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# Generalization of Szász operators involving multiple Sheffer polynomials

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**Abstract:** The present work deals with the mathematical investigation of some generalizations of the Szász operators. In this work, the multiple Sheffer polynomials are introduced. The generalization of Szász operators involving multiple Sheffer polynomials are considered. Convergence properties of these operators are verified with the help of the universal Korovkin-type result and the order of approximation is calculated by using classical modulus of continuity. Further, the convergence of these operators are also discussed in weighted spaces of functions on the positive semi-axis and estimate the approximation with the help of weighted modulus of continuity. The theoretical results are exemplified choosing the special cases of multiple Sheffer polynomials.

**Keywords:** Szász operators, Modulus of continuity, Rate of convergence, Multiple Sheffer polynomials.

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## 1 Introduction

The positive approximation processes discovered by Korovkin [16] play a central role and arise in a natural way in many problems connected with functional analysis, harmonic analysis, measure theory, partial differential equations and probability theory. In 1953, P. P. Korovkin [16] discovered perhaps, the most powerful and at the same time, simplest criterion in order to decide whether a given sequence  $(K_n)_{n \in \mathbb{N}}$  of positive linear operators on the space  $C[0, 1]$  is an approximation process, i.e.  $K_n(f) \rightarrow f$  uniformly on  $[0, 1]$  for every  $f \in C[0, 1]$ . Starting with this result, a considerable number of mathematicians have extended Korovkin's theorem to other function spaces or, more generally, to abstract spaces, such as Banach lattices, Banach algebras, Banach spaces and so on. Korovkin's work, in fact, delineated a new theory that may be called Korovkin-type approximation theory.

One of the well-known examples of positive linear operators is Szász operators [20]. Szász [20] introduced the following positive linear operators:

$$S_n(f; x) := e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \quad (1.1)$$

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where  $x \geq 0$  and  $f \in C[0, \infty)$  whenever the above sum converges. In recent years, there is an increasing interest to study linear positive operators based on certain polynomials, such as Appell polynomials, Sheffer polynomials, and Boas-Buck polynomials.

Appell polynomials  $p_k(x)$  are defined as:

$$\frac{d}{dx}p_k(x) = k p_{k-1}(x)$$

and by the following generating function:

$$g(t)e^{xt} = \sum_{k=0}^{\infty} p_k(x) \frac{t^k}{k!},$$

where  $g(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!}$ , with  $a_0 \neq 0$ .

Jakimovski and Leviatan [11] obtained a generalization of Szasz operators by means of the Appell polynomials as follows:

$$P_n(f; x) := \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right), \quad (1.2)$$

under the conditions  $|t| < R$ ,  $R > 1$  and  $g(1) \neq 0$ . For  $g(t) = 1$ , we obtain the Szász operators (1.1).

The Sheffer polynomials  $s_k(x)$  are defined by

$$g(t)e^{xH(t)} = \sum_{k=0}^{\infty} s_k(x) \frac{t^k}{k!}$$

where  $g(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!}$ , with  $a_0 \neq 0$  and  $H(t) = \sum_{k=0}^{\infty} h_k \frac{t^k}{k!}$ , with  $h_1 \neq 0$ .

Ismail [10] presented a generalization of Szasz and Jakimovski and Leviatan operators by using the Sheffer polynomials  $s_k(x)$  as

$$T_n(f; x) := \frac{e^{-nxH(1)}}{g(1)} \sum_{k=0}^{\infty} s_k(nx) f\left(\frac{k}{n}\right), \quad n \in \mathbb{N}, \quad (1.3)$$

where  $s_k(x)$ ,  $k \geq 0$  are the Sheffer polynomials under the conditions  $|t| < R$ ,  $R > 1$  and  $g(1) \neq 0$ ;  $H'(1) = 1$ . For  $g(t) = 1$  and  $H(t) = t$ , (1.3) reduces to (1.1).

In the last few years, the interest in Sheffer polynomials and their applications in different fields has significantly increased. A unified matrix representation for the Sheffer polynomials is proposed in [1]. The recurrence relations, differential equations for the Hermite-Sheffer polynomials using matrix approach have been obtained in [14]. In 2014, Costabile and Longo [5] obtained an algebraic approach to Sheffer polynomial sequences. Dattoli and his co-workers have

shown that by using operational techniques, many properties of ordinary and multi-variable special functions are simply derived and framed in a more general context, see for example [6]. The hybrid special polynomial families related to the Appell and Sheffer polynomial sequences are first introduced and studied by Khan and her co-authors, see for example [15].

In the recent past, many studies have been carried out on approximation by positive linear operators involving Sheffer polynomials; see, for example, [4, 12, 13, 19] for details. In addition, research has been undertaken in an attempt at generalization of the Szász operators associated with multiple polynomial sets [2, 21]. By a multiple polynomial system we mean a set of polynomials  $\{p_{k_1, k_2}(x)\}_{k_1, k_2=0}^{\infty}$  with degree  $k_1 + k_2$ ,  $k_1, k_2 \geq 0$ .

In [18], Lee defined the multiple Appell polynomials and found several equivalent conditions for this class of polynomials. A multiple polynomial set  $\{p_{k_1, k_2}(x)\}_{k_1, k_2=0}^{\infty}$  is called multiple Appell if there exists a generating function of the form:

$$A(t_1, t_2)e^{x(t_1+t_2)} = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} p_{k_1, k_2}(x) \frac{t_1^{k_1} t_2^{k_2}}{k_1! k_2!}, \quad (1.4)$$

where  $A(t_1, t_2)$  is defined as

$$A(t_1, t_2) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} a_{k_1, k_2} \frac{t_1^{k_1} t_2^{k_2}}{k_1! k_2!}. \quad (1.5)$$

The generalization of the Szász operators involving multiple Appell polynomials has been studied in [21]. The Jakimovski-Leviatan-Durrmeyer type operators involving multiple Appell polynomial are introduced in [2] and investigate Korovkin type approximation theorem and rate of convergence. Some properties of the generalized Szász operators by multiple Appell polynomials are given in [3] taking into consideration the power summability method.

Inspired by the above works, we construct the generalization of the Szász operators involving the multiple Sheffer polynomials. The paper is organized as follows: In Section 2, the multiple Sheffer polynomials are introduced and the positive linear operators  $G_n(f; x)$  involving multiple Sheffer polynomials are constructed. In Section 3, some auxiliary results for the operators  $G_n(f; x)$  are given. Section 4 is discusses approximation properties of the operators  $G_n(f; x)$  and the convergence theorem. In Section 5, the convergence of these operators are also discussed in weighted spaces of functions on the positive semi-axis and estimate the approximation with the help of weighted modulus of continuity. In order to show the relevance of the results, in the last section some numerical examples are given.

## 2 Multiple Sheffer polynomials

In this section, we introduce the multiple Sheffer polynomials as follows:

**Definition 2.1** *The multiple Sheffer polynomials set  $\{S_{k_1, k_2}(x)\}_{k_1, k_2=0}^{\infty}$  possesses the following generating function:*

$$A(t_1, t_2)e^{xH(t_1, t_2)} = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} S_{k_1, k_2}(x) \frac{t_1^{k_1} t_2^{k_2}}{k_1! k_2!}, \quad (2.1)$$

where  $A(t_1, t_2)$  and  $H(t_1, t_2)$  have series expansions of the form

$$A(t_1, t_2) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} a_{k_1, k_2} \frac{t_1^{k_1} t_2^{k_2}}{k_1! k_2!} \quad (2.2)$$

and

$$H(t_1, t_2) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} h_{k_1, k_2} \frac{t_1^{k_1} t_2^{k_2}}{k_1! k_2!}, \quad (2.3)$$

respectively with the conditions that

$$a_{0,0} \neq 0, \quad h_{1,1} \neq 0. \quad (2.4)$$

For  $A(t_1, t_2) = e^{\frac{\delta}{2}(t_1+t_2)^2 + \alpha_1 t_1 + \alpha_2 t_2}$  and  $H(t_1, t_2) = \delta(t_1 + t_2)$ , the multiple Sheffer polynomials  $S_{k_1, k_2}(x)$  become the multiple Hermite polynomials  $H_{k_1, k_2}^{(\alpha_1, \alpha_2)}(x)$  defined by the generating function [17]:

$$e^{\frac{\delta}{2}(t_1+t_2)^2 + \alpha_1 t_1 + \alpha_2 t_2 + \delta(t_1+t_2)x} = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} H_{k_1, k_2}^{(\alpha_1, \alpha_2)}(x) \frac{t_1^{k_1} t_2^{k_2}}{k_1! k_2!}, \quad (2.5)$$

with  $\delta < 0$  and  $\alpha_1 \neq \alpha_2$ . Note that if we take  $t_2 = 0$ , we get the generating function for the classical Hermite polynomials.

For  $A(t_1, t_2) = \frac{1}{(1-t_1-t_2)^{\alpha+1}}$  and  $H(t_1, t_2) = \frac{\beta_1 t_1 + \beta_2 t_2}{1-t_1-t_2}$ , the multiple Sheffer polynomials  $S_{k_1, k_2}(x)$  become the multiple Laguerre polynomials  $L_{k_1, k_2}^{(\alpha; \beta_1, \beta_2)}(x)$  defined by the generating function [17]:

$$\frac{1}{(1-t_1-t_2)^{\alpha+1}} e^{\left(\frac{\beta_1 t_1 + \beta_2 t_2}{1-t_1-t_2}\right)x} = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} L_{k_1, k_2}^{(\alpha; \beta_1, \beta_2)}(x) \frac{t_1^{k_1} t_2^{k_2}}{k_1! k_2!}, \quad (2.6)$$

with  $\alpha > -1$  and  $\beta_1 \neq \beta_2$ . Note that if we take  $t_2 = 0$ , we get the generating function for the classical Laguerre polynomials.

Now, we construct the positive linear operators involving multiple Sheffer polynomials  $S_{k_1, k_2}(x)$ . Throughout the paper, the following abbreviations for the partial derivatives will be used:

$$\frac{\partial A}{\partial t_i} = A_{t_i}, \quad \frac{\partial H}{\partial t_i} = H_{t_i} \quad (2.7)$$

and

$$\frac{\partial^2 A}{\partial t_i \partial t_j} = A_{t_i, t_j}, \quad \frac{\partial^2 H}{\partial t_i \partial t_j} = H_{t_i, t_j}, \quad i, j = 1, 2. \quad (2.8)$$

Let us consider the following conditions on  $S_{k_1, k_2}(x)$ :

- (i)  $S_{k_1, k_2}(x) \geq 0, k_1, k_2 \in \mathbb{N}$ ,
- (ii)  $A(1, 1) \neq 0, H_{t_1}(1, 1) = 1, H_{t_2}(1, 1) = 1$ ,
- (iii) (2.1), (2.2) and (2.3) converge for  $|t_1| < R_1, |t_2| < R_2$  ( $R_1, R_2$ )  $> 1$ .

Under the above conditions, we construct a positive linear operator involving multiple Sheffer polynomials for  $x \in [0, \infty)$  as follows:

$$G_n(f; x) = \frac{e^{-\frac{nx}{2}H(1,1)}}{A(1,1)} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{S_{k_1,k_2}(\frac{nx}{2})}{k_1! k_2!} f\left(\frac{k_1+k_2}{n}\right), \quad (2.10)$$

provided that the right-hand side of (2.10) exists.

**Remark 2.1** *It is to be noted that if we consider  $H(t_1, t_2) = t_1 + t_2$ , the multiple Sheffer polynomials  $S_{k_1,k_2}(x)$  become the multiple Appell polynomials  $p_{k_1,k_2}(x)$  defined by (1.4). Therefore, for  $H(t_1, t_2) = t_1 + t_2$  the operators (2.10) reduce to the operators involving multiple Appell polynomials defined by Varma [21].*

**Remark 2.2** *For  $A(t_1, t_2) = 1$ ,  $H(t_1, t_2) = t_1 + t_2$  and then  $t_2 = 0$ , the multiple Sheffer polynomials  $S_{k_1,k_2}(x)$  reduce to  $x^k$ . Therefore, the operators (2.10) reduce to the Szász operators (1.1).*

### 3 Auxiliary results

Note that throughout the paper we will assume that the operators  $G_n(f; x)$  are positive and we use the following test functions:

$$e_i(x) = x^i; \quad i \in \{0, 1, 2\}.$$

**Lemma 3.1** *From (2.1), the following hold:*

$$\begin{aligned} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{S_{k_1,k_2}(\frac{nx}{2})}{k_1! k_2!} &= A(1,1) e^{\frac{nx}{2}H(1,1)}, \\ \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} (k_1 + k_2) \frac{S_{k_1,k_2}(\frac{nx}{2})}{k_1! k_2!} &= [nx A(1,1) + A_{t_1}(1,1) + A_{t_2}(1,1)] e^{\frac{nx}{2}H(1,1)}, \\ \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} (k_1 + k_2)^2 \frac{S_{k_1,k_2}(\frac{nx}{2})}{k_1! k_2!} &= \left[ n^2 x^2 A(1,1) + nx \left( \frac{A(1,1)H_{t_1,t_1}(1,1) + A(1,1)H_{t_2,t_2}(1,1)}{2} \right. \right. \\ &\quad \left. \left. + A(1,1)H_{t_1,t_2}(1,1) + 2A_{t_1}(1,1) + 2A_{t_2}(1,1) + A(1,1) \right) \right. \\ &\quad \left. + A_{t_1,t_1}(1,1) + A_{t_2,t_2}(1,1) + 2A_{t_1,t_2}(1,1) \right. \\ &\quad \left. + A_{t_1}(1,1) + A_{t_2}(1,1) \right] e^{\frac{nx}{2}H(1,1)}. \end{aligned}$$

*Proof.* Taking the partial derivatives of the generating function (2.1) with respect to  $t_1$  and  $t_2$  and then letting  $t_1 = 1$ ,  $t_2 = 1$ , we obtain Lemma 3.1.

**Lemma 3.2** *The operators  $G_n$  defined by (2.10) satisfy*

$$G_n(e_0; x) = 1 \quad (3.1)$$

$$G_n(e_1; x) = x + \frac{A_{t_1}(1, 1) + A_{t_2}(1, 1)}{n A(1, 1)} \quad (3.2)$$

$$\begin{aligned} G_n(e_2; x) = & x^2 + \frac{x}{n} \left( 1 + \frac{H_{t_1, t_1}(1, 1) + H_{t_2, t_2}(1, 1)}{2} + \frac{2A_{t_1}(1, 1) + 2A_{t_2}(1, 1)}{A(1, 1)} + H_{t_1, t_2}(1, 1) \right) \\ & + \frac{A_{t_1, t_1}(1, 1) + A_{t_2, t_2}(1, 1) + 2A_{t_1, t_2}(1, 1) + A_{t_1}(1, 1) + A_{t_2}(1, 1)}{n^2 A(1, 1)}. \end{aligned} \quad (3.3)$$

*Proof.* In view of operators (2.10) and Lemma 3.1, the proof of Lemma 3.2 is straightforward.

By using the linearity property of the operators (2.10) and Lemma 3.2, it follows that:

$$G_n((e_1 - x); x) = \frac{A_{t_1}(1, 1) + A_{t_2}(1, 1)}{n A(1, 1)} \quad (3.4)$$

$$\begin{aligned} G_n((e_1 - x)^2; x) = & \frac{x}{n} \left( 1 + \frac{H_{t_1, t_1}(1, 1) + H_{t_2, t_2}(1, 1)}{2} + H_{t_1, t_2}(1, 1) \right) \\ & + \frac{A_{t_1, t_1}(1, 1) + A_{t_2, t_2}(1, 1) + 2A_{t_1, t_2}(1, 1) + A_{t_1}(1, 1) + A_{t_2}(1, 1)}{n^2 A(1, 1)}. \end{aligned} \quad (3.5)$$

## 4 Approximation results

In this section, we state our main theorem with the help of the universal Korovkin-type result and calculate the order of approximation by modulus of continuity.

P. P. Korovkin [16] has proved some remarkable results concerning the convergence of sequences  $(K_n(f, x))_{n=1}^{\infty}$ , where  $K_n(f, x)$  are positive linear operators. For example, if for  $f \in C[0, \infty)$ ,  $K_n(f, x)$  converges uniformly to  $f$  in the particular cases  $f(t) \equiv 1$ ,  $f(t) \equiv t$ ,  $f(t) \equiv t^2$ , then it does so for every continuous real  $f$ . First, we recall the following definitions and lemmas:

**Definition 4.1** *Let  $f \in \hat{C}[0, \infty)$  and  $\delta > 0$ . The modulus of continuity  $w(f; \delta)$  of the function  $f$  is defined by*

$$w(f; \delta) := \sup_{\substack{x, y \in [0, \infty) \\ |x - y| \leq \delta}} |f(x) - f(y)|, \quad (4.1)$$

where  $\hat{C}[0, \infty)$  is the space of uniformly continuous functions on  $[0, \infty)$ . Then, for any  $\delta > 0$  and each  $x \in [0, \infty)$ , it is well known that one can write

$$|f(x) - f(y)| \leq w(f; \delta) \left( \frac{|x - y|}{\delta} + 1 \right). \quad (4.2)$$

If  $f$  is uniformly continuous on  $[0, \infty)$ , then it is necessary and sufficient that

$$\lim_{\delta \rightarrow 0} w(f, \delta) = 0.$$

**Definition 4.2** The second modulus of continuity of the function  $f \in C_B[0, \infty)$  is defined by

$$w_2(f; \delta) := \sup_{0 < t \leq \delta} \|f(\cdot + 2t) - 2f(\cdot + t) + f(\cdot)\|_{C_B}, \quad (4.3)$$

where  $C_B[0, \infty)$  is the class of real-valued functions defined on  $[0, \infty)$ , which are bounded and uniformly continuous with the norm

$$\|f\|_{C_B} = \sup_{x \in [0, \infty)} |f(x)|. \quad (4.4)$$

We denote by  $C_E[0, \infty)$  the set of all continuous functions  $f$  on  $[0, \infty)$  with the property that  $|f(x)| \leq \beta e^{\alpha x}$  for all  $x \geq 0$  and some positive finite  $\alpha$  and  $\beta$ .

For  $h > 0$ , the function

$$f_h(t) = \frac{1}{h} \int_{t-\frac{h}{2}}^{t+\frac{h}{2}} f(u) \, du$$

as well as the iteratively defined functions

$$f_{h,r}(t) = \frac{1}{h} \int_{t-\frac{h}{2}}^{t+\frac{h}{2}} f_{h,r-1}(u) \, du, \quad r = 2, 3, \dots$$

$$f_{h,1}(t) = f_h(t)$$

are known as Steklov functions.

**Lemma 4.1** (Gavrea and Raşa [9]). Let  $g \in C^2[0, a]$  and  $(K_n)_{n \geq 0}$  be a sequence of positive linear operators with the property  $K_n(1; x) = 1$ . Then,

$$|K_n(g; x) - g(x)| \leq \|g'\| \sqrt{K_n((e_1 - x)^2; x)} + \frac{1}{2} \|g''\| K_n((e_1 - x)^2; x). \quad (4.5)$$

**Lemma 4.2** (Zhuk [22]). Let  $f \in C[a, b]$  and  $h \in (0, \frac{a-b}{2})$ . Let  $f_h$  be the second-order Steklov function attached to the function  $f$ . Then, the following inequalities are satisfied:

$$(i) \quad \|f_h - f\| \leq \frac{3}{4} w_2(f; h), \quad (4.6)$$

$$(ii) \quad \|f_h''\| \leq \frac{3}{2h^2} w_2(f; h). \quad (4.7)$$

**Theorem 4.1** Let  $f \in C_E[0, \infty)$ . Then

$$\lim_{n \rightarrow \infty} G_n(f; x) = f(x) \quad (4.8)$$

uniformly on each compact subset of  $[0, \infty)$ .



*Proof.* In view of Lemma 3.2, it follows that

$$\lim_{n \rightarrow \infty} G_n(e_i, x) = x^i, \quad i = 0, 1, 2 \quad (4.9)$$

uniformly on each compact subset of  $[0, \infty)$ . Application of Korovkin's theorem to (4.9) then establishes the desired result.

Next, we obtain the order of approximation of the operators  $G_n(f; x)$ .

**Theorem 4.2** *Let  $f \in C_E[0, \infty)$ . Then the operators  $G_n(f; x)$  satisfy the following inequality:*

$$|G_n(f; x) - f(x)| \leq \{1 + \sqrt{\lambda_n(x)}\} w(f; \frac{1}{\sqrt{n}}), \quad (4.10)$$

where

$$\begin{aligned} \lambda_n(x) = & x \left( 1 + \frac{H_{t_1, t_1}(1, 1) + H_{t_2, t_2}(1, 1)}{2} + H_{t_1, t_2}(1, 1) \right) \\ & + \frac{A_{t_1, t_1}(1, 1) + A_{t_2, t_2}(1, 1) + 2A_{t_1, t_2}(1, 1) + A_{t_1}(1, 1) + A_{t_2}(1, 1)}{n A(1, 1)}. \end{aligned}$$

*Proof.* In view of the modulus of continuity (4.2), we have

$$\begin{aligned} |G_n(f; x) - f(x)| & \leq \frac{e^{-\frac{nx}{2}H(1,1)}}{A(1,1)} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{S_{k_1, k_2}(\frac{nx}{2})}{k_1! k_2!} \left| f\left(\frac{k_1 + k_2}{n}\right) - f(x) \right| \\ & \leq w(f; \delta) \left\{ 1 + \frac{1}{\delta} \frac{e^{-\frac{nx}{2}H(1,1)}}{A(1,1)} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{S_{k_1, k_2}(\frac{nx}{2})}{k_1! k_2!} \left| \frac{k_1 + k_2}{n} - x \right| \right\} \quad (4.11) \end{aligned}$$

By considering the Cauchy-Schwarz inequality, we find

$$\begin{aligned} & \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{S_{k_1, k_2}(\frac{nx}{2})}{k_1! k_2!} \left| \frac{k_1 + k_2}{n} - x \right| \\ & \leq \sqrt{A(1,1)e^{xH(1,1)}} \left\{ \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{S_{k_1, k_2}(\frac{nx}{2})}{k_1! k_2!} \left( \frac{k_1 + k_2}{n} - x \right)^2 \right\}^{\frac{1}{2}} \\ & = A(1,1)e^{xH(1,1)} \left\{ \frac{x}{n} \left( 1 + \frac{H_{t_1, t_1}(1, 1) + H_{t_2, t_2}(1, 1)}{2} + H_{t_1, t_2}(1, 1) \right) \right. \\ & \quad \left. + \frac{A_{t_1, t_1}(1, 1) + A_{t_2, t_2}(1, 1) + 2A_{t_1, t_2}(1, 1) + A_{t_1}(1, 1) + A_{t_2}(1, 1)}{n^2 A(1, 1)} \right\}^{\frac{1}{2}}. \quad (4.12) \end{aligned}$$

Use of (4.12) in (4.11) then leads to

$$\begin{aligned} |G_n(f; x) - f(x)| & \leq w(f; \delta) \left\{ 1 + \frac{1}{\delta} \left[ \frac{x}{n} \left( 1 + \frac{H_{t_1, t_1}(1, 1) + H_{t_2, t_2}(1, 1)}{2} + H_{t_1, t_2}(1, 1) \right) \right. \right. \\ & \quad \left. \left. + \frac{A_{t_1, t_1}(1, 1) + A_{t_2, t_2}(1, 1) + 2A_{t_1, t_2}(1, 1) + A_{t_1}(1, 1) + A_{t_2}(1, 1)}{n^2 A(1, 1)} \right] \right\}^{\frac{1}{2}}. \quad (4.13) \end{aligned}$$

On choosing  $\delta = \frac{1}{\sqrt{n}}$ , assertion (4.10) follows.

**Theorem 4.3** *For  $f \in C[0, \alpha]$ , the following inequality is satisfied:*

$$|G_n(f; x) - f(x)| \leq \frac{2}{\alpha} \|f\| l^2 + \frac{3}{4}(\alpha + 2 + l^2)w_2(f; l), \quad (4.14)$$

where

$$l := l_n(x) = \sqrt[4]{G_n((e_1 - x)^2; x)}$$

and the second-order modulus of continuity is given by  $w_2(f; l)$  with the norm  $\|f\| = \max_{x \in [a, b]} |f(x)|$

*Proof.* Let  $f_l$  be the second-order Steklov function attached to the function  $f$ . In view of the identity (3.1), we have

$$\begin{aligned} |G_n(f; x) - f(x)| &\leq |G_n(f - f_l; x)| + |G_n(f_l; x) - f_l(x)| + |f_l(x) - f(x)| \\ &\leq 2\|f_l - f\| + |G_n(f_l; x) - f_l(x)|, \end{aligned} \quad (4.15)$$

which on using inequality (4.6) becomes

$$|G_n(f; x) - f(x)| \leq \frac{3}{2}w_2(f; l) + |G_n(f_l; x) - f_l(x)|. \quad (4.16)$$

Taking into account that  $f_l \in C^2[0, \alpha]$ , from Lemma 4.1, it follows that

$$|G_n(f_l; x) - f_l(x)| \leq \|f'_l\| \sqrt{G_n((e_1 - x)^2; x)} + \frac{1}{2} \|f''_l\| G_n((e_1 - x)^2; x), \quad (4.17)$$

which in view of inequality (4.7) becomes

$$|G_n(f_l; x) - f_l(x)| \leq \|f'_l\| \sqrt{G_n((e_1 - x)^2; x)} + \frac{3}{4l^2} w_2(f; l) G_n((e_1 - x)^2; x). \quad (4.18)$$

Further, the Landau inequality

$$\|f'_l\| \leq \frac{2}{\alpha} \|f_l\| + \frac{\alpha}{2} \|f''_l\|,$$

and Lemma 4.2 gives

$$\|f'_l\| \leq \frac{2}{\alpha} \|f\| + \frac{3\alpha}{4l^2} w_2(f; l). \quad (4.19)$$

Using inequality (4.19) in inequality (4.18) and taking  $l = \sqrt[4]{G_n((e_1 - x)^2; x)}$ , we find

$$|G_n(f_l; x) - f_l(x)| \leq \frac{2}{\alpha} \|f\| l^2 + \frac{3}{4}(\alpha + l^2)w_2(f; l). \quad (4.20)$$

Use of the inequality (4.20) in (4.16), then establishes the assertion (4.14).

## 5 Weighted Approximation

In this section, the approximation properties of the operators  $G_n(f; x)$  of the weighted spaces of continuous functions are given with exponential growth on  $\mathbb{R}_0^+ = [0, \infty)$  with the help of the weighted Korovkin type theorem established in [7, 8]. First, we consider the following weighted spaces of functions which are defined on the  $[0, \infty)$ .

Let  $\rho(x)$  be the weight function and  $M_f$  be a positive constant, then we define

$$\begin{aligned} B_\rho(\mathbb{R}_0^+) &= \{f : \mathbb{R}_0^+ \rightarrow \mathbb{R} : |f(x)| < M_f \rho(x)\} \\ C_\rho(\mathbb{R}_0^+) &= \{f \in B_\rho(\mathbb{R}_0^+) : f \text{ is continuous}\} \\ C_\rho^k(\mathbb{R}_0^+) &= \left\{f \in C_\rho(\mathbb{R}_0^+) : \lim_{n \rightarrow \infty} \frac{f(x)}{\rho(x)} = K_f < \infty\right\}. \end{aligned}$$

It should be noted that  $C_\rho^k(\mathbb{R}_0^+) \subset C_\rho(\mathbb{R}_0^+) \subset B_\rho(\mathbb{R}_0^+)$ . The space  $B_\rho(\mathbb{R}_0^+)$  is a linear normed space with the following norm:

$$\|f\|_\rho = \sup_{x \in \mathbb{R}_0^+} \frac{|f(x)|}{\rho(x)}.$$

The following results on the sequence of positive linear operators in these spaces are given [7, 8].

**Lemma 5.1** [7, 8] *The sequence of positive linear operators  $(L_n)_{n \geq 1}$  which act from  $C_\rho(\mathbb{R}_0^+)$  to  $B_\rho(\mathbb{R}_0^+)$  if and only if there exists a positive constant  $k$  such that*

$$L_n(\rho, x) \leq k\rho(x), \text{ i.e.}$$

$$\|L_n(\rho, x)\|_\rho \leq k.$$

**Theorem 5.1** [7, 8] *Let  $(L_n)_{n \geq 1}$  be the sequence of positive linear operators that act from  $C_\rho(\mathbb{R}_0^+)$  to  $B_\rho(\mathbb{R}_0^+)$  such that*

$$\lim_{n \rightarrow \infty} \|L_n(t^i, x) - x^i\|_\rho = 0, \quad i \in \{0, 1, 2\}.$$

*Then for any  $f \in C_\rho^k(\mathbb{R}_0^+)$*

$$\|L_n f - f\|_\rho = 0.$$

Now prove the following lemma:

**Lemma 5.2** *Let  $\rho(x) = 1 + x^2$  be a weight function. If  $f \in C_\rho^k(\mathbb{R}_0^+)$ , there exists a positive constant  $M$  such that*

$$\|G_n(\rho; x)\|_\rho \leq 1 + M.$$

*Proof.* From Lemma 3.2, it follows that

$$\begin{aligned} \|G_n(\rho; x)\|_\rho &= \sup_{x \geq 0} \left\{ \frac{1}{1+x^2} \left( 1+x^2 + \frac{x}{n} \Theta_{t_1, t_2} + \frac{1}{n^2} \Phi_{t_1, t_2} \right) \right\} \\ &\leq 1 + \frac{x}{n} \Theta_{t_1, t_2} + \frac{1}{n^2} \Phi_{t_1, t_2}, \end{aligned}$$

where

$$\Theta_{t_1, t_2} = 1 + \frac{H_{t_1, t_1}(1, 1) + H_{t_2, t_2}(1, 1)}{2} + \frac{2A_{t_1}(1, 1) + 2A_{t_2}(1, 1)}{A(1, 1)} + H_{t_1, t_2}(1, 1)$$

and

$$\Phi_{t_1, t_2} = \frac{A_{t_1, t_1}(1, 1) + A_{t_2, t_2}(1, 1) + 2A_{t_1, t_2}(1, 1) + A_{t_1}(1, 1) + A_{t_2}(1, 1)}{A(1, 1)}.$$

Since  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$  and  $\lim_{n \rightarrow \infty} \frac{1}{n^2} = 0$  there exists a positive constant  $M$  such that

$$G_n(\rho; x) \leq 1 + M.$$

This completes the proof.

From Lemma 5.2, it is easily verified that the operators  $G_n$  defined by (2.10) act from  $C_\rho(\mathbb{R}_0^+)$  to  $B_\rho(\mathbb{R}_0^+)$ .

**Theorem 5.2** *Let  $\rho(x) = 1 + x^2$ . Then, for each  $f \in C_\rho^k(\mathbb{R}_0^+)$ ,*

$$\lim_{n \rightarrow \infty} \|G_n(f; x) - f(x)\|_\rho = 0.$$

*Proof.* It is sufficient to show the conditions of the weighted form of Korovkin type approximation theorem proved in Theorem 5.1, are satisfied. From (3.1), it is immediately seen that

$$\lim_{n \rightarrow \infty} \|G_n(e_0; x) - e_0\|_\rho = 0. \quad (5.1)$$

Using (3.2), we have

$$\|G_n(e_1; x) - e_1\|_\rho = \frac{A_{t_1}(1, 1) + A_{t_2}(1, 1)}{n A(1, 1)},$$

which implies that

$$\lim_{n \rightarrow \infty} \|G_n(e_1; x) - e_1(x)\|_\rho = 0. \quad (5.2)$$

By means of (3.3), we get

$$\|G_n(e_2; x) - e_2(x)\|_\rho \leq \frac{x}{n} \Theta_{t_1, t_2} + \frac{1}{n^2} \Phi_{t_1, t_2},$$

where

$$\Theta_{t_1, t_2} = 1 + \frac{H_{t_1, t_1}(1, 1) + H_{t_2, t_2}(1, 1)}{2} + \frac{2A_{t_1}(1, 1) + 2A_{t_2}(1, 1)}{A(1, 1)} + H_{t_1, t_2}(1, 1)$$

and

$$\Phi_{t_1, t_2} = \frac{A_{t_1, t_1}(1, 1) + A_{t_2, t_2}(1, 1) + 2A_{t_1, t_2}(1, 1) + A_{t_1}(1, 1) + A_{t_2}(1, 1)}{A(1, 1)}.$$

Therefore,

$$\lim_{n \rightarrow \infty} \|G_n(e_2; x) - e_2(x)\|_\rho = 0. \quad (5.3)$$

From equations (5.1), (5.2) and (5.3), it follows that

$$\lim_{n \rightarrow \infty} \|G_n(e_\nu; x) - e_\nu(x)\|_\rho = 0, \quad \nu \in \{0, 1, 2\}.$$

In view of Theorem 5.1, we obtain the desired result.

## 6 Examples

**Example 6.1** The case  $\{(x+1)^{k_1+k_2}\}_{k_1, k_2=0}^\infty$

By taking  $A(t_1, t_2) = e^{t_1+t_2}$  and  $H(t_1, t_2) = t_1 + t_2$ , the multiple Sheffer polynomials  $S_{k_1, k_2}(x)$  reduce to the polynomials  $\{(x+1)^{k_1+k_2}\}_{k_1, k_2=0}^\infty$  defined by the generating function:

$$e^{(x+1)(t_1+t_2)} = \sum_{k_1=0}^\infty \sum_{k_2=0}^\infty (x+1)^{k_1+k_2} \frac{t_1^{k_1} t_2^{k_2}}{k_1! k_2!}. \quad (6.1)$$

It is clear that  $(x+1)^{k_1+k_2} \geq 0$  for  $x \in (0, \infty]$ ,  $A(1, 1) = e^2 \neq 0$  and  $H_{t_1}(1, 1) = H_{t_2}(1, 1) = 1$ . Therefore, the conditions (2.9) are satisfied.

The generalized Szász operators (2.10) involving the function  $\{(x+1)^{k_1+k_2}\}_{k_1, k_2=0}^\infty$  are obtained as follows:

$$G_n(f; x) = e^{-2-nx} \sum_{k_1=0}^\infty \sum_{k_2=0}^\infty \frac{\left(\frac{nx}{2} + 1\right)^{k_1+k_2}}{k_1! k_2!} f\left(\frac{k_1+k_2}{n}\right). \quad (6.2)$$

**Lemma 6.1** For the operators  $G_n(f; x)$  given by (6.2) and  $x \in [0, \infty)$ , we have

$$G_n(e_0; x) = 1 \quad (6.3)$$

$$G_n(e_1; x) = x + \frac{2}{n} \quad (6.4)$$

$$G_n(e_2; x) = x^2 + \frac{5x}{n} + \frac{6}{n^2}. \quad (6.5)$$

**Lemma 6.2** For the operators  $G_n(f; x)$  given by (6.2) and  $x \in [0, \infty)$ , the following identities are satisfied:

$$G_n((e_1 - 1); x) = \frac{2}{n}, \quad (6.6)$$

$$G_n((e_1 - x)^2; x) = \frac{x}{n} + \frac{6}{n^2}. \quad (6.7)$$

For  $n = 20, 30, 50$ , the convergence of the operators (6.2) to the functions

$$f(x) = \left(x - \frac{1}{2}\right) \left(x - \frac{1}{3}\right) \quad (6.8a)$$

$$f(x) = -4x^3 \quad (6.8b)$$

is illustrated in Figs. 1 and 2, respectively.

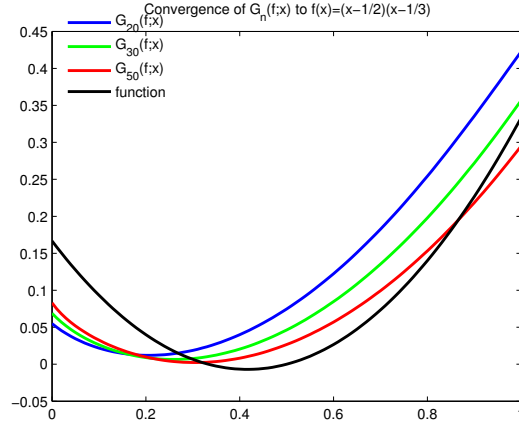


Figure 1: Convergence of the operators (6.2) to  $f(x) = \left(x - \frac{1}{2}\right) \left(x - \frac{1}{3}\right)$

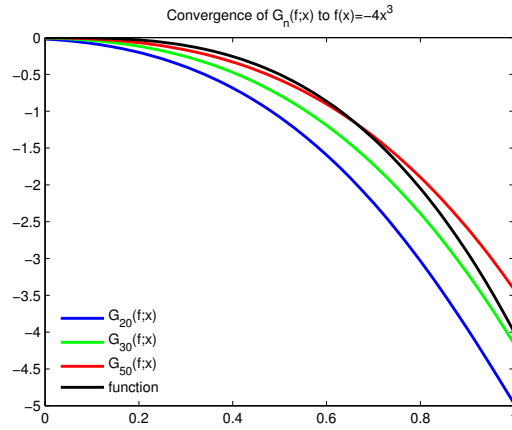


Figure 2: Convergence of the operators (6.2) to  $f(x) = -4x^3$

We now compute the error estimation by using modulus of continuity for operators (6.2) to functions (6.8a) and (6.8b) with the help of Matlab. Making use of the expression (6.7) in

Lemma 4.1, we find that

$$|G_n(f; x) - f(x)| \leq \|f'\| \sqrt{\frac{x}{n} + \frac{6}{n^2}} + \frac{1}{2} \|f''\| \left( \frac{x}{n} + \frac{6}{n^2} \right). \quad (6.9)$$

From inequality (6.9), the error bounds for the approximation by the operators (6.2) to functions (6.8a) and (6.8b) are obtained using Matlab. These error bounds are given in Tables 1 and 2.

**Table 1: Error bounds by operators (6.2) to  $(x - \frac{1}{2})(x - \frac{1}{3})$**

$n$	error bound at $x = 0.2$	error bound at $x = 0.5$	error bound at $x = 0.8$
20	0.0885	0.0608	0.2148
30	0.0600	0.0405	0.1600
50	0.0391	0.0260	0.1144

**Table 2: Error bounds by operators (6.2) to  $-4x^3$**

$n$	error bound at $x = 0.2$	error bound at $x = 0.5$	error bound at $x = 0.8$
20	0.1169	0.7025	1.9651
30	0.0748	0.5066	1.4795
50	0.0462	0.3535	1.0728

**Example 6.2** The case  $\{(k_1 + k_2)(x)^{k_1+k_2-1}\}_{k_1, k_2=1}^{\infty}$

The multiple Sheffer polynomials  $S_{k_1, k_2}(x)$  reduce to the polynomials  $\{(k_1 + k_2)(x)^{k_1+k_2-1}\}_{k_1, k_2=1}^{\infty}$  if we consider  $A(t_1, t_2) = t_1 + t_2$  and  $H(t_1, t_2) = t_1 + t_2$ , defined by the generating function:

$$(t_1 + t_2)e^{x(t_1+t_2)} = \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} (k_1 + k_2)(x)^{k_1+k_2-1} \frac{t_1^{k_1} t_2^{k_2}}{k_1! k_2!}. \quad (6.10)$$

It is clear that  $(k_1 + k_2)(x)^{k_1+k_2-1} \geq 0$  for  $x \in (0, \infty]$ ,  $A(1, 1) = 2 \neq 0$  and  $H_{t_1}(1, 1) = H_{t_2}(1, 1) = 1$ . Hence, the restrictions (2.9) are satisfied. The operators (2.10) involving  $\{(k_1 + k_2)(x)^{k_1+k_2-1}\}_{k_1, k_2=1}^{\infty}$  are given as:

$$G_n(f; x) = \frac{e^{-nx}}{2} \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \frac{\left(\frac{nx}{2}\right)^{k_1+k_2-1}}{k_1! k_2!} f\left(\frac{k_1 + k_2}{n}\right). \quad (6.11)$$

**Lemma 6.3** The operators  $G_n(f; x)$  given by (6.11) and  $x \in [0, \infty)$ , satisfy

$$G_n(e_0; x) = 1 \quad (6.12)$$

$$G_n(e_1; x) = x + \frac{1}{n} \quad (6.13)$$

$$G_n(e_2; x) = x^2 + \frac{3x}{n} + \frac{1}{n^2}. \quad (6.14)$$

**Lemma 6.4** For the operators  $G_n(f; x)$  given by (6.11) and  $x \in [0, \infty)$ , the following hold:

$$G_n((e_1 - 1); x) = \frac{1}{n}, \quad (6.15)$$

$$G_n((e_1 - x)^2; x) = \frac{x}{n} + \frac{1}{n^2}. \quad (6.16)$$

In Figs. 3 and 4 the convergence of the operators (6.11) to the functions (6.8a) and (6.8b), respectively, is illustrated.

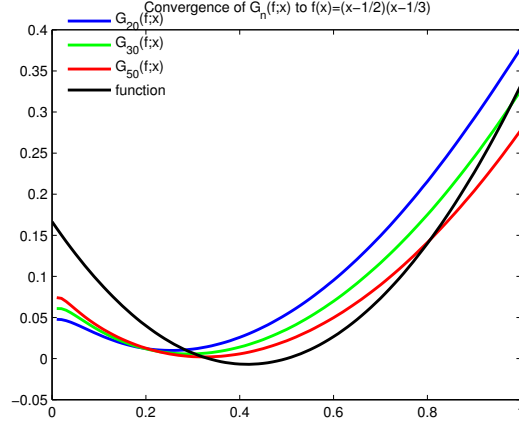


Figure 3: Convergence of the operators (6.11) to  $f(x) = (x - \frac{1}{2})(x - \frac{1}{3})$

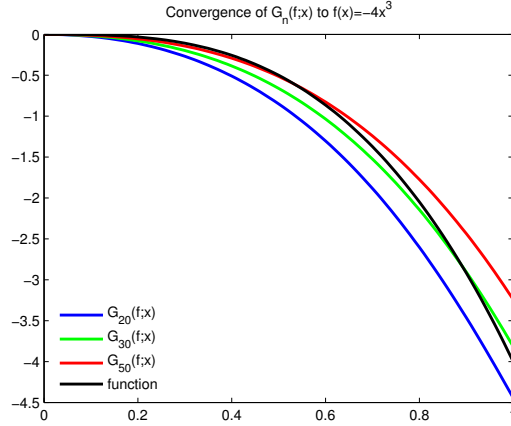


Figure 4: Convergence of the operators (6.11) to  $f(x) = -4x^3$

The following holds for (6.11):

$$|G_n(f; x) - f(x)| \leq \|f'\| \sqrt{\frac{x}{n} + \frac{1}{n^2}} + \frac{1}{2} \|f''\| \left( \frac{x}{n} + \frac{1}{n^2} \right) \quad (6.17)$$



From inequality (6.17), the error bounds for the approximation by operators (6.11) are given in Tables 3 and 4. Inspection of Tables 1 – 4 shows that the absolute error decreases with increasing  $n$ .

**Table 3: Error bounds by operators (6.11) to  $(x - \frac{1}{2})(x - \frac{1}{3})$**

$n$	error bound at $x = 0.2$	error bound at $x = 0.5$	error bound at $x = 0.8$
20	0.0559	0.0426	0.1806
30	0.0427	0.0317	0.1422
50	0.0311	0.0224	0.1066

**Table 4: Error bounds by operators (6.11) to  $-4x^3$**

$n$	error bound at $x = 0.2$	error bound at $x = 0.5$	error bound at $x = 0.8$
20	0.0647	0.5250	1.6273
30	0.0483	0.4150	1.3040
50	0.0348	0.3133	0.9954

**Note:** It can be seen that when the values of  $n$  are increasing, the graph of operators  $G_n(f; x)$  are converging to the graph of the function  $f$ . Graphical and numerical results have shown us that the proposed operators provide better approximation properties and good convergence behavior with the increased value of  $n$ .

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**Conflict of interest.** The authors declare that there is no conflict of interest.

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