A Local Adjustment Method to Improve Multiplicative Consistency of Fuzzy Reciprocal Preference Relations

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***Abstract*— Preferences that verify the transitivity property are usually referred to as rational or consistent preferences. Existent methods to improve consistency of inconsistent fuzzy reciprocal preference relations (FPRs) fail to retain the original preference values because they always derive a new FPR. This paper presents a new inconsistency identification and modification (IIM) method to detect and rectify only the most inconsistent elements of an inconsistent FPR. As such, the proposed IIM can be considered a local adjustment method to improve multiplicative consistency (MC) of FPRs. The case of inconsistent FPRs with missing preference values, i.e. incomplete FPRs, is addressed with the estimation of the missing preferences with a constrained nonlinear optimization model followed by the application of the IIM method. The implementation process of the proposed algorithms is illustrated with numerical examples. Simulation experiments and comparisons with existent methods are also included to show that the new method requires fewer iterations than existent methods to improve the MC of FPRs and achieves better MC level, while preserving the original preference information as much as possible, than existent methods. Thus, the results presented in the paper demonstrate the correctness, effectiveness, and robustness of the proposed method.**

***Index Terms*—Fuzzy reciprocal preference relation (FPR), Incomplete FPR, Inconsistency, Multiplicative consistency (MC).**

I. Introduction

PREFERENCE relations (PRs) are commonly used in multi-attribute decision making to model preferences of decision makers (DMs) on a set of alternatives. In decision making theory, pairwise comparison of alternatives may be modelled via a multiplicative PR (MPR) [1, 2] as in the analytic hierarchy process (AHP) [3], or via a fuzzy PR (FPR) to deal with uncertainty and fuzziness of human thinking [4].

Since the lack of consistency of preferences could lead to misleading outcomes, its study and analysis are key in decision making theory [5-8]. Indeed, the priorities of alternatives derived from PRs are reliable only when the PRs are of acceptable consistency. There are different approaches to improve consistency of inconsistent MPRs: there are general method that are based on the revision of all the elements of the relation [9, 10], and local methods that are based on the revision of a single element of the relation [11, 12]. However, before improving consistency, it is important to address the measuring of the level of inconsistency of the relation. Recently, some axiomatizations have been proposed [13-17] for this purpose. A self-contained exposition of the most relevant inconsistency indices is provided in [18] while a unifying approach to measure inconsistency is proposed in [19]. A general framework, based a parametric generating function, for defining inconsistency indices and deriving the actual local contributions to the inconsistency index is investigated in [20].

Traditionally, consistency of FPRs has been classified into two main categories: the ordinal consistency (OC) [21, 22] and the cardinal consistency [23]. The property of weak transitivity, on which OC is defined, is the minimum requirement for an FPR to be considered consistent [24], although cardinal consistency, a stronger concept than OC because it requires that the "actual intensity with which the preference is expressed transits through the sequence of objects in comparison" ([3], Page 7), is also required in research [25]. Two types of cardinal consistency of FPRs have been developed: the multiplicative consistency (MC) [23, 25, 26] and the additive consistency (AC) [27-31]. The AC clashes with the preference values range ([0,1]) and, although Herrera-Viedma, *et al.* [27] developed a method to fit AC outcomes within the unit interval, it is also true that some of the original preference information is lost in that fitting process. In fact, Chiclana, *et al.* [23] provided the conditions under which MC is the order isomorphic functional solution to modelling cardinal consistency of FPRs, which motivates us to focus on MC rather than AC.

It is known that an FPR that verify OC property may not verify MC or AC properties; however, an FPR that verifies MC or AC properties, it also verifies the OC property. When a PR

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does not verify a consistency property, the concept of acceptable consistent PR can be used instead: a PR is of acceptable consistency with respect to a consistency property when the corresponding consistency property based consistency degree of such PR is above a predefined threshold. An acceptable consistent PR with respect to a consistency property may not be an acceptable consistent PR for other type of consistency property. The man difference between acceptable consistency and consistency of PRs is that an acceptable consistent FPR with respect to MC or AC may as not be an acceptable consistent FPR with respect to OC. Furthermore, there is not relationship between MC and AC for an FPR and, therefore, in general only the MC or AC is considered in consistency research.

When a given FPR is not MC, methods can be applied to improve the MC, including Xu and Da [26]'s practical iterative algorithm to derive a modified FPR with acceptable MC, and Xia, *et al.* [25]'s two convergent algorithms to improve the MC and consensus levels of FPRs. However, these methods have associated the following issues, each one leading to a research question:

1) The MC of FPRs are improved mathematically at the cost of modifying most of the original judgements of the FPRs. This does not conform to real situations where only few original judgements are expected to be modified. Thus, the following research question needs to be dealt with: Is it possible to improve the MC in other ways than just mathematically so that the modified FPRs retain most of the original judgements?
2) Obtaining the FPRs with acceptable MC is computationally intensive with a high number of calculations and time required. It would be worth answering the following question: Is it possible to derive an FPR with acceptable MC more efficiently, i.e., with fewer calculation steps?
3) The modified values obtained might not be in the original scale range, leading to a distortion of the DMs' original information. The following question needs addressing: How can it be assured that the modified values are in the original scale $U_{[0.1,0.9]}=\{0.1,0.2,\ldots,0.9\}$?

This paper aims to answer the above research questions by proposing a novel local adjustment method to improve the MC of FPRs. The main research contributions of this paper are:

- A new inconsistency identification and modification (IIM) method to detect and rectify the most inconsistency elements of FPRs.
- A constrained nonlinear optimization model (CNOM) to estimate the missing values of incomplete FPRs.
- Two algorithms to improve the MC of complete and incomplete FPRs, respectively.
- In each consistency improving round, the preference value of only one pair of alternatives is modified, making this new approach a local adjustment method. The revised value is in the original scale range of preferences.
- Simulation experiments and numerical examples show that the proposed method requires fewer iterations and less information is lost than with the mentioned existent methods.

The rest of the paper is organized as follows. Section II contains a brief description and definitions of the basic concepts and known results needed throughout the paper: FPRs, incomplete FPRs, the geometric consistency index (*GCI*) and acceptable consistency. Section III presents the new multiplicative based IIM method to improve the MC of FPRs. Section IV contains the algorithms to estimate the missing values of an incomplete FPR and to improve its consistency, respectively. Numerical examples showing how the proposed method works in practice, which is complemented with an analysis and comparison with those obtained from existent methods, are included in Section V. This section also includes a comprehensive comparison with existent similar methods that evidence the advantages, effectiveness and robustness of the proposed method. Conclusions are drawn in Section VI.

II. Preliminaries

Given a set of alternatives $X=\{x_1, x_2, \ldots, x_n\}$ ($n\geq 2$), an FPR on a set of alternatives X is modelled by a membership function μ_R: $X\times X\rightarrow[0,1]$ being $\mu_R(x_i, x_j)=r_{ij}$ the preference degree of alternative x_i over alternative x_j [32]. The following interpretation is assumed: r_{ij} = 0.5 indicates indifference between alternatives x_i and x_j: $0 < r_{ij} < 0.5$ means that alternative x_j is preferred to alternative x_i; the smaller r_{ij}, the stronger the preference of alternative x_j over alternative x_i; $0.5 < r_{ij} \leq 1$ implies that alternative x_i is preferred to alternative x_j; the greater r_{ij}, the stronger the preference of alternative x_i over alternative x_j. Notice that there is an implicit reciprocity property of preferences on the above interpretation, which is summarized in the following definition:

Definition 1: [33] An FPR is represented by a matrix $R=(r_{ij})_{n\times n}$ with element $r_{ij}\in[0,1]$ verifying $r_{ij} + r_{ji} = 1$ for all $i, j \in N$, where $N=\{1, 2, \ldots, n\}$.

In the following the concept of AC of FPRs is defined:

Definition 2: [34] An FPR $R=(r_{ij})_{n\times n}$ is AC if and only if $r_{ij}=r_{ik} + r_{kj} - 0.5$, for all $i, j, k \in N$.

The above definitions of FPR and AC FPR are in conflict. Indeed, assuming that a DM states a preference degree of 0.8 of alternative x over alternative y, and a preference degree 0.9 of alternative y over alternative z, then it is not possible for this DM to provide an AC FPR because the only admissible AC preference degree of alternative x over alternative z would be 1.2, which is not in preference range [0,1]. To avoid this conflict with the unit interval, Herrera-Viedma, *et al.* [27] proposed a re-scaling method of the AC based preference values in the [0,1] scale; however, this process obviously distorts the original preference information. This issue, though, can be avoided if the following MC property is used instead.

Definition 3: [34] An FPR $R=(r_{ij})_{n\times n}$ is MC if and only if $r_{ik}r_{kj}r_{ji}=r_{ki}r_{kj}r_{ij}$, for all $i, j, k \in N$.

Given an MC FPR, $R=(r_{ij})_{n\times n}$, there exists a priority vector, $w = (w_1, w_2, \ldots, w_n)^T$ with $\sum_{i=1}^{n} w_i = 1$ and $w_i > 0$, $i \in N$, such that

$$r_{ij} = \frac{w_i}{w_i + w_j} \tag{1}$$

Algebraic manipulation leads to

$$\frac{r_{ij}}{r_{ji}} = \frac{w_i}{w_j} \Leftrightarrow \ln\frac{r_{ij}}{r_{ji}} = \ln\frac{w_i}{w_j} \Leftrightarrow \ln r_{ij} - \ln r_{ji} = \ln w_i - \ln w_j$$

(2)

FPRs rarely satisfy MC in practice; therefore, measuring the degree of consistency is useful. Xia, *et al.* [25] introduced the below geometric consistency index (GCI) to measure the MC degree of FPRs.

Definition 4: [25] The GCI of FPR $R=(r_{ij})_{n\times n}$ with priority vector $w=(w_1, w_2, \ldots, w_n)^T$ is

$$GCI(R) = \frac{2}{(n-1)(n-2)} \sum_{1\le i<j\le n} (\ln r_{ij} - \ln r_{ji} - \ln w_i + \ln w_j)^2 \tag{3}$$

When an FPR R is of MC, it is $GCI(R)=0$. So, the smaller $GCI(R)$ is, the more MC the FPR R is. Given a threshold value $\overline{GCI}$, when $GCI(R)\le \overline{GCI}$, the FPR R is said to be of acceptable consistency. In AHP, Saaty [3] originally proposed the consistency ratio (CR) to measure the consistency index (CI) of MPRs. Afterwards, Aguarón and Moreno-Jiménez [35] developed the measure of consistency proposed by Crawford and Williams [36], which is called *GCI* for MPRs. How to determine the value of $\overline{GCI}$ for FPRs is still an open issue, which needs further investigation. Generally, the value is given by the DM in advance. A highly consistent FPR is implemented with small $\overline{GCI}$ values.

Wang and Fan [37] proposed the below logarithmic least square model (LLSM) to derive the priority vector of FPRs:

(M-1) $$\min \quad J = \sum_{i=1}^{n}\sum_{j=1}^{n} (\ln r_{ij} - \ln r_{ji} - \ln w_i + \ln w_j)^2$$

(4)

$$\text{s.t.} \quad \sum_{i=1}^{n} w_i = 1,\ w_i \ge 0,\ i \in N$$

They proved the following main result:

Theorem 1: [37] The optimal solution of (M-1) is

$$w_i = \frac{\left(\prod_{j=1}^{n} \frac{r_{ij}}{r_{ji}}\right)^{1/n}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n} \frac{r_{ij}}{r_{ji}}\right)^{1/n}},\ i \in N \tag{5}$$

Lack of sufficient knowledge, limited expertise or time pressure may lead DMs to provide incomplete FPRs [38-40]. Recall that "a function f: $X\to Y$ is partial when not every element in the set X maps to an element in the set Y" [36]. Thus, incomplete FPRs are defined as follows:

Definition 5:[41] An FPR $B=(b_{ij})_{n\times n}$ on a set of alternatives X is an incomplete FPR if its membership function is partial.

Since the reciprocity property is assumed for FPRs, when b_{ij} is known, then $b_{ji}=1-b_{ij}$ is also known.

Definition 6: [39] An incomplete FPR $B=(b_{ij})_{n\times n}$ is MC if all the known elements of B satisfy $b_{ik}b_{kj}b_{ji}=b_{ki}b_{jk}b_{ij}$, for all $i, j, k\in N$.

Definition 7:[25] Let $B=(b_{ij})_{n\times n}$ be an incomplete FPR, and $w=(w_1, w_2, \ldots, w_n)^T$ the priority vector derived from B. The *GCI* of B is

$$GCI(B) = \frac{1}{s}\sum_{i=1}^{n}\sum_{j=1}^{n} \delta_{ij} (\ln b_{ij} - \ln b_{ji} - \ln w_i + \ln w_j)^2 \tag{6}$$

where δ_{ij} is the binary variable ($b_{ij}=-$ indicates elements b_{ij} is missing/unknown)

$$\delta_{ij} = \begin{cases} 0, & \text{if } b_{ij} = -, \\ 1, & \text{if } b_{ij} \ne -, \end{cases} \quad i, j \in N \tag{7}$$

and s is the number of known preference values in B

$$s = \sum_{i=1}^{n}\sum_{j=1}^{n} \delta_{ij} \tag{8}$$

If $GCI(B)=0$, then the incomplete FPR B is MC. Thus, values close to zero of $GCI(B)$ indicate high degrees of MC by B.

Xu, *et al.* [42] extended the above LLSM method to derive the weighting vector of incomplete FPRs.

Theorem 2: [42] The priority vector of an incomplete FPR B is

$$w_i = \begin{cases} \dfrac{\exp(m_i)}{\sum_{j=1}^{n-1}\exp(m_j)+1}, & i = 1,2,\ldots,n-1 \\ \dfrac{1}{\sum_{j=1}^{n-1}\exp(m_j)+1}, & i = n \end{cases} \tag{9}$$

where exp(.) is the exponential function; $M=(m_1,m_2,\ldots,m_{n-1})^T$ is

$$M = P^{-1}Q \tag{10}$$

$P=(p_{ij})_{(n-1)\times(n-1)}$ and $Q=(q_{ij})_{(n-1)\times 1}$ are matrices with elements

$$p_{ii} = \sum_{j=1,\, j\ne i}^{n} \delta_{ij},\ p_{ij} = -\delta_{ij}\ \ (i \ne j),\ i, j = 1,2,\ldots,n-1 \tag{11}$$

$$q_i = \sum_{j=1}^{n} \delta_{ij} (\ln b_{ij} - \ln b_{ji}),\ i = 1,2,\ldots,n-1 \tag{12}$$

and δ_{ij} as per (7).

Theorem 3: [42] The necessary condition for an incomplete FPR $B=(b_{ij})_{n\times n}$ to be acceptable is that at least $(n-1)$ non-diagonal elements in B are known.

III. A Local Adjustment Method Of Inconsistency Identification And Modification For FPRs

A. The Induced Matrix for Inconsistency Identification

In decision making, inconsistent conclusions are often obtained from lack of consistency [27]. In practice, it is difficult that a given PR is "fully" consistent as per Definition 3. Therefore, it is worth investigating how to get consistent PRs from inconsistent ones [3], an issue that has indeed attracted quite a lot of research efforts recently [25, 26, 29]. It is well known that in the process of constructing consistent preferences, the original preference information is usually modified.

This section proposes an MC based IIM to improve the inconsistency of FPRs practice. An induced matrix is built to identify the inconsistent elements in a given (initial/original) FPR. In order to keep the DM's original preference information

as much as possible, the preference value of only one pair of inconsistent elements is repaired at each step of the iterative improvement method.

Theorem 4: The induced matrix of the FPR $R=(r_{ij})_{n\times n}$, is the matrix, $C=(c_{ij})_{n\times n}$, with elements $c_{ij}=\sum_{k=1}^{n}\frac{r_{ik}}{r_{ki}}\frac{r_{kj}}{r_{jk}}-n\frac{r_{ij}}{r_{ji}}$, $i, j, k\in N$. If $R=(r_{ij})_{n\times n}$ is MC, then C will be the zero matrix.

Proof: If R is MC, then applying Definition 3 it is

$$r_{ik}r_{kj}r_{ji}=r_{ki}r_{kj}r_{ij}, \text{ for all } i, j, k\in N,$$

which can be re-written as

$$\frac{r_{ij}}{r_{ji}}=\frac{r_{ik}}{r_{ki}}\cdot\frac{r_{kj}}{r_{jk}}, \text{ for all } i, j, k\in N.$$

This yields

$$\sum_{k=1}^{n}\frac{r_{ik}}{r_{ki}}\frac{r_{kj}}{r_{jk}}=\sum_{k=1}^{n}\frac{r_{ij}}{r_{ji}}=n\frac{r_{ij}}{r_{ji}}$$

which implies that

$$c_{ij}=\sum_{k=1}^{n}\frac{r_{ik}}{r_{ki}}\frac{r_{kj}}{r_{jk}}-n\frac{r_{ij}}{r_{ji}}=n\frac{r_{ij}}{r_{ji}}-n\frac{r_{ij}}{r_{ji}}=0.$$

This completes the proof. □

Theorem 5: If the induced matrix of an FPR is the zero matrix, then the FPR is MC.

Proof: If $C=(c_{ij})_{n\times n}$ is the zero matrix, then

$$c_{ij}=\sum_{k=1}^{n}\frac{r_{ik}}{r_{ki}}\frac{r_{kj}}{r_{jk}}-n\frac{r_{ij}}{r_{ji}}=0 \;\Rightarrow\; \frac{1}{n}\sum_{k=1}^{n}\frac{r_{ik}}{r_{ki}}\frac{r_{kj}}{r_{jk}}\frac{r_{ji}}{r_{ij}}=1.$$

Matrix $G=(g_{ij})=(\frac{1}{n}\sum_{k=1}^{n}\frac{r_{ik}}{r_{ki}}\frac{r_{kj}}{r_{jk}}\frac{r_{ji}}{r_{ij}})$ verifies

$$g_{ij}+g_{ji}=\frac{1}{n}\sum_{k=1}^{n}\frac{r_{ik}}{r_{ki}}\frac{r_{kj}}{r_{jk}}\frac{r_{ji}}{r_{ij}}+\frac{1}{n}\sum_{k=1}^{n}\frac{r_{jk}}{r_{kj}}\frac{r_{ki}}{r_{ik}}\frac{r_{ij}}{r_{ji}}$$
$$=\frac{1}{n}\sum_{k=1}^{n}\left(\frac{r_{ik}}{r_{ki}}\frac{r_{kj}}{r_{jk}}\frac{r_{ji}}{r_{ij}}+\frac{r_{jk}}{r_{kj}}\frac{r_{ki}}{r_{ik}}\frac{r_{ij}}{r_{ji}}\right).$$

The above expression is of the type $y+\frac{1}{y}\geq 2$ (with $y>0$), with equality being true only when $y=1$. Therefore, it is $g_{ij}+g_{ji}=2$, i.e. $\forall i, j, k\in N: r_{ik}r_{kj}r_{ji}=r_{ki}r_{kj}r_{ij}$. Thus, R is MC. This completes the proof. □

Notice that when $g_{ij}<1$, it is $g_{ji}>1$. Therefore, in the presence of inconsistent preference values some g_{ij} or g_{ji} values will be greater than 1.

Remark 1. Theorems 4 and 5 are obvious results, but, the value of c_{ij} is the sum of some items, and we could check which item is responsible for the value c_{ij} being far away from 0. From this, we can locate which comparison contributes more to the inconsistency, and thus, to improve the MC locally.

B. An Algorithm to Improve MC for FPRs

The assumption of the experts being able to quantify their preferences numerically in [0, 1] instead of {0, 1}, implies that experts can accurately select from an infinite set of possible options, which in turn implies that consistency of FPRs can be mathematically modeled via a functional equation. However, due to the limitations of the human cognitive perspective and the complexity nature of decision problems, DMs may provide inconsistent FPRs. As per Theorems 4-5, the closer the induced matrix $C=(c_{ij})_{n\times n}$ is to the zero matrix, the more MC the FPR $R=(r_{ij})_{n\times n}$ will be. Therefore, MC of FPRs will increase when the distance between $(r_{ik}/r_{ki})\cdot(r_{kj}/r_{jk})$ and r_{ij}/r_{ji} approaches zero. Thus, when the absolute value of some elements in the induced matrix C is quite different to zero, there will be a remarkable deviation between $\frac{1}{n}\sum_{k=1}^{n}\left(\frac{r_{ik}}{r_{ki}}\right)\left(\frac{r_{kj}}{r_{jk}}\right)$ and r_{ij}/r_{ji}. This happens when either some of the values $(r_{ik}/r_{ki})\cdot(r_{kj}/r_{jk})$ are too large or the ratio r_{ij}/r_{ji} is too small. Since $0<r_{uv}<1$ and $r_{vu}=1-r_{uv}$, $u, v\in N$, then either r_{ik} or r_{ki} or both are too large or r_{ij} is too small. This is the idea driving the IIM method on the initial FPR, which is given in Algorithm I.

Algorithm I.

Let $R=(r_{ij})_{n\times n}$ be an FPR, t be the number of iterations, and $\overline{GCI(R)}$ be a (given) threshold value.

Step 1: Let $t=0$, $R^{(t)}=(r_{ij}^{(t)})_{n\times n}=(r_{ij})_{n\times n}$.

Step 2: Apply (5) to derive the priority vector $w^{(t)}=(w_1^{(t)}, w_2^{(t)}..., w_n^{(t)})^T$ associated with $R^{(t)}$.

Step 3: Apply (3) to compute $GCI(R^{(t)})$. If $GCI(R^{(t)})\leq\overline{GCI(R)}$, then go to Step 12. Otherwise, go to Step 4.

Step 4: Construct the induced matrix $C^{(t)}=(c_{ij}^{(t)})_{n\times n}$ associated with $R^{(t)}$:

$$c_{ij}^{(t)}=\sum_{k=1}^{n}\frac{r_{ik}^{(t)}}{r_{ki}^{(t)}}\frac{r_{kj}^{(t)}}{r_{jk}^{(t)}}-n\frac{r_{ij}^{(t)}}{r_{ji}^{(t)}}, \forall i, j, k\in N \tag{13}$$

Step 5: Identify the largest absolute value in the induced matrix $C^{(t)}$: $\left|c_{ij}^{(t)}\right|$.

Step 6: Construct the deviation vector $F^{(t)}$:

$$F^{(t)}=\left(\frac{r_{i1}^{(t)}}{r_{1i}^{(t)}}\frac{r_{1j}^{(t)}}{r_{j1}^{(t)}}-\frac{r_{ij}^{(t)}}{r_{ji}^{(t)}}, \frac{r_{i2}^{(t)}}{r_{2i}^{(t)}}\frac{r_{2j}^{(t)}}{r_{j2}^{(t)}}-\frac{r_{ij}^{(t)}}{r_{ji}^{(t)}}, ..., \frac{r_{in}^{(t)}}{r_{ni}^{(t)}}\frac{r_{nj}^{(t)}}{r_{jn}^{(t)}}-\frac{r_{ij}^{(t)}}{r_{ji}^{(t)}}\right) \tag{14}$$

Step 7: Identify the inconsistent elements in $R^{(t)}$.

Sub-step 1: If the only two values in vector $F^{(t)}$ equal to zero are the i-th and j-th components and the rest of the values are positive and close to each other, then $r_{ij}^{(t)}/r_{ji}^{(t)}$ is too small. Since $0<r_{ij}^{(t)}<1$ and $r_{ji}^{(t)}=1-r_{ij}^{(t)}$, then $r_{ij}^{(t)}$ is too small. However, if the rest of values are negative and close to each other, then $r_{ij}^{(t)}$ is too large. Go to Step 10.

Sub-step 2: Otherwise, identify the largest absolute value in vector $F^{(t)}$ and those close to it. If $(r_{ik}^{(t)}/r_{ki}^{(t)})(r_{kj}^{(t)}/r_{jk}^{(t)})-r_{ij}^{(t)}/r_{ji}^{(t)}$ is the largest positive value in $F^{(t)}$ and the rest of values are close to zero, then either $r_{ik}^{(t)}$ or $r_{kj}^{(t)}$ or both are too large. Then, go to Step 8.

Step 8: Find the values $c_{ik}^{(t)}=\sum_{l=1}^{n}\frac{r_{il}^{(t)}}{r_{li}^{(t)}}\frac{r_{lk}^{(t)}}{r_{kl}^{(t)}}-n\frac{r_{ik}^{(t)}}{r_{ki}^{(t)}}$, and $c_{kj}^{(t)}=\sum_{l=1}^{n}\frac{r_{kl}^{(t)}}{r_{lk}^{(t)}}\frac{r_{lj}^{(t)}}{r_{jl}^{(t)}}-n\frac{r_{kj}^{(t)}}{r_{jk}^{(t)}}$ in the induced matrix $C^{(t)}$. It is not possible for $c_{ik}^{(t)}>0$ and $c_{kj}^{(t)}>0$ to be true simultaneously.

Table I displays the three possible cases.

TABLE I. DIFFERENT SITUATIONS IN STEP 8.

Situation	Problem	Measure	Next step
$c_{ik}^{(t)}<0$, $c_{kj}^{(t)}>0$	$r_{ik}^{(t)}$ is too large	Decrease $r_{ik}^{(t)}$	Step 10
$c_{ik}^{(t)}>0$, $c_{kj}^{(t)}<0$	$r_{kj}^{(t)}$ is too large	Decrease $r_{kj}^{(t)}$	Step 10
$c_{ik}^{(t)}<0$, $c_{kj}^{(t)}<0$	either $r_{ik}^{(t)}$ or $r_{kj}^{(t)}$ or both are too large	Continue to identify inconsistent elements	Step 9

Step 9: Consistency test of the order reduced FPR $R^{(t)'}_{(n-1)\times(n-1)}$ formed by removing sequentially one of the alternatives x_i, x_k and x_j. Based on the above discussion, $r^{(t)}_{ik}$, $r^{(t)}_{kj}$ and $r^{(t)}_{ij}$ are likely to be the inconsistent elements, indicating that inconsistent judgments on x_i, x_k or x_j have been made. Hence, these alternatives can be removed, one after another, to check whether the corresponding order reduced FPR passes the consistency test, which is used to identify explicitly the inconsistent element.

Sub-step 1: Consistency test of order reduced FPR $R^{(t)'}_{(n-1)\times(n-1)}$ formed by removing the alternative x_i from FPR $R^{(t)}$ (delete the i-th row and the i-th column from FPR $R^{(t)}$). If the consistency test is passed, the alternative x_i is an inconsistent element and $r^{(t)}_{kj}$ is consistent. Otherwise, $r^{(t)}_{kj}$ is inconsistent. Go to Sub-step 2.

Sub-step 2: Consistency test of order reduced FPR $R^{(t)''}_{(n-1)\times(n-1)}$ formed by removing the alternative x_k from FPR $R^{(t)}$. If the consistency test is passed, $r^{(t)}_{ij}$ is consistent. Otherwise, $r^{(t)}_{ij}$ is inconsistent. Go to Sub-step 3.

Sub-step 3: Consistency test of order reduced FPR $R^{(t)'''}_{(n-1)\times(n-1)}$ formed by removing alternative x_j from FPR $R^{(t)}$. If the consistency test is passed, $r^{(t)}_{ik}$ is consistent. Otherwise, $r^{(t)}_{ik}$ is inconsistent.

TABLE II. THE IDENTIFICATION PROCESSES IN STEP 9

Remove alternative	Consistency test	Potential inconsistent elements	Consistent elements	Inconsistent elements
x_i	Yes	$r^{(t)}_{ik}$, $r^{(t)}_{ij}$	$r^{(t)}_{kj}$	
	No	$r^{(t)}_{ik}$, $r^{(t)}_{ij}$		$r^{(t)}_{kj}$
x_k	Yes	$r^{(t)}_{ik}$	$r^{(t)}_{ij}$	
	No	$r^{(t)}_{ik}$		$r^{(t)}_{ij}$
x_j	Yes		$r^{(t)}_{ik}$	
	No			$r^{(t)}_{ik}$

Table II shows the Step 9 identification process with column "Consistency test" showing whether the order reduced FPRs do pass the consistency test.

After Step 9 the inconsistent element among $r^{(t)}_{ik}$, $r^{(t)}_{kj}$ and $r^{(t)}_{ij}$ is identified. Assuming that $r^{(t)}_{ij}$ is the inconsistent value, it is denoted $r^{(t)}_{i_\tau j_\tau}$; Step 10 is activated to adjust the value of the inconsistent element in order to improve the MC of the FPR.

Step 10: Change the value of the inconsistent element $r^{(t)}_{i_\tau j_\tau}$ to the value $r^{(t+1)}_{i_\tau j_\tau}$ verifying:

$$\sum_{k=1,k\neq i_\tau,k\neq j_\tau}^{n}\frac{r^{(t)}_{i_\tau k}}{r^{(t)}_{ki_\tau}}\frac{r^{(t)}_{kj_\tau}}{r^{(t)}_{j_\tau k}}-(n-2)\frac{r^{(t+1)}_{i_\tau j_\tau}}{1-r^{(t+1)}_{i_\tau j_\tau}}=0 \tag{15}$$

Denoting $\psi=\frac{1}{n-2}\sum_{k=1,k\neq i_\tau,k\neq j_\tau}^{n}\frac{r^{(t)}_{i_\tau k}}{r^{(t)}_{ki_\tau}}\frac{r^{(t)}_{kj_\tau}}{r^{(t)}_{j_\tau k}}$, it is:

$$r^{(t+1)}_{i_\tau j_\tau}=\frac{\psi}{1+\psi} \tag{16}$$

The rest of consistent elements are not modified: $r^{(t+1)}_{uv}=r^{(t)}_{uv}$, for all $u,v\in N$, $u\neq i_\tau$ and $v\neq j_\tau$. A new FPR $R^{(t+1)}$ is computed.

Step 11: Let $t=t+1$, then go to Step 2.

Step 12: Output: Number of iterations t, the modified FPR $R^{(t)}$ and its acceptable consistency degree $GCI(R^{(t)})$.

Step 13: End.

Remark 2: The aim of Algorithm I is twofold: (1) to identify the most inconsistent elements of an FPR, and (2) to modify the identified inconsistent element. In order to achieve this, the GCI of a given FPR is computed. If it is above the MC threshold, the induced matrix is used to find the element which contributes most to its inconsistency.

Remark 3: Generally, a DM provides preference values in the scale $U_{[0.1,0.9]}=\{0.1,0.2,\ldots,0.9\}$. However, in Step 10 of Algorithm I, $r^{(t+1)}_{i_\tau j_\tau}$ as per (16) is not in $U_{[0.1,0.9]}$. In order to let the adjusted value be in this scale, it could be rounded first, i.e., $r^{(t+1)}_{i_\tau j_\tau}=\text{round}(r^{(t+1)}_{i_\tau j_\tau}\times 10)\times 10^{-1}$, and subsequently compared against 0.9: $r^{(t+1)}_{i_\tau j_\tau}=\min\{\text{round}(r^{(t+1)}_{i_\tau j_\tau}\times 10)\times 10^{-1}, 0.9\}$. However, if the aim is to improve the MC of an FPR, the value given by (16) can be implemented without further modification.

Remark 4. In Sub-step 2 of Step 7, if there are more than one largest value in $F^{(t)}$, any one of them can be chosen to compute in Step 8.

IV. A LOCAL ADJUSTMENT METHOD OF INCONSISTENCY MODIFICATION FOR INCOMPLETE FPRS

A. CNOM Estimation of Missing Values of Incomplete FPRs

Let $B=(b_{ij})_{n\times n}$ be an incomplete FPR with p missing elements. Recall that p is even; x_l and $1-x_l$ (l=1, 2, …, p/2) denote the unknown elements in B. From Theorem 4, if an FPR R is of acceptable MC, then the elements of its induced matrix $C=(c_{ij})_{n\times n}$ will be close to zero, i.e., $c_{ij}\approx 0$. This can be used as a criterion to complete $B=(b_{ij})_{n\times n}$ with acceptable MC. Thus, the following optimization problem, referred to as CNOM, is proposed:

(M-2) min $$f(b_{ij},x)=\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\sum_{k=1}^{n}\frac{b_{ik}(x)}{b_{ki}(x)}\frac{b_{kj}(x)}{b_{jk}(x)}-n\frac{b_{ij}(x)}{b_{ji}(x)}\right)^2$$

(17)

$$\text{s.t. } 0<x_l<1,\ l=1,2,\ldots,p/2$$

where $b_{ij}(x)$ represents the (i, j)-element in B. Since $f(b_{ij}, x)$ is a function of $x=(x_1, x_2,\ldots, x_{p/2})$, the Matlab™ 'syms' can be used to construct the symbolic matrix to derive $f(b_{ij},x)$, while the Matlab™ 'Fmincon' function can be utilized to obtain the optimal solution of CNOM.

B. Algorithm to Improve MC of Incomplete FPRs

As mentioned above, an incomplete FPR is usually not consistent in practical situations. Based on Algorithm I and (M-2), a method to improve the MC of incomplete FPRs is described in Algorithm II.

Algorithm II.

Let $B=(b_{ij})_{n\times n}$ be an incomplete FPR, t be the number of iterations, and $\overline{GCI(B)}$ a (given) consistency threshold value.

Step 1: Apply Theorem 3. If all non-diagonal elements in a row or column of B are unknown, then B is returned to the DM

and asked to provide a new FPR verifying Theorem 3 condition. Otherwise, the priority vector associated with B is obtained as per (9)-(12).

Step 2: Apply (6)-(8) to compute $\overline{GCI(B)}$. If $GCI(B) \le \overline{GCI(B)}$, then go to Step 11. Otherwise, continue to Step 3.

Step 3: Construct the auxiliary FPR $\bar{B}=(\bar{b}_{ij})_{n\times n}$

$$\bar{b}_{ij} = \begin{cases} b_{ij}, & \text{if } b_{ij} \ne - \\ x_l, & \text{if } b_{ij} = - \text{ and } i < j \\ 1-x_l, & \text{if } b_{ij} = - \text{ and } i > j \end{cases} \tag{18}$$

Step 4: Apply Matlab™ 'Fmincon' function to obtain the optimal solution of CNOM and derive the complete FPR $\tilde{B}=(\tilde{b}_{ij})_{n\times n}$.

Step 5: Let t=0 and $\tilde{B}^{(t)}=(\tilde{b}_{ij}^{(t)})_{n\times n}=(\tilde{b}_{ij})_{n\times n}$.

Step 6: Apply Algorithm I Steps 4-9 to identify the inconsistent element in $\tilde{B}^{(t)}$. If the identified inconsistent element is an original unknown value, then continue to check the next second absolute largest value in $C^{(t)}$, until the identified inconsistent element is one of the known values: $b_{i_\tau j_\tau}^{(t)}$ (exists because $GCI(B) \le \overline{GCI(B)}$ is not true).

Step 7: Change the inconsistent element $\tilde{b}_{i_\tau j_\tau}^{(t)}$ to the value $\tilde{b}_{i_\tau j_\tau}^{(t+1)}$ verifying:

$$\sum_{k=1,k\ne i,k\ne j}^{n} \frac{\tilde{b}_{i_\tau k}^{(t)}}{\tilde{b}_{ki_\tau}^{(t)}} \frac{\tilde{b}_{kj_\tau}^{(t)}}{\tilde{b}_{j_\tau k}^{(t)}} - (n-2)\frac{\tilde{b}_{i_\tau j_\tau}^{(t+1)}}{1-\tilde{b}_{i_\tau j_\tau}^{(t+1)}} = 0 \tag{19}$$

Denoting $\psi = \frac{1}{n-2}\sum_{k=1,k\ne i_\tau,k\ne j_\tau}^{n} \frac{\tilde{b}_{i_\tau k}^{(t)}}{\tilde{b}_{ki_\tau}^{(t)}} \frac{\tilde{b}_{kj_\tau}^{(t)}}{\tilde{b}_{j_\tau k}^{(t)}}$, it is

$$\tilde{b}_{i_\tau j_\tau}^{(t+1)} = \frac{\psi}{1+\psi} \tag{20}$$

The rest of elements of B are unchanged: $\tilde{b}_{uv}^{(t+1)}=\tilde{b}_{uv}^{(t)}$, for all u, $v \in N$, $u \ne i_\tau$ and $v \ne j_\tau$. Thus, a new modified FPR $\tilde{B}^{(t+1)}$ is constructed.

Step 8: Discard the inferred values of the unknown elements to derive the modified incomplete FPR $B^{(t+1)}$.

Step 9: Calculate $GCI(B^{(t+1)})$. If $GCI(B^{(t+1)}) > \overline{GCI(B)}$, let $t = t+1$, and go to Step 6. Otherwise, go to Step 10.

Step 10: Output: Number of iterations t, the modified incomplete FPR $B^{(t+1)}$ and its acceptable consistency degree $GCI(B^{(t+1)})$.

Step 11: End.

Remark 5: Different from Algorithm I, the first task of Algorithm II is to estimate the missing values of an incomplete FPR. Here, CNOM is applied. After the corresponding complete FPR is obtained, Algorithm I is performed to improve its MC. The modified values are from the original known values of the incomplete FPR.

V. Illustrative Examples and Comparisons

A. Illustrative Examples

Example 1 and Example 2 assume inconsistent complete FPRs, so Algorithm I is applied. Example 3 assumes an incomplete and inconsistent FPR, so Algorithm II applies. Simulation experiments and comparisons with the existent methods are also offered to show the advantages of the proposed methods.

A.1 Example 1

Algorithm I is applied to the following FPR on the set of alternatives $X=\{x_1,x_2,x_3,x_4\}$:

$$R = \begin{bmatrix} 0.5 & 0.1 & 0.8 & 0.2 \\ 0.9 & 0.5 & 0.8 & 0.7 \\ 0.2 & 0.2 & 0.5 & 0.3 \\ 0.8 & 0.3 & 0.7 & 0.5 \end{bmatrix}$$

Step 1: Let t=0, $R^{(0)}=(r_{ij}^{(0)})_{4\times4}=(r_{ij})_{4\times4}$; $\overline{GCI(R)}$=0.4.

Step 2: Applying (5), the priority vector of $R^{(0)}$ is w=(0.1065, 0.5582,0.0746, 0.2607)^T.

Step 3: Applying (3), the geometric consistency index of $R^{(0)}$ is $GCI(R^{(0)})$=0.7192>$\overline{GCI}$.

Step 4: The induced matrix $C^{(0)}$ associated with $R^{(0)}$ is:

$$C^{(0)} = \begin{bmatrix} 0 & 0.8849 & -6.9722 & 1.4735 \\ -7.6667 & 0 & 33.4444 & -0.7024 \\ 3.4643 & -0.2885 & 0 & -0.2113 \\ -3.5595 & 0.1706 & 13.0476 & 0 \end{bmatrix}$$

Step 5: The largest absolute value in $C^{(0)}$ is $\left|c_{23}^{(0)}\right| = 33.4444$.

Step 6: The obtained deviation vector

$$F^{(0)} = \left(\frac{r_{21}^{(0)}}{r_{12}^{(0)}}\frac{r_{13}^{(0)}}{r_{31}^{(0)}} - \frac{r_{23}^{(0)}}{r_{32}^{(0)}}, \frac{r_{22}^{(0)}}{r_{22}^{(0)}}\frac{r_{23}^{(0)}}{r_{32}^{(0)}} - \frac{r_{23}^{(0)}}{r_{32}^{(0)}}, \frac{r_{23}^{(0)}}{r_{32}^{(0)}}\frac{r_{33}^{(0)}}{r_{33}^{(0)}} - \frac{r_{23}^{(0)}}{r_{32}^{(0)}}, \frac{r_{24}^{(0)}}{r_{42}^{(0)}}\frac{r_{43}^{(0)}}{r_{34}^{(0)}} - \frac{r_{23}^{(0)}}{r_{32}^{(0)}} \right)$$

is $F^{(0)}$=(32, 0, 0, 1.4444).

Step 7: The largest absolute value in $F^{(0)}$ is 32, while the other values are zero or close to zero. Therefore, the inconsistent element is 32=$(r_{21}^{(0)}/r_{12}^{(0)})(r_{13}^{(0)}/r_{31}^{(0)})-r_{23}^{(0)}/r_{32}^{(0)}$, and either $r_{21}^{(0)}$ or $r_{13}^{(0)}$ or both are quite large (within [0,1]).

Step 8: Since $c_{21}^{(0)}=-7.6667$ and $c_{13}^{(0)}=-6.9722$, the third rule of Table I applies and Step 9 is applied to identify the inconsistent element.

Step 9: Removing sequentially alternatives x_1, x_2, x_3, and checking whether the corresponding order reduced preference relation passes the consistency test.

Sub-step 1: Obtain the 3×3 order matrix $R^{(0)'}$ by removing the 1st row and the 1st column from $R^{(0)}$:

$$R^{(0)'} = \begin{bmatrix} 0.5 & 0.8 & 0.7 \\ 0.2 & 0.5 & 0.3 \\ 0.3 & 0.7 & 0.5 \end{bmatrix}$$

Applying (3) and (5): w = (0.588, 01327, 0.2793)^T and $GCI(R^{(0)'})$ = 0.0317<0.4. Therefore, $R^{(0)'}$ is of acceptable consistency. From Table II rules, $r_{23}^{(0)}$ is a consistent element.

Sub-step 2: Obtain the 3×3 order matrix $R^{(0)''}$ by removing the 2nd row and the 2nd column from $R^{(0)}$

$$R^{(0)''} = \begin{bmatrix} 0.5 & 0.8 & 0.2 \\ 0.2 & 0.5 & 0.3 \\ 0.8 & 0.7 & 0.5 \end{bmatrix}$$

Applying (3) and (5): w=(0.2793,0.1327,0.5880)^T and $GCI(R^{(0)''})$ =1.2356>0.4, so $R^{(0)''}$ is not of acceptable

consistency. From Table II rules, $r_{13}^{(0)}$ is an inconsistent element and its value should be decreased.

Sub-step 3: Obtain the 3×3 order matrix $R^{(0)'''}$ by removing the 3rd row and the 3rd column from $R^{(0)}$

$$R^{(0)'''}=\begin{bmatrix}0.5 & 0.1 & 0.2\\ 0.9 & 0.5 & 0.7\\ 0.8 & 0.3 & 0.5\end{bmatrix}$$

Applying (3) and (5): w=(0.0711,0.6479,0.2810)^T and $GCI(R^{(0)'''})$ =0.0004<0.4, so $R^{(0)'''}$ is of acceptable consistency. From Table II rules, $r_{21}^{(0)}$ is a consistent element.

Table III shows the identification processes of Step 9.

Step 10: Apply (16) to find the value that the inconsistent element $r_{13}^{(0)}$ is to be modified to: $r_{13}^{(1)}$=0.3394, and $r_{31}^{(1)}$=0.6606. The new modified FPR $R^{(1)}$ is:

$$R^{(1)}=\begin{bmatrix}0.5 & 0.1 & 0.3394 & 0.2\\ 0.9 & 0.5 & 0.8 & 0.7\\ 0.6606 & 0.2 & 0.5 & 0.3\\ 0.8 & 0.3 & 0.7 & 0.5\end{bmatrix}$$

Steps 2-3: Applying (3) and (5): $GCI(R^{(1)})$=0.0111<0.4, and $R^{(1)}$ is of acceptable consistency.

In the above Step 10, the value generated by (16) is the only one automatically adopted. The new revised value $r_{13}^{(1)}$ is not in the scale $U_{[0.1,0.9]}$. As it is difficult for a DM to precisely provide the value 0.3394 in real applications, $r_{13}^{(1)}=0.3$ (the value that derives from the proposed round operation) is suggested to be the revised value while, to comply with reciprocity property, 0.7 is suggested to be the revised value $r_{31}^{(1)}$. It is worth mentioning here that Steps 10-11 with the new rounded preference values do not change the final conclusion, i.e., the new FPR is of acceptable consistency because $GCI(R^{(1)})$= 0.0161<0.4.

TABLE III. THE IDENTIFICATION PROCESSES IN STEP 9 OF EXAMPLE 1.

Removed alternative	Order reduced matrix	Weight vector	*GCI*	Potential inconsistent elements	Consistent elements	Inconsistent elements
x_1	$R^{(0)'}=\begin{bmatrix}0.5 & 0.8 & 0.7\\ 0.2 & 0.5 & 0.3\\ 0.3 & 0.7 & 0.5\end{bmatrix}$	$w=\begin{bmatrix}0.5880\\ 0.1327\\ 0.2793\end{bmatrix}$	0.0317<0.4	$r_{13}^{(0)}, r_{21}^{(0)}$	$r_{23}^{(0)}$	
x_2	$R^{(0)''}=\begin{bmatrix}0.5 & 0.8 & 0.2\\ 0.2 & 0.5 & 0.3\\ 0.8 & 0.7 & 0.5\end{bmatrix}$	$w=\begin{bmatrix}0.2793\\ 0.1327\\ 0.5880\end{bmatrix}$	1.2356 > 0.4	$r_{21}^{(0)}$		$r_{13}^{(0)}$
x_3	$R^{(0)'''}=\begin{bmatrix}0.5 & 0.1 & 0.2\\ 0.9 & 0.5 & 0.7\\ 0.8 & 0.3 & 0.5\end{bmatrix}$	$w=\begin{bmatrix}0.0711\\ 0.6479\\ 0.2810\end{bmatrix}$	0.0004 < 0.4		$r_{21}^{(0)}$	

This example shows that Algorithm I significantly improves the consistency of the FPR via the modification of just one pair of its elements. Therefore, the proposed approach preserves the majority of the original preference information, which illustrates its feasibility and effectiveness.

A.2 Example 2

The following example is adapted from Xia, *et al.* [25]. Consider the following FPR on a set of four alternatives:

$$R=\begin{bmatrix}0.5 & 0.4 & 0.7 & 0.3\\ 0.6 & 0.5 & 0.6 & 0.8\\ 0.3 & 0.4 & 0.5 & 0.3\\ 0.7 & 0.2 & 0.7 & 0.5\end{bmatrix}$$

Algorithm I is applied with the same threshold consistency value used in Xia, *et al.* [25]: $\overline{GCI(R)}$=0.4.

Step 1: Let t=0, $R^{(0)}=(r_{ij}^{(0)})=(r_{ij})_{4\times4}$.

Step 2: Applying (5), the priority vector of $R^{(0)}$ is: w=(0.2098, 0.4021,0.1373,0.2508)^T.

Step 3: Applying (3), the geometric consistency index of $R^{(0)}$ is $GCI(R^{(0)})$=0.6767> $\overline{GCI(R)}$.

Step 4: Applying (13), the induced matrix $C^{(0)}$ is:

$$C^{(0)}=\begin{bmatrix}0 & 0.3294 & -2.6667 & 2.8095\\ 6.9762 & 0 & 9.8333 & -6.7143\\ 1.1429 & -0.9405 & 0 & 1.9932\\ -3.2917 & 2.6111 & 1.1528 & 0\end{bmatrix}$$

Step 5: The largest absolute value in $C^{(0)}$ is $|c_{23}^{(0)}|$=9.8333.

Step 6: The deviation vector is: $F^{(0)}$=(2,0,0,7.8333).

Step 7: The largest absolute value in $F^{(0)}$ is 7.8333, while the rest of values are zero or close to zero. Since 7.8333= $(r_{24}^{(0)}/r_{42}^{(0)})\cdot(r_{43}^{(0)}/r_{34}^{(0)})-r_{23}^{(0)}/r_{32}^{(0)}$ then either $r_{24}^{(0)}$ or $r_{43}^{(0)}$ or both are too large (within [0,1]).

Step 8: Since $c_{24}^{(0)}$=−6.7143 and $c_{43}^{(0)}$=1.1528, the first rule of Table I applies and apply Step 10 to find the value that the identified inconsistent element is to be modified to.

Step 10: Apply (16) to find the value $r_{24}^{(1)}$ that the inconsistent element $r_{24}^{(0)}$ is to be modified to. After applying the rounding process, $r_{24}^{(1)}$ =0.4 and, by reciprocity property, $r_{42}^{(1)}$ is also modified to 0.6. The new modified FPR $R^{(1)}$ is:

$$R^{(1)}=\begin{bmatrix}0.5 & 0.4 & 0.7 & 0.3\\ 0.6 & 0.5 & 0.6 & 0.4\\ 0.3 & 0.4 & 0.5 & 0.3\\ 0.7 & 0.6 & 0.7 & 0.5\end{bmatrix}$$

Steps 2-3: Applying (3) and (5), we have $GCI(R^{(1)})$=0.1199 <0.4, and $R^{(1)}$ is of acceptable consistency.

Xia, *et al.* [25] obtained the following modified FPR after three iterations:

$$R^{(3)}=\begin{bmatrix}0.5000 & 0.3842 & 0.6754 & 0.3394\\ 0.6158 & 0.5000 & 0.6426 & 0.7574\\ 0.3246 & 0.3574 & 0.5000 & 0.3141\\ 0.6606 & 0.2426 & 0.6859 & 0.5000\end{bmatrix}$$

The geometric consistency index of $R^{(3)}$ is 0.3596, which means that the modified FPR obtained by the current proposed method is less inconsistent than with the method proposed in Xia, *et al.* [25]. Additionally, in Xia, *et al.* [25]'s method, the consistent FPR $R^{(3)}$ is obtained applying an auto-adaptive algorithm after several iterations rather than identifying and revising inconsistent pairwise values at each step. Consequently, the consistent FPR $R^{(3)}$ values are significantly different from the original preference values, which could be perceived by the DM as an untrue representation of his/her opinions, i.e., as unreliable. This obviously is avoided with the approach proposed in this article since the modification applies to only one pair of preference values, while the rest of original preference values are kept unchanged. This example also reinforces the merits of the proposed methodology.

Remark 6. Both Examples 1 and 2 use Algorithm I to identify the inconsistent elements and then revise them to be consistent. The difference of the two examples is that Example 1 uses Step 9 to determine which element is most inconsistent, while Example 2 identifies the most inconsistent elements directly.

A. 3 Example 3

Consider the following incomplete FPR on a set of six alternatives (adapted from [25, 42]):

$$B=\begin{bmatrix}0.5 & 0.4 & - & 0.3 & 0.8 & 0.3\\ 0.6 & 0.5 & 0.6 & 0.5 & - & 0.4\\ - & 0.4 & 0.5 & 0.3 & 0.6 & -\\ 0.7 & 0.5 & 0.7 & 0.5 & 0.4 & 0.8\\ 0.2 & - & 0.4 & 0.6 & 0.5 & 0.7\\ 0.7 & 0.6 & - & 0.2 & 0.3 & 0.5\end{bmatrix}$$

This incomplete FPR was first used by Xu in [39], then it was investigated by Xu, *et al.* in [42], Xu and Wang in [43], Xia *et al.* in [25]. Algorithm II is applied to improve the consistency of this incomplete FPR. To facilitate a comparison with the performance reported in [25, 42], the same geometric consistency index threshold value is used: $\overline{GCI(B)}$= 0.4.

Step 1: According to Theorem 3, B is acceptable and, therefore, it can be completed using its known elements and, after applying (9)-(12), its priority vector is obtained: w= $(0.1448,0.1807,0.141,0.2532,0.1427,0.1373)^T$.

Step 2: From (6)-(8), it is $GCI(B)$=0.4888> $\overline{GCI(B)}$.

Step 3: As per (18), the auxiliary FPR $\overline{B}$ is:

$$\overline{B}=\begin{bmatrix}0.5 & 0.4 & x_1 & 0.3 & 0.8 & 0.3\\ 0.6 & 0.5 & 0.6 & 0.5 & x_2 & 0.4\\ 1-x_1 & 0.4 & 0.5 & 0.3 & 0.6 & x_3\\ 0.7 & 0.5 & 0.7 & 0.5 & 0.4 & 0.8\\ 0.2 & 1-x_2 & 0.4 & 0.6 & 0.5 & 0.7\\ 0.7 & 0.6 & 1-x_3 & 0.2 & 0.3 & 0.5\end{bmatrix}$$

Step 4: The Matlab™ 'Fmincon' function provides the optimal solution of CNOM, which after 10 iterations, shown in Table IV and Fig. 1, is x=(0.5385, 0.5409, 0.5587). This results in the following complete FPR $\tilde{B}$:

$$\tilde{B}=\begin{bmatrix}0.5 & 0.4 & 0.5385 & 0.3 & 0.8 & 0.3\\ 0.6 & 0.5 & 0.6 & 0.5 & 0.5409 & 0.4\\ 0.4615 & 0.4 & 0.5 & 0.3 & 0.6 & 0.5587\\ 0.7 & 0.5 & 0.7 & 0.5 & 0.4 & 0.8\\ 0.2 & 0.4591 & 0.4 & 0.6 & 0.5 & 0.7\\ 0.7 & 0.6 & 0.4413 & 0.2 & 0.3 & 0.5\end{bmatrix}$$

TABLE IV. THE OBJECTIVE FUNCTION VALUES AND THE OPTIMAL SOLUTION

Iterations	Objective function value $f(x)$	Iterations	Objective function value $f(x)$	Unknown variable x_l	Optimal solution
0	1.0471×10^3	6	1.0365×10^3	x_1	0.5385
1	1.0446×10^3	7	1.0365×10^3	x_2	0.5409
2	1.0445×10^3	8	1.0365×10^3	x_3	0.5587
3	1.0416×10^3	9	1.0365×10^3		
4	1.0471×10^3	10	1.0365×10^3		
5	1.0376×10^3				

Step 5: Let t=0, and $\tilde{B}^{(0)}=(\tilde{b}_{ij}^{(0)})_{6\times6}=(\tilde{b}_{ij})_{6\times6}$.

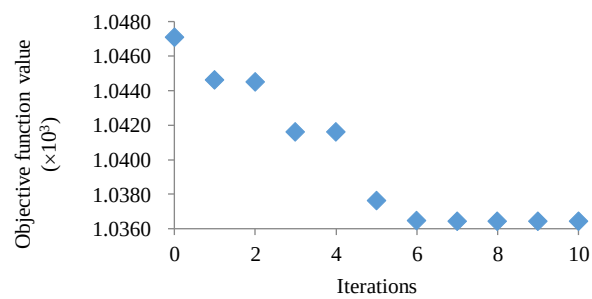


Fig. 1. The process of objective function optimization

Step 6: Algorithm I Steps 4-9 are applied to identify the inconsistent element in $\tilde{B}^{(0)}$. From (13), the induced matrix $C^{(0)}$ is obtained:

$$C^{(0)}=\begin{bmatrix}0 & 2.5777 & 0.3378 & 5.5596 & -12.9949 & 11.2550\\ -0.5311 & 0 & -0.6044 & -0.7804 & 4.4897 & 6.6243\\ 1.9010 & 1.5054 & 0 & 1.8862 & -0.9582 & 0.9619\\ 3.6664 & 5.6770 & -1.5068 & 0 & 13.0591 & -9.8237\\ 9.7889 & 2.2160 & 4.2412 & -4.1750 & 0 & -1.8163\\ -5.7159 & -3.3041 & 2.6822 & 2.4814 & 10.7378 & 0\end{bmatrix}$$

The largest absolute value in $C^{(0)}$ is $|c_{45}^{(0)}|$=13.0591; the largest absolute value in the deviation vector $F^{(0)}$=(8.6667, 0.5115, 2.8333, 0, 1.0476) is 8.6667, while the rest of values are zero or close to zero. Since 8.6667= $(\tilde{b}_{41}^{(0)}/\tilde{b}_{14}^{(0)})(\tilde{b}_{15}^{(0)}/\tilde{b}_{51}^{(0)})$ $-\tilde{b}_{45}^{(0)}/\tilde{b}_{54}^{(0)}$, then either $\tilde{b}_{41}^{(0)}$ or $\tilde{b}_{15}^{(0)}$ or both are too large (within

[0,1]). Since $c_{41}^{(0)}$=3.6664 and $c_{15}^{(0)}$=−12.9949, the second rule of Table I applies and $\tilde{b}_{15}^{(0)}$ has to be decreased.

Step 7: The inconsistent element $\tilde{b}_{15}^{(0)}$ is revised using (20). After applying the rounding process the revised values are: $\tilde{b}_{15}^{(1)}$ = 0.4 and, by reciprocity property, $\tilde{b}_{51}^{(1)}$ = 0.6. Thus, a new modified FPR $\tilde{B}^{(1)}$ is obtained:

$$\tilde{B}^{(1)} = \begin{bmatrix} 0.5 & 0.4 & 0.5385 & 0.3 & 0.4 & 0.3 \\ 0.6 & 0.5 & 0.6 & 0.5 & 0.5409 & 0.4 \\ 0.4615 & 0.4 & 0.5 & 0.3 & 0.6 & 0.5587 \\ 0.7 & 0.5 & 0.7 & 0.5 & 0.4 & 0.8 \\ 0.6 & 0.4591 & 0.4 & 0.6 & 0.5 & 0.7 \\ 0.7 & 0.6 & 0.4413 & 0.2 & 0.3 & 0.5 \end{bmatrix}$$

Step 8: Ignoring the inferred values of the unknown elements, the following modified incomplete FPR is obtained:

$$B^{(1)} = \begin{bmatrix} 0.5 & 0.4 & - & 0.3 & 0.4 & 0.3 \\ 0.6 & 0.5 & 0.6 & 0.5 & - & 0.4 \\ - & 0.4 & 0.5 & 0.3 & 0.6 & - \\ 0.7 & 0.5 & 0.7 & 0.5 & 0.4 & 0.8 \\ 0.6 & - & 0.4 & 0.6 & 0.5 & 0.7 \\ 0.7 & 0.6 & - & 0.2 & 0.3 & 0.5 \end{bmatrix}$$

Step 9: The priority vector of $B^{(1)}$ is w=(0.0974, 0.1647, 0.153, 0.2491, 0.2036, 0.1322)^T, and $GCI(B^{(1)})$=0.2345<0.4, which means that $B^{(1)}$ is of acceptable consistency.

The following provides a comparison analysis with the existent methods:

- Xu [39] proposed two non-consistency based goal programming models to derive the priority vector of incomplete FPRs. Therefore, the results of this methodology are unreliable since the considered incomplete FPR B is of not acceptable consistency.
- Since $GCI(B)$=0.4888>$\overline{GCI(B)}$, Xu, *et al.* [42] proposed the application of LLSM and eigenvector based methods to improve the MC of B, respectively. These two approaches obtained $b_{15}^{(1)}$=0.3, $b_{51}^{(1)}$=0.7, which are close but different to the values obtained with the proposed approach in this paper. However, the existent methodologies do not include any rules to guide the DM on how to identify and revise the inconsistent elements. This is not the case with the proposed approach herein, which therefore avoids arbitrariness.
  - Xu and Wang [43] also developed a method to repair the inconsistency of incomplete FPRs. They set $b_{15}^{(1)}$=0.5 and $b_{51}^{(1)}$=0.5, which leads to $GCI(B^{(1)})$=0.2648, a higher value than the value obtained with our proposed approach (see Step 9 above). Therefore, from the consistency improvement point of view, the method proposed in this paper is faster than Xu and Wang [43]'s methodology.
- Xia, *et al.* [25] developed an auto-adaptive algorithm to improve the consistency of incomplete FPRs. This algorithm obtained, after five iterations, the following modified incomplete FPR with $GCI(B^{(5)})$=0.3409:

$$B^{(5)} = \begin{bmatrix} 0.5 & 0.4182 & - & 0.3252 & 0.6952 & 0.3826 \\ 0.5818 & 0.5 & 0.5844 & 0.4654 & - & 0.4683 \\ - & 0.4156 & 0.5 & 0.3229 & 0.5584 & - \\ 0.6748 & 0.5346 & 0.6771 & 0.5 & 0.4990 & 0.7446 \\ 0.3048 & - & 0.4416 & 0.5010 & 0.5 & 0.6262 \\ 0.6174 & 0.5317 & - & 0.2554 & 0.3738 & 0.5 \end{bmatrix}$$

Again, the methodology proposed in this paper outperforms, from the consistency improving point of view, Xia, *et al.* [25]'s method, which, after five iterations, changed all the original preference values but the main diagonal values. This contrasts with our proposed methodology, which retains all original preference values but two in one iteration.

B. Comparison with Methods by Xia, et al. [25], and Xu and Da [26]

The above comparison between the proposed approach and the existent methods was based on a specific example widely used in this research area. Next, a statistical based comparative analysis is provided.

Until now, several approaches [1, 9-11, 44] have been developed to adjust inconsistency of MPRs in the AHP framework. For FPRs, different methods to identify and rectify OC, AC and MC also exist. Xu, *et al.* [21] proposed an index (OCI) to measure the OC degree, and developed an algorithm to eliminate the ordinal inconsistency of FPRs. Ma, *et al.* [29] presented a methodology to identify inconsistency and weak transitivity together with a method to repair inconsistency of FPRs. Wu and Xu [45] proposed an algorithm to modify an FPR so that AC property is verified. Xu, *et al.* [46] developed a distance-based framework to deal with ordinal and additive inconsistencies of FPRs. These methods are only able to handle the AC property. However, Chiclana, *et al.* [23] proved that cardinal consistency of FPRs is to be modelled with MC. Recently, Zhang [47] investigated the MC and consensus of FPRs, although no method to improve the MC of FPRs was proposed. As far as we know, only Xia, *et al.* [25], and Xu and Da [26] have proposed methods to improve the MC of FPRs; thus, these are the subject of the below comparative analysis with our current proposed approach.

If $GCI(R^{(t)})$> $\overline{GCI(R)}$, then the FPR is not of acceptable consistency. In this case, Xia, *et al.* [25] proposed the following equation to improve MC:

$$r_{ij}^{(t+1)} = \frac{(r_{ij}^{(t)})^{1-\theta}(w_i^{(t)})^{\theta}}{(r_{ij}^{(t)})^{1-\theta}(w_i^{(t)})^{\theta} + (1-r_{ij}^{(t)})^{1-\theta}(w_j^{(t)})^{\theta}} \tag{21}$$

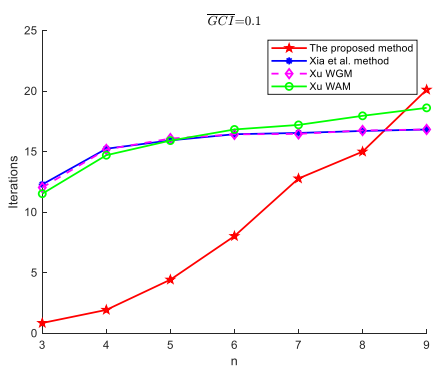
where w_i (i=1, 2,…, n) is determined as per (5) and θ is a parameter of control. Xu and Da [26] first proposed to transform FPRs into corresponding MPRs, $A=(a_{ij})_{n\times n}$ with $a_{ij}=r_{ij}/r_{ji}$, from which, if its CR is acceptable, the weighting vector $w=(w_1, w_2, \ldots, w_n)^T$ was derived using the eigenvalue method; otherwise, when CR is less than or equal to 0.1, Xu and Da [26] proposed the weighted geometric mean (WGM) and weighted arithmetic mean (WAM) methods to improve consistency. Once the revised MPR is of acceptable consistency, the corresponding revised FPR values are obtained as follows: $r_{ij}=a_{ij}/(1+a_{ji})$.

The same expression (3) is used to measure the consistency degree of FPRs when comparing the proposed method and the

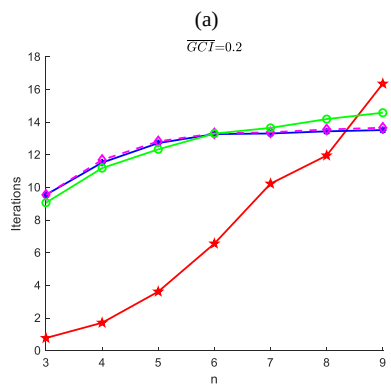
method by Xia, *et al.* [25]. Also, 1000 FPRs are randomly generated (for n=3 to n=9) when comparing the proposed method and the method by Xu and Da [26]. All the elements r_{ij} above the diagonal ($i<j$) are randomly generated ranging in (0, 1), by setting $r_{ji}=1-r_{ij}$ as well as $r_{ii}=0.5$. Average iterations needed to obtain the corresponding FPR with acceptable consistency are computed for the following different threshold values $\overline{GCI}$=0.1,0.2,0.4, respectively. Table V shows the results obtained: method by Xia, *et al.* [25] with θ=0.1 (column 3); the methods by Xu and Da [26] (WGM-column 4; WAM-column 5); the proposed method in this paper (column 6). Fig. 2 provides a graphical representation of the results in Table V.

TABLE V. THE AVERAGE ITERATIONS FOR XIA, *ET AL.* [25], XU AND DA [26] AND THE PROPOSED METHODS

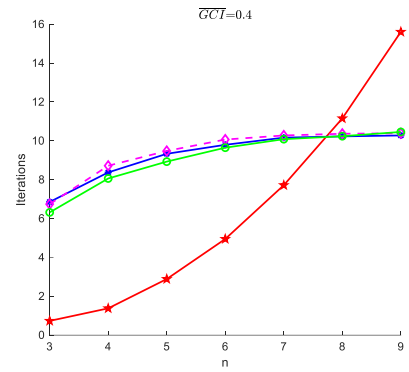
n	$\overline{GCI}$	Xia, *et al.* [25]'s method	Xu and Da [26]'s WGM	Xu and Da [26]'s WAM	The proposed method
3	0.1	12.297	12.048	11.533	0.8590
	0.2	9.557	9.551	9.068	0.779
	0.4	6.854	6.743	6.318	0.733
4	0.1	15.247	15.191	14.712	1.942
	0.2	11.528	11.692	11.19	1.713
	0.4	8.376	8.719	8.063	1.379
5	0.1	15.938	16.087	15.919	4.437
	0.2	12.723	12.829	12.341	3.621
	0.4	9.333	9.479	8.931	2.887
6	0.1	16.445	16.435	16.838	8.033
	0.2	13.275	13.319	13.307	6.564
	0.4	9.796	10.068	9.648	4.948
7	0.1	16.551	16.471	17.213	12.788
	0.2	13.312	13.384	13.656	10.236
	0.4	10.165	10.277	10.086	7.717
8	0.1	16.714	16.752	17.964	15.006
	0.2	13.444	13.56	14.194	11.962
	0.4	10.230	10.365	10.239	11.156
9	0.1	16.834	16.839	18.622	20.118
	0.2	13.521	13.665	14.584	16.354
	0.4	10.278	10.39	10.458	15.608



(a)



(b)



(c)

Fig. 2. The average iterations of Xia, *et al.* [25], Xu and Da [26] and the proposed methods for different $\overline{GCI}$, (a) $\overline{GCI}$=0.1, (b) $\overline{GCI}$=0.2, (c) $\overline{GCI}$=0.4

From Table V and Fig.2, we can see that the methods by Xia, *et al.* [25] and Xu and Da [26] perform similarly in the three considered cases $\overline{GCI}$=0.1, 0.2, 0.4. The shapes are similar at different GCI threshold levels. But the average number of iterations is different, and it decreases when $\overline{GCI}$ increases for the corresponding methods. However, these methods perform differently to the proposed method, which produces lower average iterations in all case but n=9. In general, the smaller the value of $\overline{GCI}$, the larger the average number of iterations of the four methods. However, the smaller the value of n, the larger the differences between the average number of iterations of the methods by Xia, *et al.* [25], and Xu and Da [26]. Specifically, when n=3, the differences between the average number of iterations of these methods are less than 1. The experiment shows that if an FPR is not of acceptable consistency and n = 3 or 4, our proposed method requires on average only one or two iterations to adjust an inconsistent FPR to become of acceptable consistency. However, for n=3, the methods by Xia, *et al.* [25] and Xu and Da [26] require approximately 12, 9, and 6 average iterations for $\overline{GCI}$=0.1, 0.2, 0.4, respectively. Thus, in addition to distorting less original information (see next) and improving the consistency to higher levels, this simulation shows that our proposed method is computationally more efficient than the methods by Xia, *et al.* [25], and Xu and Da [26].

Changing of preference values are required when the FPR is not of acceptable consistency. In order to show the effectiveness of the proposed method, we measure the percentage of the original preference values that are adjusted:

$$AR = \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} \upsilon_{ij}}{n(n-1)/2} \tag{22}$$

where $\upsilon_{ij} = \begin{cases} 0, r_{ij} \text{ unchanged} \\ 1, r_{ij} \text{ changed} \end{cases}$.

In the experiments, average values υ_{ij} are computed. Table VI lists the AR values of the three methods. As stated above, AR is always 100% for the method by Xia, *et al.* [25], and consequently its entries for AR for all values of n is 1. The methods by Xu and Da [26] use (21) to improve consistency and they also modify all entries of FPRs of not acceptable consistency. As we can see, the AR values of the proposed method increase steadily when n increases from 3 to 9. The values are ranged from 0.209 to 0.3784. Obviously, our proposed method keeps a high percentage of original

information unchanged in the consistency improvement process. This also shows the merits of the proposed method.

TABLE VI. THE VALUES OF AR FOR XIA, *ET AL.* [25], XU AND DA [26]'S METHODS AND THE PROPOSED METHOD WHEN $\overline{GCI}$=0.1

n	3	4	5	6	7	8	9
AR of Xia, *et al.* [25]'s method	1	1	1	1	1	1	1
AR of Xu and Da [26]'s method	1	1	1	1	1	1	1
AR of the proposed method	0.209	0.2337	0.2887	0.3138	0.3425	0.3594	0.3784

C. Comparison of the Priority Vectors Obtained by the Different Methods

We use LLSM (5) to obtain the priority vector of FPRs and compare it against the following alternative methods: eigenvector method (EM)(Xu and Wang [43]), chi-square method (CSM)(Wang, *et al.* [48]) and least square method (LSM), please see in Appendix.

To compare the robustness of the proposed method, 1,000 FPRs with different dimension, ranging from 3 to 9, are randomly generated. The above four methods are used to obtain the priority weights in Step 2 of Algorithm I. The average number of iterations of Algorithm I is provided in Table VII and plotted in Fig. 3 ($\overline{GCI}$=0.4). For *n*=3, the average number of iterations of all the methods is lower than 1, denoting that some of the randomly generated FPRs are of acceptable MC (these FPRs are not needed to be revised); otherwise, only one iteration is required to achieve FPRs of acceptable MC. From Table VII and Fig. 3, it is clear that the average number of iterations increases as the dimension *n* (number of alternatives) increases. No big differences are observed between the application of the different methods to derive the priority vector of FPRs, although the proposed LLSM method invariably required the lowest number of average iterations. However, different priority methods will derive different weights, and by Eq.(6), they will obtain different GCI values. In some cases, the GCI value for LLSM is smaller than the predefined threshold, but the GCI values for other methods are larger than the threshold, and these methods need one more iteration. Thus, the average iterations are different for these different methods. In any case, the similarity of values shown in Table VII indicates the robustness of the proposed Algorithm I, and therefore the effectiveness of the proposed method to improve MC of FPRs using different methods to derive the priority vector.

TABLE VII. THE AVERAGE ITERATIONS OF THE LLSM, EM, CSM, LSM, $CI_{CAVALLO}$, OC

n	LLSM	EM	CSM	LSM	$CI_{Cavallo}$	OC
3	0.733	0.742	0.754	0.74	0.772	0.257
4	1.379	1.455	1.4565	1.499	1.461	0.711
5	2.887	2.913	3.2184	3.273	3.081	2.124
6	4.948	5.082	5.5671	5.739	5.265	5.252
7	7.717	7.919	8.987	9.106	8.231	9.631
8	11.156	11.646	12.9639	12.846	11.813	15.142
9	15.6076	15.8617	18.4575	18.4634	16.208	21.368

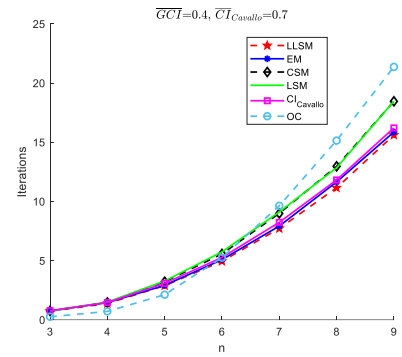


Fig. 3. The average iterations of the LLSM, EM, CSM, LSM, $CI_{Cavallo}$, and OC

D. The average iterations of inconsistency depending on the entries

In the above, we use the GCI to measure the inconsistency level, which depends on the weighting vector. Thus, when different priority methods are used, the average iterations may be different, which has been discussed in Section V.C, and shown in Fig. 3. Another important way to measure the consistency depends on entries ([49], [50]). In the following, we use Cavallo and D'Apuzzo [51], Cavallo, *et al.* [52]'s method to compute the consistency index (CI), and denoted as $CI_{Cavallo}$. Cavallo and D'Apuzzo [51] also showed that $CI_{Cavallo} \in [0.5, 1]$, and it is consistent if and only if $CI_{Cavallo}$=0.5. Ordinal consistency (OC)[53] is also another way to measure the consistency, which is independent of weighting vector. Please see in Appendix. Here, 1,000 FPRs with different dimension, ranging from 3 to 9, are also randomly generated. Algorithm I is also used to improve the MC and OC, and the difference is that we use Cavallo's CI (See Eq.(17) in [52]), OC (See Eq.(11) in [53]) to measure the consistency degree instead of GCI, and the threshold is $\overline{CI}_{Cavallo}$=0.7. The average number of iterations is also depicted in Fig. 3.

From Fig. 3, we can see that the average number of iterations of $CI_{Cavallo}$ is close to EM and LSM, which shows the effectiveness of the proposed method to improve MC. When the proposed algorithm is used to improve OC, the average number of iterations are 0.257, 0.711, and 2.124 for *n*=3, 4, 5, respectively, which are remarkable smaller than other methods. It denotes that most of the original randomly generated FPRs are of OC, or they become OC fast with less MC improving iterations. The average number of iterations is close to other methods when $n = 6$, and it is larger than other methods when $n \geq 7$. It shows that the proposed IIM method is effective for improving MC, not OC when $n \geq 7$, as OC is mainly related to transitivity or cycles of triad.

VI. Conclusions

This paper investigated the consistency of FPRs based on the multiplicative transitivity property. First, the *GCI* proposed by Xia, *et al.* [25] was applied to measure the degree of MC of a given FPR. For a complete FPR that is not of acceptable consistency, a novel IIM method to improve its consistency was proposed. For an incomplete FPR, an CNOM has been put forward to estimate its missing values before the application of the IIM method to identify and modify the inconsistent elements. Compared with the existent methods, the proposed method has the following advantages:

1) A novel local adjustment method is proposed to modify the most inconsistent element at each consistency improving round, with values belonging to the original preference scale range.
2) The proposed method is more efficient because fewer average iterations are needed for different cardinality n (n=3, 4, …, 8) as reported by simulation experiments.
3) The proposed method retains the majority of the DM's original information. This makes the results more reliable and acceptable by the DM.
4) The proposed method is also effective for improving the MC with Cavallo, *et al.* [52]'s CI.

Meanwhile, there are two interesting research directions for the future:

1) how to extend the proposed method to other types of preference relations [54, 55];
2) additive consistency is another important consistency property of FPRs; thus, it is worth investigating in the future how to improve locally the additive consistency level of FPRs.

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