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Downside Risk Neutral Probabilities

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Abstract

Risk neutral probabilities are adjusted to take into account the asset price effect of risk preferences. This paper introduces downside (respectively outer) risk neutral probabilities, which are adjusted to take into account the asset price effect of preferences for downside (resp. outer) risk and higher degree risks. Using risk preference theory, we interpret these three changes in probability measures in terms of risk substitution. With downside risk neutral probabilities, the pricing kernel is linear in wealth. Outer risk neutral probabilities can be viewed as a reasonable approximation of physical probabilities.

Keywords: downside risk, pricing kernel, prudence, risk aversion, risk neutral probabilities, risk substitution.

JEL codes: D81, G12.

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Risk preferences are not limited to risk aversion. Downside risk aversion, or "prudence", is necessary for decreasing absolute risk aversion, for "standard risk aversion" (Kimball (1993)), and it has been linked with precautionary savings (Kimball (1990)). It is thus important to take into consideration this "higher-order" risk preference. However, the incorporation of risk preferences in economic models of decision under risk often leads to a loss in tractability. In addition, it is often hard to distinguish between the effect of preferences for different degrees of risk. For example, recent studies have shown that risk aversion and downside risk aversion both matter in asset pricing. But how is downside risk aversion incorporated into asset pricing formulas?

The literature on risk preferences has identified several degrees of risk. In an expected utility framework with utility function u, the preference for each degree of risk is associated with a derivative of u of a different order.[1] An increase in second degree risk, also known as an increase in "risk", is an increase in the dispersion of a distribution in the sense of mean-preserving spreads. A risk averse agent, with $u^{\prime\prime} < 0$, is averse to such increases. An increase in third degree risk, also known as an increase in "downside risk", is a transfer of risk to the left of the distribution which leaves the mean and variance unchanged. A downside risk averse agent, with $u^{\prime\prime\prime} > 0$, is averse to such increases (Menezes, Geiss, and Tressler (1980)). An increase in downside risk implies a lower third moment of the distribution, i.e., a lower skewness (Menezes, Geiss, and Tressler (1980), Chiu (2005)). An increase in fourth degree risk, also known as an increase in "outer risk", is a transfer of risk from the center toward the tails which leaves the mean, variance, and skewness unchanged. An outer risk averse agent, with $u^{(4)} < 0$, is averse to such increases (Menezes and Wang (2005)). An increase in outer risk implies a higher fourth moment of the distribution, i.e., a higher kurtosis. These higher-order risk preferences can thus explain why especially the skewness but also the kurtosis of returns are determinants of expected returns (e.g., Harvey and Siddique (2000), Dittmar (2002)).

There are different ways to incorporate these risk preferences into an asset pricing formula. Risk neutral probabilities allow to price assets "as if" investors were risk neutral. Specifically, if investors were risk neutral (with $u^{\prime\prime} = 0$), then risk neutral probabilities would coincide with physical probabilities. If investors are not risk neutral ($u^{\prime\prime} < 0$ if they are risk averse), then risk neutral probabilities are adjusted to take into account the asset price effect of risk aversion and

[1] The signs of successive derivatives of the utility function can be directly related to the preferences for successive degrees of risk, but they cannot be directly related to preferences for moments of the distribution. Indeed, stochastic dominance criteria are related to degrees of risk rather than to moments of the distribution. For example, considering two distributions A and B with the same mean, distribution B is dominated in a second-order stochastic dominance sense if and only if it can be constructed by applying a sequence of mean-preserving spreads to distribution A (e.g. Gollier (2001)). Moreover, a change in a degree of risk implies a certain change in the corresponding moment of the distribution, but the opposite is not necessarily true. For example, a mean-preserving spread implies a higher variance, but a higher variance does not imply a mean-preserving spread of the distribution. That is, if two distributions have the same mean but different variances, the distribution with the lower variance will not necessarily be preferred by a risk averse agent. Thus, to study the effect of risk preferences on asset prices in an expected utility framework, we work with degrees of risk rather than with moments.

higher-order risk attitudes.

This paper makes three contributions. First, we introduce the concepts of downside and outer risk neutral probabilities, which are natural extensions of the concept of risk neutral probabilities. Second, we provide new asset pricing formulas based on these new probability measures. These formulas clarify the effects of risk aversion, downside risk aversion, and prudence on asset prices. Third, we provide interpretations of these changes in probability measures, as well as for the change in probability measure which yields risk neutral probabilities, in terms of risk substitution.

Downside risk neutral probabilities allow to price assets "as if" investors were averse to risk, but were neutral with respect to higher degree risks, including downside risk and outer risk. Specifically, if investors were risk averse but downside risk neutral (with $u^{II} < 0$ and $u^{III} = 0$), then downside risk neutral probabilities would coincide with physical probabilities. If investors are not downside risk neutral ($u^{III} > 0$ if they are downside risk averse), then downside risk neutral probabilities are adjusted to take into account the asset price effect of downside risk aversion and higher-order risk attitudes.[2] Due to this change in probability measure, the asset pricing formulas with downside risk neutral probabilities do not directly involve the (nonlinear) utility function. Instead, they incorporate the coefficient of absolute risk aversion evaluated at the initial level of wealth. This aspect is reminiscent of the Arrow-Pratt approximation of the risk premium, which is widely used due to its simplicity and its intuitive appeal. Yet the asset pricing formulas that involve downside risk neutral (or outer risk neutral) probabilities that we derive in this paper are not approximations: they yield the same asset prices as other asset pricing formulas.

Outer risk neutral probabilities allow to price assets "as if" investors were risk averse and downside risk averse, but were neutral with respect to higher degree risks, including outer risk. Specifically, if investors were prudent but outer risk neutral (with $u^{III} > 0$ and $u^{(4)} = 0$), then outer risk neutral probabilities would coincide with physical probabilities. If investors are not outer risk neutral ($u^{(4)} < 0$ if they are outer risk averse), then outer risk neutral probabilities are adjusted to take into account the asset price effect of outer risk aversion and higher-order risk attitudes. Due to this change in probability measure, the asset pricing formulas with outer risk neutral probabilities do not directly involve the (nonlinear) utility function. Instead, they incorporate the coefficients of absolute risk aversion and of downside risk aversion evaluated at the initial level of wealth.

We provide interpretations of the changes in probability measures in terms of risk substitution. This sheds light on the change in measure that yields the new probability measures introduced in this paper, but also on the change in measure that yields the well-known and widely used risk

[2] The change in probability measure is based on a first order Taylor expansion of marginal utility. Other papers have already used a Taylor expansion of marginal utility for asset pricing purposes. Harvey and Siddique (2000), Dittmar (2002), and Chabi-Yo (2012), among others, approximate the pricing kernel with Taylor expansions of marginal utility of order two (respectively three). Downside risk aversion and outer risk aversion, also known respectively as "prudence" and "temperance", have been defined as preferences over lotteries in Eeckhoudt and Schlesinger (2006).

neutral probabilities.

The change in probability measure which yields risk neutral probabilities is such that the first degree risk adjusts to incorporate the asset price effect of the preferences for the second and higher degree risks. This depends on the investor's rates of substitution between the second and higher degree risks on the one hand, and first degree risk on the other hand. In particular, the rate of substitution between second degree risk and first degree risk is the coefficient of absolute risk aversion, while the rate of substitution between third degree risk and first degree risk is the coefficient of downside risk aversion. The formulas in this paper show how risk neutral probabilities are adjusted to take into account the preferences for risk, but also the preferences for downside risk and for higher degree risks.

The change in probability measure which yields downside risk neutral probabilities is such that the first two degree risks adjust to incorporate the asset price effect of the preferences for the third and higher degree risks. This depends on the investor's rates of substitution between the third and higher degree risks on the one hand, and the first and second degree risks on the other hand. In particular, the rate of substitution between third degree risk and first degree risk is the coefficient of downside risk aversion, while the rate of substitution between third degree risk and second degree risk is the coefficient of absolute prudence. Finally, the change in probability measure which yields outer risk neutral probabilities is such that the first three degree risks adjust to incorporate the asset price effect of the preferences for the fourth and higher degree risks. This depends on the investor's rates of substitution between the fourth and higher degree risks on the one hand, and the first, second and third degree risks on the other hand.

The downside risk neutral probability measure can improve tractability in asset pricing models. Indeed, the pricing kernel associated with downside risk neutral probabilities is linear in future wealth. This sets our analysis apart from a number of recent papers, such as Eraker (2008) and Martin (2013), which also derive new analytical expressions for asset prices, in which the pricing kernel is not linear in state variables. These papers use physical probabilities and assume CRRA or Epstein-Zin preferences, whereas we use new probability measures and we make minimal assumptions on the utility function. Linearity in state variables is advantageous for analytical tractability, interpretation, and empirical implementation (e.g., Brandt and Chapman (2014)). A linear pricing kernel also allows to use the Capital Asset Pricing Model (CAPM) to price assets, without making strong assumptions on risk preferences or probability distributions. By contrast, with physical probabilities, a simple CAPM-like formula requires strong and potentially unrealistic assumptions. It is possible to incorporate the asset pricing effect of higher-order risk preferences with physical probabilities (e.g., equation (7a) in Harvey and Siddique (2000)). However, the formula then involves a number of additional terms, and it only holds as an approximation with a finite number of additional terms (unless the utility function is neutral with respect to some higher degree risk). With downside risk neutral probabilities, a simple CAPM-like formula holds with minimal assumptions, because the asset pricing effect of higher-order risk preferences is incorporated as a probability adjustment, and it is not an approximation.

The outer risk neutral probability measure leads to asset pricing formulas which are less tractable than under the risk neutral or the downside risk neutral probability measure, but more tractable than under the physical probability measure. Interestingly, the outer risk neutral measure could be a close approximation of the physical probability measure. Indeed, outer risk neutral probabilities coincide with physical probabilities if and only if the utility function is outer risk neutral (whether or not it is downside risk averse), a case which is not inconsistent with empirical findings on risk preferences (Deck and Schlesinger (2010)). When in addition the utility function is downside risk averse, we show that the risk neutral distribution has fat tails relative to the physical distribution, and the pricing kernel is U-shaped, consistent with the empirical evidence. In this case, we also relate the level of wealth at which the pricing kernel reaches its minimum value to the coefficient of absolute prudence.

1 The model

For simplicity, we consider an economy with two dates: $t = 0$ and $t = 1$ (in Appendix C, we consider an economy with $T \geq 1$ future dates). Current aggregate wealth in the economy is w_0, and future aggregate wealth is $\tilde{w}_1$. It is equal to w_s in state of the world s, for $s \in \{1, \ldots, S\}$, which occurs with probability $p_s \geq 0$, with $\sum_{s=1}^{S} p_s = 1$.

There is a representative agent, an expected utility maximizer with time separable preferences. They are described by a subjective discount factor $\beta \in (0, 1]$, and a utility function u such that $u' > 0$. In addition, $u'' < 0$ if the agent is risk averse, $u''' > 0$ if the agent is downside risk averse, and $u^{(4)} < 0$ if the agent is outer risk averse (see Scott and Horvath (1980)). For example, CARA and CRRA utility functions are risk averse, downside risk averse, and outer risk averse.

The payoff of an asset at time $t = 1$ is a random variable $\tilde{x}$, which is equal to x_s in state of the world s. The values of w_0, w_s and x_s are finite for any s. Using the standard stochastic discount factor formula (e.g. Hansen and Jagannathan (1991)), the price at $t = 0$ of any given asset with payoff $\tilde{x}$ is

$$P = \mathrm{E}\left[\beta \frac{u'(\tilde{w}_1)}{u'(w_0)} \tilde{x}\right], \tag{1}$$

where $\beta \frac{u'(\tilde{w}_1)}{u'(w_0)}$ is the stochastic discount factor or pricing kernel, and $\mathrm{E}[\cdot]$ is the expectation operator with respect to the physical probability measure, using the information at time 0.

2 Risk neutral probabilities

2.1 Definition and properties

Assume that u is of class C^2. For any given s, let $\eta_{2,s}$ be defined implicitly as

$$u'(w_s) \equiv \eta_{2,s} u'(w_0). \tag{2}$$

Definition 1 *Let* $v_2 \equiv \sum_{s=1}^{S} p_s \eta_{2,s}$, *and*

$$\lambda_{2,s} \equiv \frac{p_s \eta_{2,s}}{v_2} = \frac{p_s \eta_{2,s}}{\sum_{s=1}^{S} p_s \eta_{2,s}}. \tag{3}$$

The set $\{\lambda_{2,s}\}$ *is the set of risk neutral probabilities, and* Λ_2 *is the risk neutral probability measure.*

By construction, $\sum_{s=1}^{S} \lambda_{2,s} = 1$. Note that $\frac{d\lambda_{2,s}}{dp_s} = \frac{\eta_{2,s}}{v_2}$ is the Radon-Nikodym derivative of the risk neutral measure with respect to the physical measure. With linear utility, the risk neutral probability measure coincides with the physical probability measure: $u'(w_s) = u'(w_0)$ for any w_s, so that $\eta_{2,s} = \frac{\eta_{2,s}}{v_2} = 1$ for any s.

We now briefly study the determinants of the divergence between the physical and the risk neutral probability measure, i.e., we study the determinants of $\eta_{2,s}$. First, we have $\frac{d\eta_{2,s}}{dw_s} \leq 0$ if $u'' \leq 0$, with a strict inequality if $u'' < 0$.[3] Intuitively, with respect to the physical probability measure, the risk neutral probability measure overweighs bad states of the world, and underweighs good states of the world. This is illustrated in Figure 1, which depicts the determinants of $\eta_{2,s}$, namely $u'(w_s)$ and $u'(w_0)$ (cf. equation (2)) as a function of w_s. Second, if $w_s > w_0$ (respectively $w_s < w_0$), then according to the mean value theorem (cf. Simon and Blume (1994), p.825) there exists $y_s \in (w_0, w_s)$ (resp. $y_s \in (w_s, w_0)$) such that:

$$u'(w_s) = u'(w_0) + u''(y_s)\,(w_s - w_0) \tag{4}$$

Therefore, with $u' > 0$ and $u'' < 0$, we have $\eta_{2,s} > 1$ if and only if $w_s < w_0$. Moreover, given that $u'(w_s) > 0$ and $u'(w_0) > 0$, (2) implies that $\eta_{2,s} > 0$ for any s. Using (3), this in turn implies that risk neutral probabilities are positive for all s.

2.2 Risk substitution

The variable $\eta_{2,s}$ measures the divergence between the risk neutral probability and the physical probability in state s. We now study its determinants when $u'' \neq 0$, i.e., the risk neutral measure does not coincide with the physical measure. We decompose $\eta_{2,s}$ into several terms to provide an economic interpretation for the change in probability measure in terms of risk substitution.

[3] The proof immediately follows from (2) and the fact that u' is decreasing with $u'' < 0$, and constant with $u'' = 0$.

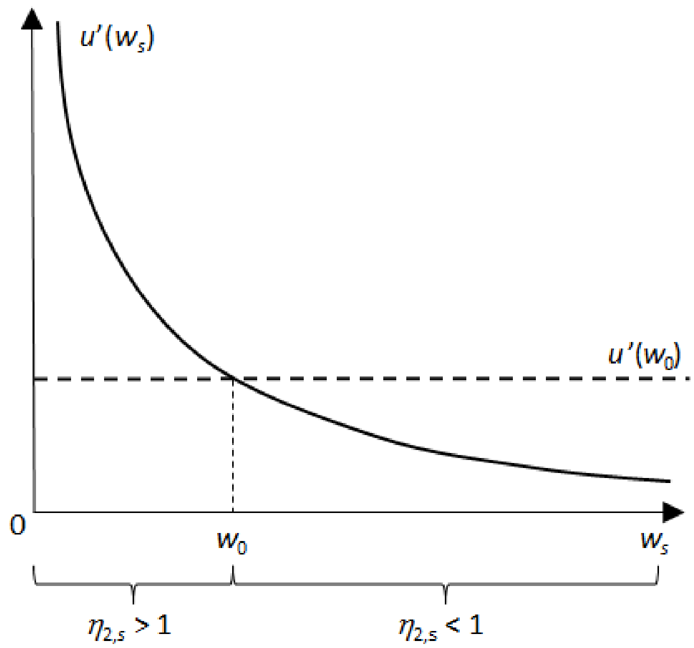

Figure 1: The derivation of risk neutral probabilities with $u'' < 0$ (cf. equations (2) and (3)).

Proposition 1 *Let i be the smallest integer such that $u^{(i)} = 0$ $(i \geq 3)$. Then*

$$\eta_{2,s} = 1 + \sum_{k=1}^{i-2} \frac{1}{k!} \frac{u^{(k+1)}(w_0)}{u'(w_0)} (w_s - w_0)^k. \tag{5}$$

Proof. With $u^{(i)} = 0$, we have

$$u'(w_s) = u'(w_0) + \sum_{k=1}^{i-2} \frac{1}{k!} u^{(k+1)}(w_0)(w_s - w_0)^k.$$

By definition $\eta_{2,s} = \frac{u'(w_s)}{u'(w_0)}$, so that

$$\eta_{2,s} = 1 + \sum_{k=1}^{i-2} \frac{1}{k!} \frac{u^{(k+1)}(w_0)}{u'(w_0)} (w_s - w_0)^k. \tag{6}$$

■

The term $\frac{u^{(k+1)}(w_0)}{u'(w_0)}$ is the coefficient of absolute preference for the $k+1$-th degree risk relative to the first degree risk at w_0. A negative preference (i.e., an aversion) for the $k+1$-th degree risk implies that $\frac{u^{(k+1)}(w_0)}{u'(w_0)} < 0$.

A first degree risk deterioration is a change in the distribution which is undesirable in the sense of first-order stochastic dominance, i.e., for all agents with increasing utility functions. A second degree risk increase is a change in the distribution which leaves the mean unchanged but is undesirable for all agents with concave utility functions (Rothschild and Stiglitz (1970)). Note that a first degree risk improvement implies a higher mean, and an increase in second degree risk implies a higher variance at constant mean.

A risk neutral investor only has preferences for the first degree risk of the distribution. The change in probability measure described in $\eta_{2,s}$ alters the first degree risk of the distribution of states of the world to incorporate the effect of higher degree risks. With i the smallest integer such that $u^{(i)} = 0$, the change in probability measure alters the first degree risk to incorporate the effect of second to $i-1$th degree risks. This change therefore depends on the relation between the aversion to second to $i-1$th degree risks, and the aversion to the first degree risk of the distribution. This explains why the terms $\frac{u^{(k+1)}(w_0)}{u'(w_0)}$, for $k+1 = \{2,\ldots,i-1\}$, appear in (5): Liu and Meyer (2013) show that $(-1)^{m-1}\frac{u^{(m)}(w_0)}{u'(w_0)}$ for $m \geq 1$, is a measure of the rate of substitution between a first degree risk increase and an increase in mth degree risk, i.e., a measure of the willingness to increase a first degree risk to avoid an increase in mth degree risk.

In particular, if $u^{(4)} = 0$, $u'(w_s)$ can be replaced by a second-order Taylor expansion about the point w_0: $u'(w_s) = u'(w_0) + u''(w_0)(w_s - w_0) + \frac{1}{2}u'''(w_s)(w_s - w_0)^2$. Using this equation and (2):

$$\eta_{2,s} = \frac{u'(w_0) + u''(w_0)(w_s - w_0) + \frac{1}{2}u'''(w_s)(w_s - w_0)^2}{u'(w_0)} \tag{7}$$

$$= 1 - A(w_0)(w_s - w_0) + \frac{1}{2}D(w_0)(w_s - w_0)^2, \tag{8}$$

where $A(w_0) \equiv -\frac{u''(w_0)}{u'(w_0)}$ is the coefficient of absolute risk aversion at w_0, $D(w_0) \equiv \frac{u'''(w_0)}{u'(w_0)}$ is the coefficient of downside risk aversion at w_0 (Modica and Scarsini (2005), Crainich and Eeckhoudt (2008), Keenan and Snow (2010)), and $P(w_0) \equiv -\frac{u'''(w_0)}{u''(w_0)}$ is the coefficient of absolute prudence at w_0 (Kimball (1990)). Equation (8) gives a simple decomposition of $\eta_{2,s}$ when the utility function is outer risk neutral – it can be risk averse and downside risk averse. With $u^{(4)} = 0$, the change in probability measure alters the first degree risk to incorporate the effect of the second and third degree risks, also known as “risk” and “downside risk”. This change therefore depends on the relation between the aversion to risk and to downside risk, and the aversion to the first degree risk of the distribution. This explains why the coefficients of absolute risk aversion and of downside risk aversion appear in (8): for $m = 2$, the term $(-1)^{m-1}\frac{u^{(m)}(w_0)}{u'(w_0)}$ mentioned above is equal to $A(w_0)$, and for $m = 3$ it is equal to $D(w_0)$.

Equation (8) also shows that, with $u^{(4)} = 0$, downside risk aversion is especially important in explaining the divergence between the risk neutral and the physical probability for levels of future wealth w_s that differ substantially from the current level of wealth w_0. More precisely, the effect of changes in downside risk aversion on $\eta_{2,s}$ dominates the effect of changes in risk aversion, in the sense that $\left|\frac{d\eta_{2,s}}{dD(w_0)}\right| > \left|\frac{d\eta_{2,s}}{dA(w_0)}\right|$ for $|w_s - w_0| > 2$.

Finally, when the utility function is downside risk averse but outer risk neutral, i.e., with $u''' > 0$ and $u^{(4)} = 0$, equation (8) shows that $\eta_{2,s}$ is quadratic in w_s, and is especially large for very high and very low values of w_s. Given that $\eta_{2,s}$ is the (scaled) ratio of the risk neutral to the physical probability in state s ($\eta_{2,s} \equiv \frac{\lambda_{2,s}}{p_s} v_2$), this in turn implies that the risk neutral probability distribution has fat tails relative to the physical distribution.

2.3 Asset pricing

As has already been shown in many papers and textbooks, the price of any asset can be expressed with risk neutral probabilities. Substituting $u'(w_s)$ from (2) in (1), the price P of an asset with stochastic payoff $\tilde{x}$ may be rewritten as:

$$P = \sum_{s=1}^{S} p_s \left(\beta \frac{\eta_{2,s} u'(w_0)}{u'(w_0)} x_s \right) = \beta v_2 \mathrm{E}^{\Lambda_2} [\tilde{x}] , \tag{9}$$

where $\mathrm{E}^{\Lambda_2}[\cdot]$ is the expectation operator with respect to the probability measure Λ_2. Since this formula must hold for any asset, including the riskfree asset with payoff $x_s = 1$ for all s and with price $P_f = \frac{1}{1+r_f}$, by definition of the riskfree rate r_f, we have $\frac{1}{1+r_f} = \beta v_2$ (the expectation of a constant under any probability measure is equal to this constant). Substituting in (9):

$$P = \frac{\mathrm{E}^{\Lambda_2} [\tilde{x}]}{1 + r_f}, \tag{10}$$

which is the standard asset pricing formula with risk neutral probabilities.

3 Downside risk neutral probabilities

3.1 Definition and properties

Assume that u is of class C^3. For any given s, let $\eta_{3,s}$ be defined implicitly as ($\eta_{3,s}$ exists generically – except for $u'(w_0) + u''(w_0)(w_s - w_0) = 0$):

$$u'(w_s) \equiv \eta_{3,s} \left[u'(w_0) + u''(w_0)(w_s - w_0)\right] . \tag{11}$$

We henceforth consider economies such that $\eta_{3,s}$ exists for all s.

Definition 2 *Let* $v_3 \equiv \sum_{s=1}^{S} p_s \eta_{3,s}$*, and*

$$\lambda_{3,s} \equiv \frac{p_s \eta_{3,s}}{v_3} = \frac{p_s \eta_{3,s}}{\sum_{s=1}^{S} p_s \eta_{3,s}}. \tag{12}$$

The set $\{\lambda_{3,s}\}$ *is the set of downside risk neutral probabilities, and* Λ_3 *is the downside risk neutral probability measure.*

By construction, $\sum_{s=1}^{S} \lambda_{3,s} = 1$. Note that $\frac{\eta_{3,s}}{v_3}$ is the Radon-Nikodym derivative of the downside risk neutral measure with respect to the physical measure. With linear utility or quadratic utility ($u''' = 0$ in both cases), the downside risk neutral probability measure coincides

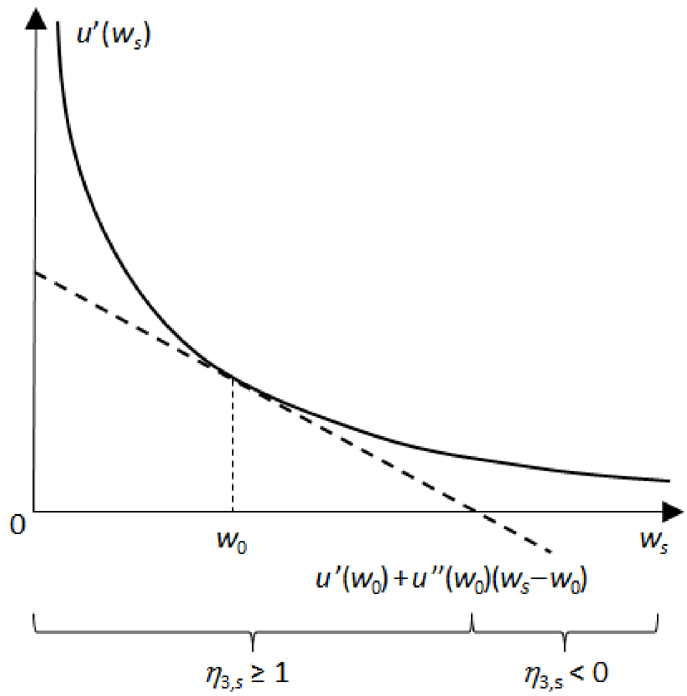

Figure 2: The derivation of downside risk neutral probabilities with $u^{II} < 0$ and $u^{IIII} > 0$ (cf. equations (11) and (12)).

with the physical probability measure: by construction, the Taylor expansion $u^I(w_0)+u^{II}(w_0)(w_s - w_0)$ is then equal to $u^I(w_s)$ for any w_s, so that $\eta_{3,s} = \frac{\eta_{3,s}}{v_3} = 1$ for any s. It is important to note that the downside risk neutral measure is *not* a risk neutral measure. For example, with quadratic utility, the risk neutral measure would not coincide with the physical measure.

We now study the determinants of the divergence between the physical and the downside risk neutral probability measure, as measured by $\eta_{3,s}$, when $u^{IIII} > 0$. The evidence suggests that absolute risk aversion $A(w)$ is nonincreasing in wealth (e.g., Levy (1994), Chiappori and Paiella (2011)), i.e., it is either constant (CARA) or decreasing (DARA) (note that CRRA utility is DARA).[4] Under this assumption, we have the following relation between $\eta_{3,s}$ and future wealth:

Claim 1 *Suppose that $u^{II} < 0$. If the utility function is CARA or DARA and if $\eta_{3,s}$ exists, then*

$$\frac{d\eta_{3,s}}{dw_s} < 0 \text{ if } w_s < w_0.$$

Proof. Rewrite (11) as

$$\eta_{3,s} = \frac{u^I(w_s)}{u^I(w_0) + u^{II}(w_0)(w_s - w_0)}, \tag{13}$$

[4]In the standard version of the portfolio choice problem with a risky asset and a riskfree asset, the dollar amount invested in the risky asset is increasing in wealth if and only if the utility function is DARA (e.g. Gollier (2001) p.59). Huang and Stapleton (2014) study a similar portfolio choice problem when the investor can also invest in an option, to establish cautiousness as a measure of skewness preference.

so that

$$\frac{d\eta_{3,s}}{dw_s} = \frac{u''(w_s)\,(u'(w_0)+u''(w_0)(w_s-w_0)) - u'(w_s)u''(w_0)}{(u'(w_0)+u''(w_0)(w_s-w_0))^{\,2}}. \tag{14}$$

The denominator in (14) is positive, and the numerator can be rearranged as

$$\underbrace{u'(w_0)u''(w_s) - u'(w_s)u''(w_0)}_{A} + \underbrace{u''(w_0)u''(w_s)(w_s - w_0)}_{B} \tag{15}$$

The sign of A is

$$\begin{aligned}
\operatorname{sign}(A) &= \operatorname{sign}\left(u'(w_0)u''(w_s) - u'(w_s)u''(w_0)\right) && (16)\\
&= \operatorname{sign}\left(\frac{u'(w_0)u''(w_s) - u'(w_s)u''(w_0)}{u'(w_0)u'(w_s)}\right) && (17)\\
&= \operatorname{sign}\left(\frac{u''(w_s)}{u'(w_s)} - \frac{u''(w_0)}{u'(w_0)}\right) && (18)
\end{aligned}$$

If the utility function is CARA, then sign(A) = 0. If the utility function is DARA, then sign(A) $<$ 0 if $w_s - w_0 < 0$. For $u'' < 0$ (whether the utility function is CARA or DARA), we have sign(B) $<$ 0 if $w_s - w_0 < 0$. ■

To better understand the determinants of the divergence between the downside risk neutral probability and the physical probability, as measured by $\eta_{3,s}$, we now study the difference between $u'(w_s)$ and the term in brackets on the right-hand-side of (11). If $w_s > w_0$ (respectively $w_s < w_0$), then according to Theorem 30.5 in Simon and Blume (1994, p.828) there exists $z_s \in (w_0, w_s)$ (resp. $z_s \in (w_s, w_0)$) such that:

$$u'(w_s) = u'(w_0) + u''(w_0)\,(w_s - w_0) + \frac{1}{2}u'''(z_s)\,(w_s - w_0)^2 \tag{19}$$

Therefore, with $u' > 0$, $u'' < 0$ and $u''' > 0$, we have $\eta_{3,s} > 1$ if $w_s < w_0$. This is because $u''' > 0$ means that u' is convex, so that u' lies above its tangents. However, we do not necessarily have $\eta_{3,s} > 1$ if $w_s > w_0$, because the term in brackets on the right-hand-side of (11) can then be negative, in which case $\eta_{3,s} < 0$.[5] This is because a first-order Taylor expansion of marginal utility is negative when w_s is high enough (in the same way that marginal utility is negative for a high enough argument of the utility function with quadratic utility), so that $\eta_{3,s}$ must also be negative for (11) to hold. This is illustrated in Figure 2, which depicts the determinants of $\eta_{3,s}$, namely $u'(w_s)$ and $u'(w_0) + u''(w_0)(w_s - w_0)$ (cf. equation (11)) as a function of w_s.

[5]According to Dirac (1942), "Negative energies and probabilities should not be considered as nonsense. They are well-defined concepts mathematically, like a negative sum of money, since the equations which express the important properties of energies and probabilities can still be used when they are negative." Like risk neutral probabilities, downside risk neutral probabilities are a mathematical construct. They are not "physical" probabilities, i.e., they do not represent the probability of occurrence of some events. Instead, their purpose is to provide alternative pricing operators – the fact that some of them can be negative is not inherently problematic in that regard.

3.2 Risk substitution

The variable $\eta_{3,s}$ measures the divergence between the downside risk neutral probability and the physical probability in state s. We now study its determinants when $u''' \neq 0$, i.e., the downside risk neutral measure does not coincide with the physical measure. We decompose $\eta_{3,s}$ into several terms to provide an economic interpretation for the change in probability measure in terms of risk substitution.

Proposition 2 *Let i be the smallest integer such that $u^{(i)} = 0$ ($i \geq 4$). Then*

$$\eta_{3,s} = \left(1 + \sum_{k=2}^{i-2} \frac{1}{k!}\left(\frac{u'(w_0)}{u^{(k+1)}(w_0)}\frac{1}{(w_s - w_0)^k} + \frac{u''(w_0)}{u^{(k+1)}(w_0)}\frac{1}{(w_s - w_0)^{k-1}}\right)^{-1}\right). \tag{20}$$

Proof. With $u^{(i)} = 0$, we have

$$u'(w_s) = u'(w_0) + \sum_{k=1}^{i-2} \frac{1}{k!} u^{(k+1)}(w_0)(w_s - w_0)^k. \tag{21}$$

By definition,

$$\eta_{3,s} = \frac{u'(w_s)}{u'(w_0) + u''(w_0)(w_s - w_0)}. \tag{22}$$

Plugging $u'(w_s)$ from (21) into (22) gives (20). ■

The term $\frac{u^{(k+1)}(w_0)}{u'(w_0)}$ is the coefficient of absolute preference for the $k+1$-th degree risk relative to the first degree risk at w_0, and $\frac{u^{(k+1)}(w_0)}{u''(w_0)}$ is the coefficient of absolute preference for the $k+1$-th degree risk relative to the second degree risk at w_0.

A downside risk neutral investor only has preferences for the first and second degree risks of the distribution. In what follows, we use the same terminology introduced in section 2.2. The change in probability measure described in $\eta_{3,s}$ alters the first and second degree risks of the distribution of states of the world to incorporate the effect of higher degree risks. With i the smallest integer such that $u^{(i)} = 0$, the change in probability measure alters the first and second degree risks to incorporate the effect of the third to $i-1$th degree risks. This change therefore depends on the relation between the aversion to the third to $i-1$th degree risks, and the aversion to the first and second degrees of risk of the distribution. This explains why the terms $\frac{u^{(k+1)}(w_0)}{u'(w_0)}$ and $\frac{u^{(k+1)}(w_0)}{u''(w_0)}$, for $k+1 = \{3, \ldots, i-1\}$, appear in (5): Liu and Meyer (2013) show that $(-1)^{m-1}\frac{u^{(m)}(w_0)}{u'(w_0)}$ for $m \geq 1$, is a measure of the rate of substitution between an increase in mth degree risk and a first degree risk increase, i.e., a measure of the willingness to increase a first degree risk to avoid an increase in mth degree risk; likewise, $-(-1)^{m-1}\frac{u^{(m)}(w_0)}{u''(w_0)}$ for $m \geq 1$, is a measure of the rate of substitution between an increase in mth degree risk and a second degree risk increase, i.e., a measure of the willingness to increase a second degree risk to avoid an increase in mth degree risk.

In particular, if $u^{(4)} = 0$, then $u'''(z_s) = u'''(w_0)$ for any z_s, and equation (20) rewrites as

$$\eta_{3,s} = 1 + \frac{1}{2}\left(\frac{1}{D(w_0)}\frac{1}{(w_s - w_0)^2} - \frac{1}{P(w_0)}\frac{1}{w_s - w_0}\right)^{-1}, \tag{23}$$

where $D(w_0) \equiv \frac{u'''(w_0)}{u'(w_0)}$ is the coefficient of downside risk aversion at w_0, and $P(w_0) \equiv -\frac{u'''(w_0)}{u''(w_0)}$ is the coefficient of absolute prudence at w_0. Equation (23) gives as simple decomposition of $\eta_{3,s}$ when the utility function is outer risk neutral. With $u^{(4)} = 0$, the change in probability measure alters the first and second degree risks to incorporate the effect of third degree risk or "downside risk". This change therefore depends on the relation between the aversion to downside risk, and the aversion to the first and second degree risks of the distribution. This explains why the coefficients of downside risk aversion and of absolute prudence appear in (23): Liu and Meyer (2013) show that $(-1)^{m-1}\frac{u^{(3)}(w_0)}{u^{(m)}(w_0)}$ for $m = 1, 2$, is a measure of the rate of substitution between an increase in third degree risk and an mth degree risk increase, i.e., a measure of the willingness to increase an mth degree risk to avoid an increase in downside risk. For $m = 1$, this term is equal to $D(w_0)$, and for $m = 2$ it is equal to $P(w_0)$.

3.3 Asset pricing

We now show how to express the price of any asset with downside risk neutral probabilities. Note that the asset price P in Propositions 1 and 2 is the same as the asset price in equations (1) and (10). With u of class C^3, the price P of an asset with stochastic payoff $\tilde{x}$ may be decomposed in the following terms:

Proposition 3

$$\begin{aligned} P &= \frac{1}{1+r_f}\hat{E}^3\left[\frac{f(w_0,\tilde{w}_1)}{\hat{E}^3[f(w_0,\tilde{w}_1)]}\tilde{x}\right] && (24)\\ &= \frac{1}{1+r_f}\left(\hat{E}^3[\tilde{x}] - \frac{A(w_0)cov^{\hat{3}}(\tilde{w}_1,\tilde{x})}{1 - A(w_0)\hat{E}^3[\tilde{w}_1 - w_0]}\right), && (25)\end{aligned}$$

where $f(w_0,\tilde{w}_1)$ is linear in $\tilde{w}_1$, and writes as $f(w_0,\tilde{w}_1) \equiv 1 - A(w_0)[\tilde{w}_1 - w_0]$.

Proof. Substituting $u'(w_s)$ from (11) in (1) gives

$$\begin{aligned} P &= \sum_{s=1}^{S} p_s\,\beta\eta_{3,s}\left(1 + \frac{u''(w_0)}{u'(w_0)}(w_s - w_0)\right)x_s && (26)\\ &= \beta v_3\hat{E}^3\left[\tilde{x} + \frac{u''(w_0)}{u'(w_0)}(\tilde{w}_1 - w_0)\tilde{x}\right] \\ &= \beta v_3\left(\hat{E}^3[\tilde{x}] - A(w_0)\hat{E}^3[(\tilde{w}_1 - w_0)\tilde{x}]\right). && (27)\end{aligned}$$

Given that equation (27) must hold for any asset, including the riskfree asset whose payoff is $x_s = 1$ for all s and whose price is by definition of the riskfree rate r_f equal to $P_f = \frac{1}{1+r_f}$, we have

$$\frac{1}{1+r_f} = \beta v_3\left(1 - A(w_0)\hat{E}^3[\tilde{w}_1 - w_0]\right).$$

Substituting in (27) gives

$$P = \frac{1}{1+r_f} \frac{E^{\Lambda_3}[\tilde{x}] - A(w_0)E^{\Lambda_3}[(\tilde{w}_1 - w_0)\tilde{x}]}{1 - A(w_0)E^{\Lambda_3}[\tilde{w}_1 - w_0]}. \tag{28}$$

This formula can be rewritten as in (24) or (25). ■

In equation (25), the utility function only (directly) enters the equation via the coefficient of absolute risk aversion evaluated at the initial level of wealth. Aversion to second degree risk, i.e., risk aversion, is captured by $u'' < 0$, and it implies $A(w_0) > 0$. Equation (25) shows that an asset whose payoff $\tilde{x}$ is positively correlated with future wealth $\tilde{w}_1$ under the downside risk neutral measure Λ_3 has a lower price.

For preferences such that $u''' = 0$, and maintaining the assumption that $u'(w_s) > 0$ for any s (with $u'' < 0$, this implies that w_s is bounded from above), Λ_3 coincides with the physical probability measure, and the utility function can without loss of generality be written as $u(w) = w - \frac{b}{2}w^2$, with $b \geq 0$ if $u'' \leq 0$. The price of any asset with stochastic payoff $\tilde{x}$ is then[6]

$$P = \frac{1}{1+r_f} E[\tilde{x}] - \frac{A(w_0)cov(\tilde{w}_1, \tilde{x})}{1 - A(w_0)E[\tilde{w}_1 - w_0]} = \frac{1}{1+r_f} E[\tilde{x}] - \frac{b\, cov(\tilde{w}_1, \tilde{x})}{1 - bE[\tilde{w}_1]}. \tag{29}$$

In formula (25), the change in probability measure also takes into account the asset price impact of preferences for downside risk and higher degree risks. It is important to note that, in (25), the expression $1 - A(w_0)E^{\Lambda_3}[\tilde{w}_1 - w_0]$ is strictly positive, as shown in the Supplementary Appendix. We have $E^{\Lambda_3}[\tilde{w}_1 - w_0] = 0$ if the expected growth in wealth under Λ_3 is nil; $E^{\Lambda_3}[\tilde{x}] = 0$ if the expected asset payoff under Λ_3 is nil; and $cov^{\Lambda_3}(\tilde{w}_1, \tilde{x}) = 0$ if the asset payoff is uncorrelated with aggregate wealth under Λ_3.

Mean-variance analysis (with physical probabilities) has at least since Rothschild and Stiglitz (1970) been criticized on the grounds that it does not take into account higher-order risk preferences, which leads to substantial pricing errors. Yet, its simplicity and intuitive appeal are such that it remains a cornerstone of finance. We now argue that downside risk neutral probabilities allow to apply mean-variance analysis in asset pricing. In equation (24), the term $\frac{1}{1+r_f}\frac{f(w_0,\tilde{w}_1)}{E^{\Lambda_3}[f(w_0,\tilde{w}_1)]}$ can be viewed as the pricing kernel associated with downside risk neutral probabilities. Comparing (25) and (29) shows that this pricing kernel corresponds to the one that would obtain with quadratic utility. That is, $\frac{1}{1+r_f}\frac{f(w_0,\tilde{w}_1)}{E[f(w_0,\tilde{w}_1)]}$ corresponds to the pricing kernel in a world where agents only have preferences about the mean and the variance of the distribution of their future wealth (with quadratic utility, the expected utility associated with any probability distribution is fully described by its mean and its variance). Thus, using downside risk neutral probabilities allows to price assets in a mean-variance framework. Moreover, equation (24) and the definition of $\boldsymbol{f}(w_0, \tilde{w}_1)$ show that the pricing kernel associated with downside risk neutral probabilities is linear in $\tilde{w}_1$, in contrast with the stochastic discount factor formula in (1).

[6]As above, the assumption that $u'(w_s) > 0$ for any s guarantees that $1 - A(w_0)E[\tilde{w}_1 - w_0]$ or equivalently $1 - bE[\tilde{w}_1]$ is strictly positive.

Comparing the expression in (25) with the one in (10) shows that using downside risk neutral probabilities (the probability measure Λ_3) instead of risk neutral probabilities (the probability measure Λ_2) results in the apparition of an additional term in the asset pricing formula. Indeed, while the risk neutral probability measure is adjusted to take into account aversion to second and higher degree risks, the downside risk neutral probability measure is only adjusted to take into account aversion to third and higher degree risks. Comparing the expression in (25) with the one in (29) shows that, with downside risk neutral probabilities, assets can be valued "as if" the representative agent is only averse to first and second degree risks but were neutral with respect to higher degree risks (including downside risk). Higher-order risk preferences such as aversion to downside risk and to outer risk are incorporated in asset prices via a change in the probability measure.

As is well-known, a linear pricing kernel allows to use the CAPM to derive the expected return on a security. Denoting by $\tilde{R}_i \equiv \frac{\tilde{x}}{P}$ the gross return on a given security i with payoff $\tilde{x}$, by $\tilde{R}_w$ the gross return on the wealth portfolio with payoff $\tilde{w}_1$, and by $\beta_i^{\Lambda_3} \equiv \frac{cov^{\Lambda_3}(R_i, R_w)}{var^{\Lambda_3}(R_w)}$ the security's CAPM beta under the downside risk neutral measure, we have:

$$E^{\Lambda_3}[\tilde{R}_i] - R_f = \beta_i^{\Lambda_3} \left(E^{\Lambda_3}[\tilde{R}_w] - R_f \right) \tag{30}$$

We refer to the Supplementary appendix for technical details. Crucially, whereas the CAPM with physical probabilities requires strong assumptions, the CAPM that can be derived with downside risk neutral probabilities requires minimal assumptions.

4 Outer risk neutral probabilities

4.1 Definition and properties

Assume that u is of class C^4. For any given s, let $\eta_{4,s}$ be defined implicitly as

$$u'(w_s) \equiv \eta_{4,s} \left(u'(w_0) + u''(w_0)(w_s - w_0) + \frac{1}{2} u'''(w_0)(w_s - w_0)^2 \right). \tag{31}$$

Definition 3 *Let* $v_4 \equiv \sum_{s=1}^{S} p_s \eta_{4,s}$, *and*

$$\lambda_{4,s} \equiv p_s \frac{\eta_{4,s}}{v_4} = \frac{p_s \eta_{4,s}}{\sum_{s=1}^{S} p_s \eta_{4,s}}. \tag{32}$$

The set $\{\lambda_{4,s}\}$ *is the set of outer risk neutral probabilities, and* Λ_4 *is the outer risk neutral probability measure.*

By construction, $\sum_{s=1}^{S} \lambda_{4,s} = 1$. Note that $\frac{\eta_{4,s}}{v_4}$ is the Radon-Nikodym derivative of the outer risk neutral measure with respect to the physical measure. With linear, quadratic or cubic utility ($u^{(4)} = 0$ in all three cases), we show below that the outer risk neutral probability measure coincides with the physical probability measure. More generally, with $u^{(4)} \leq 0$, if $w_s > w_0$

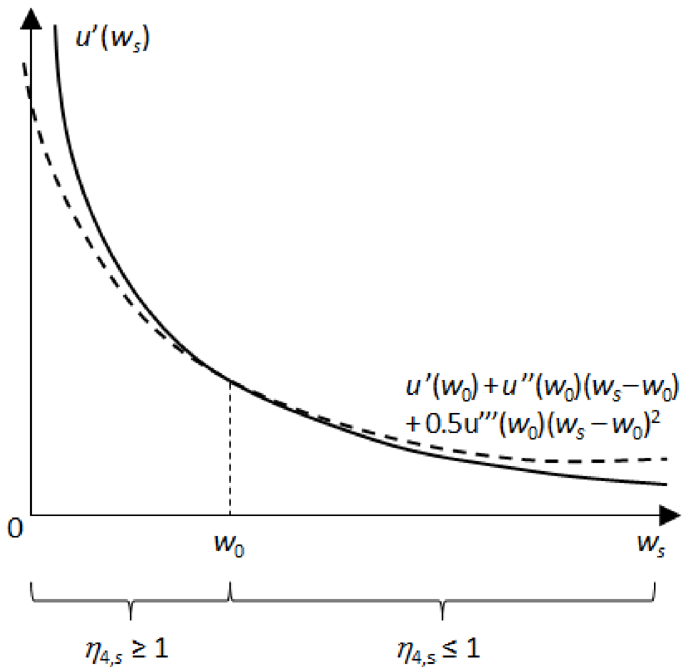

Figure 3: The derivation of outer risk neutral probabilities with $u'' < 0$, $u''' > 0$, and $u^{(4)} < 0$ (cf. equations (31) and (32))

(respectively $w_s < w_0$), then according to Theorem 30.6 in Simon and Blume (1994, p.829) there exists $\zeta_s \in (w_0, w_s)$ (resp. $\zeta_s \in (w_s, w_0)$) such that:

$$u'(w_s) = u'(w_0) + u''(w_0)(w_s - w_0) + \frac{1}{2}u'''(w_0)(w_s - w_0)^2 + \frac{1}{6}u^{(4)}(\zeta_s)(w_s - w_0)^3 \qquad (33)$$

Comparing this equality with the one in (31) yields the following result:

Claim 2 *With $u^{(4)} = 0$, $\eta_{4,s}$ exists and is equal to 1 for any s. With $u^{(4)} < 0$ and $u'' \leq 0$, $\eta_{4,s}$ exists and is strictly positive for any s, and $\eta_{4,s}$ � 1 for $w_s < w_0$.*

Proof. If $u^{(4)} = 0$, then the Taylor expansion $u'(w_0) + u''(w_0)(w_s - w_0) + \frac{1}{2}u'''(w_0)(w_s - w_0)^2$ is equal to $u'(w_s)$ (which is strictly positive by assumption) for any w_s, so that $\eta_{4,s}$ as defined in (31) exists and is equal to 1 for any s.

If $u^{(4)} < 0$ (which given the assumption $u'' \leq 0$ implies $u'' < 0$), the expression $u'(w_0) + u''(w_0)(w_s - w_0) + \frac{1}{2}u'''(w_0)(w_s - w_0)^2$, which is in brackets in (31), is quadratic in w_s, decreasing in w_s at w_0 because $u'' < 0$, and is greater than $u'(w_s)$ for $w_s > w_0$ because of $u^{(4)} < 0$ (which can be seen by comparing (31) and (33)). Because $u' > 0$, this implies that the expression $u'(w_0) + u''(w_0)(w_s - w_0) + \frac{1}{2}u'''(w_0)(w_s - w_0)^2$ in brackets in (31) is strictly positive for any w_s, so that $\eta_{4,s} > 0$ for any s. In addition, with $u^{(4)} < 0$, comparing (31) and (33) shows that $\eta_{4,s}$ � 1 for $w_s < w_0$. ■

The second part of Claim 2 is illustrated in Figure 3, which depicts the determinants of $\eta_{4,s}$, namely $u'(w_s)$ and $u'(w_0)+u''(w_0)(w_s-w_0)+\frac{1}{2}u'''(w_0)(w_s-w_0)^2$ (cf. equation (31)) as a function of w_s.

4.2 Risk substitution

The variable $\eta_{4,s}$ measures the divergence between the outer risk neutral probability and the physical probability in state s. We now study its determinants when $u^{(4)} \neq 0$, i.e., the outer risk neutral measure does not coincide with the physical measure. We decompose $\eta_{4,s}$ into several terms to provide an economic interpretation for the change in probability measure in terms of risk substitution.

Proposition 4 *Let i be the smallest integer such that $u^{(i)}=0$ ($i \geq 5$), we have*

$$\eta_{4,s} = 1+\left(\sum_{k=3}^{i-2}\frac{1}{k!}\left[\frac{u'(w_0)}{u^{(k+1)}(w_0)}\frac{1}{(w_s-w_0)^k}+\frac{u''(w_0)}{u^{(k+1)}(w_0)}\frac{1}{(w_s-w_0)^{k-1}}+\frac{1}{2}\frac{u'''(w_0)}{u^{(k+1)}(w_0)}\frac{1}{(w_s-w_0)^{k-2}}\right]^{-1}\right)^{-1}. \tag{34}$$

Proof. With $u^{(i)}=0$, we have

$$u'(w_s) = u'(w_0)+\sum_{k=1}^{i-2}\frac{1}{k!}u^{(k+1)}(w_0)(w_s-w_0)^k. \tag{35}$$

By definition,

$$\eta_{4,s} = \frac{u'(w_s)}{u'(w_0)+u''(w_0)(w_s-w_0)+\frac{1}{2}u'''(w_0)(w_s-w_0)^2}. \tag{36}$$

Plugging $u'(w_s)$ from (35) into (36) gives (34). ■

The term $\frac{u^{(k+1)}(w_0)}{u'''(w_0)}$ is the coefficient of absolute preference (or aversion) for the $k+1$-th degree risk relative to the third degree risk at w_0.

An outer risk neutral investor only has preferences for the first, second and third degree risks of the distribution. In what follows, we use the same terminology introduced in section 2.2. The change in probability measure described in $\eta_{4,s}$ alters the first, second and third degree risks of the distribution of states of the world to incorporate the effect of higher degree risks. With i the smallest integer such that $u^{(i)}=0$, the change in probability measure alters the first three degrees of risk to incorporate the effect of the fourth to $i-1$th degree risks. This change therefore depends on the relation between the aversion to the fourth to $i-1$th degree risks, and the aversion to the first three degrees of risk of the distribution. This explains why the terms $\frac{u^{(k+1)}(w_0)}{u'(w_0)}$, $\frac{u^{(k+1)}(w_0)}{u''(w_0)}$, and $\frac{u^{(k+1)}(w_0)}{u'''(w_0)}$, for $k+1=\{4,\ldots,i-1\}$, appear in (5): Liu and Meyer (2013) show that $\frac{(-1)^{m-1}}{(-1)^{n-1}}\frac{u^{(m)}}{u^{(n)}(w_0)}$ for $m \geq 1$, is a measure of the rate of substitution between an increase in mth degree risk and an increase in nth degree risk, i.e., a measure of the willingness to increase an nth degree risk to avoid an increase in mth degree risk.

In particular, if $u^{(5)} = 0$[7] we have $u^{(4)}(\zeta_s) = u^{(4)}(w_0)$ for any ζ_s, and equation (34) rewrites as

$$\eta_{4,s} = 1 + \frac{1}{6}\left(\frac{u'(w_0)}{u^{(4)}(w_0)}\frac{1}{(w_s - w_0)^3} + \frac{u''(w_0)}{u^{(4)}(w_0)}\frac{1}{(w_s - w_0)^2} + \frac{1}{2}\frac{u'''(w_0)}{u^{(4)}(w_0)}\frac{1}{(w_s - w_0)}\right)^{-1}, \tag{37}$$

where $-(-1)^{m-1}\frac{u^{(4)}(w_0)}{u^{(m)}(w_0)}$, for $m = 1, 2, 3,$ are all measures of the intensity of outer risk aversion or "temperance" (Crainich and Eeckhoudt (2011), Liu and Meyer (2013)). Equation (37) gives a simple decomposition of $\eta_{4,s}$ when the utility function is neutral with respect to fifth degree risk – it can be risk averse, downside risk averse, and outer risk averse. With $u^{(5)} = 0$, the change in probability measure alters the first three degrees of risk to incorporate the effect of fourth degree risk or "outer risk". This change therefore depends on the relation between the aversion to outer risk, and the aversion to the first three degrees of risk. This explains why several measures of the intensity of aversion to outer risk or "temperance" appear in (37): Liu and Meyer (2013) show that $-(-1)^{m-1}\frac{u^{(4)}(w_0)}{u^{(m)}(w_0)}$ for $m = 1, 2, 3$, is a measure of the rate of substitution between an increase in fourth degree risk and an mth degree risk increase, i.e., a measure of the willingness to increase an mth degree risk to avoid an increase in outer risk.

4.3 Asset pricing

We now show how to express the price of any asset with outer risk neutral probabilities. With u of class C^4, the price P of an asset with stochastic payoff $\tilde{x}$ may be decomposed in the following terms:

Proposition 5

$$P = \frac{1}{1+r_f}\hat{E}_4\left[\frac{g(w_0, \tilde{w}_1)}{\hat{E}_4[g(w_0, \tilde{w}_1)]}\tilde{x}\right] \tag{38}$$

$$= \frac{1}{1+r_f}\left(\hat{E}_4[\tilde{x}] + \frac{-A(w_0)\widehat{cov}_4(\tilde{w}_1, \tilde{x}) + \frac{1}{2}D(w_0)\widehat{cov}_4((\tilde{w}_1 - w_0)^2, \tilde{x})}{1 - A(w_0)\hat{E}_4[\tilde{w}_1 - w_0] + \frac{1}{2}D(w_0)\hat{E}_4[(\tilde{w}_1 - w_0)^2]}\right), \tag{39}$$

where $g(w_0, \tilde{w}_1) \equiv 1 - A(w_0)[\tilde{w}_1 - w_0] + \frac{1}{2}D(w_0)(\tilde{w}_1 - w_0)^2$*.*

Proof. Substituting $u'(w_s)$ from (31) in (1), the asset price P may be rewritten as:

$$\begin{aligned}
P &= \sum_{s=1}^{S} p_s\, \beta\eta_{4,s}\left(1 + \frac{u''(w_0)}{u'(w_0)}(w_s - w_0) + \frac{1}{2}\frac{u'''(w_0)}{u'(w_0)}(w_s - w_0)^2\right)x_s \\
&= \beta v_4 \hat{E}_4\left[\tilde{x} - A(w_0)(\tilde{w}_1 - w_0)\tilde{x} + \frac{1}{2}D(w_0)(\tilde{w}_1 - w_0)^2\tilde{x}\right] \\
&= \beta v_4\left(\hat{E}_4[\tilde{x}] - A(w_0)\hat{E}_4[(\tilde{w}_1 - w_0)\tilde{x}] + \frac{1}{2}D(w_0)\hat{E}_4\left[(\tilde{w}_1 - w_0)^2\tilde{x}\right]\right).
\end{aligned} \tag{40}$$

[7]The approximation of the pricing kernel in Dittmar (2002, equation (6)) considers preferences for the first four degrees of risk only, which is consistent with $u^{(5)} = 0$.

Given that equation (40) must hold for any asset, including the riskfree asset whose payoff is $x_s = 1$ for all s and whose price is by definition of the riskfree rate r_f equal to $P_f = \frac{1}{1+r_f}$, we have

$$\frac{1}{1+r_f} = \beta v_4 \left\{ 1 - A(w_0)\mathrm{E}^{\Lambda_4}[\tilde{w}_1 - w_0] + \frac{1}{2} D(w_0)\mathrm{E}^{\Lambda_4}[(\tilde{w}_1 - w_0)^2] \right\}. \tag{41}$$

Substituting (41) in (40) gives

$$P = \frac{1}{1+r_f} \frac{\mathrm{E}^{\Lambda_4}[\tilde{x}] - A(w_0)\mathrm{E}^{\Lambda_4}[(\tilde{w}_1 - w_0)\tilde{x}] + \frac{1}{2}D(w_0)\mathrm{E}^{\Lambda_4}[(\tilde{w}_1 - w_0)^2\tilde{x}]}{1 - A(w_0)\mathrm{E}^{\Lambda_4}[\tilde{w}_1 - w_0] + \frac{1}{2}D(w_0)\mathrm{E}^{\Lambda_4}[(\tilde{w}_1 - w_0)^2]}. \tag{42}$$

This formula can be rewritten as in (38) or (39). ■

The intuition behind equation (39) is that an asset whose payoff $\tilde{x}$ positively covaries with future wealth $\tilde{w}_1$ will have a lower price if the utility function is risk averse ($A(w_0) > 0$), as in the previous section. In addition, an asset whose payoff tends to be low when future wealth deviates more from its current level will have a lower price if the utility function is averse to downside risk ($D(w_0) > 0$). Consistent with the relation between downside risk and skewness, it is noteworthy that, up to a scaling factor, the term $cov((\tilde{w}_1 - w_0)^2, \tilde{x})$ can be interpreted as the coskewness of the asset (Harvey and Siddique (2000), Chabi-Yo, Leisen, and Renault (2014)). In formula (39), the opposite of this covariance measures the contribution of the asset to the downside risk of the wealth portfolio under the outer risk neutral measure (that is, a negative covariance means a positive contribution to downside risk). It is also important to note that, in (39), the expression $1 - A(w_0)\mathrm{E}^{\Lambda_4}[\tilde{w}_1 - w_0] + \frac{1}{2}D(w_0)\mathrm{E}^{\Lambda_4}[(\tilde{w}_1 - w_0)^2]$ is strictly positive, as shown in the Supplementary Appendix.

As in the previous section, in equation (38), the term $\frac{1}{1+r_f}\frac{g(w_0,\tilde{w}_1)}{\mathrm{E}^{\Lambda_4}[g(w_0,\tilde{w}_1)]}$ can be viewed as the pricing kernel associated with outer risk neutral probabilities. It is quadratic in future wealth according to Proposition 5.

We show in the Supplementary Appendix that the expected return on any asset i can be expressed as

$$\mathrm{E}^{\Lambda_4}[\tilde{R}_i] - R_f = \chi\, cov^{\Lambda_4}(\tilde{R}_w, \tilde{R}_i) + \vartheta\, cov^{\Lambda_4}(\tilde{R}^2_w, \tilde{R}_i), \tag{43}$$

for two constants χ and ϑ. This result is similar to equation (7) in Harvey and Siddique (2000), but there is an important difference. Harvey and Siddique (2000) use physical probabilities and assume a quadratic stochastic discount factor, so that their result is only an approximation if the stochastic discount factor is not quadratic in wealth. By contrast, using outer risk neutral probabilities ensures that the equation in (43) holds in any case. With $u^{(4)} = 0$, outer risk neutral probabilities coincide with physical probabilities, and our result coincides with Harvey and Siddique's. With $u^{(4)} \neq 0$, outer risk neutral probabilities will diverge from physical probabilities in such a way that (43) still holds.

In the remainder of this section, we study asset pricing when the utility function is outer risk neutral, i.e., $u^{(4)} = 0$. In this case, the outer risk neutral probability measure Λ_4 coincides with the physical probability measure (cf. Claim 2). The formulas in Proposition 5 can then be applied with physical probabilities. That is, in this case, the price of any asset with stochastic payoff $\tilde{x}$ can simply be expressed as

$$P = \mathrm{E}\left[\frac{1}{1+r_f}\frac{g(w_0,\tilde{w}_1)}{\mathrm{E}[g(w_0,\tilde{w}_1)]}\tilde{x}\right] = \frac{1}{1+r_f}\left(\mathrm{E}[\tilde{x}] + \frac{-A(w_0)cov(\tilde{w}_1,\tilde{x}) + \frac{1}{2}D(w_0)cov((\tilde{w}_1-w_0)^2,\tilde{x})}{1-A(w_0)\mathrm{E}[\tilde{w}_1-w_0]+\frac{1}{2}D(w_0)\mathrm{E}[(\tilde{w}_1-w_0)^2]}\right). \tag{44}$$

Note that the expectations and covariances in (44) are computed with physical probabilities. When the utility function is outer risk neutral, the formulas in (44) allow to disentangle between the asset price impact of risk aversion and downside risk aversion. When the utility function is outer risk averse, similar formulas in (38) and (39) apply; the probabilities are then adjusted to take into account the asset price impact of aversion to outer risk and higher degree risks.

When the utility function is outer risk neutral, the pricing kernel (with physical probabilities) can be written as $\frac{1}{1+r_f}\frac{g(w_0,w_s)}{\mathrm{E}[g(w_0,\tilde{w}_1)]}$ according to equation (44). It can then be U-shaped:

Claim 3 *When the utility function is downside risk averse but outer risk neutral, the pricing kernel is decreasing in wealth for* $w_s < w_0 + \frac{1}{P(w_0)}$*, and increasing in wealth for* $w_s > w_0 + \frac{1}{P(w_0)}$*.*

Proof. If the utility function is downside risk averse but outer risk neutral, we have $u''' > 0$ and $u^{(4)} = 0$. As shown in the Supplementary Appendix, the expression $1 - A(w_0)\mathrm{E}^{\Lambda_4}[\tilde{w}_1 - w_0] + \frac{1}{2}D(w_0)\mathrm{E}^{\Lambda_4}[(\tilde{w}_1 - w_0)^2]$, which is equal to $\mathrm{E}^{\Lambda_4}[g(w_0,\tilde{w}_1)]$, is strictly positive. Thus, with $u^{(4)} = 0$, we have $\mathrm{E}[g(w_0,\tilde{w}_1)] = \mathrm{E}^{\Lambda_4}[g(w_0,\tilde{w}_1)] > 0$. Consequently, with $1+r_f > 0$, the pricing kernel $\frac{1}{1+r_f}\frac{g(w_0,w_s)}{\mathrm{E}[g(w_0,\tilde{w}_1)]}$ is increasing (respectively decreasing) in w_s if $g(w_0,w_s)$ is increasing (resp. decreasing) in w_s. Given $g(w_0,w_s) \equiv 1 - A(w_0)[w_s - w_0] + \frac{1}{2}D(w_0)(w_s - w_0)^2$, and with $u''' > 0$ and $u' > 0$ which imply $D(w_0) > 0$, the pricing kernel is increasing in future wealth w_s if and only if $w_s > w_0 + \frac{A(w_0)}{D(w_0)}$. In addition, $\frac{A(w_0)}{D(w_0)} = \frac{-u''(w_0)/u'(w_0)}{u'''(w_0)/u'(w_0)} = \frac{1}{P(w_0)}$. ■

When the utility function is downside risk averse but outer risk neutral, Claim 3 shows how the level of wealth at which the pricing kernel reaches its minimum value is related to risk preferences. The model then predicts that the level of wealth above which the pricing kernel is increasing in wealth is larger than the current level of wealth. Moreover, for a given initial wealth w_0, the higher the coefficient of absolute prudence, the lower the level of future wealth w_s above which the pricing kernel is increasing in future wealth.

Finally, we argue in this paragraph that outer risk neutral probabilities could be viewed as a reasonable approximation of physical probabilities. Indeed, although more research is needed, the empirical evidence is not inconsistent with the hypothesis that agents are prudent but not temperant. It has often been assumed that the preferences for different degrees of risk alternate signs, and indeed they do under CRRA or CARA utility for example. Yet, contrary to what is sometimes believed, there is no clear relation between a preference for a degree of risk and another. For example, Crainich, Eeckhoudt, and Trannoy (2013) show that risk lovers can be prudent. The sign of preferences for different degrees of risk is ultimately an empirical question. There is strong direct and indirect empirical evidence for prudence (see Deck and Schlesinger

(2010, 2014) and the references therein), which suggests that it is important to consider the aversion to downside risk in asset pricing. Direct evidence regarding temperance is more recent and more mixed (Deck and Schlesinger (2010, 2014)). Noussair, Trautmann, and van de Kuilen (2014) find that temperance is associated with less risky investment portfolios and with higher risk aversion, suggesting that less temperate investors may be more active in financial markets. In addition, whereas skewness has been identified as an important and robust determinant of asset returns, the effect of kurtosis has been found to be more limited, and does not show up in all specifications (Dittmar (2002), Chang, Christoffersen, and Jacobs (2013), Amaya, Christoffersen, Jacobs, Vasquez (2015)). We also showed that, when the utility function is downside risk averse but outer risk neutral, then the risk neutral distribution has fat tails relative to the physical distribution (see section 2.2), and the pricing kernel is U-shaped (see Claim 3), consistent with the empirical evidence.

5 Conclusion

In this paper, we emphasized that the risk neutral probability measure – or second degree risk neutral probability measure – is one element of a broader family. This family of distributions notably includes third (respectively fourth) degree risk neutral probabilities, which we also referred to as downside (resp. outer) risk neutral probabilities. For $i \geq 2$, the i-th degree risk neutral probability is adjusted to take into account the asset price impact of risks of degree i and higher. We showed that the change in probability measure involves risk substitution: with the i-th degree risk neutral probability, risks of degree $i-1$ and lower are adjusted to take into account the asset price impact of risks of degree i and higher.

While risk neutral probabilities allow to value assets in a risk neutral framework, downside risk neutral probabilities allow to value assets in a simple and intuitive mean-variance framework. The pricing kernel associated with downside (respectively outer) risk neutral probabilities is linear (resp. quadratic) in future wealth. The changes in probability measure described in this paper can thus improve tractability in asset pricing models. The new formulas that we derived also further our understanding of the asset pricing effect of risk aversion, downside risk aversion, and prudence.

In Appendix A, we extend the analysis by introducing i-th degree risk neutral probabilities. In Appendix B, we plot the densities that correspond to different probability measures for illustrative purposes. In Appendix C, we extend the analysis to a multiperiod model.

6 References

Amaya, D., Christoffersen, P., Jacobs, K., Vasquez, A., 2015, Does Realized Skewness Predict the Cross-Section of Equity Returns? *Journal of Financial Economics*, 118 (1), 135-167.

Brandt, M.W., Chapman, D.A., 2014, Linear approximations and tests of conditional pricing models. Working paper, Duke University.

Chabi-Yo, F., 2012, Pricing kernels with stochastic skewness and volatility risk. *Management Science*, 58 (3), 624-640.

Chabi-Yo, F., Leisen, D.P.J., Renault, E., 2014, Aggregation of preferences for skewed asset returns. *Journal of Economic Theory*, 154, 453-489.

Chang, B.Y., Christoffersen, P., Jacobs, K., 2013, Market skewness risk and the cross section of stock returns. *Journal of Financial Economics*, 107 (1), 46-68.

Chiappori, P.A., Paiella, M., 2011, Relative risk aversion is constant: Evidence from panel data. *Journal of the European Economic Association*, 9 (6), 1021-1052.

Chiu, W.H., 2005, Skewness preference, risk aversion, and the precedence relations on stochastic changes. *Management Science*, 51 (12), 1816-1828.

Christoffersen, P., Heston, S., Jabobs, K., 2013, Capturing Option Anomalies with a Variance-Dependent Pricing Kernel. *Review of Financial Studies*, 26 (8), 1963-2006.

Crainich, D., Eeckhoudt, L., 2008, On the intensity of downside risk aversion. *Journal of Risk and Uncertainty*, 36 (3), 267-276.

Crainich, D., Eeckhoudt, L., 2011, Three measures of the intensity of temperance. Working paper, IESEG.

Crainich, D., Eeckhoudt, L., Trannoy, A., 2013. Even (Mixed) Risk Lovers Are Prudent. *American Economic Review*, 103 (4), 1529-1535.

Deck, C., Schlesinger, H., 2010, Exploring higher order risk effects. *Review of Economic Studies*, 77 (4), 1403-1420.

Deck, C., Schlesinger, H., 2014, Consistency of Higher Order Risk Preferences. *Econometrica*, 82 (5), 1913-1943.

Dirac, P.A.M., 1942, Bakerian lecture. The physical interpretation of quantum mechanics. *Proceedings of the Royal Society of London* 180 (980), 1-40.

Dittmar, R.F., 2002, Nonlinear pricing kernels, kurtosis preference, and evidence from the cross section of equity returns. *Journal of Finance*, 57 (1), 369-403.

Eeckhoudt, L., Schlesinger, H., 2006, Putting risk in its proper place. *American Economic Review*, 96 (1), 280-289.

Eraker, B., 2008, Affine general equilibrium models. *Management Science*, 54 (12), 2068-2080.

Gollier, C., 2001, *The Economics of Risk and Time*. MIT Press, Cambridge, MA.

Hansen, L.P., Jagannathan, R., 1991, Implications of security market data for models of dynamic economies. *Journal of Political Economy*, 99 (2), 225-262.

Harvey, C.R., Siddique, A., 2000, Conditional skewness in asset pricing tests. *Journal of Finance*, 55 (3), 1263-1295.

Huang, J., Stapleton, R., 2014, Cautiousness, skewness preference, and the demand for options. *Review of Finance*, 18 (6), 2375-2395.

Keenan, D.C., Snow, A., 2010, Greater prudence and greater downside risk aversion. *Journal of Economic Theory*, 145 (5), 2018-2026.

Kimball, M.S., 1990, Precautionary savings in the small and in the large. *Econometrica*, 58 (1), 53-73.

Kimball, M.S., 1993, Standard risk aversion. *Econometrica*, 61 (3), 589-611.

Levy, H., 1994, Absolute and relative risk aversion: An experimental study. *Journal of Risk and Uncertainty*, 8 (3), 289-307.

Liu, L., Meyer, J., 2013, Substituting one risk increase for another: A method for measuring risk aversion. *Journal of Economic Theory*, 148 (6), 2706-2718.

Martin, I.W.R., 2013, Consumption-based asset pricing with higher cumulants. *Review of Economic Studies*, 80 (2), 745-773

Menezes, C., Geiss, C., Tressler, J., 1980, Increasing downside risk. *American Economic Review*, 70 (5), 921-932.

Menezes, C.F., Wang, X.H., 2005, Increasing outer risk. *Journal of Mathematical Economics*, 41 (7), 875-886.

Modica, S., Scarsini, M., 2005, A note on comparative downside risk aversion. *Journal of Economic Theory*, 122 (2), 267-271.

Rothschild, M., Stiglitz, J., 1970, Increasing risk: I. A definition. *Journal of Economic Theory*, 2 (3), 225-243.

Scott, R.C., Horvath, P.A., 1980, On the direction of preference for moments of higher order than the variance. *Journal of Finance*, 35 (4), 915-919.

Simon, C.P., Blume, L., 1994, *Mathematics for Economists*. First edition, Norton, New York, NY.

A Generalization

This Appendix generalizes the analysis by introducing i-th degree risk neutral probabilities and the associated asset pricing formulas.

Assume that u is of class C^i, for an integer $i \geq 2$. For given i and s, let $\eta_{i,s}$ be defined implicitly as ($\eta_{i,s}$ exists generically – except for $\sum_{k=0}^{i-2} \frac{1}{k!} u^{(k+1)}(w_0)(w_s - w_0)^k = 0$):

$$u'(w_s) \equiv \eta_{i,s} \sum_{k=0}^{i-2} \frac{1}{k!} u^{(k+1)}(w_0)(w_s - w_0)^k. \tag{45}$$

Definition 4 *Let* $v_i \equiv \sum_{s=1}^{S} p_s \eta_{i,s}$*, and*

$$\lambda_{i,s} \equiv p_s \frac{\eta_{i,s}}{v_i} = \frac{p_s \eta_{i,s}}{\sum_{s=1}^{S} p_s \eta_{i,s}}. \tag{46}$$

The set $\{\lambda_{i,s}\}$ *is the set of i-th degree risk neutral probabilities, and* Λ_i *is the i-th degree risk neutral probability measure.*

We now show how to express the price of any asset with i-th degree risk neutral probabilities. Substituting $u'(w_s)$ from (45) in (1), the asset price P may be decomposed in the following terms:

$$\begin{aligned} P &= \sum_{s=1}^{S} p_s \left[\beta \eta_{i,s} \sum_{k=0}^{i-2} \frac{1}{k!} \frac{u^{(k+1)}(w_0)}{u'(w_0)} (w_s - w_0)^k \tilde{x} \right] \\ &= \beta v_i \mathrm{E}^{\Lambda_i} \left[\sum_{k=0}^{i-2} \frac{1}{k!} C_{k+1}(w_0)(\tilde{w}_1 - w_0)^k \tilde{x} \right] \\ &= \beta v_i \left[\sum_{k=0}^{i-2} \frac{1}{k!} C_{k+1}(w_0) \mathrm{E}^{\Lambda_i} (\tilde{w}_1 - w_0)^k \tilde{x} \right] \end{aligned} \tag{47}$$

where $C_{k+1}(w_0) \equiv \frac{u^{(k+1)}(w_0)}{u'(w_0)}$ is the coefficient of absolute preference for the $k+1$-th degree risk at w_0. A negative preference (i.e., an aversion) for the $k+1$-th degree risk implies that $C_{k+1}(w_0) < 0$. The price P of an asset with stochastic payoff $\tilde{x}$ may be decomposed in the following terms:

Proposition 6 *If u is of class* C^i*, for a given integer* $i \geq 2$*:*

$$P = \frac{1}{1+r_f} \frac{\sum_{k=0}^{i-2} \frac{1}{k!} C_{k+1}(w_0) \mathrm{E}^{\Lambda_i} \left[(\tilde{w}_1 - w_0)^k \tilde{x} \right]}{\sum_{k=0}^{i-2} \frac{1}{k!} C_{k+1}(w_0) \mathrm{E}^{\Lambda_i} \left[(\tilde{w}_1 - w_0)^k \right]} \tag{48}$$

$$= \frac{1}{1+r_f} \mathrm{E}^{\Lambda_i} \left[\frac{f_i(w_0, \tilde{w}_1)}{\mathrm{E}^{\Lambda_i}[f_i(w_0, \tilde{w}_1)]} \tilde{x} \right] \tag{49}$$

where $f_i(w_0, \tilde{w}_1) \equiv \sum_{k=0}^{i-2} \frac{1}{k!} C_{k+1}(w_0)(\tilde{w}_1 - w_0)^k$.

Proof. Given that equation (47) must hold for any asset, including the riskfree asset whose payoff is $x_s = 1$ for all s and whose price is by definition of the riskfree rate r_f equal to $P_f = \frac{1}{1+r_f}$, we have

$$\frac{1}{1+r_f} = \beta v_i \left[\sum_{k=0}^{i-2} \frac{1}{k!} C_{k+1}(w_0) E^{\Lambda_i} \left((\tilde{w}_1 - w_0)^k \right) \right]. \tag{50}$$

Substituting (50) in (47) gives (48). ■

In particular, for preferences such that $u^{(i)} = 0$, Λ_i coincides with the physical probability measure, and the price of any asset with stochastic payoff $\tilde{x}$ is

$$P = \frac{1}{1+r_f} \frac{\sum_{k=0}^{i-2} \frac{1}{k!} C_{k+1}(w_0) E\left[(\tilde{w}_1 - w_0)^k \tilde{x} \right]}{\sum_{k=0}^{i-2} \frac{1}{k!} C_{k+1}(w_0) E\left[(\tilde{w}_1 - w_0)^k \right]}. \tag{51}$$

Finally, we study the limit case as $i \to \infty$. Assume that u^I is analytic around w_0. For any given s, let $\eta_{\infty,s}$ be defined implicitly as ($\eta_{\infty,s}$ exists generically – except for $\sum_{k=0}^{\infty} \frac{1}{k!} u^{(k+1)}(w_0)(w_s - w_0)^k = 0$):

$$u^I(w_s) \equiv \eta_{\infty,s} \sum_{k=0}^{\infty} \frac{1}{k!} u^{(k+1)}(w_0)(w_s - w_0)^k. \tag{52}$$

Proposition 7 *If u^I is analytic around w_0, the i-th degree risk neutral probability measure, $\Lambda_{\infty,s}$, coincides with the physical probability measure when $i \to \infty$.*

Proof. Given that u^I is analytic around w_0, the Taylor series of u^I give:

$$u^I(w_s) = \sum_{k=0}^{\infty} \frac{1}{k!} u^{(k+1)}(w_0)(w_s - w_0)^k. \tag{53}$$

Comparing with (52), this implies that $\eta_{\infty,s} = 1$ for any s. Then Definition 4 implies that $\lambda_{\infty,s} = p_s$ for any s. ■

B Numerical examples

In this section, we present two numerical examples. In each case, we use the analog of the formulas presented in the paper for continuous distributions. First, we set $w_0 = 1$, we let $\tilde{w}_1$ be normally distributed with a mean of zero and a variance of one, and we use a CARA utility function with an absolute risk aversion of $\frac{1}{2}$.[8] Second, we set $w_0 = 10$, we let $\tilde{w}_1$ follow a gamma distribution with a shape parameter of seven and a scale parameter of one, and we use the same utility function.

Each set of four figures – figures 4 to 7, and figures 8 to 11 – depicts successively the physical density, the risk neutral density, the downside risk neutral density, and the outer risk neutral

[8]This specification was chosen in part to avoid problems that occurred with other probability distributions or utility functions when computing v_i, for $i \in \{2, 3, 4\}$.

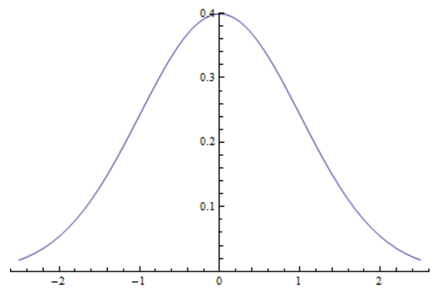

Figure 4: The physical density.

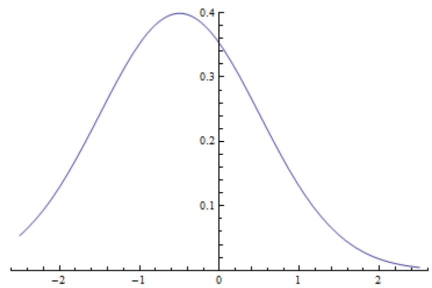

Figure 5: The risk neutral density.

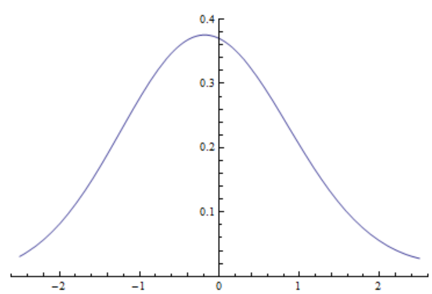

Figure 6: The downside risk neutral density.

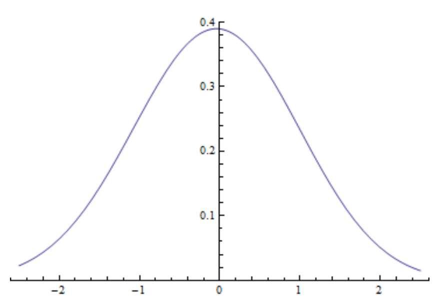

Figure 7: The outer risk neutral density.

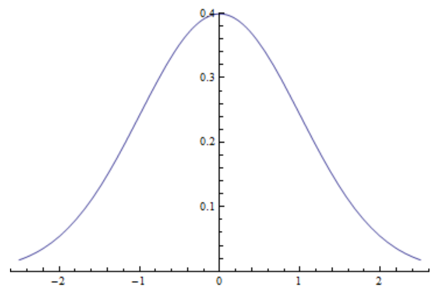

Figure 8: The physical density.

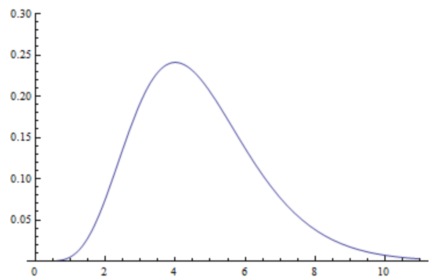

Figure 9: The risk neutral density.

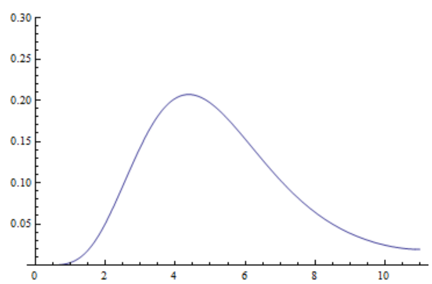

Figure 10: The downside risk neutral density.

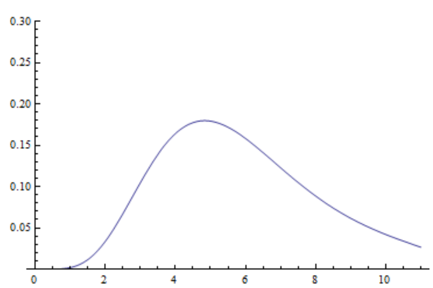

Figure 11: The outer risk neutral density.

density of states of the world. The risk neutral density is the one that differs the most from the physical density at first glance. This is expected because the risk neutral distribution is adjusted for second and higher degree risks, whereas the downside risk neutral distribution is only adjusted for third and higher degree risks, and the outer risk neutral distribution is only adjusted for fourth and higher degree risks. The outer risk neutral density is very similar to the physical density, but it is still different from the physical density because CARA utility is characterized by $u^{(4)} \neq 0$.

C A model with T future periods

This Appendix defines risk neutral, downside risk neutral, and outer risk neutral probability measures, and derives the associated asset pricing formulas, in a model with T future periods.

The assumptions are the same as in the baseline model, except that we now consider an economy with T future periods. Current aggregate wealth in the economy is w_0, and future aggregate wealth at time $t \in \{1, \ldots, T\}$ is $\tilde{w}_t$. At time t, it is equal to w_{ts} in state of the world s, for $s \in \{1, \ldots, S\}$, which occurs with probability $p_{ts} \geq 0$, with $\sum_{s=1}^{S} p_{ts} = 1$ for all t. The payoff of an asset at time $t \in \{1, \ldots, T\}$ is a random variable $\tilde{x}_t$, which is equal to x_{ts} at time t and in state of the world s. The values of w_0, w_{ts} and x_{ts} are finite for any t and any s. Using the standard stochastic discount factor formula (e.g. Hansen and Jagannathan (1991)), the price at $t = 0$ of any given asset with payoff $\{\tilde{x}_t\}$ is

$$P = \mathrm{E}\left[\sum_{t=1}^{T} \beta^t \frac{u'(\tilde{w}_t)}{u'(w_0)} \tilde{x}_t\right], \tag{54}$$

where $\beta^t \frac{u'(\tilde{w}_t)}{u'(w_0)}$ is the stochastic discount factor or pricing kernel, and $\mathrm{E}[\cdot]$ is the expectation operator with respect to the physical probability measure, using the information at time 0.

C.1 Risk neutral probabilities

Assume that u is of class C^2. For any given $\{t, s\}$, let $\eta_{2,ts}$ be defined implicitly as

$$u'(w_{ts}) \equiv \eta_{2,ts} u'(w_0). \tag{55}$$

Definition 5 *Let $v_{2t} \equiv \sum_{s=1}^{S} p_{ts}\eta_{2,ts}$, and*

$$\lambda_{2,ts} \equiv \frac{p_{ts}\eta_{2,ts}}{v_{2t}} = \frac{p_{ts}\eta_{2,ts}}{\sum_{s=1}^{S} p_{ts}\eta_{2,ts}}. \tag{56}$$

The set $\{\lambda_{2,ts}\}$ is the set of risk neutral probabilities, and Λ_{2t} is the risk neutral probability measure at time $t \in \{1, \ldots, T\}$.

The price of any asset can be expressed with risk neutral probabilities. Substituting $u'(w_{ts})$ from (55) in (54), the price P of an asset with stochastic payoff $\{\tilde{x}_t\}$ may be rewritten as:

$$P = \sum_{t=1}^{T} \sum_{s=1}^{S} p_{ts} \beta^t \eta_{2,ts} \frac{u'(w_0)}{u'(w_0)} x_{ts} = \sum_{t=1}^{T} \beta^t v_{2t} \, E^{\Lambda_{2t}}[\tilde{x}_t], \tag{57}$$

where $E^{\Lambda_{2t}}[\cdot]$ is the expectation operator with respect to the probability measure Λ_{2t}. Applying this formula to the riskfree asset of maturity τ, which pays $x_{ts} = 1$ for all s from time 1 to time τ, and denoting its price by $P_{f\tau}$:

$$P_{f\tau} = \sum_{t=1}^{\tau} \beta^t v_{2t} \qquad \text{for } \tau \in \{1, \ldots, T\}. \tag{58}$$

Denote the yield-to-maturity of a zero-coupon riskfree bond with maturity t by r_t. By definition of the riskfree rate, the price of the riskfree asset of maturity τ can also be written as:

$$P_{f\tau} = \sum_{t=1}^{\tau} \frac{1}{(1+r_t)^t} \qquad \text{for } \tau \in \{1, \ldots, T\}, \tag{59}$$

Combining (58) and (59) for $\tau \in \{1, \ldots, T\}$ gives $\beta^t v_{2t} = \frac{1}{(1+r_t)^t}$ for all $t \in \{1, \ldots, T\}$. Substituting in (57) gives

$$P = \sum_{t=1}^{T} \frac{E^{\Lambda_2}[\tilde{x}_t]}{(1+r_t)^t}, \tag{60}$$

which is the standard asset pricing formula with risk neutral probabilities.

C.2 Downside risk neutral probabilities

Assume that u is of class C^3. For any given s, let $\eta_{3,ts}$ be defined implicitly as ($\eta_{3,ts}$ exists generically – except for $u'(w_0) + u''(w_0)(w_{ts} - w_0) = 0$):

$$u'(w_{ts}) \equiv \eta_{3,ts} \left[u'(w_0) + u''(w_0)(w_{ts} - w_0)\right]. \tag{61}$$

We henceforth consider economies such that $\eta_{3,ts}$ exists for all $\{t, s\}$.

Definition 6 *Let* $v_{3t} \equiv \sum_{s=1}^{S} p_{ts}\eta_{3,ts}$, *and*

$$\lambda_{3,ts} \equiv \frac{p_{ts}\eta_{3,ts}}{v_{3t}} = \frac{p_{ts}\eta_{3,ts}}{\sum_{s=1}^{S} p_{ts}\eta_{3,ts}}. \tag{62}$$

The set $\{\lambda_{3,ts}\}$ *is the set of downside risk neutral probabilities, and* Λ_{3t} *is the downside risk neutral probability measure at time* $t \in \{1, \ldots, T\}$.

We now show how to express the price of any asset with downside risk neutral probabilities. With u of class C^3, the price P of an asset with stochastic payoff $\{\tilde{x}_t\}$ may be decomposed in the following terms:

Proposition 8

$$P = \sum_{t=1}^{T} \frac{1}{(1+r_t)^t} \hat{E}^{3t} \left[\frac{f(w_0, \tilde{w}_t)}{\hat{E}^{3t}[f(w_0, \tilde{w}_t)]} \tilde{x}_t \right] \tag{63}$$

$$= \sum_{t=1}^{T} \frac{1}{(1+r_t)^t} \left(\hat{E}^{3t} [\tilde{x}_t] - \frac{A(w_0) \widehat{cov}^{3t} (\tilde{w}_t, \tilde{x}_t)}{1 - A(w_0) \hat{E}^{3t} [\tilde{w}_t - w_0]} \right) \tag{64}$$

where $f(w_0, \tilde{w}_t)$ is linear in $\tilde{w}_1$, and writes as $f(w_0, \tilde{w}_t) \equiv 1 - A(w_0)\,[\tilde{w}_t - w_0]$.

Proof. Substituting $u'(w_{ts})$ from (61) in (54) gives

$$P = \sum_{t=1}^{T} \sum_{s=1}^{S} p_{ts} \; \beta^t \eta_{3,ts} \left(1 + \frac{u''(w_0)}{u'(w_0)} (w_{ts} - w_0) \right) x_{ts} \tag{65}$$

$$= \sum_{t=1}^{T} \beta^t v_{3t} \hat{E}^{3t} \left[\tilde{x}_t + \frac{u''(w_0)}{u'(w_0)} (\tilde{w}_t - w_0) \tilde{x}_t \right]$$

$$= \sum_{t=1}^{T} \beta^t v_{3t} \left(\hat{E}^{3t} [\tilde{x}_t] - A(w_0) \hat{E}^{3t} [(\tilde{w}_t - w_0) \tilde{x}_t] \right). \tag{66}$$

Applying the formula (66) to the riskfree asset of maturity τ, which pays $x_{ts} = 1$ for all s from time 1 to time τ:

$$P_{f\tau} = \sum_{t=1}^{\tau} \beta^t v_{3t} \left(1 - A(w_0) \hat{E}^{3t} [(\tilde{w}_t - w_0)] \right) \qquad \text{for } \tau \in \{1, \ldots, T\}. \tag{67}$$

Combining (67) and (59) for $\tau \in \{1, \ldots, T\}$ gives

$$\beta^t v_{3t} = \frac{1}{(1+r_t)^t} \frac{1}{1 - A(w_0) \hat{E}^{3t} [\tilde{w}_t - w_0]} \qquad \text{for all } t \in \{1, \ldots, T\}.$$

Substituting in (66) gives

$$P = \sum_{t=1}^{T} \frac{1}{(1+r_t)^t} \frac{\hat{E}^{3t} [\tilde{x}_t] - A(w_0) \hat{E}^{3t} [(\tilde{w}_t - w_0) \tilde{x}_t]}{1 - A(w_0) \hat{E}^{3t} [\tilde{w}_t - w_0]} \tag{68}$$

This formula can be rewritten as in (63) or (64). ■

C.3 Outer risk neutral probabilities

Assume that u is of class C^4. For any given s, let $\eta_{4,ts}$ be defined implicitly as

$$u'(w_{ts}) \equiv \eta_{4,ts} \left(u'(w_0) + u''(w_0)(w_{ts} - w_0) + \frac{1}{2} u'''(w_0)(w_{ts} - w_0)^2 \right). \tag{69}$$

Definition 7 *Let* $v_{4t} \equiv \sum_{s=1}^{S} p_{ts}\eta_{4,ts}$, *and*

$$\lambda_{4,ts} \equiv p_s \frac{\eta_{4,ts}}{v_{4t}} = \frac{p_{ts}\eta_{4,ts}}{\sum_{s=1}^{S} p_{ts}\eta_{4,ts}}. \tag{70}$$

The set $\{\lambda_{4,ts}\}$ *is the set of outer risk neutral probabilities, and* Λ_{4t} *is the outer risk neutral probability measure at time* $t \in \{1, \ldots, T\}$.

We now show how to express the price of any asset with outer risk neutral probabilities. With u of class C^4, the price P of an asset with stochastic payoff $\{\tilde{x}_t\}$ may be decomposed in the following terms:

Proposition 9

$$P = \sum_{t=1}^{T} \frac{1}{(1+r_t)^t} \mathrm{E}^{\Lambda_{4t}} \left[\frac{g(w_0, \tilde{w}_t)}{\mathrm{E}^{\Lambda_{4t}}[g(w_0, \tilde{w}_t)]} \tilde{x}_t \right] \tag{71}$$

$$= \sum_{t=1}^{T} \frac{1}{(1+r_t)^t} \left(\mathrm{E}^{\Lambda_{4t}}[\tilde{x}_t] + \frac{-A(w_0)cov^{\Lambda_{4t}}(\tilde{w}_t, \tilde{x}_t) + \frac{1}{2}D(w_0)cov^{\Lambda_{4t}}((\tilde{w}_t - w_0)^2, \tilde{x}_t)}{1 - A(w_0)\mathrm{E}^{\Lambda_{4t}}[\tilde{w}_t - w_0] + \frac{1}{2}D(w_0)\mathrm{E}^{\Lambda_{4t}}[(\tilde{w}_t - w_0)^2]} \right) \tag{72}$$

where $g(w_0, \tilde{w}_t) \equiv 1 - A(w_0)[\tilde{w}_t - w_0] + \frac{1}{2}D(w_0)(\tilde{w}_t - w_0)^2$.

Proof. Substituting $u'(w_{ts})$ from (69) in (54), the asset price P may be rewritten as:

$$\begin{aligned}
P &= \sum_{t=1}^{T} \sum_{s=1}^{S} p_{ts} \, \beta^t \eta_{4,ts} \left(1 + \frac{u''(w_0)}{u'(w_0)}(w_{ts} - w_0) + \frac{1}{2}\frac{u'''(w_0)}{u'(w_0)}(w_{ts} - w_0)^2 \right) x_{ts} \\
&= \sum_{t=1}^{T} \beta^t v_{4t} \mathrm{E}^{\Lambda_{4t}} \left[\tilde{x}_t - A(w_0)(\tilde{w}_t - w_0)\tilde{x}_t + \frac{1}{2}D(w_0)(\tilde{w}_t - w_0)^2 \tilde{x}_t \right] \\
&= \sum_{t=1}^{T} \beta^t v_{4t} \left(\mathrm{E}^{\Lambda_{4t}}[\tilde{x}_t] - A(w_0)\mathrm{E}^{\Lambda_{4t}}[(\tilde{w}_t - w_0)\tilde{x}_t] + \frac{1}{2}D(w_0)\mathrm{E}^{\Lambda_{4t}}\left[(\tilde{w}_t - w_0)^2 \tilde{x}_t\right] \right).
\end{aligned} \tag{73}$$

Applying the formula (73) to the riskfree asset of maturity τ, which pays $x_{ts} = 1$ for all s from time 1 to time τ:

$$P_{f\tau} = \sum_{t=1}^{\tau} \beta^t v_{4t} \left(1 - A(w_0)\mathrm{E}^{\Lambda_{4t}}[\tilde{w}_t - w_0] + \frac{1}{2}D(w_0)\mathrm{E}^{\Lambda_{4t}}\left[(\tilde{w}_t - w_0)^2\right] \right). \tag{74}$$

Combining (74) and (59) for $\tau \in \{1, \ldots, T\}$ gives

$$\beta^t v_{4t} = \frac{1}{(1+r_t)^t} \frac{1}{1 - A(w_0)\mathrm{E}^{\Lambda_{4t}}[\tilde{w}_t - w_0] + \frac{1}{2}D(w_0)\mathrm{E}^{\Lambda_{4t}}[(\tilde{w}_t - w_0)^2]} \quad \text{for all } t \in \{1, \ldots, T\}.$$

Substituting in (73) gives

$$P = \sum_{t=1}^{T} \frac{1}{(1+r_t)^t} \frac{\mathrm{E}^{\Lambda_{4t}}[\tilde{x}_t] - A(w_0)\mathrm{E}^{\Lambda_{4t}}[(\tilde{w}_t - w_0)\tilde{x}_t] + \frac{1}{2}D(w_0)\mathrm{E}^{\Lambda_{4t}}[(\tilde{w}_t - w_0)^2 \tilde{x}_t]}{1 - A(w_0)\mathrm{E}^{\Lambda_{4t}}[\tilde{w}_t - w_0] + \frac{1}{2}D(w_0)\mathrm{E}^{\Lambda_{4t}}[(\tilde{w}_t - w_0)^2]} \tag{75}$$

This formula can be rewritten as in (71) or (72). ■

Supplementary Appendix for Downside risk neutral probabilities
Not for publication

The expression $1 - A(w_0)\mathrm{E}^{\Lambda_3}[\tilde{w}_1 - w_0]$ **in (25)**

We show that this expression is strictly positive.

First, consider the states s such that $u'(w_0) + u''(w_0)(w_s - w_0) > 0$, and therefore $\eta_{3,s} > 0$ (given (11) and $u' > 0$), which implies $\lambda_{3,s} \geq 0$. Then we have $\lambda_{3,s}[u'(w_0) + u''(w_0)(w_s - w_0)] \geq 0$. Dividing each term by $u'(w_0)$, which is strictly positive, and summing over these states s, we have

$$\sum_{s|\eta_{3,s}>0} \lambda_{3,s}[1 - A(w_0)(w_s - w_0)] \geq 0. \tag{76}$$

Second, consider the states s such that $u'(w_0) + u''(w_0)(w_s - w_0) < 0$, and therefore $\eta_{3,s} < 0$ (given (11) and $u' > 0$), which implies $\lambda_{3,s} \leq 0$. Then we have $\lambda_{3,s}[u'(w_0) + u''(w_0)(w_s - w_0)] \geq 0$. Dividing each term by $u'(w_0)$, which is strictly positive, and summing over these states s, we have

$$\sum_{s|\eta_{3,s}<0} \lambda_{3,s}[1 - A(w_0)(w_s - w_0)] \geq 0. \tag{77}$$

Third, because $\lambda_{3,s}[1 - A(w_0)(w_s - w_0)] \geq 0$ for any s and $1 - A(w_0)(w_s - w_0) \neq 0$ (otherwise $\eta_{3,s}$ would not exist), both expressions on the left-hand-sides of (76) and (77) will be equal to zero only if $\lambda_{3,s} = 0$ for all s, which would imply $\sum_s \lambda_{3,s} \neq 1$, a contradiction. Therefore, at least one of the expressions on the left-hand-sides of (76) and (77) is strictly positive, while the other is nonnegative. Adding up these two sums then yields

$$\sum_s \lambda_{3,s}[1 - A(w_0)(w_s - w_0)] = 1 - A(w_0)\mathrm{E}^{\Lambda_3}[\tilde{w}_1 - w_0] > 0.$$

Downside risk neutral probabilities and expected returns

Let $\tilde{m} \equiv \frac{1}{1+r_f}\frac{f(w_0,\tilde{w}_1)}{\mathrm{E}^{\Lambda_3}[f(w_0,\tilde{w}_1)]}$. The pricing kernel $\tilde{m}$ in Proposition 3 is linear in $\tilde{w}_1$, and can thus be expressed as:

$$\tilde{m} = a + b\tilde{R}_w, \tag{78}$$

where $\tilde{R}_w \equiv \frac{\tilde{w}_1}{P_w}$ is the gross return on the wealth portfolio with $t = 0$ price P_w, and

$$a \equiv \frac{1}{1+r_f}\frac{1 + A(w_0)w_0}{1 - A(w_0)(\mathrm{E}^{\Lambda_3}[\tilde{w}_1] - w_0)}, \qquad b \equiv \frac{1}{1+r_f}\frac{-A(w_0)P_w}{1 - A(w_0)(\mathrm{E}^{\Lambda_3}[\tilde{w}_1] - w_0)}. \tag{79}$$

Equation (24) can thus be rewritten as $P = \mathrm{E}^{\Lambda_3}[\tilde{m}\tilde{x}]$. Dividing both sides by P gives $1 = \mathrm{E}^{\Lambda_3}[\tilde{m}\tilde{R}_i]$, where $\tilde{R}_i \equiv \frac{\tilde{x}}{P}$ is the gross return of asset i. In turn, this equation can be rewritten as

$$\frac{1}{\mathrm{E}^{\Lambda_3}[\tilde{m}]} = \mathrm{E}^{\Lambda_3}[\tilde{R}_i] + \frac{cov^{\Lambda_3}(\tilde{m}, \tilde{R}_i)}{\mathrm{E}^{\Lambda_3}[\tilde{m}]}. \tag{80}$$

Using $E^{\Lambda_3}[\tilde{m}] = \frac{1}{1+r_f} \equiv \frac{1}{R_f}$, and substituting for $\tilde{m}$, the equation above rewrites as

$$E^{\Lambda_3}[\tilde{R}_i] - R_f = -\frac{b\, cov^{\Lambda_3}(\tilde{R}_w, \tilde{R}_i)}{E^{\Lambda_3}[\tilde{m}]}. \tag{81}$$

In particular, if asset *i* is the wealth portfolio with gross return $\tilde{R}_w$, we have

$$E^{\Lambda_3}[\tilde{R}_w] - R_f = -\frac{b\, var^{\Lambda_3}(\tilde{R}_w)}{E^{\Lambda_3}[\tilde{m}]}. \tag{82}$$

Equating (81) and (82),

$$E^{\Lambda_3}[\tilde{R}_i] - R_f = \frac{cov^{\Lambda_3}(\tilde{R}_w, \tilde{R}_i)}{var^{\Lambda_3}(\tilde{R}_w)} \left(E^{\Lambda_3}[\tilde{R}_w] - R_f \right). \tag{83}$$

The expression $1 - A(w_0)E^{\Lambda_4}[\tilde{w}_1 - w_0] + \frac{1}{2}D(w_0)E^{\Lambda_4}[(\tilde{w}_1 - w_0)^2]$ **in (39)**

We show that this expression is strictly positive.

First, consider the states s such that $u'(w_0) + u''(w_0)(w_s - w_0) + \frac{1}{2}u'''(w_0)(w_s - w_0)^2 > 0$, and therefore $\eta_{4,s} > 0$ (given (31) and $u' > 0$), which implies $\lambda_{4,s} \geq 0$. Then we have $\lambda_{4,s}\left(u'(w_0) + u''(w_0)(w_s - w_0) + \frac{1}{2}u'''(w_0)(w_s - w_0)^2\right) \geq 0$. Dividing each term by $u'(w_0)$, which is strictly positive, and summing over these states s, we have

$$\sum_{s|\eta_{4,s}>0} \lambda_{4,s}\left(1 - A(w_0)(w_s - w_0) + \frac{1}{2}D(w_0)(w_s - w_0)^2\right) \geq 0. \tag{84}$$

Second, consider the states s such that $u'(w_0) + u''(w_0)(w_s - w_0) < 0$, and therefore $\eta_{4,s} < 0$ (given (31) and $u' > 0$), which implies $\lambda_{4,s} \leq 0$. Then we have

$$\lambda_{4,s}\left(u'(w_0) + u''(w_0)(w_s - w_0) + \frac{1}{2}u'''(w_0)(w_s - w_0)^2\right) \geq 0.$$

Dividing each term by $u'(w_0)$, which is strictly positive, and summing over these states s, we have

$$\sum_{s|\eta_{4,s}<0} \lambda_{4,s}\left(1 - A(w_0)(w_s - w_0) + \frac{1}{2}D(w_0)(w_s - w_0)^2\right) \geq 0. \tag{85}$$

Third, because $\lambda_{4,s}\left(1 - A(w_0)(w_s - w_0) + \frac{1}{2}D(w_0)(w_s - w_0)^2\right) \geq 0$ for any s and $1 - A(w_0)(w_s - w_0) + \frac{1}{2}D(w_0)(w_s - w_0)^2 \neq 0$ (otherwise $\eta_{4,s}$ would not exist), both expressions on the left-hand-sides of (84) and (85) will be equal to zero only if $\lambda_{3,s} = 0$ for all s, which would imply $\sum_s \lambda_{3,s} \neq 1$, a contradiction. Therefore, at least one of the expressions on the left-hand-sides of (84) and (85) is strictly positive, while the other is nonnegative. Adding up these two sums then yields

$$\sum_{s} \lambda_{4,s}\left(1 - A(w_0)(w_s - w_0) + \frac{1}{2}D(w_0)(w_s - w_0)^2\right)$$

$$= 1 - A(w_0)\mathrm{E}^{\wedge_4}[\tilde{w}_1 - w_0] + \frac{1}{2}D(w_0)\mathrm{E}^{\wedge_4}(w_s - w_0)^2 > 0.$$

Outer risk neutral probabilities and expected returns

Now let $\tilde{m} \equiv \frac{1}{1+r_f}\frac{g(w_0,\tilde{w}_1)}{\mathrm{E}^{\wedge_4}[g(w_0,\tilde{w}_1)]}$. The pricing kernel $\tilde{m}$ in Proposition 5 is quadratic in $\tilde{w}_1$, and can thus be expressed as:

$$\tilde{m} = a + b\tilde{R}_w + c\tilde{R}_w^2, \tag{86}$$

where $\tilde{R}_w \equiv \frac{\tilde{w}_1}{P_w}$ is the gross return on the wealth portfolio with $t = 0$ price P_w. Equation (38) can thus be rewritten as $P = \mathrm{E}^{\wedge_4}[\tilde{m}\tilde{x}]$. Dividing both sides by P gives $1 = \mathrm{E}^{\wedge_4}[\tilde{m}\tilde{R}_i]$, where $\tilde{R}_i \equiv \frac{\tilde{x}}{P}$ is the gross return of asset i. In turn, this equation can be rewritten as

$$\frac{1}{\mathrm{E}^{\wedge_4}[\tilde{m}]} = \mathrm{E}^{\wedge_4}[\tilde{R}_i] + \frac{cov^{\wedge_4}(\tilde{m}, \tilde{R}_i)}{\mathrm{E}^{\wedge_4}[\tilde{m}]}. \tag{87}$$

Using $\mathrm{E}^{\wedge_4}[\tilde{m}] = \frac{1}{1+r_f} \equiv \frac{1}{R_f}$, and substituting for $\tilde{m}$, the equation above rewrites as

$$\mathrm{E}^{\wedge_4}[\tilde{R}_i] - R_f = -\frac{b\, cov^{\wedge_4}(\tilde{R}_w, \tilde{R}_i) + c\, cov^{\wedge_4}(\tilde{R}_w^2, \tilde{R}_i)}{\mathrm{E}^{\wedge_4}[\tilde{m}]}. \tag{88}$$

In particular, if asset i is the wealth portfolio with gross return $\tilde{R}_w$, we have

$$\mathrm{E}^{\wedge_4}[\tilde{R}_w] - R_f = -\frac{b\, var^{\wedge_4}(\tilde{R}_w) + c\, cov^{\wedge_4}(\tilde{R}_w^2, \tilde{R}_w)}{\mathrm{E}^{\wedge_4}[\tilde{m}]}. \tag{89}$$

It follows that there exists constants χ and ϑ independent of i such that

$$\mathrm{E}^{\wedge_4}[\tilde{R}_i] - R_f = \chi\, cov^{\wedge_4}(\tilde{R}_w, \tilde{R}_i) + \vartheta\, cov^{\wedge_4}(\tilde{R}_w^2, \tilde{R}_i). \tag{90}$$