mathrm{K}_1$. I have used this extensively as a reference diagram as it allowed me to read off the order of elements in $\langle \mathbf{D}, \mathbf{P}, \mathbf{S}, \mathbf{R} \rangle$, first as elements in $\mathbf{Op}(\Omega_{\mathrm{A}_5})$, and hence also as elements of $Ext(\mathcal{N}_{\mathrm{A}_5})$ acting on a parallel product of maps $\mathcal{M}_I || \mathcal{M}_{II} || \mathcal{M}_{III}$. Also, the same information has been useful for inputting the group $\langle \mathbf{D}, \mathbf{P}, \mathbf{S}, \mathbf{R} \rangle$ as a permutation group on the 72 maps and using GAP [27] for further calculations and testing conjectures. The particulars for this are attached at the end of this Chapter, in section 3.7.

Remark 3.84. It is worth noting that the diagrams and the associated calculations refer to only one orbit of rooted fully regular maps in Ω_{A_5}, namely those with $\mathrm{K}_1 \cong \mathrm{V}_4$ as the marked edge stabilser. The other orbits share precisely the same structure. ⋆

The workings for building the upcoming diagram in Figure 3.3 are now presented, in the form of conjugation tables for involutions.

First remember that the twelve involutions outside the edge stabiliser K_1 are labelled as follows:

A	$A_2 = (15)(34)$	$A_3 = (14)(25)$	$A_4 = (13)(25)$	$A_5 = (12)(34)$
B	$B_2 = (14)(35)$	$B_3 = (15)(24)$	$B_4 = (12)(35)$	$B_5 = (13)(24)$
C	$C_2 = (13)(45)$	$C_3 = (12)(45)$	$C_4 = (15)(23)$	$C_5 = (14)(23)$

We now consider the Y-conjugates of these involutions. Notice the conjugation fixes (setwise) each isoset, which is what we should expect since conjugation by Y is, in line with Lemma 3.6, equivalent to enacting the operation $\mathbf{S}$ which is a map isomorphism. Remember that $Y = (25)(34)$ is an involution, and observe that the indices behave in a very natural way:

A^Y	$A_2^Y = (12)(34) = A_5$	$A_3^Y = (13)(25) = A_4$	$A_4^Y = A_3$	$A_5^Y = A_2$
B^Y	$B_2^Y = (13)(24) = B_5$	$B_3^Y = (12)(35) = B_4$	$B_4^Y = B_3$	$B_5^Y = B_2$
C^Y	$C_2^Y = (14)(23) = C_5$	$C_3^Y = (15)(23) = C_4$	$C_4^Y = C_3$	$C_5^Y = C_2$

The X-conjugates of the involutions are as follows. Remember that $X = (24)(35)$:

A^X	$A_2^X = (13)(25) = A_4$	$A_3^X = (12)(34) = A_5$	$A_4^X = A_2$	$A_5^X = A_3$
B^X	$B_2^X = (12)(35) = B_4$	$B_3^X = (13)(24) = B_5$	$B_4^X = B_2$	$B_5^X = B_3$
C^X	$C_2^X = (15)(23) = C_4$	$C_3^X = (14)(23) = C_5$	$C_4^X = C_2$	$C_5^X = C_3$

We may now easily check that the XY-conjugates are as shown in the following table. $XY = (23)(45)$

A^{XY}	$A_2^{XY} = A_3$	$A_3^{XY} = A_2$	$A_4^{XY} = A_5$	$A_5^{XY} = A_4$
B^{XY}	$B_2^{XY} = B_3$	$B_3^{XY} = B_2$	$B_4^{XY} = B_5$	$B_5^{XY} = B_4$
C^{XY}	$C_2^{XY} = C_3$	$C_3^{XY} = C_2$	$C_4^{XY} = C_5$	$C_5^{XY} = C_4$

The fact that in each of these tables the isosets are preserved is reminiscent of the more general truth that conjugation of a map by an element in $\langle x, y \rangle$ is a map isomorphism, as seen previously in Corollary 3.5. The above tables are thus useful for reading how $\mathbf{S}$ and its compositions with elements within $\langle\, \mathbf{D}, \mathbf{P} \,\rangle$ behave as permutations of maps in the orbit.

The only remaining generator of the permutation group is the rotational power $\mathbf{H}_2 = \mathbf{R}$ which maps z to y^z, and so we construct another set of tables for reference. When $y = Y = (25)(34)$ the table is as follows:

Y^A	$Y^{A_2} = (12)(34) = A_5$	$Y^{A_3} = (13)(25) = A_4$	$Y^{A_4} = (14)(25) = A_3$	$Y^{A_5} = (15)(34) = A_2$
Y^B	$Y^{B_2} = (15)(23) = C_4$	$Y^{B_3} = (14)(23) = C_5$	$Y^{B_4} = (13)(45) = C_2$	$Y^{B_5} = (12)(45) = C_3$
Y^C	$Y^{C_2} = (15)(24) = B_3$	$Y^{C_3} = (14)(35) = B_2$	$Y^{C_4} = (13)(24) = B_5$	$Y^{C_5} = (12)(35) = B_4$

Again, there is much structure to spot in the above table.

For the purposes of composing $\mathbf{R}$ with $\mathbf{DP}$ and $\mathbf{PD}$, it will be helpful to have the following two tables.

X^A	$X^{A_2} = (14)(23) = C_5$	$X^{A_3} = (15)(23) = C_4$	$X^{A_4} = (12)(45) = C_3$	$X^{A_5} = (13)(45) = C_2$
X^B	$X^{B_2} = (12)(35) = B_4$	$X^{B_3} = (13)(24) = B_5$	$X^{B_4} = (14)(35) = B_2$	$X^{B_5} = (15)(24) = B_3$
X^C	$X^{C_2} = (14)(25) = A_3$	$X^{C_3} = (15)(34) = A_2$	$X^{C_4} = (12)(34) = A_5$	$X^{C_5} = (13)(25) = A_4$

$(XY)^A$	$(XY)^{A_2} = B_5$	$(XY)^{A_3} = B_4$	$(XY)^{A_4} = B_3$	$(XY)^{A_5} = B_2$
$(XY)^B$	$(XY)^{B_2} = A_4$	$(XY)^{B_3} = A_5$	$(XY)^{B_4} = A_2$	$(XY)^{B_5} = A_3$
$(XY)^C$	$(XY)^{C_2} = C_3$	$(XY)^{C_3} = C_2$	$(XY)^{C_4} = C_5$	$(XY)^{C_5} = C_4$

Due to the choice of notation, there are many remarkable, but entirely explicable patterns in the above tables. When drawn altogether to get an image of the permutations of maps within the orbit of $\langle\, \mathbf{D}, \mathbf{P}, \mathbf{S}, \mathbf{R} \,\rangle$, Figure 3.3 emerges. Some information has been codified in order to simplify the diagram. The following paragraph describes how the diagram is constructed, and later there is another diagram, Figure 3.4 which omits $\mathbf{S}$, as I consider it to be easier to use.

Since the actions of $\mathbf{D}$ and $\mathbf{P}$ do not alter z, each $\langle\, \mathbf{D}, \mathbf{P} \,\rangle$ orbit is shown as a hexagon and is labelled with the name of the involution corresponding to z. Meanwhile the maps which have $x = X$ and $y = Y$ are marked by square nodes, and naturally there is one of these for each hexagon, that is for each $\langle\, \mathbf{D}, \mathbf{P} \,\rangle$ orbit. Realise that each pair of maps related to each other by the Petrie operator occur within the same blue ellipse, and it is to these ellipses which the hole operator arrows refer. Since hole

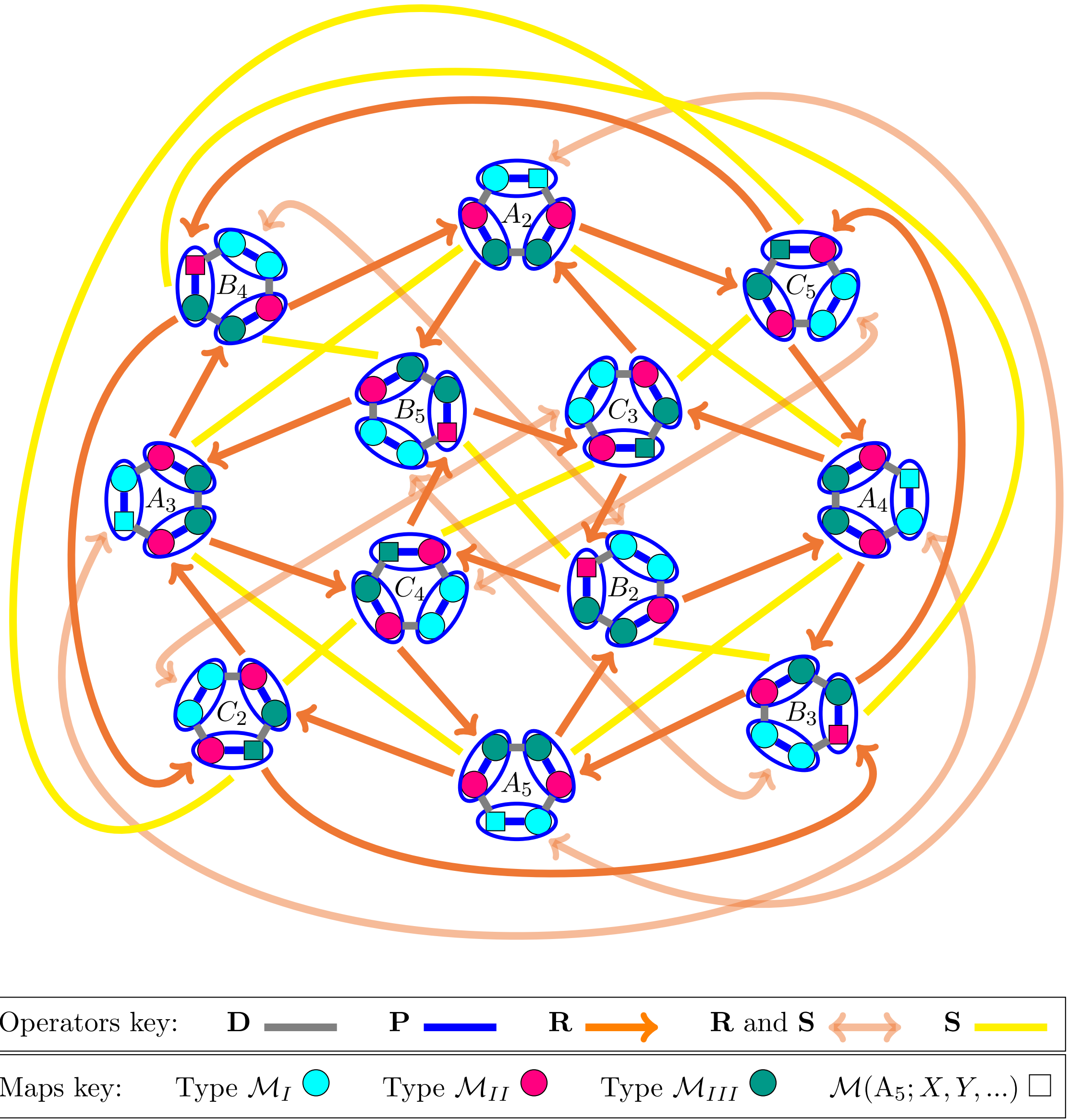

Figure 3.3: Permutations of maps in one orbit of $\langle\, \mathbf{D}, \mathbf{P}, \mathbf{S}, \mathbf{R} \,\rangle$ acting on Ω_{A_5}.

operators do not interfere with the first two generating involutions x and y, in order to establish which map is sent to which within the Petrie pairs and under the action of a hole operator, one may refer to the corresponding hexagons and make sure the two maps are in the same place relative to the square node within each $\langle\, \mathbf{D}, \mathbf{P} \,\rangle$ orbit.

At this point we make the following remarks, some of which are instantly obvious, and others less so. All of them are verifiable by reference to the diagram and/or the tables, while some of the later claims are easiest to verify using the computer.

Remark 3.85. Being connected, the diagram clearly shows that the action of $\langle\, \mathbf{D}, \mathbf{P}, \mathbf{S}, \mathbf{R} \,\rangle$ is transitive on the subset of regular maps in Ω_{A_5} consisting of all 72 maps with edge stabiliser $\langle x, y \rangle = K_1$. ⋆

Remark 3.86. Each connected component will have a given copy of V_4 as the edge-stabiliser of the distinguished edge in each component map. This can be seen by simply considering the action of each operator on the elements x, y, xy. The duality operators merely permute the elements of the edge-stabiliser, while these elements are fixed by the hole operators. Figure 3.3 shows only one connected component. ⋆

Remark 3.87. Each vertex is colour-coded according to the isomorphism class of the associated map. Respectively, the maps in $\widehat{\mathcal{M}_I}$, $\widehat{\mathcal{M}_{II}}$, and $\widehat{\mathcal{M}_{III}}$ are cyan, red and green. ⋆

Remark 3.88. The operators $\mathbf{R}$ and $\mathbf{S}$, when restricted to acting on maps of type $\mathcal{M}_I$, behave in exactly the same way, and so are indicated by just one arc. ⋆

Remark 3.89. Remembering that the hole operator $\mathbf{R}$ interchanges the isomorphism classes of $\widehat{\mathcal{M}_{II}}$ and $\widehat{\mathcal{M}_{III}}$, makes the action of $\mathbf{R}$ easier to 'read' off the diagram. Similarly for $\mathbf{S}$ which fixes the isomorphism class of any map. ⋆

Remark 3.90. It is possible to read off the order of elements in $\langle\, \mathbf{D}, \mathbf{P}, \mathbf{S}, \mathbf{R} \,\rangle$ by considering their action on a map of each isomorphism class. For example, the operator $\mathbf{DRS}$ has order twelve. ⋆

Remark 3.91. It is possible to verify identities, for example $(\mathbf{DRS})^6 = \mathbf{DSDS}$. ⋆

Remark 3.92. The operator $[\mathbf{D}, \mathbf{R}^{-1}] = [\mathbf{DP}, \mathbf{R}^{-1}]$ has order three, and cyclically permutes the three corresponding representatives for z between the isosets. For example $(X, Y, A_2) \to (X, Y, B_2) \to (X, Y, C_2) \to (X, Y, A_2)$. The natural consequence of this is that $[\mathbf{D}, \mathbf{R}^{-1}]^3$ is a relator in the operator group $\langle\, \mathbf{D}, \mathbf{P}, \mathbf{S}, \mathbf{R} \,\rangle$. ⋆

Remark 3.93. It is worth noting that the operator $\mathbf{RDR}^{-1}\mathbf{DP}$ is an involution. ⋆

Remark 3.94. $\langle\, \mathbf{D}, \mathbf{P}, \mathbf{S}, \mathbf{R} \,\rangle$ has order 2304. This is consistent with the number of fully regular map triples forming a super-symmetric parallel product $\mathcal{N}_{\mathrm{A}_5} = (\mathrm{A}_5 \times \mathrm{A}_5 \times \mathrm{A}_5; x, y, z)$ where each component map has precisely the same copy of V_4 as its edge stabiliser, that is $x = (X, X, X)$, and $y = (Y, Y, Y)$. There must be one component map in each map isomorphism class, and so remembering the isosets, there are twelve options for the first component of z, eight for the next, and four for the final component of z, while for each of these there are six permutations for the ordered pair (x, y). ⋆

Remark 3.95. The group $\langle\, \mathbf{DP}, \mathbf{SR} \,\rangle$ has order 1152 and so is normal of index two in $\langle\, \mathbf{D}, \mathbf{P}, \mathbf{S}, \mathbf{R} \,\rangle = \langle\, \mathbf{DP}, \mathbf{SR} \,\rangle \rtimes \langle\, \mathbf{P} \,\rangle$. ⋆

For ease of use, Figure 3.4 shows a diagram which has been simplified further: the action of $\mathbf{S}$ is totally suppressed. It is however easy to use since when $\mathbf{S}$ acts on $\mathcal{M}$, a map in $\widehat{\mathcal{M}_{III}}$ or $\widehat{\mathcal{M}_{III}}$, it is equivalent to the action of $\mathbf{R}^2$ on $\mathcal{M}$. Meanwhile $\mathbf{R}$ and $\mathbf{S}$ behave the same when acting on a map in $\widehat{\mathcal{M}_I}$.

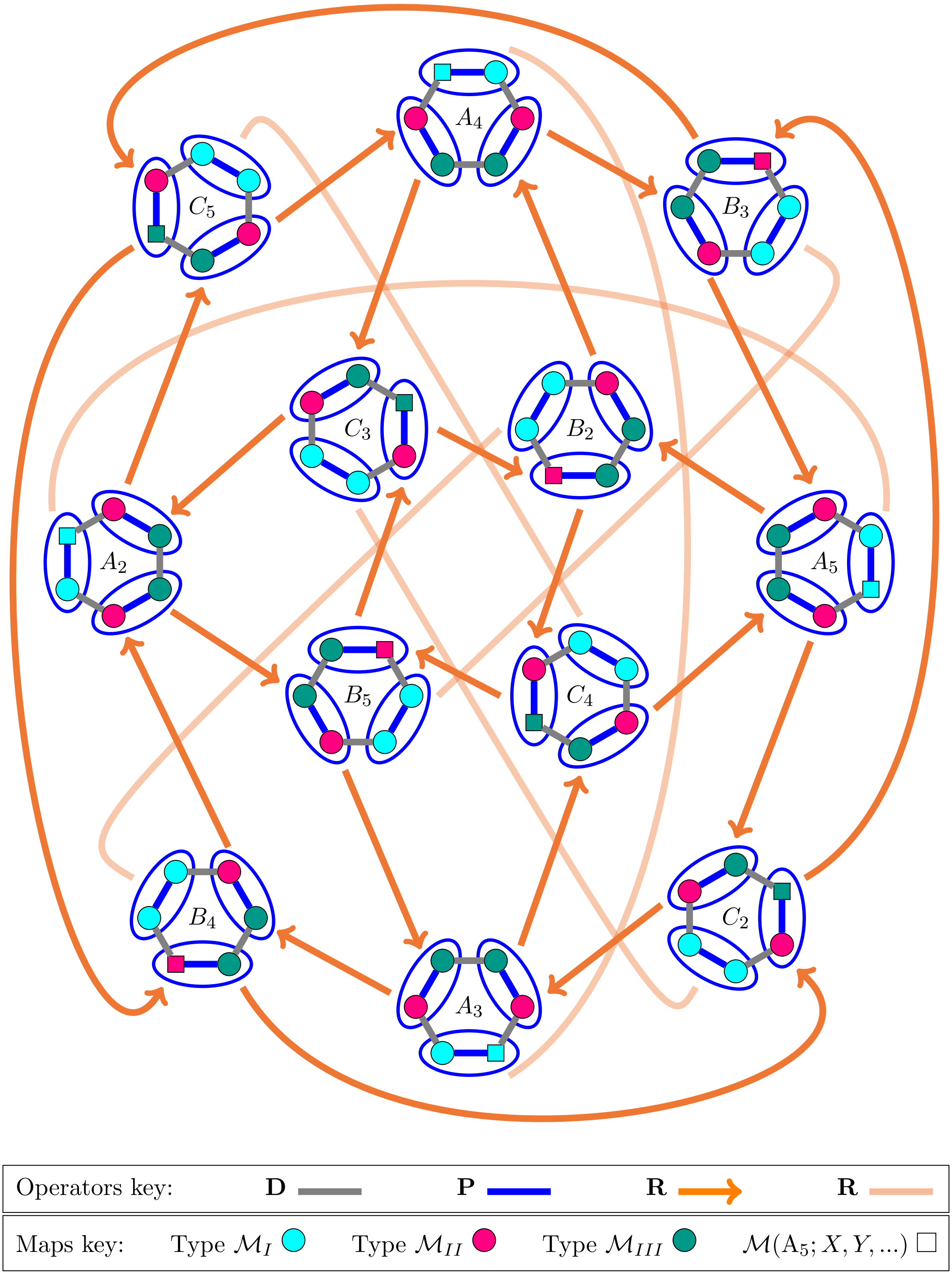

Figure 3.4: Permutations of maps within one orbit of $\langle \mathbf{D}, \mathbf{P}, \mathbf{R} \rangle$ acting on Ω_{A_5}.

3.6.3 Operators acting on $\mathbf{A}_5 \times \mathbf{A}_5 \times \mathbf{A}_5$

We now turn our attention to considering a parallel product of maps $\mathfrak{M} := \mathcal{M}_1||\mathcal{M}_2||\mathcal{M}_3$ where $\mathcal{M}_1(\mathrm{A}_5; X, Y, A_2)$, $\mathcal{M}_2(\mathrm{A}_5; X, Y, B_2)$ and $\mathcal{M}_3(\mathrm{A}_5; X, Y, C_2)$. This parallel product contains a copy of each of the three isomorphism classes of map on A_5 and so $\mathfrak{M}$ is super-symmetric. Defining $x := (X, X, X), y := (Y, Y, Y)$ and $z := (A_2, B_2, C_2)$, and recalling Lemma 3.21, we have $\mathfrak{M}(\mathrm{A}_5 \times \mathrm{A}_5 \times \mathrm{A}_5; x, y, z)$.

We saw in Lemma 3.6 that the operator $\mathbf{S}$ applied to the map $\mathcal{M}(G; x, y, z)$ is precisely $\mathcal{M}^y$, the y-conjugate of the map. Being equivalent to a conjugation by $y \in \{(X, X, X), (Y, Y, Y), (XY, XY, XY)\}$, acting (component-wise) on each (triple of) involution(s), $\mathbf{S}$ is a map isomorphism and as such will preserve the isoset of each of the z-coordinates.

When $y = (y', y', y')$, for some $y' \in \{X, Y, XY\}$ the notation is devised such that the components of z, namely A_a, B_b and C_c such that $a, b, c \in \{2, 3, 4, 5\}$, are mapped by $\mathbf{S}$ to $A_{ay'}, B_{by'}$ and $C_{cy'}$ respectively. For example, when $y = (Y, Y, Y)$ for $Y = (25)(34)$ and $z = (A_3, C_5, B_2)$, the map $\mathcal{M}(G; x, y, z)$ is mapped by $\mathbf{S}$ to $\mathcal{M}(G; x, y, (A_4, C_2, B_5))$.

The map $\mathfrak{M}$ has valency 15 and so the group of hole operators, which is isomorphic to $\mathrm{U}_{15} \cong \mathrm{C}_2 \times \mathrm{C}_4$, is generated by $\mathbf{H}_{-1}$ and $\mathbf{H}_2$, that is $\mathbf{S}$ and $\mathbf{R}$. The external symmetry group is $Ext(\mathfrak{M})$ which is thus generated by the operators $\mathbf{D}, \mathbf{P}, \mathbf{S}, \mathbf{R}$.

Meanwhile the external symmetry group of this map will necessarily be a subgroup of the automorphism group of the group of the parallel product. In this small example the automorphism group of the map is $\mathrm{A}_5 \times \mathrm{A}_5 \times \mathrm{A}_5$ and so the external symmetry group must be a subgroup of $Aut(\mathrm{A}_5 \times \mathrm{A}_5 \times \mathrm{A}_5) = (\mathrm{S}_5 \times \mathrm{S}_5 \times \mathrm{S}_5) \rtimes \mathrm{S}_3$.

All the operators setwise fix the edge-stabiliser $\langle x, y \rangle$, and so the external symmetry group must also respect this. In this small example, where the edge-stabiliser for each constituent map fixes the element labelled 1, the implication is that the external symmetry group will be isomorphic to a subgroup of $(\mathrm{S}_4 \times \mathrm{S}_4 \times \mathrm{S}_4) \rtimes \mathrm{S}_3$.

Remark 3.96. This implies that the maximum order of an operator in the group $Ext(\mathfrak{M})$ is twelve. $\star$

It is possible to track how the operators are acting on the maps by recording both the effect on the map isomorphism class for each of the components of the parallel product, as well as the effect on the generators for each component map. This can help to establish divisors of the order of an operator, as demonstrated in the table below.

Example 3.97.

	x'	y'	z'	The parallel product
$\mathfrak{M}$	x	y	z	$\mathcal{M}_I \vert\vert \mathcal{M}_{II} \vert\vert \mathcal{M}_{III}$
$\mathfrak{M}$**DRS**	y	x	$(zxz)^x$	$\mathcal{M}_{III} \vert\vert \mathcal{M}_I \vert\vert \mathcal{M}_{II}$
$\mathfrak{M}$(**DRS**)2	x	y	$((xzxzx)y(xzxzx))^y = yxzxzyzxzxy$	$\mathcal{M}_{II} \vert\vert \mathcal{M}_{III} \vert\vert \mathcal{M}_I$
$\mathfrak{M}$(**DRS**)^3	y	x	$y(zxzyzxz)x(zxzyzxz)y$	$\mathcal{M}_I \vert\vert \mathcal{M}_{II} \vert\vert \mathcal{M}_{III}$

The operator **DRS** sends the map isomorphism classes $\widehat{\mathcal{M}_I} \to \widehat{\mathcal{M}_{III}} \to \widehat{\mathcal{M}_{II}} \to \widehat{\mathcal{M}_I}$, and so three must divide its order. Meanwhile each application of the same operator interchanges x with y and so the order of **DRS** must also be even. Both of these conclusions can be drawn from the first two lines of the table, however it is instructive to note how the image of z in the following lines looks more and more complicated. •

In fact, it may be checked that this particular operator, **DRS**, has order twelve in $\mathbf{Op}(\Omega_{\mathrm{A}_5})$, and this may be verified by means of reference to the diagram in Figure 3.3. Equivalently, the operator **DRS** $\in Ext(\mathfrak{M})$ has maximal order, that is twelve.

Remark 3.98. In general, restricting this method to considering the effect of an operator on the elements x and y can be used to establish when a word *cannot* be a relator in the group $Ext(\mathcal{M})$, that is, when it cannot be equivalent to the identity operator (which by definition fixes all the generators). Since the only operators which have an effect on x and y are in $\langle\, \mathbf{D}, \mathbf{P}\, \rangle$ it can be sufficient to check that a word, when read from left to right and disregarding all the hole operators, is not a relator in the group $\langle\, \mathbf{D}, \mathbf{P}\, \rangle \cong \mathrm{S}_3$. ⋆

3.6.4 Aside: A new way of combining regular maps

The investigation of the previous subsections brought to my mind a new way of combining two fully regular maps, such that one acts on another by conjugation.

Construction 3.99. Let $\mathcal{A} = \mathcal{M}(G; x, y, z)$ and $\mathcal{B} = \mathcal{M}(G'; x', y', z')$ be two regular maps with the same automorphism group $G = G'$. We define $\mathcal{A}^{\mathcal{B}} := \mathcal{M}(G''; x'', y'', z'')$ where $x'' = x^{x'y'}$, and $y'' = y^{x'y'}$, while $z'' = z^{z'}$. ⋄

Remark 3.100. The object $\mathcal{A}^{\mathcal{B}}$ as described above is a fully regular map. The group $\langle x, y\rangle \cong \mathrm{V}_4$, and so its conjugate $\langle x, y\rangle^{x'y'}$ has the same structure, indicating that the elements x'' and y'' are two commuting involutions. Also z'' is an involution, and so by comparison with Equation (1.1) the object $\mathcal{A}^{\mathcal{B}}$ is a fully regular map. ⋆

Remark 3.101. There is no guarantee that the resulting object map $\mathcal{A}^{\mathcal{B}}$ has the same automorphism group as the original maps $\mathcal{A}$ and $\mathcal{B}$, however it is clear that the group $G'' = \langle x'', y'', z''\rangle = Aut(\mathcal{A}^{\mathcal{B}}) \leq G$. ⋆

This concept can be extended to apply to two fully regular maps which have different automorphism groups: so long as G' is isomorphic to a subgroup of the automorphism group of G, then G' may be defined to act on G. This yields a well-defined combination of the two regular maps which is itself a regular map.

Construction 3.102. The $\mathcal{B}$-conjugate of map $\mathcal{A}$: Let $\mathcal{A} = \mathcal{M}(G; x, y, z)$ and $\mathcal{B} = \mathcal{M}(G'; x', y', z')$ be two regular maps. Suppose that the group G' is isomorphic to a subgroup of $Aut(G)$ and let the automorphism in $Aut(G)$ associated with $g' \in G'$ be denoted $\psi_{g'} \in Aut(G)$. We define the *$\mathcal{B}$-conjugate of map $\mathcal{A}$* to be
$\mathcal{A}^{\mathcal{B}} := \mathcal{M}(G''; x'', y'', z'')$ where $x'' = (x)\psi_{x'y'}$, and $y'' = (y)\psi_{x'y'}$, while $z'' = (z)\psi_{z'}$. $\diamond$

Remark 3.103. The object $\mathcal{A}^{\mathcal{B}}$ is a fully regular map. The same automorphism $\psi_{x'y'}$ has been applied to both of the pair x and y to yield respectively x'' and y'' which will therefore also be a pair of commuting involutions. z'' is also an involution, and so the object is equivalent to a regular map since it is a group which is generated by three involutions, two of which commute. $\star$

The inspiration for the above constructions was the following observation.

Remark 3.104. If $\mathcal{M}(\mathrm{A}_5; x, y, z)$ is is a fully regular map, and $\mathbf{R} = \mathbf{H}_2$, then $\mathcal{M}\mathbf{R}^{-1} = \mathcal{M}^{[\mathcal{M}^{(\mathcal{M}^y)}]}$. $\star$

More generally, the following is the case.

Remark 3.105. If $\mathcal{M}(\mathrm{SL}(2, q); x, y, z)$ is is a fully regular map with $q = 2^\alpha$, and $\mathbf{R} = \mathbf{H}_2$, then $\mathcal{M}\mathbf{R}^{-1} = \mathcal{M}\mathbf{R}^{2\alpha-1} = \mathcal{M}^{[\mathcal{M}^{\cdots^{(\mathcal{M}^y)}}]}$ such that the repeated conjugation tower works from the top downwards and contains $2\alpha - 1$ copies of $\mathcal{M}$. $\star$

3.6.5 External symmetry group for the super-symmetric map $\mathcal{N}_{\mathbf{A}_5}$

The only thing which remains is to present a way of building the group $\mathbf{Op}_{15}(\Omega_{\mathrm{A}_5})$ from scratch, thereby giving direct insight into the structure of the group $Ext(\mathcal{N}_{\mathrm{A}_5})$. The generators of the permutation group are given in the following section, and with the help of a computer may be used to answer any further questions about the group.

Proposition 3.106 (Reade and Širáň)**.** *The group $Ext(\mathcal{N}_{A_5}) = \langle\, \boldsymbol{D}, \boldsymbol{P}, \boldsymbol{R}, \boldsymbol{S}, \,\rangle$ is equal to $(\langle\, \boldsymbol{D}, \boldsymbol{R}\,\rangle.\langle\, \boldsymbol{DP}\,\rangle).\langle\, \boldsymbol{SD}\,\rangle$ and isomorphic to $(((V_4 \rtimes A_4) \rtimes V_4).C_3).C_4$. The group has cardinality* 2304*, and presentation as follows:*

$$Ext(\mathcal{N}_{A_5}) = \langle\, \boldsymbol{D}, \boldsymbol{P}, \boldsymbol{R}\, \boldsymbol{S}, \mid\, \boldsymbol{D}^2, \boldsymbol{P}^2, \boldsymbol{R}^4, \boldsymbol{S}^2, (\boldsymbol{DP})^3, (\boldsymbol{PS})^2, (\boldsymbol{SD})^4, (\boldsymbol{RD})^6,$$
$$\boldsymbol{PRPR}^{-1}, \boldsymbol{SRSR}^{-1}, (\boldsymbol{SPD})^3, (\boldsymbol{DPR}^2)^6, (\boldsymbol{DR}^2)^4, (\boldsymbol{R}^2\boldsymbol{DSD})^2,$$
$$(\boldsymbol{RDR}^{-1}\boldsymbol{D})^3, (\boldsymbol{DRS})^6\boldsymbol{SDSD}, (\boldsymbol{PDRDR}^{-1})^2, (\boldsymbol{DS})^2\boldsymbol{R}(\boldsymbol{DS})^2\boldsymbol{R}^{-1}\,\rangle$$

Remark 3.107. The proof below and the proof of Theorem 3.33 both rely on the same final step, that being to introduce $\mathbf{S}$ into a group of operators (which includes the duality $\mathbf{D}$, possibly $\mathbf{P}$, and other allowable hole operators $\mathbf{H}_j$) by adjoining the operator $\mathbf{DS}$ which has order four. So long as the original group of operators has trivial intersection with $\langle\, \mathbf{DS} \,\rangle$, this is a general method which will work to give a group four times as large. The interactions between powers of $\mathbf{DS}$ with $\mathbf{D}$ and $\mathbf{P}$ have been demonstrated in the proof of Theorem 3.33, and rely on the structure of $\langle\, \mathbf{D}, \mathbf{P}, \mathbf{S} \,\rangle$ proved in Lemma 3.10. Meanwhile in Lemma 3.14 it has been proved that $(\mathbf{DS})^2\mathbf{H}_j = \mathbf{H}_j(\mathbf{DS})^2$ is true for all hole operators $\mathbf{H}_j$. Furthermore it is clear that $\mathbf{DS}.\mathbf{H}_j = \mathbf{DH}_j\mathbf{S} = \mathbf{DH}_j\mathbf{D}.\mathbf{DS}$. Finally, $(\mathbf{SD})^2\mathbf{H}_j = \mathbf{H}_j(\mathbf{DS})^2$ and hence $\mathbf{SDH}_j = \mathbf{DSH}_j^{-1}(\mathbf{DS})^2$ meaning that $(\mathbf{DS})^{-1}.\mathbf{H}_j = \mathbf{SDH}_j = \mathbf{DSH}_j^{-1}(\mathbf{DS})^2 = \mathbf{DH}_j^{-1}\mathbf{S}(\mathbf{DS})^2 = \mathbf{DH}_j^{-1}\mathbf{D}.(\mathbf{DS})^3$. From these relations we may then write down a presentation for the group. ⋆

Proof of Proposition 3.106. Sufficient staring at the diagram in Figure 3.4, to find the order of elements, establishing the size of the group itself as a permutation group, with the help of a computer and according to the data in section 3.7, along with knowledge of the interactions between elements inherited from the parent group $\mathfrak{Op}_{15} = \langle\, \mathfrak{D}, \mathfrak{P}, \mathfrak{S}, \mathfrak{R} \,\rangle$, and claims from the general case in section 3.4.2 yields the following sequence of observations.

The order of $\mathbf{R}$ is four, it may be checked that the order of $\mathbf{RD}$ is six, and clearly $\mathbf{D}$ has order two. Now, in this case, $\mathbf{Q}$ is $\mathbf{R}^2$ so the identity $(\mathbf{DQ})^4 = \mathbf{Id}$ from Theorem 3.33 implies that $(\mathbf{DR}^2)^4 = \mathbf{Id}$. Meanwhile Remark 3.92 tells us that $(\mathbf{DRDR}^{-1})^3 = \mathbf{Id}$, and computational results show that the permutation group $\langle\, \mathbf{D},\, \mathbf{R} \,\rangle$ has order 192. Moreover, it may be checked that the group with presentation $\langle\, \mathbf{D},\, \mathbf{R} \mid \mathbf{D}^2,\, \mathbf{R}^4,\, (\mathbf{RD})^6,\, (\mathbf{R}^2\mathbf{D})^4,\, (\mathbf{DRDR}^{-1})^3 \,\rangle$ also has cardinality 192.

Now $\langle\, \mathbf{D},\, \mathbf{R} \,\rangle$ is isomorphic to $(\mathrm{V}_4 \rtimes \mathrm{A}_4) \rtimes \mathrm{V}_4$, and this may be checked by computer, or by the following group building construction. The key for many of the following routine checks, are the relators in the above group. Let $a := \mathbf{R}^2$, $b := (\mathbf{DR})^2$ and $c := (\mathbf{RD})^2$. Using the fact that the order of $\mathbf{R}^2\mathbf{D}$ is four, it can be shown that bc and cb are commuting involutions. The group $\langle\, bc, cb \,\rangle$ is therefore isomorphic to V_4. Now, a is an involution and the order of b is three. Their product can be shown to have order three as well, by remembering $(\mathbf{DRDR}^{-1})^3 = \mathbf{Id}$. This implies that the group $\langle\, a, b \mid a^2, b^3, (ab)^3 \,\rangle$ is isomorphic to A_4. Then, $\langle\, bc, cb \,\rangle$ is normalised by $\langle\, a, b \,\rangle$. Taking advantage of the fact that $\mathbf{DRDR}^2$ has order four, and noting the trivial intersection between $\langle\, a, b \,\rangle$ and $\langle\, bc, cb \,\rangle$, the group $\langle\, bc, cb \,\rangle \rtimes \langle\, a, b \,\rangle$ has presentation $\langle\, a, b, c \mid a^2, b^3, c^3, (ab)^3, (ac)^3, (bc)^2, (abc)^2 \,\rangle$ and is isomorphic to $\mathrm{V}_4 \rtimes \mathrm{A}_4$. The last copy of V_4 which both normalises, and has trivial intersection with, $\langle\, a, b, c \,\rangle$ is $\langle\, \mathbf{R}^2\mathbf{D}\mathbf{R}^2, (\mathbf{DR})^3 \,\rangle$. This yields the group $\langle\, \mathbf{D}, \mathbf{R} \,\rangle$ in the promised isomorphism class.

Furthermore we may adjoin the element $\mathbf{DP}$ to the group $\langle\, \mathbf{D},\ \mathbf{R} \rangle$. By comparing the action of $\mathbf{D}$ with how $\mathbf{DP}$ acts on the first two generators x and y of the map $\mathcal{M}(G; x, y, z)$, it is clear that the intersection $\langle\, \mathbf{DP} \,\rangle \cap \langle\, \mathbf{D},\ \mathbf{R} \rangle$ is trivial. Also $\mathbf{DP}.\mathbf{R} = \mathbf{DRD}.\mathbf{DP}$ and $\mathbf{DP}.\mathbf{D} = \mathbf{D}.(\mathbf{DP})^{-1}$. Clearly $(\mathbf{DP})^{-1}.\mathbf{D}$ is $\mathbf{D}.\mathbf{DP}$ while, by Remark 3.93, $(\mathbf{DP})^{-1}.\mathbf{R} = \mathbf{PDR} = \mathbf{RDR}^{-1}\mathbf{DPRD}$ which may be written $\mathbf{RDR}^{-1}\mathbf{DR}(\mathbf{DP})^2$. Hence $\langle\ \mathbf{D},\ \mathbf{P},\ \mathbf{R}\ \rangle = \langle\ \mathbf{D},\ \mathbf{R}\ \rangle.\langle\ \mathbf{DP}\ \rangle$ is a group of size $192 \times 3 = 576$. Verification by computer confirms the group may be presented: $\langle\, \mathbf{D},\ \mathbf{R},\ \mathbf{P},\ |\ \mathbf{D}^2,\ \mathbf{P}^2,\ \mathbf{R}^4,\ (\mathbf{DP})^3,\ (\mathbf{RD})^6,\ (\mathbf{R}^2\mathbf{D})^4,\ \mathbf{PRPR}^{-1},\ (\mathbf{PDRDR}^{-1})^2 \rangle$.

Finally, adjoining the element $\mathbf{DS}$ yields the group in the proposition. First we must check that the intersection of $\langle\, \mathbf{DS} \,\rangle$ with $\langle\, \mathbf{D},\ \mathbf{R},\ \mathbf{P},\ \rangle$ is trivial, for which we may use a computer. We have already seen, in the proof of Theorem 3.33, how powers of $\mathbf{DS}$ move over finite words made from $\{\mathbf{D}, \mathbf{P}, \mathbf{Q}\}$. By Remark 3.107, powers of $\mathbf{DS}$ also permute with words over $\{\mathbf{D}, \mathbf{R}\}$. Altogether the associated relations yield the required relators to determine the structure of $\langle\, \mathbf{D},\ \mathbf{R},\ \mathbf{P},\ \rangle.\langle\, \mathbf{DS} \,\rangle$ which has cardinality $576 \times 4 = 2304$. The presentation given in the theorem is equivalent to the group as constructed above, albeit with a different set of relators. The set of relators shown are chosen for the sake of demonstrating some of the key features of the group, and as such there is naturally a degree of redundancy. $\square$

3.7 The group $\mathbf{Op}_{15}(\Omega_{\mathbf{A}_5})$ as a permutation group on 72 maps

This section details the generating permutations for the group $\mathbf{Op}_{15}(\Omega_{\mathrm{A}_5})$ which is generated by $\langle\, \mathbf{D}, \mathbf{P}, \mathbf{R}, \mathbf{S} \,\rangle$ and isomorphic to $Ext(\mathcal{N}_{A_5})$. This may be used to verify claims by computer.

Focus on a single orbit where $\langle x, y\rangle = K_1 = \langle X, Y\rangle \cong \mathrm{V}_4$. Label the maps with the numbers $\{1, 2, 3, \ldots, 72\}$ and generating triples as shown in the table below.

	$\mathcal{M}$	$\mathcal{M}\mathbf{P}$	$\mathcal{M}\mathbf{D}$	$\mathcal{M}\mathbf{DP}$	$\mathcal{M}\mathbf{DPD}$	$\mathcal{M}\mathbf{PD}$
Z	(X, Y, Z)	(XY, Y, Z)	(Y, X, Z)	(YX, X, Z)	(X, XY, Z)	(Y, XY, Z)
A_2	1	2	3	4	5	6
A_3	7	8	9	10	11	12
A_4	13	14	15	16	17	18
A_5	19	20	21	22	23	24
B_2	25	26	27	28	29	30
B_3	31	32	33	34	35	36
B_4	37	38	39	40	41	42
B_5	43	44	45	46	47	48
C_2	49	50	51	52	53	54
C_3	55	56	57	58	59	60
C_4	61	62	63	64	65	66
C_5	67	68	69	10	71	72

Then $\mathbf{Op}_{15}(\Omega_{\mathrm{A}_5}) = \langle\, \mathbf{D}, \mathbf{P}, \mathbf{R}, \mathbf{S} \,\rangle \cong Ext(\mathcal{N}_{A_5})$ where $\mathbf{D}, \mathbf{P}, \mathbf{R}, \mathbf{S}$ are as follows. The group has cardinality 2304.

$\mathbf{D}$ = (1,3)(2,6)(4,5)(7,9)(8,12)(10,11)(13,15)(14,18)(16,17)(19,21)(20,24)(22,23)(25,27)(26,30)(28,29)(31,33)(32,36)(34,35)(37,39)(38,42)(40,41)(43,45)(44,48)(46,47)(49,51)(50,54)(52,53)(55,57)(56,60)(58,59)(61,63)(62,66)(64,65)(67,69)(68,72)(70,71)

$\mathbf{P}$ = (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)(17,18)(19,20)(21,22)(23,24)(25,26)(27,28)(29,30)(31,32)(33,34)(35,36)(37,38)(39,40)(41,42)(43,44)(45,46)(47,48)(49,50)(51,52)(53,54)(55,56)(57,58)(59,60)(61,62)(63,64)(65,66)(67,68)(69,70)(71,72)

$\mathbf{R}$ = (1,19)(2,20)(3,69,15,57)(4,70,16,58)(5,47,11,41)(6,48,12,42)(7,13)(8,14)(9,63,21,51)(10,64,22,52)(17,35,23,29)(18,36,24,30)(25,61,43,55)(26,62,44,56)(27,39)(28,40)(31,67,37,49)(32,68,38,50)(33,45)(34,46)(53,59)(54,60)(65,71)(66,72)

$\mathbf{S}$ = (1,19)(2,20)(3,15)(4,16)(5,11)(6,12)(7,13)(8,14)(9,21)(10,22)(17,23)(18,24)(25,43)(26,44)(27,39)(28,40)(29,35)(30,36)(31,37)(32,38)(33,45)(34,46)(41,47)(42,48)(49,67)(50,68)(51,63)(52,64)(53,59)(54,60)(55,61)(56,62)(57,69)(58,70)(65,71)(66,72)

Part II

Edge-biregular maps

Chapter 4

Introducing edge-biregular maps

In the first two sections of this chapter we introduce maps which permit an alternate-edge-colouring, and then focus on the most symmetric examples, the edge-biregular maps. We show various approaches to studying them, by their monodromy group or by their automorphism group, stating things as they are, without proof. All this underlying structure may be formalised, and thus we have a firm foundation on which to build. See section 4.2.5 for further details, and for the moment we will adopt a more informal approach.

Much of the work in this chapter, as well as sections 5.1 and 5.2, has been published in [52].

4.1 Maps with an alternate-edge-colouring

We introduce the concept of an alternate-edge-colouring for a map, a condition which is equivalent to the medial map being bipartite. We present the corner-monodromy group for this type of map and relate it to the colour-preserving automorphism group. We then study some properties arising from the algebraic background, focussing on the subclass of these maps which have the largest possible colour-preserving automorphism group, maps which we call edge-biregular. Much of the content of this chapter (and some of the following chapter) appears in the paper with the same title [52], first published in the Journal of Algebraic Combinatorics.

4.1.1 Alternate-edge-colourings

A map has an *alternate-edge-colouring* when it is possible to colour the edge set using precisely two colours such that: consecutive edges around any given face are differently coloured; and consecutive edges in the cyclic order of edges around any vertex are assigned different colours. This property is equivalent to the map having a bipartite medial graph. In our diagrams we will use bold blue lines to denote edges of one colour (which we call shaded) and dashed red lines to indicate edges of the other colour (unshaded).

We note that a map having an alternate-edge-colouring is also equivalent to being

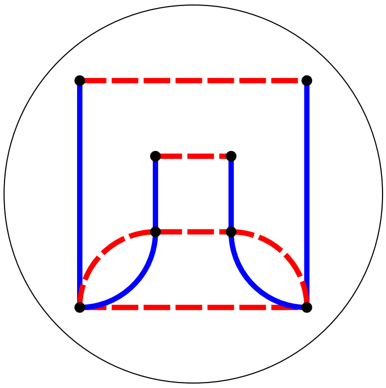

Figure 4.1: A map on a sphere with an assigned alternate-edge-colouring

able to define an orientation on each element in the set containing all corners (of faces) in the map such that adjacent corners have the opposite orientation. The orientations on the corners can then be defined to be consistent with sweeping the corners always in the same direction with respect to the colouring, for example from the shaded edge to the unshaded edge. In this last sense it is a similar definition to the pseudo-orientable maps introduced by Wilson, [64], in which orientations are assigned to the vertices of a map, rather than corners.

An example of a map with an assigned alternate-edge-colouring is shown in Figure 4.1.

In general, unless stated otherwise, we will assume that we are working with maps supported by closed surfaces, that is, maps without boundary components. In this case every vertex of a map with an alternate-edge-colouring must have even valency and each face must have an even length closed boundary walk. However, having even face lengths and valencies is not sufficient for a map to have an alternate-edge-colouring. An example of a map which has only even length faces, and only even degree vertices, is shown in the Figure 4.2. The supporting surface is a torus, for which the edges of the rectangle identified in the usual way, according to the arrows. Any attempt to form an alternate-edge-colouring will result in a contradiction. For example, the (unique) edge marked with the double arrow would "want" to be coloured with both colours, which is clearly impossible. Another way to see this is to observe that the medial graph contains at least one cycle of odd length, and therefore cannot be bipartite. Continuing with the same example, the midpoint of each edge corresponds to a vertex in the medial graph, and a dotted medial 5-cycle is shown in the diagram.

Both examples and non-examples of such maps exist on non-orientable surfaces too. If the left and right edges of the rectangle in Figure 4.2 are identified in the opposite direction from each other then we have a non-orientable map with even valency and

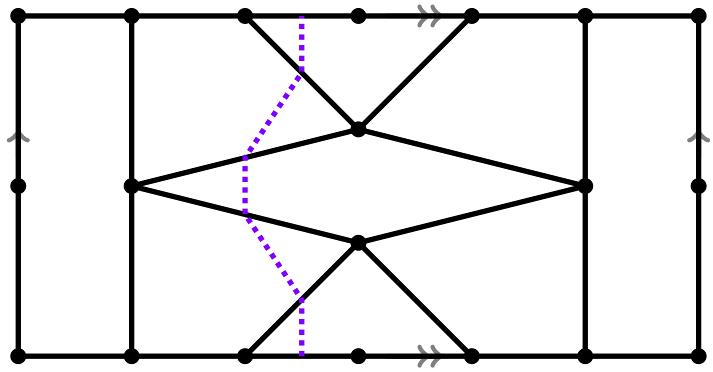

Figure 4.2: A toroidal map with no alternate-edge-colouring

even face length for which there is still no alternate-edge-colouring. However, the Klein bottle does support maps with alternate-edge-colourings, as we will see later, in section 5.1.2.3.

4.1.2 The corner-monodromy group

As before, we define *flags* of a map $\mathcal{M}$ to be the faces of its barycentric subdivision. We refer to the vertices of the barycentric subdivision by their *labels* as follows: each flag has three labels, one for each dimension $0, 1$, and 2, respectively corresponding to the vertex, edge and face of the map $\mathcal{M}$ with which that flag is associated.

In a map with an assigned alternate-edge-colouring, that is a map which has been given an alternate-edge-colouring, each flag will inherit a natural colouring, shaded or unshaded, according to the colour of the edge with which it associated. The pair of flags which together form a *corner* (that is, the two neighbouring flags from one face which meet in a natural way at a corner of that face) will therefore have one flag of each colour, and the whole map can be subdivided to yield the set $\mathcal{C}$ of ready-coloured corners, that is corners with an assigned colouring.

We now describe an analogue of the existing and well-known monodromy group for a map, see [39] for more details. The gluing instructions for the elements of the set $\mathcal{C}$ for a map with an assigned alternate-edge-colouring, can be regarded as the *ready-coloured-corner-monodromy group* $\mathcal{G}$. It is our choice that the group $\mathcal{G}$ is defined to act on the right of $\mathcal{C}$, and $\mathcal{G}$ is generated by four involutions as follows. Inspired by well-established and intuitive notation, see [38], we define the involutions $\mathcal{R}_0$ and $\mathcal{R}_2$ to be the operations which swap every shaded flag with its neighbour respectively along and across a shaded edge. The subscripts in this notation indicate the dimension of the labels of the adjacent shaded flags which are interchanged by the involution. But the action of this group is on the set of two-coloured corners, not flags, so we see $\mathcal{R}_0$ (respectively $\mathcal{R}_2$) swaps every ready-coloured-corner with its

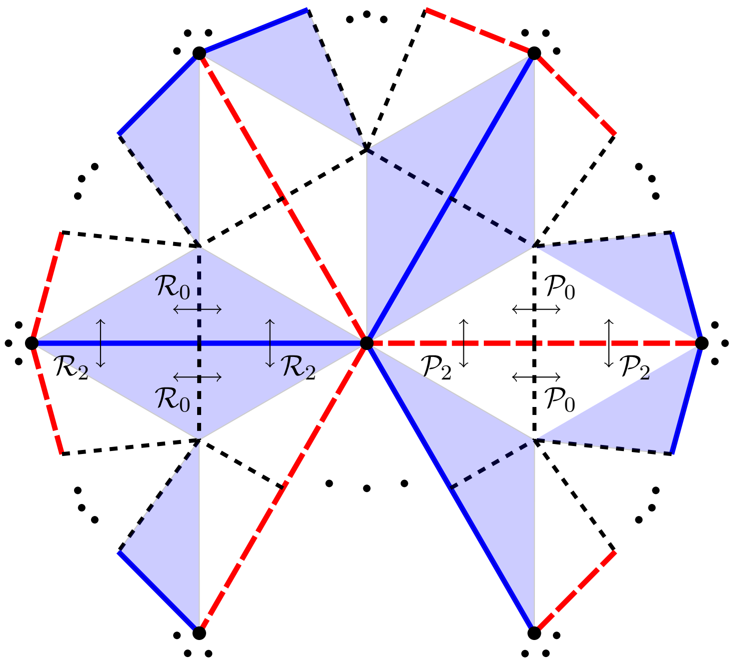

Figure 4.3: The action of $\langle \mathcal{R}_0, \mathcal{R}_2 \rangle$ applied to certain corners of a map with alternate-edge-colouring. This demonstrates an orbit which consists of the four corners associated with a given bold (blue, 'shaded') edge. Similarly the action of $\langle \mathcal{P}_0, \mathcal{P}_2 \rangle$ decomposes the corner set $\mathcal{C}$ into orbits where each orbit consists of the corners surrounding a particular dashed (red, 'unshaded') edge.

unique neighbouring ready-coloured-corner along (across) the incident shaded edge. Meanwhile we define $\mathcal{P}_0$ and $\mathcal{P}_2$ to be the involutions which interchange every ready-coloured-corner with its unique neighbour respectively along and across the incident unshaded edge.

Figure 4.3 shows the action of $\mathcal{R}_0, \mathcal{R}_2$ and $\mathcal{P}_0, \mathcal{P}_2$ on a selection of ready-coloured-corners. Each corner in the diagram is outlined by a bold line, a long-dashed line and two short-dashed lines, and consists of one flag of each colour. Given any marked corner c, the two corners c and $c\mathcal{R}_0$ are thus adjacent along a shaded edge, while c and $c\mathcal{R}_2$ are adjacent across the same shaded edge. Meanwhile c is adjacent to $c\mathcal{P}_0$ and $c\mathcal{P}_2$ respectively along and across the unshaded edge which is incident with c.

Remark 4.1. It is of course possible that some corner c is fixed by one (or more) of the permutations $\mathcal{R}_0, \mathcal{R}_2, \mathcal{P}_0, \mathcal{P}_2$. This will happen if and only if the corner c is not adjacent to another corner respectively along or across the edge of the corresponding colour. For example, given a corner c, suppose that the bold edge associated with c coincides with a boundary of the surface: then $c\mathcal{R}_2 = c$. This situation can happen only when there is a boundary. Meanwhile, the corner c is fixed by $\mathcal{R}_0\mathcal{R}_2$ if and only if $c\mathcal{R}_0 = c\mathcal{R}_2$, that is if and only if the bold edge associated with c is in fact a semi-edge. Similarly $c\mathcal{P}_0 = c\mathcal{P}_2$ if and only if the dashed edge associated with c is a semi-edge. Henceforth, unless stated otherwise, we will assume that the underlying graph of a map has no semi-edges, and it is embedded on a closed surface, that is a surface without boundary. $\star$

Since we are dealing only with maps, not hypermaps, and in light of the previous remark, each edge has precisely four corners incident with it. Each orbit of $\langle \mathcal{R}_0, \mathcal{R}_2 \rangle$ consists of the corners incident with a bold edge of the map, and so it is clear that $\mathcal{R}_0$ commutes with $\mathcal{R}_2$. Similarly $\mathcal{P}_0$ and $\mathcal{P}_2$ commute. Assuming there are no boundary components in the map, $\mathcal{R}_0, \mathcal{R}_2, \mathcal{P}_0, \mathcal{P}_2$ must be involutions, and the absence of semi-edges implies that $\mathcal{R}_0\mathcal{R}_2$ and $\mathcal{P}_0\mathcal{P}_2$ are both involutions as well.

Let $\mathcal{M}$ be a map with an assigned alternate-edge-colouring which is decomposed into its ready-coloured-corners as usual. The corner-monodromy group $\mathcal{G}$ of $\mathcal{M}$ has a presentation of the form $\mathcal{G} = \langle \mathcal{R}_0, \mathcal{R}_2, \mathcal{P}_0, \mathcal{P}_2 | \mathcal{R}_0^2, \mathcal{R}_2^2, (\mathcal{R}_0\mathcal{R}_2)^2, \mathcal{P}_0^2, \mathcal{P}_2^2, (\mathcal{P}_0\mathcal{P}_2)^2, \dots \rangle$ and as such is isomorphic to a quotient group of the free product $\Gamma :=$ $\langle \mathfrak{R}_0, \mathfrak{R}_2 \rangle * \langle \mathfrak{P}_0, \mathfrak{P}_2 \rangle = \langle \mathfrak{R}_0, \mathfrak{R}_2, \mathfrak{P}_0, \mathfrak{P}_2 | \mathfrak{R}_0^2, \mathfrak{R}_2^2, (\mathfrak{R}_0\mathfrak{R}_2)^2, \mathfrak{P}_0^2, \mathfrak{P}_2^2, (\mathfrak{P}_0\mathfrak{P}_2)^2 \rangle \cong \mathrm{V}_4 * \mathrm{V}_4$ defined by the natural epimorphism $\phi : \Gamma \to \mathcal{G}$ where $\phi : \mathfrak{R}_i \to \mathcal{R}_i$ and $\phi : \mathfrak{P}_i \to \mathcal{P}_i$. The group Γ is the ready-coloured-corner-monodromy group of the infinite universal map (which has an assigned colouring) for the class of alternate-edge-colourable maps, from which any map with this property can be determined by means of the epimorphism ϕ.

4.1.3 The colour-preserving automorphism group

Let $\mathcal{M}$ be a map with an assigned alternate-edge-colouring, and corner set $\mathcal{C}$. We consider the group H of automorphisms acting on the (left of the) set $\mathcal{C}$ which preserve both the structure and the colouring of the map $\mathcal{M}$. By this we mean all bijections of the corner set $\mathcal{C}$ onto itself which act such that the images of two neighbouring corners will share the same type of adjacency as their pre-images in $\mathcal{M}$, that is along or across either a shaded or an unshaded edge. The group H thus consists of precisely all the permutations of the set $\mathcal{C}$ which commute with all the gluing instructions in $\mathcal{G}$. Hence
$H = \{h \in Sym_{\mathcal{C}} \mid (h(c))\mathcal{R}_i = h(c\mathcal{R}_i) \text{ and } (h(c))\mathcal{P}_i = h(c\mathcal{P}_i) \text{ for all } c \in \mathcal{C}, i \in \{0, 2\}\}$, that is, H is the centraliser of $\mathcal{G}$ in the symmetric group acting on the set $\mathcal{C}$. We call the group H the *colour-preserving automorphism group* of $\mathcal{M}$.

4.2 Edge-biregular maps

We now turn our attention to the most symmetric examples of maps with a given alternate-edge-colouring, namely the edge-biregular maps. Informally, these are connected maps with an assigned alternate-edge-colouring where the colour-preserving automorphism group H is as large as possible.

4.2.1 Regularity - maximising symmetry

Before proceeding with a formal definition of an edge-biregular map, we note the following lemma.

Lemma 4.2. *Let $\mathcal{M}$ be a connected map with an assigned alternate-edge-colouring and associated corner set $\mathcal{C}$. The colour-preserving automorphism group H of $\mathcal{M}$ acts semi-regularly on the set $\mathcal{C}$.*

Proof. It is the structure-preserving condition for the map automorphisms in H which forces the (left) action of H on $\mathcal{C}$ to be semi-regular, that is fixed-point free. Indeed, let $h \in H$ such that $h(c) = c$ for some given corner $c \in \mathcal{C}$. Let $d = cR$ for some $R \in \{\mathcal{R}_0, \mathcal{R}_2, \mathcal{P}_0, \mathcal{P}_2\}$. By the condition that $h \in H$, we have $h(d) = h(cR) = h(c)R = cR = d$ so h also fixes every corner neighbouring c. The claim now follows by connectivity. □

Since we always assume that all our maps are connected, now we may also assume that the action of H on $\mathcal{C}$ is semi-regular. Moreover, this means that an element $h \in H$ is uniquely determined by the image $h(c)$ of a given corner $c \in \mathcal{C}$.

Thus the largest possible automorphism group H acting on $\mathcal{C}$, the set of (coloured) corners of a map with assigned alternate-edge-colouring, will be when H acts transitively on the set $\mathcal{C}$. When an action is both fixed-point free and transitive the action is said to be *regular*. In this case we would have $|H| = |\mathcal{C}|$ which then allows us to identify the elements of the group H with the corners in the set $\mathcal{C}$. We are now in a position to make the formal definition.

A map $\mathcal{M}$ with an assigned alternate-edge-colouring and set $\mathcal{C}$ of coloured corners is called *edge-biregular* if and only if the colour-preserving automorphism group H of $\mathcal{M}$ acts regularly on the set $\mathcal{C}$ of coloured corners.

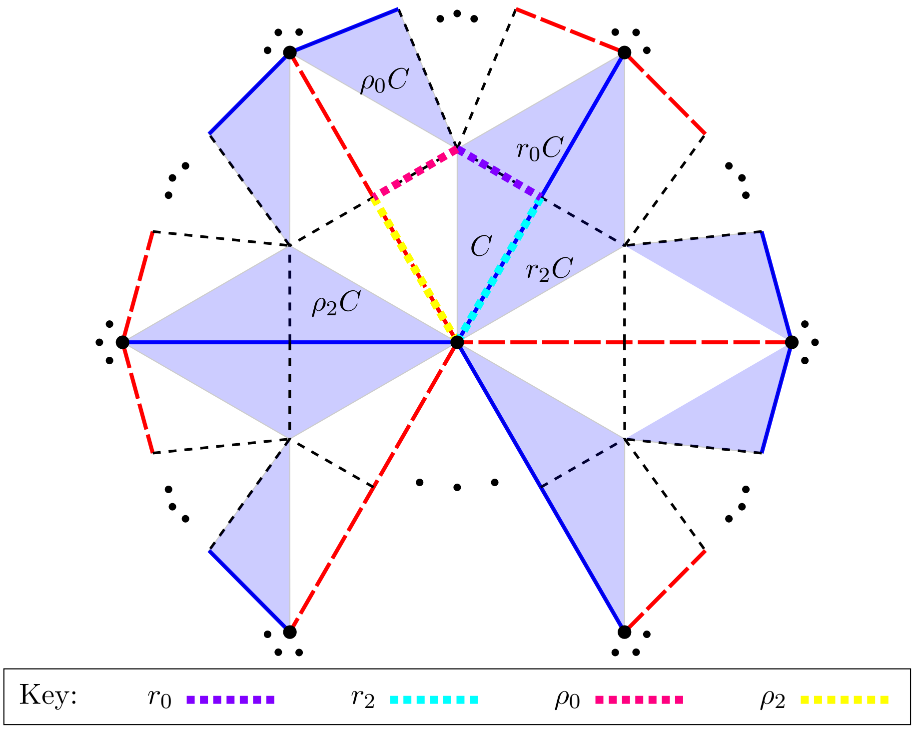

Figure 4.4: The images under r_0, r_2, ρ_0, ρ_2 of the distinguished corner C of an edge-biregular map. The thick dotted coloured lines are the corresponding local lines of reflection.

4.2.2 Canonical form $\mathcal{M} = (H; r_0, r_2, \rho_0, \rho_2)$

Henceforth we consider only edge-biregular maps. What follows is a description of these maps with notes about the group H and some of the terminology we will use. A more algebraic approach may be found in the section 4.2.5.

Let $\mathcal{M}$ be an edge-biregular map. We first mark an arbitrary corner $C \in \mathcal{C}$ which is labelled as the identity element $e \in H$. The corner C is associated with a unique vertex, a unique face, a unique bold edge and a unique dashed edge, and we refer to these structures, which are all incident to the marked corner C, as *distinguished.* The automorphisms in H are then generated by the involutions $\langle r_0, r_2, \rho_0, \rho_2 \rangle$ which act locally as reflections in the natural boundary sections of this marked corner, (respectively along or across the distinguished shaded edge and along or across the distinguished unshaded edge), while at the same time preserving all the adjacency relationships between corners. This situation is shown in Figure 4.4.

Since an automorphism respects adjacency relationships between corners, the afforementioned local lines of reflection extend through the map in a natural way. See Figure 4.5 for an example of an infinite edge-biregular map, showing the global lines of reflection corresponding to each of the generators r_0, r_2, ρ_0, ρ_2.

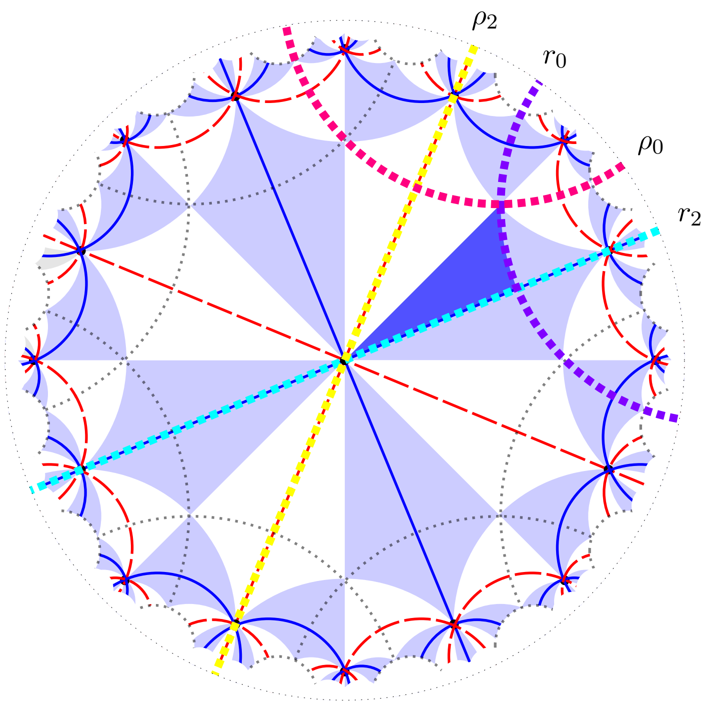

Figure 4.5: The hyperbolic lines of reflection for the generating automorphisms r_0, r_2, ρ_0 and ρ_2, shown on part of the infinite edge-biregular map (hyperbolic tessellation) of type $(8, 4)$. The marked corner, corresponding to the identity element of H, is shaded darker than the others.

Henceforth, except in some special cases, we will be considering edge-biregular maps with even valency k and even face length ℓ. Given such a map $\mathcal{M}$, with a specific marked corner C, we describe the map by using the *canonical form* $\mathcal{M} = (H; r_0, r_2, \rho_0, \rho_2)$ where the generators act as described above, and H is a group with *canonical presentation* as follows:

$$H = \langle\, r_0,\ r_2,\ \rho_0,\ \rho_2 \ \mid\ r_0^2,\ r_2^2,\ \rho_0^2,\ \rho_2^2,\ (r_0 r_2)^2,\ (\rho_0 \rho_2)^2,\ (r_2 \rho_2)^{k/2},\ (r_0 \rho_0)^{\ell/2},\ \dots \rangle. \tag{4.1}$$

The exceptional cases when the valency or face length is not even, are addressed in section 4.3. These occur when the supporting surface has non-empty boundary components and the resulting edge-biregular maps can then have non-even valency or face length. For example if, in an edge-biregular map, a vertex and its incident bold edge, and hence all bold edges, lie on the boundary of the surface, while the dashed edges do not lie on a boundary of the surface, then the vertices will necessarily have odd degree, 3. All such situations are described in detail by Bryant and Singerman [10].

There is a natural correspondence between (isomorphism classes of) edge-biregular maps (each of which has a given edge-colouring) and groups H with (the ordered set of) generators $\{r_0, r_2, \rho_0, \rho_2\}$ and canonical presentation (4.1). This allows us to study edge-biregular maps with canonical form $\mathcal{M} = (H; r_0, r_2, \rho_0, \rho_2)$ by considering groups which have presentations of the form in Equation (4.1).

4.2.3 $\mathcal{M} = (H; r_0, r_2, \rho_0, \rho_2)$ and related edge-biregular maps

Each edge-biregular map $\mathcal{M} = (H; r_0, r_2, \rho_0, \rho_2)$ has a *twin* map which is the same as the original map in every respect except the colouring of the edge orbits is switched. Since each generator is associated with one of the colours of the edges, the twin map of $\mathcal{M}$ is denoted $\mathcal{W}$ and has canonical form $\mathcal{W} = (H; \rho_0, \rho_2, r_0, r_2)$. This is just a matter of relabelling, and does not imply or demand any further relationship between the two structures.

The map automorphisms in H of an edge-biregular map are those which are generated by the four reflections which act locally along and across each of two particular adjacent edges of the map, as shown in Figure 4.5. As we have seen the group H acts regularly on the set of corners of the map. It is possible that the map has further symmetries, and the established definition for the full automorphism group of a map is defined with respect to its action on the set of flags of the map, with no reference to the colouring of edges, see [38]. When considered as acting in the natural way on the flags of the map, the group $H = \langle r_0, r_2, \rho_0, \rho_2 \rangle$ partitions the flag set of a given edge-biregular map into two orbits, one containing shaded flags, and the other containing unshaded flags. As such, it is possibly a 2-orbit map, one of those classified by Orbanić, Pellicer and Weiss [51]. However an edge-biregular map may not be a 2-orbit map, since there is no structure within our definition which disallows a colour-reversing automorphism of the map. Such an automorphism, if it exists, would fuse the two edge orbits together and the **full** automorphism group for the map would then be transitive on flags (and indeed edges), making it a fully regular map, and in this case the full automorphism group would then contain H as an index two subgroup. The edge-biregular map $\mathcal{M} = (H; r_0, r_2, \rho_0, \rho_2)$ is thus a fully regular map if and only if there is an involutory automorphism θ of the group H such that $\theta : r_i \leftrightarrow \rho_i$ for each $i \in \{0, 2\}$.

Two edge-biregular maps $\mathcal{M} = (H; r_0, r_2, \rho_0, \rho_2)$ and $\mathcal{M}' = (H'; r_0', r_2', \rho_0', \rho_2')$ are said to be *isomorphic* to each other if and only if assigning $r_i \rightarrow r_i'$ and $\rho_i \rightarrow \rho_i'$ for $i \in \{0, 2\}$ extends to an isomorphism of the group H. Thus, an edge-biregular map $\mathcal{M}$ is a fully regular map if and only if it is isomorphic to its twin $\mathcal{W}$. In the cases when $\mathcal{M}$ is not isomorphic to its twin, we say $\mathcal{M}$ is *truly* edge-biregular.

Any edge-biregular map which is not fully regular, *is* an example of a k-orbit map for $k = 2$, see [33] . The 2-orbit maps are classified in [51] where '$2_{0,2}$' is the chosen notation for the class of edge-biregular maps which are not fully regular. In [38] Jones also makes special mention of this class of maps as it is the only non-edge-transitive class which arises very naturally in the process of determining the 14 classes of edge-transtitive maps. These types of edge-transitive maps were classified in different terms by Graver and Watkins in [28] and Wilson in [64].

4.2.4 Relating monodromy with colour-preserving automorphisms

The choice of notation indicates some connection between the corner-monodromy group $\mathcal{G}$ and the colour-preserving automorphism group H, and indeed there is a very natural relationship between the two. The groups H and $\mathcal{G}$ are regular permutation representations on the set of corners of the coloured map acting respectively on the left and right. The embedded Cayley map illustrates this in Figure 4.6. This is a Cayley graph for the automorphism group H, so each of the dark vertices represents an element in H, naturally identified with the corners of the edge-biregular map, and the coloured lines are the generating automorphisms r_0, r_2, ρ_0 and ρ_2. Take the vertex in the corner marked C as the identity element of the group H, and consider the element (for example) $h = \rho_2 r_2 r_0 \rho_0 \in H$. Now, H acts on the left, so h corresponds to the flag which is the image of C after the reflections ρ_0, r_0, r_2 and ρ_2 are applied in that order. Applying these automorphisms in turn can be a somewhat laborious exercise.

However, one arrives at the same corner as when going from C to $C\mathcal{P}_2\mathcal{R}_2\mathcal{R}_0\mathcal{P}_0$. This is no coincidence. Notice that the coloured lines, when looked at in the context of the underlying map shown in gray, indicate the gluing instructions $\mathcal{R}_0, \mathcal{R}_2, \mathcal{P}_0, \mathcal{P}_2$ between the corners. An automorphism $hx_i \in H$ for some $x_i \in \{\rho_2, r_2, r_0, \rho_0\}$ acts on the marked corner C as follows. Being an automorphism, x_i commutes with the monodromy group, and C is our marked corner, so $x_i(C)\mathcal{X}_i = x_i(C\mathcal{X}_i) = C$ where $\mathcal{X}_i \in \{\mathcal{R}_0, \mathcal{R}_2, \mathcal{P}_0, \mathcal{P}_2\}$ corresponds naturally to x_i. This implies that $x_i(C) = C\mathcal{X}_i$ and hence $hx_i(C) = h(C)\mathcal{X}_i$. By induction we can conclude that, where the corner C is identified with the identity of H, the automorphism $h = x_1x_2...x_n \in H$ is identified with the flag $C\mathcal{X}_1\mathcal{X}_2...\mathcal{X}_n$.

4.2.5 Algebraic context

The extended triangle group, denoted by $T_{k,\ell}$ and also known as the full triangle group, is the group of automorphisms of the regular (k, ℓ) tessellation, which itself

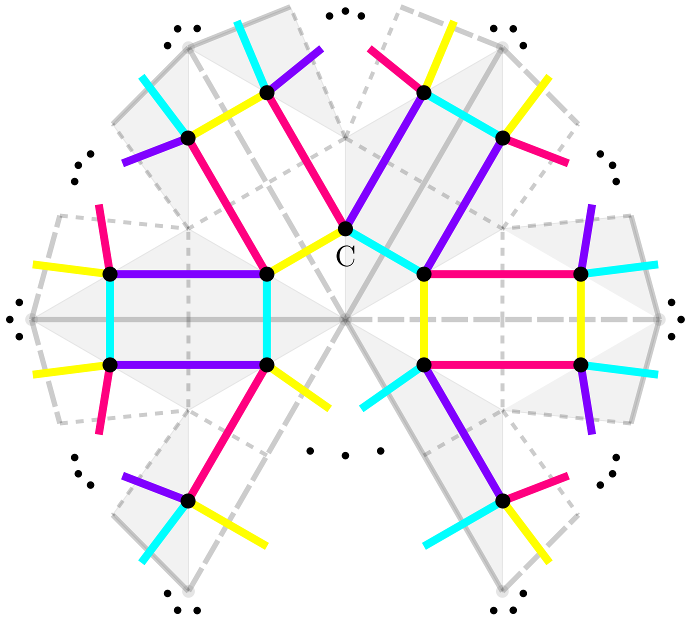

Figure 4.6: Part of the Cayley map, the embedded Cayley graph of H with connection set $\{r_0, r_2, \rho_0, \rho_2\}$.

may be spherical, Euclidean or hyperbolic. The group has the following presentation:

$$T_{k,\ell} = \langle R_0, R_2, R_1 \mid R_0^2, R_2^2, R_1^2, (R_0R_2)^2, (R_1R_2)^k, (R_0R_1)^\ell \rangle \quad (4.2)$$

The extended triangle group $T_{k,\ell}$ has up to seven index two subgroups, the best known being the ordinary triangle group $T^+_{k,\ell} = \langle R_0R_1, R_1R_2\rangle$, which is the group of orientation-preserving automorphisms. Further subgroups may be determined according to which of the three involutory generators R_0, R_1, and R_2 are included in the subgroup, and which are not: when two of them are known to be in the generating set we consider the groups $\langle R_1, R_2, R_1^{R_0}, R_2^{R_0}\rangle$, $\langle R_0, R_2, R_0^{R_1}, R_2^{R_1}\rangle$, and $\langle R_0, R_1, R_0^{R_2}, R_1^{R_2}\rangle$; when one of the elements is known to be in the subgroup we consider $\langle R_0, R_1R_2, R_0^{R_1}, R_0^{R_2}\rangle$, $\langle R_1, R_0R_2, R_1^{R_0}, R_1^{R_2}\rangle$, and $\langle R_2, R_0R_1, R_2^{R_0}, R_2^{R_1}\rangle$. Note that some of the generators in these groups are redundant, they have been retained to aid understanding. The corresponding index of the subgroup within $T_{k,\ell}$ is either one or two, and depends on the values for k and ℓ, in particular whether they are odd or even.

Given such a group T, which has index two within the extended triangle group $T_{k,\ell}$, each torsion-free normal subgroup N in of finite index in T gives rise to a finite quotient group $T/N = H$ which is naturally associated with an isomorphism class of maps of type (k, ℓ). The automorphism group of each resulting map $\mathcal{M}$ is isomorphic to the group H, and the map is described as *biregular* since the action of the group H has two (equal-sized) orbits on the flag set. The subgroup N is then analogous, within that class of biregular map, to the map subgroup of $\mathcal{M}$. If the group N is also normal within the full triangle group, then the biregular map with biregular automorphism group H is also a fully regular map with automorphism group G, and $[G : H] = 2$.

Of particular relevance to this work, when the values k and ℓ are both even, the group $T = \langle R_0, R_2, R_0^{R_1}, R_2^{R_1} \rangle$ is an index two subgroup in $T_{k,\ell}$. Relabelling $R_0^{R_1} = P_0$ and $R_2^{R_1} = P_2$, this group has presentation

$$T = \langle R_0, R_2, P_0, P_2 \mid R_0^2, R_2^2, P_0^2, P_2^2, (R_0R_2)^2, (P_0P_2)^2, (R_2P_2)^{k/2}, (R_0P_0)^{\ell/2} \rangle. \quad (4.3)$$

The edge-biregular map with canonical form and presentation
$H = \langle\, r_0,\ r_2,\ \rho_0,\ \rho_2 \ \mid\ r_0^2,\ r_2^2,\ \rho_0^2,\ \rho_2^2,\ (r_0r_2)^2,\ (\rho_0\rho_2)^2,\ (r_2\rho_2)^{k/2},\ (r_0\rho_0)^{\ell/2},\ \ldots \rangle$ then arises as a quotient of the group T by a torsion-free normal subgroup N, where N is the kernel of the natural epimorphism from T to H defined by $R_0 \to r_0$, $R_2 \to r_2$, $P_0 \to \rho_0$ and $P_2 \to \rho_2$.

4.2.6 Operators on edge-biregular maps

In light of the study of operators on regular maps in Chapter 3, it is natural to consider what operators may be defined to act on an edge-biregular map and yield an edge-biregular map as its image.

The dual and Petrie operators each have a natural analogue which act on edge-biregular maps. For example given $\mathcal{M} = (H; r_0, r_2, \rho_0, \rho_2)$, the corresponding dual map is $(H; r_2, r_0, \rho_2, \rho_0)$, and the Petrie map has canonical form $(H; r_0r_2, r_2, \rho_0\rho_2, \rho_2)$. In each case the result yields maps which are also themselves edge-biregular, and this was the property which first attracted me to this class of map.

In the case of fully regular maps, there is only one distinguished edge-stabiliser $\langle x, y\rangle$, itself a copy of V_4, and the six operators acting on it. However edge-biregular maps have two edge-stabilisers, $\langle r_0, r_2\rangle$ and $\langle \rho_0, \rho_2\rangle$, each of which is isomorphic to V_4, and so there are many more than six operators. In fact for each of the two copies of V_4, there are six operators which correspond naturally to the dual and Petrie operators and their compositions. Moffatt [47] has investigated this concept of partial duality, and the application to edge-biregular maps is clear. For example, applying the partial dual operator to only one of the edge orbits of $\mathcal{M} = (H; r_0, r_2, \rho_0, \rho_2)$, say the orbit of bold edges, yields another edge-biregular map $\mathcal{M}' = (H; r_2, r_0, \rho_0, \rho_2)$. Furthermore, one can also swap the colouring of the edges, that is equivalent to interchanging the two copies of V_4. In total, this yields 72 different operators which form a group isomorphic to the wreath product of S_3 and C_2, that is $(S_3 \times S_3) \rtimes C_2$.

Operators acting on edge-biregular maps and external symmetries of edge-biregular maps are topics for further research. Alternatively, what is the smallest edge-biregular map which has an orbit of 72 mutually non-isomorphic maps under the action of the previously mentioned 72 operators? And are there further operators, for example as an analogue to hole operators, which are well-defined on edge-biregular maps? All these questions are beyond the scope of this thesis.

4.3 Degeneracies

An edge-biregular map is usually described in terms of the four generating automorphisms r_0, r_2, ρ_0, ρ_2, but there are degenerate cases. Degeneracies occur either when one (or more) of the canonical generators is "suppressed" or "missing" (trivial and so contributes nothing to the group) or when the four involutions are not distinct. In each case the geometric interpretation may be inferred from identifying each group element with a corner, as in section 4.2.4, and by considering the implications for the H-stabilisers of the distinguished vertex, edges and face of the map, as well as their intersections. A missing generator indicates the presence of a boundary component of the surface which coincides with the corresponding boundary of the corner region. Non-distinct generators imply either the presence of an orbit of semi-edges (bold when $r_0 = r_2$ or dashed when $\rho_0 = \rho_2$), or alternatively loops (for example when $r_0 = \rho_2$), or possibly multi-edges resulting in digonal faces when $r_0 = \rho_0$, or the corresponding dual case where $r_2 = \rho_2$. These possibilities are explored in this section, informally presented with use of constructions and an emphasis on visualisation.

4.3.1 Edge-biregular maps with semi-edges

A way in which the four involutions might not be distinct would be if $r_0 = r_2$ and/or $\rho_0 = \rho_2$. In this case the corresponding edge-stabiliser, respectively $\langle r_0, r_2 \rangle$ or $\langle \rho_0, \rho_2 \rangle$, would be isomorphic to C_2 which indicates that the orbit of edges of the corresponding colour consists of semi-edges.

If both orbits of edges are semi-edges then we have a semistar. These maps have only one vertex, and, assuming there are more than two corners in the map, the group $H = \langle r_2, \rho_2 \rangle$ will be dihedral. The supporting surface, if without boundary, must be a sphere, and these even valency semistar spherical maps are both edge-biregular and also fully regular. Examples of semistar edge-biregular maps also exist on an (open)

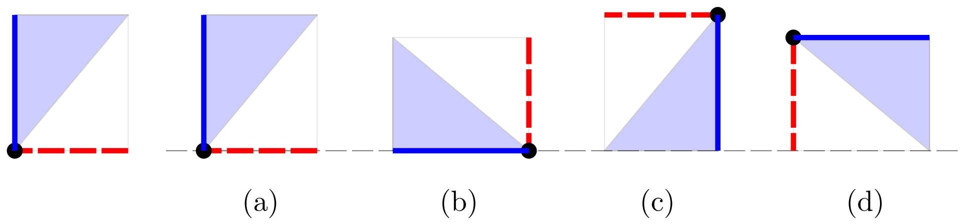

Figure 4.7: A corner, and corners meeting the surface boundary, types (a), (b), (c), and (d)

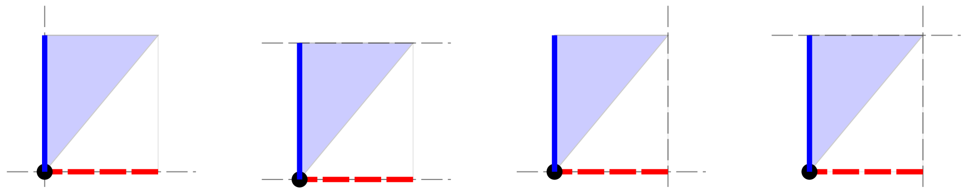

Figure 4.8: Corner regions meeting the surface boundary in two ways (up to colouring)

disc, and we explore surfaces with boundary components in the following section.

Henceforth we will assume, up to twinness, that at least one of the orbits of edges does not consist of semi-edges. When there are no semi-edges in an edge-biregular map we say it is a *proper* edge-biregular map.

4.3.2 Edge-biregular maps with boundary components

If an edge-biregular map is on a surface which is not closed, that is when the surface has non-empty boundary components, then at least one of the edges of a corner region will lie on the boundary of the supporting surface. The structure of such edge-biregular maps is thus restricted. Figure 4.7 shows a corner region, and ways that corner regions can meet the boundary of a surface. Any given corner region is the union of two adjacent flags and so can be represented by a quadrangle in the natural way. Hence a corner region could have up to four of its edges along the boundary, see Figures 4.8, 4.9. For clarity of the diagrams, we have drawn the boundary lines extending beyond the corner regions. In actuality, due to the map being on a 2-manifold with boundary (where every point on a boundary has a neighbourhood homeomorphic to a half-disc), wherever two boundaries apparently cross, the boundary will in fact be one continuous boundary around this part of the corner region.

Edge-biregular maps with boundary occur when one or more of the generators (and hence also the relators including them) collapse and hence are missing from the usual group presentation
$H = \langle\, r_0,\ r_2,\ \rho_0,\ \rho_2 \mid r_0^2,\ r_2^2,\ \rho_0^2,\ \rho_2^2\ (r_0r_2)^2,\ (\rho_0\rho_2)^2,\ (r_2\rho_2)^{k/2},\ (r_0\rho_0)^{\ell/2},\ \dots \rangle$. For

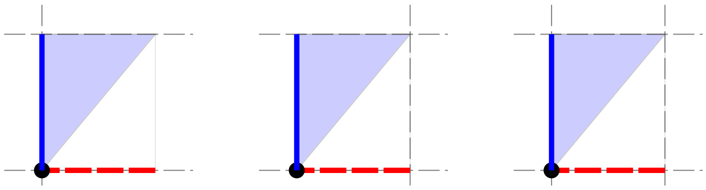

Figure 4.9: Corners meeting the surface boundary in three (up to colouring) or four ways

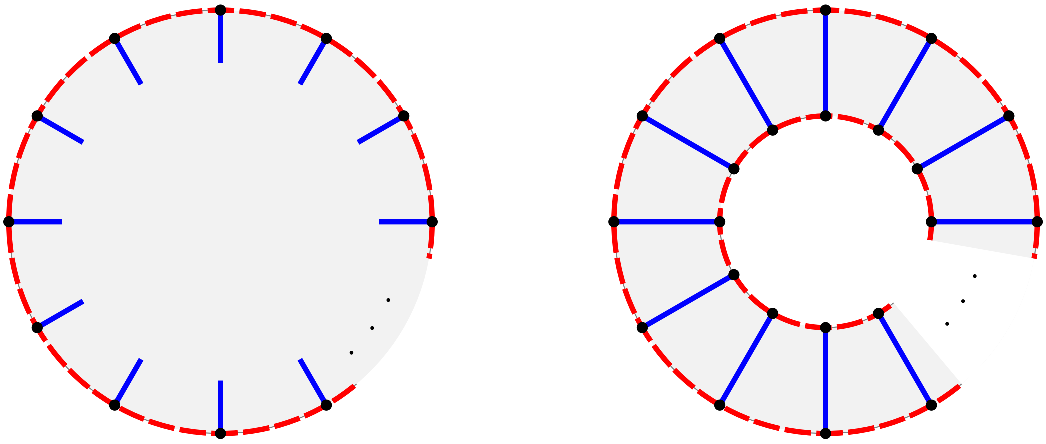

Figure 4.10: Edge-biregular maps with unshaded (red) edges along the surface boundary

some $x \in \{r, \rho\}$ if x_0 is missing then the corresponding coloured edge (and so all such edges) will in fact be a semi-edge to the boundary, while if x_2 is missing then all the edges with the corresponding colour will be along the surface boundary. If, for example, the unshaded edges are on the surface boundary, then ρ_2 is missing and we would denote the edge-biregular map by $M = (H; r_0, r_2, \rho_0, \bar{\rho_2})$.

Considering an edge-biregular map $M = (H; r_0, r_2, \rho_0, \bar{\rho_2})$, like type (a) in Figure 4.7, we have, by regularity of the action of H on corners of the map, all the unshaded edges must be on the boundary of the map. Let us assume, for the moment, that this is the only way in which corner regions meet the surface boundary. When the bold edges are semi-edges, which means that $r_0 = r_2$, the group $H = \langle r_0, \rho_0 \rangle$ is dihedral, and the edge-biregular map has a single face with the supporting surface being a disc. This is illustrated in Figure 4.10, on the left. When the bold edges are not semi-edges then $\ell = 4$ and the underlying graph is a ladder so the surface is homeomorphic to either an annulus or a Möbius strip. These maps can have any number of bold (semi)edges, and generalisations are shown in Figure 4.10, where the orientability of the second surface is dependent on the identification of edges (and the relators in the group presentation) where the dots are.

Notice that the vertices of these edge-biregular maps have odd valency. This is no surprise since having the vertices on the surface boundary destroys any hope of a non-trivial rotation around a vertex being an automorphism of the map.

Thinking about these edge-biregular maps in terms of their group presentations yields something even more interesting. Type (a) has presentation of the following form:

$$H = \langle\, r_0,\ r_2,\ \rho_0 \mid r_0^2,\ r_2^2,\ \rho_0^2,\ (r_0r_2)^2,\ (r_0\rho_0)^{\ell/2},\ \ldots \rangle$$

and we are left with H being *a group which is generated by three involutions, two of which commute.* The group H **is**, in this case, the automorphism group of a fully regular map. While the above presentation of the group H describes an edge-biregular map of type (a), it also resembles the usual presentation for the automorphism group of a fully regular map, with ρ_0 in place of r_1. This indicates a very close relationship between the two corresponding maps which is as follows.

Construction 4.3. Starting with a fully regular map $(G; r_0, r_2, r_1)$ of type (k, ℓ), see [57] for details, choose an arbitrary vertex v and cut out a small disc neighbourhood around v (the disc must be small enough not to include any other vertices nor edges that are not incident to v). Place new vertices where edges meet this new surface boundary, and draw unshaded edges between the new vertices all along this new surface boundary. Repeating this process for all vertices of the original fully regular map will yield the well-defined edge-biregular map $(H; r_0, r_2, \rho_0, \bar{\rho}_2)$ of type $(3, 2\ell)$ with $\rho_0 = r_1$. In the other direction, starting with an edge-biregular map with the unshaded edges along a surface boundary, the related regular map can be built by contracting each unshaded edge down into a single vertex. By letting all the unshaded edges (and hence flags, and indeed boundaries) disappear in the process, the reflections which used to act along the unshaded edges (conjugates of ρ_0) now simply act as the reflections across the resulting new corners (that is, conjugates of r_1) of the implicit fully regular map. ◇

An edge-biregular map with boundary of type (b) is the twin of a map of type (a), while edge-biregular maps with boundary types (c) and (d) also form twin pairs. Figure 4.11 shows some edge-biregular maps of type (c), the oppositely coloured twins of type (d) edge-biregular maps.

If one of the edges of a particular colour, let us say unshaded, meets the boundary, like (d) in Figure 4.7, then this edge, and hence all edges of this type are semi-edges to a boundary of the surface. This is an edge-biregular map $(H; r_0, r_2, \bar{\rho}_0, \rho_2)$. Assuming that a corner region meets the boundary only this way, and the shaded edges are not semi-edges (otherwise we would end up with a semistar), then each face region of the map must have as its boundary: one shaded edge; two unshaded

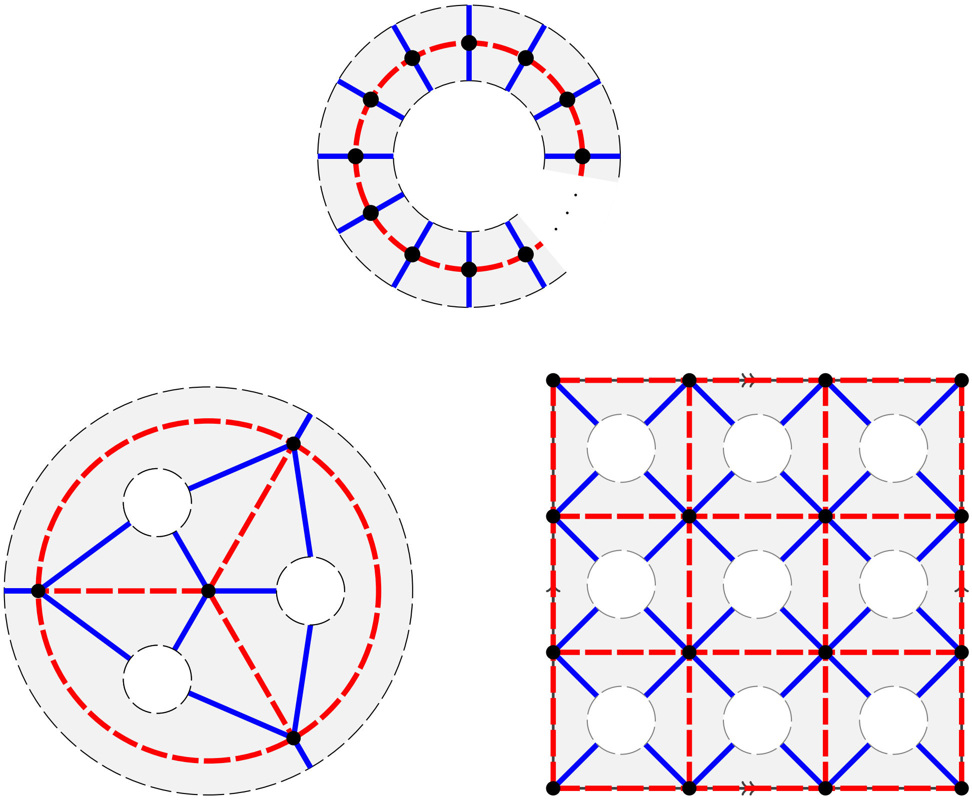

Figure 4.11: Edge-biregular maps of type (c) with bold (blue) semi-edges meeting the surface boundary components

semi-edges; and a section of the surface boundary. The automorphism group for this type (d) of edge-biregular map thus has a presentation of the form:

$$H = \langle\, r_0,\ r_2,\ \rho_2 \ \mid\ r_0^2,\ r_2^2,\ \rho_2^2,\ (r_0 r_2)^2,\ (r_2\rho_2)^{k/2},\ \ldots \rangle$$

and we are again left with H being *a group which is generated by three involutions, two of which commute.* This time we have ρ_2 instead of r_1, and this yields the following construction.

Construction 4.4. Starting from a regular map, $(G; r_0, r_2, r_1)$, drawn with all its edges shaded, we can obtain the well-defined implied edge-biregular map $(H; r_0, r_2, \bar{\rho}_0, \rho_2)$ by cutting a disc out from the interior of each face and drawing an unshaded semi-edge from this new surface boundary to each of the surrounding vertices, thereby letting $r_1 = \rho_2$. Conversely, given an edge-biregular map with unshaded semi-edges to the surface boundary, we can obtain a regular map by deleting all the unshaded semi-edges and contracting each of the boundaries in the surface to ensure the interior of each resulting face is homeomorphic to an open disc. This is valid since the boundary component which a semi-edge meets must be the same as the boundary component which the semi-edge from the adjacent corner within the same face meets. ◇

If a corner meets the boundary in more than one of the ways listed (as in the diagrams in Figures 4.8 and 4.9) then options are severely restricted: there are at most two involutory generators and so the group H is dihedral or cyclic. The interested reader may wish to verify that these include the single faced maps on a disc $H = \langle r_0, \rho_0 \rangle$, the edge-biregular maps on an orientable band $H = \langle r_2, \rho_0 \rangle$, the 4-cornered map on a disc $H = \langle r_0, r_2 \rangle \cong \mathrm{V}_4$ and the semistar on a disc $H = \langle r_2, \rho_2 \rangle$ as well as their twins, and the more trivial maps on a disc which have only one or two corners.

4.3.3 Edge-biregular maps with non-distinct generators on closed surfaces

Having addressed maps with non-empty boundary components in the previous subsection, henceforth we will assume that the supporting surfaces for our edge-biregular maps are closed, that is without boundary.

There is a further natural, if somewhat trivial, way of building an edge-biregular map from any given fully regular map, this time without introducing any boundaries, and that is as follows.

Construction 4.5. Let $(G; r_0, r_2, r_1)$ be a given fully regular map of type (k, ℓ) drawn with all its edges shaded. At every vertex draw k unshaded semi-edges, so that there is one semi-edge in every corner of the original map. This is thus the edge-biregular map $(H; r_0, r_2, \rho, \rho)$ where $\rho = \rho_0 = \rho_2 = r_1$. $\diamond$

Figure 4.12 shows a cube (a fully regular map) drawn with all shaded edges, along with the edge-biregular maps as described in the three constructions above. If you draw the embedded Cayley graphs for each of these maps, the close relationships between this collection of four maps becomes very clear. We could of course include the corresponding twin maps too, as well as the dual maps, and hence each fully regular map has many related edge-biregular maps.

There are other degeneracies arising from non-distinct generators, and we will now address these in turn. Up to duality and twinness we may assume that one of the non-distinct generators is r_0, and so as to avoid bold semi-edges, we assume the other is in $\{\rho_0, \rho_2\}$.

In the case when $r_0 = \rho_0$ we have $H = \langle r_0, r_2, \rho_2 \rangle$, the group for an edge-biregular map which has digonal faces. Maps of type $(k, 2)$ are known to be fully regular, and are supported by either the sphere or the projective plane.

Construction 4.6. The edge-biregular map $(H; \rho, r_2, \rho, \rho_2)$ where $\rho = r_0 = \rho_0$ has an implied regular map $(G; r_0, r_2, r_1)$ where $r_1 = \rho_2$ which is an embedding of the bold

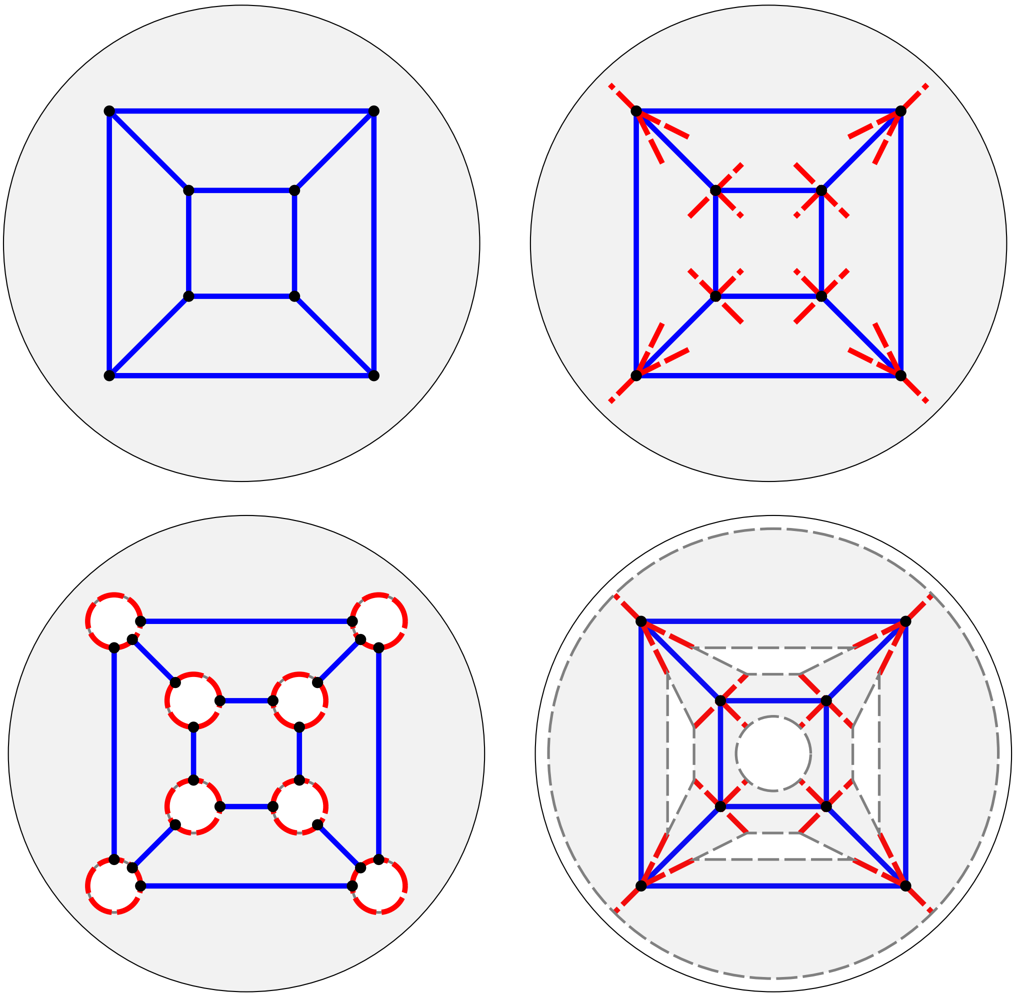

Figure 4.12: A fully regular map and some of its associated edge-biregular maps

edges of the edge-biregular map, with the dashed edges deleted. This implied fully regular map also has digonal faces. In the other direction, a fully regular map with digonal faces can be made into an edge-biregular map by the addition of unshaded edges, each cutting each original digonal face into two alternate-edge-coloured digons. $\diamond$

Remark 4.7. It may be tempting to think that any edge-biregular map $(H; \dots)$ with non-distinct generators, implies a natural regular map $(G; r_0, r_2, r_1)$ with $H \cong G$ and supported by a closely related surface: either the same closed surface; or one in which the non-empty boundary has been eliminated in the natural way by sewing a disc along each boundary component. However, this is not the case as will now become clear. $\star$

In the case where $r_0 = \rho_2 = \rho$ we have the edge-biregular map $(H; \rho, r_2, \rho_0, \rho)$ while the group H is equal to $\langle r_0, r_2, \rho_0 \rangle$. Geometrically $r_0 = \rho_2$ implies that (all) the edges in the bold orbit are loops. Also $r_0 r_2 = \rho_2 r_2$ has order dividing two and hence the vertices all have degree (2 or) 4. Similarly $\rho_0 \rho_2 = \rho_0 r_0$ has order dividing two, and so (assuming no further degeneracies occur) we have an edge-biregular map of type $(4, 4)$. Now, the tessellation of type $(4, 4)$ is Euclidean, indicating that the maps $(H; \rho, r_2, \rho_0, \rho)$ supported by a closed surface must be, depending on orientability, on either the Klein bottle or the torus. *Note: We present a classification of proper edge-biregular maps on the torus and the Klein bottle in Sections 5.1.2.2 and 5.1.2.3.*

Treating r_0 and r_2 in the usual way, we consider the implications of ρ_0 being thought of as r_1, the reflection in a corner of a regular map. Informally this is like reducing (the marked one, and hence all) the dashed edges to negligable length, thereby identifying the endpoints of each dashed edge and stitching the remaining shaded flags together in the correspondingly natural way. The resulting fully regular map would therefore have faces which are digons and the underlying graph would consist of just one vertex. Thus the underlying graph has become a bouquet of loops, **but** this process has also drastically changed the underlying surface, as we might have expected if we remembered that the Klein bottle has no regular map. In the case of the Klein bottle, the resulting surface is the projective plane. However, in the case of a toroidal map, the object created by this process (which is equivalent to contracting a non-contractible cycle) is in fact a pseudo-surface, a sphere with one pair of antipodal points identified, in which case the resulting map is an example of a regular pinched map, see [2] for further details.

This section has introduced two constructions for building edge-biregular maps with non-distinct generators, one of which results in an edge-biregular map which is not proper. In the next chapter, Theorem 5.4 is a classification of all proper edge-biregular maps which have non-distinct generators.

Having introduced the topic in this chapter, along with a notation which makes sense with respect to the context of the given edge-biregular map, from this point onwards we will use the following notation: x, y, s, t respectively denote the involutions r_0, r_2, ρ_0, ρ_2. The rationale for this is simply that it will be easier for both typing and also pronunciation during discussion. Appendix C, may be useful as a dictionary for those who find a particular notation easier to remember.

CHAPTER 5

PROPER EDGE-BIREGULAR MAPS - A PARTIAL CLASSIFICATION

This chapter presents a partial classification of proper edge-biregular maps by considering those with: supporting surfaces of small genus; dihedral colour-preserving automorphism group H; underlying surfaces of negative prime Euler characteristic.

Henceforth, for ease of reference (and typing!) we reiterate that we will be working with the following notation: $x := r_0$; $y := r_2$; $s := \rho_0$; and $t := \rho_2$. Thus, where k and ℓ are both even, we are studying edge-biregular maps of type (k, ℓ) with canonical form $\mathcal{M} = (H; x, y, s, t)$ such that H is a group with the following presentation:

$$H = \langle\, x,\, y,\, s,\, t \,|\, x^2,\, y^2,\, s^2,\, t^2,\, (xy)^2,\, (st)^2,\, (ty)^{k/2},\, (sx)^{\ell/2},\, \dots \rangle \tag{5.1}$$

We also no longer give any consideration to the degenerate cases of edge-biregular maps with boundary, and instead focus entirely on edge-biregular maps embedded on closed surfaces. Moreover, due to their correspondence with fully regular maps, as described by Construction 4.5, in each case we only briefly mention maps which have semi-edges.

5.1 Edge-biregular maps with Euler characteristic $\chi \geq 0$

Cases of small genus need to be addressed separately because they yield maps of different types, by virtue of the Euler-Poincaré formula, see [31, 59] for more background.

5.1.1 The Euler-Poincaré formula

Consider the proper edge-biregular map $\mathcal{M} = (H; x, y, s, t)$ which has type (k, ℓ), and no semi-edges. By regularity, each face has the same length boundary walk ℓ, and each vertex has the same valency k. By the alternate-edge-colouring condition, when $\mathcal{M}$ is embedded on a closed surface, both k and ℓ must be even.

Remark 5.1. We note that xs corresponds to a two-step rotation around the distinguished face, and yt a two-step rotation around the distinguished vertex. Meanwhile, both vertex and face stabilisers are dihedral groups, and the following choice of notation, D_{2n} for a dihedral group of order $2n$, is deliberate so as to avoid fractions within the subscripts. ⋆

The stabiliser in H for the distinguished face is denoted $\mathrm{D}_l := \langle x, s\rangle$ and is isomorphic to the dihedral group of order ℓ. Meanwhile we denote the H-stabiliser for the distinguished vertex $\mathrm{D}_k := \langle y, t\rangle$ and this group is isomorphic to the dihedral group of order k. The map thus has $F = \frac{|H|}{\ell}$ faces and $V = \frac{|H|}{k}$ vertices.

The stabiliser group of an edge is isomorphic to the Klein four-group $\mathrm{V}_4 \cong \mathrm{C}_2 \times \mathrm{C}_2$ and, for the distinguished shaded edge is $\langle x, y\rangle$, while for the distinguished unshaded edge the stabiliser is $\langle s, t\rangle$. Hence the map has $E = \frac{2|H|}{4}$ edges.

Supposing that the map $\mathcal{M} = (H; x, y, s, t)$ lies on a surface of Euler characteristic χ, we apply the well-known Euler-Poincaré formula which is useful when we come to classifying these maps on particular surfaces:

$$\chi = V - E + F = |H|(\frac{1}{k} - \frac{1}{2} + \frac{1}{\ell}) = |H|\frac{2k + 2\ell - k\ell}{2k\ell} = |H|\frac{4 - (k-2)(\ell-2)}{2k\ell} \quad (5.2)$$

Remark 5.2. We note that we treat semi-edges as different objects to edges. In the workings above the formula (5.2), it is clear that both orbits of edges are assumed to contain edges, not semi-edges, and $E = \frac{|H|}{2}$ denotes the number of (full) edges. Therefore in cases where semi-edges are involved, we cannot use Equation (5.2). ⋆

It is useful to note the values of k and ℓ (necessarily positive even integers) which give rise to proper edge-biregular maps of positive, zero and negative Euler characteristic. We state this as a lemma whose proof is immediate from Equation (5.2).

Lemma 5.3. *Let $\mathcal{M}$ be a proper edge-biregular map of type (k, ℓ) on a surface of Euler characteristic χ. Then:*

$\chi \geq 1$ if and only if $(k-2)(\ell-2) < 4$, which is if and only if $2 \in \{k, \ell\}$;

$\chi = 0$ if and only if $(k-2)(\ell-2) = 4$, which is if and only if $k = \ell = 4$;

$\chi \leq -1$ if and only if $(k-2)(\ell-2) > 4$, which is if and only if $k \geq 4$ and $\ell \geq 4$ and $(k, \ell) \neq (4, 4)$.

5.1.2 Edge-biregular maps of non-negative Euler characteristic

In this section, we classify all proper edge-biregular maps on surfaces for which $\chi \in \{0, 1, 2\}$. We note that Duarte's thesis [24] includes a classification of 2-restrictedly-regular edge-bipartite hypermaps, that includes all *proper* edge-biregular maps, for the sphere, the projective plane and the torus. Here we do allow for the possibility of semi-edges, and we present a different approach for Euclidean edge-biregular maps. While Duarte's work allows for many different group presentations which describe the same (isomorphism class of) Euclidean map, the approach presented here standardises the canonical presentation for such an edge-biregular map, and extends to include a classification for such maps on the Klein bottle.

We first note the following theorem which allows us to assume distinctness of generators for much of the rest of this chapter.

Theorem 5.4. *If the proper edge-biregular map $\mathcal{M} = (H; x, y, s, t)$ has non-distinct generators, then it is supported by a surface which has non-negative Euler characteristic. The supporting surfaces, and, up to duality, the corresponding maps and their groups H are listed below:*

1. *Sphere: dipoles of even degree with $H = \langle y, t\rangle \times \langle s\rangle = D_k \times \langle s\rangle$ and $x = s$.*
2. *Projective plane: single vertex maps, the degree k being a multiple of 4, with $H = \langle y, t\rangle = D_k$ and $x = s$.*
3. *Torus: Maps of type $(4, 4)$ with n quadrangular faces and (at least) one edge orbit consisting of loops such that $H = \langle y, s\rangle \times \langle x\rangle \cong D_{2n} \times C_2$ and $x = t$.*
4. *Klein bottle: Maps of type $(4, 4)$ with $n/2$ quadrangular faces and (at least) one edge orbit consisting of loops such that $H = \langle y, s\rangle \cong D_{2n}$ and $x = t$.*

Proof. Suppose we have a proper edge-biregular map $\mathcal{M} = (H; x, y, s, t)$ such that the generators x, y, s, t are not distinct. The map is assumed to be proper, so there are no semi-edges which forces $x \neq y$ and $s \neq t$. Remember that a map and its dual are both supported by the same underlying surface. Thus we may work up to duality, and so we may assume that one of the redundant generators is labelled x. This leaves two options, $x = s$, or $x = t$, which we will address in turn.

First suppose $x = s$. Then the element xs is the identity which means $\ell = 2$ and the faces of the map are digons. Also note that $H = \langle x, y, s, t\rangle = \langle y, s, t\rangle$ and we have

$[s, y] = 1$ as well as $[s, t] = 1$ so s is central. Hence either $H = \mathrm{D}_k$ or $H = \mathrm{D}_k \times \langle s \rangle$. Applying the Euler-Poincaré formula we see that these two cases correspond respectively to a single-vertex degree-k map on the projective plane (remembering $4|k$), and a degree-k dipole embedded in the sphere (for any even k).

Now suppose $x = t$. Now, $\langle x, y \rangle$ is the H-stabiliser for distinguished bold edge and $\langle t, y \rangle$ is the H-stabiliser for the distinguished vertex, and when $x = t$ these two stabilisers are the same. This tells us that the distinguished bold edge is a loop, and so by regularity all bold edges are loops. Also we have $[s, t] = [s, x] = 1$ so $\ell = 4$ and $[x, y] = [t, y] = 1$ so $k = 4$ and hence the map type is $(4, 4)$. This implies we are on a surface with Euler characteristic $\chi = 0$. If the surface is orientable then the map is on the torus, otherwise the map is supported by the Klein bottle. Going further, in the case where $x = t$, notice that x is central, $H = \langle x, y, s \rangle$, and also that $\langle y, s \rangle$ is a dihedral group, with order, say, $2n$. Now, if $x \in \langle y, s \rangle$ then $H = \langle y, s \rangle \cong \mathrm{D}_{2n}$ where n is even. Also, being central, $x = (ys)^{n/2}$ which yields a relator of odd length, forcing the supporting surface to be non-orientable, the Klein bottle. If, on the other hand, $x \notin \langle y, s \rangle$ then $H = \langle y, s \rangle \times \langle x \rangle \cong \mathrm{D}_{2n} \times \mathrm{C}_2$ and, as the direct product of a dihedral group (with the presentation generated by two involutions) and a copy of C_2, there can be no relators of odd length. These maps are therefore supported by an orientable surface of Euler characteristic zero, namely the torus. $\square$

The workings in the above proof have shown that when H has non-distinct generators x, y, s, t then the closed supporting surface for the edge-biregular map $\mathcal{M} = (H; x, y, s, t)$ has non-negative Euler characteristic $\chi \in \{0, 1, 2\}$. This yields the following corollary on which we will rely later.

Corollary 5.5. *A proper edge-biregular map $\mathcal{M} = (H; x, y, s, t)$, embedded on a surface where $\chi < 0$, has distinct generators x, y, s, t.*

We now complete the classification for surfaces of Euler characteristic $\chi \geq 0$.

5.1.2.1 Edge-biregular maps on the sphere and the projective plane

For completeness, we mention also the edge-biregular maps on these surfaces which have at least one orbit of semi-edges.

In the case where both edge orbits consist of semi-edges, we have $x = y$ and $s = t$ and the underlying graph, which has only one vertex, is called a semi-star. Hence the colour-preserving automorphism group $H = \langle x, s \rangle$ is the (colour-preserving) vertex stabiliser $H = \mathrm{D}_k$. Also we have $H = \langle y, t \rangle = \mathrm{D}_\ell$. The map has $V = 1$ and $F = 1$ while the semi-edges make no contribution to the Euler characteristic $\chi = 2$ of the supporting surface. Thus the semi-star of even valency must be embedded in the sphere to form an edge-biregular map.

An edge-biregular map with exactly one semi-edge orbit must come from a fully regular map by Construction 4.5. It is well-known that the sphere supports fully regular maps of type $(3,3)$, $(3,4)$, $(4,3)$, $(3,5)$, $(5,3)$, $(2,m)$ and $(m,2)$ and so, by attaching a semi-edge at each corner, there are edge-biregular maps on the sphere of types $(6,6)$, $(6,8)$, $(8,6)$, $(6,10)$, $(10,6)$, $(4,2m)$ and $(2m,4)$. Similarly the projective plane supports fully regular maps of type $(3,4)$, $(4,3)$, $(3,5)$, $(5,3)$, $(2,m)$ and $(m,2)$ so on this surface there will be edge-biregular maps of types $(6,8)$, $(8,6)$, $(6,10)$, $(10,6)$, $(4,2m)$ and $(2m,4)$.

Finally, if an edge-biregular map contains no semi-edges and the supporting surface is closed, then both the valency and the face length must be even. The Euler-Poincaré formula and Lemma 5.3 imply that on a surface where $\chi \geq 1$, that is on a sphere or a projective plane, a proper edge-biregular map must have type $(2,2m)$ or $(2m,2)$. These types correspond to embedded even cycles and their duals respectively. However, in order to have a valency or face length being two, the H-stabiliser for the distinguished vertex or face must be isomorphic to C_2 (not a dihedral group) meaning the map must have non-distinct generators, respectively $y = t$ or $x = s$. Thus these maps, (sometimes with an extra restriction on the value of m) are included in Theorem 5.4.

5.1.2.2 Edge-biregular maps on the torus

As before, for completeness, we briefly mention edge-biregular maps with semi-edges. Each fully regular map on the torus will have an associated edge-biregular map created by inserting a semi-edge into each corner. Thus there will be edge-biregular maps of types $(12,6)$, $(6,12)$ and $(8,8)$ built using Construction 4.5 respectively from the well-known toroidal regular maps of types $(6,3)$, $(3,6)$ and $(4,4)$.

Turning our attention to proper edge-biregular maps supported by the torus, the requirement for even valency and face length makes it clear that they must all have type $(4,4)$. The infinite group

$$\Delta \;=\; \langle\, X, Y, S, T \mid X^2, Y^2, S^2, T^2, (XY)^2, (ST)^2, (XS)^2, (YT)^2 \,\rangle$$

describes the colour-preserving automorphism group of the alternate-edge-coloured

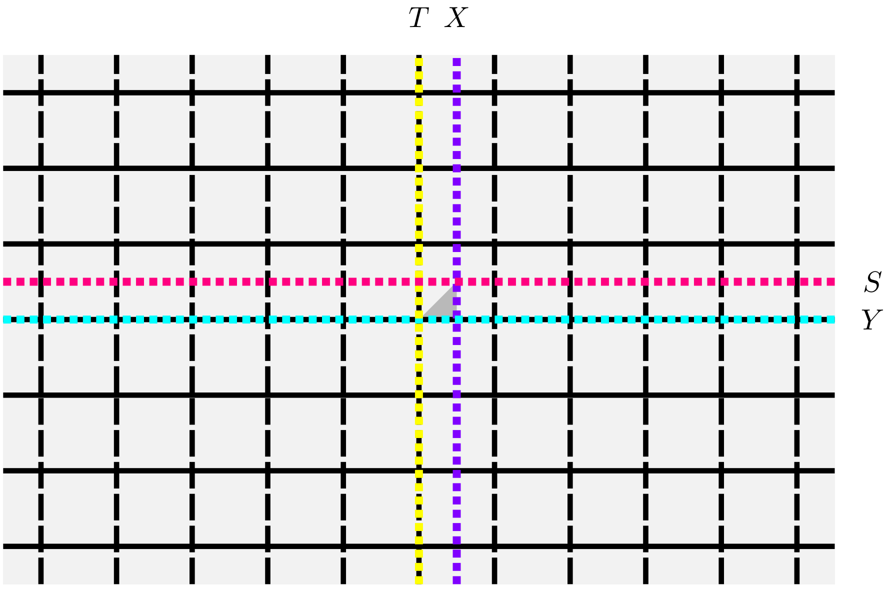

Figure 5.1: Part of the infinite Euclidean edge-biregular map of type $(4, 4)$, with the axes of reflection for X, Y, S, T shown.

infinite square grid, a tessellation of type $(4, 4)$ on the Euclidean plane.

Figure 5.1 shows the line of reflection corresponding to each of the generators of Δ. Notice that the X and T are both reflections with parallel (vertical) axes of reflection, so the composition XT is orientation preserving and corresponds to a (one-step horizontal) translation. Similarly YS will be a vertical translation. In particular the directions for these two translations are independent of each other.

Following the work in the seminal paper by Jones and Singerman [42], the colour-preserving automorphism group H for any given finite proper edge-biregular map of type $(4, 4)$ arises as a smooth quotient of Δ by a torsion-free normal subgroup N of finite index giving $H \cong \Delta/N$.

Using an analogue of the example found in Section 7 of [42], to assist in identifying subgroups of finite index in Δ, we now define $L := \langle XT, YS\rangle \leq \Delta$, noting that the two generators, each of which has infinite order, commute and so L is itself a free Abelian group of rank two, $L \cong \mathbb{Z} \times \mathbb{Z}$. Conjugation by any one of the elements X, Y, S, or T fixes the subgroup L so it is easy to see that L is normal in Δ. For example X inverts XT and fixes YS, while conjugation by Y fixes XT and inverts YS. Meanwhile the distinct reflections X and Y cannot be elements of the group L since the generators of L are orientation-preserving. This yields $\Delta = L \rtimes \langle X, Y\rangle \cong (\mathbb{Z} \times \mathbb{Z}) \rtimes \mathrm{V}_4$.

This presentation of Δ makes it clear that any rank two (torsion-free) subgroup of L will be of finite index in Δ. Geometrically speaking, the group L describes all the translations which map the aforementioned infinite square grid to itself. It will be helpful later to have the following visualisation in mind: if we mark an arbitrary vertex of the infinite grid, any subgroup of L generated by a given pair of independent translations creates a 2-dimensional lattice of equivalent vertices, all of which correspond to a single vertex on the associated toroidal map. It is well-known [21, 59] that two linearly independent translations of the plane determine a fundamental region for a torus. Drawing this all together, we have that N_L, a given rank two subgroup of L, gives rise to a proper edge-biregular map on a torus if and only if $N_L \trianglelefteq \Delta$.

Although there may be many equivalent presentations of the group H, each of which describes a particular given edge-biregular map on a torus, the following proposition seeks to identify a uniquely defined presentation corresponding to each (isomorphism class of) toroidal edge-biregular map(s).

Proposition 5.6. *Every proper toroidal edge-biregular map has type* $(4, 4)$ *and is determined by the group* H_{Rect} *or* H_{Rhomb} *with one of the two following presentations:*

$$H_{Rect} = \langle x, y, s, t \mid x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (xs)^2, (yt)^2, (xt)^a, (ys)^c \rangle$$

$$H_{Rhomb} = \langle x, y, s, t \mid x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (xs)^2, (yt)^2, (xt)^{2b}, (xt)^b(ys)^c \rangle$$

where a, b *and* c *are positive integers.*

A toroidal edge-biregular map is fully regular if and only if one of the following is the case: $H = H_{Rect}$ *and* $a = c$; *or* $H = H_{Rhomb}$ *and* $b = c$.

Proof. Suppose $\mathcal{M} = (H; x, y, s, t)$ is a given proper edge-biregular map whose supporting surface is a torus. Thus H is necessarily finite, corresponding to a map of type $(4, 4)$, and H will have canonical partial presentation:

$$H \;=\; \langle\, x, y, s, t \mid x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (xs)^2, (yt)^2, \ldots \,\rangle$$

where the dots indicate some unknown extra relators which are yet to be determined. These necessary extra relators arise from N_L, the kernel of the epimorphism $\phi : \Delta \to H$ where $\phi : X \to x$ and $\phi : Y \to y$ and $\phi : S \to s$ and $\phi : T \to t$. Below we determine all possibilities for the group N_L.

Since a map on the torus must be finite, each element of H must have finite order. Specifically xt must have finite order in H, let us say a. Thus $(xt)^a$ is a relator in the group H and since a is, by definition, as small as possible, we may force $(xt)^a$ to be one of the two uniquely defined relators which are missing in the above partial presentation. Hence $(XT)^a \in N_L$ and, by the minimality of a, we may assume it is one of a pair of generating elements in the rank two subgroup $N_L \leq L$. Geometric reasoning implies there must be another (independent, and also uniquely determined) translation generating N_L which has the form $(XT)^b(YS)^c$ where the integer $c > 0$ is minimised and then b such that $0 \leq b < a$ is also chosen to be as small as possible.

The group $N_L = \langle (XT)^a, (XT)^b(YS)^c \rangle$ is normal in the group Δ if and only if all conjugates of each of the generators of N_L are themselves members of the subgroup N_L. It is clear that this is the case for the generator $(XT)^a$. It remains to consider conjugates of the element $(XT)^b(YS)^c$ in Δ. We split this into two distinct cases. When $b = 0$, all conjugates of $(XT)^b(YS)^c = (YS)^c$ are also within N_L, the lattice of equivalent points is rectangular, and $N_L = \langle (XT)^a, (YS)^c \rangle$. This yields H_{Rect} in the statement of the Proposition. However, in the case when $b \neq 0$ then N_L is normal if and only if we have both $((XT)^b(YS)^c)^X = (XT)^{-b}(YS)^c \in N_L$ or equivalently $(XT)^{2b} \in N_L$, and also $(YS)^{2c} \in N_L$. This implies $a|2b$ but $0 < b < a$ so we must have $2b = a$, meaning that the lattice of points, which is now equivalently generated by $N_L = \langle (XT)^b(YS)^c, (XT)^b(YS)^{-c} \rangle$, is rhombic. This gives rise to the toroidal edge-biregular maps with presentations of the form H_{Rhomb}.

By construction, each of the above presentations H_{Rect} and H_{Rhomb} is uniquely determined from a given edge-biregular toroidal map which has an assigned alternate-edge-colouring, and there can be no other toroidal proper edge-biregular maps.

These maps are fully regular if and only if the lattice is in fact a square lattice, that is, in the case of H_{Rect} when $a = c$, and in the case of H_{Rhomb} when $b = c$. Informally, this can be observed by looking at Figure 5.2 and noting the need for all closed straight-ahead walks to have the same length in a fully regular map. More formally, the map is fully regular if and only if there is an automorphism of the group H which interchanges x with s and y with t. Inspection of the presentations of the groups quickly yields the same necessary and sufficient condition.

$\square$

The reason for the use of the descriptors “rectangular” and “rhombic” is illustrated with an example of each of the corresponding maps shown in the diagrams in Figure 5.2.

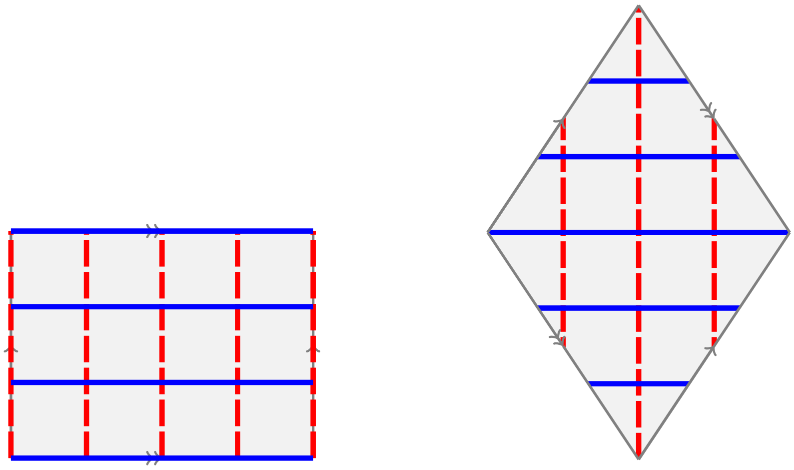

Figure 5.2: Toroidal edge-biregular maps when $a = 4, c = 3$: The lattice is rectangular, as shown on the left, when $b = 0$, and rhombic when $b = 2$.

5.1.2.3 Edge-biregular maps on the Klein bottle

It is known that the Klein bottle supports no regular maps [57], so the correspondence implied by Construction 4.5 means that an edge-biregular map on this surface cannot have any semi-edges.

Since the Klein bottle is also Euclidean, an edge-biregular map on this surface is determined by forming the quotient of the infinite group Δ by a finite-index normal subgroup which in this case we denote N_K.

Proposition 5.7. *Up to duality and twinness, edge-biregular maps on the Klein bottle have type* $(4,4)$ *and are in one-to-one correspondence with groups* H *with presentation:*

$$H_{Klein} = \langle x, y, s, t \mid x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (xs)^2, (yt)^2, (ys)^a x, (xt)^b \rangle$$

where a *is a positive integer and* $b \in \{1, 2\}$.

Outline of proof. The fundamental region for a Klein bottle is determined by a glide reflection and a translation in the direction perpendicular to the axis of reflection. See Coxeter and Moser [21] pages 40-43, or for further details of all uniform maps on this surface, see Wilson [66]. To result in an edge-biregular map, the grid must be mapped to itself by the glide reflection within N_K, maintaining the same colouring of edges. Thus the axis of reflection must be orthogonal to the square grid and so, up to duality and twinness, we may assume the glide reflection is $(YS)^a X$ and the translation is $(XT)^b$ where a and b are positive integers and as small as possible.

Suppose $N_K = \langle (YS)^a X, (XT)^b \rangle$ has finite index and is normal in Δ. When this is

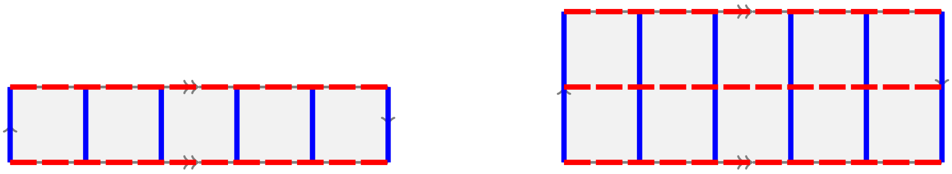

Figure 5.3: Edge-biregular maps on the Klein bottle for $a = 5$ where $b = 1$ and $b = 2$ respectively.

the case, the quotient of Δ by N_K will define an edge-biregular map on the Klein bottle. Conjugates of the translation $(XT)^b$ by elements of Δ are certainly in N_K whereas $((YS)^a X)^T = (YS)^a TXT \in N_K$ if and only if $(XT)^2 \in N_K$, that is, if and only if $b|2$. By an analogue of the proof of Proposition 5.6, for a given edge-biregular map on a Klein bottle, this yields the uniquely determined presentation for H as given in the above Proposition, with the condition that $b \in 1, 2$. □

Examples of these edge-biregular maps which are supported by the Klein bottle are shown in Figure 5.3.

In this section we have now classified all proper edge-biregular maps which are supported by surfaces of non-negative Euler characteristic. Next we turn our attention to those which have dihedral colour-preserving automorphism group, some of which have already made an appearance in this section.

5.2 Edge-biregular maps where $H \cong \mathrm{D}_{2m}$: a dihedral classification

Dihedral groups can be generated by two involutions, so it seems natural that for some edge-biregular map, the colour-preserving automorphism group H will itself be dihedral. Indeed some of the maps already discussed have dihedral H, and these are included in Table 5.1. The following theorem gives a classification, up to twinness and duality, of proper edge-biregular maps on closed surfaces when the group $H = \langle x, y, s, t\rangle \cong \mathrm{D}_{2m}$.

Theorem 5.8. *Let $\mathcal{M}(H; x, y, s, t)$ be a proper edge-biregular map such that $H \cong D_{2m}$, a dihedral group. Then m is even and, up to twinness and duality, the map is one of those in Table 5.1.*

Proof. We recall that earlier work has already resulted in a classification of proper edge-biregular maps on surfaces where $\chi \geq 0$. We note that dihedral edge-biregular maps on the sphere, projective plane, torus and Klein bottle also exist, and they are found directly in Theorem 5.4, with the Euclidean types also included in the classifications in 5.1.2.2 and 5.1.2.3. For completeness, the dihedral examples have been included in Table 5.1, while the associated reasoning, which follows easily from the earlier work, is omitted from this proof. In the remaining analysis we may assume $\chi < 0$.

Let the group $H = \langle x, y, s, t\rangle \cong \mathrm{D}_{2m}$ be the canonical presentation of the colour-preserving automorphism group for a proper edge-biregular map on a surface for which $\chi < 0$. Since we know $\mathrm{V}_4 \cong \langle x, y\rangle \leq H$ (because there are no semi-edges) it is clear that m is even. Therefore $H \cong \mathrm{D}_{2m}$ has a central involution, which we will call z. We also know that $\mathrm{V}_4 \cong \langle s, t\rangle \leq H$. Another assumption we can now make is that the generators of H, namely x, y, s, and t, are distinct by Corollary 5.5. For a dihedral group D_{2m} to have at least four distinct involutions, we must have $m \geq 4$, which forces the group H to be non-Abelian. Also, note that every copy of V_4 as a proper subgroup in a dihedral group contains the unique central involution z. Taking account of the generators being distinct, this means we must have $\mathsf{z} \in \{xy, st\}$.

By our (arbitrary) choice of colouring for the edge orbits, that is up to twinness, we may assume $\mathsf{z} = xy$. We also have $\mathsf{z} \in \{s, t, st\}$ giving options which we will address in turn.

1. Suppose $\mathsf{z} = st$.

   In this case $xy = st$ and so $sx = ty$ which forces $k = \ell$. Also $t = sxy$ so $H = \langle s, x, y\rangle = \langle s, x, xy\rangle$. But we know that $\langle s, x\rangle = \mathrm{D}_\ell$ and so this leaves us with two possibilities:

   Firstly, if $\mathsf{z} = xy \in \mathrm{D}_\ell$ then $H = \mathrm{D}_\ell = \langle s, x\rangle \cong \mathrm{D}_{2m}$ and the type of the edge-biregular map is $(2m, 2m)$. Notice, this map has just one face and one vertex, and is supported by a surface with Euler characteristic $\chi = 2 - m$. By the fact the map has a single vertex, we also have $H = \langle y, t\rangle$;

   Second, if $\mathsf{z} = xy \notin \mathrm{D}_\ell$ then, since z is central in H, we get $H \cong \mathrm{D}_\ell \times \langle \mathsf{z}\rangle \cong \mathrm{D}_{2m}$. By the uniqueness of the central involutory element, this can only happen if D_ℓ has trivial centre, that is if the order of sx is odd, which happens when $\frac{\ell}{2} = \frac{m}{2}$ is odd. By considering the order of $(sx)\mathsf{z}$, which we now know to be $\ell = m$, we see that $H = \langle s, x\mathsf{z}\rangle = \langle s, y\rangle$. We also get, by remembering $t = s\mathsf{z}$, that $H = \langle x, t\rangle$. These presentations yield a map of type (m, m) which thus has two faces and two vertices, supported by a surface where $\chi = 4 - m$.

2. Now let us suppose that $\mathsf{z} \in \{s, t\}$. Up to duality we may assume that $\mathsf{z} = s$.

   In this case we have $xy = s$ so $xs = y$ which is an involution. Thus $\ell = 4$. Also we get $H = \langle t, y, xy\rangle$. By similar reasoning to above this gives us two possibilities:

Firstly, if $\mathsf{z} \in \mathrm{D}_k$ we have $H = \mathrm{D}_k = \langle t, y \rangle \cong \mathrm{D}_{2m}$, the map has a single vertex with quadrangular faces, it is of type $(2m, 4)$ and exists on a surface where $\chi = \frac{1}{2}(2 - m)$;

Second, by the centrality of $\mathsf{z} = xy \notin \mathrm{D}_k$ we have $H = \mathrm{D}_k \times \langle \mathsf{z} \rangle = \langle y, t \rangle \times \langle \mathsf{z} \rangle$. Also, by our assumption, $H \cong \mathrm{D}_{2m}$ and so the order of yt must be odd and $H = \langle \mathsf{z}y, t \rangle = \langle x, t \rangle$. The map is of type $(m, 4)$ where $\frac{m}{2}$ is odd, and thus the map has two vertices. This map occurs on a surface with $\chi = \frac{1}{2}(4 - m)$.

Note that these (non-orientable) maps are not regular since regularity would require an automorphism of the group H which swaps s, which is central in H, with x which is not central.

These results can be tabulated to give a classification, up to duality and twinness, of edge-biregular maps on closed surfaces when the group H is dihedral, $H = \langle x, y, s, t \rangle \cong \mathrm{D}_{2m}$ and m is necessarily even. □

Remark 5.9. It is worth noting that this classification ensures that there is no closed surface which does not support edge-biregular maps. Specifically, all non-orientable surfaces with negative Euler characteristics support an edge-biregular map of type $(4(1 - \chi), 4)$, as well as its dual map of type $(4, 4(1 - \chi))$, and their twins, while the orientable surface with negative Euler characteristic χ supports an edge-biregular map of type $(2(2 - \chi), 2(2 - \chi))$. ⋆

Remark 5.10. By duality, the classification of dihedral edge-biregular maps where $H \cong \mathrm{D}_{2m}$ contains a non-Euclidean map of type $(4, 2\lambda)$ which is truly edge-biregular so long as $\lambda \geq 4$ is even. Meanwhile, (the dual of) the other type of non-Euclidean truly edge-biregular map in the dihedral classification table is of type $(4, m)$ where $m \geq 4$ and $m \equiv 2 \pmod 4$. Thus this yields truly edge-biregular maps of type $(4, 2\lambda)$ where $\lambda \geq 3$ is odd. Also, the Klein bottle provides examples of truly edge-biregular maps of type $(4, 4)$. Taken together, these facts demonstrate that truly edge-biregular maps exist for every type $(4, 2\lambda)$, where $\lambda \geq 2$, and so by duality also for every type $(2\kappa, 4)$ where $\kappa \geq 2$. ⋆

Type	$H \cong \mathrm{D}_{2m}$	Central element, z	Conditions	Surface	V	F	Regular	Notes
$(m, 2)$	$\langle y, t\rangle \times \langle s\rangle$	$x = s$	$m \equiv 2 \pmod 4$	Sphere	2	m	Yes	Dipole
$(2m, 2)$	$\langle y, t\rangle$	$x = s = (yt)^{\frac{m}{2}}$	m is even	Projective plane	1	m	Yes	Embedded bouquet
$(4, 4)$	$\langle y, s\rangle \times \langle x\rangle$	$x = t$	$m \equiv 2 \pmod 4$	Torus	$\frac{m}{2}$	$\frac{m}{2}$	Yes $\iff m = 2$	H_{Rect} with $a = 1$ and $c = \frac{m}{2}$
$(4, 4)$	$\langle y, s\rangle$	$x = t = (ys)^{\frac{m}{2}}$	m is even	Klein bottle	$\frac{m}{2}$	$\frac{m}{2}$	No	H_{Klein} with $a = \frac{m}{2}$ and $b = 1$
$(2m, 2m)$	$\langle x, s\rangle$ $\langle y, t\rangle$	$xy = st = (sx)^{\frac{m}{2}} = (yt)^{\frac{m}{2}}$	m is even	$\chi = 2 - m < 0$ Orientable	1	1	Yes	
(m, m)	$\langle y, s\rangle$ $\langle x, t\rangle$	$xy = st = (ys)^{\frac{m}{2}} = (xt)^{\frac{m}{2}}$	$m \equiv 2 \pmod 4$	$\chi = 4 - m < 0$ Orientable	2	2	Yes	
$(2m, 4)$	$\langle y, t\rangle$	$xy = s = (yt)^{\frac{m}{2}}$	m is even	$\chi = \frac{1}{2}(2 - m) < 0$ Non-orientable	1	$\frac{m}{2}$	No	
$(m, 4)$	$\langle x, t\rangle$	$xy = s = (xt)^{\frac{m}{2}}$	$m \equiv 2 \pmod 4$	$\chi = \frac{1}{2}(4 - m) < 0$ Non-orientable	2	$\frac{m}{2}$	No	

Table 5.1: Classification of proper edge-biregular maps on closed surfaces such that the colour-preserving automorphism group $H \cong \mathrm{D}_{2m}$.

5.3 Edge-biregular maps with negative prime Euler characteristic

In this section we state our classification results for edge-biregular maps on surfaces of negative prime Euler characteristic. This is joint work with my supervisor, Jozef Širáň, and has been published in [53]. For completeness I have included the full statement of Theorem 5.11, and I attach, Appendix A, the part of the proof which I credit as predominantly the work of my co-author.

We summarise our starting point by recalling, from the previous chapter, some facts about edge-biregular maps, using a slightly different (albeit equivalent) notation and reference system, which suits our purposes for both this section and for Appendix A.

5.3.1 A recap: The context and set-up using the new notation

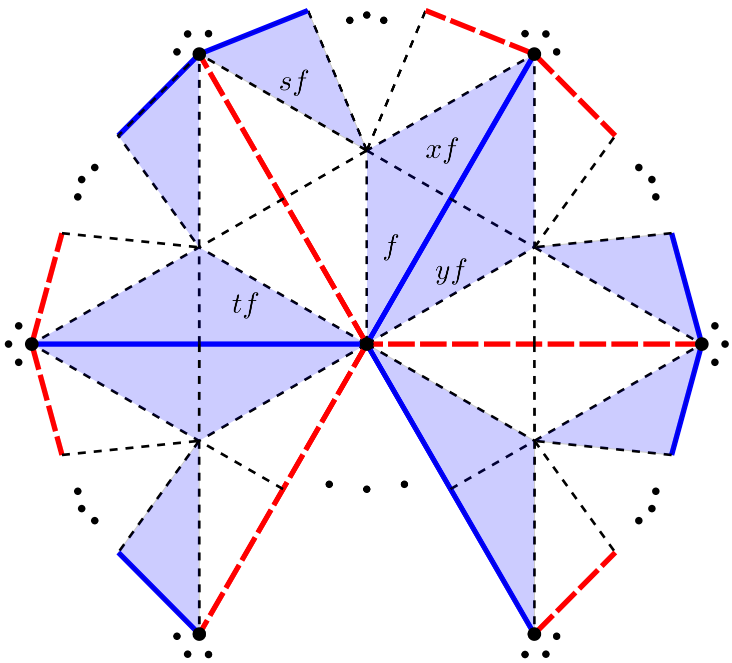

Figure 5.4: The images under x, y, s, t of the distinguished flag f of an edge-biregular map

Informally, we have a 'reference system' within the map: on the diagram one may fix and shade an arbitrarily chosen flag f, let x and y represent the reflections of H in the sides of f as depicted, and call the quadruple of flags f, xf, yf and xyf surrounding an edge *distinguished.* The pattern of the shaded flags in this diagram demonstrate how the group of automorphisms H of an edge-biregular map 'spreads around' the distinguished quadruple and hence acts transitively on faces and vertices. At the same time, Figure 5.4 displays two orbits of H on edges, those which are bold in one orbit, with dashed edges indicating the other orbit; notice also the two H-orbits on flags formed by quadruples of shaded and unshaded flags surrounding respectively bold and dashed edges.

Remember that it is a choice, consistent with the choice of notation, which orbit of edges is coloured bold, and which is dashed. As shown in Figure 5.4, the generators x and y correspond to automorphisms acting locally as reflections respectively along and across the distinguished bold edge. Meanwhile s and t correspond to reflections respectively along and across the distinguished dashed edge, that is the dashed edge which is surrounded by the quadruple of unshaded flags sharing a boundary with f. We call this, and the corresponding presentation (5.1), the *canonical form* of the edge-biregular map $\mathcal{M}$ and denote it $\mathcal{M} = (H; x, y, s, t)$. A pair of edge-biregular maps $\mathcal{M} = (H; x, y, s, t)$ and $\mathcal{M}' = (H'; x', y', s', t')$ given in the canonical form are *isomorphic* if there is a group isomorphism $H \to H'$ taking z onto z' for every $z \in \{x, y, s, t\}$.

Whether an edge-biregular map $\mathcal{M} = (H; x, y, s, t)$ is also fully regular depends on the existence (or otherwise) of an involutory automorphism of $\mathcal{M}$ lying outside H and fusing the two H-orbits of edges together. Recall from section 4.2.3 that this condition is equivalent to the existence of an automorphism of the *group* H which interchanges x with s and y with t. The map $\mathcal{M} = (H; x, y, s, t)$ is thus fully regular if and only if $\mathcal{M}$ and $\mathcal{M}' = (H; s, t, x, y)$ are isomorphic as maps, otherwise we say these maps are *twins*.

To avoid unnecessary work, all our forthcoming results will be up to duality. The dual map $\mathcal{M}^*$ of an edge-biregular map $\mathcal{M} = (H; x, y, s, t)$ is also an edge-biregular map and is formed by interchanging x with y and s with t in the presentation (5.1) for H, thereby swapping the vertices with the faces and vice versa, to give $\mathcal{M}^* = (H; y, x, t, s)$. If $\mathcal{M}$ is isomorphic to $\mathcal{M}^*$ the map is *self-dual*, which (by the map isomorphism condition) is equivalent to the existence of an automorphism of the *group* H swapping x with y and s with t.

Except for the case of characteristic -2, all our edge-biregular maps $\mathcal{M}$ in this section will be carried by non-orientable surfaces. Since each canonical generator of H corresponds to a reflection on the supporting surface, it follows that $\mathcal{M}$ is supported by an orientable surface if and only if every relator in the presentation of H has an even length in terms of x, y, s, t. This is equivalent to the statement that the carrier surface of $\mathcal{M}$ is non-orientable if and only if H is generated by any three products, of two involutions each, provided that every involution out of x, y, s, t appears in at least one of the three products.

As a result of Corollary 5.5, in this section we may assume that each of our edge-biregular maps $\mathcal{M} = (H; x, y, s, t)$ arises as a torsion-free normal quotient of the index-two subgroup $\langle R_0, R_2, R_0^{R_1}, R_2^{R_1} \rangle$ of the corresponding type of full triangle group. In particular, all the four canonical involutory generators of H are *distinct*. For completeness we address also the situations when some of the generators in the set $\{x, y, s, t\}$ are either trivial or equal to each other. These are commonly known as 'degeneracies' and have been treated in detail in Section 4.3 and Theorem 5.4; they give rise to maps on surfaces with boundary or to maps with semi-edges or to maps of spherical type with $2 \in \{k, \ell\}$ or to maps of Euclidean type with an edge orbit consisting of loops. It is clear the these degeneracies do not apply to edge-biregular maps on surfaces of negative Euler characteristic. Recall that a semi-edge has one of its endpoints attached to a vertex but the other one is not incident to any vertex; we will regard edges and semi-edges as different objects.

While maps on surfaces with boundary were dealt with in Section 4.3, edge-biregular maps with semi-edges are relevant for this classification. Leaving the trivial case of a semi-star (a spherical one-vertex map with some number of attached semi-edges) aside, suppose that an edge-biregular map of type (k, ℓ) with both entries even contains a semi-edge as well as an edge. By definition, the edges and semi-edges must then alternate around each vertex on the carrier surface of the map, and consequently the deletion of all semi-edges results in a fully regular map of type $(k/2, \ell/2)$. Conversely, every edge-biregular map containing both an edge and a semi-edge arises from a fully regular map that is not a semi-star by inserting a semi-edge into each corner.

The algebraic explanation of this edge/semi-edge phenomenon in edge-biregular maps, which corresponds to the Construction 4.5, is as follows. Consider such a map $\mathcal{M} = (H; x, y, s, t)$ in a canonical form and assume that all the dashed edges in Figure 5.4 would collapse to semi-edges. This is equivalent to identifying the flags marked sf and tf into a single flag, which means regarding s and t as identical automorphisms. It now follows from Equation (5.1) that, up to twinness, an edge-biregular map $\mathcal{M} = (H; x, y, s, t)$ of type (k, ℓ) for even k and ℓ, containing both edges and semi-edges, may be identified with its group of automorphisms H presented in the form

$$H = \langle\, x,\, y,\, s,\, t \,|\, x^2,\, y^2,\, s^2,\, t^2,\, (xy)^2,\, st,\, (ty)^{k/2},\, (sx)^{\ell/2},\, \ldots \rangle \tag{5.3}$$

Note the difference from Equation (5.1) in the power at the product st, implying $s = t$ in Equation (5.3). Conversely, let a fully regular map $\mathcal{M}$ of type $(k/2, \ell/2)$ for k, ℓ even and distinct from a semi-star be given by its (full) automorphism group, that is, by a torsion-free normal quotient of the group in Equation (4.2) presented in the form

$$\langle\, r_0,\, r_1,\, r_2 \,|\, r_0^2,\, r_1^2,\, r_2^2,\, (r_0r_2)^2,\; (r_1r_2)^{k/2},\, (r_0r_1)^{\ell/2}\; \ldots \rangle \tag{5.4}$$

Then, a presentation of the group H as in Equation (5.3) for the corresponding edge-biregular map of type (k, ℓ) arising from $\mathcal{M}$ by inserting a semi-edge into every corner of $\mathcal{M}$ is obtained from the one in (5.4) simply by letting $x = r_0$, $y = r_2$, and $s = t = r_1$.

5.3.2 The classification theorems

As already alluded to, edge-biregular maps on surfaces without boundary split into two families, according to whether they contain semi-edges or not. Also, for the purpose of this section we may disregard the semi-star maps. Let us first consider edge-biregular maps containing both edges as well as semi-edges. By the facts summed up by Construction 4.5, edge-biregular maps of type (k, ℓ) for even k and ℓ containing both edges as well as semi-edges and carried by a particular surface are in a one-to-one correspondence with fully regular maps of type $(k/2, \ell/2)$ on the same surface. It follows that a classification of all fully regular maps on a particular surface automatically implies a classification of edge-biregular maps with semi-edges on the same surface, including presentations of the corresponding groups of automorphisms. And since a classification of fully regular maps on surfaces of negative prime Euler characteristic is available from [9], a corresponding classification of edge-biregular maps with semi-edges on these surfaces follows by the outlined procedure. In the interest of saving space we will not go into any further detail and we also omit a formal statement of the related presentations, referring the reader to [9] or to [57].

We now pass onto classification of edge-biregular maps without semi-edges on surfaces with negative prime Euler characteristic. From the earlier work, we know that if such a map $\mathcal{M} = (H; x, y, s, t)$ is given in a canonical form with the group of automorphisms H presented as in (5.1), then all the four generators are *non-trivial* and *distinct*. We state our main results separately for odd primes p (in which case, of course, all the maps are carried by non-orientable surfaces) and then for $p = 2$.

Theorem 5.11 (Reade and Širáň)**.** *A surface of negative odd prime Euler characteristic $\chi = -p$ supports, up to duality and twin maps, only the following pairwise non-isomorphic edge-biregular maps, given in canonical form $\mathcal{M} = (H; x, y, s, t)$, with no semi-edges:*

1. *A single-vertex map of type $(4(p+1), 4)$ with $H = H_{p(1)} = \langle y, t\rangle$ isomorphic to*

the dihedral group of order $4(p+1)$ *with canonical presentation*

$$H_{p(1)} = \langle\, x, y, s, t \,|\, x^2,\ y^2,\ s^2,\ t^2,\ (xy)^2,\ (st)^2,\ xys,\ s(yt)^{p+1} \rangle$$

2. *A two-vertex map of type* $(2(p+2), 4)$ *with* $H = H_{p(2)} = \langle x, t \rangle$ *isomorphic to the dihedral group of order* $4(p+2)$ *and canonical presentation*

$$H_{p(2)} = \langle\, x, y, s, t \,|\, x^2,\ y^2,\ s^2,\ t^2,\ (xy)^2,\ (st)^2,\ xys,\ s(xt)^{p+2} \rangle$$

3. *The maps* $\mathcal{M}_{p,j}$ *of type* (k, ℓ) *where* $k = 4\kappa$ *and* $\ell = 2\lambda$ *for odd and relatively prime* κ *and* λ*, such that* $p = 2\kappa\lambda - 2\kappa - \lambda$*, where* $H = H_{p,j} \cong C_{\lambda\kappa} \rtimes V_4$ *is of order* $k\ell/2$ *and canonically presented as follows, for any positive integer* $j < \lambda$ *such that* $j^2 \equiv 1 \pmod{\lambda}$*, with* $a = (j-1)(\lambda+1)/2$*, and* $u = sx$*,* $v = ty$*:*

$$H_{p,j} = \langle\, x, y, s, t \,|\, x^2,\ y^2,\ s^2,\ t^2,\ (xy)^2,\ (st)^2,\ u^\lambda,\ v^{2\kappa},\ [s, v^2],\ [x, v^2],\ tutu^j,\ v^\kappa u^a s \,\rangle$$

4. *When* $p \equiv 5 \pmod 9$*, that is when* $p = 9m - 4$ *we have a map of type* $(8, 6m)$ *and the corresponding group* $H = H_p$*, which has order* $24m$ *and is an extension of* C_m *by* S_4*, has canonical presentation*

$$H_p = \langle x, y, s, t \,|\, x^2,\ y^2,\ s^2,\ t^2,\ (xy)^2,\ (st)^2,\ (sx)^{3m},\ (ty)^4,\ (sxy)^2 t,\ txty \rangle$$

5. *When* $p = 3$*, we have a map of type* $(4, 6)$ *and the canonical presentation of the corresponding group* $H = H_{(3)} \cong D_6 \times D_6$ *is*

$$H_{(3)} = \langle x, y, s, t \,|\, x^2,\ y^2,\ s^2,\ t^2,\ (xy)^2,\ (st)^2,\ (ty)^2,\ (sx)^3,\ (xyt)^3,\ (sty)^3,\ (xyst)^2 \rangle$$

Moreover, the only fully regular map in the above list is $H_{(3)}$*.*

Theorem 5.12. *A surface of Euler characteristic* $\chi = -2$ *supports, up to duality and twin maps, only the following pairwise non-isomorphic edge-biregular maps* $\mathcal{M} = (H; x, y, s, t)$ *with no semi-edges, given in their canonical forms:*

1. *One self-dual map of type* $(8, 8)$ *with a single vertex and a single face, with the group* $H = H_{2,1}$ *and canonical presentation*

$$H_{2,1} = \langle\ x, y, s, t \mid x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (ty)^4, (sx)^4, ysxs, txsx, xtyt, syty \rangle \cong D_8$$

2. *One map of type* $(4, 12)$ *with three vertices and a single face, with the group*

$H = H_{2,2}$ having canonical presentation

$$H_{2,2} = \langle\, x, y, s, t \mid x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (ty)^2, (sx)^6, xyt, t(sx)^3 \rangle \cong D_{12}$$

3. *One self-dual map of type $(6, 6)$ with two vertices and two faces, with the group $H = H_{2,3}$ having the following canonical presentation*

$$H_{2,3} = \langle\, x, y, s, t \mid x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (ty)^3, (sx)^3, styx \rangle \cong D_{12}$$

4. *Six maps of type $(4, 8)$ with four vertices and two faces, with groups $H_{2,i} \cong D_8 \times C_2$ for $i = 4, 5, 6, 7, 8, 9$ and canonical presentations*

$$\begin{aligned}
H_{2,4} &= \langle\, x, y, s, t \mid x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (ty)^2, (sx)^4, (ys)^4, (tx)^2, ysxs \rangle \\
H_{2,5} &= \langle\, x, y, s, t \mid x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (ty)^2, (sx)^4, (ys)^2, (tx)^2, y(sx)^2 \rangle \\
H_{2,6} &= \langle\, x, y, s, t \mid x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (ty)^2, (sx)^4, (ys)^2, (stx)^2, y(sx)^2 \rangle \\
H_{2,7} &= \langle\, x, y, s, t \mid x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (ty)^2, (sx)^4, (ys)^2, (tx)^2, ty(sx)^2 \rangle \\
H_{2,8} &= \langle\, x, y, s, t \mid x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (ty)^2, (sx)^4, (ys)^2, (stx)^2, tyxsx \rangle \\
H_{2,9} &= \langle\, x, y, s, t \mid x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (ty)^2, (sx)^4, (ys)^2, (stx)^2, tys \rangle
\end{aligned}$$

5. *Three maps of type $(4, 6)$ with six vertices and four faces, with groups of automorphisms $H_{2,i}$ for $i = 10, 11, 12$ given by the following canonical presentations*

$$\begin{aligned}
H_{2,10} &= \langle\, x, y, s, t \mid x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (ty)^2, (sx)^3, t(sy)^2, y(xt)^2 \rangle \cong S_4 \\
H_{2,11} &= \langle\, x, y, s, t \mid x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (ty)^2, (sx)^3, (ysx)^2, (tx)^2 \rangle \cong D_6 \times V_4 \\
H_{2,12} &= \langle\, x, y, s, t \mid x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (ty)^2, (sx)^3, (ys)^2, (tx)^2 \rangle \cong D_6 \times V_4
\end{aligned}$$

The maps supported by the groups $H_{2,1}$, $H_{2,3}$, $H_{2,4}$, $H_{2,7}$, $H_{2,11}$ and $H_{2,12}$ are orientable while the other six are not, and the only fully regular ones out of the twelve are those supported by the groups $H_{2,1}$, $H_{2,3}$, $H_{2,7}$, $H_{2,10}$ and $H_{2,12}$.

Remark 5.13. We make no claim of a lack of redundancy in these presentations - in fact quite the opposite: some of the above presentations have unnecessary relators which have been retained. This is deliberate in order to better demonstrate the interplay between the canonical generators, and also to make evident the presence, or absence, of full regularity or self-duality in the underlying map. $\star$

In order to set the stage for proofs in the following sections, throughout we let $\mathcal{M} = (H; x, y, s, t)$ be an edge-biregular map of type (k, ℓ) for even k and ℓ, with no semi-edges, and of characteristic $-p$ (meaning that the carrier surface of $\mathcal{M}$ has Euler characteristic $\chi = -p$). All our working will be up to duality, so that we may without loss of generality assume that $k \leq \ell$. We also know by Corollary 5.5 that in this situation the four generating involutions of the associated group of automorphisms H are mutually distinct. Note that this instantly implies that $k, \ell \geq 4$; moreover, since maps of type $(4, 4)$ necessarily live on surfaces with Euler characteristic 0 (see Lemma 5.3), we will assume that $k \geq 4$ and $\ell \geq 6$.

Referring to the diagram in Figure 5.4 which may be assumed to display a fragment of $\mathcal{M}$, we remind ourselves of the following facts, put into context for this part of the classification. The stabiliser in H of the distinguished face is $\langle s, x\rangle$, isomorphic to the dihedral group D_ℓ of order ℓ, while the H-stabiliser of the distinguished vertex is $\langle t, y\rangle$, isomorphic to the dihedral group D_k of order k. The map thus has $\frac{|H|}{\ell}$ faces and $\frac{|H|}{k}$ vertices. The H-stabiliser of the distinguished bold and dashed edge, respectively, is $\langle x, y\rangle$ and $\langle s, t\rangle$, in both cases isomorphic to the Klein four-group $\mathrm{V}_4 \cong \mathrm{C}_2 \times \mathrm{C}_2$; hence the map has $\frac{2|H|}{4}$ edges. The Euler-Poincaré formula applied to the number of vertices, edges and faces of $\mathcal{M}$ gives $|H|(\frac{1}{k} - \frac{1}{2} + \frac{1}{\ell}) = \chi = -p$, or, equivalently,

$$|H| = \nu p \quad \text{where} \quad \nu = \nu(k, \ell) = \frac{2k\ell}{k\ell - 2(k + \ell)} \tag{5.5}$$

It may be checked that our lower bounds $k \geq 4$ and $\ell \geq 6$ imply $\nu \leq 12$. Combining this with (5.5) one obtains the upper bound $|H| \leq 12p$ for the group of automorphisms H of the map $\mathcal{M}$ under consideration. Also, by (5.1) and distinctness of the involutory generators, the group H contains two distinct subgroups isomorphic to V_4; in particular, the order of H must be divisible by 4.

5.3.3 The proofs

From this point on we split the proof into four parts. In Section 5.3.3.1 we will consider the case when p divides the order of H, first addressing all odd primes, and then the special case when $p = 2$. Section 5.3.3.2 will deal with the foundations for the opposite case. Thenceforth the analysis will be divided according to whether the 2-part Fitting subgroup of H is cyclic or dihedral. The fine detail of this proof is joint work, not in my hand, and as such is included in Appendix A.

5.3.3.1 The case when p divides the order of H

Let us recall that we are considering an edge-biregular map $\mathcal{M} = (H; x, y, s, t)$ of type k, ℓ for even k, ℓ such that $k \le \ell$, $k \ge 4$, $\ell \ge 6$, carried by a surface of characteristic $-p$ for some prime p. Suppose p is a divisor of $|H|$. This, together with (5.5) and the facts summed up at the end of section 5.3.2, implies that $|H| = \nu p$ where $\nu \in \{4, 6, 8, 12\}$. Moreover, one may check that in our range for the even entries in the type of our biregular map $\mathcal{M}$ one has: $\nu(k, \ell) = 12$ only for $(k, \ell) = (4, 6)$; $\nu(k, \ell) = 8$ only for $(k, \ell) = (4, 8)$; $\nu(k, \ell) = 6$ only when $p = 2$ and for types $(6, 6)$ and $(4, 12)$; and finally $\nu(k, \ell) = 4$ only for (k, ℓ) equal to $(6, 12)$ or $(8, 8)$, corresponding to the cases when $p = 3$ and $p = 2$ respectively.

The case when $p = 2$ is treated as a special case in the proof of Theorem 5.12 at the end of this section. Henceforth, unless stated otherwise, we assume that p is an odd prime. We sum up the above observations for odd p in the form of a lemma.

Lemma 5.14. *If p is an odd prime which divides $|H|$, then either $|H| = 12p$ for the type $(4, 6)$, or $|H| = 8p$ for type $(4, 8)$, or else $p = 3$ and $|H| = 4p$ for type $(6, 12)$.* □

Note that our previous considerations apply also to the situation when $\chi = -1$, that is, Lemma 5.14 can be applied also to the exceptional case when $p = 1$. In fact, a more detailed look into this case will be useful later. Remembering that we need four distinct involutory generators for H if $|H| \in \{4, 8, 12\}$ this leaves us with only two options, namely, $|H| = 8$ when $(k, \ell) = (4, 8)$, and $|H| = 12$ when $(k, \ell) = (4, 6)$. If $|H| = 8$ then $\ell = 8$ and hence $H \cong \mathrm{D}_8$. If, on the other hand, $|H| = 12$ then $\ell = 6$, which means that H contains a subgroup isomorphic to D_6. There is only one such group H of order 12, namely D_{12}. By the classification Theorem 5.8 each of these dihedral groups support an edge-biregular map on a surface with $\chi = -1$. Since we do not need more detailed information about such maps, we just state the following conclusion here:

Lemma 5.15. *If $\mathcal{M} = (H; x, y, s, t)$ is an edge-biregular map on a surface of Euler characteristic $\chi = -1$, then H is isomorphic to a dihedral group of order 8 or 12.* □

We continue by showing that assuming p is a divisor of $|H|$ leads to contradictions for $p \ge 13$. To do so we first exclude the existence of maps $\mathcal{M}$ with types as above for which the group H would be a semidirect product of a particular form.

Lemma 5.16. *Let p be an odd prime which does not divide κ or $\lambda \ge 3$. Suppose $H \cong C_p \rtimes D_\nu$.*

If $\nu \equiv 4 \pmod 8$*, then* H *supports only edge-biregular maps of type* $(4, 2\lambda)$*,* $(2\lambda, 4)$*, and* $(2\lambda, 2\lambda)$*, where* $\nu = 2\lambda$*.*

If $\nu \not\equiv 4 \pmod 8$*, then* H *does not support an edge-biregular map of type* $(2\kappa, 2\lambda)$*.*

Proof. Suppose, up to duality, that an edge-biregular map of type $(2\kappa, 2\lambda)$ exists with $H \cong \mathrm{C}_p \rtimes \mathrm{D}_\nu$. With the usual notation, for any edge-biregular map we have $\langle x, y\rangle \cong \langle s, t\rangle \cong \mathrm{V}_4$, and hence 4 must divide $|\mathrm{D}_\nu|$. In particular this means that D_ν has non-trivial centre, and this leaves us with only two congruence classes to consider for ν, namely 0 modulo 8, where the central element of D_ν is a square, and 4 modulo 8, where the central element of the dihedral group is not a square.

We use additive notation in the first coordinates to indicate the Abelian cyclic group C_p and multiplicative notation for the dihedral part of the semi-direct product, where $\Phi : \mathrm{D}_\nu \to Aut(\mathrm{C}_p)$, which maps $\alpha \to \phi_\alpha$, is the associated homomorphism. Thus each element of H, and hence each of the canonical generators $\{x, y, s, t\}$, will be written (a, α) for some $a \in \mathrm{C}_p$ and some $\alpha \in \mathrm{D}_\nu$.

Since D_ν is generated by involutions, the action of any element $\phi_\alpha \in Aut(\mathrm{C}_p) \cong \mathrm{C}_{p-1}$ must have order dividing two. There are only two such elements, the identity and the unique involution, so for g, a generating element of C_p, we have ϕ_α must either fix g or invert g. Also $(a, \alpha)^2 = (a + \phi_\alpha(a), \alpha^2)$ so the element (a, α) is an involution in H if and only if both $a + \phi_\alpha(a) = 0$ and $\alpha^2 = 1$. Note that if $a \neq 0$ then, in order for (a, α) to be an involution, we must have $\phi_\alpha(a) = -a$, that is ϕ_α must be inverting.

Triples of commuting involutions, that is the non-trivial elements of a copy of V_4 in the group H, have the form: $\{(a, \alpha), (b, \beta), (a + \phi_\alpha(b), \alpha\beta)\}$ such that $a + \phi_\alpha(b) = b + \phi_\beta(a)$, and where α, β and $\alpha\beta$ are all involutions in D_ν, one of which must necessarily be central. So, if both a and b are non-zero in C_p, then $a - b = b - a$, so $2a = 2b$, and hence $a = b$, so the third element of the triple must be $(0, \alpha\beta)$. It is important to note at this point that in this case $\phi_{\alpha\beta}$ would fix any $g \in \mathrm{C}_p$. If instead we suppose that $a \neq 0$ and $b = 0$ then $a + \phi_\alpha(b) = b + \phi_\beta(a)$ becomes $a = \phi_\beta(a)$, in other words ϕ_β must fix every element of C_p.

For much of this proof we only need consider the action generated by the dihedral part of the elements in question, rather than the element itself, and so we abuse the notation to denote those in the kernel of Φ, which fix the cyclic elements by "+" and members of the other coset, whose elements invert the cyclic group, by "$-$". Hence any copy of V_4 in H contains three distinct non-identity elements written:
$(a, -), (a, -), (0, +)$ for some fixed $a \in \mathrm{C}_p$ (which could itself be zero); or,

exceptionally, $(0,+),(0,+),(0,+)$.

We now consider what combination of forms can occur within the set of four distinct canonical involutory generators for H, namely $\{x,y,s,t\}$, taking each case in turn.

It will be useful to note that assuming H is generated by any number of involutory elements, each of which has the form $(0,+)$ or $(a,-)$ for a given fixed $a \in \mathrm{C}_p$, leads to a contradiction. Indeed since $\mathrm{Ker}\,\Phi$ has index at most two in D_ν, given any two elements α and β which are not in the kernel, we have $\alpha\beta \in \mathrm{Ker}\,\Phi$. Hence the product of elements $(a,\alpha)(a,\beta)$ must have the form $(0,\alpha\beta)$ that is $(0,+)$. Also, for $\kappa \in \mathrm{Ker}\,\Phi$, the product $(0,\kappa)(a,\alpha)$ must have the form $(0+a,\kappa\alpha)$ that is $(a,-)$, and similarly $(a,\alpha)(0,\kappa)$ must have the form $(a,-)$. Regardless of the orders of any such products, this restrictive set of forms makes it clear that a is just a place-holder marking when the dihedral part of the element is not in the kernel of Φ, and as such this a cannot contribute towards generating C_p in the first coordinate. So, this is in contradiction to $H \cong \mathrm{C}_p \rtimes \mathrm{D}_\nu$.

We now highlight another property which will be of use:

Since $2\lambda > 4$, and $\langle x,s\rangle$ is a group isomorphic to $\mathrm{D}_{2\lambda}$, the central involution of the dihedral group D_ν, which we now call z, will never occur as the second coordinate of x or s. Henceforth, we identify the dihedral parts of $\langle x,s\rangle$ with the corresponding elements for a particular copy of $\mathrm{D}_{2\lambda} \leq \mathrm{D}_\nu$.

If λ is odd then $\langle x,s\rangle$ has trivial centre, and every rotation in $\mathrm{D}_{2\lambda}$ is a square element in D_ν. This means that either $\mathrm{D}_{2\lambda} \leq \mathrm{Ker}\,\Phi$ or (all of the rotations but) none of the involutions in $\mathrm{D}_{2\lambda}$ are in $\mathrm{Ker}\,\Phi$. In particular, the canonical involutions generating $\langle s,x\rangle$ must have forms with the same symbol, be that $+$ or $-$, in the second coordinate.

If λ is even then $\langle x,s\rangle$ has non-trivial centre and $\mathsf{z} \in \mathrm{D}_{2\lambda}$, and, according to the congruence class of ν modulo 8, this z may or may not be a square in D_ν.

Up to twinness of maps, we now divide the argument into two cases.

The first case to consider is when x has the form $(a,-)$, for some a which may be zero:

Suppose s has the form $(b,-)$. But sx, which has order λ, is then denoted $(b-a,+)$. Now p does not divide the face length 2λ, so this forces $b=a$, and hence s is also an $(a,-)$. Now since t commutes with s, and y commutes with x, we know that each of t and y will also have forms within the restrictive set $\{(0,+),(a,-)\}$, contradicting $H \cong \mathrm{C}_p \rtimes \mathrm{D}_\nu$.

Suppose instead that s has the form $(0,+)$. Thus λ must be even and so $\mathsf{z} \in \mathrm{D}_{2\lambda}$. This splits into two cases, according to the congruence class of ν.

When $\nu \equiv 0 \pmod 8$, z is a square element in D_ν, and hence $\mathsf{z} \in \operatorname{Ker}\Phi$, so in order to include the necessary element $(0,\mathsf{z})$, which has the form $(0,+)$, the set $\{s,t,st\}$ must be $(0,+),(0,+),(0,+)$. In this case, along with y, which is either $(0,+)$ or $(a,-)$, our canonical generators are once again all contained within the restrictive set of forms $\{(0,+),(a,-)\}$, contradicting $H \cong \mathrm{C}_p \rtimes \mathrm{D}_\nu$.

When $\nu \equiv 4 \pmod 8$, the centre $\mathsf{z} \in \mathrm{D}_{2\lambda}$ is not a square element in D_ν, and hence there is the possibility of $\mathsf{z} \notin \operatorname{Ker}\Phi$. (If the central element was in the kernel we would be in the same situation as immediately above, and so the map wouldn't exist.) The group $\langle s,x\rangle \cong \langle r,f | r^\lambda, f^2, (rf)^2\rangle = \mathrm{D}_{2\lambda} \leq \mathrm{D}_\nu$, can only have an inverting central element if $\lambda = 2(2m+1)$ and $r \notin \operatorname{Ker}\Phi$. Up to the choice of notation for f, a reflection, and r, which generates a cyclic group of order λ, this happens when $\mathrm{D}_{2\lambda} = \langle rf, f\rangle$ and exactly one of these two generating involutions is in the kernel, say $rf \in \operatorname{Ker}\Phi$, thereby ensuring r and hence $\mathsf{z} = r^{2m+1} \notin \operatorname{Ker}\Phi$. With this assumption $x = (a,f)$ and $s = (0,rf)$, which forces $y \in \{(a,\mathsf{z}),(0,f\mathsf{z})\}$ and $t \in \{(b,\mathsf{z}),(b,rf\mathsf{z})\}$. Notice in particular that t has the form $(b,-)$. If $y = (a,\mathsf{z})$ then the non-divisibility of κ by p forces $a = b$ and thus the canonical generators would be restricted to a set which cannot generate H. So, for such an edge-biregular map to survive, $y = (0,f\mathsf{z})$ and the choices for t with $b \neq a$ would give edge-biregular maps of type $(4,2\lambda)$ and $(2\lambda,2\lambda)$ respectively. In particular it is clear that no larger dihedral group than $\mathrm{D}_{2\lambda}$ can be found in the second coordinates and so $2\lambda = \nu$. Meanwhile the element xt has the form $(a-b,+)$ and hence the cyclic part C_p of H will also be generated by the set of canonical generators. These maps therefore exist, as do their twins, and since we assumed the edge-biregular map was of type $(2\kappa,2\lambda)$, the duals of these maps, with types $(2\lambda,4)$ and $(2\lambda,2\lambda)$ also exist.

Until now we have been considering the case when x has the form $(a,-)$, for some a which may be zero, and latterly when we also have s being $(0,+)$. The final case to address is when x has the form $(0,+)$:

To avoid being a twin of the maps considered immediately above, we may now assume that s also has the form $(0,+)$. But then, since p does not divide κ, the order of yt, the two canonical generators y and t introduce a maximum of one new form, say $(a,-)$, to the set of canonical generators. Hence we are once again left with an overly restrictive set of generating forms, contradicting $H \cong \mathrm{C}_p \rtimes \mathrm{D}_\nu$. $\square$

We are now in position to exclude primes $p \geq 13$ from consideration in the case of p

dividing $|H|$.

Proposition 5.17. *If p is a divisor of $|H|$, then $p \leq 11$.*

Proof. We suppose that $H = \langle x, y, s, t\rangle$ is the group for an edge-biregular map on surface of Euler characteristic $-p$ and that $p \mid |H|$ where $p \geq 13$. Combining this with Lemma 5.14 we note that the edge-biregular map must have type $(4, 8)$ or $(4, 6)$ forcing $\nu \in \{8, 12\}$ and this implies $|H| = \nu p \leq 12p < p^2$. Hence there is only one Sylow p-subgroup of H and this Sylow subgroup is cyclic, and unique (hence normal) in H.

Since we are restricted to maps of type $(4, 8)$ or $(4, 6)$, and $p \geq 13$, the factor group H/C_p forms a smooth quotient and so would correspond to an edge-biregular map of the same type as H on a surface with Euler characteristic $\chi = -1$. Applying now Lemma 5.15 one sees that H/C_p is a dihedral group of order 8 or 12.
Schur-Zassenhaus now implies that H is a (non-trivial) semi-direct product $H \cong \mathrm{C}_p \rtimes \mathrm{D}_\nu$. But by Lemma 5.16 we know H cannot have this form for these types of map. □

As the next step we extend Proposition 5.17 to all odd primes greater than 3, while the case where $p = 3$ is dealt with separately in Proposition 5.19. The results of the following two propositions, and those of Theorem 5.12, could be found using a computer, however here we adopt a more classical approach.

Proposition 5.18. *If $\mathcal{M} = (H; x, y, s, t)$ is an edge-biregular map on a surface of Euler characteristic $-p$ for some odd prime $p \geq 5$, then p is not a divisor of $|H|$.*

Proof. The conclusion of Proposition 5.17 leaves us with a few small cases we need to address, namely, those listed in Lemma 5.14 (together with the corresponding types) for $p \in \{3, 5, 7, 11\}$; here we will handle the three larger primes one by one.

- *The prime $p = 5$, type $(4, 8)$ for $|H| = 8 \times 5$ and type $(4, 6)$ for $|H| = 12 \times 5$.*

For type $(4, 8)$ with $|H| = 40$. By Sylow theorems the Sylow 5-subgroup is normal so $H \cong \mathrm{C}_5 \rtimes \mathrm{D}_8$, contrary to Lemma 5.16.

The second type to consider here, $(4, 6)$, comes with a group $|H|$ of order 60. We know H contains a (face stabiliser) subgroup isomorphic to D_6, and so H is not Abelian. Suppose that H has a unique Sylow 5-subgroup, which (by the Schur-Zassenhaus theorem) implies that $H \cong \mathrm{C}_5 \rtimes K$ for some group K of order 12. Out of the five such groups K, however, only D_{12} is generated by involutions, and the possibility $H \cong \mathrm{C}_5 \rtimes \mathrm{D}_{12}$ contradicts Lemma 5.16. It follows that H contains six Sylow 5-subgroups and so $H \cong \mathrm{A}_5$. Now A_5 contains 15 involutions and 5 Sylow 2-subgroups, meaning that the intersection of two copies of V_4 is either trivial or the two groups are the same. Hence $\langle x, y\rangle = \langle y, t\rangle = \langle s, t\rangle \cong \mathrm{V}_4$ implies $|H| = 4$, which is a contradiction.

- *The prime $p = 7$, type $(4, 8)$ for $|H| = 8 \times 7$ and type $(4, 6)$ for $|H| = 12 \times 7$.*

For the type $(4, 8)$ we have $|H| = 56$. If the Sylow 7-subgroup of H is not normal then H contains $8 \times 6 = 48$ elements of order 7, leaving only eight other elements in the group H. These eight elements must form the single (and hence normal) Sylow 2-subgroup, which is the face stabiliser, $\mathrm{D}_\ell \cong \mathrm{D}_8$. But then all the involutions of H must lie in D_ℓ and so H would not be generated by involutions. So the Sylow 7-subgroup of H is normal and hence $H \cong \mathrm{C}_7 \rtimes \mathrm{D}_8$ which contradicts Lemma 5.16 again.

The type $(4, 6)$ and $|H| = 84$ is excluded by Sylow theorems and Schur-Zassenhaus, as for the same type but in the case $p = 5$: A group of order 84 has a unique (normal) Sylow 7-subgroup, and then the unique possibility is $H \cong \mathrm{C}_7 \rtimes \mathrm{D}_{12}$, which is impossible by Lemma 5.16.

- *The prime $p = 11$, type $(4, 8)$ for $|H| = 8 \times 11$ and type $(4, 6)$ for $|H| = 12 \times 11$.*

The type $(4, 8)$ with $|H| = 88$ is excluded by observing that H contains a unique Sylow 11-subgroup, leaving only the possibility $H \cong \mathrm{C}_{11} \rtimes \mathrm{D}_8$ that contradicts Lemma 5.16.

The final case to consider is the one of the type $(4, 6)$ that comes with a group H of order 132. If the Sylow 11-subgroup in H is normal then, by a similar analysis as done for $p = 5$, we would have $H \cong \mathrm{C}_{11} \rtimes \mathrm{D}_{12}$, which is impossible by Lemma 5.16. So there are 12 Sylow 11-subgroups and there are $12 \times 10 = 120$ elements of order 11. If H contained more than one Sylow 3-subgroup then there would be 8 elements of order 3, leaving room for only one Sylow 2-subgroup. Now, H is generated by involutions and so the Sylow 2-subgroup $\mathrm{D}_k \cong \mathrm{V}_4$ cannot be unique, otherwise we would have $H = \mathrm{D}_k$. So, the Sylow 3-subgroup in H, isomorphic to C_3, must be unique and hence normal. Thus, in our case with $\ell = 6$, the subgroup $\langle sx\rangle \cong \mathrm{C}_3$ is normal in H, which implies that $ysxy, tsxt \in \{sx, xs\}$. If we suppose $tsxt = sx$ then $txt = x$ so t commutes with s and x and y. But $\langle x, y\rangle$ is a Sylow 2-subgroup of H and so, by the distinctness of the generators, we would have $t = xy$. Hence $H = \langle sx\rangle \rtimes \langle x, y, t\rangle = \langle sx\rangle \rtimes \langle x, y\rangle$ which has order 12, not 132. By symmetry, supposing that y fixes sx leads to a contradiction too. Hence we have $tsxt = xs$ and $ysxy = xs$. These give, in turn, $txt = sxs$ and $ysy = xsx$ which, when combined, give $txtysy = 1$. However, recalling that y commutes with both x and t, this yields $xs = 1$, a contradiction. This completes the proof. $\square$

We now consider the exceptional case when $p = 3$, which does indeed yield edge-biregular maps, namely those corresponding to the final part of Theorem 5.11.

Proposition 5.19. *If $\mathcal{M} = (H; x, y, s, t)$ is an edge-biregular map on a surface of Euler characteristic -3, then up to duality and twinness, H is of type $(4, 6)$ and has a presentation of the form*

$$H = \langle x, y, s, t | x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (yt)^2, (sx)^3, (xyt)^3, (sty)^3, (xyst)^2\rangle \cong D_6 \times D_6$$

Proof. When the prime $p = 3$, we have the following possibilities: type $(6, 12)$ for $|H| = 4 \times 3$; type $(4, 8)$ for $|H| = 8 \times 3$; and type $(4, 6)$ for $|H| = 12 \times 3$.

An edge-biregular map of type $(6, 12)$ on this surface would have $|H| = 4 \times 3 = 12$ and so the group H must be the face stabiliser $H = \mathrm{D}_\ell \cong \mathrm{D}_{12}$ which is dihedral. Referring to the dihedral classification of edge-biregular maps in Table 5.1 we can see that such a map does not exist.

For type $(4, 8)$ we have $|H| = 24$ and as H is generated by involutions, the Sylow 2-subgroup $\mathrm{D}_\ell \cong \mathrm{D}_8$ cannot be normal in H. This leaves only two options: $H \cong \mathrm{C}_3 \rtimes \mathrm{D}_8$, which is clearly impossible, or $H \cong \mathrm{S}_4$. Suppose $H \cong \mathrm{S}_4$ is represented as a permutation group on the set $\{1, 2, 3, 4\}$. The non-trivial elements in the three copies of V_4 in S_4 then form the sets $T_1 = \{(12)(34), (12), (34)\}$, $T_2 = \{(13)(24), (13), (24)\}$, $T_3 = \{(14)(23), (14), (23)\}$ and $T_4 = \{(12)(34), (13)(24), (14)(23)\}$. Since $\langle s, x\rangle = \mathrm{D}_\ell \cong \mathrm{D}_8$, we may assume, without loss of generality, that $\{s, x\} = \{(12)(34), (24)\}$. But then the fact that $\langle y, t\rangle = \mathrm{D}_k \cong \mathrm{V}_4$ leads us to conclude that for the four *distinct* canonical generators we must have $x, y, t, s \in T_2 \cup T_4$. But these two sets only generate D_8, not the whole of S_4, a contradiction.

We proceed with type $(4, 6)$ for a group H of order 36 and let K be its Sylow 3-subgroup of order 9.

Consider the *non-trivial and transitive* permutation representation π of H on the set H/K of the right cosets of K in H, given as follows: To each $h \in H$ we assign a permutation π_h of the set H/K mapping the coset Kx onto Kxh, for every $x \in H$.

Now, as $|H/K| = 4$, the representation π is a group homomorphism from H into S_4, and its kernel is a normal subgroup of H distinct from H; note also that π cannot be surjective (by divisibility of the source and target by 4 and 8). The only proper transitive subgroups of S_4 are A_4 and the unique subgroup isomorphic to V_4, so that $|\mathrm{Ker}\,\pi| \in \{3, 9\}$. But notice that if $|\mathrm{Ker}\,\pi| = 3$, so that H contains a normal subgroup $\mathrm{Ker}\,\pi \cong \mathrm{C}_3$, and $\mathrm{Im}\,\pi \cong \mathrm{A}_4$ then we have an immediate contradiction since A_4 is not generated by involutions.

It remains to consider the case when $|\mathrm{Ker}\,\pi| = 9$, which means that the kernel coincides with the Sylow 3-subgroup K of order 9 in H, and $H \cong K \rtimes \mathrm{V}_4$.

Suppose, for a contradiction, that $K = \mathrm{C}_9$. Now, C_3 is characteristic in C_9 and hence normal in H. There is only one copy of C_3 in H, and that is $\langle sx \rangle$. Denoting elements in $H/\langle sx \rangle$ with bar notation, we then have $\bar{s} = \bar{x}$ and hence $H/\langle sx \rangle$ is generated by three commuting involutions $\bar{x}, \bar{y}$ and $\bar{t}$. This has order at most 8, not the required 12, and so is in contradiction to the order of H.

We may consider C_3^2 as a two dimensional vector space over GF(3) with elements written as vectors. The automorphism group for C_3^2 is $GL(2, 3)$ which is known to have just one conjugacy class of subgroups isomorphic to V_4. Thus we let $x = (\underline{x}, x'), \quad y = (\underline{y}, y'), \quad t = (\underline{t}, t'), \quad s = (\underline{s}, s')$ where the underlined vectors in the first coordinates are elements of C_3^2, and $x', y', t', s' \in \mathrm{V}_4 \leq GL(2, 3)$. Now, since sx has order 3, it is clear that $s'x'$ must be the identity, and so $x' = s'$. Also, since the products xy, yt and ts must all have order 2, we certainly have $x' \neq y' \neq t' \neq x'$, and so $t' = x'y'$.

Suppose the homomorphism $\Phi : \mathrm{V}_4 \to GL(2, 3)$ associated with the semi-direct product is not injective. In any case, the kernel of Φ cannot be the whole of V_4, since this would yield a direct product $H \cong \mathrm{C}_3^2 \times \mathrm{V}_4$ which certainly cannot be generated by involutions. The only remaining option is for two of the elements in V_4 to be mapped to a given involution $\alpha \in GL(2, 3)$, while their product is mapped to the identity. Now $x' \notin \mathrm{Ker}\,\Phi$, otherwise x and s being involutions would then force $\underline{x} = \underline{s} = \underline{0}$ which is absurd since xs must have order 3. So, up to twinness, we may assume, $x'y' \in \mathrm{Ker}\,\Phi$. But then this would force $\underline{t} = \underline{0}$, meaning t is central in the group H. Now, the elements x and y are involutions so α acts to invert any non-zero parts of $\underline{x}$ and $\underline{y}$. Also, x and y commute so $\underline{x} + \alpha(\underline{y}) = \underline{y} + \alpha(\underline{x})$, that is $\underline{x} - \underline{y} = \alpha(\underline{x} - \underline{y})$, and this forces $\underline{x} = \underline{y}$. Now we have in fact $t = xy$ and so $H = \langle x, t, s \rangle = \langle x, s \rangle \langle t \rangle \cong \mathrm{D}_6 \times \mathrm{C}_2$, a contradiction.

We have now shown that the homomorphism mapping from V_4 to the associated actions in $GL(2,3)$ must be injective, since in any other case we could not generate the whole group $H \cong \mathrm{C}_3^2 \rtimes \mathrm{V}_4$ just with involutions. So we may now identify $\{x', y', t', s'\}$ with their images in $GL(2,3)$, choosing the canonical copy of V_4 in $GL(2,3)$, which consists of the diagonal matrices.

In the interests of generating the whole group $H \cong \mathrm{C}_3^2 \rtimes \mathrm{V}_4$, it can be checked that $x' = s' = -I$. To satisfy the other known order properties for the products of pairs of generating involutions, up to twinness, we find we are restricted to the set of canonical involutions as follows:

$$x = (\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}), \quad y = (\begin{pmatrix} 0 \\ x_2 \end{pmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}),$$

$$t = (\begin{pmatrix} s_1 \\ 0 \end{pmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}), \quad s = (\begin{pmatrix} s_1 \\ s_2 \end{pmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix})$$

where $x_1 \neq s_1$ and $x_2 \neq s_2$. Regardless of whichever allowable choices of values for x_i and s_i are made, it is clear that $\langle xyt, sty \rangle \cong \mathrm{C}_3^2$. Also we have $\langle y, t \rangle \cong \mathrm{V}_4$ as expected, and hence $H = \langle x, y, s, t \rangle = \langle xyt, sty \rangle \rtimes \langle y, t \rangle \cong \mathrm{C}_3^2 \rtimes \mathrm{V}_4$. Also $H = \langle xyt, xy \rangle \times \langle sty, st \rangle \cong \mathrm{D}_6 \times \mathrm{D}_6$ and a presentation for the resulting map will be as follows:

$$H = \langle x, y, s, t | x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (yt)^2, (xyt)^3, (sty)^3, (xyst)^2 \rangle \cong \mathrm{D}_6 \times \mathrm{D}_6.$$

By this analysis, the map is unique up to duality and twinness. The above group presentation clearly indicates that this edge-biregular map is isomorphic to its twin, and hence is also a fully regular map. Meanwhile the face length ℓ being six is clearer to see when the additional (consistent) relator $(sx)^3$ is incorporated into the presentation. □

The only prime left to be considered in this part is $p = 2$, and by exploring this case we will also establish validity of Theorem 5.12.

Proof of Theorem 5.12. We now assume $p = 2$. We recall the possible types of map which can occur when $\nu \in \{4, 8, 12\}$ giving $|H| = 8$, 16 and 24, respectively, for types $(8,8)$, $(4,8)$ and $(4,6)$. We will also need to address the only remaining case for even k, ℓ such that $4 \,|\, |H| = 2\nu$, namely when $\nu = 6$ and $|H| = 12$ which occurs when the type (k, ℓ) is $(4,12)$ or $(6,6)$. We proceed according to the order of the group H.

The first possibility is dealt with quickly, because in an edge-biregular map of type

$(8, 8)$ with $|H| = 8$ we *must* have $H \cong \mathrm{D}_8$ (the stabiliser of a (single) vertex, say). The group D_8 contains exactly four non-central involutions, and two subgroups isomorphic to V_4, while the central element cannot be equal to any one of the distinct canonical involutory generators. It may be checked that up to isomorphism (since the resulting map is both self-dual and fully regular) there is just one way to present H canonically, necessarily equivalent to the form given by $H_{2,1}$ in Theorem 5.12.

When we consider $\nu = 6$ for the type $(4, 12)$ we clearly have the group $H = \langle x, s\rangle \cong \mathrm{D}_{12}$, and up to twinness and duality there is only one canonical presentation. This is shown as $H_{2,2}$ in Theorem 5.12.

In the case of type $(6, 6)$, we have $\nu = 6$ and the group is again $H \cong \mathrm{D}_{12}$. This dihedral group contains two copies of D_6, one of which must be $\langle s, x\rangle$ and the other $\langle t, y\rangle$. There is only one cyclic group of order three contained in D_{12} and so there are only two possibilities for the elements of order 3, namely $sx = yt$ (which results in contradictions) or $sx = ty$ which yields the presentation $H_{2,3}$ in Theorem 5.12.

The next possibility we look at is type $(4, 8)$, with $|H| = 16 = 2 \times 8$. We know $\langle s, x\rangle = \mathrm{D}_\ell \cong \mathrm{D}_8$ is normal in H since it has index two. First note that $H/\mathrm{D}_\ell = \langle y\mathrm{D}_\ell, t\mathrm{D}_\ell\rangle$ has order 2 so either $y \in \mathrm{D}_\ell$, or $t \in \mathrm{D}_\ell$, or $y\mathrm{D}_\ell = t\mathrm{D}_\ell \neq \mathrm{D}_\ell$ and so the product ty is in D_ℓ. We deal with the latter case after addressing the first two together, since they are equivalent up to twinness. Also $\langle sx\rangle \cong \mathrm{C}_4$, being characteristic in the dihedral group, is normal in H. Conjugation by the elements x and s clearly invert sx while each of y and t must either fix or invert sx, resulting in differing canonical presentations for H.

We first suppose, by our choice of the labeling of orbits, that is up to twinness, that $y \in \mathrm{D}_\ell$. If y inverts sx then we have $y \in \{sxs, xsx\}$ and so, by the distinctness of generators, $y = sxs$, and the order of ys is four. If t also inverts sx then $tsxt = xs$. But then $y = stsxt = txt$ which implies $y = x$, a contradiction. However, if t fixes sx, which happens if and only if t fixes x, we obtain

$$\langle\, x, y, s, t \mid x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (ty)^2, (sx)^4, (ys)^4, (tx)^2, ysxs\rangle$$

which is the presentation of $H_{2,4}$ in Theorem 5.12. Notice that this map must be supported by an orientable surface since there are no odd length relators in this presentation.

If, on the other hand, y fixes sx then $y = (sx)^2$, the central element in D_ℓ, and hence $y(sx)^2$ is a relator, forcing the supporting surface to be non-orientable. Another consequence of $y = (sx)^2$ is that $(ys)^2$ may also appear as a relator. If t also fixes sx, we may arrive at the presentation

$$\langle\, x, y, s, t \mid x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (ty)^2, (sx)^4, (ys)^2, (tx)^2, y(sx)^2 \rangle$$

which is $H_{2,5}$ in Theorem 5.12. Otherwise, t inverts sx, giving

$$\langle\, x, y, s, t \mid x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (ty)^2, (sx)^4, (ys)^2, (stx)^2, y(sx)^2 \rangle$$

which is the presentation of the group $H_{2,6}$ in Theorem 5.12.

Now let us suppose that $ty \in \mathrm{D}_\ell$. As before, we consider when ty fixes sx, that is in this case $ty = (sx)^2$, the central element in D_ℓ. If one (and hence both) of y and t invert sx then $tsxt = xs$ so $txt = sxs$. This leads to $ty = sxsx = txtx$ which implies $y = xtx$ so $y = t$, a contradiction to the distinctness of the canonical involutions. In the other case both y and t fix sx. This is equivalent to y commuting with s, and t commuting with x, and this gives us the following presentation which yields a group of the right order.

$$\langle\, x, y, s, t \mid x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (ty)^2, (sx)^4, (ys)^2, (tx)^2, ty(sx)^2 \rangle$$

This is listed as $H_{2,7}$ in Theorem 5.12. One may notice this presentation has a redundant relator, which serves to make clear the underlying full regularity of the corresponding map.

On the other hand consider when ty inverts sx, and hence exactly one of y or t must fix sx. Up to twinness we may assume y fixes sx in which case y is central in H and $(ys)^2$ is a relator. In this case then t inverts sx and hence $(tsx)^2$, or equivalently $(stx)^2$, is also a relator.

For $ty \in \mathrm{D}_\ell$ to invert sx we must have $ty \in \{sxs, x, xsx, s\}$. The first two options yield a contradiciton, as we shall now see. If $ty = sxs$ then $1 = tysxs = ysys$ so $ytysx = sy$, that is $tsx = sy$ and hence $tx = y$. But then $ty = x = sxs$ which means $xs = sx$, contradicting the order of sx. If $ty = x$ then $1 = tyx = (tsx)^2$ so $y = sxts$, that is $x = yt = sxs$, the same contradiction.

It can be checked that the remaining options, namely ty is either xsx or s, avoid the contradictions which cause the order of the group H to become too small, and they yield these two remaining canonical presentations.

$$\langle\, x, y, s, t \mid x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (ty)^2, (sx)^4, (ys)^2, (stx)^2, tyxsx \rangle$$

$$\langle\, x, y, s, t \mid x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (ty)^2, (sx)^4, (ys)^2, (stx)^2, tys\rangle$$

These are listed respectively as $H_{2,8}$ and $H_{2,9}$ in Theorem 5.12. Careful comparisons of the relators in the above presentations shows that the maps are pairwise non-isomorphic.

The last possibility to address is the type $(4, 6)$ for a group H of order $3 \times 8 = 24$. Suppose the Sylow 3-subgroup is not normal in H. Then $H \in \{\mathrm{SL}(2, 3), \mathrm{S}_4, \mathrm{A}_4 \times \mathrm{C}_2\}$. Now, we know H has more than one involution, so it cannot be isomorphic to $\mathrm{SL}(2, 3)$. Also, A_4 only contains 3 involutions, all in the subgroup which is isomorphic to V_4, but H is generated by involutions, H so cannot be isomorphic to $\mathrm{A}_4 \times \mathrm{C}_2$. Thus, if the Sylow 3-subgroup is not normal in H, then $H \cong \mathrm{S}_4$.

We will now refer to the standard permutation representation of S_4 on the set $\{1, 2, 3, 4\}$ and remember the sets $T_1 - T_4$ introduced in the first part of the proof of Proposition 5.19 for $p = 3$. Since $\langle s, x\rangle = \mathrm{D}_\ell \cong \mathrm{D}_6$, we may assume, up to choice of notation and hence without loss of generality, that x and s are (12) and (23), respectively. The condition that $\langle y, t\rangle \cong \mathrm{V}_4$ means that $y = (12)(34)$, and $t = (14)(23)$, arriving at the non-orientable map given by the presentation listed as $H_{2,10}$ in Theorem 5.12:

$$\langle\, x, y, s, t \mid x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (ty)^2, (sx)^3, t(sy)^2, y(xt)^2\rangle \cong \mathrm{S}_4$$

Now suppose the Sylow 3-subgroup $\langle sx\rangle \cong \mathrm{C}_3$ is normal in H. Then conjugation by each of y and t must either fix or invert sx.

Suppose y inverts sx. Then $ysxy = xs$ so $(ysx)^2$ is a relator. If t also inverts sx then $tsxt = xs$. Then $txt = sxs$ and hence $txtx = sxsx$ but y commutes with both t and x so $txtx = ytxtxy = ysxsxy = xsxs$ is self-inverse. But $sxsx$ has order three so this is a contradiction. If, on the other hand, t fixes sx then we obtain the presentation of the group $H_{2,11}$ from Theorem 5.12:

$$\langle\, x, y, s, t \mid x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (ty)^2, (sx)^3, (ysx)^2, (tx)^2\rangle.$$

Suppose finally that y fixes sx, in which case $ysxy = sx$ so $(ys)^2$ is a relator. In the case where t also fixes sx, this leads to the presentation of the group $H_{2,12}$ from Theorem 5.12:

$$\langle\, x, y, s, t \mid x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (ty)^2, (sx)^3, (ys)^2, (tx)^2\rangle$$

Otherwise, t inverts sx, giving the presentation

$$\langle\, x, y, s, t \mid x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (ty)^2, (sx)^3, (ys)^2, (txs)^2 \rangle$$

which represents the twin map of the edge-biregular map determined by the group $H_{2,11}$.

It remains to address isomorphism type, full regularity and orientability of the twelve edge-biregular maps identified above. Obviously, $H_{2,1} \cong \mathrm{D}_8$, $H_{2,2} \cong \mathrm{D}_{12} \cong H_{2,3}$ and $H_{2,7} \cong \mathrm{S}_4$. Observe that the element $t \notin \langle s, x\rangle$ is central in $H_{2,4}$ and $H_{2,5}$ while $st \notin \langle s, x\rangle$ is central in $H_{2,6}$, and $y \notin \langle s, x\rangle$ is central in $H_{2,7}$, $H_{2,8}$ and $H_{2,9}$ and so the six groups are all isomorphic to $\mathrm{D}_8 \times \mathrm{C}_2$. Further, it follows from the derivation of the remaining two groups that $H_{2,11} = \langle s, x\rangle \rtimes \langle t, y\rangle = \langle s, x\rangle \times \langle t, xy\rangle \cong \mathrm{D}_6 \times \mathrm{V}_4$ and $H_{2,12} = \langle s, x\rangle \times \langle t, y\rangle \cong \mathrm{D}_6 \times \mathrm{V}_4$. It is easy to check that the twelve presentations are correct and complete (describing exactly the groups listed, albeit including some redundant relators). The maps defined by $H_{2,1}$, $H_{2,3}$, $H_{2,4}$, $H_{2,7}$, $H_{2,11}$ and $H_{2,12}$ are orientable because all their defining relations have even length in the generators x, y, s, t, which does not apply to the remaining six maps. Finally, the maps associated with $H_{2,1}$, $H_{2,3}$, $H_{2,7}$, $H_{2,10}$, and $H_{2,12}$ are the only fully regular ones out of the above twelve, as their groups admit an automorphism swapping x with s and y with t.

The proof of Theorem 5.12 is now complete. □

By now we have proved the existence of all of the (necessarily small) cases of edge-biregular maps $\mathcal{M}(H; x, y, s, t)$ on surfaces of negative prime characteristic $\chi = -p$ such that p divides the order of the group H, that is when $p \in \{2, 3\}$.

We may now turn our attention to the alternative case, where p does not divide $|H|$.

5.3.3.2 The case when p does not divide the order of H

(Set-up and initial steps for proof which is continued in Appendix A)

Theorem 5.11 includes two families of dihedral maps with group presentations $H_{p(1)}$ and $H_{p(2)}$, whose existence is proved in 5.8, and also the special case $H_{(3)}$ where $p = 3$ which appeared in the previous section. This subsection addresses the remaining families of maps with groups $H_{p,j}$ and H_p, by both proving their existence, and also excluding the possibility of any other maps on a surface where $\chi = -p$. This is joint work with my supervisor, Jozef Širáň. As such I will provide the foundations of the collaborative proof here, and attach the remainder of the proof (by his hand) in

Appendix A. As with the rest of the work in this section, the published work can be found in [53].

Let $\mathcal{M} = (H; x, y, s, t)$ be a (finite) edge-biregular map of type (k, ℓ) on a surface of Euler characteristic $-p$ for some prime p; in view of Theorem 5.12 we will assume that p is odd and hence the surface is non-orientable. Recall that the group H is assumed to be presented as in (5.1), and its order together with the type of the map and the characteristic of the surface are tied by the equation (5.5). We begin with an auxiliary result and omit a proof since it is almost verbatim the same as the proof of Lemma 3.2 of [18].

Lemma 5.20. *If q is a prime divisor of $|H|$ relatively prime to the Euler characteristic, then Sylow q-subgroups of H are cyclic if q is any odd prime, and dihedral if $q = 2$.* □

From this point on until the end of this section we will assume that p is *not* a divisor of the order of H. Since we are working up to duality, instead of $k \leq \ell$ we henceforth assume that *the 2-part of k is not smaller than the 2-part of ℓ*. Remember also that H contains a subgroup isomorphic to V_4 and so 4 divides $|H|$. Comparing this condition with Equation (5.5) allows us to set $k = 4\kappa$ and $\ell = 2\lambda$ for integers κ and λ. The stage will be set by proving solvability of H.

Proposition 5.21. *If p is such that $p \nmid |H|$, the group H is solvable.*

Proof. We start by proving that $|\mathrm{D}_k \cap \mathrm{D}_\ell| \leq 4$. The group $K = \mathrm{D}_k \cap \mathrm{D}_\ell$ is obviously cyclic or dihedral. Suppose for a contradiction that $|K| > 4$. Then K contains an element z of order at least 3, so that $z = (sx)^m = (yt)^n$ for some m and n. Clearly, z commutes with sx and also with yt. We also have $(xy)z(xy) = yx(sx)^m xy = y(xs)^m y = y(ty)^n y = (yt)^n = z$, so z commutes with xy as well. Now the map is carried by a non-orientable surface and so $H = \langle xy, sx, yt \rangle$. Hence z commutes with all the canonical generators, and thus it is central in H. Specifically, z is central in $\mathrm{D}_k \leq H$. It is well-known that the centre of a (non-Abelian) dihedral group is either trivial or has order 2. But, by our assumption, the order of $\langle z \rangle$ is greater than 2, a contradiction.

Next we prove that the group H is a product of D_k and D_ℓ. By (5.5) the assumption $p \nmid |H|$ implies that p must be absorbed by the denominator of $\nu(k, \ell)$, that is, $k\ell - 2(k + \ell) = rp$ for some integer r. Thus, $|H| = 2k\ell/r$, but by the first part of the proof we also have $|H| \geq |\mathrm{D}_k||\mathrm{D}_\ell|/|\mathrm{D}_k \cap \mathrm{D}_\ell| \geq k\ell/4$, implying that $r \leq 8$. Using $k = 4\kappa$ and $\ell = 2\lambda$ as agreed before, one has

$rp = k\ell - 2(k+\ell) = 8\kappa\lambda - 8\kappa - 4\lambda = 4(2\kappa\lambda - 2\kappa - \lambda) = 4cp$. Notice that this, along with the assumption $p \nmid |H|$, also implies $\gcd(2\kappa, \lambda) = c \leq 2$. Hence $|H| = k\ell/(2c)$ where $c = \gcd(2\kappa, \lambda) \in \{1, 2\}$.

If $c = 1$, then $|H| = k\ell/2$ and also λ must be odd, so that $|\mathrm{D}_k \cap \mathrm{D}_\ell| \leq 2$. We also have $|H| \geq |\mathrm{D}_k||\mathrm{D}_\ell|/|\mathrm{D}_k \cap \mathrm{D}_\ell| \geq k\ell/2$ and so equality holds throughout and hence $H = \mathrm{D}_k\mathrm{D}_\ell$ if $c = 1$. In the case when $c = 2$ we have $(2\kappa\lambda - 2\kappa - \lambda) = 2p$ where p is an odd prime, so λ must be even; also, $\gcd(\kappa, \lambda/2) = 1$. Now, $c = 2$ implies $|H| = k\ell/4$ and $|\mathrm{D}_k \cap \mathrm{D}_\ell| \leq 4$, and as we also have $|H| \geq |\mathrm{D}_k||\mathrm{D}_\ell|/|\mathrm{D}_k \cap \mathrm{D}_\ell| \geq k\ell/4$ we conclude that equality holds throughout and hence $H = \mathrm{D}_k\mathrm{D}_\ell$.

We may now complete the proof by invoking the result of Huppert [34] that the product of two dihedral groups is solvable. □

The fact that H is solvable yields that it has a non-trivial Fitting subgroup F; recall that F is the largest nilpotent normal subgroup of H. In particular, F is a direct product of its Sylow subgroups. By what we know about the Sylow subgroups of H from Lemma 5.20 we have $F = F_1 \times F_2$, where F_1 is cyclic, of odd order (possibly trivial), and F_2 (if non-trivial) is a cyclic or a dihedral 2-group; we will henceforth split our analysis according to this dichotomy.

As a general remark, observe that we may assume $F \neq H$. Indeed, if $F = H$, then F_1 would have to be trivial (otherwise F could not be generated by involutions) and F_2 would have to be non-cyclic (to contain enough distinct involutions), so that $H = F = F_2$ would have to be dihedral. But edge-biregular maps with dihedral groups H have already been classified in Theorem 5.8. Without giving details we just state that, as a consequence of the table displaying the classification results 5.1, the only edge-biregular maps of Euler characteristic $-p$ for an odd prime p determined by a dihedral group of automorphisms are the first two maps in Theorem 5.11 defined by the groups $H_{p(1)}$ and $H_{p(2)}$.

Henceforth we are thus able to dismiss any occasion when the conclusion is that the group must be dihedral, a trick which is used repeatedly in the proof of Theorem 5.11 relating to $H_{p,j}$ and H_p, as found in Appendix A.

Finally, the following remark describes a visual representation of the group H_p which relates to Figure 5.5 and reveals something of how the graph interacts with the underlying surface.

Remark 5.22. The maps $\mathcal{M}_p$ identified in the last part of our proof in Appendix A, those defined by the group H_p presented as in Equation (A.6), deserve particular

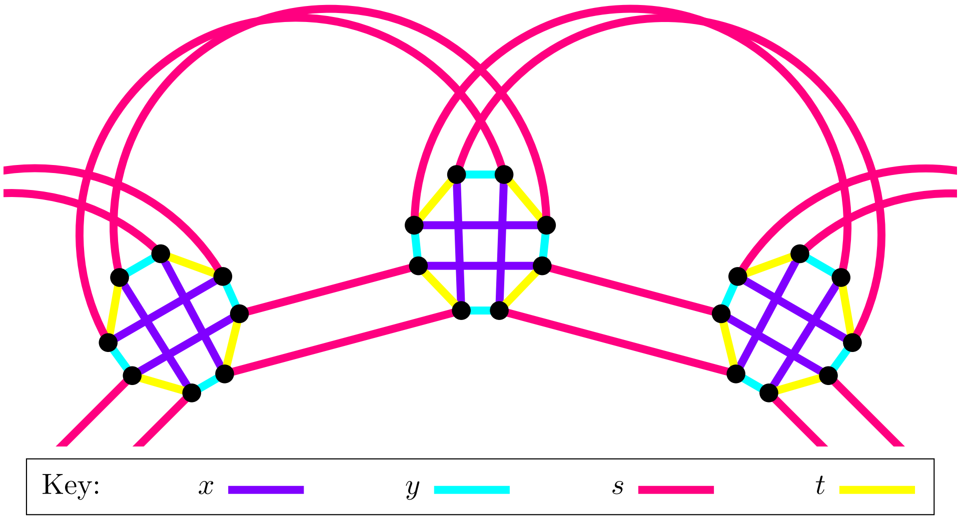

Figure 5.5: Part of the Cayley graph $\mathrm{Cay}(H_p, \{x, y, s, t\})$

attention. Their structure is best visualized by considering the associated embedded Cayley graph $\mathrm{Cay}(H_p, X)$ for the group H_p and the generating set $X = \{x, y, s, t\}$. If superimposed onto the map $\mathcal{M}_p$, the associated embedding of $\mathrm{Cay}(H_p, X)$ displays cycles labelled alternately with y and t 'around' each vertex and cycles labelled with x and s 'around' each face, while the 4-cycles labelled $xyxy$ and $stst$ 'surround', respectively, bold and dashed edges. The existence of the relator $txty$ in the group presentation (A.6), equivalent to the relation $x = tyt$, demonstrates that all the bold edges are loops. Applying this knowledge to the relator $(sxy)^2t$, which (being of odd length) signals non-orientability, one sees that the edges in the unshaded orbit partition into pairs of double-edges, each forming a 'central cycle' of a Möbius strip in the embedding. The Cayley graph $\mathrm{Cay}(H_p, X)$, part of which is shown in Figure 5.5, makes this clear. This is also an alternative way of proving non-regularity for this map. Namely, the two edge orbits contain (bold) loops on the one hand, and non-orientable (dashed) 2-cycles on the other, so there will certainly not be an automorphism of the group which interchanges these two orbits of edges. $\star$

Chapter 6

On finite genuinely edge-biregular maps of type $(2\kappa, 2\lambda)$

When first introduced to edge-biregular maps (or indeed any new concept) a natural first thought may be: Do these creatures even exist? This question is sometimes easy to answer, as in the case for edge-biregular maps, and indeed we have already seen many explicit examples, including some infinite families, and indeed classifications for certain surfaces. Furthermore, we would like to ensure that finite edge-biregular maps exist in their own right, and are not, for example, always fully regular as well. We also wish to ensure that any such examples are not in some sense degenerate. We describe a map as *genuinely edge-biregular* when it is both a proper edge-biregular map and also truly edge-biregular. Thus a genuinely edge-biregular map has no semi-edges, and is such that the embedding is not also a fully regular map. In this chapter, we seek to answer the question: Do finite, genuinely edge-biregular maps of type $(2\kappa, 2\lambda)$, always exist for any given parameters κ and λ and if not then what conditions are necessary?

One necessary condition for existence of a proper edge-biregular map on a closed surface is that the parameters in the type must both be even. It is for this reason that, in this chapter, we denote the type of an edge-biregular map by $(2\kappa, 2\lambda)$. When $\kappa \geq 2$ and $\lambda \geq 2$ are such that $\kappa + \lambda \geq 5$, if there are no further demands on the structure of the edge-biregular map (for example properties of the supporting surface), we prove that a genuinely edge-biregular map of type $(2\kappa, 2\lambda)$ exists such that H is isomorphic to either A_m or S_m for some degree m which (naturally) depends (to a certain extent) on the values of κ and λ. This follows the example of other authors in the field who, when considering the existence of a structure with given parameters, look for it on a specific type of group, for example alternating or symmetric, see [15] and [13].

In section 6.1, we approach the existence question by constructing genuinely edge-biregular maps of a given type where H is either alternating or symmetric. Section 6.2 addresses the cases where κ and λ are small. Theorem 6.40 then confirms, for almost all κ, λ, the existence of a genuinely edge-biregular map of type $(2\kappa, 2\lambda)$.

The dual of an edge-biregular map is also an edge-biregular map, and the colour-preserving automorphism groups of the resulting pair of dual maps are isomorphic to each other. Our conclusions will thus be valid up to duality, and so, without loss of generality, for most of the workings in this chapter, we assume that $\kappa \leq \lambda$. Small cases have already been dealt with earlier in Chapter 5, and so henceforth, unless stated otherwise, we will assume that $\kappa \geq 3$.

6.1 Constructions yielding H an alternating or symmetric group

This section proves that, for a given type $(2\kappa, 2\lambda)$, where κ and λ are sufficiently large, there is a *genuinely* edge-biregular map whose colour-preserving automorphism group H is isomorphic to A_m or S_m, where $m = \max\{\kappa, \lambda\}$. The proof is constructive. In the cases where the degree $\max\{\kappa, \lambda\} = m$ of the group H is prime, the resulting maps are in some sense extremal, in that the group must contain an element of order m and so the degree of the alternating or symmetric group cannot be less than (prime) m.

Throughout this section we will be considering the existence of edge-biregular maps with symmetric or alternating colour-preserving automorphism group, and this is the motivation and context for our early remarks and observations.

6.1.1 Outline of methods and useful observations with an example

We begin with an example, thereby proving the existence of genuinely edge-biregular maps of any given type $(2\kappa, 2\lambda)$ where $\lambda - 2 \geq \kappa \geq 3$ and the corresponding group is alternating or symmetric of degree $\lambda + 1$. To do this we use coset diagrams, a method which has been used in the past to prove the existence of certain types of genuinely biregular maps, namely chiral rotary maps of type (k, ℓ) in [15]. As the authors in that paper did, we make use of Jones' generalisation of Jordan's theorem, whose proof relies on the classification of finite simple groups, specifically the stated Corollary 1.3 in [35]. For future reference, we reproduce the relevant part of this powerful theorem:

Theorem 6.1. *(Jones) Let G be a primitive permutation group of finite degree n, containing a cycle with k fixed points. Then $G \geq A_n$ if $k \geq 3$.*

Later in this section we will refine and extend our claims. The first example, Construction 6.10, is left in as it was the original foundation from which I built both understanding and also the further examples. Its deficiencies, not minimising the degree of the group and at the same time requiring the difference between κ and λ to be at least two, become clear and as such it is described as an instructive example.

For some of the propositions in this chapter, namely Propositions 6.22 and 6.28 when the difference between κ and λ is respectively zero or two, we will be able to minimise the degree and avoid relying on the classification of finite simple groups by using Jordan's original theorem:

Theorem 6.2. *(Jordan) Let G be a primitive permutation group of finite degree n, containing a cycle of prime length which fixes at least three points. Then $G \geq A_n$.*

A different generalisation of Jordan's theorem (which is not dependent on the classification of finite simple groups) is thanks to Rowlinson and Williamson [55] and is as follows:

Theorem 6.3. *(Rowlinson and Williamson) Let G be a primitive permutation group of finite degree n, containing a p^r-cycle with $1 < p^r < n-2$, where p is a prime. If $p \equiv 2 \pmod 3$ then G is alternating or symmetric.*

The above statement will be sufficient for our purposes in proving Proposition 6.25 in which the difference between κ and λ is one, and we note that the above theorem was further generalised thanks to Neumann [49] to yield the same conclusion for a primitive permutation group when there is a cycle of prime power length and fixing at least three points, so long as the prime is not three. At this point in time we still rely on Theorem 6.1 (and hence also on the classification of finite simple groups) for the more general cases when the difference between κ and λ is greater than two.

An edge-biregular map $\mathcal{M}$ is also a regular map if and only if there is an automorphism of the group, $\theta : H \to H$, which interchanges x with s and y with t. In the case when $\mathcal{M}$ is in fact genuinely edge-biregular then such an automorphism does not exist.

When $m \neq 6$, all the group automorphisms of both the alternating group A_m and also the symmetric group S_m, can be identified with conjugations by elements of S_m. Conjugation naturally preserves cycle structure. Hence (in the case when $m \neq 6$) in order to demonstrate the non-existence of an assumed involutory automorphism θ as described above, it is sufficient to show that an element and its image under the assignment $x \to s$, $y \to t$, $s \to x$ and $t \to y$ have different cycle structures. We will use this method repeatedly during the course of this chapter.

When $m = 6$ there is an additional generating (outer) automorphism of the group S_6 which interchanges single transpositions with triple transpositions. This exceptional case will not trouble us too much, but we should bear it in mind.

In this preliminary section we will first note some useful lemmas and prove that (non-extremal) examples exist, thereby proving:

Proposition 6.4. *For any positive values of κ and λ such that $3 \leq \kappa \leq \lambda - 2$, there is a truly edge-biregular map of type $(2\kappa, 2\lambda)$ with H isomorphic to either S_m or A_m where $m = \lambda + 1$.*

We begin with a few observations, remembering that, in the context of permutation groups, an involution is a product of a number of disjoint transpositions.

Throughout we let permutations of a set act on the right. This is so that we may read naturally from left to right and hence, for example, 1 is mapped by (12)(23) to 3. More formally, the permutations $\pi_i \in \mathrm{S}_\Omega$ act by right multiplication so that $\omega(\pi_1\pi_2) = (\omega\pi_1)\pi_2$. All the groups we consider will act transitively on the set Ω, and this will be clear from the fact that all the diagrams resulting from our constructions are connected. Also, for the purposes of clarity, from here-on-in and until further notice, we will insert commas between the elements within each cycle of a permutation.

Notation: For the purposes of the next definition and the following constructions, we denote the set $\Omega_{(n,i)}$ which contains n elements, labelled from i up to $i + n - 1$, so that $\Omega_{(n,i)} = \{i, i+1, i+2, \ldots, i+n-1\}$. In particular, $\Omega_{(n,1)} = \{1, 2, 3, \ldots, n\}$ and $\Omega_{(n,0)} = \{0, 1, 2, \ldots, n-1\}$.

What follows is a way of defining two involutions a and b as permutations acting on a given set. For reference, an example of this constructive definition is shown in the diagram, Figure 6.1.

Definition 6.5. The *two-involution (a, b) chain on $\Omega_{(n,i)}$* : Each of a and b is a product of disjoint transpositions on the given set $\Omega_{(n,i)}$ of n distinct elements, in this case labelled $\Omega_{(n,i)} = \{i, i+1, i+2, \ldots, i+n-1\}$. The precise form of the involutions to be determined from this definition depends in a very natural way on the parity of n.

When n is even, we define:
$a = (i,\, i+1)(i+2,\, i+3)\ldots(i+2c,\, i+2c+1)\ldots(i+n-4, i+n-3)(i+n-2,\, i+n-1)$
and $b = (i+1,\, i+2)(i+3,\, i+4)\ldots(i+2c+1, i+2c+2)\ldots(i+n-3, i+n-2)$.

However when n is odd we define:
$a = (i,\, i+1)(i+2,\, i+3)\ldots(i+2c,\, i+2c+1)\ldots(i+n-3,\, i+n-2)$ and
$b = (i+1,\, i+2)(i+3,\, i+4)\ldots(i+2c+1, i+2c+2)\ldots(i+n-2, i+n-1)$.

Remark 6.6. We highlight that the permutations a and b in the ordered pair (a, b) are thus completely determined by their defining two-involution chain. Also, this definition ensures that the corresponding coset diagram is a path and not a cycle. It will be useful later to refer to a given line between two consecutive points in the

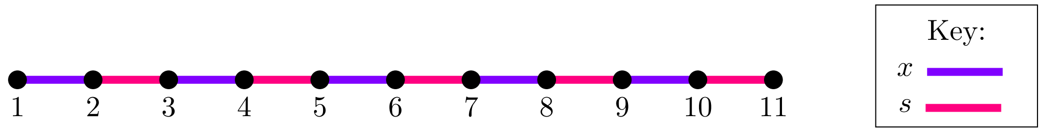

Figure 6.1: Example: a diagram of the two-involution (x, s) chain on $\Omega_{11,1}$

diagram of a chain as a *link*. Each link thus corresponds to a single transposition within precisely one of the chain generators a or b. Furthermore, (reading the corresponding diagram in ascending order) depending on the parity of n being even or odd, the two-involution (a, b) chain (which by our definition begins with an a-link) will end with an a-link or a b-link respectively. ⋆

Lemma 6.7. *Let a and b be determined by a two-involution (a, b) chain on $\Omega_{(n,i)}$, the set of n distinct points. Then the permutation ab is a single n-cycle.*

Proof. It is easy to check that when n is even we have
$ab = (i, i+2, i+4, \ldots, i+2c, \ldots i+n-2, i+n-1, i+n-3 \ldots i+3, i+1)$

while when n is odd we have

$ab = (i, i+2, i+4, \ldots, i+2c, \ldots i+n-3, i+n-1, i+n-2 \ldots i+3, i+1)$.

Thus ab is clearly a single cycle of length n. □

Moreover, we make the next easy (fence vs fencepost) observations which we will use to help us prove that our maps (which are largely constructed from combinations of these chain diagrams) are indeed genuinely edge-biregular.

Remark 6.8. A two-involution chain construction defined on a set which has even cardinality yields odd permutations. A two-involution (a, b) chain of permutations acting on $2n$ points has an odd number of links. Each involution is a product of a number of disjoint transpositions, and so the cycle structures of the two generating involutions a and b will necessarily differ by precisely one transposition. In particular, one of the involutions is an even permutation whilst the other is odd. We are thus potentially considering a symmetric group (rather than an alternating one) and clearly no automorphism sending a to b can exist. ⋆

Remark 6.9. A two-involution chain defined on a set with 1 modulo 4 elements yields even permutations. A two-involution (a, b) chain of permutations acting on a given m points has precisely $m-1$ links. By the definition of the chain, each involution is a product of a number of disjoint transpositions. In particular, both of the involutions a and b (defined by a given chain) are even permutations if and only if $m \equiv 1 \pmod 4$. We are thus potentially considering an alternating group, and if we rely solely on chain constructions to define the pairs of generators x, s and y, t, then this will be the case if and only if κ and λ are both congruent to 1 modulo 4. ⋆

We want to prove, for every (reasonable) pair of values (κ, λ), the existence of an edge-biregular map of type $(2\kappa, 2\lambda)$, whose colour-preserving automorphism group H is isomorphic to an alternating group or a symmetric group and H has the canonical presentation $H = \langle x, y, s, t | x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (ty)^\kappa, (sx)^\lambda \rangle$. We hope to construct examples for each within the set of alternating and symmetric groups.

The initial idea is to "stitch together" two two-involution chains, one (x, s) chain and one (t, y) chain, in such a way that the orders of all the elements and their products are as expected from the standard relators which occur in the canonical presentation.

From Lemma 6.7, it is clear that the number of points in a given (a, b) chain is the order of the element ab. We seek a small and uncomplicated diagram, and so investigate the following example:

Construction 6.10. Instructive example for when $6 \leq \kappa + 3 \leq \lambda + 1 = m$:

Let $\kappa \geq 3$ and $\lambda \geq \kappa + 2$ be given. We define the involutions x, y, s and t according to the two-involution (x, s) chain on $\Omega_{(\lambda,1)}$, while the two-involution (t, y) chain is defined on $\Omega_{(\kappa,0)}$, as per the above definition.

Thus when λ is even: $x = (1,\, 2)(3,\, 4) \ldots (2i-1,\, 2i) \ldots (\lambda-3, \lambda-2)(\lambda-1, \lambda)$ and $s = (2,\, 3)(4,\, 5) \ldots (2i,\, 2i+1) \ldots (\lambda-2,\, \lambda-1)$.

However when λ is odd we have: $x = (1,\, 2)(3,\, 4) \ldots (2i-1,\, 2i) \ldots (\lambda-2,\, \lambda-1)$ and $s = (2,\, 3)(4,\, 5) \ldots (2i,\, 2i+1) \ldots (\lambda-1,\, \lambda)$.

And, depending on the parity of κ, we have, when κ is even:
$t = (0,\, 1)(2,\, 3) \ldots (2i-2,\, 2i-1) \ldots (\kappa-4, \kappa-3)(\kappa-2,\, \kappa-1)$ and
$y = (1,\, 2)(3,\, 4) \ldots (2i-1,\, 2i) \ldots (\kappa-3,\, \kappa-2)$

or, when κ is odd: $t = (0,\, 1)(2,\, 3) \ldots (2i-2,\, 2i-1) \ldots (\kappa-3,\, \kappa-2)$ and
$y = (1,\, 2)(3,\, 4) \ldots (2i-1,\, 2i) \ldots (\kappa-2,\, \kappa-1)$. ⋄

An example of the corresponding diagram for the (t, y) chain on $\Omega_{(8,0)}$ is shown in Figure 6.2. Note the different labelling of the points compared to the example of an (x, s) chain on $\Omega_{(11,1)}$.

As an example, Figure 6.3 illustrates permutations on $m = \lambda + 1$ points, arranged for our convenience in a line with the points $\{0, 1, 2, \ldots, \lambda\}$ labelled in ascending order.

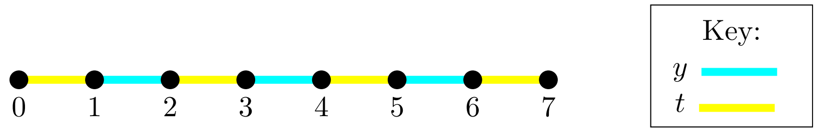

Figure 6.2: An example: the two-involution (t, y) chain on $\Omega_{(8,0)}$

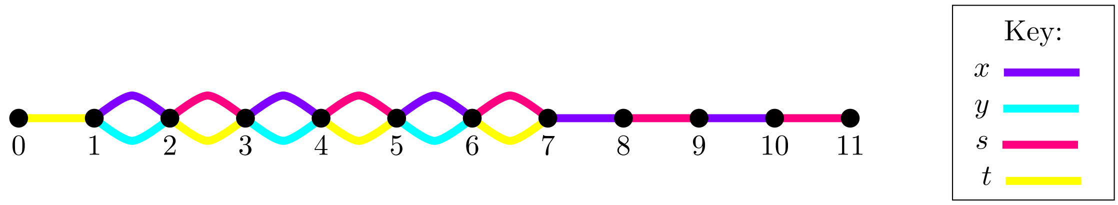

Figure 6.3: An example of Construction 6.10 where $\kappa = 8$ and $\lambda = 11$.

The permutations in question are those x, y, s, t, shown by lines in the colours violet, cyan, rose and yellow respectively, and defined by Construction 6.10 where $\kappa = 8$ and $\lambda = 11$. Realise that this is simply a natural amalgamation of examples of the two previously defined two-involution chains, while the two sets $\Omega_{(\kappa,0)}$ and $\Omega_{(\lambda,1)}$ have non-trivial intersection. In order to keep the diagram tidier, when a point is fixed by one of the involutions, we omit the loop of the corresponding colour at that point.

Remark 6.11. Interaction of commuting involutions. It is worth noting that the canonical group H for an edge-biregular map demands that x and y commute, that is $ord(xy) = 2$. This means that when constructing such diagrams for our purposes, we have only two possibilities for when a given point $\alpha \in \Omega$ is fixed by neither $x \in \mathrm{S}_\Omega$ nor $y \in \mathrm{S}_\Omega$. Either $\alpha x = \alpha y$ or $\alpha x = \beta \neq \gamma = \alpha y$ but then $\beta xyx = \alpha yx = \gamma x := \delta$ and so $\delta x = \gamma$ while $\delta y = \beta xyxy = \beta$. The same logic applies to any point which is moved (that is, not fixed) by both s and t. These two situations, which are the only possible options when a given element is moved by both of a pair of commuting involutions, are best demonstrated with diagrams, see Figure 6.4. ⋆

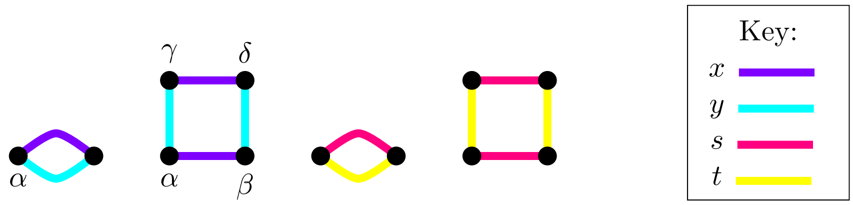

Figure 6.4: The two involutions x and y commute. Also s and t commute.

We now show that Construction 6.10 gives the group for an edge-biregular map.

Lemma 6.12. *Construction 6.10, which determines the permutations x, y, s, t on $m = \lambda + 1$ points, corresponds to the edge-biregular map $\mathcal{M} = (H; x, y, s, t)$ of type $(2\kappa, 2\lambda)$ in the natural way.*

Proof. We appeal to the equivalence between groups with the canonical presentation and edge-biregular maps. Then all that remains is to check the relators are all satisfied by referring to the canonical form (5.1). This is made clear by the construction, and so can be done by inspection. □

Next we prove the permutation group resulting from the instructive example, Construction 6.10, is primitive.

Lemma 6.13. *Construction 6.10, which defines the group action of $H = \langle x, y, s, t\rangle$ on m points, yields a primitive permutation group of degree m.*

Proof. The only thing to prove is the primitivity of the action, since H is by definition a permutation group of degree m. Consider xs, which by Lemma 6.7 forms an λ-cycle on the $m-1$ points $\{1, \dots, \lambda\}$, while leaving 0 fixed. Thus the action of the stabiliser of one point, in this case 0, is transitive on all the other points. It follows immediately that the action is primitive. □

And finally we show that, for certain values of κ and λ, we may apply the Jones version of Jordan's theorem to this construction.

Lemma 6.14. *Let $5 \leq \kappa + 2 \leq \lambda$. Construction 6.10, which defines the group action of $H = \langle x, y, s, t\rangle$ on $m = \lambda + 1$ points, represents an edge-biregular map of type $(2\kappa, 2\lambda)$ in the natural way. Furthermore, H is isomorphic to either A_m or S_m.*

Proof. We know from earlier, Lemma (6.12), that the relators are satisfied. By Lemma 6.7, ty is a κ-cycle and, by the assumption that $5 \leq \kappa + 2 \leq \lambda$, it is clear that the three points κ, $\kappa + 1$ and λ are all fixed by ty. Thus the cycle ty is of length at most $m - 3$ and now (using Lemma 6.13) we may invoke Theorem 6.1 to conclude that H is isomorphic to either A_m or S_m. □

Combining all these observations we can (up to duality) prove the following:

Lemma 6.15. *For any positive values of κ, and λ, both greater than 2 and with difference at least 2, there is a genuinely edge-biregular map of type $(2\kappa, 2\lambda)$ with H isomorphic to either S_m or A_m where $m = \lambda + 1$. In particular, if the map is built using Construction 6.10, the group $H \cong A_m$ if and only if both κ and λ are congruent to 1 modulo 4.*

This lemma relies heavily on the difference between κ and λ being large enough, in combination with the "extra" point at 0, and uses the asymmetry of the involutions acting on the point labelled 0 to prove the primitivity of the action of $H = \langle x, y, s, t\rangle$ in Construction 6.10. Ideally we would like to improve this to include small or close together cases, as well as limiting the degree to $m = \max\{\kappa, \lambda\}$. This will be addressed in the following subsection.

We now turn our attention to considering when the edge-biregular map constructed using an (x, s) chain and a (t, y) chain is genuinely edge-biregular.

Lemma 6.16. *If (at least) one of κ or λ is even, then Construction 6.10 will result in $\mathcal{M} = (H; x, y, s, t)$, a genuinely edge-biregular map whose colour-preserving automorphism group H is isomorphic to $S_{\lambda+1}$.*

Proof. By Lemma 6.12 we know that the construction gives an edge-biregular map. Also, by Lemma 6.13 and Theorem 6.1, we know that H is isomorphic to either the alternating or the symmetric of degree $\lambda + 1$.

The fact that the resulting map is genuinely edge-biregular is immediate from the evenness of at least one of κ, λ by Remark 6.8. Notice also that the group generated by x, y, s and t must be isomorphic to the full symmetric group on $\lambda + 1$ elements since there must be an odd permutation in the generating set.

Thinking about it another way, the alternating group A_m contains only even permutations, and in particular any cycle in $\mathrm{A}_{\lambda+1}$ must have odd length. So, if one of κ or λ is even then, since xs is a λ-cycle and ty is a κ-cycle, H contains a permutation which is a cycle of even length, and hence $H \cong \mathrm{S}_{\lambda+1}$. □

Proof of Proposition 6.4. Let $\mathcal{M}(H; x, y, s, t)$ be an edge-biregular map of type $(2\kappa, 2\lambda)$, defined by Construction 6.10. We remember that by Lemma 6.14 we have $\kappa + 2 \le \lambda$, with H isomorphic to either S_m or A_m where $m = \lambda + 1$.

Lemma 6.16 demonstrates that if (at least) one of the κ and λ is even then we certainly have a genuinely edge-biregular map whose colour-preserving automorphism group is $\mathrm{S}_{\lambda+1}$.

The only case which remains to be addressed is when both κ and λ are odd. In this case the associated (x, s) and (t, y) chains each have an even number of links, and so x and s have the same cycle structure, and similarly y and t have the same cycle structure.

Let both $\kappa = 2\kappa' + 1$ and λ be odd.

To prove the genuineness of the edge-biregularity, we must demonstrate that there is no automorphism θ which interchanges x with s and y with t. To prove that the map is genuinely edge-biregular, so long as $m \neq 6$, it is sufficient to show that an element of H and its image under the assignment $x \to s, y \to t, s \to x, t \to y$ have different cycle structures.

We note that, following the definitions from Construction 6.10, the involutary permutations x, y, s and t are defined on m elements $\{0, 1, \ldots, \lambda\}$, where $m = \lambda + 1 = \kappa + r + 1$.

Since κ and λ are both odd we have $r = \lambda - \kappa \geq 2$ is even:

$x = (1,\ 2)(3,\ 4)\ldots(\kappa-2,\ \kappa-1)(\kappa,\ \kappa+1)(\kappa+2,\ \kappa+3)\ldots(\kappa+r-2,\ \kappa+r-1)$

$y = (1,\ 2)(3,\ 4)\ldots(\kappa-2,\ \kappa-1)$

$s = (2,\ 3)(4,\ 5)\ldots(\kappa-3,\ \kappa-2)(\kappa-1,\ \kappa)(\kappa+1,\ \kappa+2)\ldots(\kappa+r-1,\ \kappa+r)$

$t = (0,\ 1)(2,\ 3)\ldots(\kappa-3,\ \kappa-2)$

Notice, yxs always fixes the point 0, regardless of the sizes of κ and/or λ.

The cycle structure of tsx depends on the size of κ and the difference between κ and λ, as we shall now see.

When $\kappa = 3$ we have:

$x = (1,\ 2)(3,\ 4)(\kappa+2,\ \kappa+3)\ldots(\lambda-2,\ \lambda-1)$

$y = (1,\ 2)$

$s = (2,\ 3)(\kappa+1,\ \kappa+2)\ldots(\lambda-1,\ \lambda)$

$t = (0,\ 1)$.

This yields $tsx = (0,\ 2,\ 4,\ \ldots,\ \lambda-1,\ \lambda,\ \lambda-2,\ \ldots,\ 3,\ 1)$, a single $(\lambda+1)$-cycle.

However, when $\kappa = 5$ and $\lambda \geq 7$, we have:

$x = (1,\ 2)(3,\ 4)(5\ ,6)\ldots(\lambda-2,\ \lambda-1)$

$y = (1,\ 2)(3,\ 4)$

$s = (2,\ 3)(4,\ 5)(6,\ 7)\ldots(\lambda-1,\ \lambda)$

$t = (0,\ 1)(2,\ 3)$.

Hence $tsx = (0,\ 2,\ 1)(3,\ 4,\ 6, \ldots\ \lambda-1,\ \lambda,\ \lambda-2,\ \ldots,\ 7,\ 5)$ which has a different cycle structure to the previous example, but importantly, also no fixed points.

When considering $\kappa = 7$ and $\lambda \geq 9$, we begin to see the more general structure emerging: $x = (1,\, 2)(3,\, 4)(5,\, 6)(7,\, 8)\ldots(\kappa+2,\, \kappa+3)\ldots(\lambda-2,\, \lambda-1)$

$y = (1,\, 2)(3,\, 4)(5,\, 6)$

$s = (2,\, 3)(4,\, 5)(6,\, 7)(8,\, 9)\ldots(\kappa-3,\, \kappa-2)(\kappa-1, \kappa)\ldots(\lambda-1,\, \lambda)$

$t = (0,\, 1)(2,\, 3)(4,\, 5)$.

Hence $tsx = (0,\, 2,\, 1)(3,\, 4)(5,\, 6,\, 8,\, \ldots,\, \lambda-1,\, \lambda,\, \ldots\, 7,\, 5)$ has yet another cycle structure, including transpositions, but again importantly, no fixed points.

More generally, when $\kappa = 2\kappa' + 1 \geq 9$ and $\lambda \geq 11$, we have:
$x = (1,\, 2)(3,\, 4)(5,\, 6)(7,\, 8)\ldots(\kappa-2,\, \kappa-1)(\kappa,\, \kappa+1)\ldots(\lambda-2,\, \lambda-1)$,

$y = (1,\, 2)(3,\, 4)(5,\, 6)(7,\, 8)\ldots(\kappa-2, \kappa-1)$,

$s = (2,\, 3)(4,\, 5)(6,\, 7)(8,\, 9)\ldots(\kappa-3,\, \kappa-2)(\kappa-1, \kappa)\ldots(\lambda-1,\, \lambda)$,

$t = (0,\, 1)(2,\, 3)(4,\, 5)(6,\, 7)\ldots(\kappa-3,\, \kappa-2)$ and hence

$tsx = (0, 2, 1)(3, 4)(5, 6)\ldots(\kappa-4,\, \kappa-3)(\kappa-2, \kappa-1, \kappa+1, \ldots, \lambda-1, \lambda, \lambda-2, \ldots \kappa+2, \kappa)$.

Written in the natural way, this is a 3-cycle, a number of transpositions and an $(r+3)$-cycle. This cycle structure, when $\kappa' \geq 2$, is $2^{\kappa'-2}3^1(r+3)^1$, and so notably tsx has no fixed points.

The above workings immediately contradict any claim that there is an automorphism θ, since when $m \neq 6$ the cycle structures for tsx and yxs differ, while in the exceptional case where $\kappa = 3$, $\lambda = 5$, and $m = 6$, the elements in question have different orders, six and four respectively. So we may conclude that the edge-biregular map resulting from Construction 6.10 is genuinely edge-biregular.

Thus the proposition is proved. □

When considering which group will be generated by a given construction, it is clear that we will get A_m if and only if all the generating elements x, y, s and t are even permutations. By the form of any construction consisting of precisely one chain for each of xs and yt, this can only be the case when κ and λ are both congruent to 1 modulo 4.

We do not make any such claim about possible edge-biregular maps built from another form of construction. It is easy to imagine another construction which would give the right orders for elements but look different to the union of a pair of two-involutions chains. For example: it is possible to ensure the element yt splits into disjoint cycles of lengths according to the prime power decomposition of its order, even when κ is a prime power. See Section 6.4 for further thoughts and an example.

More precisely, by our constructions, x (and likewise s) is a product of disjoint transpositions, and y and t are also both products of disjoint transpositions. The number of transpositions in each case depends on the number of links in the chain. In order to have a chance of being the alternating group, we would need κ, λ both being congruent to 1 modulo 4. Indeed, if they both are, then all the generators, and hence all the elements in the group H, are even permutations, and so $H \leq \mathrm{A}_m$. But we know from Theorem 6.1 that $H \cong \mathrm{A}_m$ or $H \cong \mathrm{S}_m$, so we conclude that in Construction 6.10, it is the case that $H \cong \mathrm{A}_m \iff \kappa, \lambda \equiv 1 \pmod 4$.

Remember: the case when at least one of κ and λ are even has been addressed with the help of a much more basic observation made in Remark 6.8.

Construction 6.10, used for the above maps, has the advantage of it being very easy to demonstrate the primitivity of the permutation group. A natural question might be:

Could similar be done with $H \cong \mathrm{A}_m$ and/or $H \cong \mathrm{S}_m$ for smaller degree $m < l + 1$?

Remark 6.17. The element $sx \in H$ has order λ. In constructions which include a λ-chain, the minimum degree is $m \geq \lambda$. Moreover, if we can minimise the degree as much as possible then $m = \lambda$. Note that, when $\lambda = m$ is prime, such a construction would yield the extremal example of an edge-biregular map of type $(2\kappa, 2\lambda)$ whose group H is isomorphic to an alternating or symmetric group. Meanwhile if λ is composite with more than one prime factor, the degree of the resulting permutation group could be minimised further, as shown in Section 6.4. $\star$

6.1.2 Limiting the degree to $m = \max\{\kappa, \lambda\}$

We still work up to duality, and so assume that $3 \leq \kappa \leq \lambda$. In each of the following subsections we address the cases when the difference between κ and λ is three or more, zero, one or two.

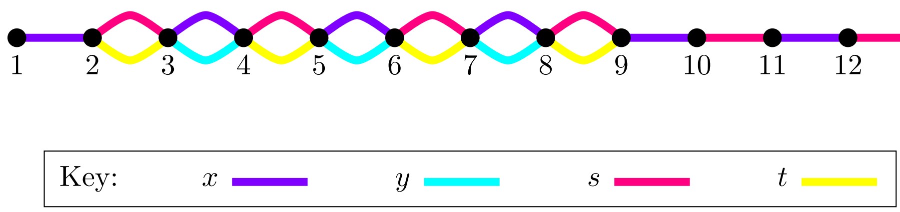

Figure 6.5: An example of Construction 6.18 where $\kappa = 8$ and $\lambda = 13$.

6.1.2.1 When $6 \leq \kappa + 3 \leq \lambda = m$

From the understanding gained in the workings of the previous subsection, it is straightforward to come up with a construction which limits the degree of the group to $max\{\kappa, \lambda\}$, so long as the difference in κ and λ is at least 3.

Construction 6.18. For when $m = \lambda = \kappa + r$ when $r \geq 3$, and $\kappa \geq 3$:

Let κ and λ be given such that $6 \leq \kappa + 3 \leq \lambda$. We define the involutions x, y, s and t according to the two-involution (x, s) chain on $\Omega_{(\lambda,1)}$, while the two-involution (t, y) chain is defined on $\Omega_{(\kappa,2)}$, as per the definition.

Thus when λ is even:

$x = (1,\ 2)(3,\ 4)\dots(2i-1,\ 2i)\dots(\lambda-3, \lambda-2)(\lambda-1, \lambda)$ and
$s = (2,\ 3)(4,\ 5)\dots(2i,\ 2i+1)\dots(\lambda-2,\ \lambda-1)$.

However when λ is odd we have:

$x = (1,\ 2)(3,\ 4)\dots(2i-1,\ 2i)\dots(\lambda-2,\ \lambda-1)$ and
$s = (2,\ 3)(4,\ 5)\dots(2i,\ 2i+1)\dots(\lambda-1,\ \lambda)$.

And, depending on the parity of κ, we have, when κ is even:
$t = (2,\ 3)\dots(2i-2,\ 2i-1)\dots(\kappa-4, \kappa-3)(\kappa-2,\ \kappa-1)(\kappa,\ \kappa+1)$ and
$y = (3,\ 4)\dots(2i-1,\ 2i)\dots(\kappa-3,\ \kappa-2)(\kappa-1,\ \kappa)$

or, when κ is odd:

$t = (2,\ 3)\dots(2i-2,\ 2i-1)\dots(\kappa-3,\ \kappa-2)(\kappa-1,\ \kappa)$ and
$y = (3,\ 4)\dots(2i-1,\ 2i)\dots(\kappa-2,\ \kappa-1)(\kappa,\ \kappa+1)$. ◇

An example of the diagram corresponding to Construction 6.18 for $\kappa = 8$ and $\lambda = 13$ is as shown in Figure 6.5.

Proposition 6.19. *Given $\kappa \geq 3$ and $\lambda \geq \kappa + 3$, Construction 6.18 yields a genuinely edge-biregular map $\mathcal{M} = (H; x, y, s, t)$ of type $(2\kappa, 2\lambda)$ where $H \cong A_\lambda$ or $H \cong S_\lambda$. Moreover $H \cong A_\lambda \iff \kappa \equiv \lambda \equiv 1 \pmod 4$.*

Proof. Since the construction defines four involutions in two commuting pairs, it certainly determines an edge-biregular map. The construction ensures that each relator of the canonical presentation is satisfied. All that remains is to prove the isomorphism class of H and determine whether the map is genuinely edge-biregular.

In order to prove the primitivity of the action of H, consider the block of imprimitivity $\mathcal{B}_1$ which contains the element labelled 1. Let $a \neq 1$ be another element such that $a \in \mathcal{B}_1$.

First suppose that $a \leq \kappa + 1$, that is a is not fixed by y or t. Then we may apply, repeatedly, y and t to show that 1 (which is fixed by both y and t) is in the same block as all points $2, 3, 4, \ldots, \kappa, \kappa + 1$. Then we may apply s and x repeatedly to show that all points are in the same block, and so $\mathcal{B}_1 = \Omega_{(\lambda,1)}$.

The same reasoning applies in any situation where an element b, which is not fixed by y or t, is in the same block as c where $2 \leq c \leq \kappa + 1$. That is, if $b, c \in \mathcal{B}_i$ are such that $b \geq \kappa + 2$ and $2 \leq c \leq \kappa + 1$, and $\mathcal{B}_i$ is a block of imprimitivity, then $\mathcal{B}_i = \Omega_{(\lambda,1)}$.

Now suppose that $a \geq \kappa + 2$. Since 1 is fixed by s, we also have $(a)s \in \mathcal{B}_1$, and we also know that either $(a)s \neq a$ or $(a)s = a = \lambda$. Where $(a)s = \kappa + 1$ the above argument applies, so we may now assume that $(a)s \geq \kappa + 3$. In particular, $1, (a)s \in \mathcal{B}_1$, and after applying x we have $2, (a)sx \in \mathcal{B}_2$, where $\mathcal{B}_2$ is a block of imprimitivity. But $(a)sx \geq \kappa + 2$ so must be fixed by y and t, whereas 2 is not, and so $\mathcal{B}_2 = \Omega_{(\lambda,1)}$ by the same argument. Hence in all cases $\mathcal{B}_1 = \Omega_{(\lambda,1)}$.

Thus any block containing more than one element is in fact the whole set $\Omega_{(\lambda,1)}$, and we may conclude that the action of H is primitive.

Due to the difference in κ and λ being at least 3, it is easy to see that ty is a single cycle with at least 3 fixed points, allowing us to invoke Theorem 6.1.

The fact the resulting map is genuinely edge-biregular follows from observing that xst and sxy have different cycle structures. In some cases it may be easy to spot that the map is genuinely edge-biregular, simply by looking at the congruences classes of k and ℓ modulo 4, but ideally we need a proof which works for every case. The workings which follow show that xst is a product of at least one transposition with precisely one longer cycle of length r or $r + 1$. Meanwhile sxy is the product of a three-cycle, a number of transpositions (non-zero so long as $\kappa \geq 5$) and an additional cycle of length r or $r + 1$.

To be precise, when κ and $\lambda = \kappa + r$ are even: $xst = (1, 2)(3, 4) \ldots (2i - 1, 2i) \ldots (\kappa - 1, \kappa)(\kappa + 1, \kappa + 3, \ldots, \kappa + 1 + 2j, \ldots, \lambda - 1, \lambda, \lambda - 2, \ldots, \lambda - 2j, \ldots, \kappa + 4, \kappa + 2)$ and

$sxy = (1,2,3)(4,5)\dots(2i,2i+1)\dots(\kappa-2,\kappa-1)(\kappa,\kappa+2,\dots,\kappa+2j,\dots,\lambda-2,\lambda,\lambda-1,\dots,\lambda-1-2j,\dots,\kappa+3,\kappa+1)$.

When κ is odd and $\lambda = \kappa + r$ is even: $xst = (1,2)(3,4)\dots(2i-1,2i)\dots(\kappa-3,\kappa-1)(\kappa,\kappa+2,\kappa+4,\dots,\kappa+2j,\dots,\lambda-1,\lambda,\lambda-2,\dots,\lambda-2j,\dots,\kappa+3,\kappa+1)$ and
$sxy = (1,2,3)(4,5)\dots(2i,2i+1)\dots(\kappa-1,\kappa)(\kappa+1,\kappa+3,\dots,\kappa+1+2j,\dots,\lambda-2,\lambda,\lambda-1,\dots,\lambda-1-2j,\dots,\kappa+4,\kappa+2)$.

When κ is even and $\lambda = \kappa + r$ is odd: $xst = (1,2)(3,4)\dots(2i-1,2i)\dots(\kappa-1,\kappa)(\kappa+1,\kappa+3,\dots,\kappa+1+2j,\dots,\lambda-2,\lambda,\lambda-1,\dots,\lambda-1-2j,\dots,\kappa+4,\kappa+2)$ and
$sxy = (1,2,3)(4,5)\dots(2i,2i+1)\dots(\kappa-2,\kappa-1)(\kappa,\kappa+2,\dots,\kappa+2j,\dots,\lambda-1,\lambda,\lambda-2,\dots,\lambda-2j,\dots,\kappa+3,\kappa+1)$.

When both κ and $\lambda = \kappa + r$ are odd: $xst = (1,2)(3,4)\dots(2i-1,2i)\dots(\kappa-3,\kappa-1)(\kappa,\kappa+2,\kappa+4,\dots,\kappa+2j,\dots,\lambda-2,\lambda,\lambda-1,\dots,\lambda-1-2j,\dots,\kappa+3,\kappa+1)$ and
$sxy = (1,2,3)(4,5)\dots(2i,2i+1)\dots(\kappa-1,\kappa)(\kappa+1,\kappa+3,\dots,\kappa+1+2j,\dots,\lambda-1,\lambda,\lambda-2,\dots,\lambda-2j,\dots,\kappa+4,\kappa+2)$.

We must remember to address the potentially awkward case of S_6: In this case $xst = (1,2)(3,5,6,4)$ and $sxy = (1,2,3)(4,6,5)$.

The differing cycle structures of xst and sxy implies that there cannot be an automorphism of the group H which interchanges x with s and y with t. Hence the map resulting from Construction 6.18 is genuinely edge-biregular.

By the assumed construction, and expanding on Remark 6.9, the group H is alternating if and only if λ and κ are both in the congruence class 1 modulo 4. □

To summarise, when the difference between κ and λ is at least 3 and $\min\{\kappa,\lambda\} \geq 3$ then (taking the dual where necessary) Construction 6.18 gives an example of a genuinely edge-biregular map of type $(2\kappa, 2\lambda)$ on symmetric or alternating group of degree $m = \max\{\kappa,\lambda\}$.

6.1.2.2 When $6 \leq \kappa = \lambda = m$

Let $\mathcal{M} = (H; x, y, s, t)$ be an edge-biregular map of type $(2\kappa, 2\kappa)$. We wish to have a construction such that H is isomorphic to either S_m or A_m, ideally where the degree m is minimised.

I was nervous of the case when $\kappa = \lambda$ for a while, wondering how I was going to manage to break the symmetry (and thus prove primitivity) with so little space, as

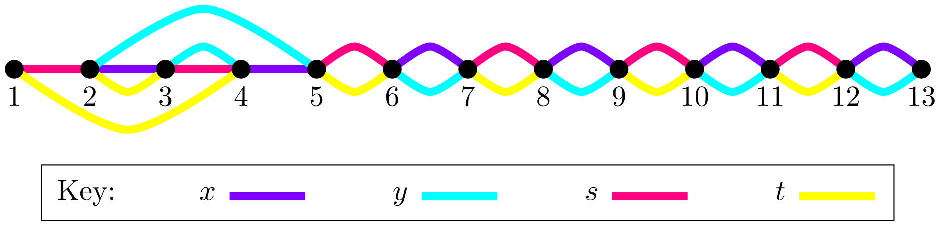

Figure 6.6: An example of Construction 6.20 where $\kappa = \lambda = 13$.

well as find a cycle with enough fixed points to be able to invoke Theorem 6.2 or one of its refinements. But it seems the following construction works, so long as $\kappa \geq 6$.

Indeed such an extremal genuinely edge-biregular maps exists in the case where κ is prime, the degree m can be as small as theoretically possible, $m = \kappa$. In the case where κ is not prime, the prime power decomposition of κ will yield a theoretical lower bound for the degree of such a group H, see Lemma 6.42 in Section 6.4. It is not proved whether this lower bound can be achieved in general. However, in order that the construction works for all cases where $\kappa = \lambda$, henceforth we wish to limit the degree to $m = \kappa$.

Construction 6.20. For when $6 \leq \kappa = \lambda = m$**:**

Let $6 \leq \kappa = \lambda$ be given. Let s and x be as determined by defining the (s, x) chain on $\Omega_{(\kappa,1)}$. Define t' and y' according to the (t', y') chain on $\Omega_{(\kappa-4,5)}$ and then let $t := (1\,4)(2\,3)t'$ and $y := (2\,5)(3\,4)y'$. $\diamond$

Example 6.21. When $\kappa = \lambda = 13$ Construction 6.20 yields:

$x = (2, 3)(4, 5)(6, 7)(8, 9)(10, 11)(12, 13)$

$y = (2, 5)(3, 4)(6, 7)(8, 9)(10, 11)(12, 13)$

$s = (1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)$

$t = (1, 4)(2, 3)(5, 6)(7, 8)(9, 10)(11, 12)$

See Figure 6.6. $\bullet$

Proposition 6.22. *When* $\kappa = \lambda \geq 6$*, Construction 6.20 yields* $\mathcal{M} = (H; x, y, s, t)$*, a genuinely edge-biregular map of type* $(2\kappa, 2\kappa)$ *where* $H \cong S_\kappa$ *or* $H \cong A_\kappa$*. Using this construction, the isomorphism class of* H *depends on the congruence class of* κ *modulo* 4*:* $H \cong A_\kappa$ *if and only if* $\kappa \equiv 1 \pmod 4$*.*

Proof. Let $\mathcal{M} = (H; x, y, s, t)$ be the edge-biregular map as defined by Construction 6.20. The relators of the canonical form are satisfied: xs and yt have the correct orders (κ), by inspection from the defining construction, and xy and st are both involutions since only the available forms as in Figure 6.4 are used.

First note that $xyst$ is a 3-cycle with all other points fixed $xyst = (1, 3, 5)(2)(4)(6)...$, so there is a cycle of prime length with at least three fixed points. Remember that we know κ is big enough since $\kappa \geq 6$.

To prove the primitivity of the action of H, consider the following permutations:
$xt = (2)(3)(1,\ 4,\ 6,\ \ldots\ 2i,\ \ldots\ \kappa,\ \ldots\ 2i-1,\ \ldots\ 5)$ while
$ys = (3)(4)(1,\ 2,\ 6,\ \ldots\ 2i,\ \ldots,\ \kappa,\ \ldots\ 2i-1,\ \ldots\ 5)$

This is an example of two permutations each with two fixed points and transitive on all other points, such that they share exactly one of the fixed points. To be explicit, both these elements stabilise the point labelled 3, and so the stabiliser in H of the point 3 is transitive on the remaining points. Hence the group is 2-transitive and thus primitive and so we can use Jordan's original theorem (Theorem 6.2) by which the group generated, $H = \langle x, y, s, t\rangle$, is isomorphic to S_κ or A_κ.

Now, when λ is even:
$stx = (1, 2, 5, 4, 3)(6, 7)(8, 9)(10, 11)(12, 13) \ldots (2j, 2j + 1) \ldots (\lambda - 2, \lambda - 1)(\lambda)$ and
$xys = (1, 2, 3, 6, 5, 4)(7, 8)(9, 10)(11, 12) \ldots (2j - 1, 2j) \ldots (\lambda - 1, \lambda)$.

Meanwhile, when λ is odd:
$stx = (1, 2, 5, 4, 3)(6, 7)(8, 9)(10, 11)(12, 13) \ldots (2j, 2j + 1) \ldots (\lambda - 1, \lambda)$ and
$xys = (1, 2, 3, 6, 5, 4)(7, 8)(9, 10)(11, 12) \ldots (2j - 1, 2j) \ldots (\lambda - 2, \lambda - 1)(\lambda)$.

Thus stx and xys have different cycle structures, indicating that there is no map isomorphism between $\mathcal{M}$ and its twin $\mathcal{M}^*$ when $m \geq 7$, and so the map $\mathcal{M}$ is genuinely edge-biregular. It is easy to show that the resulting map is genuinely edge-biregular also in the exceptional case when the degree of the permutation group is 6: the two pemutations not only have different cycle structures, but different orders too: stx has order five and xys has order six.

The only question which remains is the isomorphism class of H: Is it S_κ or A_κ? As previously, the resulting group H will be alternating if and only if all the generating involutions are even. Using the assumed construction, this is the case if and only if $\kappa \equiv 1 \pmod 4$. □

Next we will address the cases when the difference between κ and λ is 1 or 2. The principles behind each case are very similar to the reasoning in the previous subsections. The construction, an example, and a few comments in each case will be

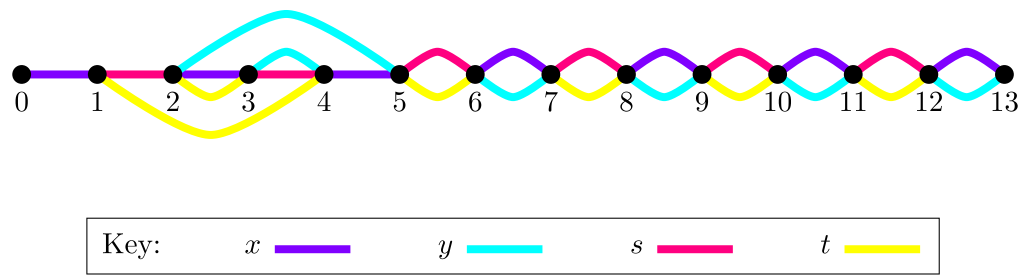

Figure 6.7: An example of Construction 6.23 where $\kappa + 1 = \lambda = 14$.

presented to provide sketch proofs of the corresponding propositions. Henceforth our constructions will require $\kappa \geq 6$, so the degree of the group will be more than six, so we no longer need to concern ourselves with the exceptional outer automorphism of the group S_6.

Ultimately we aim to prove the existence of $\mathcal{M}$, an edge-biregular map of type $(2\kappa, 2\lambda)$ where H is isomorphic to either S_m or A_m such that the degree m limited to $\max\{\kappa, \lambda\}$. We will be working up to duality, and so we continue to assume $\kappa \leq \lambda$.

6.1.2.3 When $7 \leq \kappa + 1 = \lambda = m$

In this section we consider the case when the difference between vertex valency and face length is one, that is, up to duality, when $\lambda = \kappa + 1$. The construction is remarkably similar to the previous example, Construction 6.20, with the definition for x involving an extra transposition.

Construction 6.23. For when $7 \leq \kappa + 1 = \lambda = m$:

Let $\kappa \geq 6$, and hence $\lambda = \kappa + 1$, be given. Let x and s be as determined by defining the (x, s) chain on $\Omega_{(\kappa+1,0)}$. Define t' and y' according to the (t', y') chain on $\Omega_{(\kappa-4,5)}$ and then let $t := (1\,4)(2\,3)t'$ and $y := (2\,5)(3\,4)y'$. ◇

Example 6.24. When $\kappa + 1 = \lambda = 14$ Construction 6.23 yields:

$x = (0, 1)(2, 3)(4, 5)(6, 7)(8, 9)(10, 11)(12, 13)$

$y = (2, 5)(3, 4)(6, 7)(8, 9)(10, 11)(12, 13)$

$s = (1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)$

$t = (1, 4)(2, 3)(5, 6)(7, 8)(9, 10)(11, 12)$

See Figure 6.7. •

Proposition 6.25. *Given $\kappa \geq 6$ and $\lambda = \kappa + 1$, Construction 6.23 yields a genuinely edge-biregular map $\mathcal{M}(H; x, y, s, t)$ of type $(2\kappa, 2\lambda)$ where $H \cong S_\lambda$.*

Proof. Since the construction defines four involutions in two commuting pairs, it certainly defines an edge-biregular map. All that remains is to prove the isomorphism class of H and determine whether the map is genuinely edge-biregular.

To justify that this construction yields a symmetric or alternating group, it is sufficient to note that the group generated by x, y, s, t is primitive, and has a sufficiently short cycle, this time of prime power length, so that we may once again invoke a theorem to conclude that $H \cong \mathrm{A}_\lambda$ or $H \cong \mathrm{S}_\lambda$.

Primitivity is easy to demonstrate since ty fixes one point, that labelled 0, and is transitive on the remaining points. A sufficiently short cycle still exists since $xyst = (0, 3, 5, 1)(2)(4)(6) \ldots$ is a four-cycle with at least three fixed points. Four is a prime power and in particular $4 = 2^2$ so by the Rowlinson and Williamson generalisation of Jordan's theorem (Theorem 6.3) the group generated is either symmetric or alternating of degree m. The difference of one between κ and λ ensures that one of them is even, forcing the isomorphism class of H to be the symmetric group S_λ, rather than the alternating group of the same degree.

It may be checked by inspection that the elements x, y, s, t satisfy the relators in the canonical presentation of the group H for an edge-biregular map $\mathcal{M}(H; x, y, s, t)$, and all we desire now is that the map can be shown to be genuinely edge-biregular. For this we again recall that κ and λ have a difference of 1 implying that one of them is even. The cycle structures of the corresponding pair of involutions (either x and s for when λ is even, or y and t when κ is even) must then be different by Remark 6.8. This precludes the existence of an involutory automorphism mapping $x \to s$ and $y \to t$, and so we may conclude that the resulting map is genuinely edge-biregular. □

6.1.2.4 When $8 \leq \kappa + 2 = \lambda = m$

Finally we consider the case when the difference between κ and λ is 2. Naturally we are still working up to duality and so assume $\lambda = \kappa + 2$. Again the construction is built on the previous one, Construction 6.23, which itself came from the one before, Construction 6.20.

Construction 6.26. For when $8 \leq \kappa + 2 = \lambda = m$:

Let $\kappa \geq 6$, and hence $\lambda = \kappa + 2$, be given. Let x and s be as determined by defining the (s, x) chain on $\Omega_{(\kappa+2,-1)}$. To help with ease of reading the notation, we denote the

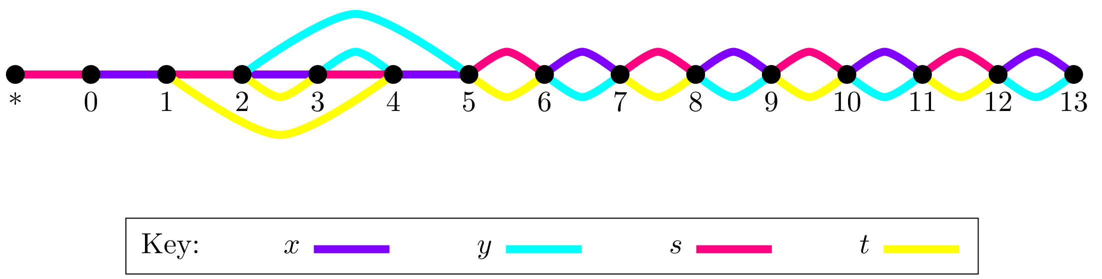

Figure 6.8: An example of Construction 6.26 where $\kappa + 2 = \lambda = 15$.

point labelled -1 by $*$. Define t' and y' according to the (t', y') chain on $\Omega_{(\kappa-4,5)}$ and then define $t = (1\,4)(2\,3)t'$ while $y = (2\,5)(3\,4)y'$. ◇

Example 6.27. When $\kappa + 2 = \lambda = 15$ Construction 6.26 gives:

$x = (0, 1)(2, 3)(4, 5)(6, 7)(8, 9)(10, 11)(12, 13)$

$y = (2, 5)(3, 4)(6, 7)(8, 9)(10, 11)(12, 13)$

$s = (*, 0)(1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)$

$t = (1, 4)(2, 3)(5, 6)(7, 8)(9, 10)(11, 12)$

See Figure 6.8. •

Proposition 6.28. *Given $\kappa \geq 6$ and $\lambda = \kappa + 2$, Construction 6.26 yields a genuinely edge-biregular map $\mathcal{M} = (H; x, y, s, t)$ of type $(2\kappa, 2\lambda)$ where $H \cong S_\lambda$.*

Proof. As previously, the construction clearly defines four involutions in two commuting pairs, so it certainly defines an edge-biregular map $\mathcal{M} = (H; x, y, s, t)$. All that remains is to prove the isomorphism class of H and determine whether the map is genuinely edge-biregular.

Notice that yt fixes $*$ and 0 and acts transitively on the rest of the points. Meanwhile $xytx$ fixes $*$ and 1. Also $xytx$ must have the same cycle structure as its conjugate yt, and so is transitive on the other points. Hence the stabiliser of the point labelled $*$ is transitive on all other points. This implies that the group is primitive.

So long as $\lambda \geq 8$, which it is since we have assumed $\kappa \geq 6$, then $xyst = (*, 0, 3, 5, 1)$ is a sufficiently short cycle of prime length to satisfy the remaining condition for Theorem 6.2. Hence $H \cong \mathrm{A}_\lambda$ or $H \cong \mathrm{S}_\lambda$.

Now, since the difference in κ and λ is exactly two, we know they have the same parity. In both cases the result is a genuinely edge-biregular map with $H \cong \mathrm{S}_\lambda$. For a contradiction, assume there is an involutory automorphism of the group H called θ such that $\theta : x \to s$ and $\theta : y \to t$.

1. When both κ and λ are even, the construction includes odd permutations (by the observation made in Remark 6.8) and hence $H \cong \mathrm{S}_\lambda$. Meanwhile, by the same observation, the construction includes even permutations. Note that, by the construction using the two-involution (x, s)-chain, the mapping θ would map an even permutation to an odd and vice versa. Hence we have a contradiction and may conclude that the map $\mathcal{M}$ is genuinely edge-biregular in this case.

2. When κ and λ are both odd, since they have a difference of precisely two, one of them is congruent to 1 modulo 4 and the other is congruent to 3 modulo 4. Thus, by Remark 6.9 the two generating involutions whose product has order 3 modulo 4, are both odd permutations, leading us to conclude that $H \cong \mathrm{S}_\lambda$. Meanwhile, $stx = (*, 1, 2, 5, 4, 3, 0)(6, 7) \dots (\kappa - 1, \kappa)$ has a 7-cycle (and no fixed points) while $xys = (*, 0, 2, 3, 6, 5, 4, 1)(7, 8) \dots (\kappa - 2, \kappa - 1)(\kappa)$ does not have a 7-cycle (but it does have a fixed point in this case). The difference in these cycle structures contradicts the claim that there is an automorphism θ. Hence the map $\mathcal{M}$ is genuinely edge-biregular. □

We now summarise what we have proved so far.

6.1.3 Theorem from the constructions

Taken altogether, the previously introduced constructions, along with the associated propositions prove:

Proposition 6.29. *Given $\lambda \geq \kappa \geq 6$ there exists a genuinely edge-biregular map $\mathcal{M}(H; x, y, s, t)$ of type $(2\kappa, 2\lambda)$ whose colour-preserving automorphism group H is isomorphic to either the alternating or the symmetric group of degree λ.*

Proof. Given $\lambda \geq \kappa \geq 6$, let $\lambda - \kappa = d$. When $d \geq 3$ use Construction 6.18 and Proposition 6.19; for $d = 2$ use Construction 6.26 and Proposition 6.28; if $d = 1$ use Construction 6.23 and Proposition 6.25; and finally in the cases where $\kappa = \lambda$ use Construction 6.20 and Proposition 6.22. □

The fact that the dual of an edge-biregular map is also edge-biregular means that we can generalise as follows.

Proposition 6.30. *Given $\lambda \geq \kappa \geq 6$ there exists a genuinely edge-biregular map*

$\mathcal{M}'(H; x', y's', t')$ of type $(2\lambda, 2\kappa)$ whose colour-preserving automorphism group H is such that $H \cong A_\lambda$ or $H \cong S_\lambda$.

Proof. The dual of an edge-biregular map $\mathcal{M}(H; x, y, s, t)$ of type (k, ℓ) is $\mathcal{M}^*(H; y, x, t, s) = \mathcal{M}'(H; x', y's', t')$ which is also edge-biregular and has type (ℓ, k). So the existence of an edge-biregular map of type $(2\lambda, 2\kappa)$ with $H \cong \mathrm{A}_\lambda$ or $H \cong \mathrm{S}_\lambda$ is immediate from the previous Proposition. The absence of a homomorphism mapping $x \to s$ and $y \to t$ in the map $\mathcal{M}$ of type (k, ℓ) translates in the dual map $\mathcal{M}^*$ to the absence of a homomorphism mapping $y' \to t'$ and $x' \to s'$, which is equivalent to the map $\mathcal{M}'$ being genuinely edge-biregular. $\square$

Combining these two propositions proves the following theorem, leaving only small examples still to address.

Theorem 6.31. *Given $\kappa, \lambda \geq 6$ there exists a genuinely edge-biregular map $\mathcal{M}(H; x, y, s, t)$ of type $(2\kappa, 2\lambda)$ whose colour-preserving automorphism group H is isomorphic to either the alternating or the symmetric group of degree max$\{\kappa, \lambda\}$.*

We have, in fact, proved more, as the condition that $\kappa \geq 6$ only applies to the constructions where the difference between κ and λ is less than three. The less restrictive $\kappa \geq 3$ condition applies to the cases where $\lambda \geq \kappa + 3$. Next we aim to fill in as many gaps as possible to give a complete existence theorem.

6.2 The question of existence when κ and λ are small

The last two paragraphs of the previous section summarise progress so far towards an existence theorem for genuinely edge-biregular maps. Now we turn our attention to filling the remaining gaps, regardless of the isomorphism class of the underlying finite group H. To continue the theme of the chapter, where possible, we find a symmetric or alternating example and work up to duality, and so we assume $\kappa \leq \lambda$.

Thus far, in this chapter, we have assumed $\kappa \geq 3$, and focussed on cases where H is isomorphic to an alternating or symmetric group. For completeness, we now refer to edge-biregular maps with small values of κ which have been investigated elsewhere in this thesis.

When considering these small cases for given types, $(2\kappa, 2\lambda)$, so as to avoid confusion with the notation describing the type of a map, we revert to the traditional cycle notation for permutations, that is with no commas.

6.2.1 When $\kappa \leq 2$

The biggest gaps, being infinitely large, are when $\kappa = 1$ or $\kappa = 2$ for types $(2\kappa, 2\lambda)$. We do know, by Theorem 5.4, that these types of edge-biregular maps exist for $\kappa = 1$, but they coincide with the fully regular maps of these types, and so are not genuinely edge-biregular. However, by Remark 5.10, when $\kappa = 2$ genuinely edge-biregular maps do exist.

$\kappa = 1$ These edge-biregular maps of type $(2, 2\lambda)$ are addressed in Theorem 5.4 and Table 5.1. The underlying graphs are clearly cycles, and the corresponding maps are all known to be fully regular [57], with supporting surface of characteristic $\chi \geq 1$.

Note: The associated group $H = \langle x, y, s, t\rangle$ contains the dihedral face stabiliser subgroup of order 2λ which has index 1 or 2 in the group H. If we then also demand that H is isomorphic to an alternating or symmetric group then that is seriously limiting: $H \cong \mathrm{D}_6 \cong \mathrm{S}_3$ is the only candidate. Reference to the dihedral classification Table 5.1, or simply remembering that H for a proper edge-biregular map has a subgroup isomorphic to V_4, shows that there is no corresponding proper edge-biregular map.

$\kappa = 2$ Truly edge-biregular maps of types $(4, 2\lambda)$ exist for any $\lambda \geq 2$ by Remark 5.10 in the dihedral classification. In particular, when $\lambda = 2$ Euclidean examples exist, and when $\kappa = 2$ and $\lambda \geq 3$, Theorem 5.8 gives an example of a single-faced genuinely edge-biregular map of type $(4, 2\lambda)$ with $H \cong \mathrm{D}_{2\lambda}$ for even values of λ, and a two-faced genuinely edge-biregular map of type $(4, 2\lambda)$ with $H \cong \mathrm{D}_{4\lambda}$ for odd values of λ. We will now turn our attention to genuinely edge-biregular maps of these types on symmetric and alternating groups.

To continue the theme of this chapter, we present a construction for $\mathcal{M} = (H; x, y, s, t)$ in the case when $\kappa = 2$ and $\lambda \geq 7$, such that the colour-preserving automorphism group H is symmetric, with degree λ.

Construction 6.32. For when $\kappa = 2$ and $\lambda = m \geq 7$:

Let $\kappa = 2$, and $\lambda \geq 7$, be given. Let x and s be as determined by defining the (x, s) chain on $\Omega_{(\lambda,1)}$. Define $y := (12)$ and $t := (47)(56)$. $\diamond$

See Figure 6.9 for an example.

Proposition 6.33. *Given $\lambda \geq 7$, Construction 6.32 yields $\mathcal{M} = (H; x, y, s, t)$, a genuinely edge-biregular map of type $(4, 2\lambda)$ where $H \cong S_\lambda$.*

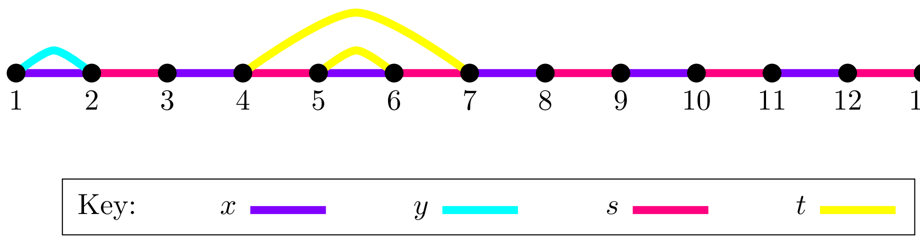

Figure 6.9: An example of Construction 6.32 where $\lambda = 13$.

Type	x	y	s	t	$H \cong$
$(4,6)$	$(12)(67)$	$(17)(26)(45)$	$(23)(56)$	(17)	S_7
$(4,8)$	$(12)(34)$	(12)	(23)	(45)	S_5
$(4,10)$	$(12)(34)$	$(14)(23)$	$(23)(45)$	(23)	S_5
$(4,12)$	$(12)(45)$	$(15)(24)$	(23)	(15)	S_5

Table 6.1: Generators for examples of extremal genuinely edge-biregular maps of type $(4, 2\lambda)$ on alternating and symmetric groups. See Figure 6.10.

Proof (sketch). By referring to equation (5.1), the canonical form of the presentation for H, it is clear that the construction describes an edge-biregular map of type $(4, 2\lambda)$. By virtue of the different cycle structures of y and t, the resulting map is genuinely edge-biregular in all cases.

It remains to address the isomorphism class which can be shown to be symmetric as follows. sx is a single transitive cycle and y is a single 2-cycle transposing two consecutive points within the sx-cycle, namely 1 and 2. Conjugating y by powers of sx shows that all transpositions of adjacent elements within the sx-cycle are therefore within the group $H = \langle x, y, s, t\rangle$ and hence $H = Sym(\Omega_{(\lambda,1)}) \cong S_\lambda$. □

We note, using the same reasoning as previously, that if λ is prime then Construction 6.32 will yield a genuinely edge-biregular map $\mathcal{M} = (H; x, y, s, t)$ of type $(4, 2\lambda)$ which is extremal in the sense that the degree m of $H \cong S_m$ is minimised, $m = \lambda$.

This leaves under question, for $\kappa = 2$, the small cases of symmetric or alternating genuinely edge-biregular maps of type $(4, 2\lambda)$ where $2 \leq \lambda \leq 6$. A map of type $(4, 4)$ is Euclidean, and the structure of the groups for Euclidean edge-biregular maps are not symmetric or alternating, see Sections 5.1.2.2 and 5.1.2.3. To address the other cases, Table 6.1 lists an extremal example of a genuinely edge-biregular map, with symmetric colour-preserving automorphism group, for every type $(4, 2\lambda)$ where $3 \leq \lambda \leq 6$.

Lemma 6.34. *The examples of genuinely edge-biregular maps $\mathcal{M} = (H; x, y, s, t)$ given by the generators in Table 6.1 are extremal with respect to the degree of the symmetric permutation group H.*

Proof. We first consider the maps of type $(4, 6)$.

Let $\mathcal{M} = (H; x, y, s, t)$ be a genuinely edge-biregular map of type $(4, 6)$ such that H is symmetric or alternating. The degree of the permutation group must be at least four, since S_3 contains no copy of V_4. The order of sx is three and so either there is at least one two-involution (x, s) chain on three points, or cycle of length six which splits into two 3-cycles. Up to labelling, that is without loss of generality, we may assume that the transposition (12) occurs in the involution x while (23) occurs in s.

Suppose the degree of H is 4. In order to maintain the order of xs (which is three), x and s must both fix 4, so in fact we have $x = (12)$ and $s = (23)$. Now we know that y and x commute, and y is a transposition, so $y = (34)$ or $y = (12)(34)$. Meanwhile, t, an involution, must commute with s and y, so there is a contradiction.

Suppose now that the degree of H is five. Respecting the order of xs (which is three) leads to two cases which we dismiss in turn.

Case A: $x = (12), s = (23)$

Now, $y \neq x$ and y commutes with x, so, up to labelling of elements, $y \in \{(12)(34), (34), (12)(45), (45)\}$. Meanwhile $t \in \{(23)(14), (23)(15), (23)(45), (14), (15), (45)\}$. The only options for a pair of involutions y, t which don't both leave the point labelled 5 fixed and also have a product of order 2 are: $y = (34), t = (15)$ which yields a fully regular map; alternatively $y = (12)(45), t = (45)$ or $y = (45), t = (23)(45)$, neither of which yields a transitive permutation group of degree five, in contradiction to the assumption that H is isomorphic to a symmetric group.

Case B: $x = (12)(45), s = (23)(45)$

Now, y commutes with x so $y \in \{(12), (45), (14)(25), (15)(24)\}$ and t commutes with s so $t \in \{(23), (45), (34)(25), (35)(24)\}$. Also, the two permutations y and t must commute since the valency of the map is 4. So either $y = (12), t = (45)$, or $y = (45), t = (23)$. In each case, the permutation group is not transitive, and so cannot be symmetric or alternating.

For further contradictions, we now suppose that the degree of the permutation group H is six. There are then four different possibilities for the two involutions s, x such that their product has order three. Without loss of generality (that is, up to labelling of the six elements in the set of objects being permuted) the cases are listed here. In order to avoid tedious checking, in each case we give a sketch proof as to how/ why the group generated is not S_6, with reference to diagrams in Figure 6.10. We work up to twinness of maps, which means that in each case we first consider the possibilities for y, followed by conclusions we may draw about the involution t.

Case 1: $x = (12), s = (23)$

Since we require the diagram to be connected, at least one of y and t must be a double (or triple) transposition. Assume, up to twinness, that y is not a single transposition, and as such it corresponds to one of the diagrams, Case 1(a), 1(b), 1(c) or 1(d), in the first row of Figure 6.10. From inspection, the only options for an involution t which commutes with both y and s, leave the diagram still not connected. Hence the group is not transitive, contrary to our assumption that H is symmetric or alternating of degree six.

Case 2: $x = (12)(45), s = (23)(45)$

Again we require the diagram to be connected, specifically to the point labelled α. Using y as is our custom (that is, up to twinness), since $y \neq x$ commutes with x, there is only one option to connect α, and this is to the component containing the three points $1, 2, 3$. This step is shown as Case 2(i) in Figure 6.10. At this stage it is impossible to find t which satisfies the required relations and connects the diagram, so we conclude y must be the triple transposition as shown in the diagram labelled Case 2(ii). [**Note**: The labels for 4 and 5 may be interchanged if necessary, the labels in the diagram are omitted to allow for this.] In order to satisfy the relators in equation (5.1) for a map of type $(4, 6)$, the transposition t must commute with both s and y. Hence t must be as shown in Case 2(iii). At a glance this may look promising, but $\mathcal{B}_1 := \{1, 2, 3\}$ is a block of imprimitivity, contradicting the assumption that the group H is symmetric or alternating.

Case 3: $x = (12)(56), s = (23)(45)$

In this case we assume (up to twinness) that y connects the two components of the diagram. The options are shown in Figure 6.10. In each case, the sets defined by $\mathcal{B}_i := \{i, 7 - i\}$ for each $i \in \{1, 2, 3\}$ form a block system with respect to the group $\langle x, y, s\rangle$. The inclusion of t (which must be an involution and also commute with both s and y) fails to break this block system, meaning that H is not primitive, and so we have a contradiction.

Case 4: $x = (12)(34)(56), s = (16)(23)(45)$

Remembering that $y \neq x$ yields (up to the labelling of points which has been deliberately omitted) six cases as shown in Figure 6.10. As before, in each case there is a block system which partitions the diagram into three "levels", as shown in the diagram for Case 4(a) in Figure 6.10, each consisting of a pair of points. By virtue of t being an involution which commutes with both y and s, the inclusion of t cannot destroy the blocks of imprimitivity. This is the final contradiction.

Remark 6.35. The extra-ordinary outer automorphism of S_6 swaps single transpositions with triple transpositions. In this respect, Case 4 is equivalent to Case 1, and we could have disregarded it on those grounds. $\star$

It is straightforward to check that the map of type $(4, 6)$ in Table 6.1 is genuinely edge-biregular with $H \cong \mathrm{S}_7$ and, by the above reasoning, extremal with respect to the degree of the permutation group. To avoid excessive tedium, we leave it to the reader to convince themselves that the colour-preserving automorphism group for a genuinely edge-biregular map of type $(4, 6)$ cannot be isomorphic to A_7, noting only that in such a group, all involutions and hence x, y, s and t are double transpositions, which turns out to be simply too limiting when attempting to satisfy the required relators from the canonical form (5.1) for an edge-biregular map of this type.

Turning our attention to the remaining rows of Table 6.1, it is easy to check that the conditions for a presentation of the form (5.1) are satisfied and also that the maps of type $(4, 10)$ and $(4, 12)$ are genuinely edge-biregular, while a quick computer check will yield the isomorphism classes. Considering the requirement to have elements of order five and six respectively, it is clear that the degree of the groups (five in both cases) has been minimised, giving, with respect to the degree of the permuatation group, an extremal example for each type.

It remains to consider the map of type $(4, 8)$. Again, it is simple to check the required relators are satisfied, and the only question still unanswered is whether there might be an example on a symmetric group of smaller degree than five.

Suppose $\mathcal{M} = (H; x, y, s, t)$, a genuinely edge-biregular map of type $(4, 8)$, exists on S_4. Then we may, up to twinness, and without loss of generality, assume that $x = (12)(34)$ and $s = (23)$. The map is proper so $x \neq y$, and x and y commute so either y is a single transposition, which renders the required existence of t impossible, or $y = (14)(23)$. Now $t \neq s$ so $t = (14)$ or $t = (14)(23) = y$. In either case $\{1, 4\}$ and $\{2, 3\}$ is a block system, and so the group generated by the canonical involutions is only a subgroup of S_4, not the full symmetric group.

We may now conclude that the examples in Table 6.1 are genuinely edge-biregular maps, with symmetric colour-preserving automorphism group of minimal degree. $\square$

To summarise the work in this subsection, we now have the following.

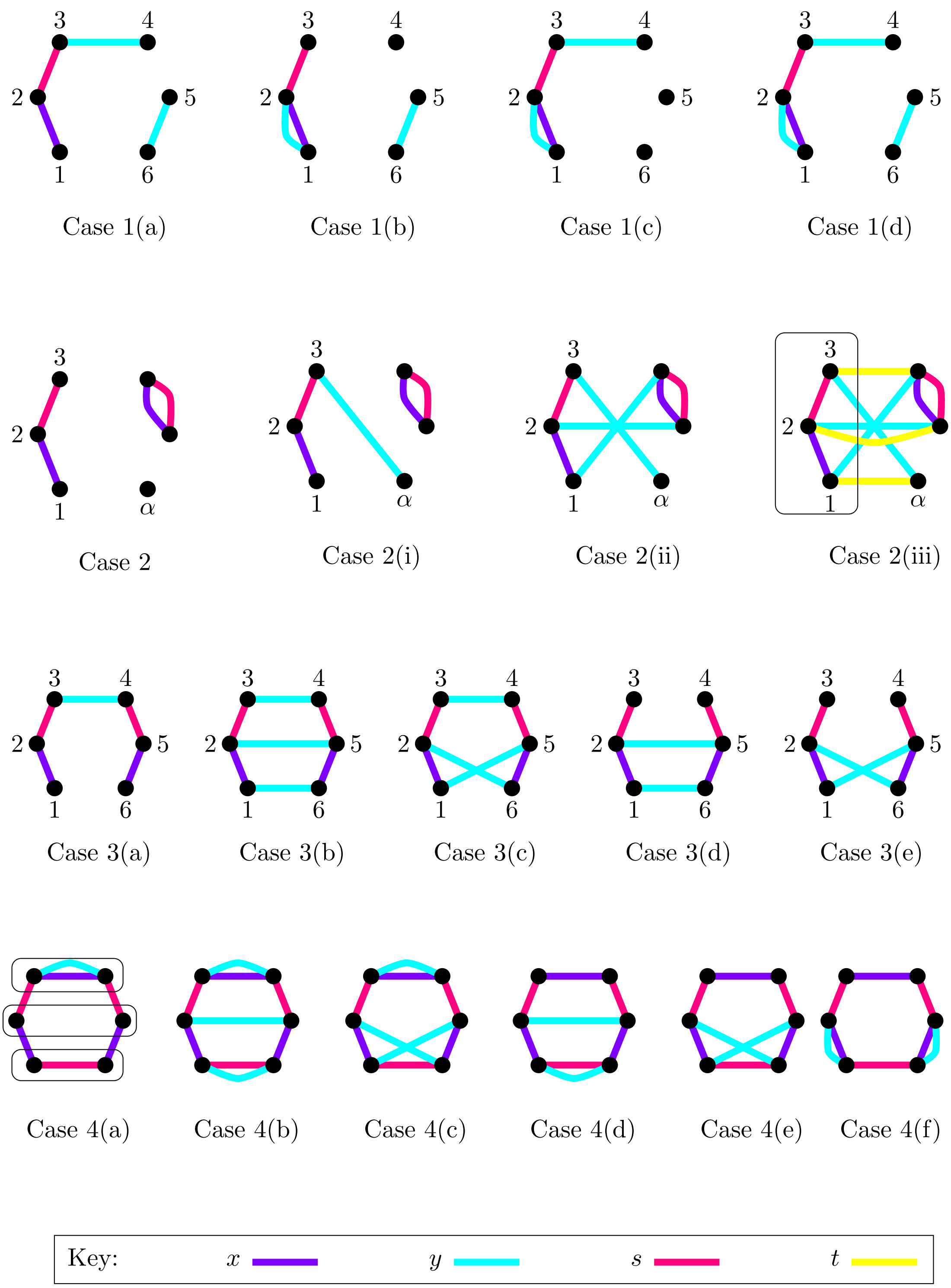

Figure 6.10: Diagrams to assist in excluding $H \cong \mathrm{S}_6$ for $\mathcal{M} = (H; x, y, s, t)$, a genuinely edge-biregular map of type $(4, 6)$. See Lemma 6.34.

Type	x	y	s	t	$H \cong$
$(6,6)$	(12)	(34)	(23)	(45)	S_5
$(6,8)$	$(12)(34)$	(12)	(23)	(14)	S_4
$(6,10)$	$(12)(34)$	(12)	$(23)(45)$	(23)	S_5
$(8,8)$	$(12)(34)$	(34)	(23)	$(14)(23)$	S_4
$(8,10)$	$(12)(34)$	$(13)(24)$	$(23)(45)$	(23)	S_5
$(8,12)$	$(12)(34)$	$(14)(23)$	(45)	(13)	S_5
$(10,10)$	$(12)(34)$	$(13)(24)$	$(23)(45)$	$(34)(25)$	A_5
$(10,12)$	(12)	$(12)(34)$	$(25)(34)$	$(23)(45)$	S_5
$(10,14)$	$(12)(34)(56)$	$(13)(24)$	$(23)(45)(67)$	$(23)(45)$	S_7

Table 6.2: Generators for extremal examples of genuinely edge-biregular maps on alternating and symmetric groups. See Figure 6.11.

Lemma 6.36. *Given an integer $\lambda \geq 2$, there exists a genuinely edge-biregular map of type $(4, 2\lambda)$. Moreover, for each type where $\lambda \geq 3$, the colour-preserving automorphism group H can be symmetric.*

Proof (sketch). The Klein bottle supports genuinely edge-biregular maps of type $(4, 4)$, see Theorem 5.7. For $3 \leq \lambda \leq 6$, the maps described in Table 6.1 are all genuinely edge-biregular with symmetric colour-preserving automorphism group. Proposition 6.33 then covers all remaining types $(4, 2\lambda)$ for $\lambda \geq 7$, yielding $H \cong S_\lambda$. □

6.2.2 When $3 \leq \kappa \leq 5$ and $\kappa \leq \lambda \leq \kappa + 3$

Thus the existence question remains open only for genuinely edge-biregular maps of the following types: $(6,6); (6,8); (6,10); (8,8); (8,10); (8,12); (10,10); (10,12);$ and $(10,14)$. In finding these, permutation groups come to the rescue again, as they provide an easy way of building small examples of edge-biregular maps. All we require for an edge-biregular map of type $(2\kappa, 2\lambda)$ is four involutory permutations x, y, s, t such that: x, y commute; s, t commute; yt has order κ; and xs has order λ. In order to yield a proper edge-biregular map we also need $x \neq y$ and $s \neq t$, and to ensure the corresponding maps are genuinely edge-biregular we may apply the similar reasoning as repeatedly used earlier in this chapter. In each case I have taken the opportunity to look for an extremal example with smallest possible degree of the permutation group. Hence there is a commentary on the situation for type $(6, 6)$ and the cases where genuineness is not clear from inspection. There is, in Table 6.2, an example of the generating involutions x, y, s, t and the isomorphism class of $H = \langle x, y, s, t \rangle$ for each remaining type. The corresponding permutation diagrams are shown in Figure 6.11.

Drawing together all these examples proves the following Lemma.

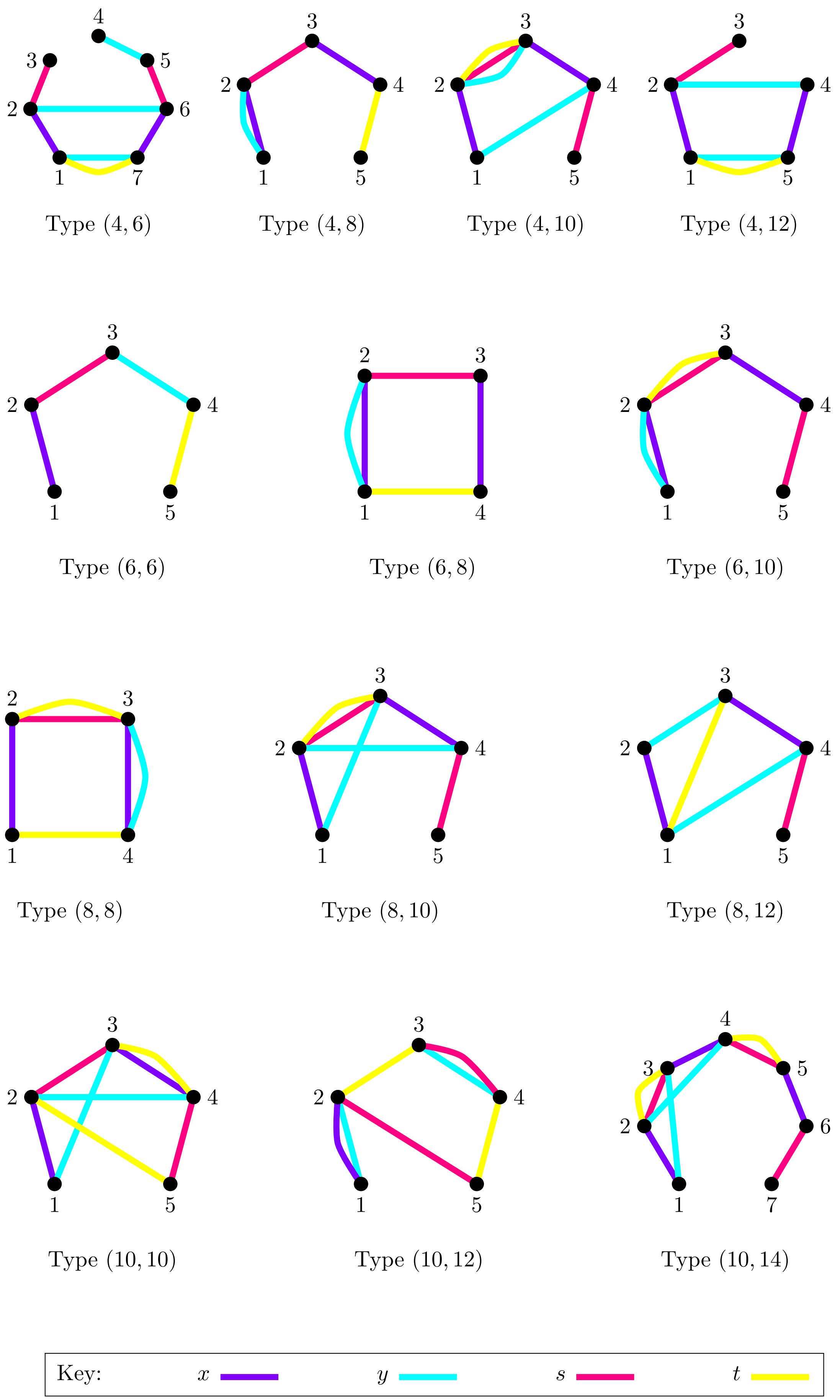

Figure 6.11: Diagrams corresponding to the maps in Tables 6.1 and 6.2.

Lemma 6.37. *Genuinely edge-biregular maps of type* $(2\kappa, 2\lambda)$ *exist for all* $3 \leq \kappa \leq 5$ *and* $\lambda = \kappa + r$ *where* $0 \leq r \leq 2$. *Moreover, the colour-preserving automorphism group* H *may be alternating or symmetric, as shown by the examples given in Table 6.2 which are extremal with respect to the degree of the permutation group.*

Proof. Let $\mathcal{M} = (H; x, y, s, t)$ be one of the groups described by the generators in a row of Table 6.2. The types cover all the ordered pairs of values $(2\kappa, 2\lambda)$ from the claim. For each row it is easy to check that the relators in the group presentation (5.1) are satisfied for the relevant values of k and ℓ, and so $\mathcal{M}$ is an edge-biregular map of the required type.

With the exception of the map of type $(6, 6)$, it is easy to verify (by considering the need for elements of orders κ and λ) that the examples given are extremal in that the degree of the permutation group is minimised in each case. Also, it may be checked that the isomorphism classes are as shown.

For five of the rows in Table 6.2, the resulting map is genuinely edge-biregular by inspection: the different cycle structures of x and s, or in the case of type $(8, 10)$ the different cycle structures of y and t, imply that there can be no automorphism interchanging x with s and y with t. The special case for permutation groups of degree six, where there is the extra outer automorphism, does not trouble us in this reasoning.

For the map of type $(6, 10)$, there can be no automorphism of the group H which interchanges x with s and y with t since $stx = (12)(345)$ has order six while $xys = (1)(2354)$ has order five. Hence the map is genuinely edge-biregular. Meanwhile, in the map of type $(10, 10)$, the cycle structure of $stx = (12354)$ and $xys = (154)$ are different, so that map is also genuinely edge-biregular. Finally, the map of type $(10, 14)$ is genuinely edge-biregular since $stx = (12)(34)(567)$ while $xys = (15764)(2)(3)$.

The map of type $(6, 6)$ is a special case. Now, considering just the order of elements xs and yt, three is the minimum degree of the permutation group for this type of map. However, S_3 is too small, as already discussed: it contains no copy of V_4.

Four is too small a degree as well, as we now see: Assume the permutation group for a proper edge-biregular map of type $(6, 6)$ has degree four. Up to labelling of elements, (without loss of generality) $\mathrm{ord}(sx) = 3$ forces $x = (12)$ and $s = (23)$ to be single transpositions, and similarly y and t must be single transpositions with one element in common. But to be a proper edge-biregular map $x \neq y$ and $s \neq t$. Meanwhile x

and y commute, so y must be disjoint from x, that means the only option is $y = (34)$. Similarly $t = (14)$. Now (13) is a conjugator which maps $x \leftrightarrow s$ and $y \leftrightarrow t$, meaning that it is (fully regular and so) not a genuinely edge-biregular map.

The example in the table for type $(6, 6)$ is genuinely edge-biregular since xys has order four and stx has order six. The isomorphism class for the group is S_5.

Finally, to complete our detailed investigation of the exceptional case, we convince ourselves that the group cannot be A_5 for a genuinely edge-biregular map of type $(6, 6)$: Let us assume that five is the degree of the alternating permutation group for a proper edge-biregular map of type $(6, 6)$. Five is too small to have two disjoint three-cycles, and the order of xs is three, so (up to labelling of elements) we must have $x = (12)(45)$ and $s = (23)(45)$. Bearing in mind that x and y must commute, and $x \neq y$ in order to be proper, we must have $y = (15)(24)$ (interchanging the labelling of the points 4 and 5 if necessary). Now the challenge is to find even t such that s and t commute, and yt has order three. The only option is $t = (24)(35)$. However, the element (13) applied as a conjugator swaps x with s and y with t, indicating that the map is fully regular, not genuinely edge-biregular. □

Remark 6.38. There are no examples of self-dual maps in Table 6.2. However, for example, conjugation by $(12)(34)$ applied to the map of type $(6, 8)$ in Table 6.2 is a map isomorphism corresponding to the partial dual operator $s \leftrightarrow t$ which acts on the dashed edges. It is not surprising to find some examples of maps with external symmetries in this list since there are so many operators on edge-biregular maps, (there are seventy-two such operators - see section 4.2.6), and, informally, in small examples there is not a lot of "room" to hide asymmetry. ⋆

It is beyond the scope of this thesis, but I believe the following may be the case.

Conjecture 6.39. *Let H be the colour-preserving automorphism group of a genuinely edge-biregular map of given type $(2\kappa, 2\lambda)$. For sufficiently large κ and λ, the group H may be alternating or symmetric such that the degree of the permutation group is arbitrarily large.*

For the moment, however, we may combine the results of this chapter to draw the following conclusion.

6.3 Existence theorem

As promised, genuinely edge-biregular maps exist for nearly all feasible types.

Theorem 6.40. *For any given pair of integers $\kappa \geq 2$ and $\lambda \geq 2$, there exists $\mathcal{M} = (H; x, y, s, t)$, a genuinely edge-biregular map of type $(2\kappa, 2\lambda)$.*

When $\kappa + \lambda = 4$ the group H is not isomorphic to a symmetric or alternating group, but when $\kappa + \lambda \geq 5$, the group H may be isomorphic to a symmetric group $H \cong S_m$, or an alternating group $H \cong A_m$. In particular, extremal examples having minimal degree m are as follows:

1. *When $\kappa + \lambda = 5$, the extremal examples are such that $H \cong S_7$.*
2. *When $\kappa + \lambda = 6$, the extremal examples are such that $H \cong S_5$.*
3. *When $\kappa + \lambda \geq 7$, the colour-preserving automorphism group H may be isomorphic to a symmetric or an alternating group with minimal degree no more than max$\{\kappa, \lambda\}$.*

Proof. Up to duality we only need to prove the claim for $\kappa \leq \lambda$.

For $\kappa + \lambda = 4$, that is $\kappa = \lambda = 2$, the infinite families of edge-biregular maps of type $(4, 4)$ on the Klein bottle provide examples to support the claim. Genuinely edge-biregular maps of type $(4, 4)$ must be Euclidean and examples can be generated by an appropriate choice of the parameters in the classification theorems. For example, a toroidal map H_{Rect} as found in Theorem 5.6 and such that $a \neq c$ is genuinely edge-biregular, as are all of the maps in the classification for the Klein bottle, Theorem 5.7. However, the corresponding colour-preserving automorphism groups are not symmetric or alternating.

When $\kappa + \lambda = 5$, the example for type $(4, 6)$ in Table 6.1 has been proved to be an extremal symmetric example by Lemma 6.34.

Meanwhile for $\kappa + \lambda = 6$, the example for type $(4, 8)$ in Table 6.1 has been proved to be an extremal symmetric example by Lemma 6.34, and the example of type $(6, 6)$ has been addressed as the special case in Lemma 6.37, resulting in the generators shown in Table 6.2. In each of these cases the conclusion is that the minimal degree for the permutation group is five, and $H \cong \mathrm{S}_5$.

When $\kappa = 2$ and $\lambda \geq 7$, Proposition 6.33 guarantees the existence of a genuinely edge-biregular map of type $(4, 2\lambda)$ where the colour-preserving automorphism group is isomorphic to a symmetric group of degree equal to max$\{\kappa, \lambda\}$.

In the cases where $\kappa \geq 3$ and $\lambda = \kappa + r$ for $r \geq 3$, Proposition 6.19 ensures that the claim is true. Meanwhile Theorem 6.31 proves the claim for all cases where $\kappa \geq 6$.

This only leaves some small cases, all of which are covered by Lemma 6.37. □

The above existence theorem leaves open the question as to how the degree of the permutation group might be minimised in the last case. The following section gives a discussion regarding such extremal examples.

6.4 Chasing extremes: with respect to the degree m of the permutation group H

Thus far we have focussed on edge-biregular maps of a given type $(2\kappa, 2\lambda)$ and limiting the degree of the alternating or symmetric group to $m = \max\{\kappa, \lambda\}$. However, so long as $\max\{\kappa, \lambda\}$ is not prime, it is possible that an edge biregular map of type $(2\kappa, 2\lambda)$ will exist on a permutation group(s) of degree less than $\max\{\kappa, \lambda\}$.

Indeed, if both κ and λ are composite numbers, then it may be possible to find an example of an edge-biregular map of type $(2\kappa, 2\lambda)$ on a symmetric group of degree "much smaller" than $\max\{\kappa, \lambda\}$, as the following example demonstrates.

Example 6.41. Let $\kappa = 20 = 4 \times 5$ and $\lambda = 21 = 3 \times 7$. Define the involutions $x := (12)(34)(56)(90)$, $y := (12)(34)(78)(90)$, $s := (23)(45)(67)(89)$, and $t := (23)(67)(89)$. Then $\mathcal{M} = (H; x, y, s, t)$ is a genuinely edge-biregular map of type $(2\kappa, 2\lambda)$, that is $(40, 42)$, whose colour-preserving automorphism group H is isomorphic to S_{10}. •

Example 6.41 is extremal in the sense that the degree m of the permutation group, that is the number of elements in the set upon which $H = \langle x, y, s, t\rangle$ acts, is minimised. Since $21 = 1 \times 21 = 3 \times 7$, the cycle structure of a permutation of order 21 is restricted to cycles of length $1, 3, 7$, and 21. Hence, in order to contain an element of order 21, the symmetric group S_m must be such that the degree m is at least $3 + 7 = 10$. The group $H = \langle x, y, s, t\rangle$ generated by the above example is clearly contained in S_{10}. In fact it is easy to check, by computer or otherwise, that in this case $H \cong \mathrm{S}_{10}$.

The question "What is the minimum degree of a symmetric or alternating group which will support a (genuinely) edge-biregular map of given type $(2\kappa, 2\lambda)$?" is thus closely related to the question: "What is the minimium degree of the symmetric group which contains an element of a given order?" Although we must remember that our requirements are slightly more restrictive than the scope of this more general question, it is easy to imagine that it may be possible to combine (preserving transitivity) two sets (with non-trivial intersection) such that the permutations

x, y, s, t satisfy all the conditions of equation (5.1), and the resulting group is contained in the symmetric group with the theoretic minimal degree. The group structure would clearly depend on the construction of the involutory permutations x, y, s and t, and so the isomorphism class of H will not necessarily be obvious. At this juncture, this is merely vague speculation, but some small concrete (extremal) examples which demonstrate the idea are included in Tables 6.1 and 6.2.

Given the above discourse, it seems reasonable to conjecture that, for an edge-biregular map of type $(2\kappa, 2\lambda)$, the minimal degree of a symmetric colour-preserving automorphism group is related to the prime power decompositions of κ and λ. In fact the above considerations yield an easy lower bound for the minimum degree of such a permutation group.

Lemma 6.42. *Suppose $\mathcal{M} = (H; x, y, s, t)$ is an edge-biregular map of type $(2\kappa, 2\lambda)$ such that $H \cong S_m$ or $H \cong A_m$. Let $\kappa = p_{k_1}^{a_{k_1}} p_{k_2}^{a_{k_2}} \dots p_{k_i}^{a_{k_i}}$ and $\lambda = p_{\ell_1}^{a_{\ell_1}} p_{\ell_2}^{a_{\ell_2}} \dots p_{\ell_j}^{a_{\ell_j}}$ be the prime power decompositions for κ and λ.*

Then $m \geq min\{\sum_{n=1}^{i} p_{k_n}^{a_{k_n}}, \sum_{n=1}^{j} p_{\ell_n}^{a_{\ell_n}}\}$.

At this point in time we provide no guarantee that this lower bound can be attained in every case. There may be intricacies involved in stitching together various (x, s), (s, x), (y, t) and (t, y) chains while both respecting the required relators from (5.1), and keeping the diagram connected, all of which may force m to be greater than this lower bound.

What follows is a further example of an extremal edge-biregular map, designed to illustrate the case in point, that is the minimal degree may be much smaller than $\max\{\kappa, \lambda\}$, however it also raises the question of whether there is a construction which would work in general.

Example 6.43. Let $\kappa = 2^2 \times 3 \times 5$ while $\lambda = 2 \times 3 \times 7$. Find $\mathcal{M} = (H; x, y, s, t)$, an edge-biregular map of type $(2\kappa, 2\lambda)$ such that $H \cong \mathrm{S}_{12}$.

This example may lull us into a false sense of security. The most natural first guess works: define two-involution chains using x and s for each prime power according to the values of λ and repeat this for κ, each using the same set of integer-labelled points starting at 1. See Figure 6.12. It can be checked that $H \cong \mathrm{S}_{12}$. •

On a set of points labelled $0, 1, 2, 3, \dots$, it is clearly always possible to arrange consecutive but disjoint (y, t) and (t, y) chains such that the transpositions for t are always of the form $(2b, 2b + 1)$, and similarly consecutive but disjoint (s, x) and (x, s)

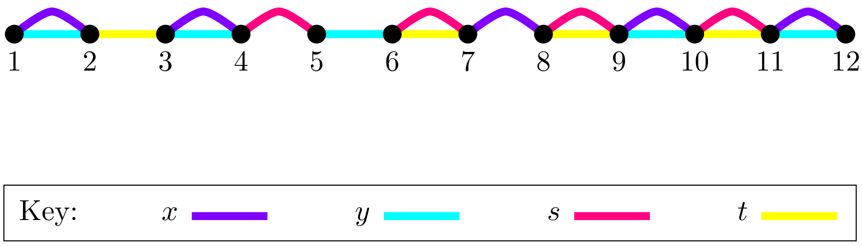

Figure 6.12: A solution to Example 6.43

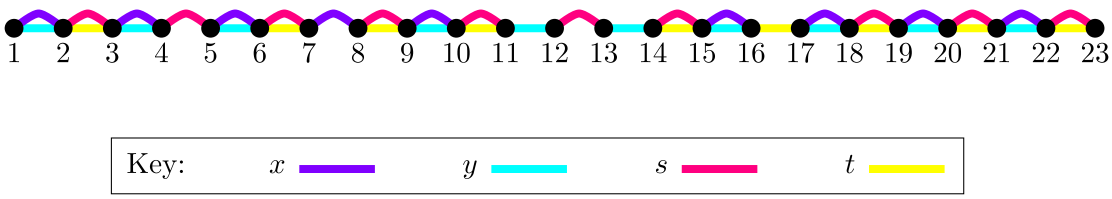

Figure 6.13: A solution to Example 6.44

chains such that the transpositions for s match those for t in that they are of the form $(2b, 2b+1)$. However, using a first-come-first-served style algorithm as above will not always yield the solution we seek, as we demonstrate in the following example.

Example 6.44. Let $\kappa = 2^2 \times 3 \times 5 \times 11$ while $\lambda = 2 \times 3 \times 7 \times 11$. Find $\mathcal{M} = (H; x, y, s, t)$, an edge-biregular map of type $(2\kappa, 2\lambda)$ such that $H \cong \mathrm{S}_{23}$.

Similarity with the previous example indicates that laying out the required chains efficiently and in order of the prime powers listed will cause a problem. Indeed the resulting diagram is not connected. Since $2^2 + 3 + 5 = 2 + 3 + 7 = 12$ there would be a gap between the points labelled 12 and 13 meaning that the group is not transitive on the $m = 23$ elements. As such the four involutions defined in this way will certainly not yield $H \cong \mathrm{S}_{23}$ or $H \cong \mathrm{A}_{23}$.

In this case, it is easy to see that changing the order of the list of prime power factors for λ yields a solution which is transitive, see Figure 6.13. A computer check confirms that $H \cong \mathrm{S}_{23}$. •

In order to prove this kind of method will work in general we need to be able to order the prime powers within the two sums, $\sum_{n=1}^{i} p_{k_n}^{a_{k_n}}$ and $\sum_{n=1}^{j} p_{\ell_n}^{a_{\ell_n}}$, such that at no point are the two partial sums ever equal. It is also imperative that the diagram is connected, so, depending on the values κ, λ, we may need to include further examples of cycles (which must still respect the orders of the elements) to link otherwise disjoint cycles. This is all rather vague, but may highlight why it is not so obvious how to create a general construction of minimal degree. I conjecture that the theoretical minimal degree can be obtained, with the exception of small cases, some (possibly all?) of which we have already seen, for example type $(4,6)$. Maybe there is some nice known fact from number theory which ensures the lower bound can always be attained...? Even so, after finding a solution in the form of a diagram which works, there will still be further work to do to in order to find which isomorphism class of group is generated.

Minimising the degree of the underlying permutation group is one approach, and this opens up the question for the other extreme: maximising the degree, and indeed dealing with everything in between! We already have that, so long as the difference in κ and λ is greater than one, and $\min\{\kappa,\lambda\} \geq 3$, the degree can be one more than $\max\{\kappa,\lambda\}$, by the original instructive example, Construction 6.23. For a given κ,λ, what other values could the degree m be? With a caveat which may exclude some small cases, my sneaking suspicion is: however big one likes... and this, too, is a topic for further research.

Chapter 7

Conclusion

We draw this thesis to a close by highlighting related topics for further research, starting with directions for further research from Part I.

Chapter 2 presents a standalone theorem which leaves little room for extension. However it does inspire the still open question about the possible odd degrees for super-symmetric maps, which was the motivation for the work in Chapter 3, which itself leaves open more questions than it answers! There is obvious work to be done to improve the understanding of orbit structures from section 3.5, in particular by proving exactly how the orbit decomposition works. Currently it is not always clear that each of the sets in question constitutes an orbit, and not a union of distinct orbits.

Beyond this, there are then two natural directions of further research stemming from Part I: Regular maps.

1. Super-symmetric maps and the question of their existence for given odd degree.
2. Further understanding of the structure of external symmetry groups for super-symmetric maps. Generalities in this field are difficult, and so this is likely to be most tangible when considering specific constructions for super-symmetric maps.

Naturally, questions for further research also arise from Part II: Edge-biregular maps. After the introductory material in Chapter 4, the work in Chapter 5 is a partial classification of edge-biregular maps on certain groups and certain surfaces. Meanwhile Chapter 6 has largely focussed on genuinely edge-biregular maps where the colour-preserving group H is isomorphic to an alternating or symmetric group. There is still much more work to be done in the field of edge-biregular maps. Allowing ourselves to consider other groups and/or surfaces as well, there are many further challenges, which may not always be possible!

1. Given the type $(2\kappa, 2\lambda)$ for an edge-biregular map $\mathcal{M} = (H; x, y, s, t)$, minimise the degree m of the group H such that $H \cong \mathrm{S}_m$.

2. Given the type $(2\kappa, 2\lambda)$ for an edge-biregular map $\mathcal{M} = (H; x, y, s, t)$, minimise the degree m of the group H such that $H \cong \mathrm{A}_m$.

3. Prove that, given the type $(2\kappa, 2\lambda)$ with κ and λ being sufficiently large, an edge-biregular map $\mathcal{M} = (H; x, y, s, t)$ exists such that $H \cong \mathrm{S}_m$ (or A_m), and the degree m may be arbitrarily large.

4. Classify and/or enumerate the edge-biregular maps which are realisable on a symmetric or alternating group of given degree m.

5. Given a (class of) group(s) H, say for example $H = \mathrm{PSL}(2, q)$ where q is a prime power, find what edge-biregular maps $\mathcal{M} = (H; x, y, s, t)$ are supported by this group. Even better would be to find explicit constructions for all such maps.

6. Alternatively, given a type $(2\kappa, 2\lambda)$, establish whether an edge-biregular map $\mathcal{M} = (H; x, y, s, t)$ of this type exists such that $H \cong PGL(2, q)$, and determine what values q may take.

7. Extend the classification of edge-biregular maps for surfaces of Euler characteristic $-2p$ for prime p.

8. Explore the existence of external symmetries of edge-biregular maps.

9. Determine the existence or otherwise of further 'natural' operators acting on edge-biregular maps.

Each of these projects can easily be narrowed to address *genuinely* edge-biregular maps, although they are stated above in the broadest terms.

Much of the contents of the above list is beyond the scope of this thesis, although the first two items have been partially addressed. For example, for sufficiently large κ and λ, Theorem 6.31 has already limited the degree to $m = \max\{\kappa, \lambda\}$, and Remark 6.17 shows this is extremal in the case when m is prime.

There is plenty more to do, and I am looking forward to it... :-)

Appendix A

The remaining proofs for Chapter 5

This is the fine detail of what remains for the proof of Theorem 5.11. This paragraph excepted, it is written by my supervisor, Professor Jozef Širáň, and results from joint work to determine the parts of the classification of edge-biregular maps on surfaces of negative Euler characteristic corresponding to $H_{p,j}$ and H_p. I have included this as an appendix for reference and completeness. We begin with a reminder of what has been set up so far.

From this point on until the end of this section we will assume that p is *not* a divisor of the order of H. Since we are working up to duality, instead of $k \leq \ell$ we henceforth assume that *the* 2*-part of k is not smaller than the* 2*-part of ℓ*. Remember also that H contains a subgroup isomorphic to V_4 and so 4 divides $|H|$. Comparing this condition with Equation (5.5) allows us to set $k = 4\kappa$ and $\ell = 2\lambda$ for integers κ and λ.

The fact that H is solvable yields that it has a non-trivial Fitting subgroup F; recall that F is the largest nilpotent normal subgroup of H. In particular, F is a direct product of its Sylow subgroups. By what we know about the Sylow subgroups of H from Lemma 5.20 we have $F = F_1 \times F_2$, where F_1 is cyclic, of odd order (possibly trivial), and F_2 (if non-trivial) is a cyclic or a dihedral 2-group; we will henceforth split our analysis according to this dichotomy.

As a general remark, observe that we may assume $F \neq H$. Indeed, if $F = H$, then F_1 would have to be trivial (otherwise F could not be generated by involutions) and F_2 would have to be non-cyclic (to contain enough distinct involutions), so that $H = F = F_2$ would have to be dihedral. But edge-biregular maps with dihedral groups H have already been classified in Theorem 5.8. Without giving details we just state that, as a consequence of Table 5.1 displaying the dihedral classification results, the only edge-biregular maps of Euler characteristic $-p$ for an odd prime p determined by a dihedral group of automorphisms are the first two maps in Theorem 5.11 defined by the groups $H_{p(1)}$ and $H_{p(2)}$.

A.1 The case when the Fitting subgroup is cyclic

From now on we will *assume that F is cyclic*. In such a case F contains either no involution (if $|F|$ is odd) or a unique involution (if $F_2 \cong \mathrm{C}_2$). This implies that F can contain at most one of the four involutions s, t, x, y.

We will show that F cannot have index 2 in H. Indeed, suppose $[H : F] = 2$. If F contains no generating involution from the set $\{s, t, x, y\}$, then the index-2 condition implies that $sx, xy, yt \in F$. But since $H = \langle sx, xy, yt\rangle$ we would have $F = H$, a contradiction. Thus, let one of $\{s, t, x, y\}$ be the unique generating involution of H contained in F. We may without loss of generality assume that this element is y, as all the other cases are handled by symmetries in the forthcoming argument. Now $\langle y\rangle$ is characteristic in F and therefore normal in H, and so y, being an involution, is also a central in H. Further, from $s, t \notin F$ we have $st \in F$, and by uniqueness of the involution in F it follows that $y = st$.

As now $s, x \notin F$, we have $sx \in F$ and so F contains the cyclic group $\langle sx\rangle$. If its order is odd, then F also contains the cyclic group $K = \langle sx\rangle\langle y\rangle$. But then, recalling centrality of y in H, conjugation by, say, s inverts every element of K and so $L = K \rtimes \langle s\rangle$ is a dihedral group. From $s, xs, y \in L$ and $y = st$ we also have $t, x \in L$ and so $L = H$ is dihedral, the case we have already disposed of. If the order of $\langle sx\rangle$ is even, then $y = (sx)^j$ for j equal to half of the order of sx, since both elements are an involution in the cyclic group F. But we saw earlier that $y = st$, giving $st = (sx)^j$ and hence $t = s(sx)^j$. This, however, with $y = (sx)^j$ shows that $H = \langle s, x\rangle$, which again means that H is dihedral.

Altogether, we have shown that for non-dihedral H and for a cyclic F we must have $[H : F] > 2$. Now, by the available theory H/F embeds in $\mathrm{Aut}(F)$, and as the latter is Abelian if F is cyclic, we conclude that H/F is Abelian. But H/F is generated by four elements of order at most 2, so that $H/F \cong \mathrm{C}_2^m$ for some m such that $2 \le m \le 4$. Further, since $H = \langle st, sx, xy\rangle = \langle st, ty, xy\rangle$ it follows that $m \in \{2, 3\}$ and both sxF and tyF have order at most 2, so that $(sx)^2, (ty)^2 \in F$.

From earlier calculations we recall that both k, ℓ are even and greater than 2, and assuming that the 2-part of k is not smaller than the 2-part of ℓ we have the following: *If $c = \gcd(k/2, \ell/2)$, then $c \in \{1, 2\}$, k is a multiple of 4, and $|H| = k\ell/(2c)$.* In particular, note that $8 \nmid \ell$, and $c = 1$ or 2 according as $\ell/2$ is odd or even.

Under these conditions we first show that $ty \notin F$. Indeed, suppose that $ty \in F$. Observe that the order of $(sx)^c$ is $\ell/(2c)$ and hence odd, so that $(sx)^2 \in F$ implies

$(sx)^c \in F$ also for $c = 1$. As $ty, (sx)^c \in F$ and the orders of the two elements, $k/2$ and $\ell/(2c)$, are relatively prime, it follows that
$(k/2)(\ell/(2c)) \le |F| = |H|/[H : F] \le k\ell/(2c[H : F])$, which yields $[H : F] \le 2$, a contradiction.

It follows that $ty \notin F$. But we know that $(ty)^2 \in F$, and we may use essentially the same chain of inequalities as above, with $k/2$ replaced by $k/4$ (which is the order of $(ty)^2$), to conclude that $[H : F] \le 4$. We saw, however, that $[H : F]$ is a power of 2 and greater than 2, so that $[H : F] = 4$, and we must have equalities in the above chain throughout. In more detail, and using the fact that F is cyclic, we have
$F = \langle (sx)^c\rangle\langle (ty)^2\rangle = \langle (sx)^c(ty)^2\rangle \cong \mathrm{C}_n$ for $n = k\ell/(8c)$, with $ty \notin F$. Observe that the order of $(sx)^2$ is odd in both cases for $c \in \{1, 2\}$, and is and relatively prime to $k/4$ (the order of $(ty)^2$).

We show that F contains none of the generating involutions s, t, x, y of H, and at most one of the involutions st, xy. For if F contained one of t, y, then this element would have to coincide with the central involution $(ty)^{k/4}$, but such an equality quickly gives $t = 1$ or $y = 1$. If $s \in F$ (and the case $x \in F$ is done similarly), then s would commute with $(sx)^2$, which is equivalent to $(sx)^4 = 1$. Then, $\ell/2$ would have to divide 4 and since $8 \nmid \ell$ we would have $\ell = 4$. But this would give $H = k\ell/4 = k$, so that H would be dihedral.

To address the remaining part, note that by non-orientability we know that $H = \langle st, xy, yt\rangle$. From $H/F = \langle stF, xyF, ytF\rangle$ and $yt \notin F$ while $(yt)^2 \in F$, together with $[H : F] = 4$, it follows that at most one of st, xy can be contained in F.

In what follows we will without loss of generality assume that $st \notin F$. As F and $\langle s, t\rangle$ now intersect trivially, with the help of the above finding this means that the semi-direct product

$$F \rtimes \langle s, t\rangle = \langle (sx)^c(ty)^2\rangle \rtimes \langle s, t\rangle \cong \mathrm{C}_n \rtimes \mathrm{V}_4 \tag{A.1}$$

for $n = k\ell/(8c)$ has order $4n = |H|$ and so $H = F \rtimes \langle s, t\rangle \cong \mathrm{C}_n \rtimes \mathrm{V}_4$. If $n = |F|$ is even, then the unique non-trivial involution $(ty)^4 \in F$ generates a subgroup isomorphic to C_2 that is characteristic in F and hence normal in H, which means that $(ty)^4$ is central in H. But then H would contain the subgroup $\langle (ty)^{k/4}\rangle \times \langle s, t\rangle \cong \mathrm{C}_2^3$, contrary to the fact that H has dihedral Sylow 2-subgroups. It follows that n is odd and so is $\kappa = k/4$ (and we know the same about $\ell/(2c) = \lambda/c$); also, both xy and st must lie outside F, and the Sylow 2-subgroups of H are isomorphic to V_4.

In the proof of solvability of H we have encountered the equation $2\kappa\lambda - 2\kappa - \lambda = cp$ for some $c \in \{1, 2\}$. If $c = 2$, then λ is exactly divisible by 2, and then oddness of κ with $2\kappa(\lambda - 1) = 2p + \lambda$ gives a contradiction as the right-hand side is divisible by 4 while the left-hand side is not. It follows that $c = 1$, and so $sx \in F$; observe that then $sy \notin F$ as in the opposite case we would have $(xs)(sy) = xy \in F$ which has already been excluded.

We now let $u = sx$ and $v = ty$; note that our cyclic group F, of order a product of two odd and relatively prime numbers $\ell/2$ and $k/4$ (the orders of u and v^2), is generated e.g. by uv^2. We saw that $y, ys, yt \notin F$, so that by (A.1) for $c = 1$ we must have $y = wst$ for some $w \in F$. The fact that $w = yts$ commutes with $u = sx$ is equivalent to $ty(sx)yt = xs = u^{-1}$, so that conjugation by v inverts u. Similarly, $w = yts$ commutes with $v^2 \in F$, which translates to $[s, v^2] = 1$, and as $u = sx$ commutes with v^2 we also have $[x, v^2] = 1$.

In somewhat more detail, let $w = u^a(v^2)^b = u^a v^{2b}$ for uniquely determined integers a, b such that $0 \le a < \ell/2$ and $0 \le b < k/4$. Using the facts that s inverts u and commutes with v^2 and $[u, v^2] = 1$, from $y = wst$ it follows that $v^{-2} = (yt)^2 = (u^a v^{2b} s)^2 = v^{4b}$, so that $v^{4b+2} = 1$, and for b in the above range we have $4b + 2 = k/2$ (the order of v) and so $b = (k - 4)/8$. Normality of $\langle u \rangle$ in H (being characteristic in F) further implies that $yuy = u^j$ for for some j, $1 \le j < \ell/2$, such that $j^2 \equiv 1 \bmod \ell/2$. Since v inverts u, we obtain $u^{-1} = tyuyt = tu^j t$, which implies $tut = u^{-j}$. Observing now that conjugation by st maps u^a onto u^{aj} and inverts v^2, from $y = u^a v^{2b} st$ we obtain $1 = (u^a v^{2b} st)^2 = u^{a(j+1)}$, so that $a(j + 1) \equiv 0 \bmod \ell/2$.

Going one step further and using properties derived above, from $y = u^a v^{2b} st$ we have $xy = xu^a sv^{2b} t = u^{-a-1} v^{2b} t$, and as conjugation by t inverts v^2 and maps u onto u^{-j} one obtains $1 = (u^{-a-1} v^{2b} t)^2 = u^{(a+1)(j-1)}$, which gives $(a + 1)(j - 1) \equiv 0 \bmod \ell/2$. Subtracting the last congruence from $a(j + 1) \equiv 0 \bmod \ell/2$ yields $2a + 1 \equiv j \bmod \ell/2$. As $2^{-1} = (\ell + 2)/4 \bmod \ell/2$ (which is odd), it follows that $a \equiv (j - 1)2^{-1} = (j - 1)(\ell + 2)/4 \bmod \ell/2$, giving a unique value of $a \in \{0, 1, \ldots, \ell/2 - 1\}$. Observe also that for this value of a and for any j such that $j^2 \equiv 1 \bmod \ell/2$ one has $(j + 1)a = (j^2 - 1)2^{-1} \equiv 0 \bmod \ell/2$, which is the congruence obtained earlier.

The last step will be reintroducing the notation $k = 4\kappa$ and $\ell = 2\lambda$ for odd and relatively prime κ and λ, and observing that $2b + 1 = \kappa$ and so $1 = tyv^{2b}u^a s = v^\kappa u^a s$. The last relation is equivalent to $v^{\kappa-1} u^a = v^{-1} s$, and from $[u, v^2] = 1$ (a consequence of $[s, v^2] = 1 = [x, v^2]$) and oddness of κ it follows that $1 = [u, v^{\kappa-1} u^a] = [u, v^{-1} s]$ and commutation of u and $v^{-1}s$ is equivalent to u being inverted by conjugation by v.

Summing up, the above facts well-define a group $H = H_{p,j}$ of order $k\ell/2$ generated by four involutions s, t, x, y and presented as follows, with $u = sx$ and $v = ty$:

$$H_{p,j} = \langle\, x, y, s, t \,|\, x^2,\, y^2,\, s^2,\, t^2,\, (xy)^2,\, (st)^2,\, u^\lambda,\, v^{2\kappa},\, [s, v^2],\, [x, v^2],\, tutu^j,\, v^\kappa u^a s\,\rangle \tag{A.2}$$

for a non-negative integer $j < \ell/2$ such that $j^2 \equiv 1 \bmod \lambda$, with $a = (j-1)(\lambda+1)/2$. This is the presentation appearing as a third item in Theorem 5.11. Note that here $F = \langle u, v^2\rangle = \langle uv^2\rangle$ is cyclic, of order $\kappa\lambda = k\ell/8$, and $H_{p,j} = F \rtimes \langle s, t\rangle$, as derived earlier.

Proof of correctness of the presentation (A.2)

Let $G_0 = \langle s, x \,|\, s^2, x^2, (sx)^\lambda\rangle \cong \mathrm{D}_\ell$ and let $u = sx$. Let us introduce a pair of automorphisms θ and τ of G_0 completely defined by letting $\theta(s) = s$ and $\theta(u) = u^{-j}$ for some j such that $j^2 \equiv 1 \bmod \lambda$, and $\tau(u) = u^j$, $\tau(x) = x$. These definitions imply, for example, that $\theta(x) = su^{-j}$ and $\tau(s) = u^j x$. It can be easily verified that θ and τ commute and both are of order two. It follows that we have a well-defined split extension of G_0 by the subgroup $\mathrm{V}_4 \cong \langle\theta, \tau\rangle < \mathrm{Aut}(G_0)$; the order of the extension is 4ℓ. By general knowledge on split extensions the new group has an equivalent representation in the form $G_1 = \langle s, x\rangle \rtimes \langle t, y\rangle$, where t, y are two commuting involutions acting on G_0 by conjugation the same way as the two earlier automorphisms do, that is, $\theta(z) = tzt$ and $\tau(z) = yzy$ for every $z \in G_0$. Note that this also implies that the subgroups G_0 and $\langle t, y\rangle$ intersect trivially. Further, using $u = sx$ and $v = ty$ it can be verified that in G_1 one has the relations $uvu = v$; moreover, we have $[s, t] = [x, y] = 1$ by the definition of the two automorphisms and their conversion to conjugations. It follows that the group G_1 has a presentation of the form

$$G_1 = \langle x, y, s, t \,|\, x^2,\, y^2,\, s^2,\, t^2,\, (xy)^2,\, (st)^2,\, u^\lambda,\, v^2,\, tutu^j,\, uvuv^{-1}\rangle \tag{A.3}$$

where, again, $u = sx$, $v = ty$, and j is an integer such that $j^2 \equiv 1 \bmod \lambda$.

Next, let us consider the group G_2 generated by the same involutions as G_1 but with a presentation obtained from that of G_1 by omitting the relator v^2 and adding the conditions of v^2 commuting with s and x:

$$G_2 = \langle x, y, s, t \,|\, x^2,\, y^2,\, s^2,\, t^2,\, (xy)^2,\, (st)^2,\, uvuv^{-1},\, u^\lambda,\, tutu^j,\, [s, v^2],\, [x, v^2]\rangle \tag{A.4}$$

The relators $uvuv^{-1}$ and $tutu^j$ imply $yuy = u^j$ and $txt = su^{-j}$, and these together with $tut = u^{-j}$ and $yxy = x$ show that $\langle u, x\rangle = \langle s, x\rangle$ is a normal subgroup of G_2. By inspection, in the absence of any condition involving the element v the presentation of $G_2/\langle s, x\rangle$ reduces to $\langle t, y \,|\, t^2, y^2\rangle$, which is an infinite dihedral group; hence G_2 is infinite. The subgroup $N = \langle v^2\rangle$ is obviously normal in G_2 and as $G_2/N \cong G_1$ we conclude that N is isomorphic to an infinite cyclic group. The subgroup $\kappa N = \{(v^{2\kappa})^i;\ i \in \mathbb{Z}\}$ is characteristic in N and so normal in G_2. Applying the Third isomorphism theorem we obtain $G_2/N \cong (G_2/\kappa N)/(N/\kappa N)$ and so $|G_2/\kappa N| = \kappa|G_2/N|$. Since $G_2/N \cong G_1$, the new group $G_3 = G_2/\kappa N$ of order $8\kappa\lambda$ has a presentation obtained from that of (A.4) by adding the relator $v^{2\kappa}$:

$$G_3 = \langle x, y, s, t \,|\, x^2,\ y^2,\ s^2,\ t^2,\ (xy)^2,\ (st)^2,\ uvuv^{-1},\ u^\lambda,\ tutu^j,\ [s, v^2],\ [x, v^2],\ v^{2\kappa}\rangle \tag{A.5}$$

The last step is to consider the element $z = v^\kappa u^a s \in G_3$, where $a = 2^{-1}(j-1) \bmod \lambda$ as before. For the calculations that follow it is useful to observe that from the earlier congruence $a(j+1) \equiv 0 \bmod \lambda$ we have $aj \equiv -a \bmod \lambda$, so that $tu^a t = u^a$ and $yu^a y = u^{-a}$; note also that $u^{2a+1} = u^j$. We begin by showing that z is an involution. Indeed, using $[u, v^2] = 1$, the facts that both s and v invert u, and the properties of u listed before one obtains $(u^a s v^\kappa)^2 = u^{2a} s v^{-1} v^{\kappa+1} s v^\kappa = u^\lambda u^{-1} s v^{-1} s v = u^\lambda y u^{-1} y = 1$, which yields $z^2 = 1$.

We show even more; namely, that z is a central involution of G_3. To show that z commutes with s, note that the above implies that $zs = v^\kappa u^a$ is an involution and so $(sz)^2 = s(v^\kappa u^a)^2 s = 1$, so that $[s, z] = 1$. With the help of the fact that conjugation by t preserves u^a and v^κ (note that $v^\kappa = v^{-\kappa}$) we obtain $[z, t] = v^\kappa u^a st \cdot tv^\kappa u^a s = z^2 = 1$. As κ is odd and so v^κ inverts u, it follows that $[x, z] = xv^\kappa u^a(sx)v^\kappa u^a s = xv^\kappa u^{a+1} v^\kappa u^{a+1} x = 1$. Last, using inversion of u^a and preservation of v^κ by conjugation by y, as well as preservation of u by conjugation by sv, one has $[z, y] = v^\kappa u^a s v^\kappa u^{-a} y s y = v^\kappa u^a v^{\kappa-1} u^{-a} s v y s y = v^{-1} s v y s y = 1$.

We have proved that for $z = v^\kappa u^a s$ the subgroup $\langle z\rangle \cong \mathrm{C}_2$ is central in G_3. But this means that the group $G_3/\langle z\rangle$, which is isomorphic to $H_{p,j}$, has order $|G_3|/2 = 4\kappa\lambda = k\ell/2$, as claimed. This completes the proof of correctness of the presentation (A.2); the relator $uvuv^{-1}$ can be omitted since it is a consequence of the remaining ones, as shown in the paragraph immediately preceding the presentation (A.2).

For completeness we show that none of the edge-biregular maps with automorphism groups $H_{p,j}$ presented as in (A.2) are fully regular. Indeed, in the opposite case $H_{p,j}$ would have to admit an automorphism interchanging s with x and t with y. In such a case, however, along with $s = v^{\kappa}u^{a}$ the group $H_{p,j}$ would also have to admit the relation $x = v^{-\kappa}u^{-a} = u^{a}v^{\kappa}$. This would imply $xs = u^{2a}$ and hence $u^{2a+1} = 1$, and as $2a+1 \equiv j \bmod \lambda$ we would have $u^{j} = 1$, contrary to $j^{2} \equiv 1 \bmod \lambda$.

Finally, the Euler-Poincaré formula yields that $p = 2\kappa\lambda - 2\kappa - \lambda$ so $p+1 = (2\kappa-1)(\lambda-1)$. We know κ and λ are both odd so letting $p+1 = 2^{\alpha}bd'$, where $b \equiv 1 \bmod 4$, and d' is odd, then we have $2\kappa - 1 = b$ and $\lambda - 1 = 2^{\alpha}d' = d$, which, so long as κ and λ are coprime, will yield an edge-biregular map as described above. Hence, for every factorisation $p+1 = bd$ such that $b \equiv 1 \bmod 4$ and $\gcd(b+1, d+1) = 1$ we have such a map of type $(2(b+1), 2(d+1))$, completing the analysis related to the third item of Theorem 5.11.

A.2 The case when the 2-part of the Fitting subgroup is dihedral

Recall that we are investigating an edge-biregular map $M = (H; x, y, s, t)$ of type (k, ℓ) with both entries even, on a surface of Euler characteristic $-p$ for some odd prime p; at the beginning of section 5.3.3.2 we also made the assumption that the 2-part of k is not smaller than the 2-part of ℓ and we had $k = 4\kappa$ and $\ell = 2\lambda$. By Proposition 5.21 we know that H is a solvable group and so has a non-trivial Fitting subgroup F, of which we may assume that $F \neq H$. By earlier results and observations we also know that either F is cyclic, or F has a dihedral 2-part, denoted F_2. We have dealt with the first possibility in subsection A.1 and *from now on we will assume that F_2 is dihedral*, which of course means that $|F_2| \geq 4$. Our next result places a substantial restriction on F_2 and D_k (the vertex-stabilizer in M).

Proposition A.1. ***(Reade, Širáň)*** *If F_2 is dihedral, then D_k is a 2-group, $[D_k : F_2] = 2$, and k is a multiple of 8 while $\ell/(2c)$ is odd.*

Proof. We have assumed that the 2-part of k is not smaller than that of ℓ, and we know that $|H| = k\ell/(2c)$ for $c = 1, 2$. Analysis of equation (5.5) with these conditions yields that a Sylow 2-subgroup of H is contained in D_k. As F_2 is the Sylow 2-subgroup of F, normality of F in H implies that F_2 is a subgroup of D_k.

Suppose $F_2 = \mathrm{D}_k = \langle t, y \rangle$. Then H/F_2 is generated by sF_2 and xF_2, each of which has order less than or equal to two. But $|H|/|F_2| = |H|/k = \ell/(2c) = \lambda/c$ and since $c = \gcd(2\kappa, \lambda) \leq 2$ it follows that λ/c must be an odd integer and hence the group H/F_2 does not contain any involution. Hence $s, x \in F_2$ and so $H = \mathrm{D}_k$, contrary to our assumption that H is not dihedral.

To show that $[\mathrm{D}_k : F_2] \leq 2$, up to the choice of labelling of orbits we may assume

$F_2 = \langle y, (ty)^\mu \rangle$ for some μ [the alternative option would be $F_2 = \langle t, (ty)^\mu \rangle$]. Now, F_2 is normal in H (because it is characteristic in F) and so $y^{ty} = tyyyt = tyt \in F_2$. But $y \in F_2$ also so this means $(ty)^2 \in F_2$, which implies $|\mathrm{D}_k : F_2| \le 2$. Hence D_k is a 2-group; the conclusions about k and ℓ are obvious from the above. □

Since F_2 is normal in H and hence also in $\langle t, y \rangle \cong \mathrm{D}_k$, we have established that there are only three possibilities for a dihedral F_2 if $k \ge 8$: either $F_2 = \langle t, y \rangle$, or F_2 is one of $\langle t, (ty)^2 \rangle$, $\langle y, (ty)^2 \rangle$; in particular, k must be a power of 2. In the first case we have H/F_2 generated by (at most) four involutions, but oddness of $\ell/(2c) = |H/F_2|$ implies that the generating involutions s, t, x, y all belong to F_2 and hence $H = F_2 \cong \mathrm{D}_k$; such maps with a dihedral automorphism group have already been sorted out. In what follows we will assume that $F_2 = \langle z, (ty)^2 \rangle$ for some $z \in \{t, y\}$.

If $k \ge 16$, the cyclic subgroup $\langle (ty)^2 \rangle$ of F_2, of order $k/4$, is characteristic in F_2 and therefore normal in H. Note that for $k = 8$ and $F_2 \cong \mathrm{C}_2 \times \mathrm{C}_2$ this need not be valid. For now we will *assume* that $\langle (ty)^2 \rangle$ is normal in H also for $k = 8$ and we will return to the opposite case later.

Before proceeding we make a remark about the case $c = 2$. By the proof of solvability of H, for $c = 2$ the subgroups D_k and D_ℓ intersect in a subgroup isomorphic to V_4. For $k \ge 16$ the subgroup $\langle (ty)^{k/4} \rangle \cong \mathrm{C}_2$ generated by the centre of F_2 is characteristic in F_2 and hence normal in H, which means that $(ty)^{k/4}$ is a central involution also in H. Note that this is now also valid for $k = 8$ because of the assumption made in the previous paragraph. If $c = 2$ we may also assume that $\ell \ge 12$ (as for $\ell = 4$ we would have H a dihedral group), and so for the central element of $\langle s, x \rangle \cong \mathrm{D}_\ell$ we have $(sx)^{\ell/4} = (ty)^{k/4}$.

Normality of $\langle (ty)^2 \rangle$ of F_2 implies that $s(ty)^2 s = (ty)^{2i}$ and $x(ty)^2 x = (ty)^{2j}$ for some integers i, j such that $i^2 \equiv j^2 \equiv 1 \bmod k/4$; the exponents 2 in the congruences match the orders of s and x. It follows that $sx(ty)^2 xs = s(ty)^{2j} s = (ty)^{2ij}$. But as $(sx)^{\ell/(2c)}$ is either the identity or a central element of H, we must have $(ij)^{\ell/(2c)} \equiv 1 \bmod k/4$. In view of the previous two congruences, oddness of $\ell/(2c)$ implies that $1 \equiv (ij)^{\ell/(2c)} \equiv ij$, so that sx commutes with $(ty)^2$. Since k and $\ell/(2c)$ are relatively prime and $k/4$ is even while $\ell/(2c)$ is odd, it follows that the subgroup $J = \langle sx, (ty)^2 \rangle$ of H is cyclic, of order $k\ell/(8c)$, and generated by the product $(sx)(ty)^2$.

From the fact that sx commutes with $(ty)^2$ we have $s(ty)^2 s = x(ty)^2 x$ and this element is in $\langle (ty)^2 \rangle$ by normality of this subgroup in H. Note that we cannot have $x \in J$; in the opposite case x would have to be equal to $(ty)^{k/4}$, the unique involution in J, but then x would commute with sx and hence with s, giving $(sx)^2 = 1$, contrary to $\ell \geq 6c$. Thus, the semi-direct product $K = J \rtimes \langle x \rangle$ is a subgroup of H of order $k\ell/(4c)$, and hence normal (of index 2) in H.

The subgroup $\langle sx \rangle < J$ as the unique cyclic subgroup of K of order $\ell/2$ is characteristic in K and hence normal in H. Thus, $y(sx)y = (sx)^a$ and $t(sx)t = (sx)^b$ for some positive integers $a, b < \ell/2$, with both a, b odd if $c = 2$. By $[x, y] = 1 = [s, t]$ we then have $ysy = s(xs)^{a-1}$ and $(sy)^2 = (xs)^{a-1}$; similarly, $txt = (xs)^{b-1}x$ and $(tx)^2 = (xs)^{b-1}$. By normality of F_2 in H, the element sys or xtx (depending on whether $y \in F_2$ or $t \in F_2$) is equal to either the central element $(ty)^{k/4}$ or an involution of the form $(ty)^{2j}z$ for some $j < k/4$. In the first case, either sys or xtx would commute with xs, which is easily seen to be equivalent to $(sy)^2 = 1$ or $(tx)^2 = 1$. In the second case, either $(sy)^2$ or $(tx)^2$ are a power of $(ty)^2$ and so their order is a power of 2. But we have also established that $(sy)^2 = (xs)^{a-1}$ or $(tx)^2 = (xs)^{b-1}$. Both $(xs)^{a-1}$ and $(xs)^{b-1}$ have, however, odd order; this is obvious for $c = 1$ and for $c = 2$ it follows from oddness of a and b. The two order parities can be matched only if $(sy)^2 = 1$ or $(tx)^2 = 1$. In both cases we have established that, depending on whether $y \in F_2$ or $t \in F_2$, we have $(sy)^2 = 1$ or $(tx)^2 = 1$, i.e., $[s, y] = 1$ or $[t, x] = 1$.

Next, we show that $ty \notin K$. Indeed, if $ty \in K$, then K would contain the cyclic group $\langle ty \rangle \cong \mathrm{C}_{k/2}$ and also the dihedral group $\langle s, x \rangle \cong \mathrm{D}_\ell$. By comparing their orders with $|K| = k\ell/(4c)$ it follows that the two groups intersect non-trivially. If $c = 2$ then the two groups would have to intersect in a group of order 4, which would have to be cyclic and dihedral at the same time, a contradiction. It follows that $c = 1$ and the only non-trivial element in their intersection is an involution. Since the only involution contained in the cyclic group is the central one, we would have $(ty)^{k/4} = s(sx)^j$ for some j. But as the central element commutes with sx, the same must hold for $s(sx)^j$ and this is easily seen to be equivalent to $(sx)^2 = 1$, contrary to the bound on ℓ. Thus, $ty \notin K$, as claimed. But then, from $H/K = \langle tK, yK \rangle \cong \mathrm{C}_2$ it follows that either $t \in K$ or $y \in K$. This means that K contains a dihedral subgroup of order $k/2$, which is in fact the unique Sylow 2-subgroup F_2 of F. Hence $F_2 < K$ and, moreover, from $[t, x] = 1$ or $[s, y] = 1$ it follows that $K = \langle sx \rangle \cdot F_2 \cong \mathrm{C}_{\ell/2} \times \mathrm{D}_{k/2}$. (In fact, at this stage it follows that K is the Fitting subgroup F of H; to see this one only needs to see that $F \neq H$ but in the opposite case F/F_2 would be trivial (being generated by involutions) and we would be back in the case $F = \mathrm{D}_k$. However, in the light of the conclusion we will arrive at, the fact that $K = F$ will turn out to be irrelevant.)

To finish this part of our argument we explore the fact that $x \in K$. By the above structural information this means that $x \in F_2$, and so x would commute with sx, which is equivalent to $(sx)^2 = 1$, contrary to our bound on ℓ.

It thus remains to investigate the case when $k = 8$ and the subgroup $\langle (ty)^2\rangle \cong \mathrm{C}_2$ of $F_2 = \langle z, (ty)^2\rangle \cong \mathrm{C}_2 \times \mathrm{C}_2$ for some $z \in \{t, y\}$ is not a normal subgroup of H, with $|H| = k\ell/(2c) = 4\ell/c$ (note that now 8 exactly divides the order of H). For definiteness we will assume that $t \in F_2$; the case when $y \in F_2$ is done in a completely analogous way by replacing s with x and t with y in the subsequent arguments.

Thus, let $F_2 = \{1, t, yty, (ty)^2\}$; by our assumption $y \notin F_2$. If x was in F_2 then clearly $x \neq 1, t$, and as $x = yty$ implies ty has order two, the only possibility would be $x = (ty)^2$. This would mean that $F_2 = \langle t, x\rangle$; by normality in H, conjugation by s preserves F_2. We cannot have $[s, x] = 1$ as this contradicts our bound on ℓ, and since $[s, t] = 1$ it follows that $sxs = xt$ and hence $(sx)^4 = 1$, contrary to $\ell \geq 6c$. Therefore $x \notin F$, and, as we know, $y \notin F$ either, but note that $xy \in F$. Indeed, in the opposite case, by normality of F_2 and its trivial intersection with $\langle x, y\rangle$, the subgroup $F_2 \rtimes \langle x, y\rangle$ of H would have order 16, contrary to 8 exactly dividing the order of H. A calculation as above shows that the only option for $xy \in F$ is $xy = (ty)^2$ which is equivalent to $txty = 1$ (hence tx has order 4). Observe that this relation also shows that conjugation of F_2 by x fixes $(ty)^2$, and commutativity of $xy, t \in F_2$ implies $xtx = yty$.

Since F_2 is normal in H, it is preserved by conjugation by s, which fixes t. If s fixed $(ty)^2$, then with x fixing $(ty)^2$ the subgroup $\langle (ty)^2\rangle$ would be normal in H, contrary to our assumption made in this special case $k = 8$. It follows that conjugation by s induces an automorphism of F_2 fixing t and transposing yty with $(ty)^2$; the relation $s(yty)s = (ty)^2$ is equivalent to $(syty)^2t = 1$. The composition of the two conjugations, namely, $w \mapsto (xs)w(sx)$ for $w \in F_2 \cong \mathrm{C}_2 \times \mathrm{C}_2$, induces and automorphism in $\mathrm{Aut}(\mathrm{C}_2 \times \mathrm{C}_2) \cong \mathrm{S}_3$ represented by the cycle $t \mapsto yty \mapsto (ty)^2 \mapsto t$ of length 3. In particular, conjugation by $(sx)^3$ centralizes t. If now $\ell/2 = 3q + r$ for $r = \pm 1$, then $[(sx)^3, t] = 1$ implies $[(sx)^r, t] = 1$, contrary to the fact established earlier that conjugation by sx does not fix t. It follows that $r = 0$ and hence $\ell/2$ is a multiple of 3.

We are now approaching derivation of a presentation of H. Observe first that the relation $xtx = yty$ derived earlier simplifies $(syty)^2t = 1$ to $(sxy)^2t = 1$. From $y = txt$ and the fact that $(sx)^3$ centralizes F_2 we obtain $y(sx)^3y = (xs)^3$, so that $\langle (sx)^3\rangle$ is a normal cyclic subgroup of H. If $c = 2$, then, by the findings in the previous paragraph, the odd integer $\ell/4$ is a multiple of 3 and so $(sx)^{\ell/4}$ is a central element of H as obviously commutes with s and x, and also with t (because $(sx)^3$ does) and hence also with $y = txt$. But then $\langle (sx)^{\ell/4}\rangle$ must be a subgroup of F_2, otherwise its product with F_2 would be isomorphic to C_2^3, contrary to the Sylow 2-subgroups of H being dihedral. We cannot have $(sx)^{\ell/4}$ equal to t or yty because the two elements do not commute with y, so that the only option is $(sx)^{\ell/4} = (ty)^2$, but while the element on the left commutes with s the one on the right does not. It follows that $c = 1$ and hence $|H| = 4\ell$.

We note that normality of $\langle (sx)^3 \rangle$ cannot be extended to normality of $\langle sx \rangle$, as otherwise we would have $t(sx)t = (sx)^d$ and then $(xt)^2 = (sx)^{d-1}$ for some d, which are elements of orders of different parity (4 versus some (odd) divisor of $\ell/2$). This shows, as an aside, that the Fitting subgroup of H is $F \cong \langle (sx)^3 \rangle \times F_2 \cong \mathrm{C}_{\ell/6} \times \mathrm{V}_4$. It may also be useful to note that $y = txt$ implies that $sy = stxt = t(sx)t$, showing that the orders of sx and sy are the same, namely, $\ell/2$.

This way, in the case when $k = 8$ and $F_2 = \langle t, yty \rangle$, with $\ell/2 = 3m$ for some odd $m \geq 1$, we have arrived at a presentation of $H = H_p$ of the form

$$H_p = \langle x, y, s, t \,|\, x^2,\ y^2,\ s^2,\ t^2,\ (xy)^2,\ (st)^2,\ (sx)^{3m},\ (ty)^4,\ (sxy)^2t,\ txty \rangle \tag{A.6}$$

representing the edge-biregular map $M = M_p$ of type $(k, \ell) = (8, 6m)$ that appears in item 4 of Theorem 5.11. The corresponding dual map is obtained by interchanging x with y and s with t, while the twin map is found by interchanging x with s and y with t.

Proof of correctness of the presentation (A.6)

It remains to prove that the presentation (A.6) indeed defines a group of order $k\ell/2 = 24m$. For this we begin with a 'universal' group U with relators as in (A.6) but with $(sx)^{3m}$ omitted:

$$U = \langle x, y, s, t \,|\, x^2,\ y^2,\ s^2,\ t^2,\ (xy)^2,\ (st)^2,\ (ty)^4,\ (sxy)^2t,\ txty \rangle \tag{A.7}$$

We show that $N = \langle (sx)^3 \rangle$ is a normal subgroup of U. For this it is sufficient to prove that N is invariant under conjugation by t, as from $y = txt$ and automatic invariance of N under conjugation by s and x it then follows that $yNy = N$. From the last two relators in (A.7) we have $1 = (sxy)^2t = (s(ty)^2)^2t$, which, using $(ty)^4 = 1$, gives $s(ty)^2s = yty$. It follows that conjugation by s fixes t (by the relation $(st)^2 = 1$) and interchanges yty with $(ty)^2$. On the other hand, as $x = tyt$, one similarly obtains that conjugation by x fixes $(ty)^2$ and interchanges t with yty. It follows that the composition of the two conjugations, that is, the mapping $z \mapsto (xs)z(sx)$, induces a 3-cycle $t \mapsto yty \mapsto (ty)^2 \mapsto t$, so that t commutes with $(sx)^3$ and hence preserves N.

(This is what we saw before but now we needed to make sure that it was established solely from the presentation (A.7).)

By the Reidemeister-Schreier theory implemented in MAGMA in the form of its Rewrite command one may check that N is a free group, that is, N is infinite cyclic. Also, by MAGMA one can check that $U/N \cong \mathrm{S}_4$, of order 24. Now, for an arbitrary integer $m \geq 1$ let $N_m = \langle (sx)^{3m} \rangle$ be the cyclic subgroup of N of index m. Since N_m is characteristic in N and so normal in U, we may use the Third isomorphism theorem to write $U/N \cong (U/N_m)/(N/N_m)$ and as $U/N_m \cong H$, $N/N_m \cong \mathrm{C}_m$ and $U/N \cong \mathrm{S}_4$ it follows that $|H| = 24m$, as claimed. This proves correctness of the presentation (A.6).

We conclude by showing that none of the edge-biregular maps given by the group H_p from (A.6) is regular. For such a map to be regular there would have to be an automorphism of H of order 2 interchanging s with x and t with y. If this is the case then H would also contain the relator $ysyt$. From the relator $txty$ of H we have $y = txt$, which, when substituted into $ysyt = 1$ and canceling terms, gives $txsx = 1$. Combining this with $tysy = 1$ yields $xsx = ysy$, or, equivalently, $(sxy)^2 = 1$. But this in combination with the relator $(sxy)^2t$ of (A.6) gives $t = 1$, a contradiction. This completes both the analysis related to the fourth item of Theorem 5.11 as well as our proofs of the main results from section 5.3.2. $\square$

Appendix B

Glossary

Alternate-edge-colouring. A colouring assigned to the edge set of a map using precisely two colours such that: consecutive edges around any given face are differently coloured; and consecutive edges in the cyclic order of edges around any vertex are assigned different colours. This property is equivalent to the map having a bipartite medial graph.

Automorphism of a map. A structure-preserving bijection from a set to itself such that all adjacencies and types of adjacency are preserved. In fully regular maps, the set is usually the set of flags, in edge-biregular maps the set is often $\mathcal{C}$, the set of ready-coloured-corners.

Base map $\mathcal{B}_\gamma$. A map with trace triple $(1, \gamma, 1+\gamma)$ and automorphism group $\mathrm{SL}(2, 2^\alpha)$.

Canonical form for an edge-biregular map and the **Canonical presentation for an edge-biregular map.** Given an edge-biregular map $\mathcal{M}$, with a specific marked corner C, we describe the map by using the *canonical form* $\mathcal{M} = (H; r_0, r_2, \rho_0, \rho_2)$ where H is a group with *canonical presentation* of the form

$$H = \langle\, r_0,\ r_2,\ \rho_0,\ \rho_2 \ \mid\ r_0^2,\ r_2^2,\ \rho_0^2,\ \rho_2^2,\ (r_0r_2)^2,\ (\rho_0\rho_2)^2,\ (r_2\rho_2)^{k/2},\ (r_0\rho_0)^{\ell/2},\ \ldots\rangle. \tag{B.1}$$

Equivalently $\mathcal{M} = (H; x, y, s, t)$ where H is a group with canonical presentation

$$H = \langle\, x,\ y,\ s,\ t\ \mid\ x^2,\ y^2,\ s^2,\ t^2,\ (xy)^2,\ (st)^2,\ (yt)^{k/2},\ (xs)^{\ell/2},\ \ldots\rangle. \tag{B.2}$$

Colour-preserving automorphism group, H. The colour-preserving automorphism group, H, of an edge-biregular map $\mathcal{M}$ is the centraliser of the monodromy group G in the symmetric group acting on the set of ready-coloured-corners $\mathcal{C}$. It is such that $H = \langle\, r_0,\ r_2,\ \rho_0,\ \rho_2\rangle$.

Corner. The pair of flags which together form a *corner* are the two neighbouring flags from one face which meet in a natural way at a corner of that face. In the regular map $\mathcal{M} = (G; x, y, z)$, for $g \in G$, any two-element subset $\{g, gz\}$ is a corner. In an edge-biregular map, each corner has one flag of each colour, and the whole map can be decomposed into the set $\mathcal{C}$ of ready-coloured corners, that is corners with the given assigned colouring.

Dart. A directed edge.

Distinguished. The corner C is associated with a unique vertex, a unique face, a unique bold edge and a unique dashed edge, and we refer to these structures, which are all incident to the marked corner C, as *distinguished*. The automorphisms in H are then generated by the involutions $\langle r_0, r_2, \rho_0, \rho_2 \rangle$ which act locally as reflections in the natural boundary sections of this distinguished corner, (respectively along or across the distinguished shaded edge and along or across the distinguished unshaded edge), while at the same time preserving all the adjacency types and relationships between corners.

Dual. The dual map $\mathcal{M}^*$ of an edge-biregular map $\mathcal{M} = (H; x, y, s, t)$ is also an edge-biregular map and is formed by interchanging x with y and s with t in the presentation (5.1) for H, thereby swapping the vertices with the faces and vice versa, to give $\mathcal{M}^* = (H; y, x, t, s)$.

Edge-biregular map. A map $\mathcal{M}$ with an assigned alternate-edge-colouring and set $\mathcal{C}$ of coloured corners is called *edge-biregular* if and only if the colour-preserving automorphism group H of $\mathcal{M}$ acts regularly on the set $\mathcal{C}$ of coloured corners.

Exponent of a group. In a finite group G the exponent a is defined to be the least common multiple of all the orders of elements of the group.

Exponent of a map. The map $\mathcal{M}$ has j as an exponent if and only if $\mathcal{M}\mathbf{H}_j \cong \mathcal{M}$.

Extended type (k, ℓ, m) of a map. A fully regular map of extended type (k, ℓ, m) has valency k, face length ℓ and Petrie polygons which are m-gons.

External symmetry. When a map $\mathcal{M}$ is isomorphic to its image under a given operator, for example the dual operator, or the Petrie operator, the map $\mathcal{M}$ is said to have, respectively duality, or Petrie-duality, as an external symmetry.

External symmetry group of $\mathcal{M}$. The group, denoted $Ext(\mathcal{M})$, of operators under composition consisting of those operators which are map isomorphisms for $\mathcal{M}$.

Face length. The length of the shortest non-empty closed boundary walk around the given face of a map. When a map is described as having face length ℓ, in this work it is assumed that every face of the map has the same face length.

Flag. The *flags* of a map $\mathcal{M}$ are the faces of the map's barycentric subdivision.

Frorbit. An orbit of elements in a finite field GF(p^α) under the action of the Frobenius field automorphism $x \to x^p$.

Fully regular map. A fully regular map is a map for which its automorphism group acts regularly (fixed-point free and transitive) on the set of flags.

Genuinely edge-biregular map. A genuinely edge-biregular map is a proper edge-biregular map which is truly edge-biregular, that is, it has no semi-edges and the embedding is not also a fully regular map.

Hole operator $\mathbf{H}_j$. Also known as the jth rotational power. When $(k, j) = 1$, the operator $\mathbf{H}_j$ is defined on the regular map $\mathcal{M} = (G; x, y, z)$ of valency k such that $\mathcal{M}\mathbf{H}_j = (G; x, y, (zy)^j y)$.

Isomorphic edge-biregular maps. A pair of edge-biregular maps $\mathcal{M} = (H; x, y, s, t)$ and $\mathcal{M}' = (H'; x', y', s', t')$ given in the canonical form are *isomorphic* if there is a group isomorphism $H \to H'$ taking z onto z' for every $z \in \{x, y, s, t\}$.

Isoset. The *isoset* for the map $\mathcal{M} = (G; x, y, z)$ is the set $\mathcal{Z}$ of involutions such that $z' \in \mathcal{Z}$ if and only if $\mathcal{M} = (G; x, y, z')$ is isomorphic to the map $\mathcal{M} = (G; x, y, z)$.

Kaleidoscopic map. A fully regular map $\mathcal{M}$ of valency k is kaleidoscopic if and only if $\mathcal{M}\mathbf{H}_j \cong \mathcal{M}$ for every j coprime to k.

Link. A given line between two consecutive points in the diagram of a two-involution chain. A link corresponds to a single transposition within an involutory permutation.

Medial. Given a map, its medial graph is the 4-valent graph whose vertices correspond to the edges of the original map, and whose edges connect the neighbouring vertices in the natural way. In this way the medial map has a face corresponding to each original vertex, as well as, for each original face with face-length $/ell$, a face with face-boundary walk of length $/ell$.

Monodromy group for an edge-biregular map. The gluing instructions for the elements of the set of ready-coloured corners $\mathcal{C}$ for a map with an assigned alternate-edge-colouring is called the *ready-coloured-corner-monodromy group* $G = \langle \mathcal{R}_0, \mathcal{R}_2, \mathcal{P}_0, \mathcal{P}_2 \rangle$. It is our choice that the group G is defined to act on the right of $\mathcal{C}$.

Natural parallel product of regular maps with automorphism group G. Denoted $\mathcal{N}_G$, this is a parallel product of fully regular maps with automorphism group G featuring one map from each map isomorphism class. It is super-symmetric by definition.

Non-involution exponent of the group G. Denoted $k(G)$, this is the least common multiple of the valencies over all regular maps $\mathcal{M}(G; x, y, z)$.

Opposite map. The opposite map of $\mathcal{M}$ is $\mathcal{M}\mathbf{DPD}$.

Proper edge-biregular map. When there are no semi-edges in an edge-biregular map it is a *proper* edge-biregular map.

Ready-coloured corner. The set $\mathcal{C}$ of ready-coloured corners, consists of the corners of a given edge-biregular map $\mathcal{M} = (H; x, y, s, t)$ the flags of which have an assigned colouring which is naturally inherited from the colour of the incident edge.

Reflexible. A map is described as reflexible if it is fully regular.

Rooted map. A map which is studied with respect to a distinguished marked flag or ready-coloured-corner.

Self-dual. If $\mathcal{M}$ is isomorphic to $\mathcal{M}^*$ the map is *self-dual*, which (by the map isomorphism condition) is equivalent to the existence of an automorphism of the *group* H swapping x with y and s with t.

Self-opposite. When the map $\mathcal{M}$ is isomorphic to $\mathcal{M}\mathbf{DPD}$.

Self-Petrie. If $\mathcal{M}$ is isomorphic to $\mathcal{M}\mathbf{P}$ then the map is *self-Petrie.*

Self-triality. *Self-triality* is demonstrated when $\mathcal{M} \cong \mathcal{M}\mathbf{DP} \cong \mathcal{M}\mathbf{PD}$. The map $\mathcal{M}$ may then be referred to as *self-trial.*

Semi-edge. A free edge with precisely one end having an associated vertex. A semi-edge is incident to precisely two flags, and so has stabiliser isomorphic to C_2.

Super-symmetric map. A fully regular map is super-symmetric if and only if it is both kaleidoscopic and also has Trinity symmetry.

Trace triple. The ordered triple $(\omega_k, \omega_\ell, \omega_m)$ for a fully regular map on $\mathrm{SL}(2, 2^\alpha)$.

Triality. The concept which relates the three maps $\mathcal{M}$, $\mathcal{M}\mathbf{DP}$ and $\mathcal{M}\mathbf{PD}$ to each-other.

Trinity symmetry. A fully regular map has Trinity symmetry if and only if it is both self-dual and self-Petrie.

Triple trace ratio. The ratio $\omega_k : \omega_\ell : \omega_m$ for a fully regular map on $\mathrm{SL}(2, 2^\alpha)$.

Truly edge-biregular. The map $\mathcal{M}$ is *truly* edge-biregular when it is not isomorphic to its twin.

Twin. Each edge-biregular map $\mathcal{M} = (H; r_0, r_2, \rho_0, \rho_2)$ has a *twin* map which is the same as the original map in every respect except the colouring of the edge orbits is switched. Since each generator in the canonical form is associated with one of the colours of the edges, the twin map of $\mathcal{M}$ is denoted $\mathcal{W}$ and has canonical form $\mathcal{W} = (H; \rho_0, \rho_2, r_0, r_2)$.

Two-involution (a, b) chain. The *two-involution (a, b) chain on $\Omega_{(n,i)}$* is defined as follows: Each of a and b is a product of disjoint transpositions on the given set $\Omega_{(n,i)}$ of n distinct elements, in this case labelled $\Omega_{(n,i)} = \{i, i+1, i+2, \ldots, i+n-1\}$. The involutions to be determined from this definition depend on the parity of n. See Definition 6.5 for precise details.

Type (k, ℓ) of a map. We denote a fully regular or edge-biregular map as having type (k, ℓ) where k is the valency and ℓ is the face-length of the map.

Usual setup in Chapter 2. We focus on regular maps of type (k, ℓ) where the associated group $G = \langle x, y, z \rangle$ is isomorphic to $\mathrm{PSL}(2, q)$ or $\mathrm{PGL}(2, q)$ where q is a power of a given odd prime p. By [19], each of k and ℓ are either equal to p or divide one of $q-1$ or $q+1$. In particular, for the case of $\mathrm{PSL}(2, q)$ where q is odd, each of k and ℓ are either equal to p or divide one of $\frac{q-1}{2}$ or $\frac{q+1}{2}$. Following the convention and notation of [19], in the latter two cases we let ξ_κ and ξ_λ be primitive $2k$th and 2ℓth roots of 1 in the field as follows: if k or ℓ divides $q-1$ then the corresponding primitive root is in the field $\mathrm{GF}(q)$; otherwise it is in the unique quadratic extension $\mathrm{GF}(q^2)$. We also define $\omega_i = \xi_i + \xi_i^{-1}$ for $i \in \{\kappa, \lambda\}$. Note that ω_i is thus in the field $\mathrm{GF}(q)$. We too assume that (k, ℓ) is a hyperbolic pair, that is $1/k + 1/\ell < 1/2$. This implies that $k \geq 3$ and $\ell \geq 3$. The conditions in this paragraph are what we refer to as *the usual setup.*

Appendix C

Notation

Notation	Meaning
$\langle a, b \mid, a^2, b^3 \rangle$	The group generated by a and b such that $a^2 = b^3 = identity$.
$\langle a, b \mid, a^2, b^3, \dots \rangle$	A group generated by a and b such that $a^2 = b^3 = identity$. The dots signify that the group may have further defining relators which are either unknown or not listed.
$\cong$	Symbol denoting isomorphism. This may be between a pair of groups or between a pair of maps.
$\diamond$	The end of a construction.
$\bullet$	The end of an example.
$\star$	The end of a remark.
$\beta, \gamma, \delta, \dots \zeta, \xi, \omega$	Lower case Greek letters often denote an element of $\mathrm{GF}(q)$.
A_n	The alternating group of degree n.
$Aut(\mathcal{M})$	The group of automorphisms of the map $\mathcal{M}$.
$Aut^+(\mathcal{M})$	The group of orientation-preserving automorphisms of the map $\mathcal{M}$.
$\mathcal{B}_\gamma$	The base map with trace triple $(1, \gamma, 1+\gamma)$.
$\mathfrak{B}_\gamma$	The set containing base maps $\mathcal{B}_\epsilon$ where $\epsilon \in \{\gamma, \gamma^{-1}, 1+\gamma, 1+\gamma^{-1}, (1+\gamma)^{-1}, \gamma(1+\gamma)^{-1}\}$.
$[\beta^i]$	The Frorbit containing the field element β^i, with reference to a given primitive element β.
C_n	The cyclic group of order n.
$\mathcal{C}$	The set of ready-coloured corners of an edge-biregular map.
$D := \omega_\kappa^2 + \omega_\lambda^2 - 4$	Definition from [19].
$\mathbf{D}$	The dual operator.
D_{2n}	The dihedral group of order $2n$.
$\mathrm{D}_k := \langle y, t \rangle$	Vertex stabiliser for an edge-biregular map.
$\mathrm{D}_\ell := \langle x, s \rangle$	Face stabiliser for an edge-biregular map
$\Delta = \langle X, Y, S, T \rangle$	The colour-preserving automorphism group for an edge-biregular map of type $(4,4)$ on the Euclidean plane.
$Ext(\mathcal{M})$	The group of external symmetries of the map $\mathcal{M}$.
$\eta := (\xi_\kappa - \xi_\kappa^{-1})^{-1}$	Definition from [19].
$G = G(\mathcal{M})$	The full automorphism group for some given map $\mathcal{M}$. When $\mathcal{M}$ is fully regular, $G = \langle r_0, r_1, r_2 \rangle = \langle x, y, z \rangle$. When $\mathcal{M}$ is fully regular and non-orientable, $G = \langle R, S \rangle = \langle yz, zx \rangle$.
$\mathcal{G} = \mathcal{G}(\mathcal{M})$	The ready-coloured-corner-monodromy group for $\mathcal{M}$, some given edge-biregular map $\mathcal{G} = \langle \mathcal{R}_0, \mathcal{R}_2, \mathcal{P}_0, \mathcal{P}_2 \rangle$.

$\Gamma := \langle \mathfrak{R}_0, \mathfrak{R}_2 \rangle * \langle \mathfrak{P}_0, \mathfrak{P}_2 \rangle$	The universal parent group for edge-biregular maps.
$\mathrm{GF}(q)$	The finite field with q elements.
$H = H(\mathcal{M})$	The colour-preserving automorphism group for $\mathcal{M}$, some given edge-biregular map where $H = \langle r_0, r_2, \rho_0, \rho_2 \rangle = \langle x, y, s, t \rangle$.
$\mathbf{H}_j$	The jth hole operator, the jth rotational power.
k	The valency (degree) of a vertex.
ℓ	The face length.
$(k, j) = 1$	k and j are coprime.
(k, ℓ)	The type of a map of valency k and face length ℓ.
(k, ℓ, m)	The extended type of a map.
$k(G)$	Non-involution exponent of the group G.
$\mathcal{M}$	A map.
$\mathcal{M}^*$	The dual of map $\mathcal{M}$.
$\mathcal{M}_\beta$	A map on $\mathrm{SL}(2, 2^\alpha)$ with trace triple $(\beta, 1, 1 + \beta)$.
$\mathcal{M} = (G; r_0, r_2, r_1)$	A fully regular map.
$\mathcal{M} = (G; x, y, z)$	A fully regular map. Also $\mathcal{M}(G; x, y, z)$.
$\mathcal{M} = (G; R, S)$	A non-orientable fully regular map. Also $\mathcal{M}(G; R, S)$.
$\mathcal{M} = (H; r_0, r_2, \rho_0, \rho_2)$	An edge-biregular map.
$\mathcal{M} = (H; x, y, s, t)$	An edge-biregular map. Also $\mathcal{M}(H; x, y, s, t)$.
$\mathcal{M}(\omega_k, \omega_\ell, \omega_m)$	The regular map on $\mathrm{SL}(2, 2^\alpha)$ with trace triple $(\omega_k, \omega_\ell, \omega_m)$.
$\widehat{\mathcal{M}_I}$	The isomorphism class of fully regular maps on A_5 of extended type $(3, 5, 5)$.
$\widehat{\mathcal{M}_{II}}$	The isomorphism class of fully regular maps on A_5 of extended type $(5, 3, 5)$.
$\widehat{\mathcal{M}_{III}}$	The isomorphism class of fully regular maps on A_5 of extended type $(5, 5, 3)$.
$\mathcal{M}\mathbf{T}$	Image map of the operator $\mathbf{T}$ acting on map $\mathcal{M}$. Also $(\mathcal{M})\mathbf{T}$.
$\mathcal{N}_G$	The natural parallel product of regular maps with automorphism group G, such that each map isomorphism class is represented.
$N(g)$	Norm of g.
$\mathcal{O}$	The ring of algebraic integers of a field.
$\mathfrak{Op}_k$	The universal parent group for operators for given valency k. See section 3.2.1.
$\mathbf{Op}_k(\Omega_G)$	The group of operators $\langle \mathbf{D}, \mathbf{P}, \mathbf{H}_j \mid (j, k) = 1 \rangle$ defined as permutations of the set Ω_G. See section 3.2.2.
$Orb(\mathcal{M})$	The orbit under $\mathbf{Op}_k(\Omega_G)$ of the fully regular map $\mathcal{M}(G; x, y, z)$.
ω	A third root of 1 in a finite field.
$\omega_\kappa := \xi_\kappa + \xi_\kappa^{-1}$	The trace of R.
$\omega_\lambda := \xi_\lambda + \xi_\lambda^{-1}$	The trace of S.
$(\omega_k, \omega_\ell, \omega_m)$	Trace triple for a regular map on $SL(2, 2^\alpha)$.
Ω_G	The set of all rooted fully regular maps $\mathcal{M} = (G; x, y, z)$.
$\Omega_{(n,i)}$	The set $\{i, i+1, i+2, \ldots, i+n-1\}$.

$\mathrm{ord}(ab)$	The order of the group element ab.
$\mathbf{P}$	The Petrie operator.
$\mathrm{PGL}(2,q)$	The general linear group of degree 2 over $\mathrm{GF}(q)$.
$\mathrm{P\Gamma L}(2,q)$	The semi-linear group of degree 2 over $\mathrm{GF}(q)$ for $q=p^\alpha$ where p is prime. The group acts as the automorphism group for both $\mathrm{PGL}(2,q)$ and $\mathrm{PSL}(2,q)$ while $\mathrm{P\Gamma L}(2,q) \cong \mathrm{PGL}(2,q) \rtimes \mathrm{C}_\alpha$. See section 2.1.2.
$\mathrm{PSL}(2,q)$	The projective special linear group of degree 2 over $\mathrm{GF}(q)$.
φ	Euler totient function.
ϕ_j	Repeated Frobenius field automorphism: $\phi_j : x \to x^r$ where $r=p^j$.
$q=p^\alpha$	A prime power.
$\mathbb{Q}$	The set of rational numbers.
$\mathbf{Q}$	The hole operator $\mathbf{H}_q$ for fully regular maps on $SL(2,q)$.
$R:=yz$	The rotation around a vertex of $\mathcal{M}(G;R,S)$.
R_i	An automorphism of the universal regular tessellation of type (k,ℓ), and a generator of the full triangle group for $i \in \{0,1,2\}$.
$\mathcal{R}_i$	The gluing instruction permutation which interchanges each ready-coloured-corner with its adjacent neighbour respectively along or across a bold edge, where $i \in \{0,2\}$. See section 4.1.2.
$r_0 = x$	The automorphism of a symmetric map $\mathcal{M}$ acting locally as a reflection along the distinguished (bold) edge.
$r_1 = z$	The automorphism of a fully regular map acting locally as a reflection across the distinguished corner. See section 1.3.
$r_2 = y$	The automorphism of a symmetric map $\mathcal{M}$ acting locally as a reflection across the distinguished (bold) edge.
$\mathcal{P}_i$	The gluing instruction permutation which interchanges each ready-coloured-corner with its adjacent neighbour respectively along or across a dashed edge for $i \in \{0,2\}$. See section 4.1.2.
$\rho_0 = s$	The automorphism of the edge-biregular map $\mathcal{M}$ acting locally as a reflection along the distinguished dashed edge. See section 4.2.2.
$\rho_2 = t$	The automorphism of the edge-biregular map $\mathcal{M}$ acting locally as a reflection across the distinguished dashed edge. See section 4.2.2.
$S:=zx$	Rotation around a face in $\mathcal{M}(G;R,S)$.
S_n	The symmetric group of degree n.
$\mathcal{T}$	The tetrahedral regular map.
$T_{k,\ell} := \langle R_0,R_1,R_2\rangle$	The group $T_{k,\ell}$ is the full (or extended) triangle group of type (k,ℓ) with group presentation defined to be $\langle\, R_0,R_2,R_1 \mid R_0^2,R_2^2,R_1^2,(R_0R_2)^2,(R_1R_2)^k,(R_0R_1)^\ell \,\rangle$.
$T^+_{k,\ell} := \langle R_0R_1, R_2R_1\rangle$	The (ordinary) triangle group of type (k,ℓ).
U_k	The group of units modulo k.

V_4	The Klein four-group.
(X, Y, Z)	Generating triple for a regular map.
ξ_κ	A primitive κth root of unity in a finite field.
z	The central element of a group.
$\mathbb{Z}$	The set of integers.

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