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INVOLVING NIKIFOROV-UVAROV METHOD IN SCHRODINGER EQUATION OBTAINING HARTMANN POTENTIAL[†]

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The total wave function and the bound state energy are investigated by involving Nikiforov-Uvarov method to Schrodinger equation in spherical coordinates employing Hartmann Potential (HP). The HP is considered as non-central potential that is mostly recognized in nuclear field potentials. Every wave function is specified by principal quantum number n, angular momentum number l, and magnetic quantum number m. The radial part of the wave function is obtained in terms of the associated Laguerre polynomial, using the coordinate transformation $x = \cos\theta$ to obtain the angular wave function that depends on inverse associated Legendre polynomials.
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INTRODUCTION

The Hartmann Potential is a kind of non-central potentials that have been studied in nuclear physics field, which consider as the coulomb potential surrounded by the ring-shaped inverse square potential. Organic molecules such as cyclic polyenes and benzene, handling this potential since 1972 [1-2]. In spherical coordinate the Hartmann potential is coulomb potential adding a one potential proportional to $(r\tan\theta)^{-2}$. So, it defined by [3-4]:

$$V(r,\theta) = -\frac{\alpha}{r} + \frac{\beta\cos^2\theta}{r^2\sin^2\theta}. \tag{1}$$

Where α, β are constants that consider as positive real numbers. Obtaining this HP in Schrodinger equation, assuming $(\mu = \hbar = 1)$ [5]:

$$\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2}{\partial\phi^2}\right) + \frac{2\alpha}{r} - 2\beta\frac{\cos^2\theta}{r^2\sin^2\theta}\right\}\psi(r,\theta,\varphi) + 2E\psi(r,\theta,\varphi) = 0. \tag{2}$$

The equation (2) depends on the total coordinates in spherical coordinates (r,θ,φ), to find the total wave function we need apply separation of variables method on equation (2). The main object in this work is to determine the bound state energy and the wave function.

SEPARATION OF VARIABLES METHOD

Obviously, in spherical potential, we let [6]:

$$\psi(r,\theta,\varphi) = \frac{U(r)}{r}\,H(\theta)\,\phi(\varphi). \tag{3}$$

Now, by Separating variables in equation (2), we hold that:

The Radial Part

$$\frac{d^2U(r)}{dr^2} + \left[2E + 2\frac{\alpha}{r} - \frac{\lambda}{r^2}\right]U(r) = 0,$$

where λ is separation constant. And, to solve the equation, we let

$$\lambda = l`(l` + 1) - 2\beta.$$

$$\beta = \frac{m`^2 - m^2}{2}.$$

Where $l`$ and $m`$ are positive integers or zero.

$$L = \frac{1}{2}\left\{\sqrt{1 + 4\left[\left(k + \sqrt{2\beta + m^2}\right)\left(k + \sqrt{2\beta + m^2} + 1\right) - 2\beta\right]} - 1\right\}. \tag{4}$$

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So, one can easily show that $\lambda = L(L+1)$ (see Appendix I)

The radial Schrodinger equation can displayed as:

$$\frac{d^2U(r)}{dr^2} + \left[2E + 2\frac{\alpha r}{r} - \frac{L(L+1)}{r^2}\right]U(r) = 0. \tag{5}$$

Where the angular part can displayed as:

The Angular part

Insert a new variable $x = \cos\theta$ and using Chain rules technique, to get the angular Schrodinger equation.

$$\frac{d^2H(x)}{dx^2} - \frac{2x}{1-x^2}\frac{dH(x)}{dx} + \frac{1}{1-x^2}\left(\lambda + 2\beta - \frac{m^2+2\beta}{1-x^2}\right)H(x) = 0. \tag{6}$$

And we let, $\lambda + 2\beta = l`(l`+1)$, $m`^2 = m^2 + 2\beta$ and $v` = l`(l`+1)$.

After applying all assumes into equation (6), we get

$$\frac{d^2H(x)}{dx^2} - \frac{2x}{1-x^2}\left\{\frac{dH(x)}{dx}\right\} + \frac{(-v`x^2+v`-m`^2)}{(1-x^2)^2}H(x) = 0\,. \tag{7}$$

The Azimuthal Part

$$\frac{1}{\phi(\varphi)}\left(\frac{d^2}{d\phi^2}\right)\phi(\varphi) = m^2. \tag{8}$$

Where m is the magnetic quantum number.

NIKIFOROV-UVAROV METHOD

The Nikiforov-Uvarov method is a one of the methods used to predict the solutions of generalized second order liner differential equation like Schrodinger equation with particular orthogonal function, we could be obtaining the solution by NU-method when make some transforming to Schrodinger equation to be the same of the below equation [7]

$$u``(z) + \frac{\check{\tau}(z)}{\sigma(z)}u`(z) + \frac{\breve{\sigma}(z)}{\sigma^2(z)}u(z) = 0. \tag{9}$$

Where equation (9) is considering the standard form of NU-method. Where, $\sigma(z)$ and $\breve{\sigma}(z)$ are polynomials with a maximum degree of 2; $\check{\tau}(z)$is polynomial with a maximum degree of 1; $u(z)$ is a hypergeometric function type, and the primes intending the derivatives respect to z. by supposing that:

$$u(z) = \phi(z)X(z). \tag{10}$$

The equation (9) become as hypergeometric form:

$$\sigma(z)X``(z) + \tau(z)X`(z) + \lambda\,X(z) = 0. \tag{11}$$

Where

$$\tau(z) = 2\pi(z) + \check{\tau}(z) \qquad , \frac{d}{dz}\tau(z) < 0. \tag{12}$$

Where $\pi(z)$ is a parameter of 1st polynomial degree and introduces by equation (13):

$$\pi(z) = \frac{\sigma` - \check{\tau}}{2} \pm \sqrt{\left(\frac{\sigma` - \check{\tau}}{2}\right)^2 - \hat{\sigma}(z) + k\sigma}. \tag{13}$$

While λ is introduced by equation (14)

$$\lambda = k + \pi`(z). \tag{14}$$

Since $\pi(z)$ is 1st degree polynomial, this implies that second order function under square root must be equal to zero, then the quadratic equation can determine k.

To obtain $\phi(z)$ we can use the integral below equation:

$$\frac{\phi`(z)}{\phi(z)} = \frac{\pi(z)}{\sigma(z)}. \tag{15}$$

And the parameter λ in equation (14) defined by;

$$\lambda = \lambda_n = -n\tau`(z) - \frac{n(n-1)}{2}\sigma``(z). \tag{16}$$

The weight function $\rho(z)$ is obtained in (Eq. 17).

$$\frac{d}{dz}[\sigma(z)\,\rho(z)] = \tau(z)\,\rho(z). \tag{17}$$

While the Rodrigue relation (Eq. 18) is used to determine $X_n(z)$.

$$X_n(z) = \frac{B_n}{\rho(z)} \frac{d^n}{dz^n}[\sigma^n \rho(z)]. \tag{18}$$

Where $B_n$is normalization constant, now Substitute $X_n(z)$ and $\phi(z)$ into equation (10) to find $u(z)$ [8-12].

DEVELOPING HARTMANN POTENTIAL IN SCHRODINGER EQUATION

The Radial Schrodinger Equation

From equation (5) we can write the radial part by the below form

$$\frac{d^2U(r)}{dr^2} + \frac{1}{r^2}[2Er^2 + 2\alpha r - L(L+1)]U(r) = 0. \tag{19}$$

By comparing equation (9) by equation (19) to obtain the NU-Coefficients, we get

$$\check{\tau}\,(r) = 0, \tag{20-a}$$

$$\sigma(r) = r, \tag{20-b}$$

$$\breve{\sigma}\,(r) = 2Er^2 + 2\alpha r - L(L+1). \tag{20-c}$$

Using equations (20-a) (20-b) and (20-c) into equation (13) to get the parameter $\pi(r)$

$$\pi(r) = \frac{1}{2} \pm \frac{1}{2}\sqrt{-8Er^2 + 4(k-2\alpha)r + 4L(L+1) + 1}. \tag{21}$$

Now, taking this quadratic equation's discriminant equal zero, then the value of constant k could be determined.

$$k^2 - 4\alpha k + 4\alpha^2 + 2E\{4L(L+1)+1\} = 0. \tag{22}$$

The quadratic equation (22) provides two roots for k

$$k_\pm = 2\alpha \pm \sqrt{-2E\{4L(L+1)+1\}}. \tag{23}$$

Substituting equation (23) into equation (21), to get:

$$\pi_1(r) = \pi_{++}(r) = \frac{1}{2} + \left\{\sqrt{-2E}r + \left(L + \frac{1}{2}\right)\right\}, \tag{24-a}$$

$$\pi_2(r) = \pi_\pm(r) = \frac{1}{2} + \left\{\sqrt{-2E}r - \left(L + \frac{1}{2}\right)\right\}, \tag{24-b}$$

$$\pi_3(r) = \pi_\mp(r) = \frac{1}{2} - \left\{\sqrt{-2E}r + \left(L + \frac{1}{2}\right)\right\}, \tag{24-c}$$

$$\pi_4(r) = \pi_{--}\,(r) = \frac{1}{2} - \left\{\sqrt{-2E}r - \left(L + \frac{1}{2}\right)\right\}. \tag{24-d}$$

Taking $\pi_4(r)$ where τ(r) is negative in equation (12) to hold the well value by NU method; so:

$$\tau_4(r) = 1 - 2\left(\sqrt{-2E}r - \left(L + \frac{1}{2}\right)\right). \tag{25}$$

Returning to the equations (14) and (16) respectively, and developing equation (25) we get:

$$\lambda = 2\alpha - (2\mathrm{L}+2)\sqrt{-2E}. \tag{26}$$

$$\lambda_n = 2n\left(\sqrt{-2E}\right). \tag{27}$$

Comparing equations (26) and (27) one can predict bound state energy.

$$E = -\frac{\alpha^2}{2(L+n+1)^2}. \tag{28}$$

Where L is given by equation (4).

Depending on previous result especially equation 25 we can hold the function $\phi(r)$ and the weight function $\rho(r)$ that in equations (15) and (17) in a new form:

$$\phi(r) = r^{(L+1)} e^{-\sqrt{-2E}r} \tag{29}$$

$$\rho(r) = e^{-2\sqrt{-2E}r} r^{(2L+1)} \tag{30}$$

By obtaining equation (18) and (30); one can establish the polynomial $X_n(r)$:

$$X_n(r) = B_n e^{2\sqrt{-2E}r} r^{-(2L+1)} \frac{d^n}{dz^n}\left[e^{-2\sqrt{-2E}r} r^{(2L+1+n)}\right] \tag{31}$$

Also, the wave function $U(r)$ is hold by multiply $X_n(r)$ with $\phi(r)$.

$$U_{nl}(r) = B_n\, r^{(L+1)} e^{-\sqrt{-2E}r} e^{2\sqrt{-2E}r} r^{-(2L+1)}\ \frac{d^n}{dz^n}\left[e^{-2\sqrt{-2E}r} r^{(2L+1+n)}\right].$$

Involving associated Laguerre polynomial in equation [4] then $U_{nl}(r)$ can be defined as:

$$U_{nl}(r) = B_n\, r^{(L+1)} e^{-\sqrt{-2E}r}\, L_n^{2L+1}\left(2\sqrt{-2E}r\right). \tag{32}$$

By substituting $\sqrt{-2E}$ as in equation (28) where $\alpha = ze^2$, then the radial wave function $R(r)$, which defined as $R(r) = \frac{U(r)}{r}$ is obtained as:

$$R_{ln}(r) = B_n r^L e^{-\frac{ze^2}{(L+n+1)}r}\, L_n^{2L+1}\left(\frac{2ze^2}{(L+n+1)} r\right). \tag{33}$$

Where, B_n is the normalized constant for orthogonally associated Laguerre polynomial. So, the normalized constant equal

$$B_n = \sqrt{\frac{n!}{\Gamma(2L+n+2)}\, \frac{A^{2L+3}}{\{2n+2L+2\}}} \tag{34}$$

Substituting equation (34) to write the final form of radial Schrodinger equation

$$R_{nl}(r) = \sqrt{\frac{n!}{\Gamma(2L+n+2)}\, \frac{\frac{2ze^2}{(L+n+1)}^{2L+3}}{2\{n+L+1\}}}\, (r)^L e^{-\frac{\frac{2ze^2}{(L+n+1)}}{2}r}\, L_n^{2L+1}\left(\frac{2ze^2}{(L+n+1)} r\right) \tag{35}$$

The Angular Schrodinger Equation

Now, to determine the angular wave function $H(x)$; compare equation (7) with equation (9) to obtain

$$\check{\tau}\,(x) = -2x \tag{36-a}$$

$$\sigma(x) = 1 - x^2 \tag{36-b}$$

$$\check{\sigma}\,(x) = -v`x^2 + v` - m`^2 \tag{36-c}$$

Substitute equations $(36-a)$, $(36-b)$and $(36-c)$ into (13), then one can conclude $\pi(x)$:

$$\pi(x) = \pm\sqrt{(v` - k)x^2 - v` + m`^2 + k} \tag{37}$$

following the NU-method technic, both values of constant k.

$$k_1 = v` - m`^2\ ,\ k_2 = v` \tag{38}$$

So, the values of parameter $\pi(x)$ giving by;

$$\pi_{11}(x) = \sqrt{(m`^2)x^2} = m`x \tag{39-a}$$

$$\pi_{12}(x) = -\sqrt{(m`^2)x^2} = -m`x \tag{39-b}$$

$$\pi_{21}(x) = \sqrt{m`^2} = m` \tag{39-c}$$

$$\pi_{22}(x) = -\sqrt{m`^2} = -m` \tag{39-d}$$

Substituting the four values of $\pi(x)$ into equation (14) and equation (36-a), we obtain $\tau(z)$; where $\tau`_2(r) < 0$

$$\tau_2(x) = -2x - 2m`x \tag{40}$$

We have obtained the constants λ and λ_n from the equations (14) and (16) respectively.

$$\lambda = v` - m`^2 - m \tag{41}$$

$$\lambda_n = -n(-2 - 2m`) + n(n-1) \tag{42}$$

Comparing equation (41) with equation (42), we get:

$$l` = -\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\left\{n + \sqrt{m^2 + 2\beta}\right\}\left(n + \sqrt{m^2 + 2\beta} + 1\right)\}} \tag{43}$$

Now, depending on the upon result we return to use equations (15) and (17) to determine the function $\phi(r)$ and weight function $\rho(r)$

$$\frac{\phi`(x)}{\phi(x)} = \frac{-m`x}{1-x^2}$$

$$\frac{\rho`(r)}{\rho(r)} = \frac{\{-2x(1+m`)+2x\}}{1-x^2}$$

By integration the above equations we get;

$$\phi(x) = (1-x^2)^{\frac{m`}{2}} \tag{44}$$

$$\rho(x) = (1-x^2)^{m} \tag{45}$$

Now we can determine the polynomial $X_n(r)$ by equation (18) and (45).

$$X_n(x) = B_n(1-x^2)^{-m`} \frac{d^n}{dx^n}[(1-x^2)^{m`} \ (1-x^2)^n] \tag{46}$$

And by using $H(x) = X_n(x)\,\phi(x)$ that are defined by equations (46) and (44) and where

$$n + m` = l`$$

$$H_n(x) = (-1)^{l`} B_n\,(1-x^2)^{-\frac{m`}{2}} \ \frac{d^{l`-m`}}{dx^{l`-m`}}[(x^2-1)^{l`}] \tag{47}$$

By use some relations in associated Legendre polynomials

$$p_l^{-m`} = \frac{(-1)^{m`}}{2^l\,l!}(1-x^2)^{-\frac{m`}{2}}\,\frac{d^{l-m`}}{dx^{l-m`}}(x^2-1)^l = (-1)^{m`}\frac{(l-m`)!}{(l+m`)!}\,p_l^{m`}(x)$$

$$\frac{d^{l-m`}}{dx^{l-m`}}(x^2-1)^l = (-1)^{m`}\frac{(l-m`)!}{(l+m`)!}\,(1-x^2)^{m`}\,\frac{d^{l+m`}}{dx^{l+m`}}(x^2-1)^l \tag{48}$$

Now apply the equation (48) into equation (47);

$$H_n(x) = (-1)^{l`} B_n (1-x^2)^{-\frac{m`}{2}}\left\{c_{lm}(1-x^2)^{m`}\,\frac{d^{l+m`}}{dx^{l+m}}(x^2-1)^l\right\} \tag{49}$$

Where:

$$c_{lm`} = (-1)^{m`}\frac{(l-m`)!}{(l+m`)!} \tag{50}$$

Where associated Legendre polynomials is giving by equation [13];
So, equation (49) become:

$$H_n(x) = (-1)^{l`} B_n\,c_{lm`}\,\frac{2^l l!}{(-1)^{m`}}\,p_l^{m`}(x) \tag{51}$$

Where the normalization constant is

$$N_{lm`} = (-1)^{l`} B_n\,c_{lm`}\,\frac{2^l l!}{(-1)^{m`}} \tag{52}$$

So $H_n(x)$become $H_n(x) = N_{lm`}\,p_l^{m`}(x)$.
To find the normalized constant, use the normalized condition $\int_{-1}^{1} H_n^2(x)dx = 1$.
By use associated Legendre polynomials orthogonally [13], we get

$$N_{lm`} = \sqrt{\frac{2l+1}{2}\,\frac{(l`-m`)!}{(l`+m`)!}} \tag{53}$$

So, the angular wave function become:

$$H_n(x) = \sqrt{\frac{2l+1}{2}\,\frac{(l`-m`)!}{(l`+m`)!}}\;p_l^{m`}(x)$$

Where associated Legendre polynomials equal;[14]

$$p_l^{m`}(x) = (1-x^2)^{\frac{m`}{2}}\sum_{v=0}^{\frac{l`-m`}{2}}\frac{(-1)^v\Gamma(2l`-2v+1)}{2^{l`}v!\,(l`-m`-2v)!\,\Gamma(l`-v+1)}\;x^{l`-m`-2v}$$

So, after replacing $x = \cos\theta$ the angular wave function equal;

$$H_n(x) = \sqrt{\frac{2l+1}{2}\frac{(l`-m`)!}{(l`+m`)!}}\sin^{m`}\theta \times \sum_{v=0}^{\frac{l`-m`}{2}} \frac{(-1)^v\Gamma(2l`-2v+1)}{2^{l`}v!(l`-m`-2v)!\Gamma(l`-v+1)}(\cos\theta)^{l`-m`-2v} \tag{54}$$

The Azimuthal Wave Function

From equation (8) we can easily determine the azimuthal part equation [15]:

$$\phi(\varphi) = \frac{1}{\sqrt{2\pi}}e^{-im\varphi} \tag{55}$$

by including the equations (35), (54) and (55), then the total wave function $\psi(r,\theta,\varphi)$ can be expressed as:

$$\psi(r,\theta,\varphi) = \sqrt{\frac{n!}{\Gamma(2L+n+2)}\frac{\frac{2ze^2}{(L+n+1)}^{2L+3}}{2\{n+L+1\}}}(r)^L e^{-\frac{\frac{2ze^2}{(L+n+1)}}{2}r} L_n^{2L+1}\left(\frac{2ze^2}{(L+n+1)}r\right) \times$$
$$\sqrt{\frac{2l+1}{2}\frac{(l`-m`)!}{(l`+m`)!}}\sin^{m`}\theta \sum_{v=0}^{\frac{l`-m`}{2}} \frac{(-1)^v\Gamma(2l`-2v+1)}{2^{l`}v!(l`-m`-2v)!\Gamma(l`-v+1)}(\cos\theta)^{l`-m`-2v} \times \frac{1}{\sqrt{2\pi}}e^{-im\varphi} \tag{56}$$

CONCLUSIONS

The total wave function and bound state energy using Hartmann potential are determined explicitly where they show a great similarity with other studies. The total wave function depends firstly, on associated Laguerre functions in the radial part, secondly, on the value of cosine in the angular part, and lastly, on the exponential function in the azimuthal part. The number of states n and quantum numbers l, m are also appeared and established.

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Appendix (I)

$$\frac{d^2U(r)}{dr^2} + \left[2E + 2\frac{\alpha}{r} - \frac{\lambda}{r^2}\right]U(r) = 0$$

$$\lambda = l`(l`+1) - 2\beta \tag{1}$$

$$\beta = \frac{m`^2 - m^2}{2} \tag{2}$$

$$L = \frac{1}{2}\left\{\sqrt{1 + 4\left[\left(k+\sqrt{2\beta+m^2}\right)\left(k+\sqrt{2\beta+m^2}+1\right) - 2\beta\right]} - 1\right\}$$

Substation equation (2) into equation (1):

$$\lambda = l`(l` + 1) - m`^2 + m^2 \quad (3)$$

$$\text{Let } l` = k + m`$$

$$k = l` - m`$$

From equation (2), we get:

$$2\beta + m^2 = m`^2$$

From equation (3), we get:

$$\lambda = k^2 + 2km` + m`^2 + k + m` + l` - m`^2 + m^2$$

$$\lambda = k^2 + 2k\sqrt{2\beta + m^2} + \sqrt{2\beta + m^2} + 2k + \sqrt{2\beta + m^2} + m^2$$

$$\lambda = k^2 + 2k\sqrt{2\beta + m^2} + 2\sqrt{2\beta + m^2} + 2k + m^2$$

To proof $\mathrm{L(L + 1) = \lambda}$, we use Some simple calculations.

$$L^2 + L = \lambda$$

$$\frac{1}{4}\left\{\sqrt{1 + 4\left[\left(k + \sqrt{2\beta + m^2}\right)\left(k + \sqrt{2\beta + m^2} + 1\right) - 2\beta\right]} - 1\right\}^2 + \frac{1}{2}\left\{\sqrt{1 + 4\left[\left(k + \sqrt{2\beta + m^2}\right)\left(k + \sqrt{2\beta + m^2} + 1\right) - 2\beta\right]} - 1\right\} = \lambda \quad (4)$$

The value that under square root in equation (4):

$$\left(k + \sqrt{2\beta + m^2}\right)\left(k + \sqrt{2\beta + m^2} + 1\right) = k^2 + 2km` + k + m`^2 + m`$$

Equation (4) become:

$$\frac{1}{4}\left\{\sqrt{1 + 4[k^2 + 2km` + k + m`^2 + m` - 2\beta]} - 1\right\}^2 + \frac{1}{2}\left\{\sqrt{1 + 4[k^2 + 2km` + k + m`^2 + m` - 2\beta]} - 1\right\} = \lambda$$

After use equation (2), we get:

$$\frac{1}{4}\left\{\sqrt{1 + 4[k^2 + 2km` + k + m`^2 + m` - m`^2 + m^2]} - 1\right\}^2 + \frac{1}{2}\left\{\sqrt{1 + 4[k^2 + 2km` + k + m`^2 + m` - m`^2 + m^2]} - 1\right\} = \lambda \quad (5)$$

Substation equation (1) into equation (5), and simple mathematics.

$$\frac{1}{4}\left\{\sqrt{1 + 4[(l` - m`)^2 + 2(l` - m`)m` + l` + m^2]} - 1\right\}^2 + \frac{1}{2}\left\{\sqrt{1 + 4[(l` - m`)^2 + 2(l` - m`)m` + l` + m^2]} - 1\right\} = l`(l` + 1) - m`^2 + m^2$$

by expand the Quadratic Arc, we get:

$$\frac{1}{4} + [(l` - m`)^2 + 2(l` - m`)m` + l` + m^2] - \frac{1}{2}\sqrt{1 + 4[(l` - m`)^2 + 2(l` - m`)m` + l` + m^2]} + \frac{1}{4} + \frac{1}{2}\sqrt{1 + 4[(l` - m`)^2 + 2(l` - m`)m` + l` + m^2]} - \frac{1}{2} = l`(l` + 1) - m`^2 + m^2$$

$$\frac{1}{4} + [(l` - m`)^2 + 2(l` - m`)m` + l` + m^2] - \frac{1}{4} = l`(l` + 1) - m`^2 + m^2$$

$$(l` - m`)^2 + 2(l` - m`)m` + l` + m^2 = l`(l` + 1) - m`^2 + m^2$$

$$l^{2`} - 2m`l` + m`^2 + 2l`m` - 2m`^2 + l` + m^2 = l`(l` + 1) - m`^2 + m^2$$

$$l^{2`} + l` - m`^2 + m^2 = l`(l` + 1) - m`^2 + m^2$$

$$l`(l` + 1) - m`^2 + m^2 = l`(l` + 1) - m`^2 + m^2$$

ЗАСТОСУВАННЯ МЕТОДУ НІКІФОРОВА-УВАРОВА ДО РІВНЯННЯ ШРЕДІНГЕРА, З ВИКОРИСТАННЯМ ПОТЕНЦІАЛУ ГАРТМАНА

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Повна хвильова функція та енергія зв'язаного стану досліджуються методом Нікіфорова-Уварова до рівняння Шредінгера в сферичних координатах з використанням потенціалу Гартмана (HP). HP вважається нецентральним потенціалом, який в основному визнається в потенціалах ядерного поля. Кожна хвильова функція визначається головним квантовим числом n, числом кутового моменту *l* і магнітним квантовим числом *m*. Радіальну частину хвильової функції отримано через асоційований поліном Лагерра за допомогою перетворення координат $x = \cos\theta$ для отримання кутової хвильової функції, яка залежить від обернених асоційованих поліномів Лежандра.

Ключові слова: *рівняння Шредінгера; метод Нікіфорова-Уварова; потенціал Гартмана*