공학박사학위논문

CONSENSUS OF LINEAR TIME INVARIANT MULTI-AGENT SYSTEMS OVER MULTILAYER NETWORK

다층레이어 네트워크 구조를 가지는 선형 시불변 다개체 시스템의 상태일치

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이 승 준의 공학박사 학위논문을 인준함.

2021년 2월

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ABSTRACT

Consensus of Linear Time Invariant Multi-agent Systems over Multilayer Network

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Traditionally, the consensus of multi-agent systems is often studied by assuming that there is a single network consisting of a single type of interaction. Recently, such an assumption is being challenged due to its limitation in representing more complex interactions. In this thesis, we consider the case where each agent is interacting using multiple, different types of output information. In order to model such interactions, the concept of a multilayer graph is employed. A novel necessary and sufficient condition is proposed for the existence of a dynamic coupling law to achieve state consensus for a multi-agent system over an undirected network. Specifically, the proposed condition couples graph theoretic conditions with system theoretic conditions and highlights the interplay between the communication network and information exchange between agents. Furthermore, based on the proposed condition, an observer-based dynamic controller is designed to achieve state consensus over an undirected network.

The main results are then extended to output consensus problem over a directed network. Unfortunately, the proposed conditions are no longer necessary and sufficient and the challenge is illustrated through various examples. Nevertheless, additional assumptions are made on the dynamics of the agent to recover the equivalence for output consensus over the undirected multilayer network. A sufficient condition is also given for output consensus problem over the directed network and the corresponding controller design is presented.

The effectiveness of the work is shown by a series of applications of the main results. First, the distributed state estimation problem is formulated into a consensus problem over a multilayer network. The proposed approach allowed us to develop a novel design for a distributed observer that communicates less information with its neighbors compared to existing designs. Secondly, the main results are applied to the formation control problem. Specifically, we consider the case when the desired formation is given by a combination of relative positional constraint and bearing constraint. Using the proposed approach, a dynamic controller is designed to achieve the desired formation while organically scaling the overall size of the formation. Finally, a multilayer network is also applied to the distributed optimization problem. Through multilayer networks, a communication-efficient algorithm is proposed which only communicates a part of the decision vector at each time instant.

Keywords: consensus, linear homogeneous multi-agent systems, multilayer network, formation control, distributed estimation, distributed optimization

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Symbols and Acronyms

$\mathbb{R}$	field of real numbers
$\mathbb{C}$	field of complex numbers
$\mathbb{N}_0$	set of natural numbers including 0
$\mathbb{R}^n$	Euclidean space of dimension n
$\mathbb{R}^{m\times n}$	space of $m \times n$ matrices with real entries
$\mathbb{R}_{>0}$	set of positive real numbers
$\mathbb{R}_{\geq 0}$	set of non-negative real numbers
$\mathrm{Re}\,(s)$	real part of complex number s
$\mathrm{Im}\,(s)$	imaginary part of complex number s
1_n	$n \times 1$ column vector of all ones
I_n	$n \times n$ identity matrix
0_n	$n \times 1$ column vector having all elements equal to 0
$0_{m\times n}$	$m \times n$ matrix having all elements equal to 0
A^{-1}	inverse of the nonsingular matrix A
$A^\top$	transpose of the matrix A
$\mathrm{diag}(A_1, \ldots, A_k)$	block diagonal matrix whose i-th block diagonal is A_i
$[A_1; \cdots ; A_N]$	stack of matrices A_i
$A \otimes B$	Kronecker product of matrices A and B

$\sum_{i=m}^{n} x_i$	summation of the sequence x_i; i.e., $x_m + x_{m+1} + \cdots + x_n$ if $m < n$, x_m if $m = n$, and 0 if $m > n$
$[a, b]$	interval of real numbers a and b; i.e., $\{x \in \mathbb{R} : a \leq x \leq b\}$
$\lvert X \rvert$	cardinal number of the set X
$\max\{a_1, \ldots, a_n\}$	maximum value among a_1, a_2, $\ldots$, a_n
$\min\{a_1, \ldots, a_n\}$	minimum value among a_1, a_2, $\ldots$, a_n
$\lvert x \rvert$	Euclidean norm of the vector $x \in \mathbb{R}^n$
$\lvert x \rvert_\infty$	maximum norm of the vector $x \in \mathbb{R}^n$
$\lvert A \rvert$	induced norm of the matrix $A \in \mathbb{R}^{m\times n}$
$\lvert A \rvert_\infty$	induced maximum norm of the matrix $A \in \mathbb{R}^{m\times n}$
$\operatorname{im} A$	image space of the matrix $A \in \mathbb{R}^{m\times n}$
$\ker A$	kernel of the matrix $A \in \mathbb{R}^{m\times n}$
$\dim \mathcal{X}$	dimension of vector space $\mathcal{X}$
$\lambda_{\min}(A)$	the eigenvalue of A with the minimum real value
$\lambda_{\max}(A)$	the eigenvalue of A with the maximum real value
$\sigma_{\min}(A)$	the minimum singular value of A
$\sigma_{\max}(A)$	the maximum singular value of A
$\mathcal{X}_\lambda(A)$	generalized eigenspace of matrix A corresponding to eigenvalue λ
$\mathcal{X}^u(A)$	unstable generalized eigenspace of matrix A
$\mathcal{X}^s(A)$	stable generalized eigenspace of matrix A
$A > 0$ $(A \geq 0)$	symmetric matrix A is positive definite (positive semidefinite)
$\lvert x \rvert_\Xi$	distance from a point x to a set Ξ; i.e., $\inf_{y\in\Xi} \lVert x - y \rVert$
$f\colon A \to B$	f is a function on the set A into the set B
∇f	gradient of function f

$\nabla^2 f$	Hessian of function f
$\mathfrak{C}^i$	i-th times continuously differentiable
$\lfloor x \rfloor$	floor of a real number x
$x \bmod y$	remainder of x divided by y
$:=$	defined as
$\Rightarrow$	implies
$\forall$	for all
$\Diamond$	end of theorems, lemmas, propositions, assumptions, remarks, and so on
$\square$	end of proof

- A square matrix A is said to be Hurwitz (matrix) if every eigenvalue λ of A has strictly negative real parts, i.e., $\mathrm{Re}(\lambda) < 0$ or equivalently $\lambda_{\max}(A) < 0$.
- For any state variable $x(t)$, its initial condition will be denoted by $x(0)$.
- For simplicity, we often use I_n, 0_n, and $0_{m\times n}$ without subscripts if their dimensions are obvious from the context.

Acronyms

ARE	Algebraic Riccati equation
DEP	Distributed state estimation problem
LTI	Linear time-invariant
MAS	Multi-agent system

Chapter 1

Introduction

1.1 Research Background

Consensus and synchronization of multi-agent systems have been studied extensively for past decades and found a wide range of applications. In particular, application of consensus includes formation control (e.g., [OPA15, ZZ17]), distributed state estimation (e.g., [KLS20, MS18]) and distributed optimization problem (e.g., [NO09, LS19]) to name a few. (More details can be found in [RBA07, OSM04, Wie10] and references therein.) As consensus of multi-agent systems has been essential in many of the applications, various studies are done in the literature. Most of the results developed for the consensus of multi-agent systems can be categorized based on the following characteristics:

1. **Model:** Complexity of the model for each agent. Is it homogeneous or heterogeneous? Is it linear or nonlinear? Is it continuous or discrete?

2. **Communication Network:** Structure of the communication network. Is it directed or undirected? Is it connected or disconnected? Does communication have a delay or only happens at discrete times?

3. **Information:** Details on *how and what* information is exchanged. Do agents communicate absolute or relative information? Is it the output or state information of each agent? Is there any additional information communicated among agents?

Before moving further into the details of each topic and related literature, let us briefly discuss and emphasize a particularly important aspect of the consensus problem for the multi-agent systems (which is not only limited to the consensus problem over multilayer network) that is assumed throughout this dissertation. Throughout this dissertation, we suppose that only the relative output information (as opposed to absolute output information) is available to each agent for control. This is motivated by the physical limitations in practical scenarios. For instance, consider the consensus problem of drones. Then, it is easy to measure and obtain a relative position between drones but much harder to obtain an absolute position of itself. Hence, there is a need for a control algorithm which only utilizes the relative information (which is easily available) to achieve its goal. Due to this reason, the majority of the literature on the consensus problem of multi-agent systems only utilize the relative output information, which we will follow as well. If the absolute output information is available to each agent, then the consensus problem becomes less challenging as the problem becomes how each agent control itself, which is similar to the classical problems in the control theory. Additionally, the control using the absolute information is also less challenging in the sense that the same technique developed for the consensus using relative information can be used directly since relative information can be easily obtained from absolute information. Nonetheless, there are few works which use absolute information for consensus problem and we refer to works such as [KKB+16, YSSG11] for more details.

Now, a brief overview of the various studies done for the consensus is given. As the consensus problem of multi-agent systems received a considerable amount of attention from various research communities, numerous aspects of the consensus problem have been studied. Among those, the consensus of linear, homogeneous agents is a particularly well-studied problem due to rich results from linear system theory. As mentioned above, the result for the linear homogeneous agents can be further classified based on communication networks and information. Early studies for the consensus problem were done for agents of integrator (e.g., [RB05]) or double-integrators (e.g., [RB08]), while the early survey can be found in [RBA05]. Then the consensus problem is solved for more general linear systems. For instance, the LQR-based approach is used in [Tun08] to solve consensus problems using static diffusive state coupling over undirected networks and LMI-based approach is proposed in [WKA11] for directed graphs. Dynamic controllers are studied in [SS09] for switching networks and observer-based controllers are proposed in [ZLD11, WLH09]. The aforementioned observer-based controllers are often required to communicate additional information (on top of relative output information) such as the state of the dynamic controller (e.g., the estimate of observers). This challenge has been first resolved by [SSB09] using the low-gain based controllers which do not require any additional information other than the relative output information. Improvements to the low-gain based controllers are made in the following works such as [WSS+13] which applied a similar approach under communication delay. Finally, the aforementioned works can be viewed as finding a sufficient condition (and the corresponding controller design) to achieve consensus. Conversely, studies are also done to find a necessary condition for the existence of such controllers. For instance, necessary conditions for the continuous-time multi-agent system are discussed in [MZ10] and a similar result is also studied in the discrete-time domain [GML12, YX11].

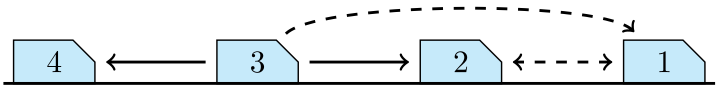
The consensus problem is also extensively studied for systems with more complex models such as nonlinear systems. However, the details are not discussed as a nonlinear system is not the main focus of the dissertation. For the interested readers, refer to works such as [Kim16, Lee19, PL17, WSA11, MBA15, KSS11, LYS18] and references therein.

The majority of the studies mentioned so far were focused on *models* and *communication network* aspect of the multi-agent system. Consequently, less work investigated more complex structure for *information*. In particular, previous works often assume agents are interacting with only a single type of output information over a single network. However, this paradigm has been challenged in recent years due to the limitation of a single network to represent more complex types of interactions and relationships. For example, consider the multi-agent system of cyber-physical systems where agents may have *physical* interaction as well as *cyber* interaction (e.g., wireless communication) with other agents [WCL17]. Most importantly, since physical and cyber interactions have different characteristics, each agent may have *different* neighbors for each type of interaction. For instance, an agent may communicate with other agents despite having no direct physical interactions or an agent may not communicate with other agents, yet still have physical interactions.

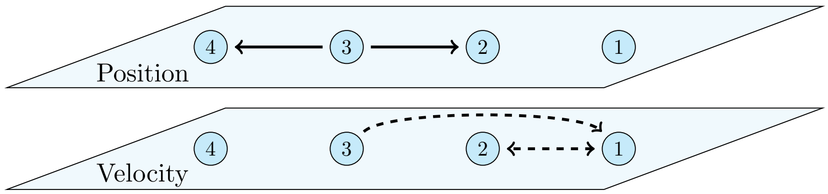
In light of this, *multilayer network*[1], a more general and complex model of the network is studied in various fields [BBC+14, DDSRC+13]. A multilayer network is a collection of multiple *layers* of graphs, each sharing the same node set but with a different edge set (with formal definition to be given in the following chapters). Specifically, each layer of the graph is used to represent a single type of interaction. The possibility of multiple layers of graphs and different edge sets for each layer allows more complex interactions among agents.

Additional complexity provided by the multilayer network is shown to be essential in various fields especially when studying and modeling the dynamical systems [DDGPA16]. For example, in opinion dynamics, people may interact with others in various methods such as physical contact or through online services [SLT10, ZSL17]. In biological systems, neurons have both electrical coupling and chemical coupling, each having different characteristic [APD11]. Schooling of fish also has multiple layers of interaction including vision and lateral line with different neighbors for each interaction [PP80]. In the study of epidemic spreading, authors of [GGA13, ZWL+14] employed the multilayer network to model various interactions including transmission among different communities and awareness of the disease. The transportation system is a classic example that can be represented using the multilayer graph, with each layer modeling different means of transportation such as automobile, train, or plane [CGGZ+13]. The electrical power network can also be described as a consensus problem with both physical couplings by transmission lines and cyber coupling through wireless communication [WCL17, MDP17]. In all of the examples mentioned so far, each agent interacts with others through multiple types of interaction. Therefore, it is challenging to model and analyze the dynamics of such a system using conventional

[1]The exact definition of the multilayer network varies from work to work and other vocabularies such as *multiplex network* is also used to denote the similar structure (e.g., see [BBC+14] for details). In this dissertation, the term multilayer network is used for consistency, and its exact definition is presented in Chapter 2



(a) Communication network. Solid lines denote relative position measurement and dashed lines denote relative velocity measurement.



(b) Multilayer graph representing the communication network.

Figure 1.1: Example of multilayer network for the platooning problem. Arrows denote the flow of information.

approaches. However, the multilayer network can be used to conveniently model the aforementioned examples with each layer representing a specific type of interaction.

For a more concrete example, the vehicle platooning problem is given to motivate the usefulness of the multilayer graphs. Suppose that the dynamics of each vehicle is described by

$$\dot{x}_i = v_i, \quad \dot{v}_i = u_i, \quad \forall i = 1, \dots, 4,$$

where x_i is the position, v_i is the velocity and u_i is the acceleration of the vehicle. The goal of the platooning problem is to control the vehicles to maintain the distance $d^* > 0$ and to travel at the same velocity, i.e., $x_i(t) - x_{i+1}(t) \to d^*$ for all $i = 1, \dots, 3$ and $v_j(t) - v_i(t) \to 0$ for all $i, j \in \{1, 2, 3, 4\}$ as $t \to \infty$. Suppose that each vehicle has different measurement capabilities. Specifically, vehicles may either measure relative position or relative velocity with its neighbors as shown in Fig. 1.1(a).

For instance, vehicle 2 has a sensor at the back and vehicle 4 has a sensor at the front to measure the relative distance to vehicle 3 (note that the arrows in Fig. 1.1 represent the flow of information). On the other hand, vehicles 2 and 3 receive velocity of vehicle 1 via wireless communication while vehicle 2 is capable of transmitting its velocity back to the vehicle 1. This allows vehicles to obtain relative velocity. Then, the vehicle platooning problem can be formulated into a consensus problem with multiple different interactions, where the interactions can be represented as shown in Fig. 1.1(b). Let $d_i^* = (i-1) \cdot d^*$ be the desired distance from vehicle 1 and let $e_i := x_i + d_i^*$ for $i = 1, \ldots, 4$. Then, it follows that

$$\dot{e}_i = v_i, \quad \dot{v}_i = u_i, \quad \forall i = 1, \ldots, 4.$$

Hence, it is easily verified that achieving consensus of the state $[e_i(t) \ v_i(t)]^\top$ implies $x_i(t) - x_{i+1}(t) \to d^*$ for all $i = 1, \ldots, 3$ and $v_i(t) - v_j(t) \to 0$ for all $1 \leq i, j \leq 4$.

Despite the flexibility of the multilayer graphs for modeling the multi-agent systems, there are only a few works studying such a system to the best of the author's knowledge. For example, distributed proportional-integral-derivative (PID) control is studied in [LDB16] where proportional and integral actions are represented as separate layers. However, the difference between the layer is only on the mean to compute the action (e.g., proportional versus integral) and the same output information is used on each layer. In [HCH$^+$17, Sor12], a consensus problem over the multilayer network is studied where the diffusive coupling is used with different output on each layer and [FDLR18] studied a similar problem for double integrators. However, these works often consider a multilayer network with 2 layers and assume simultaneous diagonalization (or triangularization) of Laplacian matrices of each layer, which is a restrictive assumption. Authors of [WCL17] study consensus problem for two layers with non-commuting Laplacian matrices, but full state information is used and assumed connectivity of the corresponding layer. Master stability function is employed in [GGGBB16] to study nonlinear systems, but only numerical results are obtained. Moreover, most of the works mentioned only provide sufficient conditions for achieving consensus.

More recent works done by the control community include [PLSJ19] which studies the observability of dynamical systems with a multilayer structure and [ZSL17] which applied multilayer network to linear threshold model. However, these works studied the structural properties of the system and not the consensus problem in particular.

The line of work that is perhaps closest to the results of the dissertation is the works such as [Tun16, Tun17, Tun19, Tun20, TNLA18]. The authors proposed a concept of *matrix weighted Laplacian* and analyzed various properties relating to detectability while also proposing controllers to achieve consensus (e.g., dynamic controller in [Tun16] and static controllers in [Tun19, Tun20, TNLA18]). In fact, it can be shown that the multilayer framework proposed in the dissertation is equivalent to the framework using the matrix weighted Laplacian as in [Tun17] (e.g., see Appendix A.2 more discussion). However, the work of [Tun16, Tun17, Tun19, Tun20, TNLA18] are limited to undirected and state consensus problem. Additionally, it only provides a sufficient condition to design a dynamic controller in the discrete-time domain while using a continuous-time operation at each time step. Hence, it is not known if the similar design can be implemented in the continuous-time domain. The main contribution of the dissertation compared to these works include proposing a sufficient condition and designing a controller purely in the continuous-time domain. Moreover, we also consider the multilayer network for more complex problems such as output consensus, directed graphs, or switching networks. Finally, various nontrivial applications of the multilayer network are also presented using the proposed approach.

1.2 Contributions and Outline of Dissertation

In this section, an outline of the dissertation is presented and the main contributions are summarized.

Chapter 2. Preliminaries on Graph Theory and Convex Optimization

In this chapter, we review the basics of algebraic graph theory and convex optimization. The concept of multilayer graphs is also presented and related concepts are developed. The main contributions of this chapter are:

- We formerly define multilayer graph and present related concepts from algebraic graph theoretical.

- Preliminary results from algebraic graph theory and convex optimization are summarized.

Chapter 3. Consensus Problem over the Multilayer Network

In this chapter, we formulate the consensus problem over a multilayer network, where each agent exchanges various *different* output information over *different* communication network. We start by first addressing the state consensus problem over an undirected multilayer network. Using an intuitive understanding of the consensus problem, a novel necessary condition is proposed. Specifically, the proposed condition involves both graph theoretical concepts such as connectivity along with system theoretical concepts such as detectability. It is also illustrated through simple examples that such a combination is necessary to achieve consensus over a multilayer network.

Based on the necessary condition, a novel dynamic controller is proposed to achieve consensus. In particular, the proposed design is motivated by observer-based approaches for the single-layer network, but with an additional component to overcome challenges faced in the multilayer network. From the proposed design, it is also established that the proposed condition is necessary and sufficient for state consensus over an undirected network. Contents of this chapter are mainly based on [LS20d]. The main contributions can be listed as below:

- We give a novel necessary and sufficient condition for state consensus problem over multilayer network.
- Various equivalent conditions for the proposed necessary condition are made and interpretations are given relating to the traditional concept of detectability and connectedness.
- An observer-based controller is proposed to achieve the state consensus over an undirected multilayer network.

Chapter 4. Extension to Output Consensus over Directed Network

In this chapter, the problem of consensus over the multilayer network is extended to output consensus over a directed network. Unfortunately, the proposed conditions from Chapter 3 is no longer necessary and sufficient and challenges are illustrated through various examples (and counterexamples). Nonetheless, additional assumptions are made to recover equivalence for output consensus problem over the undirected network and sufficient condition is given for output consensus problem over the directed network.

- Results developed for state consensus over undirected multilayer network are extended to output consensus problem over directed multilayer network.
- Counterexamples are given to illustrate challenges for finding necessary and sufficient conditions.
- Additional assumptions are made which recover equivalence for output consensus problem over the undirected network.
- A sufficient condition for output consensus problem over the directed network is proposed and an intuitive explanation is given.

Chapter 5. Application to the Distributed State Estimation Problem

In this chapter, we apply the main results developed to the distributed state estimation problem. The distributed state estimation problem is where a number of agents wishes to estimate the state of the plant only using local measurement and communication with neighbor agents. It is supposed that no single agent may recover the state of the plant by itself and hence communication with neighboring agents is necessary.

We show that the distributed state estimation problem can be formulated into a consensus problem over a multilayer network. Consequently, the proposed necessary and sufficient conditions apply. In particular, a novel design is proposed for observers that communicate less information compared to existing designs. Furthermore, results are extended to switching networks such that the communication among observers as well as local measurement may be exposed to communication losses. Contents of this chapter are based on [LS20e]. The main contributions of this chapter are:

- We formulate the distributed state estimation problem into the consensus problem over a multilayer network.
- A novel design for the observer is proposed for a marginally stable plant.
- It is shown that the proposed design exchanges less communication compared with existing designs and in fact, exchanges the minimum amount of information.
- Analysis is extended to show that the proposed distributed observer design achieves state estimation under the switching network.

Chapter 6. Application to the Formation Control Problem

In this chapter, we present the application of consensus over a multilayer network to the formation control problem. Specifically, we consider the problem where the desired formation is given by a combination of relative positional constraints and bearing constraints. Similar to the previous chapter, it is shown that the formation control problem can be formulated into a consensus problem over a multilayer network. Consequently, a dynamic controller is designed to achieve the desired formation. The efficacy of the proposed approach is shown by easily scaling the overall size of the formation in a distributed manner. Contents of this chapter are discussed in [LS20d]. The main contribution can be listed as follows:

- We formulate the formation control problem into a consensus problem over a multilayer network. Specifically, we consider the scenario where the constraints are given both by relative position and bearing.
- Dynamic controller is designed to achieve the desired formation.

Chapter 7. Application to the Distributed Optimization Problem

In this chapter, we present application of the multilayer network to the distributed optimization problem. Traditional consensus algorithms from single-layer networks are recently applied to solve the distributed optimization problem. Consequently, theories developed for the consensus over the multilayer network can be similarly applied to the distributed optimization problem. In particular, by considering the switching multilayer network, communication-efficient algorithms are developed which only communicate part of the decision variable at each time instant. This chapter also contains the analysis and construction of continuous-time distributed algorithms using the blended dynamics approach motivated by [LS20b]. The contents of this chapter are based on [LS20c]. The main contributions of this chapter are:

- We apply the switching multilayer networks to the distributed optimization algorithm to propose a communication-efficient algorithm.
- A general tool is proposed to achieve asymptotic consensus of heterogeneous multi-agent systems.
- The proposed approach is used to analyze and construct novel algorithms which combine consensus algorithm with accelerated methods such as distributed heavy-ball method.

Chapter 2

Preliminaries on Graph Theory and Convex Optimization

This chapter provides a brief introduction to graph theory and convex optimization theory used throughout the dissertation.

2.1 Graph Theory and Consensus Problem

For the modeling of the interactions between multi-agent systems, graph theory is used extensively. In this section, graph theory and relevant definitions are summarized. For more details, refer to textbooks such as [GR01].

2.1.1 Basic Definitions

Definition 2.1.1. A time-varying graph is a triple $G(t) = (\mathcal{N}, \mathcal{E}(t), \mathfrak{A}(t))$ consisting of a nonempty finite set of nodes $\mathcal{N} = \{1, \ldots, N\}$, an edge set of ordered pairs of nodes $\mathcal{E}(t) \subseteq \mathcal{N} \times \mathcal{N}$, and a weighted adjacency matrix $\mathfrak{A}(t) = [\alpha_{ij}(t)] \in \mathbb{R}_{\geq 0}^{N \times N}$, where $t \in \mathbb{R}$ represents time, satisfying the following properties:

1. The graph contains no self-loops, i.e., $(i, i) \notin \mathcal{E}(t)$ and $\alpha_{ii}(t) = 0$ for all $t \in \mathbb{R}$ and $i \in \mathcal{N}$.

2. Each element of the weighted adjacency matrix $\alpha_{ij} : \mathbb{R} \to \mathbb{R}_{\geq 0}$ are non-negative, piecewise continuous, and bounded functions of time.

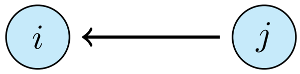


Figure 2.1: Node i and j with the edge (j, i).

3. An edge $(j, i) \in \mathcal{E}(t)$ at time t if and only if $\alpha_{ij}(t) > \omega$ for some fixed threshold $\omega > 0$ and $\alpha_{ij}(t) = 0$ otherwise. $\Diamond$

An edge $(j, i) \in \mathcal{E}$ is represented by an arrow tailed at the node j and headed towards the node i and it is related with the adjacency matrix $\mathfrak{A}(t)$ by the rule that $\alpha_{ij}(t) > \omega$ if and only if $(j, i) \in \mathcal{E}(t)$ and $\alpha_{ij}(t) = 0$ otherwise. The edge $(j, i) \in \mathcal{E}(t)$ indicates that information of agent j is passed to agent i at time t, which is shown in Fig. 2.1. From the definition of time-varying graphs, the following special cases can be derived.

Definition 2.1.2. We define the following special cases.

1. A graph $G(t)$ is *fixed* (or time-invariant) if it does not change over time t. In this case, one can simply write $G = (\mathcal{N}, \mathcal{E}, \mathfrak{A})$.

2. A graph $G(t)$ is *undirected* if $\alpha_{ij}(t) = \alpha_{ji}(t)$ for all $t \geq 0$ and $i, j \in \mathcal{N}$.

3. A graph $G(t)$ is *unweighted* if $\alpha_{ij}(t) \in \{0, 1\}$ for all $t \geq 0$ and $i, j \in \mathcal{N}$. In this case, one can simply write $G = (\mathcal{N}, \mathcal{E}(t))$. $\Diamond$

For each definition, a graph that is not fixed, not undirected, or not unweighted is defined as time-varying, directed, and weighted, respectively. Note that each of the special case is independent of each other, e.g., a graph can be time-invariant, undirected, and weighted all at once.

2.1.2 Connectedness of the Graph

One of the most important properties of the graph is its connectedness. Recall that the flow of the information is represented by the graph. Hence, it is intuitively clear that information must propagate throughout edges and reach all agents in order to reach consensus. The concept of the connectedness of a graph and related concepts are defined formally as follows.

Definition 2.1.3. Consider a time-varying graph $G(t) = (\mathcal{N}, \mathcal{E}(t), \mathfrak{A}(t))$ and some fixed time t.

1. A sequence of edges $(v_1, v_2), \ldots, (v_l, v_{l+1})$ is a *directed path* if $(v_j, v_{j+1}) \in \mathcal{E}(t)$, and a *weak path* if $(v_j, v_{j+1}) \in \mathcal{E}(t)$ or $(v_{j+1}, v_j) \in \mathcal{E}(t)$ for all $j = 1, \ldots, l$.

2. *In-neighbors* of node i is a set defined as $\mathcal{N}_i^{\text{in}}(t) := \{j \in \mathcal{N} \mid (j,i) \in \mathcal{E}(t)\}$ and *out-neighbors* of node i is a set defined as $\mathcal{N}_i^{\text{out}}(t) := \{j \in \mathcal{N} \mid (i,j) \in \mathcal{E}(t)\}$.

3. *In-degree* and *out-degree* of node $i \in \mathcal{N}$ is defined as $d_i^{\text{in}}(t) := |\mathcal{N}_i^{\text{in}}(t)|$ and $d_i^{\text{out}} := |\mathcal{N}_i^{\text{out}}(t)|$.

4. Graph is *strongly connected (weakly connected)* if any two distinct nodes can be joined by a directed path (weak path) [GR01].

5. Graph *contains a rooted spanning tree* if there exists a node $i \in \mathcal{N}$ denoted as the *root* such that there exists a path from node i to node j for all $j \in \mathcal{N} \backslash \{i\}$. ◊

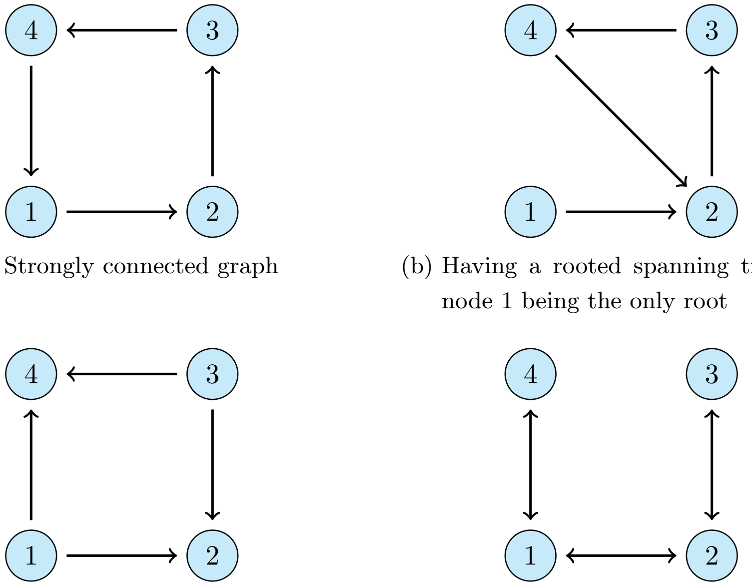
For undirected graphs, the graph is denoted simply as *connected* (at time t) if there exists a path[1] between any two nodes $i \neq j \in \mathcal{N}$. Also, we drop dependence of time t if the graph is time-invariant.

Concepts of the connectedness introduced in Definition 2.1.3 deal with connectivity at a specific time t. However, connectedness can be also defined while considering the evolution of the graph over time. For this, define T-averaged adjacency matrix as

$$\bar{\mathfrak{A}}_T(t) = [\bar{\alpha}_{ij}(t)] := \frac{1}{T} \int_t^{t+T} \mathfrak{A}(\tau) d\tau,$$

where $T > 0$ is a given time horizon. Similarly, define $\bar{\mathcal{E}}_T(t)$ such that $(j,i) \in \bar{\mathcal{E}}_T(t)$ if and only if $\bar{\alpha}_{ij}(t) > \omega$. Then the graph $\bar{G}_T(t) := (\mathcal{N}, \bar{\mathcal{E}}_T(t), \bar{\mathfrak{A}}_T(t))$ can be

[1] Notice that for undirected graphs, directed and weak path is equivalent and is simply denoted as path.



(a) Strongly connected graph

(b) Having a rooted spanning tree with node 1 being the only root

(c) Weakly connected graph

(d) Connected graph

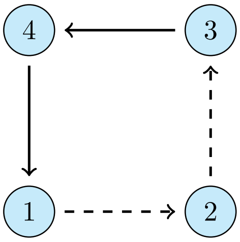
Figure 2.2: Examples of various time-invariant graphs and the corresponding connectedness.

defined as the union graph of intervals of length T. Then the following definition is proposed in [Mor04, Lin06].

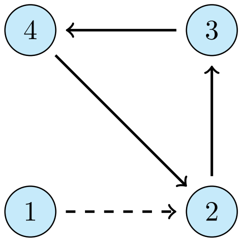
Definition 2.1.4. A time-varying graph $G(t)$ is *uniformly connected (strongly connected, weakly connected, contains a rooted spanning tree)* if there exists $T > 0$ such that $\bar{G}_T(t)$ is connected (strongly connected, weakly connected, contains a rooted spanning tree) for all time t, respectively. $\lozenge$

Uniform connectedness in Definition 2.1.4 extends the concept of connectedness to time-varying graphs.

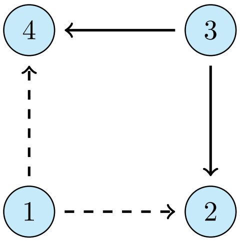
Examples of the various concepts of connected graphs for time-invariant graphs are shown in Fig. 2.2 and time-varying graphs are shown in Fig. 2.3. We would like to remark that the concept of uniform connectedness is weaker than the concept of connectedness. In particular, uniformly connected graphs may not be connected at any time instance t. For example, consider the graph given by Fig. 2.3(a). At any time instance, only two edges are present in the graph, e.g., only the solid arrows or only the dashed arrows. Therefore, the graph is not strongly



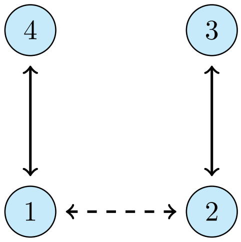
(a) Uniformly strongly connected graph



(b) Having a rooted spanning tree uniformly with node 1 being the root



(c) Uniformly weakly connected graph



(d) Uniformly connected graph

Figure 2.3: Examples of various time-varying graphs and the corresponding uniform connectedness with $T = 2$. An edge (j, i) denoted as solid lines are the ones with $\alpha_{ij}(t) = 1$ for $t \bmod 2 \in [0, 1)$ and edge (j, i) denoted with dashed lines are the ones with $\alpha_{ij}(t) = 1$ for $t \bmod 2 \in [1, 2)$.

connected for any time instance. However, with $T = 2$, it can be checked that the union graph $\bar{G}_T(t)$ is time-invariant and has the same structure as the graph in Fig. 2.2(a). Since $\bar{G}_T(t)$ is a time-invariant graph that is strongly connected, $G(t)$ is uniformly strongly connected despite not being connected at any time instance.

2.1.3 Laplacian Matrix and Its Properties

Given a time-invariant graph G, define the Laplacian matrix of the graph G as follows.

Definition 2.1.5. Laplacian matrix $\mathfrak{L} \in \mathbb{R}^{N \times N}$ of a graph G is defined as $\mathfrak{L} := D^{\text{in}} - \mathfrak{A}$. ◊

The eigenvalues of the Laplacian matrix have a special property related to the structure of the graph. First, it is well-known that eigenvalues have positive real parts as stated below.

Lemma 2.1.1. Eigenvalues of the Laplacian matrix $\mathfrak{L}$ lie on closed right half plane, i.e., $\mathrm{Re}\,(\lambda_i(\mathfrak{L})) \geq 0$ for all $i = 1, \dots, N$. $\Diamond$

Proof. Proof follows directly from applying Geršgorin disc theorem [RAH19, Thm. 6.1.1]. □

The locations of the eigenvalues are characterized in Lemma 2.1.1. In addition to this, the following property connecting the algebraic property of the Laplacian matrix with the connectivity of the graph is well-known, whose proof can be found in, for instance, at [RBM04].

Lemma 2.1.2. The graph G contains a rooted spanning tree if and only if $\mathrm{Re}\,(\lambda_2(\mathfrak{L})) > 0$. $\Diamond$

For the undirected graphs, $\lambda_2(\mathfrak{L})$ is known as the *algebraic connectivity* (or also as the Fiedler eigenvalue) of the graph [Fie73]. It characterizes how *well* the graph is connected and it is related to other concepts of connectivity such as the diameter of the graph [Moh91][2]. In particular, it can be shown (for instance by using Weyl's inequality) that algebraic connectivity is a non-decreasing function of weights or the addition of edges, i.e., more edges in the graph imply larger algebraic connectivity. This does not necessarily hold for directed graphs and more discussion can be found in [Wie10, Ch. 2].

In order to introduce the main technical lemma, let us first define the concept of unweighted induced subgraphs, whose definition is taken from [Wie10].

Definition 2.1.6. Given a graph $G(t) = (\mathcal{N}, \mathcal{E}, \mathfrak{A}(t))$, consider a subset $\tilde{\mathcal{N}} \subseteq \mathcal{N}$ of the nodes of $G(t)$. Then the graph $\tilde{G}(t) = (\tilde{\mathcal{N}}, \tilde{\mathcal{E}})$ is the induced graph of $G(t)$ by $\tilde{\mathcal{N}}$ where $\tilde{\mathcal{E}}(t) := \{(v, w) \in \mathcal{E}(t) \mid v, w \in \tilde{\mathcal{N}}\}$. $\Diamond$

The induced graph $\tilde{G}(t)$ can be simply obtained by removing all the nodes in $\mathcal{N} \backslash \tilde{\mathcal{N}}$ along with edges starting or ending at the removed nodes. With induced subgraphs, we can find connected component of a graph as follows.

[2] Specifically, it is shown in [Moh91, Thm 4.2] that $\lambda_2(\mathfrak{L}) \geq 4/(N \cdot \mathrm{diam}(\mathfrak{L}))$, where N is the number of nodes in a graph and $\mathrm{diag}(\mathfrak{L})$ is the diameter.

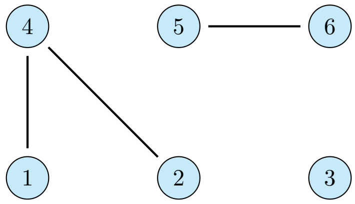


Figure 2.4: An example of a graph. Connected components of the graph are given by subgraphs induced by $\tilde{\mathcal{N}}_1 = \{1, 2, 4\}$, $\tilde{\mathcal{N}}_2 = \{3\}$, and $\tilde{\mathcal{N}}_3 = \{5, 6\}$.

Definition 2.1.7. Consider a time-invariant and undirected graph G. Then a connected component of the graph G is an induced subgraph $\tilde{G} = (\tilde{\mathcal{N}}, \tilde{\mathcal{E}})$ which is maximal, subject to being connected. That is, $\tilde{G}$ is connected and the unweighted graph induced by any set $\tilde{\mathcal{N}} \subseteq \hat{\mathcal{N}} \subseteq \mathcal{N}$ is connected if and only if $\hat{\mathcal{N}} = \tilde{\mathcal{N}}$. ♢

From the definition of a connected component, a graph G can always be decomposed as a set of connected components whose node sets are distinct and having no edges between any two connected components. An example of connected components of a graph is shown in Fig. 2.4. It is also easy to show that if a graph is connected, then it only has a single connected component. Finally, a well known result relating the algebraic property of the Laplacian matrix with the connectivity is given below [GR01, Lem. 13.1.1].

Lemma 2.1.3. Let $\mathfrak{L} \in \mathbb{R}^{N\times N}$ be a Laplacian matrix for an undirected graph G with c connected components. Then, it holds that $\mathrm{rank}(\mathfrak{L}) = N - c$. ♢

From Lemma 2.1.3, it follows that the kernel of the Laplacian matrix has a dimension of c, i.e., the dimension of the kernel characterize the number of connected components of the graph. Using this fact, a useful transformation can be found, which extends the result of [KYS$^+$16].

Theorem 2.1.4. Let $\mathfrak{L} \in \mathbb{R}^{N\times N}$ be a Laplacian matrix of an undirected graph with $c \geq 1$ connected components $G_i' = (\mathcal{N}_i', \mathcal{E}_i')$. Then, there exists a nonsingular matrix $W \in \mathbb{R}^{N\times N}$ such that

$$W\mathfrak{L}W^{-1} = \begin{bmatrix} 0_{c\times c} & 0 \\ 0 & \Lambda \end{bmatrix}$$

where $\Lambda = \mathrm{diag}(\lambda_{c+1}(\mathfrak{L}), \ldots, \lambda_N(\mathfrak{L})) \in \mathbb{R}^{(N-c)\times(N-c)}$. In particular, the matrices W and W^{-1} can be written as

$$W = \begin{bmatrix} v^\top \\ R^\top \end{bmatrix}, \quad W^{-1} = \begin{bmatrix} p & R \end{bmatrix},$$

where $v, p \in \mathbb{R}^{N\times c}$ are defined as

$$p = [p_{ij}] := \begin{cases} 1 & \text{if } i \in \mathcal{N}'_j \\ 0 & \text{if } i \notin \mathcal{N}'_j \end{cases}, \quad v := p \cdot \begin{bmatrix} \frac{1}{|\mathcal{N}'_1|} & & \\ & \ddots & \\ & & \frac{1}{|\mathcal{N}'_c|} \end{bmatrix}$$

and $R \in \mathbb{R}^{N\times(N-c)}$ are real matrix with following properties:

1. $R^\top R = I_{N-c}$ and $|R| = 1$
2. $R^\top p = 0$, $v^\top R = 0$
3. $R^\top \mathfrak{L} = \Lambda R^\top$, $\mathfrak{L} R = R\Lambda$
4. $RR^\top = I_N - pv^\top$. ◊

Proof. The proof extends the argument of [Kim16, Thm. 2.2.4]. Using the Schur decomposition and the fact that $\mathfrak{L}$ is symmetric, there exists an orthonormal matrix $U \in \mathbb{R}^{N\times N}$ such that

$$\mathfrak{L} = U \begin{bmatrix} 0_{c\times c} & \\ & \Lambda \end{bmatrix} U^\top.$$

Without loss of generality, first c rows of U can be written as

$$\begin{bmatrix} \frac{1}{\sqrt{|\mathcal{N}'_1|}} & & \\ & \ddots & \\ & & \frac{1}{\sqrt{|\mathcal{N}'_c|}} \end{bmatrix} \cdot p^\top$$

such that it spans $\ker \mathfrak{L}$. Define the matrix W as

$$W := \begin{bmatrix} \frac{1}{\sqrt{|\mathcal{N}_1'|}} & & & \\ & \ddots & & \\ & & \frac{1}{\sqrt{|\mathcal{N}_c'|}} & \\ & & & I_{N-c} \end{bmatrix} U.$$

Then it follows that

$$W = \begin{bmatrix} v^\top \\ R^\top \end{bmatrix}, \quad W^{-1} = U^\top \cdot \begin{bmatrix} \sqrt{|\mathcal{N}_1'|} & & & \\ & \ddots & & \\ & & \sqrt{|\mathcal{N}_c'|} & \\ & & & I_{N-c} \end{bmatrix} = \begin{bmatrix} p & R \end{bmatrix},$$

where R is a real matrix of the size $N \times (N - c)$. Since U is unitary, it is easy to verify that the rest of the properties hold. □

For the connected graphs, we only have a single connected component. Hence, it follows that

$$W = \begin{bmatrix} \frac{1}{N} 1_N^\top \\ R^\top \end{bmatrix}, \quad W^{-1} = \begin{bmatrix} 1_N & R \end{bmatrix},$$

which is often used for the analysis.

Although the Laplacian matrix is the most common matrix used for the analysis of a multi-agent system, its decomposition using the concept of incidence matrix is also useful. For this, let $G = (\mathcal{N}, \mathcal{E}, \mathfrak{A})$ be an undirected graph where the edge set is labeled[3] as $\mathcal{E} = \{e_1, \ldots, e_M\}$ with $M := |\mathcal{E}|$. Then the incidence matrix is defined as follows (e.g., see [KP17, GR01]).

Definition 2.1.8. *Incidence matrix* $\mathfrak{B} = [b_{ig}] \in \mathbb{R}^{N \times M}$ of a graph $G = (\mathcal{N}, \mathcal{E}, \mathfrak{A})$

[3] Since we are only considering the undirected graphs, we only count the edge (i, j) with $i > j$ and do not count (j, i) with $i \leq j$. This coincides with the notation typically used in the literature.

is defined by

$$b_{ig} = \begin{cases} -\sqrt{\alpha_{ij}}, & \text{if } e_g = (i,j), \\ \sqrt{\alpha_{ij}}, & \text{if } e_g = (j,i), \\ 0 & \text{otherwise.} \end{cases}$$

Using the incidence matrix, the Laplacian matrix can be decomposed as stated below [GR01, Lemma 8.3.2].

Proposition 2.1.5. Suppose G is an undirected graph. Then, it holds that

$$\mathfrak{L} = \mathfrak{B}\mathfrak{B}^\top. \qquad \Diamond$$

Proposition 2.1.5 states that for undirected graphs, the Laplacian matrix can be decomposed into the product of incidence matrix $\mathfrak{B}$ and its transpose. This property is useful when dealing with quantity for each edge, instead of each node, as done by the Laplacian matrix. To the best of the author's knowledge, no similar decomposition exists for the directed graphs.

2.2 Multilayer Graph Theory

A brief introduction to the multilayer graph is presented in this section. For an extensive overview and usage of multilayer graphs, see [BBC$^+$14] and [DDSRC$^+$13].

Definition 2.2.1. Time-varying multilayer graph with N nodes and L layers is defined as

$$\mathcal{G}(t) := \left(\mathcal{N}, \{\mathcal{E}^l(t)\}_{l\in\mathcal{L}}, \{\mathfrak{A}^l(t)\}_{l\in\mathcal{L}}\right),$$

where $\mathcal{N} = \{1, 2, \dots, N\}$ and $\mathcal{E}^l(t) \subseteq \mathcal{N} \times \mathcal{N}$ for each $l \in \mathcal{L} := \{1, \dots, L\}$. The graph $\mathcal{G}^l(t) := (\mathcal{N}, \mathcal{E}^l(t), \mathfrak{A}^l(t))$ is assumed to satisfy the properties of Definition 2.1.1 and we denote $\mathcal{G}^l(t)$ as the l-th layer of $\mathcal{G}$ and $\mathcal{E}^l(t)$ the edge set of the l-th layer of $\mathcal{G}$. $\Diamond$

The main feature of a multilayer graph is the existence of *multiple layers* of the edge sets denoted as $\mathcal{E}^l(t)$. In fact, a multilayer graph can be visualized as graphs stacked on top of each other as shown in Fig. 2.5(a). In particular, if $L = 1$, a multilayer graph is identical to the conventional definition of a graph, and we call it the *single-layer graph.* Many properties of the graph can be easily generalized to the multilayer graphs by considering each property in a *layer-wise* fashion. For example, common graph theoretical concepts are extended as follows.

Definition 2.2.2. Consider a multilayer graph $\mathcal{G}(t)$.

1. Adjacency matrices of $\mathcal{G}(t)$ are given by a set of matrices $\mathfrak{A}^1(t), \ldots, \mathfrak{A}^L(t)$ where $\mathfrak{A}^l(t)$ is the adjacency matrix of the graph $\mathcal{G}^l(t)$.
2. Similarly, Laplacian matrices of $\mathcal{G}(t)$ are given by a set of matrices $\mathfrak{L}^1(t), \ldots,$ $\mathfrak{L}^L(t)$ where $\mathfrak{L}^l(t)$ is the Laplacian matrix of the graph $\mathcal{G}^l(t)$.
3. Multilayer graph $\mathcal{G}(t)$ is *undirected* if $\mathcal{G}^l(t)$ is an undirected graph for all $l \in \mathcal{L}$ and *directed* otherwise.
4. *In-neighbors* of agent i on layer l is defined as $\mathcal{N}_i^l(t) := \{j \in \mathcal{N} \mid (j, i) \in \mathcal{E}^l(t)\}$. ◊

Finally, a useful concept for the multilayer graph [BBC+14, Section 2.1.1] is defined below.

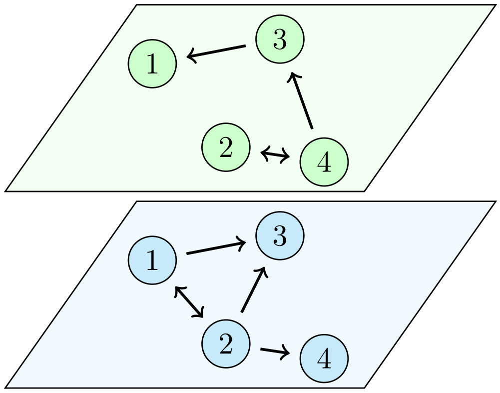
Definition 2.2.3. Projection graph of a multilayer graph $\mathcal{G}(t)$ is defined as

$$\operatorname{proj}(\mathcal{G})(t) := (\mathcal{N}, \mathcal{E}^{\mathrm{p}}(t), \mathfrak{A}^{\mathrm{p}}(t)),$$

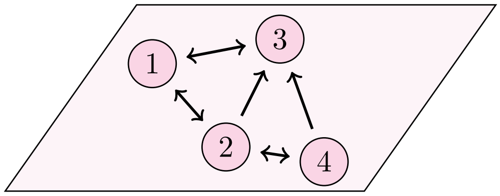
where $\mathcal{E}^{\mathrm{p}}(t) := \bigcup_{l \in \mathcal{L}} \mathcal{E}^l(t)$ and $\mathfrak{A}^{\mathrm{p}}(t) = [\alpha_{ij}^{\mathrm{p}}(t)] := \sum_{l=1}^{L} \mathfrak{A}^l(t)$. ◊

Note that the projection graph is a single-layer graph. An example of a multilayer graph with $N = 4$ and $L = 2$ is shown in Fig. 2.5(a) and its projection graph is shown in Fig. 2.5(b).

Remark 2.2.1. Multilayer graph defined in this section can be further generalized as in [BBC+14] to contain edges connecting nodes *between* layers. However, for our purposes, nodes from different layers represent the same agent. Hence, the physical meaning of such edge is less clear and thus not investigated in this dissertation. ◊



(a) Multilayer graph $\mathcal{G}$ with $N = 4$ and $L = 2$.



(b) Projection graph of $\mathcal{G}$.

Figure 2.5: Example of a multilayer graph $\mathcal{G}$ and $\text{proj}(\mathcal{G})$.

2.3 Convex Optimization

In this section, preliminary convex analysis and convex optimization are discussed. Basic definitions of the convex function as well as useful properties used for the main results are presented. We also discuss the continuous-time algorithms to minimize the convex function and analyze its convergence rates. For example, heavy-ball algorithms are introduced which have accelerated convergence rate. Only a selection of results directly related to the contents of the dissertation is presented. For a more general overview, refer to textbooks such as [Ber03, Ber16, Nes04, SPB19]

2.3.1 Convex Functions and Useful Properties

Basic definitions and fundamental properties are summarized from [Ber16].

Definition 2.3.1. A differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is *convex* if

$$f(y) \geq f(x) + (y - x)^\top \nabla f(x)$$

for all $x, y \in \mathbb{R}^n$ and *strictly convex* if the above inequality is strict whenever $y \neq x$. ◊

For convex functions, the following proposition lists useful relations.

Proposition 2.3.1. Let f be a convex function. For a scalar $L > 0$, the following properties are equivalent:

1. Globally Lipschitz gradient (or equivalently L-Smoothness):

$$|\nabla f(x) - \nabla f(y)| \leq L|x - y|$$

   for all $x, y \in \mathbb{R}^n$.

2. $f(x) + \nabla f(x)^\top (y - x) + \frac{1}{2L}|\nabla f(x) - \nabla f(y)|^2 \leq f(y)$, for all $x, y \in \mathbb{R}^n$.

3. Co-coercivity:

$$(\nabla f(x) - \nabla f(y))^\top (x - y) \geq \frac{1}{L}|\nabla f(x) - \nabla f(y)|^2$$

   for all $x, y \in \mathbb{R}^n$.

4. Quadratic Upper Bound:

$$f(y) \leq f(x) + \nabla f(x)^\top (y - x) + \frac{L}{2}|y - x|^2$$

   for all $x, y \in \mathbb{R}^n$. ◊

Proof. See [Ber16, Proposition B.3]. □

The first condition of Proposition 2.3.1 supposes the global Lipschitzness of the gradient of the function f, which is often assumed for the optimization problem. Properties listed in Proposition 2.3.1 can be simplified further for specific points in consideration. For instance, by letting $x = x^*$ where x^* is the optimal point of function f, condition 4 becomes

$$f(y) - f(x^*) \leq \frac{L}{2}|y - x^*|^2 \tag{2.3.1}$$

where we used $\nabla f(x^*) = 0$. This implies that the distance between the value of the function can be bounded by the distance between solutions.

Definition 2.3.2. A function $f : \mathbb{R}^n \to \mathbb{R}$ is α–*strongly convex* if

$$f(y) \geq f(x) + \nabla f(x)^\top (y - x) + \frac{\alpha}{2}|x - y|^2 \tag{2.3.2}$$

for all $x, y \in \mathbb{R}^n$. ◊

Note that if $\alpha = 0$, then the definition is equivalent to convex function. In addition, by substituting x^* into x in (2.3.2), we obtain

$$\frac{\alpha}{2}|y - x^*|^2 \leq f(y) - f(x^*), \quad \forall y \in \mathbb{R}^n. \tag{2.3.3}$$

Combining (2.3.3) with (2.3.1), it follows that for L–smooth and α–strongly convex function, distance in the value of function is equivalent to the distance in the values. (It is also evident that $\alpha \leq L$ must hold.)

Similar to that of convex function, strongly convex function satisfies the following properties.

Proposition 2.3.2. Let f be a α–strongly convex function. Then

$$(\nabla f(x) - \nabla f(y))^\top (x - y) \geq \alpha|x - y|^2, \quad \forall x, y \in \mathbb{R}^n.$$ ◊

Proof. Since f is α–strongly convex, it follows from (2.3.2) that

$$f(y) \geq f(x) + \nabla f(x)^\top (y - x) + \frac{\alpha}{2}|x - y|^2$$
$$f(x) \geq f(y) + \nabla f(y)^\top (x - y) + \frac{\alpha}{2}|y - x|^2.$$

Adding two inequalities and using $|x - y|^2 = |y - x|^2$, the result follows. □

Next, we introduce Polyak-Łojasiewicz inequality (PL inequality) first proposed in [Pol63, Thm. 4].

Definition 2.3.3. The function f satisfies PL inequality with $\mu > 0$ if

$$\frac{1}{2}|\nabla f(x)|^2 \geq \mu(f(x) - f^*)$$

where $f^* := \min_{x \in \mathbb{R}^n} f(x)$. ♢

PL condition is strictly weaker than strong convexity as shown below.

Proposition 2.3.3. Suppose f is α–strongly convex. Then f satisfies

$$\frac{1}{2}|\nabla f(x)|^2 \geq \alpha(f(x) - f^*)$$

where $f^* := \min_{x \in \mathbb{R}^n} f(x)$. ♢

Proof. Proof follows by taking minimum with respect to y the both sides of the inequality (2.3.2). First, the left-hand side becomes f^*. On the other hand, since the right-hand side is quadratic in y, it holds that the minimum value happens when

$$\nabla f(x) + \alpha(x - y) \cdot -1 = 0.$$

Thus, the right-hand side of (2.3.2) takes the minimum value when $y = x - (1/\alpha)\nabla f(x)$. Substituting this results in

$$\begin{aligned} f^* &\geq f(x) + \nabla f(x) \cdot \left(x - \frac{1}{\alpha}\nabla f(x) - x\right) + \frac{\alpha}{2}\left(x - x + \frac{1}{\alpha}\nabla f(x)\right)^2 \\ &= f(x) - \frac{1}{\alpha}|\nabla f(x)|^2 + \frac{1}{2\alpha}|\nabla f(x)|^2 \\ &= f(x) - \frac{1}{2\alpha}|\nabla f(x)|^2. \end{aligned}$$

Therefore, it follows immediately that f satisfies PL inequality. □

As shown in Proposition 2.3.3, PL condition is weaker than strong convexity. In fact, function may not even be convex even if it satisfies the PL inequality. Examples of function satisfying PL condition (but not convex) and strongly convex function are shown in Fig. 2.6.

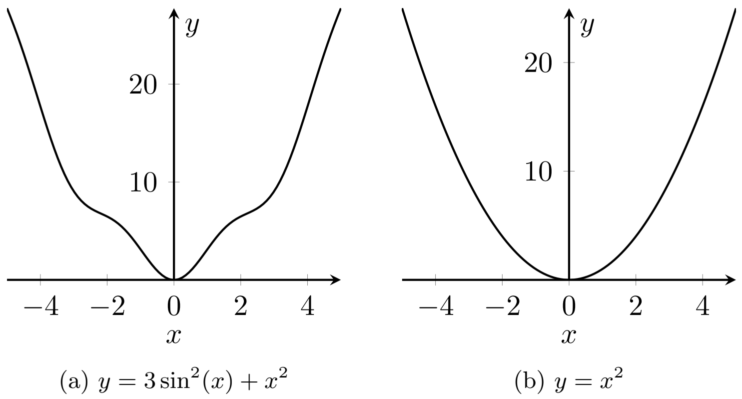


Figure 2.6: (a) An example of function satisfying PL inequality but not convex. (b) An example of a strongly convex function.

2.3.2 Optimization Algorithms

In this subsection, continuous-time optimization algorithms and their convergence properties are discussed. Throughout the discussion, suppose that f is a convex function and x^* be an optimal point with the value $f^* := f(x^*)$.

2.3.2.1 Continuous-time Gradient Descent Algorithm

The most basic continuous-time gradient descent is given by

$$\dot{x} = -\gamma \nabla f(x), \tag{2.3.4}$$

where $\gamma > 0$ is a gain. Its convergence properties are presented in following result.

Lemma 2.3.4. Consider the gradient descent algorithm given by (2.3.4). If f is convex and L-smooth, then it holds that

$$f(x(t)) - f^* \leq \frac{1}{t} \cdot \frac{1}{2\gamma} |x(0) - x^*|^2.$$

If $f \in \mathfrak{C}^1$ and satisfies PL inequality with μ instead, then

$$f(x(t)) - f^* \leq e^{-2\gamma\mu t} |f(0) - f^*|.$$

Hence, exponential convergence is achieved if f satisfies PL inequality. $\Diamond$

Proof. Proof is taken from [WSC16] and presented here for completeness. For the convergence when f is convex and L-smooth, consider the functional

$$E(t) := \gamma t(f(x) - f(x^*)) + \frac{1}{2}|x - x^*|^2.$$

Then, its time-derivative along (2.3.4) becomes

$$\begin{aligned}
\dot{E}(t) &= \gamma(f(x) - f(x^*)) + \gamma t \nabla f(x)^\top \cdot \dot{x} + (x - x^*)^\top \dot{x} \\
&= \gamma(f(x) - f(x^*)) - \gamma^2 t \nabla f(x)^\top \nabla f(x) - \gamma(x - x^*)^\top \nabla f(x) \\
&= \gamma(f(x) - f(x^*) - \nabla f(x)^\top (x - x^*)) - \gamma^2 t \nabla f(x)^\top \nabla f(x) \\
&\leq -\gamma^2 t \nabla f(x)^\top \nabla f(x) \\
&\leq 0,
\end{aligned}$$

that is, $E(t)$ is non-increasing. Since $|x - x^*|^2 \geq 0$, we obtain

$$\gamma t(f(x) - f(x^*)) \leq E(t) \leq E(0) = \frac{1}{2}|x(0) - x^*|^2,$$

which implies

$$f(x(t)) - f(x^*) \leq \frac{1}{t} \cdot \frac{1}{2\gamma}|x(0) - x^*|^2.$$

In case when f satisfies the PL inequality with μ, consider the function $V(x) = f(x) - f^*$. Then the time-derivative of V along (2.3.4) becomes

$$\begin{aligned}
\dot{V} &= \nabla f(x)^\top \cdot -\gamma \nabla f(x) \\
&\leq -2\gamma\mu(f(x) - f^*) \\
&= -2\gamma\mu V,
\end{aligned}$$

where we used the definition of the PL inequality. This results in

$$f(x(t)) - f^* \leq e^{-2\gamma\mu t}(f(x(0)) - f^*). \qquad \square$$

Result of Lemma 2.3.4 establishes the convergence rate of gradient descent algorithm. Note that the convergence rate is defined in terms of the value of the function $f(x) - f^*$ and how fast it decays. For the general convex function, this does not necessarily translate into the convergence rate of the optimal solution, i.e., $x - x^*$. However, if the function is α–strongly convex, then the under bound (2.3.3) can be used to obtain the convergence rate in terms of $|x - x^*|$. For instance, using Proposition 2.3.3 and Lemma 2.3.4, it is easy to see that

$$\frac{\alpha}{2}|x(t) - x^*|^2 \leq f(x(t)) - f^* \leq \frac{L}{2}e^{-2\gamma\alpha t}|x(0) - x^*|^2,$$

which implies

$$|x(t) - x^*| \leq \frac{L}{\alpha}e^{-\gamma\alpha t}|x(0) - x^*|. \tag{2.3.5}$$

Remark 2.3.1. Seeing (2.3.5) from nonlinear system theory perspective, we may say that $x(t)$ converges to x^* exponentially with rate $\gamma\alpha$ with the gain L/α [CG98]. Now imagine the convergence rate as $\alpha \to 0$. It is natural that the convergence rate gets slower (in fact linearly since we are using gradient descent method). Meanwhile, we also see that the gain L/α *increases* as $\alpha \to 0$. This means that the upper bound provided by (2.3.5) gets worse for small time t. For general nonlinear system, this is a typical phenomena, that is as we accurately model convergence rate, the gain gets larger (see [CG98] for more discussion). However, for simple gradient descent method, by using $V(x - x^*) = (1/2)(x - x^*)^\top(x - x^*)$ as the Lyapunov function, it is easy to show

$$|x(t) - x^*| \leq e^{-\gamma\alpha t}|x(0) - x^*|.$$

However, such a tight upper bound is hard to obtain for more complex algorithms. ◊

2.3.2.2 Heavy-ball Method

As seen from Lemma 2.3.4, the convergence rate for general convex function is in order of $1/t$ and exponential with the rate of α for α–strongly convex function. However, it has been proven that the convergence rate for L-smooth and $\alpha-$strongly convex function using discrete-time algorithm can be improved and the corresponding algorithms achieving the optimal rate have been developed [Nes04]. Such algorithms achieving the optimal rate are often called *accelerated gradient methods* and the main technique used for improving the convergence rate is often termed as *momentum*. The most well-known algorithms for achieving the accelerated convergence rate are Nesterov's gradient descent and Polyak's heavy-ball method. These algorithms are conventionally studied in the discrete-time domain with the corresponding analysis for their convergence rates. In this dissertation, the continuous-time heavy-ball method is mainly used for the development and hence the rest of the section focuses on the continuous-time version of the heavy-ball method.

For an α–strongly convex function, the continuous-time heavy-ball method can be written [Qia99] as

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} P_1 z \\ -P_2 z - P_3 \nabla f(x) \end{bmatrix}, \tag{2.3.6}$$

where $P_1, P_2, P_3 > 0$ are symmetric positive definite matrices. Then, the following convergence result can be shown.

Proposition 2.3.5. (Asymptotic convergence of the general heavy-ball algorithm) Consider the heavy-ball method (2.3.6). Suppose that f is convex and the sub-level sets of f are compact. Then the solution of (2.3.6) converges to $(x^*, 0)$ if there exists $\Phi > 0$ such that

$$\Phi P_3 = P_1, \quad \Phi P_2 + P_2 \Phi > 0 \tag{2.3.7}$$

holds. $\Diamond$

Proof. Let the Lyapunov function be

$$V(x, z) = f(x) - f^* + \frac{1}{2} z^\top \Phi z.$$

Then, its time-derivative along (2.3.6) becomes

$$\begin{aligned}\dot{V} &= \nabla f(x)^\top \cdot P_1 z - \frac{1}{2} z^\top (\Phi P_2 + P_2 \Phi) z - z^\top \Phi P_3 \nabla f(x) \\ &= -\frac{1}{2} z^\top (\Phi P_2 + P_2 \Phi) z \\ &\leq 0,\end{aligned}$$

where we used (2.3.7). Hence, the solution is bounded. Applying LaSalle's invariance principle [Kha02, Thm. 4.4], it follows that the solution converges to the set

$$E := \{(x, z) \mid z^\top (\Phi P_2 + P_2 \Phi) z = 0\}.$$

Since $\Phi P_2 + P_2 \Phi > 0$, it follows that $z = 0$ and consequently implies $\dot{z} = 0$ and $\dot{x} = 0$ within E. However, this implies

$$0 = -P_2 z - P_3 \nabla f(x) \implies \nabla f(x) = 0.$$

Thus x converges to an optimal point. □

Remark 2.3.2. The condition (2.3.7) includes a few common cases for heavy-ball method. For example, $P_3 = P_1$ and $Q = I_n$ satisfies (2.3.7) which results in

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} P_1 z \\ -P_2 z - P_1 \nabla f(x) \end{bmatrix}.$$

On the other hand, if $P_2 = P_3 = I_n$ and $Q = P_1$ results in

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} P_1 z \\ -z - \nabla f(x) \end{bmatrix}. \qquad \Diamond$$

Result of Proposition 2.3.5 proves the asymptotic convergence for the general heavy-ball algorithm under mild condition on the function f. However, the convergence rate cannot be characterized due to the usage of LaSalle's principle. For the strongly convex functions, accelerated convergence can be shown for a particular choice of P_1, P_2 and P_3.

Theorem 2.3.6. (Accelerated convergence for heavy-ball method) Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is α–strongly convex, L–smooth and consider the heavy-ball algorithm (2.3.6) with $P_1 = P_3 = I_n$ and $P_2 = 2\sqrt{\alpha} I_n$, i.e.,

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} z \\ -2\sqrt{\alpha} z - \nabla f(x) \end{bmatrix}. \tag{2.3.8}$$

Then the solution of (2.3.8) converges to $(x^*, 0)$ exponentially fast. In particular, with $\psi := [x - x^*; z]$, it holds that

$$|\psi(t)| \leq M e^{-\frac{\sqrt{\alpha}}{2} t} |\psi(0)|$$

where the constant $M > 0$ is defined as

$$M := \sqrt{\frac{1}{\lambda_{\min}(Q)} \cdot \max\left(\frac{3L}{2}, 1\right)}, \quad Q := \begin{bmatrix} \alpha & \frac{\sqrt{\alpha}}{2} \\ \frac{\sqrt{\alpha}}{2} & \frac{1}{2} \end{bmatrix}$$

and $\lambda_{\min}(Q) = \frac{1}{2}\left(\alpha + \frac{1}{2} - \sqrt{\alpha^2 + \frac{1}{4}}\right)$. ◊

Proof. Proof is based on [Sie19, Thm. 2.2] (also see [WRJ16]) and here we extend the analysis to the case when $z(0) \neq 0$. Consider the Lyapunov function

$$V(x, z) := f(x(t)) - f(x^*) + \frac{1}{2}\left|\sqrt{\alpha}(x(t) - x^*) + z(t)\right|^2.$$

It is easy to see that $V(x, z)$ is positive definite. Consequently, quadratic upper and lower bound for $V(x, z)$ can also be found. For the upper bound, we have

$$\begin{aligned} V(x, z) &\leq \frac{L}{2}|x - x^*|^2 + \frac{1}{2}\alpha|x - x^*|^2 + \frac{1}{2}|z|^2 + \sqrt{\alpha}|x - x^*||z| \\ &\leq \left(\frac{L + \alpha}{2}\right)|x - x^*|^2 + \frac{1}{2}|z|^2 + \frac{\alpha}{2}|x - x^*|^2 + \frac{1}{2}|z|^2 \\ &= \left(\frac{L + 2\alpha}{2}\right)|x - x^*|^2 + |z|^2 \\ &\leq \max\left(\frac{3L}{2}, 1\right) \cdot \left|\begin{bmatrix} x - x^* \\ z \end{bmatrix}\right|^2, \end{aligned}$$

where we used quadratic upper bound property for L–smooth function (2.3.3), $\alpha \leq L$ and Young's inequality[4]. On the other hand, we can obtain the under bound of $V(x, z)$ as

$$\begin{aligned} V(x,z) &\geq \frac{\alpha}{2}|x - x^*|^2 + \frac{1}{2}\left|\sqrt{\alpha}(x - x^*) + z\right|^2 \\ &= \begin{bmatrix} x - x^* \\ z \end{bmatrix} \begin{bmatrix} \alpha & \frac{\sqrt{\alpha}}{2} \\ \frac{\sqrt{\alpha}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x - x^* \\ z \end{bmatrix} \\ &=: \psi^\top Q \psi, \end{aligned}$$

where we used the quadratic under bound property for strongly convex functions (2.3.3). Applying Schur's complement to the matrix Q, it holds that

$$\frac{1}{2} - \frac{\alpha}{4} \cdot \frac{1}{\alpha} = \frac{1}{2} - \frac{1}{4} > 0.$$

Hence, $Q > 0$ and $V(x, z)$ has an quadratic under bound. Moreover, the eigenvalues of Q are given by

$$\lambda_{\min}(Q) = \frac{1}{2}\left(\alpha + \frac{1}{2} - \sqrt{\alpha^2 + \frac{1}{4}}\right), \quad \lambda_{\max}(Q) = \frac{1}{2}\left(\alpha + \frac{1}{2} + \sqrt{\alpha^2 + \frac{1}{4}}\right).$$

The time-derivative of $V(x, z)$ along (2.3.8) becomes

$$\begin{aligned} \dot{V} &= \nabla f(x)^\top \cdot z + (-\sqrt{\alpha} z - \nabla f(x))^\top \cdot (\sqrt{\alpha}(x - x^*) + z) \\ &= -\sqrt{\alpha} \nabla f(x)^\top (x - x^*) - \alpha z^\top (x - x^*) - \sqrt{\alpha} z^\top z \end{aligned}$$

Here we use definition of strong convexity (2.3.2) with $y = x^*$ to obtain

$$-\nabla f(x)^\top (x - x^*) \leq -1 \cdot \left(f(x) - f^* + \frac{\alpha}{2}|x - x^*|^2\right).$$

Therefore, we obtain

$$\dot{V} = -\sqrt{\alpha}\left(f(x) - f^* + \frac{\alpha}{2}|x - x^*|^2\right) - \alpha z^\top (x - x^*) - \sqrt{\alpha} z^\top z$$

[4]Here we use the version of Young's inequality which states that for any $a, b \geq 0$ and $\varepsilon \geq 0$, $ab \leq \frac{1}{2\varepsilon}|a|^2 + \frac{\varepsilon}{2}|b|^2$ holds.

$$= -\sqrt{\alpha}\left(f(x) - f^* + \frac{\alpha}{2}|x - x^*|^2 + \sqrt{\alpha}z^\top(x - x^*) + \frac{1}{2}z^\top z + \frac{1}{2}z^\top z\right)$$
$$= -\sqrt{\alpha}\left(f(x) - f^* + \frac{1}{2}\left|\sqrt{\alpha}(x - x^*) + z\right|^2\right) - \frac{\sqrt{\alpha}}{2}z^\top z$$
$$\leq -\sqrt{\alpha}V.$$

Hence, it follows that $V(t) \leq e^{-\sqrt{\alpha}t}V(0)$. From the upper and lower bounds of $V(x,z)$, we obtain

$$\lambda_{\min}(Q)|\psi(t)|^2 \leq \psi(t)^\top Q\psi(t) \leq V(t) \leq e^{-\sqrt{\alpha}t}V(0)$$
$$\leq \max\left(\frac{3L}{2}, 1\right) e^{-\sqrt{\alpha}t}|\psi(0)|^2.$$

This implies

$$|\psi(t)| \leq \sqrt{\frac{1}{\lambda_{\min}(Q)} \cdot \max\left(\frac{3L}{2}, 1\right)} e^{-\frac{\sqrt{\alpha}}{2}t}|\psi(0)|,$$

which completes the proof. □

Although both gradient descent algorithm (2.3.4) and heavy-ball method (2.3.8) achieve exponential convergence for strongly convex functions, their convergence rates are different. Specifically, the gradient descent method converges with a rate of α whereas the heavy-ball method achieves the rate of $\sqrt{\alpha}/2$. In fact, it is observed in the practice that gradient descent achieves a faster convergence rate when α is high (i.e., when $\alpha > 1/2$). However, if α is low (i.e., when $0 < \alpha < 1/2$), then Theorem 2.3.6 states that the heavy-ball method converges faster. The following example illustrates this for a simple function.

Example 2.3.1. Consider $f(x) = (1/2)ax^2$ which is a–strongly convex. Consider the gradient descent algorithm given by

$$\dot{x} = -\nabla f(x) = -ax \tag{2.3.9}$$

and the heavy-ball method given by

$$\begin{aligned} \dot{x} &= z \\ \dot{z} &= -2\sqrt{a}z - ax. \end{aligned} \tag{2.3.10}$$

Since both gradient descent and heavy-ball method are linear systems, convergence rates can be computed easily by investigating its eigenvalues. Specifically, eigenvalue of the system (2.3.9) is given by $-a$, where as the eigenvalues of the system (2.3.10) are given by $\{-\sqrt{a}, -\sqrt{a}\}$. Hence, the heavy-ball method converges faster for $0 < a < 1$ and the gradient descent method converges faster for $a > 1$. ♢

Remark 2.3.3. By applying state transformation to (2.3.8), exponential convergence can be shown for the slightly modified dynamics. Specifically, consider the state transformation given by

$$\begin{bmatrix} \xi \\ x \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ 0 & I \end{bmatrix} \begin{bmatrix} z \\ x \end{bmatrix}, \quad \begin{bmatrix} z \\ x \end{bmatrix} = \begin{bmatrix} A_1^{-1} & A_1^{-1}A_2 \\ 0 & I \end{bmatrix} \begin{bmatrix} \xi \\ x \end{bmatrix}.$$

where A_1 is an invertible matrix. Then we have

$$\begin{aligned} \dot{\xi} &= A_1\dot{z} + A_2\dot{x} \\ &= A_1(-2\sqrt{\alpha}z - \nabla f(x)) + A_2 z \\ &= A_1(-2\sqrt{\alpha}(A_1^{-1}\xi + A_1^{-1}A_2 x)) - A_1\nabla f(x) + A_2(A_1^{-1}\xi + A_1^{-1}A_2 x) \\ &= \left[-2\sqrt{\alpha}I + A_2A_1^{-1}\right]\xi + \left[A_2 + A_2A_1^{-1}A_2\right]x - A_1\nabla f(x) \\ \dot{x} &= z \\ &= A_1^{-1}\xi + A_1^{-1}A_2 x. \end{aligned}$$

The transformed dynamics seems much more complex, but exponential convergence is conserved since the transformation is linear. ♢

Theorem 2.3.6 studied the exponential convergence for a specific set of parameters. A more general convergence result is shown in the following theorem.

Theorem 2.3.7. (Exponential convergence of the general heavy-ball algorithm) Consider the general heavy-ball algorithm (2.3.6) where $P_1, P_2, P_3 > 0$. Then the solution converges exponentially if $P_1 P_3^{-1}$ is a symmetric matrix and

$$P_1 P_3^{-1} > 0, \quad \Psi := (P_1 P_3^{-1}) P_2 + P_2 (P_1 P_3^{-1}) > 0 \tag{2.3.11}$$

are satisfied. ◊

Proof. Let the Lyapunov function be

$$V(x, z) := \gamma \left(f(x(t)) - f^* \right) + \begin{bmatrix} x - x^* \\ z \end{bmatrix}^\top Q \begin{bmatrix} x - x^* \\ z \end{bmatrix},$$

where the matrix $Q > 0$ is in form of

$$Q := \begin{bmatrix} Q_1 & Q_2 \\ Q_2 & Q_3 \end{bmatrix}$$

where each element Q_i is to be determined. Note that using (2.3.3) and (2.3.1), it is easy to verify

$$\begin{aligned} \min(\gamma, \lambda_{\min}(Q)) |\psi|^2 &\leq \min(\gamma, \lambda_{\min}(Q)) \left[(f - f^*) + |\psi|^2 \right] \\ &\leq V(x, z) \\ &\leq \max(\gamma, \lambda_{\max}(Q)) \left[f - f^* + |\psi|^2 \right] \leq \left(\lambda_{\max}(Q) + \frac{\gamma L}{2} \right) |\psi|^2, \end{aligned}$$

where $\psi := [x; z]$ and let $\alpha_1 := \min(\gamma, \lambda_{\min}(Q))$, $\alpha_2 := \max(\gamma, \lambda_{\max}(Q))$.

Taking time-derivative of V along (2.3.6), it follows that

$$\begin{aligned} \dot{V} &= \gamma \nabla f(x)^\top P_1 z + 2(x - x^*)^\top Q_1 P_1 z \\ &\quad + z^\top (-Q_3 P_2 - P_2 Q_3) z - 2 z^\top Q_3 P_3 \nabla f(x) \\ &\quad -2(x - x^*)^\top Q_2 P_2 z - 2(x - x^*)^\top Q_2 P_3 \nabla f(x) + 2 z^\top P_1 Q_2 z \\ &= z^\top \left(\gamma P_1 - 2 Q_3 P_3 \right) \nabla f(x) + (x - x^*)^\top \left(2 Q_1 P_1 - 2 Q_2 P_2 \right) z \\ &\quad + z^\top \left(-Q_3 P_2 - P_2 Q_3 + 2 P_1 Q_2 \right) z + (x - x^*)^\top \cdot -2 Q_2 P_3 \nabla f(x). \end{aligned} \tag{2.3.12}$$

Now, for further analysis we let

$$Q_3 = \frac{\gamma}{2} P_1 P_3^{-1}, \quad Q_2 = \frac{1}{2} P_3^{-1} \tag{2.3.13}$$

where $P_1 P_3^{-1}$ is symmetric positive definite matrix due to assumption. Then, the first and last term of (2.3.12) satisfies

$$\gamma P_1 - 2Q_3 P_3 = \gamma P_1 - 2\frac{1}{2} P_1 P_3^{-1} P_3 = 0, \quad 2Q_2 P_3 = I.$$

Then, (2.3.12) becomes

$$\begin{aligned}\dot{V} &= (x - x^*)^\top \left(2Q_1 P_1 - P_3^{-1} P_2\right) z + z^\top \left(-Q_3 P_2 - P_2 Q_3 + P_1 P_3^{-1}\right) z \\ &\quad - (x - x^*)^\top \nabla f(x).\end{aligned}$$

Recall from (2.3.2) that

$$-(x - x^*)^\top \nabla f(x) \leq -1 \cdot \left(f(x) - f^* + \frac{\alpha}{2} |x - x^*|^2\right).$$

Hence, we obtain

$$\begin{aligned}\dot{V} &\leq (x - x^*)^\top \left(2Q_1 P_1 - P_3^{-1} P_2\right) z + \left(-\frac{\gamma}{2} \lambda_{\min}(\Psi) + \lambda_{\max}(P_1 P_3^{-1})\right) z^\top z \\ &\quad - \left(f(x) - f^* + \frac{\alpha}{2} |x - x^*|^2\right) \\ &= -(f(x) - f^*) \\ &\quad + \begin{bmatrix} x - x^* \\ z \end{bmatrix}^\top \begin{bmatrix} -\frac{\alpha}{2} & Q_1 P_1 - \frac{1}{2} P_3^{-1} P_2 \\ P_1 Q_1 - \frac{1}{2} P_2 P_3^{-1} & (-\frac{\gamma}{2} \lambda_{\min}(\Psi) + \lambda_{\max}(P_1 P_3^{-1}))I \end{bmatrix} \begin{bmatrix} x - x^* \\ z \end{bmatrix} \\ &=: -(f(x) - f^*) - \psi^\top \Delta \psi\end{aligned}$$

From Schur's complement, it holds that $-\Delta < 0$ if

$$-\frac{\gamma}{2} \lambda_{\min}(\Psi) + \lambda_{\max}(P_1 P_3^{-1}) + (P_1 Q_1 - \frac{1}{2} P_2 P_3^{-1}) \cdot \frac{\alpha}{2} \cdot (Q_1 P_1 - \frac{1}{2} P_3^{-1} P_2) < 0.$$

Hence let $\gamma_1 > 0$ be sufficiently large such that

$$\frac{2}{\lambda_{\min}(\Psi)} \cdot \left[I + (P_1 Q_1 - \frac{1}{2} P_2 P_1^{-1}) \cdot \frac{\alpha}{2} \cdot (Q_1 P_1 - \frac{1}{2} P_1^{-1} P_2) \right] < \gamma_1.$$

Then, it holds that

$$\dot{V} \leq -(f(x) - f^*) - \lambda_{\min}(\Delta)|\psi|^2 \leq -\lambda_{\min}(\Delta)|\psi|^2.$$

Next, from (2.3.13), matrix Q used in the definition of V becomes

$$Q = \begin{bmatrix} Q_1 & \frac{1}{2} P_3^{-1} \\ \frac{1}{2} P_3^{-1} & \frac{\gamma}{2} P_1 P_3^{-1} \end{bmatrix}.$$

Hence $Q > 0$ if and only if

$$\frac{\gamma}{2} P_1 P_3^{-1} - \frac{1}{4} P_3^{-1} Q_1^{-1} P_3^{-1} > 0.$$

Thus, let $\gamma_2 > 0$ such that

$$\gamma_2 > \frac{1}{2} \frac{\lambda_{\max}(P_3^{-1} Q_1^{-1} P_3^{-1})}{\lambda_{\min}(P_1 P_3^{-1})}.$$

Finally, choose $\gamma > \max(\gamma_1, \gamma_2)$ such that V is positive definite while its time-derivative is negative definite. Specifically, the Lyapunov function $V(x, z)$ satisfies

$$\begin{aligned} \min(\gamma, \lambda_{\min}(Q))|\psi(t)|^2 \leq V(x(t), z(t)) &\leq V(x(0), x(0)) \cdot e^{-\lambda_{\min}(\Delta)t} \\ &\leq \left(\lambda_{\max}(Q) + \frac{\gamma L}{2} \right) |\psi(0)|^2 \cdot e^{-\lambda_{\min}(\Delta)t}. \end{aligned}$$

In conclusion,

$$|\psi(t)|^2 \leq \frac{\lambda_{\max}(Q) + \frac{\gamma L}{2}}{\min(\gamma, \lambda_{\min}(Q))} |\psi(0)|^2 \cdot e^{-\lambda_{\min}(\Delta)t},$$

which completes the proof. □

Remark 2.3.4. It can be easily seen that the condition (2.3.11) is equivalent to the condition (2.3.7) used for asymptotic convergence. In order for P_1, P_2 and

P_3 to satisfy the condition (2.3.11), P_1 and P_3 must commute so that $P_1P_3^{-1}$ is a symmetric matrix. This is easily done if either $P_1 = P_3$, or either P_1 or P_3 is identity matrix. If $P_1 = P_3$, then we obtain $\Psi = P_2 + P_2 > 0$. Instead, if $P_3 = I$, we get $\Psi = P_1P_2 + P_2P_1$. Thus, $\Psi > 0$ easily holds if $P_2 = I$ and $P_1 > 0$. $\lozenge$

Remark 2.3.5. From the proof of Theorem 2.3.7, it can be checked that the convergence rate for $f(x(t))-f^*$ is linear in α. This is in contrast to Theorem 2.3.6 which had convergence rate of $\sqrt{\alpha}$. Hence, although the exponential convergence is proven for general parameters, it does not prove acceleration. $\lozenge$

Despite the long history of the optimization theory and various algorithms, most discussions are done in the discrete-time domain due to its practicality. Analysis of the corresponding continuous-time algorithms are only started gaining attention again in recent years. For continuous-time algorithms with strongly convex functions, Theorem 2.3.7 proves global exponential convergence of the heavy-ball method. However, it did not show an accelerated convergence rate. Accelerated convergence rate can be seen in Theorem 2.3.6 for specific set of parameters which is shown recently in [Sie19, WRJ16]. For general convex functions, non-ergodic rate of $\mathcal{O}(1/t)$ is shown in [SYL$^+$19] under additional constraint that $|\ddot{x}(t)| = |\dot{z}(t)| \leq \theta|z(t)|$ for some $\theta > 0$. On the other hand, the continuous-time Nesterov's gradient method received more attention and it has been proved to achieve convergence rate proportional to $1/t^2$ for general convex functions [WSC16] and exponential convergence with the rate proportional to $\sqrt{\alpha}$ for strongly convex functions [SDSJ19, SDJS18]. Accelerated convergence rate of $\sqrt{\alpha}$ is particularly impactful in the machine learning context as α depends on the inverse of the sample size [Bub15, Sec. 3.6].

Chapter 3

Consensus Problem over the Multilayer Network

This chapter presents the main results on the consensus of MAS over the multilayer network.

3.1 Problem Formulation

Consider the multi-agent system with N agents and the time-invariant multilayer graph $\mathcal{G}$ with L layers. We suppose that the dynamics of each agent is given by

$$\begin{aligned} \dot{x}_i &= Ax_i + Bu_i, \\ y_i^l &= C^l x_i, \\ \zeta_i &= Rx_i, \quad i \in \mathcal{N},\ l \in \mathcal{L}, \end{aligned} \tag{3.1.1}$$

where $x_i \in \mathbb{R}^n$, $y_i^l \in \mathbb{R}^{q^l}$ and $u_i \in \mathbb{R}^p$ are state, output and input respectively. The common output $\zeta_i \in \mathbb{R}^q$ is the signal to be synchronized among agents. We also suppose throughout the dissertation that (A, B) is stabilizable.

Note that each agent has L different outputs over L layers, and the output matrix corresponding to each layer is denoted as C^l. Thus, the output of agent i on layer l is defined as[1] $y_i^l = C^l x_i$ (which is null if y_i^l is not available to agent i).

[1]Throughout the chapter, the superscript (subscript) is often used to denote the index of a layer (agent) respectively.

Furthermore, we suppose that the communication among agents is described by the multilayer graph $\mathcal{G}$. Specifically, each agent only has access to the relative output information δ_i over L layers which is defined as

$$\delta_i := \begin{bmatrix} \delta_i^1 \\ \vdots \\ \delta_i^L \end{bmatrix} = \begin{bmatrix} \sum_{j\in\mathcal{N}_i^1}(y_j^1 - y_i^1) \\ \vdots \\ \sum_{j\in\mathcal{N}_i^L}(y_j^L - y_i^L) \end{bmatrix} \in \mathbb{R}^{\bar{q}},$$

where $\mathcal{N}_i^l := \{j \in \mathcal{N} \mid (j,i) \in \mathcal{E}^l\}$ denotes the set of neighbors of agent i on layer l and $\bar{q} := \sum_{l=1}^{L} q^l$. Finally, we assume that each agent is equipped with a dynamic controller of the form

$$\begin{aligned} \dot{\xi}_i &= f_{\mathrm{c},i}\big(\delta_i, \chi_i\big), \\ u_i &= h_{\mathrm{c},i}\big(\delta_i, \chi_i\big), \end{aligned} \tag{3.1.2}$$

where $\xi_i \in \mathbb{R}^\nu$ is the state of the controllers and χ_i represents the communication of ξ_i between controllers. In particular, we consider the following cases for χ_i:

1. Long-range communication of controller state, i.e., χ_i is a stack of ξ_k for $k \in \mathcal{P}_i^{\mathrm{p}} \cup \{i\}$, where $\mathcal{P}_i^{\mathrm{p}}$ is the set of nodes having a directed path to node i on $\mathrm{proj}\,(\mathcal{G})$. This is used when we derive a necessary condition to achieve consensus under a general form of controller.

2. Local communication of controller state, i.e., χ_i is stack of ξ_k for $k \in \mathcal{N}_i^{\mathrm{p}} \cup \{i\}$, where $\mathcal{N}_i^{\mathrm{p}} := \{j \in \mathcal{N} \mid (j,i) \in \mathcal{E}^{\mathrm{p}}\}$. This is used for the construction of dynamic controller.

3. No communication between controllers, i.e., $\chi_i = \xi_i$. Controller of this form is presented in Section 4.4.

Regardless of the exact definition of χ_i, each agent combines available information (e.g., δ_i and χ_i) to compute the control input u_i. The functions $f_{\mathrm{c},i}$ and $h_{\mathrm{c},i}$ are assumed to be locally Lipschitz functions satisfying $f_{\mathrm{c},i}(0,0) = 0$ and $h_{\mathrm{c},i}(0,0) = 0$. Moreover, it is supposed that the solution of the overall system (3.1.1)–(3.1.2) is well defined for all $t \geq 0$ from any initial condition.

Remark 3.1.1. If the communication structure is given by a single-layer graph, i.e., $L = 1$, then the dynamic controller (3.1.2) includes distributed, linear controllers proposed in the literature to solve the classical consensus problem. For example, if $h_{\mathrm{c},i}(\delta_i, \chi_i) = h_{\mathrm{c},i}(\delta_i)$, i.e., each control input only depends on δ_i, then (3.1.2) represents static diffusive output controller which is studied in [ZLD11, MZ10, YRWS11]. If $h_{\mathrm{c},i}(\delta_i, \chi_i) = h_{\mathrm{c},i}(\delta_i, \xi_i)$ and $f_{\mathrm{c},i}(\delta_i, \chi_i) = f_{\mathrm{c},i}(\delta_i, \xi_i)$, then (3.1.2) becomes a dynamic controller only using relative output information as in [SSB09, WSS+13]. If $h_{\mathrm{c},i}(\delta_i, \chi_i) = h_{\mathrm{c},i}(\delta_i, \{\xi_j\}_{j\in\mathcal{N}_i\cup\{i\}})$ and $f_{\mathrm{c},i}(\delta_i, \chi_i) = f_{\mathrm{c},i}(\delta_i, \{\xi_j\}_{j\in\mathcal{N}_i\cup\{i\}})$, then states of the controllers are received from its neighbors which include observer-based controllers proposed in [ZLD11, WLH09]. ◊

In this dissertation, necessary conditions to achieve consensus for the multi-agent systems over the multilayer network is studied. To be precise, the notion of the consensusability is defined as follows.

Definition 3.1.1. System (3.1.1) is *output consensusable* with an (nonzero) output matrix $R \in \mathbb{R}^{q\times n}$, if there exists a controller of the form (3.1.2) such that for all initial conditions $x_i(0)$ and $\xi_i(0)$, it holds that

$$\lim_{t\to\infty} |\zeta_i(t) - \zeta_j(t)| = 0, \quad \forall i, j \in \mathcal{N}.$$

In addition, the system (3.1.1) is *state consensusable* if $R = I_n$. ◊

If the matrix A in (3.1.1) is Hurwitz, then the system (3.1.1) is trivially consensusable. Thus, this case is excluded in the study and the following assumption is made.

Assumption 3.1.1. The system (3.1.1) satisfies $\mathcal{X}^u(A) \not\subseteq \langle \ker R \mid A\rangle$. ◊

If Assumption 3.1.1 does not hold, i.e., $\mathcal{X}^u(A) \subseteq \langle \ker R \mid A\rangle$, then it can be verified that letting $u_i(t) = 0$ for all $t \geq 0$ results in $\lim_{t\to\infty} |\zeta_i(t) - \zeta_j(t)| = 0$ from any initial condition. Hence, (3.1.1) is trivially output consensusable.

For the analysis of the multi-agent systems, detectability decomposition is used extensively. Consider the dynamics of an agent given by (3.1.1) and let y_i^l as the output of the system. Then, there exists an invertible matrix $(T^l)^{-1} \in \mathbb{R}^{n\times n}$ in form of

$$T^l := \begin{bmatrix} Y^l & U^l \end{bmatrix}, \quad (T^l)^{-1} = \begin{bmatrix} (Z^l)^\top \\ (W^l)^\top \end{bmatrix} \tag{3.1.3}$$

such that

$$(T^l)^{-1}AT^l = \begin{bmatrix} (Z^l)^\top AY^l & 0 \\ \star & (W^l)^\top AU^l \end{bmatrix} =: \begin{bmatrix} A_d^l & 0 \\ \star & A_{\bar{d}}^l \end{bmatrix},$$

and

$$C^lT^l = \begin{bmatrix} C^lY^l & 0 \end{bmatrix} =: \begin{bmatrix} C_d^l & 0 \end{bmatrix}$$

for each $l \in \mathcal{L}$, where the pair $A_d^l \in \mathbb{R}^{(n-\nu^l)\times(n-\nu^l)}$ and $C_d^l \in \mathbb{R}^{q^l\times(n-\nu^l)}$ are such that (C_d^l, A_d^l) is detectable and the asterisk denotes the elements that are not of our interest. Specifically, it holds that $Y^l \in \mathbb{R}^{n\times(n-\nu^l)}$, $U^l \in \mathbb{R}^{n\times\nu^l}$, $Z^l \in \mathbb{R}^{n\times(n-\nu^l)}$ and $W^l \in \mathbb{R}^{n\times\nu^l}$. Also recall that $\ker (Z^l)^\top$ is the undetectable subspace of the pair (C^l, A), i.e., $\ker (Z^l)^\top = \langle \ker C^l \,|\, A\rangle \cap \mathcal{X}^u(A)$ [TSH12, Thm. 5.15], and that $\nu^l = \dim\ker(Z^l)^\top$ is the dimension of undetectable subspace. Similarly, define $Z_R \in \mathbb{R}^{n\times(n-\nu^R)}$ such that $\ker (Z_R)^\top = \langle \ker R \,|\, A\rangle \cap \mathcal{X}^u(A)$ where $\nu^R := \dim\ker Z_R^\top$ and Y_R accordingly.

Remark 3.1.2. The consensus problem over multilayer network defined on this section strictly generalizes the classical consensus problem over single-layer network. Namely, multilayer network may represent the single-layer network, but the converse cannot be done. Furthermore, even if one uses heterogeneous agents over the single-layer network (which is more general compared to the homogeneous single-layer network), it still cannot represent the consensus problem over multilayer network. On the other hand, it is possible to adopt the concept of the matrix-weighted graphs proposed recently in [Tun17] to arrive at the equivalent formulation for the problem. More detailed discussions and comparisons of different approaches are described in Appendix A.2. ◊

3.2 A Necessary and Sufficient Condition for State Consensus over Undirected Network

From the study of the multi-agent systems with a single-layer network, it is well known that system theoretic conditions as well as graph theoretic conditions are involved. Specifically, stabilizability, detectability, and connectedness of the communication network are necessary conditions for state consensusability (e.g., see [MZ10]). Therefore, it is natural to suspect that necessary conditions for the consensusability over multilayer networks are also related to the system theoretic and graph theoretic conditions. As a motivating example, consider the state consensusability of the multi-agent systems over a multilayer network. As a candidate for a necessary condition, suppose that (C^l, A) is detectable for each $l \in \mathcal{L}$ and that every layer contains a rooted spanning tree. Then, the problem becomes trivial and degenerates into the classical state consensus problem as any single layer is sufficient to design a controller (e.g., use results of [SSB09, ZLD11]). On the other hand, suppose that (C, A) is detectable where $C := [C^1; \ldots; C^L]$ and that the projection graph contains a rooted spanning tree. Then, the following example presents a case where the system is trivially *not* state consensusable.

Example 3.2.1. Consider the system with 2 layers such that $\mathcal{E}^1 = \emptyset$ and $\mathcal{E}^2$ is the edge set of the complete graph, while $C^1 = I_n$ and $C^2 = 0$. Then (C, A) is detectable and the projection graph contains a rooted spanning tree, yet it is trivial to see that no information is exchanged among agents. Hence, state consensus cannot be achieved. ♢

The discussion so far suggests that if graph theoretic properties and system theoretic properties are considered separately, then the resulting conditions are either too strong or too weak for consensusability. Motivated by this, we will couple these conditions to develop an appropriate necessary condition. For notational purpose, let $\mathbf{A} := I_N \otimes A$ and define $\bar{\mathcal{K}} \subseteq \mathbb{R}^{Nn}$ as

$$\bar{\mathcal{K}} := \bigcap_{l=1}^{L} \ker \mathfrak{L}^l \otimes (Z^l)^\top,$$

where $\mathfrak{L}^1, \ldots, \mathfrak{L}^l$ are Laplacian matrices of $\mathcal{G}$.

The following theorem proposes a necessary and sufficient condition for the state consensusability of the system (3.1.1) over an undirected network, whose proof is presented in the following sections.

Theorem 3.2.1. Suppose that Assumption 3.1.1 holds and that $\mathcal{G}$ is undirected. Then the following statements are equivalent.

1. System (3.1.1) is state consensusable.

2. The detectability-like condition

$$\bigcap_{l=1}^{L} \langle \ker (\mathfrak{L}^l \otimes C^l) \mid \mathbf{A} \rangle \cap \mathcal{X}^u(\mathbf{A}) \subseteq \mathcal{S}_n^N \cap \mathcal{X}^u(\mathbf{A}). \tag{3.2.1}$$

3. The geometric condition

$$\bigcap_{l=1}^{L} \ker \mathfrak{L}^l \otimes (Z^l)^\top = \mathcal{S}_n^N. \tag{3.2.2}$$

4. The algebraic condition given by

$$\lambda_{n+1} \left(\sum_{l=1}^{L} \mathfrak{L}^l \otimes (Z^l)(Z^l)^\top \right) > 0. \tag{3.2.3}$$

◊

Theorem 3.2.1 states that the three conditions (3.2.1)–(3.2.3) are equivalent and that they are necessary and sufficient condition for state consensusability of the system over a multilayer network. Detailed discussions and proof of necessity and sufficiency of three conditions with state consensusability are deferred to Section 3.3 and Section 3.4. Instead, let us investigate (3.2.1)–(3.2.3) and how these three statements are related to each other and to the consensus problem over the multilayer network.

Proof of the equivalence among (3.2.1), (3.2.2) *and* (3.2.3)

((3.2.2) ⇒ (3.2.1)) Suppose that (3.2.2) holds and let $x = [x_1; \cdots ; x_N] \in \mathbb{R}^{Nn}$

be such that

$$x \in \langle \ker(\mathfrak{L}^l \otimes C^l) \,|\, \mathbf{A} \rangle \cap \mathcal{X}^u(\mathbf{A}), \quad \forall l \in \mathcal{L}.$$

Then, it follows from the definition of unobservable subspace and the properties of the Kronecker product that

$$x \in \bigcap_{k=0}^{Nn-1} \ker(\mathfrak{L}^l \otimes C^l)(I_N \otimes A)^k = \bigcap_{k=0}^{Nn-1} \ker(I_N \otimes C^l A^k)(\mathfrak{L}^l \otimes I_n).$$

Hence, it follows that

$$(\mathfrak{L}^l \otimes I_n)x \in \bigcap_{k=1}^{Nn-1} \ker(I_N \otimes C^l A^k), \quad \forall l \in \mathcal{L}.$$

Since $x_i \in \mathcal{X}^u(A)$, we obtain $\sum_{j \in \mathcal{N}_i^l}(x_j - x_i) \in \langle \ker C^l \,|\, A \rangle \cap \mathcal{X}^u(A) = \ker(Z^l)^\top$ for all $l \in \mathcal{L}$. Thus,

$$x \in \bigcap_{l=1}^{L} \ker \mathfrak{L}^l \otimes (Z^l)^\top = \mathcal{S}_n^N$$

holds from (3.2.2). Since $x \in \mathcal{S}_n^N$ and $x \in \mathcal{X}^u(\mathbf{A})$, the result holds.

((3.2.2) $\Leftarrow$ (3.2.1)) Conversely, suppose that (3.2.1) holds and let

$$x = [x_1; \cdots; x_N] \in \bigcap_{l=1}^{L} \ker \mathfrak{L}^l \otimes (Z^l)^\top.$$

Decompose x_i as $x_i = x_i^u + x_i^s$, where $x_i^u \in \mathcal{X}^u(A)$, $x_i^s \in \mathcal{X}^s(A)$ and define $x^u := [x_1^u; \cdots; x_N^u]$, $x^s := [x_1^s; \cdots; x_N^s]$. Then, for all $l \in \mathcal{L}$, it holds that

$$\sum_{j \in \mathcal{N}_i^l}(x_j^u - x_i^u) + \sum_{j \in \mathcal{N}_i^l}(x_j^s - x_i^s) \in \langle \ker C^l \,|\, A \rangle \cap \mathcal{X}^u(A). \tag{3.2.4}$$

Hence, we obtain $(\mathfrak{L}^l \otimes I_n)x^u \in \langle \ker(I_N \otimes C^l) \,|\, \mathbf{A} \rangle$ for all $l \in \mathcal{L}$. Then, it can be verified that

$$x^u \in \bigcap_{l=1}^{L} \langle \ker \mathfrak{L}^l \otimes C^l \mid \mathbf{A} \rangle \cap \mathcal{X}^u(\mathbf{A}) \subseteq \mathcal{S}_n^N.$$

On the other hand, (3.2.4) also implies that $(\mathfrak{L}^l \otimes I_n)x^s = 0$ for all $l \in \mathcal{L}$.

Now, we claim the projection graph contains a rooted spanning tree so that $\cap_{l=1}^{L} \ker \mathfrak{L}^l \otimes I_n = \mathcal{S}_n^N$. Suppose that the projection graph does not contain a rooted spanning tree. Let $x^* := s^* \otimes w^*$, where $s^* \in \mathbb{R}^N$ is such that $\mathfrak{L}^l s^* = 0$ for all $l \in \mathcal{L}$ and $s^* \notin \mathcal{S}_1^N$, while $w^* \in \mathbb{R}^n$ is such that $w^* \in \mathcal{X}^u(A)$ and $w^* \neq 0$. Then, $x^* \in \cap_{l=1}^{L} \langle \ker (\mathfrak{L}^l \otimes C^l) \mid \mathbf{A} \rangle \cap \mathcal{X}^u(\mathbf{A})$ but $x^* \notin \mathcal{S}_n^N \cap \mathcal{X}^u(\mathbf{A})$, which contradicts since (3.2.1) is assumed. Hence the claim is proven. In conclusion, it follows that

$$x^s \in \bigcap_{l=1}^{L} \ker (\mathfrak{L}^l \otimes I_n) = \mathcal{S}_n^N.$$

This completes the proof.

((3.2.2) $\Leftrightarrow$ (3.2.3)) We end the proof by showing (3.2.2) is equivalent to (3.2.3). Equivalence of (3.2.2) and the state consensusability of (3.1.1) is presented in the next section.

Since $\mathfrak{L}^l$ is symmetric, $\sum_{l=1}^{L} \mathfrak{L}^l \otimes (Z^l)(Z^l)^\top$ is symmetric and positive semidefinite matrix. In addition, it is trivial to see that $\mathcal{S}_n^N \subseteq \ker (\sum_{l=1}^{L} \mathfrak{L}^l \otimes (Z^l)(Z^l)^\top)$. Hence, (3.2.3) is equivalent to

$$\ker \left(\sum_{l=1}^{L} \mathfrak{L}^l \otimes (Z^l)(Z^l)^\top \right) = \mathcal{S}_n^N.$$

On the other hand, by defining $Q := [(\mathfrak{B}^1)^\top \otimes (Z^1)^\top; \cdots ; (\mathfrak{B}^L)^\top \otimes (Z^L)^\top]$ and $\bar{\mathcal{K}} = \cap_{l=1}^{L} \mathfrak{L}^l \otimes (Z^l)^\top$, it follows from $\mathfrak{L}^l = \mathfrak{B}^l (\mathfrak{B}^l)^\top$ that

$$\ker \left(\sum_{l=1}^{L} \mathfrak{L}^l \otimes (Z^l)(Z^l)^\top \right) = \ker Q^\top Q = \ker Q = \bar{\mathcal{K}}.$$

Therefore, it follows that (3.2.2) is equivalent to (3.2.3). □

First, notice that the condition (3.2.1) resembles the detectability of the pair

(C, A). Specifically, recall that the detectability of the pair (C, A) can be written [TSH12, Thm. 5.16] as

$$\langle \ker C \mid A \rangle \cap \mathcal{X}^u(A) \subseteq \{0\}$$

and that the set $\langle \ker C \mid A \rangle \cap \mathcal{X}^u(A)$ represents the undetectable subspace of the pair (C, A). Moreover, the undetectability subspace is a set where the output converges to zero, while the state does not. Hence, the detectability condition is stating that if output converges to zero, then so does the state.

With the meaning of detectability condition in mind, it can be seen that the set on the left of (3.2.1) denotes the intersection of undetectable subspace of the pair $(\mathfrak{L}^l \otimes C^l, \mathbf{A})$ for all layers $l \in \mathcal{L}$. Hence, the condition (3.2.1) can be interpreted as saying that if $\sum_{j \in \mathcal{N}_i^l}(y_j^l(t) - y_i^l(t)) \to 0$, then the corresponding state $x(t)$ must be in the synchronization space $\mathcal{S}_n^N$. Such a perspective of the multi-agent system is also investigated in [Tun17] and more detailed discussion is present in Appendix A.3.

The condition (3.2.2) also has a similar interpretation as (3.2.1). It is trivial to see that

$$\mathcal{S}_n^N \subseteq \bigcap_{l=1}^{L} \ker \mathfrak{L}^l \otimes (Z^l)^\top.$$

Therefore, condition (3.2.2) is in fact requiring $\bigcap_{l=1}^{L} \ker \mathfrak{L}^l \otimes (Z^l)^\top \subseteq \mathcal{S}_n^N$. The set on the left of (3.2.2) characterize the set of states where

$$\sum_{j \in \mathcal{N}_i^l} (Z^l)^\top (x_j(t) - x_i(t)) \equiv 0, \quad \forall i \in \mathcal{N}, l \in \mathcal{L}.$$

Hence, if the above condition holds, then $x(t)$ must be in the synchronization space. It is natural to suspect that the above is a necessary condition for consensusability since the controller primarily uses the relative output information. Thus, if relative output information is identically zero, then there is no additional information to utilize to achieve consensus, implying that the state consensus should have been achieved already.

The conditions (3.2.1) and (3.2.2) are not easy check computationally and it must compare relations between the sets. Instead, (3.2.3) gives an equivalent algebraic condition to check (3.2.2), which can be easily checked numerically. Notice that (3.2.3) resembles the necessary conditions used for the single-layer network. For instance, if $L = 1$, then (3.2.3) becomes $\lambda_{n+1}(\mathfrak{L}^1 \otimes (Z^1)(Z^1)^\top) > 0$, which holds if only if algebraic connectivity of $\mathfrak{L}^1$ is strictly positive and (C^1, A) is detectable. These are exactly the conditions for the single-layer network and hence (3.2.3) can be seen as a natural generalization of this fact to multilayer graphs.

As evident from (3.2.1)–(3.2.3), we would like to emphasize that the proposed conditions involve both the graph theoretic as well as system theoretic concepts and combine these into a single statement. This is in contrast to the consensus problem over a single-layer graph whose conditions are independent of each other as discussed at the start of this section. Specifically, it can be shown that (3.2.2) implies the following *decoupled* conditions.

Proposition 3.2.2. Suppose that $\mathcal{G}$ is undirected and let $C := [C^1; \cdots; C^L]$. If the necessary condition (3.2.2) holds, then the followings hold.

1. $\mathrm{proj}\,(\mathcal{G})$ is connected.

2. $\mathcal{X}^u(A) \cap \langle \ker C \,|\, A \rangle = \{0\}$, i.e., (C, A) is detectable. $\Diamond$

Proof. For a proof by contradiction, first suppose that the projection graph is not connected. Then, without loss of generality, nodes can be relabeled such that $\mathfrak{L}^l = \mathrm{diag}(\mathfrak{L}^l_1, \mathfrak{L}^l_2)$ for all $l \in \mathcal{L}$, where $\mathfrak{L}^l_i \in \mathbb{R}^{N_i \times N_i}$ for $i = 1, 2$ are the Laplacian matrices with suitable size. Choose any nonzero vector $x' \neq 0$. Let $x^* := [1_{N_1} \otimes x'; -1_{N_2} \otimes x'] \in \mathbb{R}^{Nn}$. Then, $x^* \in \bar{\mathcal{K}}$ follows from the construction. However, it can be checked that $x^* \notin \mathcal{S}^N_n$. Thus, $\bar{\mathcal{K}} \not\subseteq \mathcal{S}^N_n$ which leads to a contradiction.

Next, suppose $\mathcal{X}^u(A) \cap \langle \ker C \,|\, A \rangle \not\subseteq \{0\}$. Then there exists a nonzero vector $h^* \in \mathbb{R}^n$ such that $h^* \in \mathcal{X}^u(A) \cap \langle \ker C \mid A \rangle$ but $h^* \neq 0$. Let $x' := [\alpha_1 h^*; \cdots; \alpha_N h^*] \in \mathbb{R}^{Nn}$ where $\alpha_k \neq 0$ are distinct scalars. By construction, it follows that $x' \notin \mathcal{S}_n^N$. However,

$$h^* \in \langle \ker C \mid A \rangle \cap \mathcal{X}^u(A) \subset \langle \ker C^l \mid A \rangle \cap \mathcal{X}^u(A) = \ker (Z^l)^\top$$

for all $l \in \mathcal{L}$. Therefore, this implies $(I_N \otimes (Z^l)^\top)x' = 0$. Hence, we obtain $x' \in \bar{\mathcal{K}}$, while $x' \notin \mathcal{S}_n^N$. This leads to a contradiction which completes the proof. □

The decoupled conditions are also sufficient in case of the single-layer network, but they are not sufficient for multilayer network.

Remark 3.2.1. We may match each statement of Theorem 3.2.1 to common conditions for the observability of linear system. For example, consider the linear system given by

$$\dot{x} = Ax$$
$$y = Cx,$$

and let $\mathcal{O} := [C; CA; \cdots; CA^{n-1}]$. Then , recall that the following statements are equivalent for linear system.

1. (C, A) is observable
2. $\langle \ker C \mid A \rangle = \{0\}$
3. $\ker \mathcal{O} = \{0\}$
4. $\lambda_1 \left(\mathcal{O}^\top \mathcal{O}\right) > 0$

Each condition stated above for the observability corresponds to the conditions stated in Theorem 3.2.1. In fact, the results of Theorem 3.2.1 extend these concepts to the consensus problem by combining with the graph theoretical concepts. ◊

3.3 Proof of Necessity

Before presenting the proof of Theorem 3.2.1, the following lemma presents a few properties of $\bar{\mathcal{K}}$.

Lemma 3.3.1. The following properties hold for $\bar{\mathcal{K}} = \cap_{l=1}^{L} \mathfrak{L}^l \otimes (Z^l)^\top$.

1. $\bar{\mathcal{K}}$ is $\mathbf{A}$-invariant.
2. If $x \in \bar{\mathcal{K}}$, then $\delta_i^l = 0$ for all $i \in \mathcal{N}$ and $l \in \mathcal{L}$. ◊

Proof. For $\mathbf{A}$-invariance of $\bar{\mathcal{K}}$, let $x' \in \bar{\mathcal{K}}$ and notice that

$$(\mathfrak{L}^l \otimes (Z^l)^\top)\mathbf{A}x' = (I_N \otimes A_d^l)(\mathfrak{L}^l \otimes (Z^l)^\top)x' = 0,$$

where we used properties of Kronecker product and the fact that $(Z^l)^\top A = A_d^l (Z^l)^\top$. Therefore, $\bar{\mathcal{K}}$ is $\mathbf{A}$-invariant.

For the second statement, Let $x := [x_1; \cdots; x_N] \in \bar{\mathcal{K}}$. By definition, it holds that

$$\sum_{j \in \mathcal{N}_i^l} \alpha_{ij}^l (Z^l)^\top (x_j - x_i) = 0, \quad \forall i \in \mathcal{N}, l \in \mathcal{L}.$$

Then, it follows from the definition of δ_i^l and detectability decomposition that

$$\delta_i^l = C_d^l \sum_{j \in \mathcal{N}_i^l} \alpha_{ij}^l (Z^l)^\top (x_j - x_i) = 0,$$

which completes the proof. □

Now, we show (3.2.2) holds if the system (3.1.1) is state consensusable. Let $\mathcal{X}^e := \mathbb{R}^{Nn+N\nu}$ be the extended state space and denote $x^e \in \mathcal{X}^e$ as $x^e = [x; \xi] \in \mathbb{R}^{Nn+N\nu}$ where $x = [x_1; \cdots; x_N] \in \mathbb{R}^{Nn}$ and $\xi = [\xi_1; \cdots; \xi_N] \in \mathbb{R}^{N\nu}$. Furthermore, let

$$\mathcal{V} := \{x^e \in \mathcal{X}^e \mid x \in \bar{\mathcal{K}},\ \xi = 0\}. \tag{3.3.1}$$

Then, $\mathcal{V} \subseteq \mathcal{X}^e$ is invariant under the dynamics (3.1.1)–(3.1.2). In fact, if the initial condition satisfies $x^e(0) \in \mathcal{V}$, we obtain

$$\begin{aligned} \dot{x}_i &= Ax_i + Bh_{\mathrm{c},i}\big(0, 0\big) = Ax_i, \\ \dot{\xi}_i &= f_{\mathrm{c},i}\big(0, 0\big) = 0, \end{aligned} \tag{3.3.2}$$

where we used Lemma 3.3.1 with the properties of $h_{\mathrm{c},i}$ and $f_{\mathrm{c},i}$. Since $\bar{\mathcal{K}}$ is **A**-invariant, it follows that $x(t) \in \bar{\mathcal{K}}$ and $\xi(t) = 0$ for all $t \geq 0$ and that $\mathcal{V}$ is invariant. The set $\mathcal{V}$ defined in (3.3.1) represents the states of the multi-agent system where no input is applied to the agents. In particular, $x(t) \in \bar{\mathcal{K}}$ implies that the relative output information satisfies $\delta_i^l(t) = 0$. Thus, combined with $\xi_i(t) = 0$, this implies $u_i(t) = 0$.

We first claim that the projection graph of $\mathcal{G}$ is connected. For a contradiction, suppose that $\mathrm{proj}\,(\mathcal{G})$ is not connected. Then, without loss of generality, agents can be relabeled such that the Laplacian matrices are given by

$$\mathfrak{L}^l = \begin{bmatrix} \mathfrak{L}_1^l & & \\ & \ddots & \\ & & \mathfrak{L}_c^l \end{bmatrix}, \quad \forall l \in \mathcal{L},$$

where $c \geq 2$ is the number of connected component of $\mathrm{proj}\,(\mathcal{G})$. Specifically, $\mathfrak{L}_k^l \in \mathbb{R}^{N_k \times N_k}$ is the Laplacian matrix for $k = 1, \ldots, c$ where $N_k \geq 1$ [Wie10, Section 2.2.2]. Let $p' \in \mathbb{R}^n$ be a nonzero vector such that $p' \in \mathcal{X}^u(A)$ whose existence follows from Assumption 3.1.1. Define $p^* \in \mathbb{R}^{Nn}$ as

$$p^* := \left[\alpha_1(1_{N_1} \otimes p'); \quad \cdots \quad \alpha_c(1_{N_c} \otimes p')\right],$$

where $\alpha_k \neq 0$ are distinct scalars. Consider the solution of (3.1.1) from the initial condition given by $x(0) = p^*$ and $\xi(0) = 0$. Since $[x(0); \xi(0)] \in \mathcal{V}$, invariance of $\mathcal{V}$ implies

$$\dot{x}_i = Ax_i, \quad \dot{\xi}_i = 0, \quad \forall i \in \mathcal{N}.$$

Define $e(t) := x_{k_1}(t) - x_{k_2}(t)$ where $k_1 \in \mathcal{N}$ is the index of a node from the first connected component, and $k_2 \in \mathcal{N}$ is the index of a node from the second connected component. Then, $e(0) = (\alpha_1 - \alpha_2)p' \in \mathcal{X}^u(A)$. Hence, it follows that

$$\dot{e} = Ae.$$

By the definition of $e(0)$, it holds that $|e(t)| \not\to 0$ as $t \to \infty$. This implies that the consensus is not achieved. Since it was assumed that the system (3.1.1) is consensusable, this leads to a contradiction. Therefore, proj $(\mathcal{G})$ is connected.

Now, it will be shown that (3.2.2) holds if (3.1.1) is state consensusable. For a contradiction, suppose that (3.2.2) does not hold. Then, it is left to show there exists an initial condition such that consensus is not achieved.

Let $x^* := [x_1^*; \cdots ; x_N^*] \in \mathbb{R}^{Nn}$ be a vector such that $x^* \in \bar{\mathcal{K}}$ but $x^* \notin \mathcal{S}_n^N$. Suppose that the initial condition of the system (3.1.1)–(3.1.2) is given by $x(0) = x^*$ and $\xi(0) = 0$. Since $[x^*; 0] \in \mathcal{V}$, it follows from invariance of $\mathcal{V}$ that $\xi(t) = 0$ and $x(t) \in \bar{\mathcal{K}}$ for $t \geq 0$.

In addition, we claim that there exist indices $i^* \in \mathcal{N}$ and $l^* \in \mathcal{L}$ such that $\sum_{j \in \mathcal{N}_{i^*}^{l^*}} \alpha_{i^*j}^{l^*}(x_j^* - x_{i^*}^*) \neq 0$. To see this, suppose that $\sum_{j \in \mathcal{N}_i^l} \alpha_{ij}^l (x_j^* - x_i^*) = 0$ for all $i \in \mathcal{N}$ and $l \in \mathcal{L}$. Then, it follows that

$$\left(\sum_{l=1}^{L} \mathfrak{L}^l \otimes I_n \right) x^* = 0.$$

Since the projection graph is connected, $\sum_{l=1}^{L} \mathfrak{L}^l$ is a Laplacian matrix of a connected graph. Therefore, it follows that $x^* \in \mathcal{S}_n^N$ which leads to a contradiction since $x^* \notin \mathcal{S}_n^N$. Hence the claim is proven.

Finally, define the error variable as $e' := \sum_{j \in \mathcal{N}_{i^*}^{l^*}} \alpha_{i^*j}^{l^*}(x_j - x_{i^*})$ where $e'(0) = \sum_{j \in \mathcal{N}_{i^*}^{l^*}} \alpha_{i^*j}^{l^*}(x_j^* - x_{i^*}^*) \neq 0$ by definition of i^* and l^*. Recall that due to the invariance of $\mathcal{V}$, it holds that $u_i(t) = 0$ for all $t \geq 0$. Thus, the dynamics of e' can be written as

$$\dot{e}' = Ae'.$$

Moreover, it follows from $x^* \in \bar{\mathcal{K}}$ that $x^* \in \ker \mathfrak{L}^{l^*} \otimes (Z^{l^*})^\top$. Hence, we obtain $e'(0) \in \ker (Z^{l^*})^\top \subseteq \mathcal{X}^u(A)$, i.e., $e'(0) \in \mathcal{X}^u(A)$ and $e'(0) \neq 0$. Therefore, we obtain $|e'(t)| \not\to 0$ as $t \to \infty$ and consensus is not achieved when $x(0) = x^*$ and $\xi(0) = 0$. This leads to a contradiction since it is assumed that the system is consensusable. This completes the proof that (3.2.2) is a necessary condition for achieving state consensus. □

It can be deduced from the proof of Theorem 3.2.1 that $\bar{\mathcal{K}} = \cap_{l=1}^{L} \mathfrak{L}^l \otimes (Z^l)^\top$ is related to an invariant set where no control input is applied. Intuitively speaking, the condition (3.2.2) can be interpreted as stating that the consensus is achieved within an invariant set.

Remark 3.3.1. Necessary condition for consensusability of the system and invariance of the set $\mathcal{V}$ in the proof can be related to the similar concepts in [WWA13]. The work of [WWA13] studied necessary conditions for achieving consensus of heterogeneous nonlinear multi-agent systems over a single-layer network. In words, the necessary condition can be interpreted as requiring that an invariant set must be contained within the synchronization space. The same concept arises naturally for multilayer network as well through statements such as the invariance of $\mathcal{V}$ and $\bar{\mathcal{K}} = \mathcal{S}_n^N$. ♢

Finally, we end the discussion regrading the necessary conditions with yet another interpretation of (3.2.2). For the statement, denote m distinct eigenvalues of A as $\lambda_i^{\mathrm{d}}(A)$ such that

$$\mathrm{Re}\left(\lambda_1^{\mathrm{d}}(A)\right) \leq \ldots \leq \mathrm{Re}\left(\lambda_m^{\mathrm{d}}(A)\right).$$

Also define m_s be the number of stable eigenvalues. Then the following result states that the condition (3.2.2) can be decomposed into each *mode* of the system.

Proposition 3.3.2. Let $\bar{\mathcal{K}}_k := \bigcap_{l=1}^{L} \ker \mathfrak{L}^l \otimes (Z_k^l)^\top$ where Z_k^l is defined such that $\ker (Z_k^l)^\top = \langle \ker C^l \mid A \rangle \cap \mathcal{X}_{\lambda_k^{\mathrm{d}}}(A)$ for all $k = 1, \ldots, m$. Then, the condition (3.2.2) holds if and only if the set $\bar{\mathcal{K}}_k$ satisfies

$$\bar{\mathcal{K}}_k \subseteq \mathcal{S}_n^N, \quad \forall k = m_s + 1, \ldots, m. \tag{3.3.3}$$

♢

Proof. (3.2.2) ⇒ (3.3.3): Let $x^* = [x_1^*; \cdots; x_N^*] \in \bar{\mathcal{K}}_k$ for some k such that $m_s + 1 \leq k \leq m$. From the definition of $\bar{\mathcal{K}}_k$, it follows that $\sum_{j \in \mathcal{N}_i^l} \alpha_{ij}^l (x_j^* - x_i^*) \in \ker (Z_k^l)^\top$ for all $l \in \mathcal{L}$ and $i \in \mathcal{N}$. Since $\mathcal{X}_{\lambda_k}(A) \subseteq \mathcal{X}^u(A)$, this implies that

$$\sum_{j \in \mathcal{N}_i^l} \alpha_{ij}^l (x_j^* - x_i^*) \in \langle \ker C^l \mid A \rangle \cap \mathcal{X}_{\lambda_k^{\mathrm{d}}}(A) \subseteq \ker (Z^l)^\top$$

for all $i \in \mathcal{N}$ and $l \in \mathcal{L}$. Hence, $x^* \in \bar{\mathcal{K}} = \mathcal{S}_n^N$ which completes the proof.

(3.3.3) $\Rightarrow$ (3.2.2): Recalling from Lemma 3.3.1 that $\bar{\mathcal{K}}$ is $\mathbf{A}$-invariant, it follows from [Won74, Prop. 0.4] that

$$\bar{\mathcal{K}} = \big(\bar{\mathcal{K}} \cap \mathcal{X}_{\lambda_1^{\mathrm{d}}}(\mathbf{A})\big) \oplus \cdots \oplus \big(\bar{\mathcal{K}} \cap \mathcal{X}_{\lambda_m^{\mathrm{d}}}(\mathbf{A})\big),$$

where $\oplus$ denotes the direct sum between vector spaces. Therefore $x^* \in \bar{\mathcal{K}}$ can be written as $x^* = \sum_{k'=1}^{m} x^{k'}$, where $x^{k'} := [x_1^{k'}; \cdots ; x_N^{k'}] \in \bar{\mathcal{K}} \cap \mathcal{X}_{\lambda_{k'}^{\mathrm{d}}}(A)$. Hence, it is sufficient to show $x^{k'} \in \mathcal{S}_n^N$ for all $k' = 1, \ldots, m$. From the definitions, it follows that for each $k' = 1, \ldots, m$,

$$\sum_{j \in \mathcal{N}_i^l} \alpha_{ij}^l (x_j^{k'} - x_i^{k'}) \in \langle \ker C^l \mid A \rangle \cap \mathcal{X}^u(A) \cap \mathcal{X}_{\lambda_{k'}^{\mathrm{d}}}(A) \tag{3.3.4}$$

for all $i \in \mathcal{N}$ and $l \in \mathcal{L}$. For all $k' = m_s + 1, \ldots, m$, we obtain $\mathcal{X}^u(A) \cap \mathcal{X}_{\lambda_{k'}^{\mathrm{d}}}(A) = \mathcal{X}_{\lambda_{k'}^{\mathrm{d}}}(A)$. Hence, this implies $x^{k'} \in \bar{\mathcal{K}}_k \subseteq \mathcal{S}_n^N$.

On the other hand, $\mathcal{X}^u(A) \cap \mathcal{X}_{\lambda_{k'}^{\mathrm{d}}}(A) = \{0\}$ for $k' = 1, \ldots, m_s'$. Thus, it follows from (3.3.4) that $x^{k'} \in \bigcap_{l=1}^{L} \ker \mathfrak{L}^l \otimes I_n$ for all $k' = 1, \ldots, m_s'$. However, this implies $x^{k'} \in \bar{\mathcal{K}}_k \subseteq \mathcal{S}_n^N$ for any $k \in \{m_s + 1, \ldots, m\}$ and for all $k' = 1, \ldots, m_s'$. Thus, we obtain $x^{k'} \in \mathcal{S}_n^N$ for $k' = 1, \ldots, m$ and this completes the proof. □

Following corollary provides a physical interpretation of the necessary condition (4.1.1) and connects graph theoretic condition to each eigenvalue of the system.

Corollary 3.3.3. For the system (3.1.1), suppose that the multilayer graph $\mathcal{G}$ is undirected and that the geometric multiplicity of $\lambda_k(A)$ is 1. Define an index set $\mathcal{I}_{\lambda_k} \subseteq \mathcal{L}$ as

$$\mathcal{I}_{\lambda_k} := \left\{ l \in \mathcal{L} \;\middle|\; \operatorname{rank} \begin{pmatrix} A - \lambda_k(A) I_n \\ C^l \end{pmatrix} = n \right\}. \tag{3.3.5}$$

If the system (3.1.1) is state consensusable, then for all unstable eigenvalues of A (i.e., $\lambda_k(A)$, the projection graph of $\mathcal{G}_{\lambda_k} := (\mathcal{N}, \{\mathcal{E}^l\}_{l \in \mathcal{I}_{\lambda_k}})$ is connected. ♢

Proof. For the proof by contrapositive, suppose that there exists an index k^* such that $\mathrm{Re}\,(\lambda_{k^*}(A)) \geq 0$, but the $\mathrm{proj}\,(\mathcal{G}_{\lambda_{k^*}})$ is not connected. Then, without loss of generality, it holds that

$$\mathfrak{L}^l = \mathrm{diag}(\mathfrak{L}_1^l, \mathfrak{L}_2^l), \quad \forall l \in \mathcal{I}_{\lambda_{k^*}},$$

where $\mathfrak{L}_i^l \in \mathbb{R}^{N_i \times N_i}$ for $i = 1, 2$ are Laplacian matrices. Let $v \in \mathcal{X}_{\lambda_{k^*}}(A)$ be a nonzero vector, and define $x^* := [1_{N_1} \otimes v; -1_{N_2} \otimes v]$. Then, it can be verified that $x^* \notin \mathcal{S}_n^N$. Now, it will be proved that $x^* \in \bar{\mathcal{K}}_{k^*}$. For $l \in \mathcal{I}_{\lambda_{k^*}}$, we obtain

$$(\mathfrak{L}^l \otimes (Z_k^l)^\top) x^* = \begin{bmatrix} (\mathfrak{L}_1^l \cdot 1_{N_1}) \otimes (Z_k^l)^\top v \\ (\mathfrak{L}_2^l \cdot -1_{N_2}) \otimes (Z_k^l)^\top v \end{bmatrix} = 0.$$

On the other hand, for all $l \in \mathcal{L} \setminus \mathcal{I}_{\lambda_{k^*}}$,

$$\mathrm{rank} \begin{pmatrix} A - \lambda_{k^*}(A) I_n \\ C^l \end{pmatrix} < n.$$

Since geometric multiplicity of $\lambda_{k^*}(A)$ is 1 by the assumption, this implies $C^l v = 0$. Thus, it follows that $v \in \langle \ker C^l \mid A \rangle \cap \mathcal{X}_{\lambda_{k^*}}(A) = \ker\,(Z_{k^*}^l)^\top$ for all $l \in \mathcal{L} \setminus \mathcal{I}_{\lambda_{k^*}}$. Therefore, it is easy to see that for all $l \in \mathcal{L} \setminus \mathcal{I}_{\lambda_{k^*}}$,

$$(\mathfrak{L}^l \otimes (Z_{k^*}^l)^\top x^* = (\mathfrak{L}^l \otimes I_n)(I_N \otimes (Z_{k^*}^l)^\top) x^* = 0.$$

Hence, it holds that $x^* \in \bar{\mathcal{K}}_{k^*}$ while $x^* \notin \mathcal{S}_n^N$. This proves the negation of (3.3.3) and completes the proof. □

Note that the condition used in (3.3.5) is exactly the PBH test for $\lambda_k(A)$ with the output matrix C^l. Therefore, the set $\mathcal{I}_{\lambda_k}$ represents indices of all layers where $\lambda_k(A)$ is an *observable* eigenvalue by the corresponding output matrix C^l. Thus the necessary condition (3.2.2) can be interpreted as follows. For each unstable eigenvalue $\lambda_{k^*}(A)$, the projection graph constructed among layers having $\lambda_{k^*}(A)$ as an observable eigenvalue must be connected. In other words, unstable eigenvalues must be *connected*, in a sense that its projection graph corresponding to the multilayer graph $\mathcal{G}_{\lambda_{k^*}} = (\mathcal{N}, \{\mathcal{E}^l\}_{l \in \mathcal{I}_{\lambda_{k^*}}})$ is connected. This also recovers our previous result [LS17]. Although the discussion is done for the state consensus problem over undirected multilayer networks, interpretation provided by Corollary 3.3.3 extends to the output consensus problem over directed multilayer networks. Detailed discussions on this topic can be found in Chapter 4.

3.4 Proof of Sufficiency

In this section, sufficiency of the condition (3.2.2) is proved by constructing a dynamic controller of the form (3.1.2) with χ_i being the stack of ξ_k for $k \in \mathcal{N}_i^{\mathrm{p}} \cup \{i\}$. Recall from the consensus of multi-agent systems over a single-layer network that a common approach for solving the consensus problem is to use the observer-based dynamic controller. Specifically, a dynamics is constructed to estimate the relative state information (e.g., $\sum_{j \in \mathcal{N}_i} \alpha_{ij}(x_j - x_i)$), which is then used to compute suitable control action. However, for the multilayer network, relative state information cannot be estimated directly as each layer contains much less information. For instance, each layer may not be detectable or even connected. Hence, information from each layer must be appropriately combined to compute control input. In order to achieve this, we propose a hierarchical structure which first estimates the partial information from each layer and then combines partial information over multiple layers to obtain the desired result.

First, we propose a dynamics given by

$$\dot{\xi}_i^l = A_d^l \xi_i^l + G^l \Big[\sum_{j \in \mathcal{N}_i^l} \alpha_{ij}^l C_d^l (\xi_j^l - \xi_i^l) - \alpha_{ij}^l (y_j^l - y_i^l) \Big] + (Z^l)^\top B u_i, \tag{3.4.1}$$

for all $i \in \mathcal{N}, l \in \mathcal{L}$, where $\xi_i^l \in \mathbb{R}^{n-\nu^l}$. Then the following result holds.

Lemma 3.4.1. Consider the dynamics given by (3.1.1) and (3.4.1). Then for each $l \in \mathcal{L}$, there exists G^l such that

$$\xi_j^l(t) - \xi_i^l(t) \to (Z^l)^\top (x_j(t) - x_i(t)) \tag{3.4.2}$$

as $t \to \infty$ for all nodes i and j belonging to the same connected component of $\mathcal{G}^l$. ◊

Proof. Proof follows by applying Lemma A.1.3 of the Appendix to detectable part of the system (3.1.1) via C^l and the fact that $\mathfrak{L}^l = \mathfrak{B}^l(\mathfrak{B}^l)^\top$, where $\mathfrak{B}^l$ is the incidence matrix of $\mathcal{G}^l$. (For instance, $\hat{z}_i$ of Lemma A.1.3 is ξ_i^l and $\varepsilon_i(t) = 0$.) $\square$

Result of Lemma 3.4.1 states that the proposed dynamics (3.4.1) acts like a partial observer. The state ξ_i^l is an estimate by agent i on layer l and recovers as much partial relative state information possible from l-th layer. In particular, by computing the difference of ξ_i^l with its neighboring agents on l-th layer, an agent may obtain the partial relative state information. However, since each layer is not necessarily detectable nor connected, (3.4.1) only recovers the detectable part of the state and the convergence only holds for the agents within the same connected component of $\mathcal{G}^l$. Such a challenge was not present in case of single layer network as connectivity and detectability is assumed.

Nonetheless, by using ξ_i^l for all $l \in \mathcal{L}$, relative state difference can be obtained. For this, we propose an additional estimator given by

$$\dot{\hat{x}}_i = A\hat{x}_i + Bu_i + \gamma \sum_{l=1}^{L} \sum_{j\in\mathcal{N}_i^l} \Big[\alpha_{ij}^l (Z^l)(Z^l)^\top (\hat{x}_j - \hat{x}_i) - \alpha_{ij}^l (Z^l)(\xi_j^l - \xi_i^l)\Big], \quad (3.4.3)$$

where $\gamma > 0$ is a gain to be designed. Then, the following lemma states that (3.4.3) recovers the relative state difference.

Lemma 3.4.2. Consider the dynamics (3.4.3) and suppose that the necessary condition (3.2.2) holds. Then there exists $\gamma^* > 0$ such that for all $\gamma > \gamma^*$,

$$\hat{x}_j(t) - \hat{x}_i(t) \to x_j(t) - x_i(t), \quad \forall i, j \in \mathcal{N},$$

as $t \to \infty$. $\Diamond$

Proof. Let $e_i := \hat{x}_i - x_i$, $e := [e_1; \cdots; e_N]$, $x := [x_1; \cdots; x_N]$ and $\xi^l := [\xi_1^l; \cdots; \xi_N^l]$. Then, the dynamics of $\hat{x}$ becomes

$$\dot{\hat{x}} = (I_N \otimes A)\hat{x} + (I_N \otimes B)u - \gamma \sum_{l=1}^{L} \mathfrak{L}^l \otimes (Z^l)(Z^l)^\top \hat{x} + \gamma \sum_{l=1}^{L} (\mathfrak{L}^l \otimes Z^l)\xi^l.$$

Hence, it follows that

$$\begin{aligned}\dot{e} &= \dot{\hat{x}} - \dot{x} \pm \sum_{l=1}^{L} \mathfrak{L}^l \otimes (Z^l)(Z^l)^\top x \\ &= \left[(I_N \otimes A) - \gamma \sum_{l=1}^{L} \mathfrak{L}^l \otimes Z^l (Z^l)^\top \right] e + \gamma \Delta(t), \end{aligned} \tag{3.4.4}$$

where $\Delta(t) := \sum_{l=1}^{L} \big(\mathfrak{L}^l \otimes (Z^l)\big) \big(\xi^l - (I_N \otimes (Z^l)^\top)x\big)$.

Let $W = [(1/N)1_N^\top; R^\top] \in \mathbb{R}^{N\times N}$ be an orthogonal matrix given by Theorem 2.1.4 applied to a connected Laplacian matrix. Apply the state transformation given by $[\bar{e}; \tilde{e}] := (W \otimes I_n)e$, where $\bar{e} \in \mathbb{R}^n$ and $\tilde{e} \in \mathbb{R}^{(N-1)n}$. Then, (3.4.4) can be written as

$$\begin{aligned}\dot{\bar{e}} &= A\bar{e}, \\ \dot{\tilde{e}} &= \left[(I_{N-1} \otimes A) - \gamma \sum_{l=1}^{L} R^\top \mathfrak{L}^l R \otimes (Z^l)(Z^l)^\top \right] \tilde{e} + \gamma (R^\top \otimes I_n)\Delta(t).\end{aligned}$$

Since $\mathrm{im}\,(R) = (\mathcal{S}_1^N)^\perp$, it follows from the algebraic condition (3.2.3) that

$$\sum_{l=1}^{L} R^\top \mathfrak{L}^l R \otimes (Z^l)(Z^l)^\top > 0.$$

Hence, γ can be chosen sufficiently large such that

$$(I_{N-1} \otimes A) - \gamma \sum_{l=1}^{L} R^\top \mathfrak{L}^l R \otimes (Z^l)(Z^l)^\top$$

is Hurwitz. In particular, let $\gamma > 0$ such that

$$\gamma^* := \frac{\lambda_n(A + A^\top)}{2\lambda_1\big(\sum_{l=1}^{L} R^\top \mathfrak{L}^l R \otimes (Z^l)(Z^l)^\top\big)} < \gamma.$$

In addition, $\Delta(t)$ is exponentially decaying by Lemma 3.4.1. Hence, it holds that $\tilde{e}(t) \to 0$, i.e., $e_j(t) - e_i(t) \to 0$. Thus, $\hat{x}_j(t) - \hat{x}_i(t) \to x_j(t) - x_i(t)$ for all $i, j \in \mathcal{N}$ which completes the proof. □

The proposed dynamics (3.4.3) with (3.4.1) estimates the relative state difference between any two agents in the network by appropriately combining partial information from each layer. Using such an estimate, control inputs have been designed for the single-layer network which can be analogously used for the multilayer network. For the multilayer network, two designs are presented.

1. Design using $\hat{x}_j - \hat{x}_i$

Let $\mathfrak{L}^{\mathrm{p}}$ be the Laplacian matrix of $\mathcal{G}^{\mathrm{p}} = \mathrm{proj}\,(\mathcal{G})$ and design the control input (e.g., see [WLH09]) as

$$u_i = B^\top P \sum_{j \in \mathcal{N}_i^{\mathrm{p}}} \alpha_{ij}^{\mathrm{p}} (\hat{x}_j - \hat{x}_i), \tag{3.4.5}$$

where $P > 0$ is the unique solution of

$$A^\top P + PA - \lambda_2(\mathfrak{L}^{\mathrm{p}}) PBB^\top P = -I_n. \tag{3.4.6}$$

Then the overall system can be written as

$$\dot{x} = \mathbf{A}x - (I_N \otimes B)(\mathfrak{L}^{\mathrm{p}} \otimes B^\top P)\hat{x}.$$

Apply the transformation $[\bar{e}; \tilde{e}] := (U \otimes I_n)x$ where W is defined as in Theorem 2.1.4 applied to $\mathfrak{L}^{\mathrm{p}}$ such that $W\mathfrak{L}^{\mathrm{p}}W^\top = \mathrm{diag}(0, \lambda_2(\mathfrak{L}^{\mathrm{p}}), \dots, \lambda_N(\mathfrak{L}^{\mathrm{p}})) =: \mathrm{diag}(0, \Lambda^{\mathrm{p}})$. Then, it follows that

$$\begin{aligned} \dot{\bar{e}} &= A\bar{e}, \\ \dot{\tilde{e}} &= \Big[(I_{N-1} \otimes A) - (\Lambda^{\mathrm{p}} \otimes BB^\top P)\Big]\tilde{e} + (R^\top \otimes BB^\top P)(\mathfrak{L}^{\mathrm{p}} \otimes I_n)(x - \hat{x}), \end{aligned} \tag{3.4.7}$$

where x and $\hat{x}$ are stack of x_i and $\hat{x}_i$ respectively. Note that $(\mathfrak{L}^{\mathrm{p}} \otimes I_n)(x - \hat{x})$ is exponentially decaying due to Lemma 3.4.2 and $(I_{N-1} \otimes A) - (\Lambda^{\mathrm{p}} \otimes BB^\top P)$ is Hurwitz due to (3.4.6). Therefore, it follows that the consensus is achieved.

2. Design using $\hat{x}_i$

Alternatively, control input can also be designed as

$$u_i = K\hat{x}_i, \tag{3.4.8}$$

where K such that $A + BK$ is Hurwitz (e.g., see [LDCH10, ZXD14]). Then the system (3.1.1) with (3.4.8) can be written as

$$\dot{x} = (I_N \otimes A)x + (I_N \otimes BK)\hat{x}.$$

Applying the transformation $[\bar{e}; \tilde{e}] := (W \otimes I_n)x$ with W from Theorem 2.1.4 applied to $\mathfrak{L}^{\mathrm{p}}$ such that $W\mathfrak{L}^{\mathrm{p}}W^\top = \mathrm{diag}(0, \Lambda^{\mathrm{p}})$, we obtain

$$\begin{aligned} \dot{\bar{e}} &= A\bar{e} + \frac{1}{N}(1_N^\top \otimes BK)\hat{x}, \qquad (3.4.9)\\ \dot{\tilde{e}} &= (I_{N-1} \otimes (A + BK))\tilde{e} + (I_N \otimes BK)(R^\top \otimes I_n)(\hat{x} - x). \end{aligned}$$

From Lemma 3.4.2 and the definition of R, it follows that $(\mathfrak{L}^{\mathrm{p}} \otimes I_n)(\hat{x} - x) = (R\Lambda^{\mathrm{p}}R^\top \otimes I_n)(\hat{x} - x) \to 0$ and that $R\Lambda^{\mathrm{p}}$ has full column rank. Hence we obtain $(R^\top \otimes I_n)(\hat{x} - x) \to 0$. Since $A + BK$ is Hurwitz, this implies $\tilde{e} \to 0$, i.e., consensus is achieved.

In conclusion, dynamic controller given by (3.4.1), (3.4.3) with either (3.4.5) or (3.4.8) achieves consensus from arbitrary initial condition. Note that the crucial assumption required for the proposed controller to work is the algebraic condition (3.2.3), which is used to prove Lemma 3.4.2. Since the algebraic condition (3.2.3) is a necessary condition for consensusability, this completes the proof of Theorem 3.2.1.

Remark 3.4.1. The two designs for the control input each have a different property. For instance, the dynamic controller given by (3.4.1), (3.4.3) and (3.4.5) achieves *average consensus*. That is, the trajectories satisfy

$$\lim_{t\to\infty} |x_i(t) - s(t)| = 0, \quad \forall i \in \mathcal{N},$$

where $s(t)$ is the solution of

$$\dot{s} = As, \quad s(0) = \frac{1}{N}\sum_{i=1}^{N} x_i(0).$$

(This easily follows from (3.4.7).) Specifically, the converged trajectory only depends on the initial conditions of the plant $x_i(0)$ and not on $\xi_i^l(0)$ or $\hat{x}_i(0)$.

On the other hand, the dynamic controller given by (3.4.1), (3.4.3) and (3.4.8) does not achieve average consensus as one can verify from (3.4.9). However, (3.4.8) achieves consensus with (3.4.3) being a stable system. To see this, note that (3.4.3) becomes

$$\dot{\hat{x}}_i = (A + BK)\hat{x}_i + \gamma \sum_{l=1}^{L} \sum_{j \in \mathcal{N}_i^l} \Big[\alpha_{ij}^l (Z^l)(Z^l)^\top (\hat{x}_j - \hat{x}_i) - \alpha_{ij}^l (Z^l)(\xi_j^l - \xi_i^l)\Big].$$

Using Lemma 3.4.1 and Lemma 3.4.2, it can be verified that $\hat{x}_i(t) \to 0$. $\Diamond$

3.4.1 Additional Considerations for the Controllers

For the rest of the chapter, we briefly discuss the performance of the proposed controllers as well as some design methodologies.

3.4.1.1 Performance of the Proposed Controllers

Performance of the proposed controller can be also easily seen from the analysis. In particular, by defining $e^l := (I_N \otimes (Z^l)^\top)x - \xi^l$ and $e := \hat{x} - x$, overall dynamics with (3.4.5) can be written as

$$\left[\begin{array}{c} \dot{e}^1 \\ \vdots \\ \dot{e}^L \\ \hline \dot{e} \\ \dot{x} \end{array}\right] = \left[\begin{array}{ccc} (I_N \otimes A_d^1 - \mathfrak{L}^1 \otimes G^1 C^1) & & \\ & \ddots & \\ & & (I_N \otimes A_d^L - \mathfrak{L}^L \otimes G^L C^L) \\ \hline \gamma \mathfrak{L}^1 \otimes (Z^1) & \cdots & \gamma \mathfrak{L}^L \otimes (Z^L) \\ 0 & 0 & 0 \end{array}\right.$$

$$\left.\begin{array}{cc} & \\ & \\ \hline I_N \otimes A - \gamma \sum_{l=1}^{L} \mathfrak{L}^l \otimes (Z^l)(Z^l)^\top & 0 \\ \mathfrak{L}^{\mathrm{p}} \otimes BK & I_N \otimes A - \mathfrak{L}^{\mathrm{p}} \otimes BK \end{array}\right]\left[\begin{array}{c} e^1 \\ \vdots \\ e^L \\ \hline e \\ x \end{array}\right]$$

which has a block lower triangular structure. Therefore, it can be easily seen that the performance depends on the eigenvalue of the matrices $(I_N \otimes A_d^l - \mathfrak{L}^l \otimes G^l C^l)$ for all $l \in \mathcal{L}$, $I_N \otimes A - \gamma \sum_{l=1}^{L} \mathfrak{L}^l \otimes (Z^l)(Z^l)^\top$, and $I_N \otimes A - \mathfrak{L}^{\mathrm{p}} \otimes BK$.

If additional assumptions hold, then the convergence rate can be made arbitrarily fast. For instance, suppose (A, B) is controllable. Then choose the gain $K = B^\top P$, where $P > 0$ is the solution of

$$A^\top P + PA - \lambda_2(\mathfrak{L}^{\mathrm{p}}) PBB^\top P = -\theta P \tag{3.4.10}$$

for some $\theta > -2\mathrm{Re}\,(\lambda_{\min}(A))$. It is shown in [ZDL08] that the solution P for (3.4.10) exists, unique and positive definite. Additionally, we can show that the real part of eigenvalues of $A - \lambda_i(\mathfrak{L}^{\mathrm{p}})BB^\top$ less than $-\theta$ for $i = 2, \dots, N$. For this, consider the Lyapunov function defined as $V(x) = x^\top P x$. Then it holds that

$$\begin{aligned}
\frac{d}{dt}V(x) &= x^\top (A^\top P + PA - \lambda_i(\mathfrak{L}^{\mathrm{p}}) PBB^\top P) x \\
&= x^\top (A^\top P + PA - \lambda_2(\mathfrak{L}^{\mathrm{p}}) PBB^\top P + (\lambda_2(\mathfrak{L}^{\mathrm{p}}) - \lambda_i(\mathfrak{L}^{\mathrm{p}})) PBB^\top P) x \\
&= -\theta x^\top P x + (\lambda_2(\mathfrak{L}^{\mathrm{p}}) - \lambda_i(\mathfrak{L}^{\mathrm{p}})) x^\top PBB^\top P x \\
&\le -\theta x^\top P x \\
&= -\theta V(x).
\end{aligned}$$

Since $V(x)$ decays at rate θ, real part of eigenvalues of $A - \lambda_i(\mathfrak{L}^{\mathrm{p}})BB^\top P$ is at least $-\theta/2$. Thus convergence rate can be assigned arbitrarily by choosing $\theta > 0$ sufficiently large[2].

Similarly, we can obtain arbitrarily fast convergence rate for the partial ob-

[2] If the input is given by (3.4.8), a simple pole placement can be used to obtain the same conclusion.

servers using the similar argument. In particular, by using observability decomposition (instead of detectable decomposition), we can always make (C_d^l, A_d) an observable pair. Finally, since $\sum_{l=1}^{L} \mathfrak{L}^l \otimes (Z^l)(Z^l)^\top$ is positive definite, the desired eigenvalues of $I_N \otimes A - \gamma \sum_{l=1}^{L} \mathfrak{L}^l \otimes (Z^l)(Z^l)^\top$ can be made arbitrarily negative by choosing sufficiently large γ.

3.4.1.2 Design for the Worst-case Scenario

For the single-layer consensus problem, the design parameters can often be designed a priori to the operation such that it is robust to changes in the network structure (e.g., with fixed environment such as A, B or number of agents in the system). In particular, the algebraic connectivity of the graph (i.e., the second smallest eigenvalue of the Laplacian matrix) plays a crucial role, and the parameters such as feedback gain can be designed by considering the smallest possible algebraic connectivity (e.g., see works such as [Tun08, SSB09] for more details).

Similar result can be obtained for the proposed controller design over a multilayer network. Recall that the design parameters of the proposed controller are:

1. Gain for the partial observers G^l.
2. Gain for the state observer γ.
3. Gain for the state feedback (i.e., P in (3.4.5) or K in (3.4.8)).

The design procedure of the gains G^l and P are identical to the one from the single-layer network, and hence these can be designed if the algebraic connectivity of the network is known. For example, for a fixed number of agents and with unweighted graphs, the algebraic connectivity is given by $2(1 - \cos(\pi/N))$, i.e., the algebraic connectivity of a path graph. The gain K in (3.4.8) can be designed using pole placement which is independent of the structure of the communication network. Finally, the γ can be chosen given the value of $\lambda_1 \left(\sum_{l=1}^{L} R^\top \mathfrak{L}^l R \otimes (Z^l)(Z^l)^\top \right) > 0$. One can find the minimum value by enumerating all possible combinations of the multilayer network. Existence of a simpler characterization similar to the algebraic connectivity of a path graph (which is used in the design for the single-layer graph) is an open question[3].

In conclusion, the design designed prior to the operation such that it is robust to the changes of the network structure.

[3]Nonetheless, we conjecture that there exists a constant c, which only depends on the number of agents N and the number of layers L, such that

$$\lambda_1\left(\sum_{l=1}^{L} R^\top \mathfrak{L}^l R \otimes (Z^l)(Z^l)^\top\right) > c \cdot \lambda_2(\mathfrak{L}^{\mathrm{P}}).$$

Chapter 4

Extension to the Output Consensus Problem over Directed Network

In the previous chapter, the state consensus problem is studied over an undirected multilayer network. This chapter studies extension of previous results to output consensus problem over a directed multilayer network.

4.1 Necessary Conditions for the Output Consensus Problem

In this section, necessary conditions developed for the state consensus problem are extended to the output consensus problem over a directed multilayer network. Recalling the interpretation that (3.2.2) implies that the invariant set must be contained in the consensus space (e.g., see Section 3.3 or Remark 3.3.1), an analogous condition for output consensus problem is proposed as

$$\bar{\mathcal{K}} \subseteq \ker \Pi \otimes (Z_R)^\top, \tag{4.1.1}$$

where $\bar{\mathcal{K}}$ is defined using the Laplacian matrices of directed graphs. In particular, the set $\ker \Pi \otimes (Z_R)^\top$ is where the difference of x_i belongs to undetectable subspace of the pair (R, A) when $u_i(t) \equiv 0$.

The following result presents that (4.1.1) indeed a necessary conditions for the output consensus problem over directed multilayer networks.

Lemma 4.1.1. Suppose that Assumption 3.1.1 holds. If the system (3.1.1) is output consensusable, then (4.1.1) holds. In addition, (4.1.1) is equivalent to

$$\bar{\mathcal{K}}_k \subseteq \ker \Pi \otimes (Z_R)^\top, \quad \forall k = m_s + 1, \dots, m, \tag{4.1.2}$$

where $\bar{\mathcal{K}}_k$ is defined in Corollary 3.3.2. ◊

Proof. Proof of Lemma 4.1.1 closely follows the proof of similar results from Theorem 3.2.1 (which was presented in Section 3.3). Hence, only the difference is highlighted and details are omitted.

To show (4.1.1) is a necessary condition for the output consensus problem, we first claim that the projection graph of $\mathcal{G}$ contains a rooted spanning tree. For a contradiction, suppose that the proj $(\mathcal{G})$ does not contain a rooted spanning tree. Then, without loss of generality, agents can be relabeled such that the Laplacian matrices are given by

$$\mathfrak{L}^l = \begin{bmatrix} \mathfrak{L}_1^l & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & \mathfrak{L}_c^l & 0 \\ * & \cdots & * & \mathfrak{L}_{c+1}^l \end{bmatrix}, \quad \forall l \in \mathcal{L},$$

where $c \geq 2$ is the number of independently strongly connected component (iSCC) of proj $(\mathcal{G})$. Specifically, $\mathfrak{L}_k^l \in \mathbb{R}^{N_k \times N_k}$ is a Laplacian matrix for $k = 1, \dots, c$ where $N_k \geq 1$ [Wie10, Section 2.2.2]. Let $p' \in \mathbb{R}^n$ be a nonzero vector such that $p' \in \mathcal{X}^u(A)$ but $p' \notin \langle \ker R \,|\, A \rangle$ whose existence follows from Assumption 3.1.1. Define $p^* \in \mathbb{R}^{Nn}$ as

$$p^* := \begin{bmatrix} \alpha_1(1_{N_1} \otimes p'); & \cdots & \alpha_c(1_{N_c} \otimes p'); & * \end{bmatrix},$$

where $\alpha_k \neq 0$ are distinct scalars and the asterisk denotes the elements without any interest for the result. Consider the initial condition given by $x(0) = p^*$ and $\xi(0) = 0$. Next, restrict the attention to the nodes in the k-th iSCC. Then, using the similar argument as in the proof of Theorem 3.2.1 and the fact that χ_i only consists of ξ_k for $k \in \mathcal{P}_i^{\mathrm{p}} \cup \{i\}$, it can be obtained that

$$\dot{x}_i = Ax_i, \quad \dot{\xi}_i = 0 \tag{4.1.3}$$

for all nodes i belonging to the k-th iSCC. Define $e(t) := x_{k_1}(t) - x_{k_2}(t)$ where k_1 is the index of a node from the first iSCC, and k_2 is the index of a node from the second iSCC. Then, $e(0) = (\alpha_1 - \alpha_2)p' \in \mathcal{X}^u(A)$ and $e(0) \notin \langle \ker R \,|\, A \rangle$. Since (4.1.3) holds for any iSCC, it follows that

$$\dot{e} = Ae, \quad \psi = Re,$$

where $\psi := \zeta_{k_1} - \zeta_{k_2}$ is the relative output error. By the definition of $e(0)$, it holds that $|\psi(t)| \not\to 0$ as $t \to \infty$. This implies that the output consensus is not achieved. Since it was assumed that the system (3.1.1) is output consensusable, this leads to a contradiction. Therefore, $\mathrm{proj}\,(\mathcal{G})$ contains a rooted spanning tree.

Now, let $x^* := [x_1^*; \cdots; x_N^*] \in \bar{\mathcal{K}}$ but $x^* \notin \ker \Pi \otimes (Z_R)^\top$. Then, using the fact that $\mathrm{proj}\,(\mathcal{G})$ has a rooted spanning tree, it can be proven that there exists indices $i^* \in \mathcal{N}$ and $l^* \in \mathcal{L}$ such that $\sum_{j \in \mathcal{N}_{i^*}^{l^*}} \alpha_{i^*j}^{l^*}(x_j^* - x_{i^*}^*) \notin \ker (Z_R)^\top$. Let $x(0) = x^*$, $\xi(0) = 0$ and define $e' := \sum_{j \in \mathcal{N}_{i^*}^{l^*}} \alpha_{i^*j}^{l^*}(x_j - x_{i^*})$. Then we obtain

$$\dot{e}' = Ae', \quad \psi' = Re',$$

where $e'(0) \in \mathcal{X}^u(A)$ and $e'(0) \notin \ker (Z_R)^\top$. Thus, $|\psi'(t)| \not\to 0$ and hence output consensus is not achieved when $x(0) = x^*$ and $\xi(0) = 0$. This leads to a contradiction since it is assumed that the system is output consensusable.

Proof for the equivalence of (4.1.1) and (4.1.2) can be obtained by following the proof of Proposition 3.3.2. In particular, the same argument can be applied by simply replacing $\mathcal{S}_n^N$ with $\ker \Pi \otimes (Z_R)^\top$ and hence omitted. □

Lemma 4.1.1 extends the necessary condition (3.2.2) and the results of Proposition 3.3.2 to output consensus over directed graphs. Specifically, (4.1.1) becomes (3.2.2) and (4.1.2) becomes (3.3.3) since $Z_R = I_n$ for state consensus problem.

For the algebraic condition (3.2.3), we consider an extension given by

$$\mathrm{Re}\left(\lambda_{n-\nu^R+1}\big(\sum_{l=1}^{L}\mathfrak{L}^l\otimes(Y_R)^\top(Z^l)(Z^l)^\top Y_R\big)\right)>0, \tag{4.1.4}$$

where Y_R is defined in (3.1.3). The proposed condition (4.1.4) is a valid extension since $Z_R = Y_R = I_n$ in case of state consensus problem and hence it becomes (3.2.3) if network is undirected. Relation between the geometric condition (4.1.1) and the algebraic condition (4.1.4) is stated below.

Lemma 4.1.2. Following statements hold.

1. If $\mathcal{G}$ is undirected, then (4.1.1) implies (4.1.4).
2. If $R = I_n$, then (4.1.4) implies (4.1.1). $\Diamond$

Proof. (4.1.1) $\Rightarrow$ (4.1.4): Suppose that $\mathcal{G}$ is undirected and (4.1.1) holds. Then using the fact that $\mathfrak{L}^l = \mathfrak{B}^l(\mathfrak{B}^l)^\top$, (4.1.1) is equivalent to

$$((\mathfrak{B}^l)^\top\otimes(Z^l)^\top)x=0,\ \forall l\in\mathcal{L} \implies (\Pi\otimes Z_R^\top)x=0 \tag{4.1.5}$$

for all $x\in\mathbb{R}^{Nn}$. Meanwhile, it can be checked that (4.1.4) is equivalent to showing

$$\bigcap_{l=1}^{L}\ker\big((\mathfrak{B}^l)^\top\otimes((Z^l)^\top Y_R)\big)\subseteq\mathcal{S}^N_{n-\nu^R}.$$

Let $z\in\mathbb{R}^{N(n-\nu^R)}$ be such that

$$((\mathfrak{B}^l)^\top\otimes(Z^l)^\top Y_R)z=0,\quad \forall l\in\mathcal{L}.$$

However, this means $((\mathfrak{B}^l)^\top\otimes(Z^l)^\top)((I_N\otimes Y_R)z)=0$. Hence, it follows from (4.1.5) that $(\Pi\otimes Z_R^\top)(I_N\otimes Y_R)z=0$. Since $Z_R^\top Y_R = I_{n-\nu^R}$, we obtain $z\in\mathcal{S}^N_{n-\nu^R}$ which completes the proof.

(4.1.4) $\Rightarrow$ (4.1.1): Suppose that $R = I_n$ and (4.1.4) holds. Let $x\in\bar{\mathcal{K}}$, i.e., $(\mathfrak{L}^l\otimes(Z^l)^\top)x=0$ for all $l\in\mathcal{L}$. This implies $(\mathfrak{L}^l\otimes(Z^l)(Z^l)^\top)x=0$ and hence $\sum_{l=1}^{L}(\mathfrak{L}^l\otimes(Z^l)(Z^l)^\top)x=0$. However, (4.1.4) implies $\ker\big(\sum_{l=1}^{L}\mathfrak{L}^l\otimes(Z^l)(Z^l)^\top\big)=$

$\mathcal{S}_n^N$ and hence $x \in \mathcal{S}_n^N$. This completes the proof since $\ker \Pi \otimes (Z_R)^\top = \mathcal{S}_n^N$ if $R = I_n$. □

4.2 Challenges for the Output Consensus Problem over Directed Multilayer Networks

Unfortunately, Lemma 4.1.2 cannot be further extended as in Theorem 3.2.1 to establish the equivalence. For instance, (4.1.1) is not equivalent to (4.1.4) in general. Furthermore, (4.1.1) is also clearly not sufficient to achieve output consensus. These are illustrated through the following examples.

Example 4.2.1. Consider the system given by

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, R = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, C^1 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$$

with $L = 1, N = 3$ and $\mathcal{G}^1$ being a complete graph. By defining Z_R and W_R as

$$Z_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad W_R = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

while Z^1 and W^1 are defined as

$$Z^1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad W^1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

Hence the transformation matrix are orthogonal and it follows that $Y_R = Z_R$ and $U_R = W_R$. Then it can be computed that

$$\lambda_2 \left(\mathfrak{L}^1 \otimes (Y_R)^\top (Z^l)(Z^l)^\top Y_R \right) = 3.0 > 0,$$

that is (4.1.4) holds. However, it can be verified that $x^* := [1; 2; 3] \otimes [1; -1; 0] \in \bar{\mathcal{K}}$

yet $x^* \notin \ker \Pi \otimes (Z_R)^\top$. $\Diamond$

Example 4.2.1 depicts a case where the algebraic condition (4.1.4) holds, yet (4.1.1) does not. In particular, this example illustrates a scenario where the available information δ_i^l are all identically zero, but output consensus is not reached.

Next example shows that the converse is also not true in general, i.e., (4.1.1) holds yet the algebraic condition (4.1.4) may not.

Example 4.2.2. Let $A = 0_{3\times3}$, $L = 3$, $N = 4$, $n = 3$ and suppose that $\mathfrak{L}^l \in \mathbb{R}^{4\times4}$ and $C^l \in \mathbb{R}^{2\times3}$ are given by

$$\mathfrak{L}^1 = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathfrak{L}^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix}, \quad \mathfrak{L}^3 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

while the output matrices are given by

$$C^1 = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \end{bmatrix}, \quad C^2 = \begin{bmatrix} 4 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}, \quad C^3 = \begin{bmatrix} 1 & 4 & 2 \\ 4 & 3 & 4 \end{bmatrix}, \quad R = I_3.$$

Then, it can be checked that $Z^l = (C^l)^\top$ for all $l \in \mathcal{L}$ and

$$\mathrm{Re}\left(\lambda_4\Big(\sum_{l=1}^{L} \mathfrak{L}^l \otimes (Z^l)(Z^l)^\top\Big)\right) = 0,$$

while $\bar{\mathcal{K}} = \mathcal{S}_3^4$. In particular, we have

$$\mathrm{Re}\left(\lambda_1\Big(\sum_{l=1}^{L} \mathfrak{L}^l \otimes (Z^l)(Z^l)^\top\Big)\right) \approx -1.3542 < 0. \qquad \Diamond$$

Example 4.2.2 highlights a challenge for generalizing the algebraic condition (3.2.3) to directed graphs. Specifically, Example 4.2.2 shows that the eigenvalues of $\sum_{l=1}^{L} \mathfrak{L}^l \otimes (Z^l)(Z^l)^\top$ are not necessarily on the closed right-half plane. This leads to difficulties since algebraic condition (3.2.3) is utilized when constructing the dynamic controller.

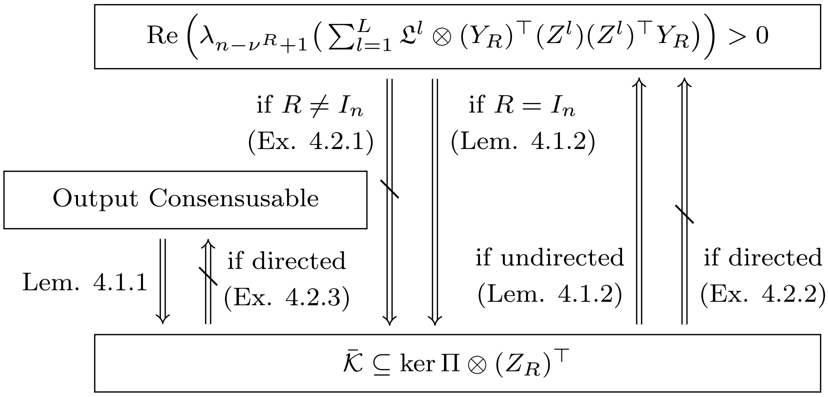


Figure 4.1: Relationships between various conditions for output consensus problem over directed graph and corresponding counterexamples.

Finally, the following example reveals that the proposed necessary condition (4.1.1) is in fact *not sufficient* for directed multilayer graphs.

Example 4.2.3. Suppose that $A = 0$, $L = 2$, $N = 3$, $n = 1$ and $R = C^1 = C^2 = 1$, where the Laplacian matrices are given by

$$\mathfrak{L}^1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \quad \mathfrak{L}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

Then, it can be checked that $\bar{\mathcal{K}} = \mathcal{S}_1^3$. However, it follows from the proof of Lemma 4.1.1 that state consensus cannot be achieved since $\mathrm{proj}\,(\mathcal{G})$ does not contain a rooted spanning tree. $\diamondsuit$

Summary of various conditions discussed so far and its relations are shown in Fig. 4.1. For the state consensus problem over the undirected network, Theorem 3.2.1 state that all statements in Fig. 4.1 are equivalent. Unfortunately, a similar result does not hold for the output consensus problem over a directed network.

One of the fundamental limitations of the condition (4.1.1) can be seen from its physical interpretation. Recall from the detectability interpretation of the condition that (4.1.1) is saying that if the relative output information $\sum_{j \in \mathcal{N}_i^l} y_j^l - y_i^l \equiv 0$, then the consensus of ζ_i must be achieved. However, it is possible that the desired output achieves consensus (i.e., $\zeta_i = \zeta_i$), while the relative output difference over the multilayer network (i.e., $y_j^l - y_i^l$) is not zero.

4.3 Controller Design for the Output Consensus Problem

Discussions so far illustrated the difficulties of extending the previous results to the output consensus problem over directed multilayer networks. Nevertheless, this section designs a dynamic controller to achieve output consensus by imposing additional assumptions on the system. From Chapter 3, we have seen that the algebraic condition (3.2.3) is integral to designing a dynamic controller. Similarly, a dynamic controller will be designed using algebraic condition (4.1.4) to achieve output consensus. However, notice from Fig. 4.1 that even if $\mathcal{G}$ is undirected, algebraic condition (4.1.4) alone is clearly not sufficient (as it contradicts with Example 4.2.1 if (4.1.4) implies output consensus). Therefore, we make additional assumptions such that the dynamic controller can be designed.

4.3.1 Controller Design under System Theoretic Constraint

In order to design a dynamic controller for a directed multilayer network, we make the following assumption (which is discussed in [LS20b] for unobservable subspaces) to design a dynamic controller.

Assumption 4.3.1. There exists a basis $\{v_1, \dots, v_n\}$ of $\mathbb{R}^n$ such that every undetectable subspace $\langle \ker C^1 \mid A \rangle \cap \mathcal{X}^u(A), \dots, \langle \ker C^L \mid A \rangle \cap \mathcal{X}^u(A)$ and $\langle \ker R \mid A \rangle \cap \mathcal{X}^u(A)$ is a span of a subset of the basis. $\Diamond$

Assumption 4.3.1 holds if the characteristic polynomial of A is same as the minimal polynomial of A, or if each distinct unstable eigenvalue only has a single Jordan block. For example, suppose that A consists of a single Jordan block with a real unstable eigenvalue. Then $\langle \ker C^l \mid A \rangle \cap \mathcal{X}^u(A) = \{0\}$ if the first column of C^l is nonzero. If the first q columns of C^l are zero, then $\langle \ker C^l \mid A \rangle \cap \mathcal{X}^u(A) = \operatorname{span}\{e_1, \dots, e_q\}$. Hence, Assumption 4.3.1 holds with $v_k = e_k$ for $k = 1, \dots, n$. There are cases when Assumption 4.3.1 holds even when an eigenvalue has more than a single block. For more details, we refer to [LS20b, Appendix B].

Before proceeding further, few notations and intermediate results are introduced. Under Assumption 4.3.1, let the indicator $s_k^l \in \{0,1\}$ be

$$s_k^l = \begin{cases} 1, & \text{if } v_k \notin \ker (Z^l)^\top = \langle \ker C^l \mid A \rangle \cap \mathcal{X}^u(A), \\ 0, & \text{if } v_k \in \ker (Z^l)^\top = \langle \ker C^l \mid A \rangle \cap \mathcal{X}^u(A) \end{cases}$$

for all $k = 1, \ldots, n$ and $l \in \mathcal{L}$, while $s_k^R \in \{0,1\}$ is defined similarly. By defining $V := [v_1 \ \cdots \ v_n] \in \mathbb{R}^{n \times n}$, there exists $h_k \in \mathbb{R}^n$ such that

$$H^\top V := \begin{bmatrix} h_1 & \cdots & h_n \end{bmatrix}^\top \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} = I_n,$$

so that $H^\top = V^{-1}$. Now, apply the transformation

$$\rho_i = H^\top x_i, \quad V\rho_i = x_i, \tag{4.3.1}$$

where $\rho_i \in \mathbb{R}^n$. Then the following property holds.

Lemma 4.3.1. Suppose that Assumption 4.3.1 holds and consider the system (3.1.1) under the transformation (4.3.1), which can be written as

$$\begin{aligned} \dot{\rho}_i &= H^\top A V \rho_i + H^\top B u_i, \\ y_i^l &= C^l V \rho_i, \\ \zeta_i &= R V \rho_i, \end{aligned} \tag{4.3.2}$$

for all $i \in \mathcal{N}$. Then the transformation matrix $T^l \in \mathbb{R}^{n \times n}$ for the detectable decomposition of (4.3.2) is a permutation matrix for all $l \in \mathcal{L}$. $\Diamond$

Proof. Let $e_k \in \mathbb{R}^n$ be the elementary vector where k-th element is 1 and the rest of the elements are zero. Then for each $l \in \mathcal{L}$, let $(T^l)^\top$ be a permutation matrix given by

$$\check{\rho}_i^l = (T^l)^\top \rho_i =: \begin{bmatrix} Z_\text{o}^l & W_\text{o}^l \end{bmatrix}^\top \rho_i,$$

where the columns of $Z_\text{o}^l \in \mathbb{R}^{n \times (n-\nu^l)}$ are e_k for all k such that $s_k^l = 1$ and the columns of $W_\text{o}^l \in \mathbb{R}^{n \times \nu^l}$ are e_k such that $s_k^l = 0$. Since the columns of V spans the undetectable subspaces by the definition and $H^\top = V^{-1}$, it can be verified that the dynamics of $\check{\rho}_i^l$ is in form of the detectability decomposition. □

In order to improve the clarity, without loss of generality, suppose that v_i are such that $s_1^R = \cdots = s_{n-\nu^R}^R = 1$ and $s_{n-\nu^R+1}^R = \cdots = s_n^R = 0$. Accordingly, we have $Z_{R,\mathrm{o}} = [I_{n_o}; 0_{\nu^R \times n_o}] \in \mathbb{R}^{n \times n_o}$ and $W_{R,\mathrm{o}} = [0_{n_o \times \nu^R}; I_{\nu^R}] \in \mathbb{R}^{n \times \nu^R}$ where $n_o := n - \nu^R$.

Under Assumption 4.3.1, a dynamic controller can be designed to achieve consensus. For this, we choose Z^l, W^l, Z_R, and W_R as

$$Z^l = HZ_{\mathrm{o}}^l,\ W^l = HW_{\mathrm{o}}^l,\ Z_R = HZ_{R,\mathrm{o}},\ W_R = HW_{R,\mathrm{o}}. \tag{4.3.3}$$

Consequently, it can be checked that

$$Y^l = VZ_{\mathrm{o}}^l,\ U^l = VW_{\mathrm{o}}^l,\ Y_R = VZ_{R,\mathrm{o}},\ U_R = VW_{R,\mathrm{o}} \tag{4.3.4}$$

and that these matrices form a transformation which results in undetectable decomposition.

Overall structure is similar to the controller developed for the state consensus problem. Specifically, the same partial observer (3.4.1) is used, which is rewritten as

$$\dot{\xi}_i^l = A_d^l \xi_i^l + G^l \Big[\sum_{j \in \mathcal{N}_i^l} \alpha_{ij}^l C_d^l (\xi_j^l - \xi_i^l) - \alpha_{ij}^l (y_j^l - y_i^l) \Big] + (Z^l)^\top B u_i. \tag{4.3.5}$$

With (4.3.5), relative difference of detectable part by R is estimated via

$$\begin{aligned} \dot{\hat{\eta}}_i = A_d^R \hat{\eta}_i + \gamma \sum_{l=1}^{L} \sum_{j \in \mathcal{N}_i^l} \Big[&\alpha_{ij}^l (Z_R)^\top V V^\top (Z^l)(Z^l)^\top V V^\top Z_R (\hat{\eta}_j - \hat{\eta}_i) \\ &- \alpha_{ij}^l (Z_R)^\top V V^\top (Z^l)(\xi_j^l - \xi_i^l) \Big] + B_d^R u_i, \end{aligned} \tag{4.3.6}$$

where $A_d^R := (Z_R)^\top A Y_R$, $B_d^R := (Z_R)^\top B$ and $\gamma > 0$. By letting $\hat{\eta} := [\hat{\eta}_1; \cdots; \hat{\eta}_N]$,

$\xi^l := [\xi_1^l; \cdots \xi_N^l]$ and $u := [u_1; \cdots; u_N]$, we obtain

$$\begin{aligned}\dot{\hat{\eta}} = (I_N \otimes A_d^R)\hat{\eta} - \gamma \sum_{l=1}^{L} (\mathfrak{L}^l \otimes Z_R^\top V V^\top Z^l (Z^l)^\top V V^\top Z_R)\hat{\eta} \\ + \gamma \sum_{l=1}^{L} (\mathfrak{L}^l \otimes Z_R^\top V V^\top Z^l)\xi^l + (I_N \otimes B_d^R)u\end{aligned} \quad (4.3.7)$$

Then the following result holds.

Lemma 4.3.2. Consider the dynamics given by (3.1.1) and (3.4.1). Then for each $l \in \mathcal{L}$, there exists G^l such that

$$\sum_{j \in \mathcal{N}_i^l} \alpha_{ij}^l (\xi_j^l(t) - \xi_i^l(t)) \to \sum_{j \in \mathcal{N}_i^l} \alpha_{ij}^l (Z^l)^\top (x_j(t) - x_i(t)), \quad (4.3.8)$$

for all $i \in \mathcal{N}$ and $l \in \mathcal{L}$ as $t \to \infty$. Furthermore, suppose that Assumption 4.3.1 and (4.1.4) holds. Then there exists $\gamma^* > 0$ such that for all $\gamma > \gamma^*$,

$$\hat{\eta}_j(t) - \hat{\eta}_i(t) \to (Z_R)^\top (x_j(t) - x_i(t)), \quad \forall i, j \in \mathcal{N}, \quad (4.3.9)$$

as $t \to \infty$. ◊

Proof. Proof of (4.3.8) follows directly by applying Lemma A.1.3 to the detectable part of the system (3.1.1) via C^l.

Proof of (4.3.9) is similar to the proof of Lemma 3.4.2. Hence only the difference is highlighted and details are omitted. Let $e_i := \hat{\eta}_i - (Z_R)^\top x_i$, and $e := [e_1; \cdots; e_N]$. Then, using (4.3.7) and (3.1.1), the dynamics of e is given by

$$\begin{aligned}\dot{e} &= \dot{\hat{\eta}} - (I_N \otimes Z_R^\top)\dot{x} \\ &= \Bigg[(I_N \otimes A_d^R)\hat{\eta} - \gamma \sum_{l=1}^{L} (\mathfrak{L}^l \otimes Z_R^\top V V^\top Z^l (Z^l)^\top V V^\top Z_R)\hat{\eta} \\ &\quad + \gamma \sum_{l=1}^{L} (\mathfrak{L}^l \otimes Z_R^\top V V^\top Z^l)\xi^l + (I_N \otimes B_d^R)u\Bigg] \\ &\quad - (I_N \otimes Z_R^\top)\Big[(I_N \otimes A)x + (I_N \otimes B)u\Big]\end{aligned}$$

$$
\begin{aligned}
&= \Big[(I_N \otimes A_d^R)\hat{\eta} - (I_N \otimes Z_R^\top A)x\Big] - \gamma \sum_{l=1}^{L} (\mathfrak{L}^l \otimes Z_R^\top V V^\top Z^l (Z^l)^\top V V^\top Z_R)\hat{\eta} \\
&\quad + \gamma \sum_{l=1}^{L} (\mathfrak{L}^l \otimes Z_R^\top V V^\top Z^l)\xi^l \\
&= (I_N \otimes A_d^R)e - \gamma \sum_{l=1}^{L} (\mathfrak{L}^l \otimes Z_{R,o}^\top Z_o^l (Z_o^l)^\top Z_{R,o})\hat{\eta} + \gamma \sum_{l=1}^{L} (\mathfrak{L}^l \otimes Z_{R,o}^\top Z_o^l)\xi^l,
\end{aligned} \tag{4.3.10}
$$

where we used $Z_R^\top A = A_d^R (Z_R)^\top$. To proceed further, we use $x = U_R W_R^\top x + Y_R Z_R^\top x = V W_{R,o} W_R^\top x + V Z_{R,o} Z_R^\top x$ and add $\pm\gamma \sum_{l=1}^{L} \mathfrak{L}^l \otimes Z_R^\top V V^\top (Z^l)(Z^l)^\top x$ to (4.3.10). Specifically, we have

$$
\begin{aligned}
& -\gamma \sum_{l=1}^{L} (\mathfrak{L}^l \otimes Z_R^\top V V^\top Z^l (Z^l)^\top V V^\top Z_R)\hat{\eta} + \gamma \sum_{l=1}^{L} \mathfrak{L}^l \otimes Z_R^\top V V^\top (Z^l)(Z^l)^\top x \\
= & -\gamma \sum_{l=1}^{L} (\mathfrak{L}^l \otimes Z_{R,o}^\top Z_o^l (Z_o^l)^\top Z_{R,o})\hat{\eta} \\
& \quad + \gamma \sum_{l=1}^{L} \mathfrak{L}^l \otimes Z_{R,o}^\top (Z_o^l)(Z_o^l)^\top H^\top (V W_{R,o} W_R^\top + V Z_{R,o} Z_R^\top)x \\
= & -\gamma \sum_{l=1}^{L} (\mathfrak{L}^l \otimes Z_{R,o}^\top Z_o^l (Z_o^l)^\top Z_{R,o})e + \gamma \sum_{l=1}^{L} \mathfrak{L}^l \otimes Z_{R,o}^\top (Z_o^l)(Z_o^l)^\top W_{R,o} W_R^\top x
\end{aligned}
$$

and

$$
\begin{aligned}
& \gamma \sum_{l=1}^{L} (\mathfrak{L}^l \otimes Z_R^\top V V^\top Z^l)\xi^l - \gamma \sum_{l=1}^{L} \mathfrak{L}^l \otimes Z_R^\top V V^\top (Z^l)(Z^l)^\top x \\
&= \gamma \sum_{l=1}^{L} (\mathfrak{L}^l \otimes Z_{R,o}^\top Z_o^l)\xi^l - \gamma \sum_{l=1}^{L} \mathfrak{L}^l \otimes Z_{R,o}^\top (Z_o^l)(Z^l)^\top x \\
&= \gamma \sum_{l=1}^{L} (I_N \otimes Z_{R,o}^\top Z_o^l)(\mathfrak{L}^l \otimes I_n)\big(\xi^l - (I_N \otimes (Z^l)^\top)x\big).
\end{aligned}
$$

Therefore, the dynamics of e becomes

$$\begin{aligned}\dot{e} = &\left[(I_N \otimes A_d^R) - \gamma \sum_{l=1}^{L} \mathfrak{L}^l \otimes (Z_{R,\mathrm{o}})^\top Z_\mathrm{o}^l (Z_\mathrm{o}^l)^\top Z_{R,\mathrm{o}}\right] e \\ &+ \sum_{l=1}^{L} \mathfrak{L}^l \otimes (Z_{R,\mathrm{o}})^\top Z_\mathrm{o}^l (Z_\mathrm{o}^l)^\top W_{R,\mathrm{o}} W_R^\top x + \gamma (I_N \otimes Z_{R,o}^\top)\Delta(t),\end{aligned} \tag{4.3.11}$$

where $\Delta(t) := \sum_{l=1}^{L} \left(\mathfrak{L}^l \otimes Z_\mathrm{o}^l\right)\left(\xi^l - (I_N \otimes (Z^l)^\top)x\right)$. From Lemma 3.4.1, it follows that $\Delta(t)$ is exponentially decaying signal. Moreover, due to the particular choice of transformation matrices,

$$\lambda_{n-\nu^R+1}\left(\sum_{l=1}^{L} \mathfrak{L}^l \otimes Y_R^\top Z^l (Z^l)^\top Y_R\right) = \lambda_{n-\nu^R+1}\left(\sum_{l=1}^{L} \mathfrak{L}^l \otimes (Z_{R,\mathrm{o}})^\top Z_\mathrm{o}^l (Z_\mathrm{o}^l)^\top Z_{R,\mathrm{o}}\right) > 0$$

follows directly from (4.1.4).

Finally, we claim that

$$(Z_{R,\mathrm{o}})^\top Z_\mathrm{o}^l (Z_\mathrm{o}^l)^\top W_{R,\mathrm{o}} = 0, \quad \forall l \in \mathcal{L}.$$

To see this, note that $Z_\mathrm{o}^l (Z_\mathrm{o}^l)^\top$ is a diagonal matrix and hence

$$(Z_{R,\mathrm{o}})^\top Z_\mathrm{o}^l (Z_\mathrm{o}^l)^\top W_{R,\mathrm{o}} = \begin{bmatrix} I_{n_o} & 0_{\nu^R \times n_o} \end{bmatrix} \begin{bmatrix} \star & 0 \\ 0 & \star \end{bmatrix} \begin{bmatrix} 0_{n_o \times \nu^R} \\ I_{\nu^R} \end{bmatrix} = 0.$$

Therefore, we obtain

$$\dot{e} = \left[(I_N \otimes A_d^R) - \gamma \sum_{l=1}^{L} \mathfrak{L}^l \otimes (Z_{R,\mathrm{o}})^\top Z_\mathrm{o}^l (Z_\mathrm{o}^l)^\top Z_{R,\mathrm{o}}\right] e + \gamma (I_N \otimes Z_{R,o}^\top)\Delta(t),$$

Hence, following the similar argument as in the proof of Lemma 3.4.2, there exists γ^* such that $e_j(t) - e_i(t) \to 0$ for all $\gamma > \gamma^*$. This implies $\hat{\eta}_j(t) - \hat{\eta}_i(t) \to (Z_R)^\top (x_j(t) - x_i(t))$ which completes the proof. □

With the dynamics constructed as (3.4.1) and (4.3.6), control input can be designed analogous to the state consensus problem. Specifically, let the control input be

$$u_i = (B_d^R)^\top P \sum_{j \in \mathcal{N}_i^{\mathrm{p}}} \alpha_{ij}^{\mathrm{p}} (\hat{\eta}_j - \hat{\eta}_i), \tag{4.3.12}$$

where $P > 0$ is the unique solution of

$$(A_d^R)^\top P + P(A_d^R) - \mathrm{Re}\,(\lambda_2(\mathfrak{L}^{\mathrm{p}}))\, P(B_d^R)(B_d^R)^\top P = -I_{n-\nu^R}.$$

In particular, existence of P follows from stabilizability of (A_d^R, B_d^R) which holds due to the stabilizability of (A, B).

Alternatively, input can be also designed as

$$u_i = K\hat{\eta}_i, \tag{4.3.13}$$

where K is such that $A_d^R + B_d^R K$ is Hurwitz. Then the following result shows that the output consensus is achieved.

Theorem 4.3.3. Suppose that Assumptions 3.1.1 and 4.3.1 hold. If the algebraic condition (4.1.4) holds, then the proposed controllers (4.3.5), (4.3.6) with either (4.3.12) or (4.3.13) achieve output consensus. ♢

Proof. Proof can be obtained by using Lemma 4.3.2 and similar arguments as in the proof of Theorem 3.2.1 to the system (3.1.1) obtained via detectability decomposition using the output matrix R. Hence, the details are omitted. □

Theorem 4.3.3 provides a sufficient condition for the output consensus problem over a directed multilayer network. In case of undirected graphs, Theorem 4.3.3 recovers the equivalence as below.

Corollary 4.3.4. Suppose that Assumptions 3.1.1 and 4.3.1 hold and that $\mathcal{G}$ is undirected. Then the system (3.1.1) is output consensusable if and only if the geometric condition for output consensusability (4.1.1) holds. In addition, geometric condition (4.1.1), geometric condition for each mode (4.1.2) and algebraic condition (4.1.4) are all equivalent. ♢

Proof. Proof follows from Lemma 4.1.1, Lemma 4.1.2 and Theorem 4.3.3. □

For directed graphs under Assumption 4.3.1, following proposition provides an intuitive explanation of the sufficient condition.

Proposition 4.3.5. Suppose that the transformation matrices are chosen as (4.3.3) and (4.3.4). Define the index set $\mathcal{I}_k := \{l \in \mathcal{L} \mid s_k^l = 1\}$ and the corresponding multilayer graph $\mathcal{G}_k := (\mathcal{N}, \{\mathcal{E}^l\}_{l\in\mathcal{I}_k}, \{\mathcal{A}^l\}_{l\in\mathcal{I}_k})$ for $k = 1, \dots, n$. Then

$$\lambda_{n-\nu^R+1}\left(\sum_{l=1}^{L} \mathfrak{L}^l \otimes (Z_{R,\mathrm{o}})^\top (Z_\mathrm{o}^l)(Z_\mathrm{o}^l)^\top Z_{R,\mathrm{o}}\right) > 0$$

holds if and only if $\mathrm{proj}\,(\mathcal{G}_k)$ has a rooted spanning tree for all $k = 1, \dots, n_o$. $\diamondsuit$

Proof. Consider the permutation $\pi\colon \{1, \dots, Nn_o\} \to \{1, \dots, Nn_o\}$ given by

$$\pi(z) = ((z-1) \bmod n_o) \cdot N + ((z-1) \div n_o) + 1$$

and permutation matrix $M := [e_{\pi(1)}^\top; \cdots; e_{\pi(Nn_o)}^\top] \in \mathbb{R}^{Nn_o \times Nn_o}$. Then, noting that $Z_{R,o}^\top Z_o^l (Z_o^l)^\top Z_{R,o} = \mathrm{diag}(s_1^l, \dots, s_{n_o}^l)$, it can be checked that

$$M\left(\sum_{l=1}^{L} \mathfrak{L}^l \otimes (Z_{R,\mathrm{o}})^\top (Z_\mathrm{o}^l)(Z_\mathrm{o}^l)^\top Z_{R,\mathrm{o}}\right)M^\top = \mathrm{diag}\left(\sum_{l=1}^{L} \mathfrak{L}^l s_1^l, \dots, \sum_{l=1}^{L} \mathfrak{L}^l s_{n_o}^l\right).$$

However, $\sum_{l=1}^{L} \mathfrak{L}^l s_k^l = \sum_{l\in\mathcal{I}_k} \mathfrak{L}^l$ is the Laplacian matrix of $\mathrm{proj}\,(\mathcal{G}_k)$. Therefore, the result of the theorem follows since $\mathrm{Re}\left(\lambda_2\left(\sum_{l=1}^{L} \mathfrak{L}^l s_k^l\right)\right) > 0$ if and only if $\mathrm{proj}\,(\mathcal{G}_k)$ has a rooted spanning tree. $\square$

Denoting k-th mode of x_i as $h_k^\top x_i$, $s_k^l = 1$ if k-th mode is detectable (i.e., $h_k^\top x_i$ can be estimated via C^l). Hence, $\mathcal{I}_k$ is the index set of layers which can estimate $h_k^\top x_i$. Thus, Proposition 4.3.5 is stating that the algebraic condition (3.2.3) is equivalent to saying unstable and observable modes via R must be *connected*, in a sense that $\mathrm{proj}\,(\mathcal{G}_k)$, which only includes the edge set of the detectable layers of k-th mode, must have a rooted spanning tree. This also extends the result of Corollary 3.3.3 to the output consensus problem over directed multilayer networks.

4.3.2 Controller Design under Information Structural Constraint

Previous section proposed a controller under Assumption 4.3.1, which limits the class of system based on the matrices relationships among A, C^l and R. In this section, the following assumption is proposed to obtain the convergence.

Assumption 4.3.2. The system satisfies

$$\ker \Pi \otimes Z_R^\top \subseteq \bar{\mathcal{K}}. \qquad \lozenge$$

Assumption 4.3.2 recovers the physical interpretation used in the state consensus problem. Specifically, it says that if the desired output is in consensus, then the relative output information is identically zero. In general, this limits the what *kind* of information can be communicated between agents based on the desired output. Consequently, it can be shown that the same dynamic controller proposed earlier achieves output consensus. For this, suppose without loss of generality that the detectability decomposition is given by orthonormal matrices. Then consider the observers (which is similar to (4.3.5) and (4.3.6)) given by

$$\dot{\xi}_i^l = A_d^l \xi_i^l + G^l \Big[\sum_{j \in \mathcal{N}_i^l} \alpha_{ij}^l C_d^l (\xi_j^l - \xi_i^l) - \alpha_{ij}^l (y_j^l - y_i^l) \Big] + (Z^l)^\top B u_i, \tag{4.3.14a}$$

$$\begin{aligned} \dot{\hat{\eta}}_i = A_d^R \hat{\eta}_i + \gamma \sum_{l=1}^{L} \sum_{j \in \mathcal{N}_i^l} \Big[&\alpha_{ij}^l (Z_R)^\top (Z^l)(Z^l)^\top Z_R (\hat{\eta}_j - \hat{\eta}_i) \\ &- \alpha_{ij}^l (Z_R)^\top (Z^l)(\xi_j^l - \xi_i^l) \Big] + B_d^R u_i. \end{aligned} \tag{4.3.14b}$$

Also suppose that the control input is given by (4.3.12) or (4.3.13). Then the consensus is shown in the following result.

Theorem 4.3.6. Consider the overall system given by the system (3.1.1), dynamic controller (4.3.14) and control input (4.3.12) or (4.3.13). Suppose that Assumptions 3.1.1 and 4.3.2 hold. If the algebraic condition (4.1.4) holds, then the output consensus is achieved. $\lozenge$

Proof. Proof of the convergence is similar to Theorem 4.3.3 presented in Section 4.3.1. The only difference arises in the proof of Lemma 4.3.2 which shows the convergence of $\hat{\eta}$. Specifically, proof of Lemma 4.3.2 is similar up to (4.3.11). From there, we claim

$$\sum_{l=1}^{L} \mathfrak{L}^l \otimes Z_R^\top Z^l (Z^l)^\top W_R W_R^\top x = 0, \quad \forall x \in \mathbb{R}^{Nn}.$$

From the assumption, we have $\ker \Pi \otimes Z_R^\top \subseteq \bar{\mathcal{K}}$. First, observe that

$$(\Pi \otimes Z_R^\top)(I_N \otimes W_R W_R^\top)x = (\Pi \otimes Z_R^\top W_R W_R^\top)x = 0$$

where we used $Z_R^\top W_R = 0$ which holds since matrices are orthogonal by definition. Thus,

$$(I_N \otimes W_R W_R^\top)x \in \ker \Pi \otimes Z_R^\top \subseteq \bar{\mathcal{K}}.$$

Then by the definition of $\bar{\mathcal{K}}$,

$$\left(\sum_{l=1}^{L} \mathfrak{L}^l \otimes (Z^l)(Z^l)^\top \right) (I_N \otimes W_R W_R^\top)x = 0.$$

Therefore, the claim holds. The rest of the proof is similar to Lemma 4.3.2 and hence omitted. □

For undirected multilayer graphs, by supposing Assumption 4.3.2, equivalence can be recovered as in Corollary 4.3.4 as stated below.

Corollary 4.3.7. Suppose that Assumptions 3.1.1 and 4.3.2 hold and that $\mathcal{G}$ is undirected. Then the system (3.1.1) is output consensusable if and only if the geometric condition for output consensusability (4.1.1) holds. In addition, geometric condition (4.1.1), geometric condition for each mode (4.1.2) and algebraic condition (4.1.4) are all equivalent. ◊

Proof. Proof follows from Lemma 4.1.1, Lemma 4.1.2 and Theorem 4.3.6. □

4.4 Static Output Diffusive Coupling

In previous sections, a dynamic controller motivated by the observer-based feedback is designed to achieve consensus over a multilayer network. Instead of a dynamic controller, this section presents a simple static controller to achieve the output consensus.

In order to design a static controller, we make the following assumption on the dynamics of each agent and the multilayer network.

Assumption 4.4.1. There exists a positive definite matrix $P \in \mathbb{R}^{n\times n}$ such that the system satisfies[1] $A^\top P + PA \le 0$, $B = I_n$ and the multilayer graph is undirected. $\Diamond$

The following result proposes a static controller.

Theorem 4.4.1. Suppose that A is not Hurwitz and that Assumption 4.4.1 hold. Then, the system (3.1.1) is output consensusable if and only if the geometric condition (4.1.1) holds. In particular, output consensus is achieved with

$$u_i = \gamma_i P_i^{-1} \sum_{l=1}^{L} \sum_{j\in\mathcal{N}_i^l} \alpha_{ij}^l (C^l)^\top (y_j^l - y_i^l),$$

for any $\gamma_i > 0$ and $P_i > 0$ such that $A^\top P_i + P_i A \le 0$. $\Diamond$

Proof. For sufficiency, let $x = [x_1; \dots; x_N] \in \mathbb{R}^{Nn}$. Then the overall dynamics can be written as

$$\dot{x} = (I_N \otimes A)x - QMx,$$

where $M := \sum_{l=1}^{L} \mathfrak{L}^l \otimes (C^l)^\top C^l$ is a positive semidefinite matrix and $Q := \mathrm{diag}(\gamma_1 P_1^{-1}, \dots, \gamma_N P_N^{-1})$ is positive definite. Let Lyapunov function be $V(x) = x^\top Q^{-1} x$. Then, its time derivative along (3.1.1) becomes

$$\begin{aligned}\dot{V} &= x^\top \left(Q^{-1}(I_N \otimes A) + (I_N \otimes A^\top)Q^{-1}\right) x - 2x^\top Mx \\ &= \sum_{i=1}^{N} \frac{1}{\gamma_i} x_i^\top (P_i A + A^\top P_i) x_i - 2x^\top Mx\end{aligned}$$

[1]Such assumption on A is also known as neutral stability of A.

$$\leq 0.$$

Hence the solution is bounded. Moreover, it follows from LaSalle's invariance principle that the state trajectories approach to the largest invariant set contained in $E := \{x \in \mathbb{R}^{Nn} \mid x^\top M x = 0\}$. Since $\langle \ker C^l \mid A \rangle$ is A-invariant and $\mathfrak{L}^l$ is symmetric, it can be verified that

$$\bar{E} := \left\{ x \in \mathbb{R}^{Nn} \mid x^\top \left(\sum_{l=1}^{L} (\mathfrak{L}^l)^\top \mathfrak{L}^l \otimes (\mathcal{O}^l)^\top \mathcal{O}^l \right) x = 0 \right\}$$

is the largest invariant set in E where $\mathcal{O}^l$ is defined such that $\ker \mathcal{O}^l = \langle \ker C^l \mid A \rangle$. Let $x(t)$ be the solution that belongs identically to $\bar{E}$. Then the solution can be written as $x(t) = x^u(t) + x^s(t)$ where $x^u(t) \in \bar{E} \cap \mathcal{X}^u(\mathbf{A})$ and $x^s(t) \in \bar{E} \cap \mathcal{X}^s(\mathbf{A})$. In addition, it can be checked that $\bar{E} \cap \mathcal{X}^u(\mathbf{A}) \subseteq \bar{\mathcal{K}}$. Since $\bar{\mathcal{K}} \subseteq \ker \Pi \otimes Z_R^\top$ by the assumption, output consensus is achieved. □

For the systems satisfying the assumptions of Theorem 4.4.1, it follows that the proposed necessary condition $\bar{\mathcal{K}} \subseteq \ker \Pi \otimes Z_R^\top$ is indeed a necessary and sufficient condition for achieving output consensus.

Remark 4.4.1. The assumption that A is neutrally stable is a restrictive assumption. Comparing the result of Theorem 4.4.1 to results from the single-layer consensus problem, it corresponds to the result of [SS09] which considered consensus of MAS when A is marginally stable and $B = I_n$. For the single-layer consensus, an extension is made in [Tun08], which used LQR-based gain to achieve consensus for general plant A. Extension of Theorem 4.4.1 seems challenging as such technique cannot be applied directly to multilayer networks but it is an interesting direction for future research as one may draw motivations from these developments. ◊

Although the result of Theorem 4.4.1 is restrictive, its usage will be presented in Chapter 5 to solve the distributed estimation problem. In fact, an extension of Theorem 4.4.1 is developed for a class of directed multilayer network with switching topology.

Assumptions	Controller Type	Single-layer	Multilayer
A marginally stable, $B = I_n$	Static	[SS09]	Thm. 4.4.1
(A, B) stab., $C = I_n$	Static	[Tun08]	Not applicable
(A, B, C) stab. and detect.	Dyn. w. comm.	[WLH09]	Thm. 4.3.3 Thm. 4.3.6
(A, B, C) stab., detect. and $\mathrm{Re}\,(\lambda_i(A)) \leq 0$	Dyn. w.o. comm.	[SSB09]	Open

Table 4.1: Summary of controller designs for the single-layer system and corresponding designs for the multilayer network proposed in this chapter.

4.5 Summary of Results

Before moving onto the next few chapters, which discuss the application of the consensus problem over multilayer networks, we end this chapter by summarizing the main results. Similar results from the single-layer consensus problem are also briefly discussed and compared.

4.5.1 Comparison with Single-layer Consensus Problem

Controller designs developed in Chapters 3 and 4 can be compared with its corresponding designs for the single-layer system, which is summarized in Table 4.1. The static feedback controller developed in Theorem 4.4.1 is an extension of the work such as [SS09] which studied a similar problem for the single-layer network. Both of these controllers use static feedback and assume the marginal stability of the system matrix. Classical work [Tun08] does not have corresponding work in this dissertation as $C = I_n$ is not applicable to multilayer networks. However, observer-based controller developed in works such as [WLH09] are extended to the multilayer network in Theorem 4.3.3 and Theorem 4.3.6. Notice that our results not only generalize these results to the multilayer network but also extends to the output consensus problem over directed networks.

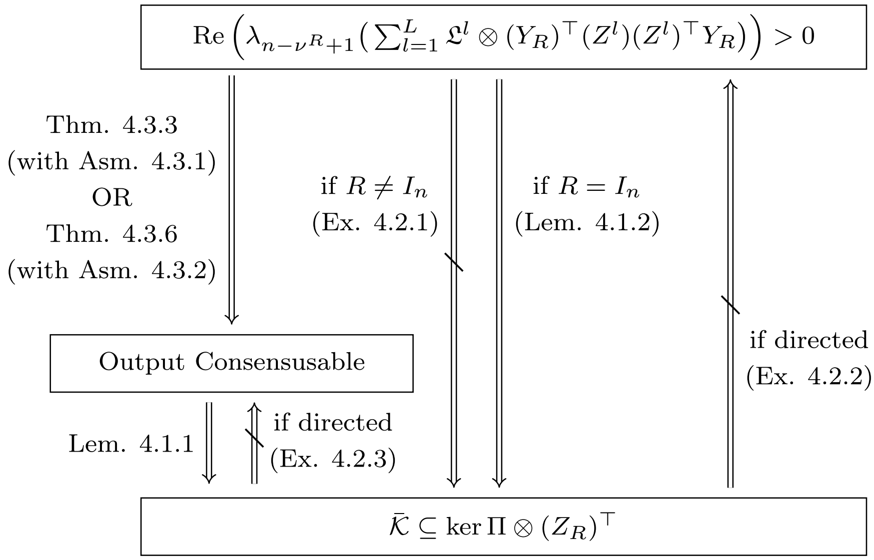


Figure 4.2: Relation for directed multilayer network.

4.5.2 Relation between Necessary and Sufficient Conditions

We end this chapter by showing figures which illustrate the relations between main results for the design of the controllers.

First, results for the directed multilayer network is shown in Fig. 4.2. It can be seen that the algebraic condition along with Assumption 4.3.1 or Assumption 4.3.2 implies the output consensus, while the geometric condition is a necessary condition for output consensusability. Unfortunately, we cannot close the loop since the geometric condition does not imply the algebraic condition in general.

Relations for undirected multilayer network is shown in Fig. 4.3. Notice that geometric condition implies the algebraic condition in undirected networks. Hence necessary and sufficient conditions can be found under additional assumptions. Specifically, dynamic controller can be designed under Assumption 4.3.1 or Assumption 4.3.2 and static controller under Assumption 4.4.1.

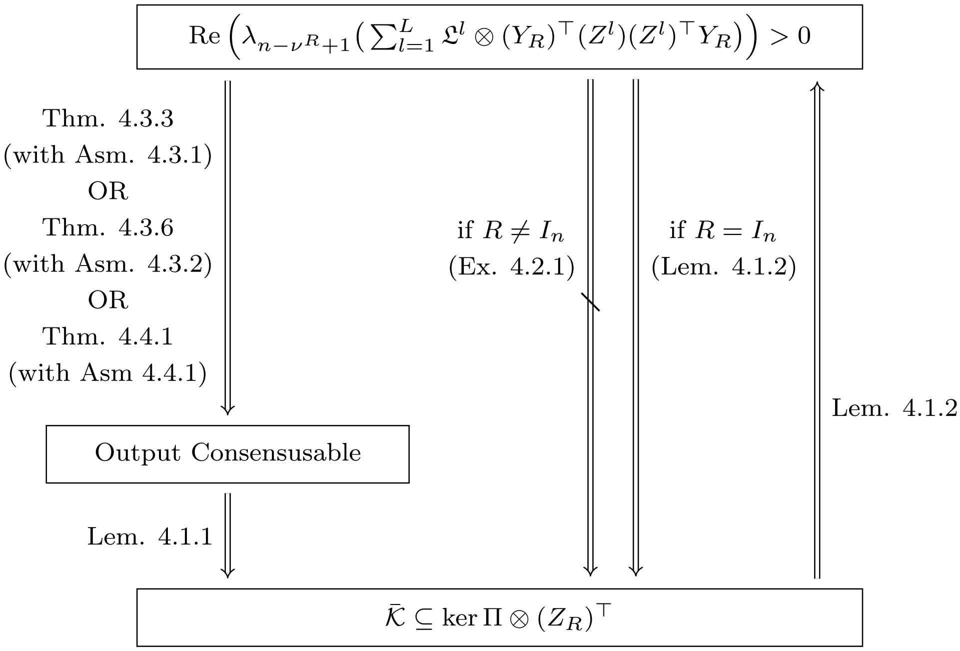


Figure 4.3: Relation for undirected multilayer network.

Chapter 5

Application to the Distributed State Estimation Problem

In this chapter, results developed for the consensus over the multilayer network is applied to the distributed state estimation problem (DEP). It is established that the DEP can be formulated into a consensus problem over a multilayer network. Through this approach, results from Chapter 3 are applied to obtain a necessary condition for the solvability of DEP. Moreover, a novel design is proposed which reduces the communication burden compared with the existing designs in the literature.

5.1 Problem Formulation

Consider the system in form[1] of

$$\dot{\rho} = A\rho + Bu(t), \tag{5.1.1a}$$

$$\omega = \begin{bmatrix} \omega_2 \\ \vdots \\ \omega_{N+1} \end{bmatrix} = \begin{bmatrix} H_2 \\ \vdots \\ H_{N+1} \end{bmatrix} \rho =: H\rho, \tag{5.1.1b}$$

where $\rho \in \mathbb{R}^n$ is the state of the plant to be estimated, $u(t) \in \mathbb{R}^p$ is the control input applied to the plant and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times q}$ are the system matrices. It

[1] Index of observers start from 2. This is not conventional but used to make notations clear for the main results.

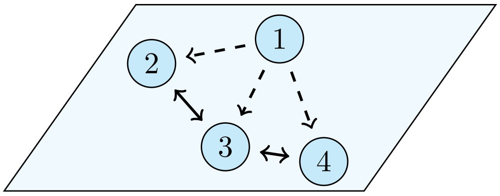
is supposed that there are N agents where $\omega_i \in \mathbb{R}^{q^i}$ for $i = 2, \ldots, N+1$ is the measurement available to agent i and let $H_1 = 0$. The objective of the DEP is for each agent to estimate the state of the plant $\rho(t)$ only using local measurement ω_i, control input $u(t)$ and communication with its neighbors. Specifically, we consider the case when (H_i, A) is not necessarily detectable, but (H, A) is detectable. For a more detailed discussion of the DEP, see [KSC16, MS18].

Recall that the classical centralized state estimation problem can be treated as a consensus problem between two homogeneous agents, the plant and an observer. Similarly, the DEP can be viewed as a consensus problem of $N+1$ agents consisting of a single plant and N observers. However, since each observer receives different output information from the plant (as the measurement), the DEP cannot be represented as a consensus over a single-layer network. Therefore, we propose to use the multilayer network to represent each output measurement as a separate layer. Then, the DEP can be formulated into an equivalent consensus problem over a multilayer network.

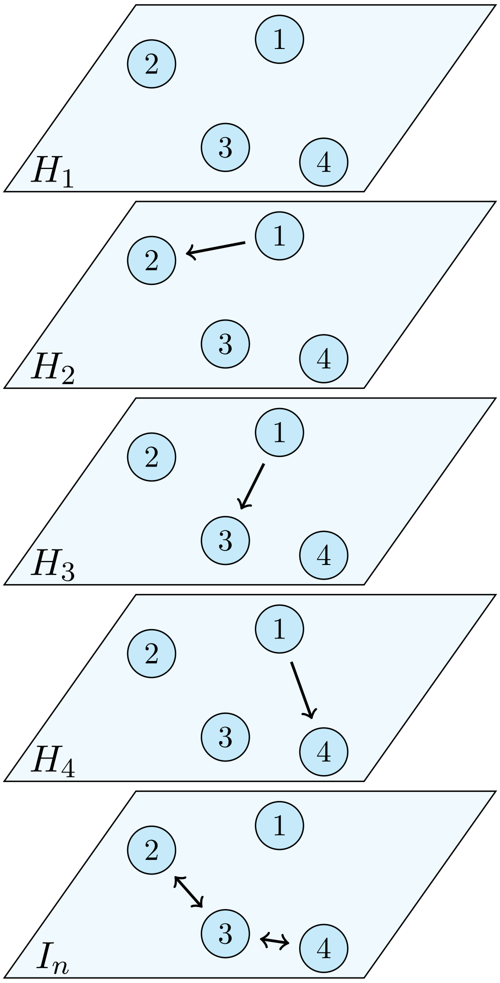
For illustration, consider the DEP consisting of a single plant and 3 agents with its communication structure shown in Fig. 5.1(a). Specifically, dashed arrows denote the output measurement of each agent and solid arrows denote the communication between agents. Then, the DEP can be interpreted as a multi-agent system consisting of 4 agents where the communication structure is given as in Fig. 5.1(b). In particular, the first layers 2 to $N+1$ represent the output measurement of each agent. First layer is added for the sake of consistency with indices but it does not contain any edge and $H_1 = 0$ (that is, the problem is equivalent without this layer). Note that each agent uses a *different* output matrix H_i to measure the output which corresponds to the output matrix of the layer. Finally, the last layer represents the cooperation between agents, i.e., communication among neighboring agents.

Therefore, the DEP can be formulated into an equivalent state consensus problem of the system

$$\begin{aligned} \dot{\rho} &= A\rho + Bu(t), \\ \dot{\hat{\rho}}_i &= A\hat{\rho}_i + Bu(t) + h_{c,i}(\delta_i), \quad i = 2, \ldots, 4, \end{aligned} \tag{5.1.2}$$



(a) Communication network for the DEP with agent 1 being the plant. Solid arrows denote communication among observers, and dashed arrows denote output measurement.



(b) Equivalent multilayer network. Layers 2 to 4 represent output measurement of each observer, while layer 5 represents the communication among observers.

Figure 5.1: An example of the distributed estimation problem where agent 1 is the plant and nodes 2 to 4 are the observers.

where $\hat{\rho}_i \in \mathbb{R}^n$ is the state of each observer. Additionally, δ_i is a stack of relative information obtained from neighboring agents (and plant) over the multilayer network and $h_{\mathrm{c},i}$ represents the error injection term.

The dynamics represented by (5.1.2) can be compared with the system (3.1.1) and the controller (3.1.2) to obtain appropriate relationships between the two problems. Specifically, (5.1.2) is using static output feedback to achieve state consensus. In general, the DEP can be seen as solving the consensus problem of $N+1$ agents with L $(\geq N+2)$ layers using static output feedback. Specifically, the first $N+1$ layers represent the output measurement, whereas the layers $N+2$ to L represent the communication among observers.

Since the DEP is equivalent to a state consensus problem, results from Chapter 3 and 4 can be applied to obtain a necessary condition for solving the DEP. Moreover, it can be shown that the necessary condition is also sufficient under additional assumptions.

5.2 Distributed State Estimation with Reduced Communication over Static Network

Before presenting the main results, let us introduce some notations. Suppose that the multilayer network $\mathcal{G} = (\{1, \ldots, N+1\}, \{\mathcal{E}^l\}_{l \in \mathcal{L}})$ with $N+1$ nodes and L layers represents the communication among N observers and the plant. For $l = N+2, \ldots, L$, define $\mathcal{G}^l_{\mathrm{obs}} = (\{2, \ldots, N+1\}, \mathcal{E}^l_{\mathrm{obs}})$ as the single-layer graph representing the communication among N observers via output C^l. For $l = N+2, \ldots, L$, define $\mathfrak{L}^l_{\mathrm{obs}} \in \mathbb{R}^{N \times N}$ such that

$$\mathfrak{L}^l = \begin{bmatrix} 0_1 & 0_{1 \times N} \\ \star & \mathfrak{L}^l_{\mathrm{obs}} \end{bmatrix},$$

where $\mathfrak{L}^l$ is the Laplacian matrix of $\mathcal{G}^l$. In particular, $\mathfrak{L}^l_{\mathrm{obs}}$ is a Laplacian matrix representing the communication among observers (with appropriate indices). Then, we make the following assumption.

Assumption 5.2.1. Communication among observers are bidirectional, i.e., $\mathfrak{L}^l_{\mathrm{obs}}$

is undirected for $l = N+2, \dots, L$. $\Diamond$

We also define Z^k as

$$\ker(Z^k)^\top = \begin{cases} \langle \ker H_k \mid A \rangle \cap \mathcal{X}^u(A), & \text{if } k = 1, \dots, N+1 \\ \langle \ker C^k \mid A \rangle \cap \mathcal{X}^u(A), & \text{if } k = N+2, \dots, L, \end{cases}$$

so that $\ker\,(Z^k)^\top$ is undetectability subspace of the output matrix corresponding to the l-th layer.

Now, we propose a distributed observer in form of

$$\begin{aligned} \dot{\hat{\rho}}_i &= A\hat{\rho}_i + Bu(t) \\ &\quad + \gamma_i K_i \underbrace{\sum_{l=N+2}^{L} \sum_{j \in \mathcal{N}_i^l} (C^l)^\top (\hat{y}_j^l - \hat{y}_i^l)}_{\text{diffusive coupling}} + \gamma_i \alpha_i K_i \underbrace{H_i^\top (w_i - H_i \hat{\rho}_i)}_{\text{output injection}}, && (5.2.1a) \\ \hat{y}_i^l &= C^l \hat{\rho}_i, \qquad \forall l = N+2, \dots, L,\ i = 2, \dots, N+1, && (5.2.1b) \end{aligned}$$

where $\gamma_i > 0$, $K_i \in \mathbb{R}^{n \times n}$ are the coupling gains to be designed and $\alpha_i = 1$ if observer i measures the output of the plant and 0 otherwise. Note that each observer applies diffusive output coupling using information obtained from its neighboring observers as well as the output injection term using the local output measurement.

By defining $\hat{\rho} := [\hat{\rho}_2; \cdots ; \hat{\rho}_{N+1}] \in \mathbb{R}^{Nn}$, it can be verified that the plant (5.1.1) and the distributed observer (5.2.1) can be written as

$$\dot{\rho} = A\rho + Bu(t), \tag{5.2.2a}$$

$$\dot{\hat{\rho}} = (I_N \otimes A)\hat{\rho} - \Gamma L^* \hat{\rho} + \Gamma G^* \big((1_N \otimes I_n)\rho - \hat{\rho}\big) + (1_N \otimes B)u(t), \tag{5.2.2b}$$

where $\Gamma := \mathrm{diag}(\gamma_2 K_2, \dots, \gamma_{N+1} K_{N+1}) \in \mathbb{R}^{Nn \times Nn}$ and $L^* \in \mathbb{R}^{Nn \times Nn}$ is defined as

$$L^* := \sum_{l=N+2}^{L} \mathfrak{L}_{\mathrm{obs}}^l \otimes (C^l)^\top C^l$$

and $G^* \in \mathbb{R}^{Nn \times Nn}$ is defined as

$$G^* := \operatorname{diag}\left(\alpha_2 \cdot (H_2)^\top H_2, \ldots, \alpha_{N+1} \cdot (H_{N+1})^\top H_{N+1}\right).$$

Note that both L^* and G^* are symmetric, positive semidefinite matrix. For the convergence of the proposed design the following assumption is made.

Assumption 5.2.2. The matrix A is marginally stable, i.e., there exists a positive definite matrix $P \in \mathbb{R}^{n \times n}$ such that

$$A^\top P + PA \leq 0. \qquad \diamondsuit$$

Remark 5.2.1. Marginal stability of A_p allows harmonic oscillators and the single integrator. However, double integrators do not satisfy Assumption 5.2.2. $\diamondsuit$

Now, define the set

$$\bar{\mathcal{K}}^{N+1} := \bigcap_{l=1}^{L} \ker \mathfrak{L}^l \otimes (Z^l)(Z^l)^\top = \bigcap_{l=2}^{L} \ker \mathfrak{L}^l \otimes (Z^l)(Z^l)^\top,$$

where we use $\bar{\mathcal{K}}^{N+1}$ to emphasize that the set is constructed from $N+1$ agents including the plant. Then the following theorem states the convergence of the proposed distributed observer.

Theorem 5.2.1. Suppose that A is not Hurwitz and Assumptions 5.2.1 and 5.2.2 hold. Then design observer gain γ_i to be any positive number for $i = 2, \ldots, N+1$ and let $K_i = P_i^{-1}$ where $P_i > 0$ is such that

$$A^\top P_i + P_i A \leq 0,$$

whose existence follows from Assumption 5.2.2. Then the proposed distributed observer (5.2.1) solves the DEP, i.e.,

$$\lim_{t \to \infty} |\hat{\rho}_i(t) - \rho(t)| = 0, \quad \forall i = 2, \ldots, N+1,$$

if and only if

$$\bar{\mathcal{K}}^{N+1} = \mathcal{S}_n^{N+1}. \qquad \lozenge$$

Proof. Necessity follows directly from Lemma 4.1.1. Specifically, since the estimation problem can be interpreted as the state consensus problem over a (directed) multilayer network, it follows that

$$\bar{\mathcal{K}}^{N+1} \subseteq \ker \Pi \otimes Z_R^\top = \mathcal{S}_n^{N+1}.$$

The converse relation $\mathcal{S}_n^{N+1} \subseteq \bar{\mathcal{K}}^{N+1}$ is trivial to show.

For sufficiency, define the estimation error as $e := \hat{\rho} - (1_N \otimes I_n)\rho \in \mathbb{R}^{Nn}$. Then, its dynamics becomes

$$\dot{e} = \Big((I_N \otimes A) - \Gamma(L^* + G^*)\Big)e, \tag{5.2.3}$$

where we used the fact that $L^*(1_N \otimes I_n) = 0$. Let Lyapunov function be

$$V(e) = e^\top \Gamma^{-1} e = e^\top \left(\begin{bmatrix} \frac{1}{\gamma_2} P_2 & & \\ & \ddots & \\ & & \frac{1}{\gamma_{N+1}} P_{N+1} \end{bmatrix} \right) e.$$

Then it holds that

$$\begin{aligned} \dot{V}(e) &= e^\top \left(\Gamma^{-1}(I_N \otimes A) + (I_N \otimes A^\top)\Gamma^{-1}\right) e - 2e^\top (L^* + G^*)e \\ &= e^\top \begin{bmatrix} \frac{1}{\gamma_2}(P_2 A + A^\top P_2) & & \\ & \ddots & \\ & & \frac{1}{\gamma_{N+1}}(P_{N+1}A + A^\top P_{N+1}) \end{bmatrix} e - 2e^\top (L^* + G^*)e \\ &\leq -2e^\top (L^* + G^*)e. \end{aligned}$$

Hence, the solution of (5.2.3) is bounded. Applying the LaSalle's invariance principle [Kha02, Thm. 4.4], it follows that $e(t)$ converges to the largest invariant set in $E := \{e \mid e^\top (L^* + G^*)e = 0\}$. Since $\mathfrak{L}_{\text{obs}}^l$ is symmetric, similar arguments as in the proof of Theorem 4.4.1 shows that the solution converges to

$$\bar{E} := \left\{ e \in \mathbb{R}^{Nn} \mid e^\top (L_z^* + G_z^*) e = 0 \right\},$$

where $L_z^* := \sum_{l=N+2}^{L} (\mathfrak{L}_{\mathrm{obs}}^l)^\top \mathfrak{L}_{\mathrm{obs}}^l \otimes (Z^l)(Z^l)^\top$ and

$$G_z^* := \mathrm{diag}\left(a_2 (Z^2)(Z^2)^\top, \ldots, a_{N+1}(Z^{N+1})(Z^{N+1})^\top \right).$$

Next, it is shown that $L_z^* + G_z^*$ is positive definite if $\bar{\mathcal{K}}^{N+1} = \mathcal{S}_n^{N+1}$. To see this, recalling that $\ker \mathfrak{L}^l \otimes (Z^l)(Z^l)^\top = \ker (\mathfrak{L}^l)^\top (\mathfrak{L}^l) \otimes (Z^l)(Z^l)^\top$, it holds that $\bar{\mathcal{K}}^{N+1}$ can be written as

$$\begin{aligned}
\bar{\mathcal{K}}^{N+1} &= \ker \left(\sum_{l=1}^{L} (\mathfrak{L}^l)^\top \mathfrak{L}^l \otimes (Z^l)(Z^l)^\top \right) \\
&= \ker \begin{bmatrix} \sum_{k=2}^{N+1} \alpha_k (Z^k)(Z^k)^\top & -(1_N \otimes I_n)^\top G_z^* \\ -G_z^* (1_N \otimes I_n) & L_z^* + G_z^* \end{bmatrix} =: \ker \hat{\mathfrak{L}}.
\end{aligned}$$

In particular, we have used $\mathfrak{L}^l = e_{l+1} e_{l+1}^\top - e_{l+1} e_1^\top$ for all $l = 1, \ldots, N$, which results in

$$(\mathfrak{L}^l)^\top \mathfrak{L}^l = \left[\begin{array}{c|c|c|c} 1 & & -1 & \\ \hline & & & \\ \hline -1 & & 1 & \\ \hline & & & \end{array} \right].$$

Since $\bar{\mathcal{K}}^{N+1} = \mathcal{S}_n^{N+1}$, it follows from Lemma A.1.1 that $w^\top \hat{\mathfrak{L}} w > 0$ for all $w \notin \mathcal{S}_n^{N+1}$. Hence, by letting $w := [0_n; w']$, it can be shown that $L_z^* + G_z^*$ is positive definite. In conclusion, we obtain $\lim_{t\to\infty} e(t) = 0$, i.e., it holds that $\lim_{t\to\infty} |\hat{\rho}_i(t) - \rho(t)| = 0$. □

Theorem 5.2.1 states that the proposed observer (5.2.1) achieves the distributed state estimation. Similar to the result of Theorem 4.4.1, it shows that the necessary condition is also sufficient under additional assumptions.

The major difference of the proposed design compared with typical designs in the literature (e.g., [KSC16, MS18, WM18]) is the amount of information communicated between observers. In fact, it is often assumed that the full state estimation $\hat{\rho}_i$ is communicated with neighboring agents to achieve the distributed state estimation. This corresponds to the case when $L = N+2$ and $C^L = I_n$. Instead, the proposed observer (5.2.1) is designed for general multilayer communication network, and specifically, it allows each observer to communicate part of the state estimation via output matrix C^k for $k = N+2, \ldots, L$. The following example illustrates this feature of the proposed design.

Example 5.2.1. Consider a plant with 3 agents where the system and measurement matrices are given by

$$A_{\rm p} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{bmatrix}, \quad B_{\rm p} = 0$$
$$H_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \quad H_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}, \quad H_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}.$$

Also, the communication among observers are given by

$$C^5 = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}, \quad \mathfrak{L}^5_{\rm obs} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Then, it can be checked that all assumptions of Theorem 5.2.1 hold with $P = I_4$ and that $\bar{\mathcal{K}}^{N+1} = \mathcal{S}_n^{N+1}$. Hence, the observer (5.2.1) is given by

$$\begin{aligned}
\dot{\hat{\rho}}_2 &= A_{\rm p}\hat{\rho}_2 + (C^5)^\top(C^5\hat{\rho}_3 - C^5\hat{\rho}_2) + H_2^\top(w_2 - H_2\hat{\rho}_2), \\
\dot{\hat{\rho}}_3 &= A_{\rm p}\hat{\rho}_3 + (C^5)^\top\big((C^5\hat{\rho}_2 - C^5\hat{\rho}_3) + (C^5\hat{\rho}_4 - C^5\hat{\rho}_3)\big) + H_3^\top(w_3 - H_3\hat{\rho}_3), \\
\dot{\hat{\rho}}_4 &= A_{\rm p}\hat{\rho}_4 + (C^5)^\top(C^5\hat{\rho}_3 - C^5\hat{\rho}_4),
\end{aligned}$$

where $\gamma_i = 1$ and $P_i = I_4$ are used. Thus, it follows from Theorem 5.2.1 that $\hat{\rho}_i(t)$ recovers the state $\rho(t)$. In this example, note that observers only communicate the scalar variable $C^5\hat{\rho}_i \in \mathbb{R}^1$ with its neighbors instead of the full state estimate. Nonetheless, state estimation is still achieved. Numerical simulation is presented in Section 5.4. ◊

It also follows from Theorem 5.2.1 that the design parameters γ_i and P_i can be determined in a completely decentralized manner. Each agent can locally solve for P_i and choose any positive γ_i to design its local observer.

We end this section by investigating the proposed condition $\bar{\mathcal{K}}^{N+1} = \mathcal{S}_n^{N+1}$ when observers communicate the full state estimation, i.e., when $C^{N+2} = I_n$ and $L = N+2$.

Theorem 5.2.2. Consider the DEP and suppose that $L = N+2$ and $C^{N+2} = I_n$. Then, $\bar{\mathcal{K}}^{N+1} = \mathcal{S}_n^{N+1}$ if and only if

$$\bigcap_{i \in \widetilde{\mathcal{V}}_k} \ker\,(Z^i)^\top = \{0\}, \quad \forall k = 1, \dots, c \tag{5.2.4}$$

where $c \geq 1$ is the number of connected components of the graph $\mathcal{G}_{\text{obs}}^{N+1}$ and $\widetilde{\mathcal{V}}_k \subseteq \mathcal{N} = \{2, \dots, N+1\}$ is the node set of the k-th connected component. ◊

Proof. ($\implies$) Suppose that $\bar{\mathcal{K}} = \mathcal{S}_n^{N+1}$ and let $x \in \bar{\mathcal{K}}$ where $x = [x_1; \dots; x_{N+1}] \in \mathbb{R}^{(N+1)n}$. It follows from the assumption that $Z^{N+2} = I_n$. Therefore, it holds that

$$(\mathfrak{L}^{N+2} \otimes I_n)x = 0, \tag{5.2.5}$$

where $\mathfrak{L}^{N+2} = \text{diag}(0, \mathfrak{L}_{\text{obs}}^{N+2}) \in \mathbb{R}^{(N+1)\times(N+1)}$. Then it follows from (5.2.5) that

$$x_i = x_j, \quad \forall i, j \in \widetilde{\mathcal{V}}_k \tag{5.2.6}$$

for all $k = 1, \dots, c$. Hence, define η_k as the value of x_i belonging to the k-th connected component.

Now, note that there is a single directed edge from the plant to an individual agent from layer 2 to $N+1$. Therefore, it follows from $x \in \ker \mathfrak{L}^i \otimes (Z^i)^\top$ for $i = 2, \dots, N+1$ that

$$x_1 - x_i \in \ker(Z^i)^\top, \quad \forall i = 2, \dots, N+1. \tag{5.2.7}$$

In particular, from (5.2.6) and (5.2.7), we obtain

$$x_1 - \eta_k \in \ker(Z^i)^\top, \quad \forall i \in \widetilde{\mathcal{V}}_k$$

for $k = 1, \dots, c$. Therefore, it follows that

$$x_1 - \eta_k \in \bigcap_{i \in \widetilde{\mathcal{V}}_k} \ker(Z^i)^\top, \quad \forall k = 1, \dots, c.$$

We claim $\cap_{i \in \widetilde{\mathcal{V}}_k} \ker(Z^i)^\top = \{0\}$ for all $k = 1, \dots, c$. For the contradiction, suppose that there exists $k' \in \{1, \dots, c\}$ such that

$$\bigcap_{i \in \widetilde{\mathcal{V}}_{k'}} \ker(Z^i)^\top \neq \{0\}.$$

Then it follows that $x_1 - \eta_{k'} \neq 0$. (If $x_0 = \eta_{k'}$, then a nonzero constant $\Delta \in \cap_{i \in \widetilde{\mathcal{V}}_{k'}} \ker(Z^i)^\top$ can be added to x_i for all $i \in \{1, \dots, N+1\} \setminus \widetilde{\mathcal{V}}_{k'}$.) Hence, $x \notin \mathcal{S}_n^{N+1}$, which leads to a contradiction since $x \in \bar{\mathcal{K}} \subseteq \mathcal{S}_n^{N+1}$ by the assumption. Thus, (5.2.4) holds.

($\Longleftarrow$) Let $x \in \bar{\mathcal{K}}$ where $x := [x_1; \dots; x_{N+1}] \in \mathbb{R}^{(N+1)n}$. Then, following the arguments of the necessity part of the proof, it holds that

$$x_1 - \eta_k \in \bigcap_{i \in \widetilde{\mathcal{V}}_k} \ker(Z^i)^\top, \quad \forall k = 1, \dots, c.$$

Therefore, it follows from (5.2.4) that $x_1 = \eta_1 = \dots = \eta_c$. Since every node (other than node 1, i.e., the plant) belongs to a connected component of $\mathcal{G}_{\text{obs}}^{N+1}$, it follows that $x \in \mathcal{S}_n^{N+1}$. □

In fact, the condition (5.2.4) recovers the necessary condition proposed for the DEP in the literature (e.g., see [KSC16]). For example, it says that the stack of output matrix (H, A) should be detectable when the communication network is connected. Moreover, in this particular case, the necessary condition (5.2.4) is shown to be sufficient for the general plant A. For instance, a static output controller is proposed in [KSC16, LS20b]. A solution to the DEP is proposed in

[KSC16] as

$$\dot{\hat{\rho}}_i = A\hat{\rho}_i + G_i(\omega_i - H_i\hat{\rho}_i) + kW_iW_i^\top \sum_{j\in\mathcal{N}_i}(\hat{\rho}_j - \hat{\rho}_i)$$

for $i = 2, \ldots, N+1$, where G_i and W_i are appropriate gain matrices. It can be verified that this is exactly in form of (5.1.2).

5.2.1 Design Procedures

A necessary and sufficient condition for solving the distributed estimation problem is given in the previous section. Specifically, it has been shown that the state estimation is achieved (along with appropriate assumptions) if and only if

$$\bar{\mathcal{K}}^{N+1} = \mathcal{S}_n^{N+1},$$

while the corresponding gains can be designed in a decentralized manner. Given a multilayer network describing the distributed observer, the above condition can be checked by counting the number of 0 eigenvalues of the matrix

$$\sum_{l=1}^{L} \mathfrak{L}^l \otimes (Z^l)(Z^L)^\top.$$

However, one often needs to *design* the multilayer network (e.g., $\mathcal{G}^l_{\text{obs}}$ or C^l for $l = N+2, \ldots, L$) such that the above condition holds. In this section, we present a few observations that are tailored towards the DEP to assist the design procedure. Additionally, a simple design method is proposed to ensure DEP is solved while communicating the minimum amount of information. First, the following sufficient condition is obtained.

Lemma 5.2.3. Let $H := [H_2; \cdots; H_{N+1}]$. If (H, A) is detectable and

$$\bar{\mathcal{K}}^N := \bigcap_{l=N+2}^{L} \mathfrak{L}^l_{\text{obs}} \otimes (Z^l)(Z^l)^\top = \mathcal{S}_n^N,$$

then $\bar{\mathcal{K}}^{N+1} = \mathcal{S}_n^{N+1}$, i.e., the DEP can be solved. $\Diamond$

Proof. It is sufficient to show $\bar{\mathcal{K}}^{N+1} \subseteq \mathcal{S}_n^{N+1}$. Let $x = [x_1; x_1; \cdots; x_{N+1}] := [x_1; x_{\text{obs}}] \in \bar{\mathcal{K}}^{N+1}$, i.e.,

$$x \in \bigcap_{l=1}^{L} \mathfrak{L}^l \otimes (Z^l)(Z^l)^\top.$$

Then it follows from the structure of the multilayer graph that

$$(Z^i)^\top (x_i - x_1) = 0, \quad \forall i = 2, \dots, N+1, \tag{5.2.8}$$

and

$$\mathfrak{L}^l_{\text{obs}} \otimes (Z^l)(Z^l)^\top x_{\text{obs}} = 0, \quad \forall l = N+2, \dots, L.$$

This implies that $x_{\text{obs}} \in \bar{\mathcal{K}}^N$. Since $\bar{\mathcal{K}}^N = \mathcal{S}_n^N$ by the assumption, it follows that $x_{\text{obs}} \in \mathcal{S}_n^N$. Hence, by letting $x_{\text{obs}} = (1_N \otimes z)$, it follows from (5.2.8) that

$$\begin{bmatrix} (Z^2)^\top \\ \vdots \\ (Z^{N+1})^\top \end{bmatrix} (z - x_1) = 0 \implies z - x_1 \in \langle \ker H \mid A \rangle \cap \mathcal{X}^u(A).$$

Due to te detectability of pair (H, A), it holds that $z = x_1$. Hence, this implies $x \in \mathcal{S}_n^{N+1}$, which completes the proof. □

From the proof of Lemma 5.2.3, it follows easily that detectability (H, A) is a necessary condition for solving DEP. Once the detectability condition is met, it is sufficient to design communication between observers can be such that $\bar{\mathcal{K}}^N = \mathcal{S}_n^N$. A simple method to satisfy this condition is to design communication among observers such that $L = N+2$ and design C^{N+2} such that (C^{N+2}, A) is detectable. This is shown in the following theorem.

Theorem 5.2.4. Consider the DEP and suppose that (H, A) is detectable. Construct the distributed observer of the form (5.2.1) where the multilayer graph with $L = N + 2$ and $N + 2$-th layer is a connected graph with (C^{N+2}, A) being a detectable pair. Then, (5.2.1) solves the DEP. ♢

Proof. Using Lemma 5.2.3, it is sufficient to show $\bar{\mathcal{K}}^N = \mathcal{S}_n^N$ holds. Due to the structure of the communication network, it holds that

$$\bar{\mathcal{K}}^N = \ker \mathfrak{L}_{\text{obs}}^{N+2} \otimes (Z^{N+2})(Z^{N+2})^\top.$$

Since $(Z^{N+2})(Z^{N+2})^\top > 0$ due to detectability and $\mathfrak{L}_{\text{obs}}^{N+2}$ is a Laplacian matrix of a connected graph, it is trivial to check

$$\lambda_{n+1}\left(\mathfrak{L}_{\text{obs}}^{N+2} \otimes (Z^{N+2})(Z^{N+2})^\top\right) > 0,$$

which implies $\bar{\mathcal{K}}^N = \mathcal{S}_n^N$. Thus, the result follows. □

Theorem 5.2.4 proposes a design of the distributed observer where each agent communicate $C^{N+2}x_i \in \mathbb{R}^{q^{N+2}}$ to its neighbors. Notice that the design procedure of the proposed design is rather simple. In particular, it only requires detectability of (H, A), connectedness of the communication network of the observers, and detectability of (C^{N+1}, A). Specifically, required conditions such as connectivity and detectability are completely decoupled and hence can be designed separately.

Compared with the existing designs with $C^{N+2} = I_n$, we instead assume detectability of the pair (C^{N+2}, A). Hence, in order to reduce the communication load, C^{N+2} should be designed to minimize q^{N+2}. Given a real matrix A with m distinct eigenvalues, by investigating its Jordan form [Che99], it can be verified that the minimum q^* is given by

$$q^* = \max_{k \text{ s.t. } \mathrm{Re}(\mu_k(A)) \geq 0} \mu_k(A) \leq n,$$

where $\mu_k(A)$ is the geometric multiplicity of $\lambda_k(A)$. Some examples are given below.

1. If the system matrix $A \in \mathbb{R}^{n \times n}$ has n distinct eigenvalues, then

$$q^* = \max(1, \ldots, 1) = 1.$$

Hence, it is sufficient for observers to only communicate a single dimensional information.

2. Similarly, if the system matrix A is chain of integrator given by

$$A = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix}.$$

   Then, it can be checked that $q^* = 1$ and hence communicating single dimensional information is sufficient.

3. On the other hand, if $A = 0_n$, then

$$q^* = \max(n) = n.$$

   Thus, no improvements can be made as observers must communicate full n-dimensional information.

Remark 5.2.2. A design with less communicational load may exist for a particular system. For example, if every observer measures a common mode, then this information is not required to be communicated to neighboring agents (that is $\bar{\mathcal{K}}^N = \mathcal{S}_n^{N+1}$ but (C^{N+1}, A) not necessarily detectable). However, such design depends on global information such as the exact measurement structure of agents and the structure of the communication network. ♢

5.3 Distributed State Estimation with Reduced Communication over Switching Network

This section presents extension of the results from the previous section to switching networks. Most of the notations are identical to Section 5.2. Consider again the distributed estimation problem with the plant

$$\dot{\rho} = A\rho + Bu(t), \tag{5.3.1a}$$

$$\omega = \begin{bmatrix} \omega_2 \\ \vdots \\ \omega_{N+1} \end{bmatrix} = \begin{bmatrix} \alpha_2(t)H_2 \\ \vdots \\ \alpha_{N+1}(t)H_{N+1} \end{bmatrix} \rho =: \Lambda(t)H\rho, \tag{5.3.1b}$$

where $\alpha_i(t) \in \{0,1\}$ is a piecewise constant signal denoting the switching of the output matrix and $\Lambda(t) := \mathrm{diag}(s_2(t), \ldots, s_{N+1}(t))$. Also recall that the distributed observer is given by

$$\begin{aligned} \dot{\hat{\rho}}_i &= A\hat{\rho}_i + Bu(t) \\ &\quad + \gamma_i K_i \sum_{l=N+2}^{L} \sum_{j\in\mathcal{N}_i^l(t)} (C^l)^\top (\hat{y}_j^l - \hat{y}_i^l) + \gamma_i \alpha_i(t) K_i (H_i)^\top (w_i - H_i\hat{\rho}_i) \end{aligned} \tag{5.3.2a}$$

$$\hat{y}_i^l = C^l \hat{\rho}_i, \qquad \forall l = N+2, \ldots, L,\ i = 2, \ldots, N+1, \tag{5.3.2b}$$

Define $G^*(t) \in \mathbb{R}^{Nn\times Nn}$ and $L^*(t) \in \mathbb{R}^{Nn\times Nn}$ as

$$G^*(t) := \mathrm{diag}\left(\alpha_2(t)H_2^\top H_2, \ldots, \alpha_{N+1}(t)H_{N+1}^\top H_{N+1}\right)$$

and

$$L^*(t) := \sum_{l=N+2}^{L} \mathfrak{L}_{\mathrm{obs}}^l(t) \otimes (C^l)^\top C^l.$$

Similarly, define

$$G_z^*(t) := \mathrm{diag}\left(\alpha_2(t)Z^2(Z^2)^\top, \ldots, \alpha_{N+1}(t)Z^{N+1}(Z^{N+1})^\top\right)$$

and

$$L_z^*(t) := \sum_{l=N+2}^{L} \mathfrak{L}_{\mathrm{obs}}^l(t)^\top \mathfrak{L}_{\mathrm{obs}}^l(t) \otimes (Z^l)(Z^l)^\top.$$

Suppose for simplicity that graphs are piece-wise constant, with switching times denoted as $\{t_q\}_{q=1,2,\ldots}$. It is also supposed that the switching signals have dwell time as stated below.

Assumption 5.3.1. Signals $\alpha_i(t)$ and Laplacian matrices $\mathfrak{L}^l(t)$ are piece-wise constant with dwell time $\delta > 0$. ◊

For the analysis of the switched systems, we first introduce the following result from [SH12, Lem. 1].

Lemma 5.3.1. Given the sequence $\{t_q\}$ with $t_0 = 0$ and $t_{q+1} - t_q \geq \delta > 0$ for all $q \in \mathbb{N}_0$, suppose that $V : [0, \infty) \to \mathbb{R}$ satisfies:

1. $\lim_{t\to\infty} V(t)$ exists.
2. $V(t)$ is twice differentiable[2] on each interval $[t_q, t_{q+1})$.
3. There exists a positive constant K such that

$$\sup_{t_q \leq t < t_{q+1}, q \in \mathbb{N}_0} |\ddot{V}(t)| \leq K. \tag{5.3.3}$$

Then, $\lim_{t\to\infty} \dot{V}(t) = 0$. ◊

Using the previous lemma, extension of Theorem 5.2.1 to switching networks is shown below.

Theorem 5.3.2. Suppose that Assumptions 5.2.1, 5.2.2, 5.3.1 hold and consider the distributed observer (5.3.2) with $K_i = P_i^{-1}$. Then for any $\gamma > 0$, (5.3.2) solves the distributed estimation problem if and only if C^l are chosen such that there exists $\nu > 0$ and a subsequence $\{q_k\}$ of $\{q \mid q \in \mathbb{N}_0\}$ satisfying $t_{q_{k+1}} - t_{q_k} < \nu$ such that

$$\sum_{j=0}^{q_{k+1}-q_k-1} L_z^*(t_{q_k+j}) + G_z^*(t_{q_k+j}) > 0 \tag{5.3.4}$$

for all $k \in \mathbb{N}_0$. ◊

[2]With abuse of notation, we use upper Dini derivatives for $V(t)$ at points t_q. For more details, readers are referred to [SH12, Jia09].

Proof. Following the proof of Theorem 5.2.1, dynamics of the estimation error becomes

$$\dot{e} = \Big((I_N \otimes A) - \Gamma(L^*(t) + G^*(t))\Big)e, \tag{5.3.5}$$

where $\Gamma = \mathrm{diag}(\gamma_2 K_2, \ldots, \gamma_{N+1} K_{N+1})$. For the necessity, suppose that (5.3.4) does not hold. Then for any subsequence $\{q_k\}$ and $\nu > 0$ with $t_{q_{k+1}} - t_{q_k} < \nu$, it holds that

$$\sum_{j=0}^{q_{k+1}-q_k-1} L_z^*(t_{q_k+j}) + G_z^*(t_{q_k+j})$$

is positive semidefinite (and not positive definite) for all $k \in \mathbb{N}_0$. Then by choosing $\nu > \delta$ and $q_k = k$, it follows that

$$\ker(L_z^*(t_q) + G_z^*(t_q)) \neq \{0\}, \quad \forall q \in \mathbb{N}_0,$$

i.e., $L_z^*(t) + G_z^*(t)$ is not positive definite for all $t \geq 0$.

Now we claim there exists a nonzero vector e^* such that

$$e^* \in \ker\left(L_z^*(t) + G_z^*(t)\right), \quad \forall t \geq 0.$$

By contradiction, if for all $e \in \mathbb{R}^{Nn}$, there exists t^* such that

$$e \notin \ker(L_z^*(t^*) + G_z^*(t^*)),$$

which implies $L_z^*(t^*) + G_z^*(t^*)$ is positive definite. However, this contradicts since $L_z^*(t^*) + G_z^*(t^*)$ is not positive definite.

Now, it can be checked that $\ker\left(L_z^*(t) + G_z^*(t)\right)$ is $(I_N \otimes A)$–invariant for each time period. Thus, by investigating the system (5.3.5) with $e(0) = e^*$, it follows that $\lim_{t\to\infty} e(t) \not\to 0$ and the estimation is not achieved. Hence, (5.3.4) is a necessary condition to achieve the distributed state estimation.

For the sufficiency, we mostly follow the arguments of [SH12, Thm. 1]. Using the Lyapunov function $V(e)$ from Theorem 5.2.1, it can be verified that

$$\dot{V} \leq -2\gamma e^\top (L^*(t) + G^*(t))e \leq 0. \tag{5.3.6}$$

Hence, $V(t)$ is positive and bounded from above, which implies $\lim_{t\to\infty} V(t)$ exists. On the other hand, boundedness of $e(t)$ also follows from (5.3.6) and (5.3.5) implies that $\ddot{V}$ satisfies the last condition Lemma 5.3.1. Therefore, $\lim_{t\to\infty} \dot{V}(t) = 0$ by Lemma 5.3.1. Thus, (5.3.6) results in

$$\lim_{t\to\infty} e(t)^\top (L^*(t) + G^*(t))e(t) = 0,$$

which in turn implies

$$\lim_{t\to\infty} e(t)^\top L^*(t)e(t) = 0, \quad \lim_{t\to\infty} e(t)^\top G^*(t)e(t) = 0 \tag{5.3.7}$$

since $L^*(t)$ and $G^*(t)$ are positive semidefinite matrices.

Next, we claim (5.3.7) implies

$$\lim_{t\to\infty} L_z^*(t)e(t) = 0, \quad \lim_{t\to\infty} G_z^*(t)e(t) = 0.$$

For this, we apply Lemma 5.3.1 to $\eta(t) := L^*(t)e(t)$. First, since $L^*(t)$ is symmetric, it follows from (5.3.7) that $\lim_{t\to\infty} \eta(t)$ exists (and equal to 0). Twice differentiability of $\eta(t)$ on intervals $[t_q, t_{q+1})$ follows from definitions and Assumption 5.3.1. Finally, the last condition of Lemma 5.3.1, i.e., (5.3.3), holds since $L^*(t)$, $G^*(t)$ and $e(t)$ are all bounded. Thus, we obtain $\lim_{t\to\infty} \dot{\eta}(t) = 0$. Therefore, it follows from (5.3.5) and (5.3.7) that

$$\begin{aligned}
\lim_{t\to\infty} L^*(t)(I_N \otimes A)e(t) &= \lim_{t\to\infty} L^*(t)\Big(\dot{e}(t) + \Gamma(L^*(t) + G^*(t))e(t)\Big) \\
&= \lim_{t\to\infty} L^*(t)\dot{e}(t) + \lim_{t\to\infty} L^*(t) \cdot \Gamma\big(L^*(t)e(t) + G^*(t)e(t)\big) \\
&= \lim_{t\to\infty} L^*(t)\dot{\eta}(t) + \lim_{t\to\infty} L^*(t) \cdot \Gamma\big(L^*(t)e(t) + G^*(t)e(t)\big) \\
&= 0.
\end{aligned}$$

Now, suppose that $\lim_{t\to\infty} L^*(t)(I_N \otimes A)^k e(t) = 0$. Then, it follows from induction that $\lim_{t\to\infty} L^*(t)(I_N \otimes A)^{k+1}e(t) = 0$. Hence, using the fact that $\mathfrak{L}^l_{\text{obs}}$ is symmetric, it can be verified that

$$\lim_{t\to\infty} e(t)^\top L^*_z(t)e(t) = 0. \tag{5.3.8}$$

In addition, similar argument can be used to obtain $\lim_{t\to\infty} e(t)^\top G^*_z(t)e(t) = 0$.

Rest of the proof will now show $\lim_{t\to\infty} e(t) = 0$. First, recall that

$$e(t+T_0) = e^{(I_N\otimes A)T_0}e(t) + \Delta(t,T_0) \tag{5.3.9}$$

where $T_0 > 0$ is a constant and

$$\Delta(t,T_0) := \Gamma\int_t^{t+T_0} e^{(I_N\otimes A)(t+T_0-\tau)}(-L^*(\tau) - G^*(\tau))e(\tau)d\tau.$$

By letting $\bar{\Delta} := \max_{t\in[0,T_0]} |e^{(I_N\otimes A)t}| \cdot |\Gamma|$, it follows that

$$|\Delta(t,T_0)| \le \bar{\Delta}\int_t^{t+T_0} |(L^*(\tau) + G^*(\tau))e(\tau)|d\tau.$$

Thus, we obtain $\lim_{t\to\infty} \Delta(t,T_0) = 0$ by using (5.3.7). (Also see [SH12, Lem. 5]).)

Next, we claim that for each $j = 0,\dots,q_{k+1} - q_k - 1$,

$$\lim_{k\to\infty} e(t_{q_k})^\top L^*_z(t_{q_k+j})e(t_{q_k}) = 0. \tag{5.3.10}$$

Since it follows from (5.3.9) that $e(t) = e^{-(I_N\otimes A)T_0}(e(t+T_0)-\Delta(t,T_0))$, by letting $t = t_{q_k}$ and $T_0 = t_{q_k+j} - t_{q_k}$, we have

$$e(t_{q_k}) = e^{-(I_N\otimes A)T_0}\big(e(t_{q_k+j}) - \Delta(t_{q_k},T_0)\big)$$

for each $j = 0,\dots,q_{k+1} - q_k - 1$.

Hence, substituting expression of $e(t_{q_k})$ to (5.3.10), it follows that

$$\begin{aligned}&e(t_{q_k})^\top L^*_z(t_{q_k+j})e(t_{q_k})\\&= \Big(e^{-(I_N\otimes A)T_0}\big(e(t_{q_k+j}) - \Delta(t_{q_k},T_0)\big)\Big)^\top L^*_z(t_{q_k+j})\left(e^{-(I_N\otimes A)T_0}\big(e(t_{q_k+j}) - \Delta(t_{q_k},T_0)\big)\right)\end{aligned}$$

$$
\begin{aligned}
&= e(t_{q_k+j})^\top e^{-(I_N\otimes A^\top)T_0} L_z^*(t_{q_k+j}) e^{-(I_N\otimes A)T_0} e(t_{q_k+j}) \\
&\quad - 2\Delta(t_{q_k}, T_0)^\top L_z^*(t_{q_k+j}) e^{-(I_N\otimes A)T_0} e(t_{q_k+j}) + \Delta(t_{q_k}, T_0)^\top L_z^*(t_{q_k+j})\Delta(t_{q_k}, T_0) \\
&=: e(t_{q_k+j})^\top e^{-(I_N\otimes A^\top)T_0} L_z^*(t_{q_k+j}) e^{-(I_N\otimes A)T_0} e(t_{q_k+j}) + g(\Delta(t_{q_k}, T_0)),
\end{aligned}
$$

where $\lim_{k\to\infty} g(\Delta(t_{q_k}, T_0)) = 0$ for some function $g\colon \mathbb{R}^n \to \mathbb{R}$. Thus, proving (5.3.10) is equivalent to showing

$$
\lim_{k\to\infty} e(t_{q_k+j})^\top e^{-(I_N\otimes A^\top)T_0} L_z^*(t_{q_k+j}) e^{-(I_N\otimes A)T_0} e(t_{q_k+j}) = 0. \tag{5.3.11}
$$

To prove (5.3.11), first note that (5.3.8) implies

$$
\alpha_{ji}^l(t)(e_j(t) - e_i(t)) \to \ker (Z^l)^\top, \quad \forall (i,j) \in \mathcal{E}_{\text{obs}}^l, l = N+2, \dots, L.
$$

Since $\ker (Z^l)^\top$ is e^{-AT_0} invariant, it follows that

$$
\lim_{t\to\infty} (Z^l)^\top e^{-AT_0} \alpha_{ji}^l(t)(e_j(t) - e_i(t)) = 0, \quad \forall (i,j) \in \mathcal{E}_{\text{obs}}^l, l = N+2, \dots, L.
$$

By stacking, it can be verified that the above implies

$$
\begin{aligned}
&\lim_{t\to\infty} \left(\mathfrak{L}_{\text{obs}}^l(t) \otimes (Z^l)(Z^l)^\top\right)(I_N \otimes e^{-AT_0}) e(t) \\
&= \lim_{t\to\infty} \left(\mathfrak{L}_{\text{obs}}^l(t) \otimes (Z^l)(Z^l)^\top\right) e^{-(I_N\otimes A)T_0} e(t) \\
&= 0.
\end{aligned}
$$

Thus, (5.3.11) and equivalently (5.3.10) holds.

Now, by summing (5.3.10) for $j = 0, 1, \dots, q_{k+1} - q_k - 1$, it follows that

$$
\lim_{k\to\infty} e(t_{q_k})^\top \left(\sum_{j=0}^{q_{k+1}-q_k-1} L_z^*(t_{q_k+j}) \right) e(t_{q_k}) = 0. \tag{5.3.12}
$$

Using the similar arguments, we may also obtain (5.3.12) with $L_z^*(t_{q_k+j})$ replaced with $G_z^*(t_{q_k+j})$. Since $\sum_{j=0}^{q_{k+1}-q_k-1} L_z^*(t_{q_k+j}) + G_z^*(t_{q_k+j}) > 0$ due to assumption, we obtain

$$\lim_{k\to\infty} |e(t_{q_k})| = 0.$$

Since $|e(t_{q_k})|$ is non-increasing, it follows that $\lim_{t\to\infty} |e(t)| = 0$. □

As mentioned, Theorem 5.3.2 is extension of Theorem 5.2.2 to switching networks. Hence, similar design procedures can be obtained as in the static network. In particular, by letting $L = N+2$, we obtain following condition which extends results of Theorem 5.2.4.

Corollary 5.3.3. Suppose that (C^{N+2}, A) is detectable, $\mathcal{G}_{\rm obs}(t)$ is uniformly connected and

$$\left(\left[\bar{H}_2; \cdots ; \bar{H}_{N+1}\right], A\right) \tag{5.3.13}$$

is detectable, where $\bar{H}_i := \sum_{j=0}^{t_{q_{k+1}}-t_{q_k}-1} \alpha_i(t_{q_k+j}) H_i$ for all $k = 0, 1, \ldots$. Then (5.3.4) holds. ◊

Proof. Overall proof is similar to the proof of Theorem 5.2.4, hence only the difference is highlighted. By direct calculation, it holds that

$$\sum_{j=0}^{q_{k+1}-q_k-1} L_z^*(t_{q_k+j}) + G_z^*(t_{q_k+j}) = Q^\top Q, \tag{5.3.14}$$

where $Q := [Q_1; Q_2]$ is defined as

$$Q_1 := \begin{bmatrix} (\mathfrak{L}^{N+2})(t_{q_k}) \otimes (Z^{N+2})^\top \\ \vdots \\ (\mathfrak{L}^{N+2})(t_{q_{k+1}-1}) \otimes (Z^{N+2})^\top \end{bmatrix},$$

$$Q_2 := \begin{bmatrix} \operatorname{diag}\big(\alpha_2(t_{q_k}) Z_2^\top, \ldots, \alpha_{N+1}(t_{q_k}) Z_{N+1}^\top\big) \\ \vdots \\ \operatorname{diag}\big(\alpha_2(t_{q_{k+1}-1}) Z_2^\top, \ldots, \alpha_{N+1}(t_{q_{k+1}-1}) Z_{N+1}^\top\big) \end{bmatrix}.$$

From uniform connectedness of $\mathcal{G}(t)$ and detectability of (C^{N+2}, A), it can be verified that $\ker Q_1 = \mathcal{S}_n^N$. Consequently, it follows from (5.3.13) that $Q_2(1_N \otimes$

$\bar{x}) = 0$ for all $\bar{x} \in \mathbb{R}^n$. Therefore, $\ker(Q) = \ker(Q^\top Q) = \{0\}$, and hence $\sum_{j=0}^{q_{k+1}-q_k-1} L_z^*(t_{q_k+j}) + G_z^*(t_{q_k+j})$ is positive definite. Then, the result follows from Theorem 5.3.2. □

Similar to the result of Theorem (5.2.4), Corollary 5.3.3 proposes a simple, decomposed sufficient conditions for the convergence of the proposed distributed observer. It requires the detectability of the pair $\bar{H}, A)$ (which can be interpreted as the uniform detectability of $(\Lambda(t)H, A)$), uniform connectivity of the communication network among observers $\mathcal{G}_{\text{obs}}(t)$ and detectability of the pair (C^{N+1}, A). Additionally, similar arguments for reducing the communication load between observers via C^{N+2} also applies.

5.4 Simulation Results

Numerical simulation is done to illustrate the effectiveness of the proposed design. Consider a plant with 3 agents where the system and measurement matrices are given by

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{bmatrix}, \quad \begin{matrix} H_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} \\ H_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \\ H_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \\ H_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix}, \quad B = 0.$$

The switching signal $\sigma(t)$ for the communication network is given by[3]

$$\sigma(t) = \begin{cases} 1 & 0 \le \lfloor t \rfloor \bmod 6 < 2, \\ 2 & 2 \le \lfloor t \rfloor \bmod 6 < 4, \\ 3 & 4 \le \lfloor t \rfloor \bmod 6 < 6, \end{cases}$$

i.e., $\sigma(t)$ cycles through each mode every 2 seconds. Corresponding network is given by switching graph as depicted in Fig. 5.2. In particular, node 1 denotes

[3]We use $\lfloor \cdot \rfloor$ to denote floor function and mod to denote the remainder.

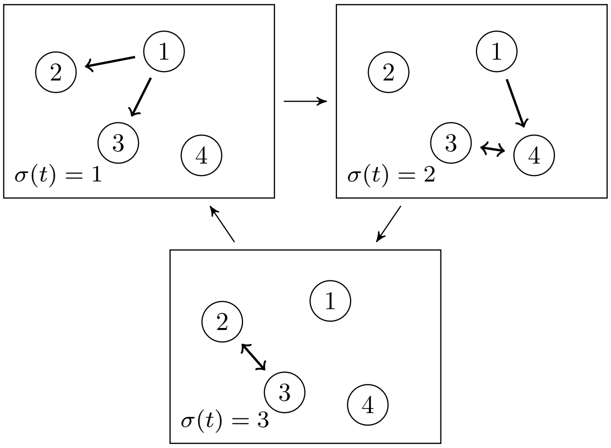


Figure 5.2: Graph structure used for the simulation. Each mode is switched every 2 seconds.

the plant, arrows from node 1 to node i denote the output measurement of node i and arrows between agents denote the communication between observers. For example, $\alpha_1(t) = \alpha_2(t) = 1$, $\alpha_3(t) = 0$ and $\mathfrak{L}_1 = 0$ for $t \in [0, 2)$. It can be checked that $s^* = 1$ and hence let $C^5 \in \mathbb{R}^{1\times 4}$ be

$$C^5 = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}$$

such that (C^5, A) is detectable. Then, it can be verified that the conditions of Corollary 5.3.3 hold.

The proposed observer (5.1.2) becomes

$$\dot{\hat{\rho}}_i = A\hat{\rho}_i + \alpha_i(t)H_i^\top(y_i - H_i\hat{\rho}_i) + (C^5)^\top \sum_{j\in\mathcal{N}_i(t)} (C^5\hat{\rho}_j - C^5\hat{\rho}_i),$$

where $P_i^{-1} = I_4$ and $\gamma_i = 1$ is used.

Simulation results are shown in Fig. 5.3, which verify the convergence of the proposed design under the switching network. Estimation errors are also plotted in Fig. 5.4. Notice that the proposed observers only communicate the scalar variable $C\hat{x}_i \in \mathbb{R}^1$ with its neighbors instead of the full state estimate. Nonetheless, state estimation is still achieved.

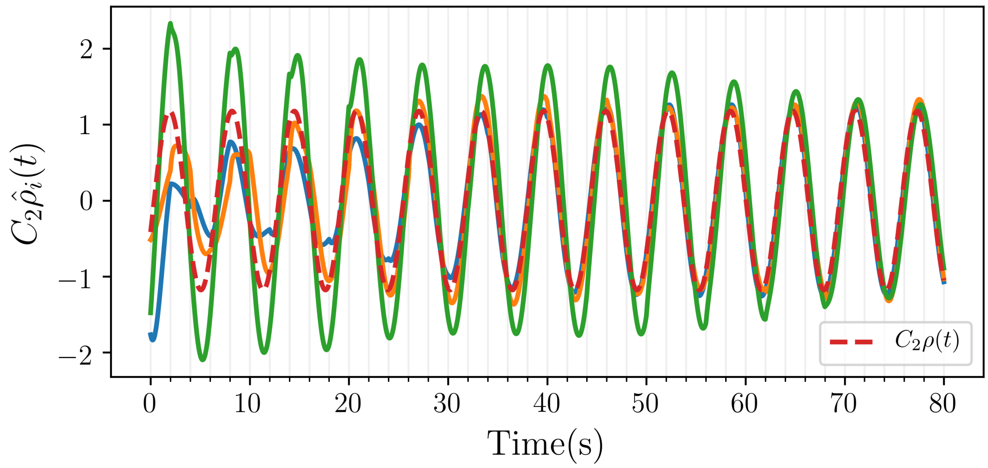


Figure 5.3: Simulation result for distributed state estimation. Dashed line denote the state of the plant and solid lines denote the estimate of each agent.

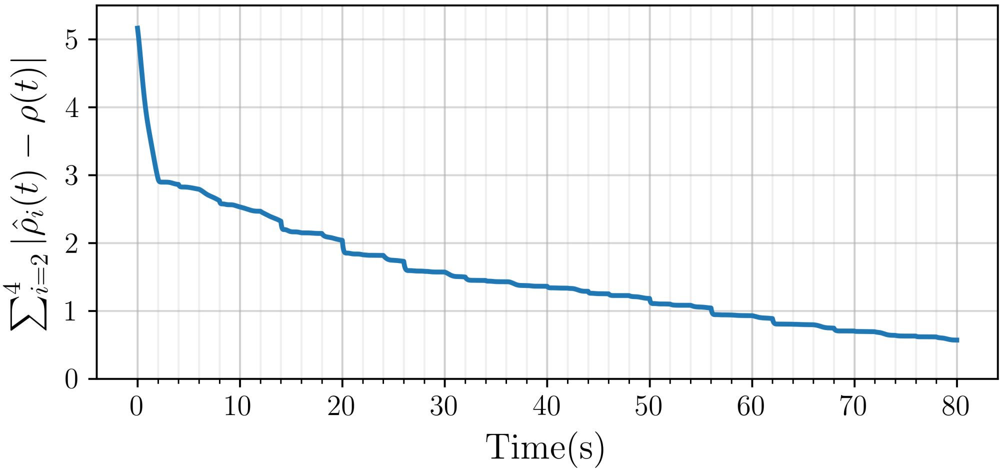


Figure 5.4: Plot of estimation error $\sum_{i=2}^{4} |\hat{\rho}_i(t) - \rho(t)|$.

Chapter 6

Application to the Formation Control Problem

In this section, the formation control problem is formulated into a consensus problem over a multilayer network. A dynamic controller is developed using the results from the previous chapter to achieve the desired formation.

6.1 Problem Formulation

Consider a network of N agents where $p_i \in \mathbb{R}^n$ and $v_i \in \mathbb{R}^n$ denote the position and velocity of each agent in $\mathbb{R}^n$, respectively. The dynamics of each agent is described by

$$\dot{p}_i = v_i, \quad \dot{v}_i = u_i, \quad i \in \mathcal{N}, \tag{6.1.1}$$

where $u_i \in \mathbb{R}^n$ is the control input. Define $x_i := [p_i; v_i] \in \mathbb{R}^{2n}$ to obtain

$$\dot{x}_i = \begin{bmatrix} 0_n & I_n \\ 0_n & 0_n \end{bmatrix} x_i + \begin{bmatrix} 0_n \\ I_n \end{bmatrix} u_i =: Ax_i + Bu_i.$$

The information flow among agents is given by a graph $G = (\mathcal{N}, \mathcal{E})$ and assume that each agent may measure the relative position to its neighbors, e.g., $p_j - p_i$ for all $j \in \mathcal{N}_i$. Also suppose that the measurement is bidirectional and that the two agents measuring the relative position may also communicate with each other.

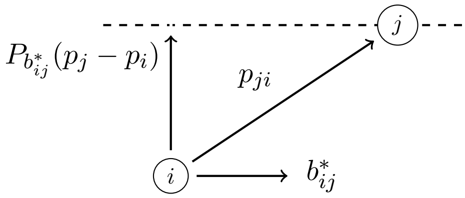


Figure 6.1: Illustration of relative position and $P_{b_{ij}^*}(p_j - p_i)$.

Objective of the formation control problem is to design a controller such that the agents form a desired formation.

In order to describe the desired formation, define the *bearing* as follows.

Definition 6.1.1. The *bearing* of agent j relative to agent i is defined as

$$b_{ji} := \frac{p_j - p_i}{|p_j - p_i|} \in \mathbb{R}^n.$$

Also let $P_{b_{ij}} := I_n - b_{ij}b_{ij}^\top \in \mathbb{R}^{n\times n}$ as the orthogonal projection matrix onto $\operatorname{im}(b_{ij})^\perp$. $\Diamond$

Note that it follows from the definitions that $b_{ij} = -b_{ji}$ and $P_{b_{ij}} = P_{b_{ji}}$. The bearing contains information about the direction (or angle) between two agents, but not the distance between two agents. In particular, two agents may be apart by different distances yet still have the same bearing. An illustration of relative position and $P_{b_{ij}^*}$ is shown in Fig. 6.1.

We consider the problem of controlling each agent to form a desired formation, where the desired formation is given by a combination of desired relative positions and desired bearings between a pair of agents. Specifically, the desired formation is given by a set of desired relative positions $\{p_{ij}^*\}_{(i,j)\in\mathcal{E}_p}$ and a set of desired bearings $\{b_{ij}^*\}_{(i,j)\in\mathcal{E}_b}$, where p_{ij}^*, b_{ij}^* are n-dimensional vectors and $\mathcal{E}_p, \mathcal{E}_b \subseteq \mathcal{E}$ are the edge sets specifying the pair of agents for each constraint. We say that the agents achieve the desired formation if the followings hold as $t \to \infty$:

1. $p_j(t) - p_i(t) \to p_{ji}^*$ for all $(i,j) \in \mathcal{E}_p$ and
2. $\dfrac{p_j(t) - p_i(t)}{|p_j(t) - p_i(t)|} \to b_{ji}^*$ for all $(i,j) \in \mathcal{E}_b$.

It is also our objective to achieve velocity consensus, i.e.,

$$\lim_{t\to\infty} v_j(t) - v_i(t) = 0, \quad \forall i, j \in \mathcal{N}.$$

For more details on the formation control problem, refer to works such as [ZZ17, OPA15, TNLA18] and references therein.

6.2 Formation Control Problem using Multilayer Network

The formation control problem with the desired formation described only by the relative position can be easily solved using the consensus protocols [OPA15]. Studies are also done recently to solve the formation control problem with bearing measurements only. However, the problem becomes more challenging if the desired formation involves both the relative position and the bearing. Nonetheless, the formation control problem with relative position and bearing constraints can be equivalently formulated into a consensus problem over a multilayer network and dynamic controllers are designed using the previous results.

In order to derive an equivalent consensus problem, the following assumption is made throughout the section.

Assumption 6.2.1. Given a desired formation $\{p^*_{ij}\}_{(i,j)\in\mathcal{E}_p}$ and $\{b^*_{ij}\}_{(i,j)\in\mathcal{E}_b}$, there exists p^*_i such that $p^*_j - p^*_i = p^*_{ji}$ for all $(i,j) \in \mathcal{E}_p$ and $(p^*_j - p^*_i)/|p^*_j - p^*_i| = b^*_{ji}$ for all $(i,j) \in \mathcal{E}_b$. $\diamondsuit$

Remark 6.2.1. Assumption 6.2.1 supposes that a valid formation exists which satisfies the desired relative position and bearing constraints. The existence and uniqueness of p^*_i is a fundamental question studied in the formation control problem and it is out of the scope of this dissertation. For more details, we refer to [OPA15] for a survey and [ZZ17] for bearing-constrained formation. $\diamondsuit$

Under Assumption 6.2.1, let $x^*_i := [p^*_i; 0] \in \mathbb{R}^{2n}$ and define the error as $e_i := x_i - x^*_i$. Then, achieving consensus of e_i implies $x_j - x_i = p^*_j - p^*_i$ and $v_i = v_j$ for all $i, j \in \mathcal{N}$. Therefore, it follows from Assumption 6.2.1 that the desired formation is achieved with the same velocity. Consequently, rest of the section focuses on designing a controller such that e_i achieves consensus.

For the controller design, recall that agent i is only aware of its local constraints. Then, we model relative position and bearing constraints using the multilayer graph $\mathcal{G}_f$ with $L := |\mathcal{E}_b| + 1$ layers. Construction of the multilayer network is as follows.

1. For the relative position constraints, let $\mathfrak{L}^1 \in \mathbb{R}^{N\times N}$ be the Laplacian matrix of the graph $G^1 = (\mathcal{N}, \mathcal{E}_p)$ and let $C^1 = [I_n \; 0_n] \in \mathbb{R}^{n\times 2n}$ be the corresponding output matrix. On layer 1, agents measure the relative position and will use this information for the local feedback.

2. For the l-th bearing constraint with an edge $(i, j) \in \mathcal{E}_b$, let $\mathfrak{L}^{l+1} \in \mathbb{R}^{N\times N}$ be the Laplacian graph of the graph $G^{l+1} = (\mathcal{N}, \{(i, j), (j, i)\})$ and let $C^{l+1} := [P_{b^*_{ij}} \; 0] \in \mathbb{R}^{n\times 2n}$ be the output matrix of $l + 1$-th layer. On layers 2 to L, agents compute the bearing error, e.g., $P_{b^*_{ji}}(p_j - p_i)$, and use this information for the control.

Remark 6.2.2. Construction of the multilayer graph used to represent constraints required $|\mathcal{E}_b| + 1$ layers. In particular, each bearing constraint is represented as a separate layer. However, if some bearing constraints are identical, e.g., $P_{b^*_{ij}} = P_{b^*_{i'j'}}$, then (i, j) and (i', j') can be represented in the same layer as two different edges. ♢

Finally, since $Ax^*_i = 0$, the dynamics of e_i can be written as

$$\begin{aligned} \dot{e}_i &= Ae_i + Bu_i, \\ y^l_i &= C^l e_i, \quad \forall i \in \mathcal{N}, l \in \mathcal{L}, \end{aligned} \tag{6.2.1}$$

with the multilayer graph $\mathcal{G}_f$. Then the following result follows directly from Theorem 3.2.1.

Proposition 6.2.1. Consider the system (6.2.1) with multilayer graph $\mathcal{G}_f$ and suppose that the necessary condition (3.2.2) holds. Then, a dynamic controller can be constructed such that e_i achieves consensus. Equivalently, (6.1.1) achieves the desired formation and velocity consensus achieved. ♢

Proof. Consensus of e_i follows directly from Theorem 3.2.1 and controller design in Section 3.4. Since consensus of e_i implies $x_j - x_i = p_j^* - p_i^*$ and $v_j - v_i$, the result follows. □

6.3 Simulation Results

In this section, we provide simulation results for the formation control problems. Section 6.3.1 presents a dynamic controller for static formation in 2-D. Results are also applied to the formation control problem in 3-D in Section 6.3.2 where we also show how the proposed method is capable of easily scaling the desired formation.

6.3.1 Achieving a Static Formation

For the numerical simulation, consider (6.1.1) with $N = 4$ and $n = 4$. The desired formation is given by $p_{12}^* = p_{34}^* = -p_{21}^* = -p_{43}^* := [0; -1]$, $b_{32}^* := [1; 0]$ and $b_{42}^* := [-1/\sqrt{2}; -1/\sqrt{2}]$, while $\mathcal{E}_p = \{(1,2),(2,1),(3,4),(4,3)\}$ and $\mathcal{E}_b = \{(2,3),(3,2),(2,4),(4,2)\}$. In particular, it can be checked that the square formation with length 1, which is shown in Fig. 6.2(a), is the only formation satisfying the constraints. Equivalent multilayer graph $\mathcal{G}_f$ with corresponding output matrices is shown in Fig. 6.2(b), where $C^l \in \mathbb{R}^{2\times 4}$ are

$$C^1 = \begin{bmatrix} I_2 & 0_2 \end{bmatrix}, C^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, C^3 = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 \end{bmatrix}$$

and the Laplacian matrices $\mathfrak{L}^l \in \mathbb{R}^{4\times 4}$ are given by

$$\mathfrak{L}^1 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \quad \mathfrak{L}^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathfrak{L}^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix},$$

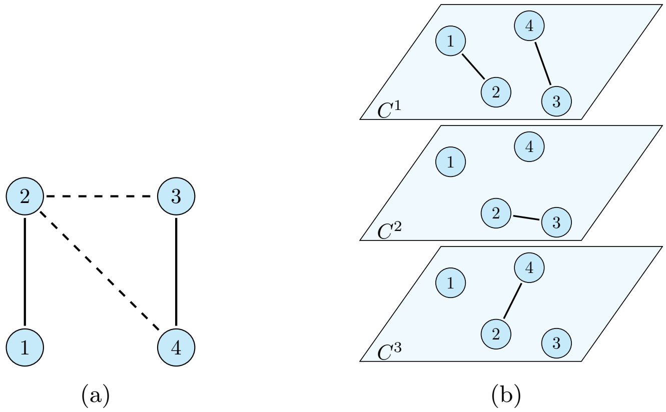


Figure 6.2: (a) Desired formation used for the simulation. Solid line denotes the relative position constraint and dashed line denotes the bearing constraint. (b) Equivalent multilayer network.

For the transformation matrices Z^l and W^l, we have used

$$Z^1 = I_4, \quad Z^2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad Z^3 = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} \end{bmatrix}$$

and

$$W^1 = \text{null}, \quad W^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad W^3 = \begin{bmatrix} -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & 0 \\ 0 & -\frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} \end{bmatrix}.$$

Hence, it follows that $\nu^1 = 0$, $\nu^2 = 3$ and $\nu^3 = 2$. Also, it can be checked that Z^l and W^l forms an orthonormal basis for each of the undetectable subspace.

For the controller, first let $\psi^1_{ji} := p^*_{ji}$ and $\psi^2_{ji} = \psi^3_{ji} = 0$. Then, following the result of Section 3.4, dynamic controller for each agent can be designed as

$$\dot{\xi}^l_i = A^l_d \xi^l_i + G^l \Big[\sum_{j \in \mathcal{N}^l_i} C^l_d(\xi^l_j - \xi^l_i) - (C^l_d(x_j - x_i) - \psi^l_{ji}) \Big] + (Z^l)^\top B u_i,$$

$$\dot{\hat{x}}_i = A\hat{x}_i + Bu_i + \gamma \sum_{l=1}^{L} \sum_{j\in\mathcal{N}_i^l} \Big[(Z^l)(Z^l)^\top(\hat{x}_j - \hat{x}_i) - (Z^l)(\xi_j^l - \xi_i^l)\Big],$$

$$u_i = BK\hat{x}_i.$$

Specifically, let $\tau = 0.5$ such that τ is less than nonzero eigenvalues of $\mathfrak{L}^l$ for all $l \in \mathcal{L}$. Then, $G^l = P^l(C^l)^\top$ where $P^l > 0$ is the solution of the ARE given by

$$(A^l)^\top P^l + P^l(A^l)^\top - \tau P^l(C^l)^\top(C^l)P^l = -I_{n-\nu^l}, \quad \forall l \in \mathcal{L}.$$

Numeric values for G^l can be computed as

$$G^1 = \begin{bmatrix} 1.9566 & 0 \\ 0 & 1.9566 \\ 1.0 & 0 \\ 0 & 1.0 \end{bmatrix}, \quad G^2 = \begin{bmatrix} 0 & 1.9566 \\ 0 & 1.0 \end{bmatrix}, \quad G^3 = \begin{bmatrix} 1.3836 & 1.3836 \\ 0.7071 & 0.7071 \end{bmatrix}.$$

The control gain K is designed as

$$K = -\begin{bmatrix} 1.0 & 0 & 1.7321 & 0 \\ 0 & 1.0 & 0 & 1.7321 \end{bmatrix}$$

such that $A+BK$ is Hurwitz and $\gamma = 5$ is used. Moreover, it can be verified that (3.2.2) holds.

Simulation results are shown in Fig. 6.3, which depicts the trajectories of agents. Red dots represent the position of each agent at time $t = 0, 2, 15, 20$. It can be seen that the desired formation is achieved under the proposed controller. Agents also achieve velocity consensus and travel in the same direction while maintaining the desired formation. Relative position error and bearing error are also plotted in Fig. 6.4.

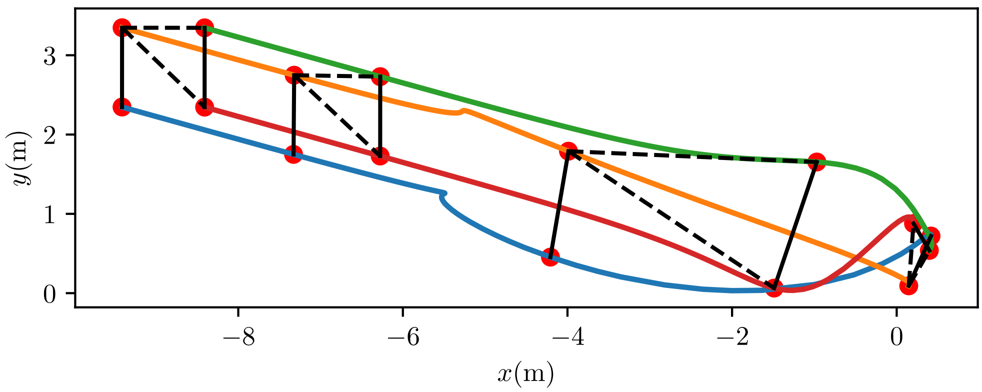


Figure 6.3: Trajectories of agents forming the desired formation.

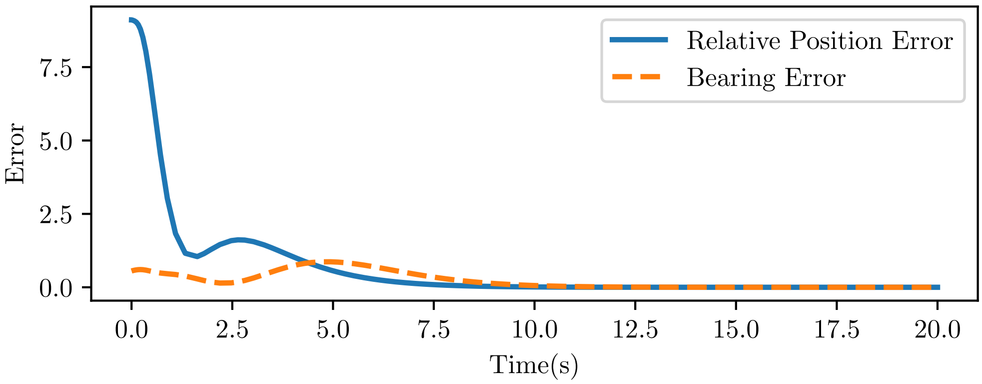


Figure 6.4: Plot of relative position error $\sum_{(i,j)\in\mathcal{E}_p} |(p_j - p_i) - p^*_{ji}|^2$ and bearing error $\sum_{(i,j)\in\mathcal{E}_b} |P_{b^*_{ij}}(p_i - p_j)|^2$.

6.3.2 Scaling Formation via Multilayer Network

In this section, we apply the proposed result to agents moving in 3-D with a single leader agent. The desired formation (given by both bearing and relative position constraints) is shown in Fig. 6.5. Note that the leader agent is denoted as agent 0 and its direct neighbors (e.g., agents 1 to 4) have a relative positional constraint. Agents 1 to 8 have the bearing constraints between them and the bearing constraints alone define a unique formation up to scaling and translation. Therefore, combined with the positional constraints, it can be verified that the desired formation indeed defines a unique formation (up to translation).

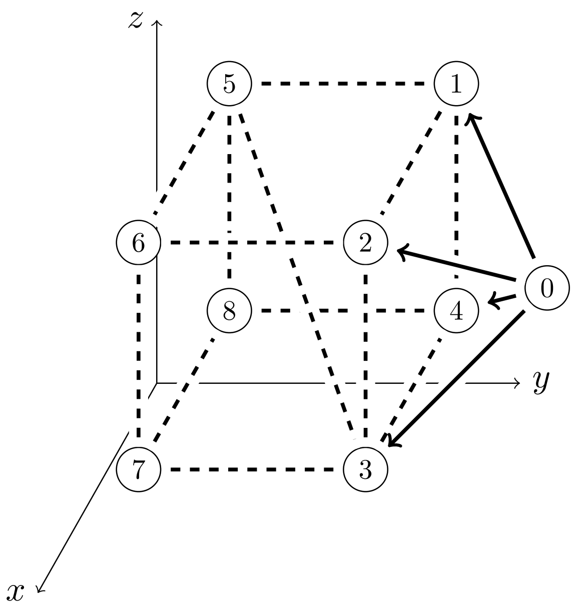


Figure 6.5: Desired formation and constraints for 3-D formation. Solid lines denote the relative position constraint and dashed lines denote the bearing constraints.

Corresponding multilayer network representing the desired formation can be constructed as described in the previous section. The multilayer graph is shown in Fig. 6.6 where C^l are given by

$$C^1 = \begin{bmatrix} I_3 & 0_3 \end{bmatrix}, \quad C^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad C^3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$
$$C^4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad C^5 = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \end{bmatrix}.$$

Specifically, C^1 represent the relative position constraint, C^2 is the bearing constraint in y-direction, C^3 is the bearing constraint in x-direction, C^4 is the bearing constraint in z-direction and finally C^5 is the bearing constraint between agent 3 and 5. The dynamic controller is designed similar to the previous section and hence details are not presented. In particular, (3.4.5) is employed and hence the control input is designed as

$$u = (\mathfrak{L}^{\mathrm{p}} \otimes K)\hat{x} \tag{6.3.1}$$

for the appropriate value of K.

For the simulation, we have chosen an initial condition such that the leader agent travels in y-direction. Moreover, scale of the desired formation changes at $t = 50s$ and $t = 100s$. In particular, scaling is done by agents 1 to 4 by scaling vectors p^*. In practice, the leader agent may send an appropriate signal to its immediate followers to scale the size of the relative position vector to stay closer to the leader. This simulates the scenario where agents must stay closer in order to avoid obstacles. As formation passes through the obstacle, it recovers the original formation.

Simulation results are shown in Fig. 6.7, which plots the trajectory of the agents. Red dots depict the position of each agent at a fixed time interval, the red line denotes the trajectory of the leader, and solid lines denote the trajectory of followers. Bearing constraints and position constraints are shown in black dotted and solid lines, respectively. First, notice that the leader agent travels straight in y-direction. This is consequence of the input (6.3.1), which achieves

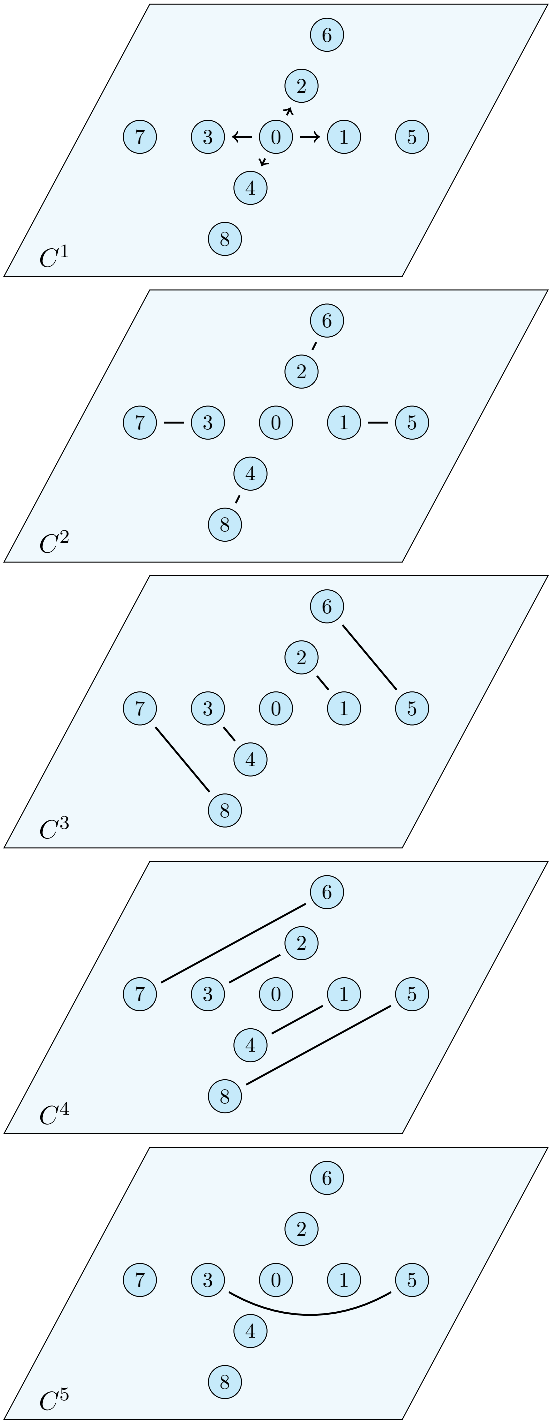


Figure 6.6: Corresponding multilayer graph for 3-D formation problem. Overall constraints can be represented using 5 layers.

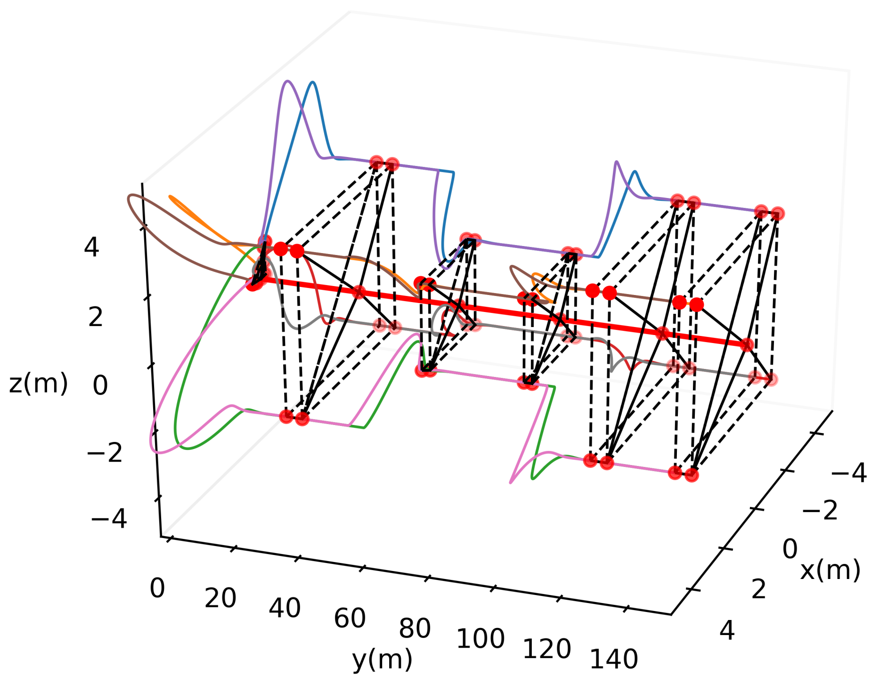


Figure 6.7: Simulation result for formation control in 3-D. Figure shows agents shrinks and expands scale of the formation.

(weighted) average consensus. (Specifically, we have a rooted spanning tree with the leader agent being a root.) Secondly, from the random initial condition, we see that agents achieve the desired formation around the leader agent. Finally, as the scale of the desired formation shrinks or expands, followers automatically converge to a new formation.

Remark 6.3.1. Compared to works such as [ZZ17], the proposed controllers only require relative position measurement, and information about the velocity is not required. Moreover, by using directed graphs and a combination of bearing with position constraints, scaling of the formation is easily done by a single leader agent. ◊

Chapter 7

Application to the Distributed Optimization Problem

This chapter studies the distributed optimization problem by formulating it into an output consensus problem of heterogeneous agents. Additionally, it is supposed that the communication structure between agents is given by a multilayer network. Specifically, a new type of algorithm is proposed to solve the distributed optimization problem in a more efficient manner.

7.1 Problem Formulation

We consider the distributed optimization problem of N agents given by

$$\min_{w\in\mathbb{R}^n} F(w) := \frac{1}{N}\sum_{i=1}^{N} f_i(w), \tag{7.1.1}$$

where $f_i\colon \mathbb{R}^n \to \mathbb{R}$ is differentiable and $F(w)$ is strongly convex. The main objective of the distributed optimization problem is to design an algorithm such that each agent finds the global minimizer of $F(w)$ only using its local cost function f_i and communication with their neighbors. Distributed optimization problem has received much attention due to various applications such as resource allocation problem including economic dispatch problem [LS19, YSA19], distributed state estimation [LS20a] or distributed machine learning.

In order to solve (7.1.1), consider the multi-agent system consisting of N nodes where dynamics of each agent is given by

$$\begin{aligned}\dot{x}_i &= h_i(x_i) + u_i \\ \zeta_i &= E x_i, \\ y_i^l &= C_{\mathrm{p}}^l \zeta_i = C_{\mathrm{p}}^l E x_i,\end{aligned} \tag{7.1.2}$$

where $x_i \in \mathbb{R}^n$ is state, $h_i : \mathbb{R}^n \to \mathbb{R}^n$ is vector field, $u_i \in \mathbb{R}^n$ is input and $\zeta_i \in \mathbb{R}^q$ is output of each agent to be synchronized. It is assumed that the output matrix $E \in \mathbb{R}^{q \times n}$ has full row rank.

The communication network among agent is given by a multilayer graph $\mathcal{G}_{\mathrm{p}}$, and the output of i-th agent on l-th layer is given by

$$y_i^l = C_{\mathrm{p}}^l \zeta_i = C_{\mathrm{p}}^l E x_i,$$

where $y_i^l \in \mathbb{R}^{n^l}$ denotes the partial information of the output ζ_i communicated to its neighbors. Specifically, y_i^l denote the information of agent i on layer l, where $C_{\mathrm{p}}^l \in \mathbb{R}^{n^l \times q}$ is the corresponding output matrix[1]. We suppose that the communication structure of y_i^l is given by an undirected multilayer graph $\mathcal{G}_{\mathrm{p}}$ with L_{p} layers. By defining $\mathcal{L}_{\mathrm{p}} := \{1, \ldots, L_{\mathrm{p}}\}$, the multilayer graph is defined as $\mathcal{G}_{\mathrm{p}} := (\mathcal{N}, \{\mathcal{E}_{\mathrm{p}}^l\}_{l \in \mathcal{L}_{\mathrm{p}}})$. Specifically, agent i communicates y_i^l to its neighbors on l-th layer $\mathcal{N}_i^l$.

The main objective of the problem studied in this chapter is to design the dynamics (7.1.2) (e.g., h_i, E and C_{p}^l) and input $u_i(t)$ (only using $y_j^l - y_i^l$) such that the output of (7.1.2) converges to the optimal solution of (7.1.1), i.e.,

$$\lim_{t \to \infty} \zeta_i(t) = w^*, \quad \forall i \in \mathcal{N}, \tag{7.1.3}$$

where $w^* := \operatorname{argmin}_w F(w)$. Note that output consensus is also achieved since $\zeta_i(t)$ converges to the same value.

[1] In this chapter, subscript p is used to denote that a variable is related to proportional feedback (and not related to the projection graph as in the previous chapters).

7.2 Distributed PI Algorithm

7.2.1 Distributed PI Algorithm under Static Network

In this section, we propose a PI based algorithm which achieves asymptotic output consensus for system (7.1.2) under static network. For the control input, first let $\xi_i \in \mathbb{R}^q$ as the integral state with its dynamics given by

$$\dot{\xi}_i = -\gamma K \sum_{l=1}^{L_\mathrm{p}} (C_\mathrm{p}^l)^\top \Bigg\{ \sum_{j \in \mathcal{N}_{i,\mathrm{p}}^l} \Big(y_j^l - y_i^l \Big) \Bigg\}, \tag{7.2.1}$$

where $\mathcal{N}_{i,\mathrm{p}}^l := \{j \in \mathcal{N} \mid (j,i) \in \mathcal{E}_\mathrm{p}^l\}$ is the neighbors of agent i on layer l of the multilayer graph $\mathcal{G}_\mathrm{p}$ with L_p layers. For the control input, ξ_i is exchanged with neighboring agents over the communication network given by a multilayer graph $\mathcal{G}_\mathrm{I}$ with L_I layers (which may be different from $\mathcal{G}_\mathrm{p}$). Then, the control input of each agent is designed as

$$u_i = k_\mathrm{p} E^\top \Bigg[\sum_{l=1}^{L_\mathrm{p}} (C_\mathrm{p}^l)^\top \Bigg\{ \sum_{j \in \mathcal{N}_{i,\mathrm{p}}^l} \Big(y_j^l - y_i^l \Big) \Bigg\} \Bigg] + k_\mathrm{I} E^\top \Bigg[\sum_{l=1}^{L_\mathrm{I}} (C_\mathrm{I}^l)^\top \Bigg\{ \sum_{j \in \mathcal{N}_{i,\mathrm{I}}^l} C_\mathrm{I}^l \left(\xi_j - \xi_i \right) \Bigg\} \Bigg], \tag{7.2.2}$$

where $\mathcal{N}_{i,\mathrm{I}}^l$ is defined similarly as $\mathcal{N}_{i,\mathrm{p}}^l$. The matrix $K \in \mathbb{R}^{q \times q}$ in (7.2.1) and the constants $k_\mathrm{p}, k_\mathrm{I}, \gamma$ are the design parameters of the control input. Meanwhile, we also consider the control input in from of

$$\begin{aligned} u_i = k_\mathrm{p} E^\top \Bigg[\sum_{l=1}^{L_\mathrm{p}} (C_\mathrm{p}^l)^\top \Bigg\{ \sum_{j \in \mathcal{N}_{i,\mathrm{p}}^l} \Big(y_j^l - y_i^l \Big) \Bigg\} \Bigg] - k_\mathrm{I} E^\top \sum_{l=1}^{L_\mathrm{I}} (C_\mathrm{I}^l)^\top C_\mathrm{I}^l \xi_i \\ \text{with } \sum_{i=1}^{N} \xi_i(0) = 0. \end{aligned} \tag{7.2.3}$$

Note from (7.2.1) that ξ_i can be regarded as the integral of the relative output error $y_j^l - y_i^l$. Therefore, both the input given by (7.2.2) and (7.2.3) are in form of the proportional-integral (PI) controller. Observe that the control input (7.2.2) communicate both y_i^l and ξ_i while (7.2.3) only requires communication of y_i^l.

However, the control input (7.2.3) requires initial conditions of the integrator must be such that $\sum_{i=1}^{N}\xi_i(0) = 0$. In this sense (7.2.1) and (7.2.2) are called *initialization-free* and suitable in cases where agents may join or leave the system during the operation.

Define $x \in \mathbb{R}^{Nn}$, $\zeta \in \mathbb{R}^{Nq}$ and $\xi \in \mathbb{R}^{Nq}$ as the stack of x_i, ζ_i and ξ_i respectively. Then the dynamics (7.1.2) with controller (7.2.1) and (7.2.2) can be written as

$$\dot{x} = h(x) - k_{\mathrm{p}}(I_N \otimes E)^\top \mathfrak{L}_{\mathrm{p}}(I_N \otimes E)x - k_{\mathrm{I}}(I_N \otimes E)^\top \mathfrak{D}_{\mathrm{I}}\xi \tag{7.2.4a}$$

$$\dot{\xi} = \gamma(I_N \otimes K)\mathfrak{L}_{\mathrm{p}}(I_N \otimes E)x, \tag{7.2.4b}$$

where $h(x) := [h_1(x_1); \cdots ; h_N(x_N)] \in \mathbb{R}^{Nn}$. The matrices $\mathfrak{L}_{\mathrm{p}}, \mathfrak{D}_{\mathrm{I}} \in \mathbb{R}^{Nq\times Nq}$ are defined as

$$\mathfrak{L}_{\mathrm{p}} := \sum_{l=1}^{L_{\mathrm{p}}} \mathfrak{L}_{\mathrm{p}}^l \otimes (C_{\mathrm{p}}^l)^\top C_{\mathrm{p}}^l$$

$$\mathfrak{D}_{\mathrm{I}} := \begin{cases} \sum_{l=1}^{L_{\mathrm{I}}} \mathfrak{L}_{\mathrm{I}}^l \otimes (C_{\mathrm{I}}^l)^\top C_{\mathrm{I}}^l & \text{if (7.2.2) is used} \\ \sum_{l=1}^{L_{\mathrm{I}}} I_N \otimes (C_{\mathrm{I}}^l)^\top C_{\mathrm{I}}^l & \text{if (7.2.3) is used} \end{cases}$$

where $\mathfrak{L}_{\mathrm{p}}^l \in \mathbb{R}^{N\times N}$ and $\mathfrak{L}_{\mathrm{I}}^l \in \mathbb{R}^{N\times N}$ are the Laplacian matrices of $\mathcal{G}_{\mathrm{p}}$ and $\mathcal{G}_{\mathrm{I}}$ respectively.

We make the following assumption which is related to the structure of the multilayer graph.

Assumption 7.2.1. For $\mathfrak{L}_{\mathrm{p}}$ and $\mathfrak{D}_{\mathrm{I}}$, it holds that

1. $\lambda_{q+1}(\mathfrak{L}_{\mathrm{p}}) > 0$.

2. $\lambda_{q+1}(\mathfrak{D}_{\mathrm{I}}) > 0$ if (7.2.2) is used and $\mathfrak{D}_{\mathrm{I}} > 0$ if (7.2.3) is used. ◊

Recall that for an undirected graph (i.e., a multilayer graph with $L = 1$) with $\mathfrak{L}$ as the Laplacian matrix, the connectivity is equivalent to $\lambda_2(\mathfrak{L}) > 0$. In a similar fashion, the Assumption 7.2.1 can be regarded as a generalization of this statement to the multilayer graphs. These conditions require that the output information to be *well-connected in some sense*, such that the consensus can be achieved. Notice that it is similar to the algebraic conditions studied in Chapter 3.

Remark 7.2.1. The proposed controller (7.2.4) includes the classical PI algorithms studied in the literature [HCIL18, LCH19, KCM15, WE10]. Specifically, if $K = C^l = E = I_n$, $L_{\rm p} = L_{\rm I} = 1$, $\mathcal{G}_{\rm p} = \mathcal{G}_{\rm I}$ and (7.2.2) is used, then the overall system (7.2.4) becomes

$$\begin{aligned} \dot{x} &= h(x) - k_{\rm p}(\mathfrak{L} \otimes I_n)x - k_{\rm I}(\mathfrak{L} \otimes I_n)\xi \\ \dot{\xi} &= \gamma(\mathfrak{L} \otimes I_n)x, \end{aligned} \tag{7.2.5}$$

where $h(x) := [\nabla f_1(x_1); \cdots ; \nabla f_N(x_N)]$ and $\mathfrak{L} \in \mathbb{R}^{N\times N}$ denotes the Laplacian matrix of the communication network. Then the (7.2.5) is exactly the continuous-time distributed PI algorithm studied in the literature. ♢

Remark 7.2.2. The dynamics (7.2.5) with (7.2.2) also admit an another interpretation from the optimization theory. Note that the optimization problem (7.1.1) can be written equivalently as

$$\begin{aligned} \min_{w_i \in \mathbb{R}^n} F(w_1, \ldots, w_N) &:= \frac{1}{N}\sum_{i=1}^{N} f_i(w_i), \\ (\mathfrak{L} \otimes I_n)\begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix} &= 0. \end{aligned}$$

Then the augmented Lagrangian of the optimization problem can be defined as

$$\mathcal{L}(w_1, \ldots, w_N, \xi) = \frac{1}{N}\sum_{i=1}^{N} f_i(w_i) + k\xi^\top \mathfrak{L}\bar{w} + k\bar{w}^\top \mathfrak{L}\bar{w},$$

where ξ is the dual variable and $\bar{w} := [w_1; \cdots ; w_N]$. Then the primal-dual algorithm [CGC17] is given by

$$\begin{aligned} \dot{\bar{w}} &= -\nabla_{\bar{w}}\mathfrak{L}(\bar{w}, \xi) \\ &= -\nabla F(\bar{w}) - (\mathfrak{L} \otimes I_n)\bar{w} - (\mathfrak{L} \otimes I_n)\xi \end{aligned}$$

$$\dot{\xi} = (\mathfrak{L} \otimes I_n)\bar{w}.$$

It can be checked that the primal-dual algorithm is identical to (7.2.5). However, such interpretation is no longer valid when only output is communicated or h_i is more complex. ◊

Remark 7.2.3. The main objective of achieving $\zeta_i(t) \to w^*$ for all $i \in \mathcal{N}$ can be seen as output consensus of nonlinear, heterogeneous agents. A necessary condition for consensus of a nonlinear heterogeneous agent is studied in [WWA13], which states that a common internal model must be present. In this regard, it can be seen that adding an integrator (7.2.1) introduces a common internal model such that asymptotic consensus to a constant value can be achieved despite heterogeneity. ◊

7.2.2 State Transformation for Analysis

In this section, a series of state transformations motivated by [LS20b] are introduced which is used for the analysis of the proposed algorithm. First, apply the transformation

$$z_i = Z^\top x_i \in \mathbb{R}^{n-q}, \quad w_i = W^\top x_i \in \mathbb{R}^q, \tag{7.2.6}$$

where $T := [Z\ W] \in \mathbb{R}^{n\times n}$ is an orthonormal matrix such that $\mathrm{im}\,(Z) = \ker(E)$ and $\mathrm{im}\,(W) = \ker(E)^\perp = \mathrm{im}\,(E^\top)$. Hence, it holds that $EZ = 0$ and $x_i = Zz_i + Ww_i$. Then by defining $z := [z_1; \cdots; z_N] \in \mathbb{R}^{N(n-q)}$ and $w := [w_1; \cdots; w_N] \in \mathbb{R}^{Nq}$, the overall dynamics (7.2.4) becomes

$$\begin{aligned}
\dot{z} &= (I_N \otimes Z)^\top h(x) \\
\dot{w} &= (I_N \otimes W)^\top h(x) - k_{\mathrm{p}}(I_N \otimes EW)^\top \mathfrak{L}_{\mathrm{p}}(I_N \otimes EW)w - k_{\mathrm{I}}(I_N \otimes EW)^\top \mathfrak{D}_{\mathrm{I}}\xi \\
\dot{\xi} &= \gamma(I_N \otimes K)\mathfrak{L}_{\mathrm{p}}(I_N \otimes EW)w.
\end{aligned} \tag{7.2.7}$$

Next, let $U \in \mathbb{R}^{N\times N}$ be the matrix for some connected Laplacian matrix given by Theorem 2.1.4 and let w, ξ be the stack of w_i and ξ_i. Now, apply the state transformation to w and ξ as

$$\begin{bmatrix} \bar{w} \\ \tilde{w} \end{bmatrix} = (U \otimes I_q)w, \quad \begin{bmatrix} \bar{\xi} \\ \tilde{\xi} \end{bmatrix} = (U \otimes I_q)\xi, \tag{7.2.8}$$

where $\bar{w}, \bar{\xi} \in \mathbb{R}^q$ and $\tilde{w}, \tilde{\xi} \in \mathbb{R}^{(N-1)q}$. Specifically, we have $w = (1_N \otimes I_q)\bar{w} + (R \otimes I_q)\tilde{w}$, $\xi = (1_N \otimes I_q)\bar{\xi} + (R \otimes I_q)\tilde{\xi}$, and hence $x = Z_\otimes z + (1_N \otimes W)\bar{w} + (R \otimes W)\tilde{w}$. In addition, define $\Lambda_\mathrm{p}, \Lambda_\mathrm{I} \in \mathbb{R}^{(N-1)q \times (N-1)q}$ as

$$\Lambda_\mathrm{p} := (R \otimes I_q)^\top \mathfrak{L}_\mathrm{p}(R \otimes I_q) \tag{7.2.9}$$

$$\Lambda_\mathrm{I} := (R \otimes I_q)^\top \mathfrak{D}_\mathrm{I}(R \otimes I_q). \tag{7.2.10}$$

Using Assumption 7.2.1 and the definitions of R, it can be easily verified that Λ_p and Λ_I are symmetric and positive definite.

Using the transformation (7.2.8), dynamics of $\bar{\xi}$ and $\tilde{\xi}$ transform into

$$\begin{aligned} \dot{\bar{\xi}} &= 0 \\ \dot{\tilde{\xi}} &= \gamma(R^\top \otimes K)\mathfrak{L}_\mathrm{p}(R \otimes EW)\tilde{w}. \end{aligned}$$

Meanwhile, the dynamics of $\bar{w}$ and $\tilde{w}$ becomes

$$\begin{aligned} \dot{\bar{w}} &= \frac{1}{N}(1_N \otimes W)^\top h(x) \\ \dot{\tilde{w}} &= (R \otimes W)^\top h(x) - k_\mathrm{p}(R \otimes EW)^\top \mathfrak{L}_\mathrm{p}(R \otimes EW)\tilde{w} \\ &\quad - k_\mathrm{I}(R \otimes EW)^\top \mathfrak{D}_\mathrm{I}(R \otimes I_q)\tilde{\xi} \\ &= (R \otimes W)^\top h(x) - k_\mathrm{p}(I_{N-1} \otimes EW)^\top \Lambda_\mathrm{p}(I_{N-1} \otimes EW)\tilde{w} - k_\mathrm{I}(I_{N-1} \otimes EW)^\top \Lambda_\mathrm{I}\tilde{\xi}, \end{aligned}$$

where we used $(1_N \otimes I_q)^\top \mathfrak{L}_\mathrm{p} = 0$, $(1_N \otimes I_q)^\top \mathfrak{D}_\mathrm{I}\bar{\xi}(t) = 0$ (regardless of whether (7.2.2) or (7.2.3) is used), $\mathfrak{L}_\mathrm{p}(1_N \otimes I_q) = 0$ and $(R^\top \otimes I_q)\mathfrak{D}_\mathrm{I}(1_N \otimes I_q) = 0$.

Finally, it follows that the system (7.2.4) becomes

$$\dot{z} = (I_N \otimes Z)^\top h(x) \tag{7.2.11a}$$

$$\dot{\bar{w}} = \frac{1}{N}(1_N \otimes W)^\top h(x) \tag{7.2.11b}$$

$$\dot{\tilde{w}} = (R \otimes W)^\top h(x) - k_{\mathrm{p}}(I_{N-1} \otimes EW)^\top \Lambda_{\mathrm{p}}(I_{N-1} \otimes EW)\tilde{w} \\ - k_{\mathrm{I}}(I_{N-1} \otimes EW)^\top \Lambda_{\mathrm{I}}\tilde{\xi} \tag{7.2.11c}$$

$$\dot{\bar{\xi}} = 0 \tag{7.2.11d}$$

$$\dot{\tilde{\xi}} = \gamma(R^\top \otimes K)\mathfrak{L}_{\mathrm{p}}(R \otimes EW)\tilde{w}. \tag{7.2.11e}$$

Note that (7.2.11d) shows $\bar{\xi}$ (i.e., the average of ξ_i) is constant and hence the trajectory of $\bar{\xi}$ is determined by the initial conditions of $\xi_i(0)$.

Now, we investigate the equilibrium point of the system (7.2.11). For this, we make the following assumption.

Assumption 7.2.2. The dynamical system

$$\dot{z} = (I_N \otimes Z)^\top h\Big((I_N \otimes Z)z + (1_N \otimes W)\bar{w}\Big) \\ \dot{\bar{w}} = \frac{1}{N}(1_N \otimes W)^\top h\Big((I_N \otimes Z)z + (1_N \otimes W)\bar{w}\Big), \tag{7.2.12}$$

which is exactly the system (7.2.11a) and (7.2.11b) with $\tilde{w}(t) \equiv 0$, has a unique equilibrium point $(z^*, \bar{w}^*)$ that is globally exponentially stable with the rate $\mu > 0$. That is, there exists $c > 0$ such that

$$\left\| \begin{bmatrix} z(t) - z^* \\ \bar{w}(t) - w^* \end{bmatrix} \right\| \leq \bar{c}e^{-\mu t} \left\| \begin{bmatrix} z(0) - z^* \\ \bar{w}(0) - w^* \end{bmatrix} \right\|. \qquad \Diamond$$

Assumption 7.2.2 only supposes existence of the equilibrium point of the system (7.2.12). However, this implies that the overall system (7.2.11) also has an equilibrium point which is stated below.

Lemma 7.2.1. Suppose that Assumptions 7.2.1 and 7.2.2 hold. Then, the overall system (7.2.11) has an equilibrium point given by

$$\begin{bmatrix} z^* \\ \bar{w}^* \\ \tilde{w}^* \\ \bar{\xi}^* \\ \tilde{\xi}^* \end{bmatrix} = \begin{bmatrix} z^* \\ \bar{w}^* \\ 0 \\ \frac{1}{N}\sum_{i=1}^N \xi_i(0) \\ \tilde{\xi}^* \end{bmatrix}.$$

Proof. First, we claim that an equilibrium point of (7.2.11) takes the form of

$$\left(z^*, \bar{w}^*, 0, \frac{1}{N}\sum_{i=1}^{N} \xi_i(0), \tilde{\xi}^*\right), \tag{7.2.13}$$

where z^* and $\bar{w}^*$ are from Assumption 7.2.2, $\xi_i(0)$ is the initial condition of ξ_i and $\tilde{\xi}^*$ is to be determined. From the definitions, it can be verified that $\dot{z} = 0$, $\dot{\bar{w}} = 0$, $\dot{\bar{\xi}} = 0$ and $\dot{\tilde{\xi}} = 0$ at the proposed equilibrium point (7.2.13). Hence, it is left to show there exists $\tilde{\xi}^*$ satisfying

$$(R \otimes W)^\top h(x^*) - k_\mathrm{I}(I_{N-1} \otimes EW)^\top \Lambda_\mathrm{I} \tilde{\xi}^* = 0 \tag{7.2.14}$$

so that $\dot{\tilde{w}} = 0$.

Since Λ_I is positive definite and EW is invertible, $\tilde{\xi}^*$ can be written as

$$\tilde{\xi}^* := \frac{1}{k_\mathrm{I}} \left((I_{N-1} \otimes EW)^\top \Lambda_\mathrm{I}\right)^{-1} (R \otimes W)^\top h(x^*),$$

which completes the proof. □

Since Lemma 7.2.1 states the existence of the equilibrium point for the overall system (7.2.11), apply the final transformation given by $\delta z = z - z^*$, $\delta \bar{w} = \bar{w} - \bar{w}^*$, and $\delta\tilde{\xi} = \tilde{\xi} - \tilde{\xi}^*$. The state x is written as

$$x = (I_N \otimes Z)(\delta z + z^*) + (1_N \otimes W)(\delta \bar{w} + \bar{w}^*) + (R \otimes W)\tilde{w}$$

and define $x^* := [x_1^*; \cdots ; x_N^*] \in \mathbb{R}^{Nn}$ as

$$x^* := (I_N \otimes Z)z^* + (1_N \otimes W)\bar{w}^*. \tag{7.2.15}$$

Then (7.2.11) becomes

$$\dot{\delta z} = (I_N \otimes Z)^\top h(x) \tag{7.2.16a}$$

$$\dot{\delta \bar{w}} = \frac{1}{N}(1_N \otimes W)^\top h(x) \tag{7.2.16b}$$

$$\dot{\tilde{w}} = (R \otimes W)^\top (h(x) - h(x^*)) - k_\mathrm{p}(I_{N-1} \otimes EW)^\top \Lambda_\mathrm{p} (I_{N-1} \otimes EW)\tilde{w} \\ - k_\mathrm{I}(I_{N-1} \otimes EW)^\top \Lambda_\mathrm{I} \delta\tilde{\xi} \tag{7.2.16c}$$

$$\dot{\bar{\xi}} = 0 \tag{7.2.16d}$$

$$\delta\dot{\tilde{\xi}} = \gamma(R^\top \otimes K)\mathfrak{L}_\mathrm{p}(R \otimes EW)\tilde{w}, \tag{7.2.16e}$$

where we used the fact that

$$0 = (R \otimes W)^\top h(x^*) - k_\mathrm{I}(I_{N-1} \otimes EW)^\top \Lambda_\mathrm{I} \tilde{\xi}^*.$$

For notational purpose, also let $e := [\delta z; \delta\bar{w}; \tilde{w}; \delta\tilde{\xi}] \in \mathbb{R}^{Nn-q}$.

7.3 Convergence Analysis for the PI Algorithm

In this section, convergence of the system (7.2.4) to the equilibrium point given by Lemma 7.2.1 is shown. For the proof, additional assumptions are made either on the dynamics of each agent or on the stability property of the equilibrium point.

7.3.1 Convergence with Weak Coupling

For the completeness of the results, we first consider the case when each h_i satisfies following assumption.

Assumption 7.3.1. Function h_i is locally Lipschitz and satisfies

$$(x - y)^\top (h_i(x) - h_i(y)) < 0$$

for all $x \neq y$. ◊

For instance, Assumption 7.3.1 holds if $h_i(x_i) = -\nabla f_i(x_i)$ where $f_i(x_i)$ is a strictly convex function.

Theorem 7.3.1. Suppose that Assumptions 7.2.1, 7.2.2 and 7.3.1 hold. In addition, suppose that $\mathfrak{L}_\mathrm{p} = \mathfrak{D}_\mathrm{I}$ and let $K = I_q$. Then for any $k_\mathrm{I} > 0$, $\gamma > 0$ and

$k_\mathrm{p} \geq 0$,

$$\lim_{t\to\infty} x_i(t) = x_i^*, \quad \forall i \in \mathcal{N},$$

where x_i^* is an equilibrium point defined in (7.2.15). Moreover, it follows that the output consensus is achieved, i.e.,

$$\lim_{t\to\infty} |\zeta_i(t) - \zeta_j(t)| = 0, \quad \forall i, j \in \mathcal{N}.$$

Proof. From the state transformations, it is sufficient to show the stability of (7.2.16). Define $V(\delta z, \delta\bar{w}, \tilde{w}, \delta\tilde{\xi})$ as

$$V = \frac{1}{2N}\delta z^\top \delta z + \frac{1}{2}\delta\bar{w}^\top \delta\bar{w} + \frac{1}{2}\tilde{w}^\top \tilde{w} + \frac{\sigma}{2}\delta\tilde{\xi}^\top \delta\tilde{\xi}.$$

be the Lyapunov function for (7.2.16) where $\sigma := k_\mathrm{I}/\gamma > 0$. Then the time-derivative of V along (7.2.16) becomes

$$\begin{aligned}\dot{V} \leq{}& \frac{1}{N}\delta z^\top (I_N \otimes Z)^\top h(x) + \frac{1}{N}\delta\bar{w}^\top (1_N \otimes W)^\top h(x)\\ &+ \tilde{w}^\top (R \otimes W)^\top (h(x) - h(x^*)) - k_\mathrm{p}\tilde{w}^\top (I_{N-1} \otimes EW)^\top \Lambda_\mathrm{p}(I_{N-1} \otimes EW)\tilde{w},\end{aligned}$$

where we used the fact that

$$k_\mathrm{I}\tilde{w}^\top (R \otimes EW)^\top \mathfrak{D}_\mathrm{I}(R \otimes I_q)\delta\tilde{\xi} = (\delta\tilde{\xi})^\top \cdot \sigma\gamma(R^\top \otimes I)\mathfrak{L}_\mathrm{p}(R \otimes EW)\tilde{w}.$$

Now, recall from Assumption 7.2.2 that $(1_N \otimes W)^\top h(x^*) = 0$ and $(I_N \otimes Z)^\top h(x^*) = 0$. Hence, we obtain

$$\begin{aligned}\dot{V} \leq{}& -k_\mathrm{p}\tilde{w}^\top (I_{N-1} \otimes EW)^\top \Lambda_\mathrm{p}(I_{N-1} \otimes EW)\tilde{w}\\ &+ \frac{1}{N}\Big[(I_N \otimes Z)\delta z + (1_N \otimes W)\delta\bar{w} + (R \otimes W)\tilde{w}\Big]^\top (h(x) - h(x^*)). \quad (7.3.1)\end{aligned}$$

However, note it can be checked that

$$(I_N \otimes Z)\delta z + (1_N \otimes W)\delta\bar{w} + (R \otimes W)\tilde{w} = x - x^*.$$

Hence, (7.3.1) becomes

$$\dot{V} \leq -k_{\mathrm{p}}\tilde{w}^\top (I_{N-1} \otimes EW)^\top \Lambda_{\mathrm{p}}(I_{N-1} \otimes EW)\tilde{w} + \frac{1}{N}(x - x^*)^\top (h(x) - h(x^*)) \leq 0,$$

where the last inequality follows from Assumption 7.3.1.

Using the LaSalle's invariance principle, states converge to the set given by

$$\mathcal{E} = \{x \in \mathbb{R}^{Nn} \mid (x - x^*)^\top (h(x) - h(x^*)) = 0\}.$$

Thus, it follows from Assumption 7.3.1 that $\lim_{t\to\infty} x(t) = x^*$.

To show output consensus is achieved, recall that $x^* = (I_N \otimes Z)z^* + (1_N \otimes W)\bar{w}^*$. Consequently, we obtain

$$\lim_{t\to\infty} |\zeta_j(t) - \zeta_i(t)| = \lim_{t\to\infty} \left|E(Zz_i^* - Zz_j^*)\right| = 0, \qquad \square$$

since $EZ = 0$.

Theorem 7.3.1 states that the solution of the proposed algorithm (7.2.4) converges to x^* as long as $k_{\mathrm{p}} \geq 0$. Specifically, if $k_{\mathrm{p}} = 0$, control input only uses the integral coupling and no proportional feedback is applied.

Remark 7.3.1. Similar to how the traditional PI solves the consensus optimization problem, the dynamics (7.2.4) in fact solves a *output consensus optimization problem.* For instance, consider the scenario where multilayer graph is singlelayer and $h_i(x_i) = -\nabla f_i(x_i)$ where f_i is a strictly convex function. Then, (7.2.4) solves the optimization problem given by

$$\begin{aligned} &\min_{x_1,\ldots,x_N} \sum_{i=1}^{N} f_i(x_i) \\ &\text{s.t.} \quad (\mathfrak{L} \otimes E)x = 0. \end{aligned}$$

It can be easily checked that (7.2.4) is the saddle point dynamics for output consensus optimization problem. Thus it follows that the solution converges to an optimal point (e.g., see [CGC17]). However, it will be shown in the following sections that the proposed dynamics extend further than the output consensus optimization problem. ◊

7.3.2 Convergence with Strong Coupling

In the previous section, convergence has been shown when Assumption 7.3.1 holds. Assumption 7.3.1 has been used extensively to establish the synchronization of multi-agent systems. For instance, [HCIL18] and [KCM15] supposed h_i satisfies Assumption 7.3.1 where $h_i = -\nabla f_i$ where f_i is a convex function. Practical synchronization is also studied with a similar assumption [MBA15]. Instead of Assumption 7.3.1, this section studies the convergence of the proposed algorithm by using high proportional gain and exponential stability of the (7.2.12). As a motivating example, the following example presents a case that does not satisfy Assumption 7.3.1.

Example 7.3.1. Suppose that the dynamics of an agent is given by

$$\begin{bmatrix} \dot{w}_i \\ \dot{z}_i \end{bmatrix} = \begin{bmatrix} z_i \\ -2\sqrt{\alpha} z_i - \nabla f_i(w_i) \end{bmatrix} =: h_i(x_i). \tag{7.3.2}$$

In particular, (7.3.2) is applying the heavy-ball method for each agent. However, $h_i(x_i)$ is not incrementally passive. To see this, let $x^k := [w^k; z^k]$ and note we have

$$\begin{aligned}
&(x^1 - x^2)^\top (h_i(x^1) - h_i(x^2)) \\
&= \begin{bmatrix} w^1 - w^2 \\ z^1 - z^2 \end{bmatrix}^\top \begin{bmatrix} z^1 - z^2 \\ -2\sqrt{\alpha} z^1 - \nabla f_i(w^1) + 2\sqrt{\alpha} z^2 + \nabla f_i(w^2) \end{bmatrix} \\
&= (w^1 - w^2)^\top (z^1 - z^2) + (z^1 - z^2)^\top (-2\sqrt{\alpha} z^1 + 2\sqrt{\alpha} z^2 - \nabla f_i(w^1) + \nabla f_i(w^2)) \\
&= (w^1 - w^2)^\top (z^1 - z^2) - 2\sqrt{\alpha} \left| z^1 - z^2 \right|^2 + (z^1 - z^2)^\top (-\nabla f_i(w^1) + \nabla f_i(w^2)).
\end{aligned}$$

Suppose that z^k and w^k are scalar. Then by letting $\nabla f_i(w) = Lw$, it follows that

$$\begin{aligned}
&(x^1 - x^2)^\top (h_i(x^1) - h_i(x^2)) \\
&= (w^1 - w^2)(z^1 - z^2) - 2\sqrt{\alpha} \left| z^1 - z^2 \right|^2 - L(z^1 - z^2)(w^1 - w^2) \\
&= (1 - L)(w^1 - w^2)(z^1 - z^2) - 2\sqrt{\alpha} \left| z^1 - z^2 \right|^2 .
\end{aligned}$$

Now, consider the case when $z^1 - z^2 > 0$ is fixed, $L > 1$ and w^k are such that $w^1 - w^2 = -k(z^1 - z^2)$, where $k > 2\sqrt{\alpha}/(L-1)$. Then we obtain

$$\begin{aligned}(x^1 - x^2)^\top (h_i(x^1) - h_i(x^2)) &= (1-L)(w^1 - w^2)(z^1 - z^2) - 2\sqrt{\alpha}\left|z^1 - z^2\right|^2 \\ &= \left((L-1)k - 2\sqrt{\alpha}\right)(z^1 - z^2)^2 \\ &> 0.\end{aligned}$$

Thus, h_i does not satisfy Assumption 7.3.1. ♢

Example 7.3.1 illustrates a system where dynamics of each agent is not incrementally passive. Therefore, a different approach is needed to analyze the convergence of such system. For this we first make following assumption.

Assumption 7.3.2. Functions h_i are globally Lipschitz with Lipschitz constant given by L_i. ♢

Using the Lipschitzness of h_i, Lipschitz constant for $h(x)$ can be found as $\bar{L} := \max L_i$. Following theorem states the exponential convergence of the proposed algorithm.

Theorem 7.3.2. (Exponential convergence without an explicit rate.) Consider the dynamics given by (7.2.4) and suppose that Assumptions 7.2.1, 7.2.2 and 7.3.2 hold. Let $k_{\mathrm{I}}, \gamma > 0$ be arbitrary positive scalars and x^*, ξ^* be given as in Lemma 7.2.1. Then with

$$K := EWW^\top E^\top > 0,$$

there exists $k_{\mathrm{p}}^* > 0$, such that for all $k_{\mathrm{p}} > k_{\mathrm{p}}^*$, the equilibrium point (x^*, ξ^*) is exponentially stable. ♢

Proof. Consider the system (7.2.4) transformed into (7.2.16). Then, it is sufficient to prove that the origin of (7.2.16) without (7.2.16d) is exponentially stable.

The first step is to obtain a Lyapunov function for the blended dynamics (7.2.12) that can characterize the convergence rate of μ. For this purpose, we employ the converse Lyapunov theorem by [CG98, Thm. 1], which states under

Assumption 7.2.2 that, for any $0 < \upsilon < \mu$, there exists $c_1, c_2, c_3 > 0$ such that $\bar{V}(\delta)$ satisfies and

$$c_1|\delta|^2 \le \bar{V}(\delta) \le c_2|\delta|^2$$
$$\frac{\partial \bar{V}}{\partial \delta}\begin{bmatrix} (I_N \otimes Z)^\top h\left((I_N \otimes Z)z + (1_N \otimes W)\bar{w}\right) \\ \frac{1}{N}(1_N \otimes W)^\top h\left((I_N \otimes Z)z + (1_N \otimes W)\bar{w}\right) \end{bmatrix} \le -2\cdot\left(\mu - \frac{\upsilon}{2}\right)\cdot \bar{V}(\delta),$$
$$\left\|\frac{\partial \bar{V}}{\partial \delta}\right\| \le c_3|\delta|.$$

where $\delta := [\delta z; \delta\bar{w}]$.

Now, we let Lyapunov function be

$$\begin{aligned} V(\delta z, \delta\bar{w}, \tilde{w}, \delta\tilde{\xi}) &= \eta\bar{V}(\delta) + \frac{1}{2}\begin{bmatrix} \tilde{w} \\ \delta\tilde{\xi} \end{bmatrix}^\top \begin{bmatrix} aX & \epsilon Y \\ \epsilon Y^\top & b\Lambda_{\mathrm{I}} \end{bmatrix}\begin{bmatrix} \tilde{w} \\ \delta\tilde{\xi} \end{bmatrix} \\ &=: \eta\bar{V}(\delta) + V_2(\tilde{w}, \delta\tilde{\xi}), \end{aligned} \tag{7.3.3}$$

where

$$\begin{aligned} X &:= (I_{N-1} \otimes EW)^\top \Lambda_{\mathrm{p}}(I_{N-1} \otimes EW) \\ Y &:= (I_{N-1} \otimes W^\top E^\top)\Lambda_{\mathrm{I}}. \end{aligned}$$

The positive scalars a, b, ϵ, and η are to be determined.

First of all, we choose $\epsilon > 0$ such that V_2 is positive definite. From Schur's complement, V_2 is positive definite if and only if

$$aX - \frac{\epsilon^2}{b}Y\Lambda_{\mathrm{I}}^{-1}Y^\top > 0. \tag{7.3.4}$$

Using the definitions of X and Y, we obtain

$$\begin{aligned} & aX - \frac{\epsilon^2}{b}Y\Lambda_{\mathrm{I}}^{-1}Y^\top \\ &= aX - \frac{\epsilon^2}{b}(I_N \otimes W^\top E^\top)\Lambda_{\mathrm{I}}\Lambda_{\mathrm{I}}^{-1}\Lambda_{\mathrm{I}}(I_N \otimes EW) \\ &= aX - \frac{\epsilon^2}{b}(I_N \otimes W^\top E^\top)\Lambda_{\mathrm{I}}(I_N \otimes EW). \end{aligned}$$

Hence the inequality (7.3.4) is satisfied if

$$\begin{aligned}
\epsilon &< \sqrt{ab} \cdot \sqrt{\frac{\lambda_{\min}(X)}{\lambda_{\max}\left((I_N \otimes W^\top E^\top)\Lambda_{\mathrm{I}}(I_N \otimes EW)\right)}} \\
&= \sqrt{ab} \cdot \sqrt{\frac{\lambda_{\min}(\Lambda_{\mathrm{p}})}{\lambda_{\max}(\Lambda_{\mathrm{I}})} \cdot \frac{\sigma_{\min}(E)}{\sigma_{\max}(E)}}, \qquad (7.3.5)
\end{aligned}$$

where we used $\sigma_i(EW) = \sigma_i(E)$. Thus, if $\epsilon < \epsilon_1$, then V is positive definite.

Now we compute the derivative of V along (7.2.16). For the ease of presentation, each component is done separately.

Derivative of $\eta\bar{V}(\delta)$:

First, recalling that $x = (I_N \otimes Z)(\delta z + z^*) + (1_N \otimes W)(\delta\bar{w} + \bar{w}^*) + (R \otimes W)\tilde{w}$, the time-derivative of $\eta\bar{V}(\delta)$ along (7.2.16) is given by

$$\begin{aligned}
\eta\dot{\bar{V}}(\delta) &= \eta\frac{\partial\bar{V}}{\partial\delta} \cdot \dot{\delta} \\
&= \eta\frac{\partial\bar{V}}{\partial\delta} \cdot \begin{bmatrix} (I_N \otimes Z) \\ \frac{1}{N}(1_N \otimes W) \end{bmatrix}^\top h(x) \\
&= \eta\frac{\partial\bar{V}}{\partial\delta} \cdot \begin{bmatrix} (I_N \otimes Z)^\top \\ \frac{1}{N}(1_N \otimes W)^\top \end{bmatrix} h(x) \pm \eta\frac{\partial\bar{V}}{\partial\delta} \cdot \begin{bmatrix} (I_N \otimes Z)^\top \\ \frac{1}{N}(1_N \otimes W)^\top \end{bmatrix} h\big(x - (R \otimes W)\tilde{w}\big) \\
&\le -2(\mu - \upsilon)\eta\bar{V}(\delta) + \eta\left\|\frac{\partial V}{\partial\delta}\right\| \cdot \left\|\begin{bmatrix} (I_N \otimes Z) \\ \frac{1}{N}(1_N \otimes W) \end{bmatrix}\right\| \cdot \bar{L} \cdot \Big|(R \otimes W)\tilde{w}\Big| \\
&\le -2\left(\mu - \frac{\upsilon}{2}\right)\eta\bar{V}(\delta) + \eta c_3\bar{L}|W||R| \cdot |\delta| \cdot |\tilde{w}| \\
&\le -2\left(\mu - \upsilon\right)\eta\bar{V}(\delta) - \upsilon\eta c_1|\delta|^2 + \eta c_3\bar{L}|\delta||\tilde{w}| \qquad (7.3.6)
\end{aligned}$$

where we used $|Z| = |W| = 1$ and $|R| = 1$.

Derivative of $V_2(\tilde{w}, \delta\tilde{\xi})$:

For the derivative of V_2 along (7.2.16), note that

$$\begin{aligned}
|h(x) - h(x^*)| &\le \bar{L}\,|(I_N \otimes Z)\delta z + (1_N \otimes W)\delta\bar{w} + (R \otimes W)\tilde{w}| \\
&\le 2\bar{L}\sqrt{N}|\delta| + L|\tilde{w}|.
\end{aligned}$$

With this, we first investigate the diagonal terms. The derivative of $(a/2)\tilde{w}^\top X\tilde{w}$ results in

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{a}{2}\tilde{w}^\top X\tilde{w}\right)\\
&= a\tilde{w}^\top X\left[(R^\top \otimes W^\top)(h(x)-h(x^*)) - k_{\mathrm{p}}X\tilde{w} - k_{\mathrm{I}}(I_{N-1}\otimes W^\top E^\top)\Lambda_{\mathrm{I}}\delta\tilde{\xi}\right]\\
&= a\tilde{w}^\top X(R^\top \otimes W^\top)(h(x)-h(x^*)) - ak_{\mathrm{p}}\tilde{w}^\top X^2\tilde{w}\\
&\qquad\qquad - ak_{\mathrm{I}}\tilde{w}^\top X(I_{N-1}\otimes W^\top E^\top)\Lambda_{\mathrm{I}}\delta\tilde{\xi}.
\end{aligned} \tag{7.3.7}$$

On the other hand, derivative of $(b/2)\cdot(\delta\tilde{\xi})^\top\Lambda_{\mathrm{I}}\delta\tilde{\xi}$ becomes

$$\frac{d}{dt}\left(\frac{b}{2}(\delta\tilde{\xi})^\top\Lambda_{\mathrm{I}}\delta\tilde{\xi}\right) = b(\delta\tilde{\xi})^\top\Lambda_{\mathrm{I}}\left[\gamma(R^\top\otimes K)\mathfrak{L}_{\mathrm{p}}(I_N\otimes EW)(R\otimes I_q)\tilde{w}\right]. \tag{7.3.8}$$

However, note that the last term of (7.3.7) and the transpose of (7.3.8) satisfies

$$\begin{aligned}
&- ak_{\mathrm{I}}\tilde{w}^\top X(I_{N-1}\otimes W^\top E^\top)\Lambda_{\mathrm{I}}\delta\tilde{\xi} + b\gamma\tilde{w}^\top(R\otimes EW)^\top\mathfrak{L}_{\mathrm{p}}(R\otimes K^\top)\Lambda_{\mathrm{I}}\delta\tilde{\xi}\\
=\; &- ak_{\mathrm{I}}\tilde{w}^\top XY\delta\tilde{\xi} + b\gamma\tilde{w}^\top(R^\top\otimes EW)^\top\mathfrak{L}_{\mathrm{p}}(R\otimes EWW^\top E^\top)\Lambda_{\mathrm{I}}\delta\tilde{\xi}\\
=\; &(-ak_{\mathrm{I}}+b\gamma)\tilde{w}^\top XY\delta\tilde{\xi}
\end{aligned}$$

In summary, we have

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{a}{2}\tilde{w}^\top X\tilde{w} + \frac{b}{2}(\delta\tilde{\xi})^\top\Lambda_{\mathrm{I}}\delta\tilde{\xi}\right)\\
&= a\tilde{w}^\top X(R^\top\otimes W^\top)(h(x)-h(x^*)) - ak_{\mathrm{p}}\tilde{w}^\top X^2\tilde{w} + (-ak_{\mathrm{I}}+b\gamma)\tilde{w}^\top XY\delta\tilde{\xi}\\
&\le a|X|\left(2\bar{L}\sqrt{N}|\delta| + \bar{L}|\tilde{w}|\right)|\tilde{w}| - ak_{\mathrm{p}}\tilde{w}^\top X^2\tilde{w} + (-ak_{\mathrm{I}}+b\gamma)\tilde{w}^\top XY\delta\tilde{\xi}\\
&= a2|X|\bar{L}\sqrt{N}|\delta||\tilde{w}| + a|X|L|\tilde{w}|^2 - ak_{\mathrm{p}}\tilde{w}^\top X^2\tilde{w} + (-ak_{\mathrm{I}}+b\gamma)\tilde{w}^\top XY\delta\tilde{\xi},
\end{aligned} \tag{7.3.9}$$

Finally, the derivative of $\tilde{w}^\top\cdot\epsilon Y\cdot\delta\tilde{\xi}$ becomes

$$\begin{aligned}
&\frac{d}{dt}\left(\tilde{w}^\top\cdot\epsilon Y\cdot\delta\tilde{\xi}\right)\\
&= \left[(R^\top\otimes W^\top)(h(x)-h(x^*)) - k_{\mathrm{p}}X\tilde{w} - k_{\mathrm{I}}Y\delta\tilde{\xi}\right]^\top\epsilon Y\delta\tilde{\xi} + \tilde{w}^\top\epsilon Y\gamma(I_{N-1}\otimes EW)X\tilde{w}\\
&\le \left(2\bar{L}\sqrt{N}|\delta| + \bar{L}|\tilde{w}|\right)|\epsilon Y||\delta\tilde{\xi}| - k_{\mathrm{p}}\epsilon\tilde{w}^\top XY\delta\tilde{\xi} - \epsilon k_{\mathrm{I}}\delta\tilde{\xi}^\top Y^\top Y\delta\tilde{\xi}
\end{aligned}$$

$$
\begin{aligned}
&+\epsilon\gamma\tilde{w}^\top Y(I_{N-1}\otimes EW)X\tilde{w}\\
&=2\bar{L}\sqrt{N}\epsilon|Y||\delta||\delta\tilde{\xi}|+\epsilon\bar{L}|Y||\tilde{w}||\delta\tilde{\xi}|-k_{\mathrm{p}}\epsilon\tilde{w}^\top XY\delta\tilde{\xi}-\epsilon k_{\mathrm{I}}\delta\tilde{\xi}^\top Y^\top Y\delta\tilde{\xi}\\
&+\epsilon\gamma\tilde{w}^\top Y(I_{N-1}\otimes EW)X\tilde{w}.
\end{aligned}
\tag{7.3.10}
$$

Finally, combining (7.3.6), (7.3.9) and (7.3.10), the derivative of V along (7.2.16) satisfies

$$
\begin{aligned}
\dot{V}\leq&-2(\mu-\upsilon)\eta\bar{V}(\delta)-\upsilon\eta c_1|\delta|^2+\left(\eta c_3\bar{L}+a2|X|\bar{L}\sqrt{N}\right)|\delta||\tilde{w}|\\
&+a|X|L|\tilde{w}|^2-ak_{\mathrm{p}}\tilde{w}^\top X^2\tilde{w}+\epsilon\gamma\tilde{w}^\top Y(I_{N-1}\otimes EW)X\tilde{w}\\
&+2\bar{L}\sqrt{N}\epsilon|Y||\delta||\delta\tilde{\xi}|+\epsilon\bar{L}|Y||\tilde{w}||\delta\tilde{\xi}|-\epsilon k_{\mathrm{I}}\delta\tilde{\xi}^\top Y^\top Y\delta\tilde{\xi}\\
&+(-ak_{\mathrm{I}}+b\gamma-k_{\mathrm{p}}\epsilon)\tilde{w}^\top XY\delta\tilde{\xi}\\
=&-2(\mu-\upsilon)\eta\bar{V}(\delta)-\upsilon\eta c_1|\delta|^2-\frac{a}{2}k_{\mathrm{p}}\tilde{w}^\top X^2\tilde{w}-\frac{1}{2}\epsilon k_{\mathrm{I}}\delta\tilde{\xi}^\top Y^\top Y\delta\tilde{\xi}\\
&+\left(\eta c_3\bar{L}+a2|X|\bar{L}\sqrt{N}\right)|\delta||\tilde{w}|+a|X|L|\tilde{w}|^2-\frac{a}{2}k_{\mathrm{p}}\tilde{w}^\top X^2\tilde{w}\\
&+\epsilon\gamma\tilde{w}^\top Y(I_{N-1}\otimes EW)X\tilde{w}+2\bar{L}\sqrt{N}\epsilon|Y||\delta||\delta\tilde{\xi}|+\epsilon\bar{L}|Y||\tilde{w}||\delta\tilde{\xi}|\\
&-\frac{1}{2}\epsilon k_{\mathrm{I}}\delta\tilde{\xi}^\top Y^\top Y\delta\tilde{\xi}+(-ak_{\mathrm{I}}+b\gamma-k_{\mathrm{p}}\epsilon)\tilde{w}^\top XY\delta\tilde{\xi}\\
=:&-2(\mu-\upsilon)\eta\bar{V}(\delta)-\frac{a}{2}k_{\mathrm{p}}\tilde{w}^\top X^2\tilde{w}-\frac{1}{2}\epsilon k_{\mathrm{I}}\delta\tilde{\xi}^\top Y^\top Y\delta\tilde{\xi}\\
&-\upsilon\eta c_1|\delta|^2+F(|\delta|,|\tilde{w}|,|\delta\tilde{\xi}|).
\end{aligned}
\tag{7.3.11}
$$

Now, to show the exponential convergence, rest of the proof will show $-\upsilon\eta c_1|\delta|^2+F(\cdot)\leq 0$ hold. For this, let

$$
\epsilon=\frac{b\gamma}{k_{\mathrm{p}}}
$$

such that $-ak_{\mathrm{I}}+b\gamma-k_{\mathrm{p}}\epsilon=-ak_{\mathrm{I}}$ and notice that applying the Young's inequality to the terms of $F(\cdot)$ result in

$$
\eta c_3\bar{L}|\delta||\tilde{w}|\leq\frac{\upsilon\eta c_1}{3}|\delta|^2+\frac{3\eta c_3^2\bar{L}^2}{4\upsilon c_1}|\tilde{w}|^2 \tag{7.3.12a}
$$

$$
a2|X|\bar{L}\sqrt{N}|\delta||\tilde{w}|\leq\frac{\upsilon\eta c_1}{3}|\delta|^2+\frac{3a^2|X|^2\bar{L}^2N}{\upsilon c_1\eta}|\tilde{w}|^2 \tag{7.3.12b}
$$

$$2\bar{L}\sqrt{N}\epsilon|Y||\delta||\delta\tilde{\xi}| \leq \frac{\upsilon\eta c_1}{3}|\delta|^2 + \frac{3\bar{L}^2N\epsilon^2|Y|^2}{\upsilon c_1\eta}|\delta\tilde{\xi}|^2 \tag{7.3.12c}$$

$$\epsilon\bar{L}|Y||\tilde{w}||\delta\tilde{\xi}| \leq \epsilon k_{\mathrm{I}}\frac{1}{8}\lambda_{\min}(Y^\top Y)|\delta\tilde{\xi}|^2 + \frac{2\epsilon\bar{L}^2|Y|^2}{k_{\mathrm{I}}\lambda_{\min}(Y^\top Y)}|\tilde{w}|^2 \tag{7.3.12d}$$

$$\epsilon k_{\mathrm{p}}|X||Y||\tilde{w}||\delta\tilde{\xi}| \leq \epsilon k_{\mathrm{I}}\frac{1}{8}\lambda_{\min}(Y^\top Y)|\delta\tilde{\xi}|^2 + \frac{2\epsilon k_{\mathrm{p}}^2|X|^2|Y|^2}{k_{\mathrm{I}}\lambda_{\min}(Y^\top Y)}|\tilde{w}|^2 \tag{7.3.12e}$$

Then we obtain

$$\begin{aligned}
&-\upsilon\eta c_1|\delta|^2 + F(|\delta|,|\tilde{w}|,|\delta\tilde{\xi}|) \\
&= \left(\frac{3\eta c_3^2\bar{L}^2}{4\upsilon c_1} + \frac{3a^2|X|^2\bar{L}^2N}{\upsilon c_1\eta} + \frac{2b\gamma\bar{L}^2|Y|^2}{k_{\mathrm{p}}k_{\mathrm{I}}\lambda_{\min}(Y^\top Y)} + \frac{2b\gamma k_{\mathrm{p}}|X|^2|Y|^2}{k_{\mathrm{I}}\lambda_{\min}(Y^\top Y)} + a|X|\bar{L}\right)|\tilde{w}|^2 \\
&\quad - \frac{a}{2}k_{\mathrm{p}}\tilde{w}^\top X^2\tilde{w} + b\frac{\gamma^2}{k_{\mathrm{p}}}\tilde{w}^\top Y(I_{N-1}\otimes EW)X\tilde{w} \\
&\quad + \frac{b\gamma}{k_{\mathrm{p}}}\left(-k_{\mathrm{I}}\frac{1}{4}\lambda_{\min}(Y^\top Y) + \frac{3\bar{L}^2Nb\gamma|Y|^2}{k_{\mathrm{p}}\upsilon c_1\eta}\right)|\delta\tilde{\xi}|^2
\end{aligned} \tag{7.3.13}$$

Now, it is easy to see from (7.3.13) that for any given γ and k_{I}, k_{p} can be chosen sufficiently large such that

$$-\upsilon\eta c_1|\delta|^2 + F(|\delta|,|\tilde{w}|,|\delta\tilde{\xi}|) \leq 0,$$

while $\epsilon < \epsilon_1$.

In conclusion, we obtain

$$\dot{V} \leq -2(\mu-\upsilon)\eta\bar{V}(\delta) - \frac{a}{2}k_{\mathrm{p}}\tilde{w}^\top X^2\tilde{w} - \frac{b\gamma}{2k_{\mathrm{p}}}k_{\mathrm{I}}\delta\tilde{\xi}^\top Y^\top Y\delta\tilde{\xi}.$$

Then, it follows that the system (7.2.16) is exponentially stable, i.e., $\lim_{t\to\infty} x(t) = x^*$ and $\lim_{t\to\infty}\xi(t) = \xi^*$ exponentially. In addition, since $x^* = (I_N\otimes Z)z + (1_N\otimes I_q)\bar{w}$, it follows that ζ_i also achieve consensus exponentially. □

Result of Theorem 7.3.2 holds under the assumption that the system (7.2.12) has exponentially stable equilibrium point. The dynamics (7.2.12) has reduced dimension of $Nq+(n-1)$ compared with the overall dynamical system in dimension Nn. Hence, it can be regarded as an *emergent behavior*, that is it originates due to the coupling among agents and is different from the dynamics of any individual agent. This also implies that the individual dynamics is less important as long as the collective behavior is desirable.

Remark 7.3.2. Note from the proof that the dynamics converges for any $k_{\mathrm{I}} > 0$ and $\gamma > 0$. In particular, this implies that the integral gain k_{I} only contributes to the asymptotic convergence of the agents. On the other hand, overall stability is ensured with the sufficiently high proportional gain k_{p}. ◊

Theorem 7.3.2 shows that the system achieves exponential convergence. The convergence rate is characterized in the following result.

Theorem 7.3.3. (Exponential convergence with an explicit rate.) Suppose that the assumptions of Theorem 7.3.2 hold and let $K := EWW^\top E^\top > 0$ and $0 < \upsilon < \mu$. Then the following results hold.

1. For any positive k_{I} and γ satisfying $2\gamma > k_{\mathrm{I}}$, there exists $k_{\mathrm{p},1}^*$ such that for all $k_{\mathrm{p}} \geq k_{\mathrm{p},1}^*$, the solution is exponentially stable with a rate no less than

$$2\min\left(\mu-\upsilon, \frac{a}{4\rho_2}k_{\mathrm{p}}\lambda_{\min}(\Lambda_{\mathrm{p}})^2\sigma_{\min}(E)^4, \frac{1}{4\rho_2}\frac{2\gamma-k_{\mathrm{I}}}{k_{\mathrm{p}}}k_{\mathrm{I}}\lambda_{\min}(\Lambda_{\mathrm{I}})^2\sigma_{\min}(E)^2\right),$$

   where $\rho_2 = \max(a\lambda_{\max}(\Lambda_{\mathrm{p}}) \cdot \sigma_{\max}(E)^2, b\lambda_{\max}(\Lambda_{\mathrm{I}}))$.

2. There exists $\theta^* > 0$ and $k_{\mathrm{p},2}^*$ such that for all $k_{\mathrm{p}} \geq k_{\mathrm{p},2}^*$ and $k_{\mathrm{I}} = \gamma = \theta^* k_{\mathrm{p}}$, the solution converges exponentially fast with a rate $\mu - \upsilon$. ◊

Proof. Proof is similar to the proof of Theorem 7.3.2 and in particular, we will use the same Lyapunov function V from (7.3.3), but now with

$$\epsilon = \frac{2\gamma - k_{\mathrm{I}}}{k_{\mathrm{p}}}.$$

Then, it follows from (7.3.5) that V_2 is positive definite if

$$\frac{2\gamma - k_{\mathrm{I}}}{k_{\mathrm{p}}} < \sqrt{ab} \cdot \sqrt{\frac{\lambda_{\min}(\Lambda_{\mathrm{p}})}{\lambda_{\max}(\Lambda_{\mathrm{I}})}} \cdot \frac{\sigma_{\min}(E)}{\sigma_{\max}(E)} =: \theta_1.$$

On the other hand, we can also find an upper and lower bound of the function $V_2(\tilde{w}, \delta\tilde{\xi})$. For the upper bound, note

$$\begin{aligned} V_2(\tilde{w}, \delta\tilde{\xi}) &\leq \left(\frac{a}{2}\lambda_{\max}(X) + \frac{(2\gamma - k_{\mathrm{I}})|Y|}{2k_{\mathrm{p}}}\right)|\tilde{w}|^2 + \left(\frac{b}{2}\lambda_{\max}(\Lambda_{\mathrm{I}}) + \frac{(2\gamma - k_{\mathrm{I}})|Y|}{2k_{\mathrm{p}}}\right)|\delta\tilde{\xi}|^2 \\ &\leq \left(\max\left(\frac{a}{2}\lambda_{\max}(X), \frac{b}{2}\lambda_{\max}(\Lambda_{\mathrm{I}})\right) + \frac{(2\gamma - k_{\mathrm{I}})|Y|}{2k_{\mathrm{p}}}\right)\left(|\tilde{w}|^2 + |\delta\tilde{\xi}|^2\right) \end{aligned}$$

In order to obtain an upper bound which remains constant with respect to k_{p}, suppose that

$$\begin{aligned} \frac{2\gamma - k_{\mathrm{I}}}{k_{\mathrm{p}}} &\leq \frac{2}{|Y|}\max\left(\frac{a}{2}\lambda_{\max}(X), \frac{b}{2}\lambda_{\max}(\Lambda_{\mathrm{I}})\right) \\ &= \frac{1}{\lambda_{\max}(\Lambda_{\mathrm{I}})\sigma_{\max}(E)} \cdot \max\left(a\lambda_{\max}(\Lambda_{\mathrm{p}})\sigma_{\max}(E)^2, b\lambda_{\max}(\Lambda_{\mathrm{I}})\right) \\ &= \max\left(\frac{a\lambda_{\max}(\Lambda_{\mathrm{p}})}{\lambda_{\max}(\Lambda_{\mathrm{I}})}\sigma_{\max}(E), \frac{b}{\sigma_{\max}(E)}\right) =: \theta_2. \end{aligned}$$

Then V_2 is upper bounded by

$$\begin{aligned} V_2 &\leq \max(a\lambda_{\max}(X), b\lambda_{\max}(\Lambda_{\mathrm{I}}))(|\tilde{w}|^2 + |\delta\tilde{\xi}|^2) \\ &= \max(a\lambda_{\max}(\Lambda_{\mathrm{p}}) \cdot \sigma_{\max}(E)^2, b\lambda_{\max}(\Lambda_{\mathrm{I}}))(|\tilde{w}|^2 + |\delta\tilde{\xi}|^2) \\ &=: \rho_2(|\tilde{w}|^2 + |\delta\tilde{\xi}|^2). \end{aligned} \tag{7.3.14}$$

Note here that ρ_2 is independent of the gains if $(2\gamma - k_{\mathrm{I}})/k_{\mathrm{p}} \leq \theta_2$.

Next, for the lower bounds of V_2, we can write V_2 as

$$V_2(\tilde{w}, \delta\tilde{\xi}) = \frac{1}{2}\begin{bmatrix}\tilde{w} \\ \delta\tilde{\xi}\end{bmatrix}^\top \left(\begin{bmatrix} aX & 0 \\ 0 & b\Lambda_{\mathrm{I}}\end{bmatrix} + \epsilon\begin{bmatrix} 0 & Y \\ Y^\top & 0\end{bmatrix}\right)\begin{bmatrix}\tilde{w} \\ \delta\tilde{\xi}\end{bmatrix}$$

and notice that $\mathrm{diag}(aX, b\Lambda_{\mathrm{I}})$ is a positive definite matrix. Therefore, finding the lower bound of V_2 is equivalent to finding a minimum eigenvalue of the matrix

$$\begin{bmatrix} aX & \epsilon Y \\ \epsilon Y^\top & b\Lambda_{\mathrm{I}}\end{bmatrix},$$

which is a perturbation of $\mathrm{diag}(aX, b\Lambda_{\mathrm{I}})$. Using results such as [RAH19, Thm.

6.3.2], we may obtain[2]

$$\begin{aligned}
\lambda_{\min}\left(\begin{bmatrix} aX & \epsilon Y \\ \epsilon Y^\top & b\Lambda_{\mathrm{I}} \end{bmatrix}\right) &\geq \min\left(a\lambda_{\min}(X), b\lambda_{\min}(\Lambda_{\mathrm{I}})\right) - \kappa\left(\begin{bmatrix} aX & 0 \\ 0 & b\Lambda_{\mathrm{I}} \end{bmatrix}\right) \cdot \epsilon \left\|\begin{bmatrix} 0 & Y \\ Y^\top & 0 \end{bmatrix}\right\| \\
&\geq \frac{1}{2}\min\left(a\lambda_{\min}(X), b\lambda_{\min}(\Lambda_{\mathrm{I}})\right) \\
&= \frac{1}{2}\min\left(a\lambda_{\min}(\Lambda_{\mathrm{p}}) \cdot \sigma_{\min}(E)^2, b\lambda_{\min}(\Lambda_{\mathrm{I}})\right) \\
&=: \rho_1
\end{aligned}$$

for sufficiently small ϵ, where $\kappa(X) := |X^{-1}||X|$ is the condition number. Specifically, we choose $\epsilon = (2\gamma - k_{\mathrm{I}})/k_{\mathrm{p}}$ such that

$$\begin{aligned}
\frac{2\gamma - k_{\mathrm{I}}}{k_{\mathrm{p}}} &\leq \frac{1}{2}\min\left(a\lambda_{\min}(X), b\lambda_{\min}(\Lambda_{\mathrm{I}})\right) \cdot \kappa\left(\begin{bmatrix} aX & 0 \\ 0 & b\Lambda_{\mathrm{I}} \end{bmatrix}\right)^{-1} \cdot \left\|\begin{bmatrix} 0 & Y \\ Y^\top & 0 \end{bmatrix}\right\|^{-1} \\
&= \frac{1}{2}\min\left(a\lambda_{\min}(X), b\lambda_{\min}(\Lambda_{\mathrm{I}})\right) \cdot \frac{\lambda_{\max}(X)}{\lambda_{\min}(X)} \cdot \frac{1}{\sigma_{\max}(Y)} \\
&= \frac{1}{2}\min\left(a\lambda_{\min}(\Lambda_{\mathrm{p}})\sigma_{\min}(E)^2, b\lambda_{\min}(\Lambda_{\mathrm{I}})\right) \cdot \frac{\lambda_{\min}(\Lambda_{\mathrm{p}})\sigma_{\min}(E)^2}{\lambda_{\max}(\Lambda_{\mathrm{p}})\sigma_{\max}(E)^2} \cdot \frac{1}{\lambda_{\max}(\Lambda_{\mathrm{I}})\sigma_{\max}(E)} \\
&=: \theta_3.
\end{aligned}$$

With these in mind, recall from (7.3.11) that

$$\begin{aligned}
\dot{V} \leq& -2\left(\mu - \upsilon\right)\eta\bar{V}(\delta) - \frac{a}{2}k_{\mathrm{p}}\tilde{w}^\top X^2\tilde{w} - \frac{1}{2}\epsilon k_{\mathrm{I}}\delta\tilde{\xi}^\top Y^\top Y\delta\tilde{\xi} \\
&- \upsilon\eta c_1|\delta|^2 + F(|\delta|, |\tilde{w}|, |\delta\tilde{\xi}|).
\end{aligned}$$

Using the fact that $-ak_{\mathrm{I}} + b\gamma - \epsilon k_{\mathrm{p}} = 0$ and inequalities (7.3.12a)–(7.3.12d), it follows from (7.3.13) that

$$\begin{aligned}
\dot{V} \leq& -2\left(\mu - \upsilon\right)\eta\bar{V}(\delta) - \frac{a}{2}k_{\mathrm{p}}\tilde{w}^\top X^2\tilde{w} - \frac{1}{2}\frac{2\gamma - k_{\mathrm{I}}}{k_{\mathrm{p}}}k_{\mathrm{I}}\delta\tilde{\xi}^\top Y^\top Y\delta\tilde{\xi} \\
&- \upsilon\eta c_1|\delta|^2 + F(|\delta|, |\tilde{w}|, |\delta\tilde{\xi}|)
\end{aligned}$$

[2]One may obtain tighter bounds using the properties of the considered matrices. However, a general approach is introduced here for simplicity.

$$
\begin{aligned}
&= -2\,(\mu - \upsilon)\,\eta \bar{V}(\delta) - \frac{a}{2} k_{\mathrm{p}} \tilde{w}^\top X^2 \tilde{w} - \frac{1}{2}\frac{2\gamma - k_{\mathrm{I}}}{k_{\mathrm{p}}} k_{\mathrm{I}} \delta\tilde{\xi}^\top Y^\top Y \delta\tilde{\xi} \\
&\quad + \underbrace{\left(-\frac{a}{4} k_{\mathrm{p}} \lambda_{\min}(X^2) + \frac{3\eta c_3^2 \bar{L}^2}{4\upsilon c_1} + \frac{3a^2 |X|^2 \bar{L}^2 N}{\upsilon c_1 \eta} + \frac{2b(2\gamma - k_{\mathrm{I}})\bar{L}^2 |Y|^2}{k_{\mathrm{p}} k_{\mathrm{I}} \lambda_{\min}(Y^\top Y)} + a|X|\bar{L}\right)}_{\text{Term 1}} |\tilde{w}|^2 \\
&\quad \underbrace{- \frac{a}{4} k_{\mathrm{p}} \tilde{w}^\top X^2 \tilde{w} + b\frac{(2\gamma - k_{\mathrm{I}})\gamma}{k_{\mathrm{p}}} \tilde{w}^\top Y (I_{N-1} \otimes EW) X \tilde{w}}_{\text{Term 2}} \\
&\quad + \frac{b(2\gamma - k_{\mathrm{I}})}{k_{\mathrm{p}}} \underbrace{\left(-k_{\mathrm{I}} \frac{3}{8} \lambda_{\min}(Y^\top Y) + \frac{3\bar{L}^2 N b (2\gamma - k_{\mathrm{I}})|Y|^2}{k_{\mathrm{p}} \upsilon c_1 \eta}\right)}_{\text{Term 3}} |\delta\tilde{\xi}|^2.
\end{aligned}
$$

We claim that the terms 1,2, and 3 can be made positive with sufficiently large k_{p}. First it can be checked that the term 1 is positive if

$$
\frac{a}{4} k_{\mathrm{p}}^2 \lambda_{\min}(X^2) - \left(\frac{3\eta c_3^2 \bar{L}^2}{4\upsilon c_1} + \frac{3a^2 |X|^2 \bar{L}^2 N}{\upsilon c_1 \eta} + a|X|\bar{L}\right) k_{\mathrm{p}} - \frac{2b(2\gamma - k_{\mathrm{I}})\bar{L}^2 |Y|^2}{k_{\mathrm{I}} \lambda_{\min}(Y^\top Y)} > 0.
$$

Now, seeing above as the quadratic equation, and using the fact that $\sqrt{c^2 + d^2} \leq |c| + |d|$ for all $c, d \in \mathbb{R}$, we obtain that the term 1 is positive if

$$
\begin{aligned}
k_{\mathrm{p}} &\geq \frac{4}{a \lambda_{\min}(X^2)} \cdot \left(\frac{3\eta c_3^2 \bar{L}^2}{4\upsilon c_1} + \frac{3a^2 |X|^2 \bar{L}^2 N}{\upsilon c_1 \eta} + a|X|\bar{L}\right) + \sqrt{\frac{2\gamma - k_{\mathrm{I}}}{k_{\mathrm{I}}} \frac{4b\bar{L}^2 |Y|^2}{\lambda_{\min}(Y^\top Y)}} \\
&=: \kappa_1.
\end{aligned}
$$

Secondly, the term 2 is positive if

$$
\begin{aligned}
\frac{\sqrt{(2\gamma - k_{\mathrm{I}})\gamma}}{k_{\mathrm{p}}} \leq \sqrt{\frac{a}{4b} \frac{\lambda_{\min}(X)^2}{|Y||E||X|}} &= \sqrt{\frac{a}{4b}} \cdot \frac{\lambda_{\min}(\Lambda_{\mathrm{p}})}{\sqrt{\Lambda_{\max}(\Lambda_{\mathrm{I}}) \lambda_{\max}(\Lambda_{\mathrm{p}})}} \cdot \frac{\sigma_{\min}(E)^2}{\sigma_{\max}(E)^2} \\
&=: \theta_4.
\end{aligned}
$$

Finally, the term 3 is positive if

$$
k_{\mathrm{p}} \cdot \frac{k_{\mathrm{I}}}{2\gamma - k_{\mathrm{I}}} \geq \frac{8\bar{L}^2 N b |Y|^2}{\upsilon c_1 \eta \lambda_{\min}(Y^\top Y)} = \frac{8\bar{L}^2 N b}{\upsilon c_1 \eta} \frac{\lambda_{\max}(\Lambda_{\mathrm{I}})^2}{\lambda_{\min}(\Lambda_{\mathrm{I}})^2} \frac{\sigma_{\max}(E)^2}{\sigma_{\min}(E)^2} =: \kappa_2.
$$

Now, for some given gains k_{I} and γ, let

$$k_{\mathrm{p},1}^* := \max\left(\kappa_1, \kappa_2\frac{2\gamma - k_{\mathrm{I}}}{k_{\mathrm{I}}}, \frac{2\gamma - k_{\mathrm{I}}}{\theta_1}, \frac{2\gamma - k_{\mathrm{I}}}{\theta_2}, \frac{2\gamma - k_{\mathrm{I}}}{\theta_3}, \frac{\sqrt{(2\gamma - k_{\mathrm{I}})\gamma}}{\theta_4}\right).$$

Then, for any $k_{\mathrm{p}} \geq k_{\mathrm{p},1}^*$, the terms 1,2, and 3 are positive and we obtain

$$\begin{aligned}
\dot{V} &\leq -2\left(\mu - \upsilon\right)\eta\bar{V}(\delta) - \frac{a}{2}k_{\mathrm{p}}\tilde{w}^\top X^2\tilde{w} - \frac{1}{2}\frac{2\gamma - k_{\mathrm{I}}}{k_{\mathrm{p}}}k_{\mathrm{I}}\delta\tilde{\xi}^\top Y^\top Y\delta\tilde{\xi} \\
&\leq -2\left(\mu - \upsilon\right)\eta\bar{V}(\delta) - \min\left(\frac{a}{2}k_{\mathrm{p}}\lambda_{\min}(X^2), \frac{1}{2}\frac{2\gamma - k_{\mathrm{I}}}{k_{\mathrm{p}}}k_{\mathrm{I}}\lambda_{\min}(Y^\top Y)\right)\left(|\tilde{w}|^2 + |\delta\tilde{\xi}|^2\right) \\
&\leq -2\left(\mu - \upsilon\right)\eta\bar{V}(\delta) - \frac{1}{\rho_2}\min\left(\frac{a}{2}k_{\mathrm{p}}\lambda_{\min}(X^2), \frac{1}{2}\frac{2\gamma - k_{\mathrm{I}}}{k_{\mathrm{p}}}k_{\mathrm{I}}\lambda_{\min}(Y^\top Y)\right)V_2(\tilde{w}, \delta\tilde{\xi}) \\
&\leq -\min\left(2\left(\mu - \upsilon\right), \frac{a}{2\rho_2}k_{\mathrm{p}}\lambda_{\min}(X^2), \frac{1}{2\rho_2}\frac{2\gamma - k_{\mathrm{I}}}{k_{\mathrm{p}}}k_{\mathrm{I}}\lambda_{\min}(Y^\top Y)\right)V,
\end{aligned}$$

where ρ_2 is defined in (7.3.14). That is, the Lyapunov function V decays with the rate

$$\begin{aligned}
&\min\left(2\left(\mu - \upsilon\right), \frac{a}{2\rho_2}k_{\mathrm{p}}\lambda_{\min}(X^2), \frac{1}{2\rho_2}\frac{2\gamma - k_{\mathrm{I}}}{k_{\mathrm{p}}}k_{\mathrm{I}}\lambda_{\min}(Y^\top Y)\right) \\
&= \min\left(2\left(\mu - \upsilon\right), \frac{a}{2\rho_2}k_{\mathrm{p}}\lambda_{\min}(\Lambda_{\mathrm{p}})^2\sigma_{\min}(E)^4, \frac{1}{2\rho_2}\frac{2\gamma - k_{\mathrm{I}}}{k_{\mathrm{p}}}k_{\mathrm{I}}\lambda_{\min}(\Lambda_{\mathrm{I}})^2\sigma_{\min}(E)^2\right).
\end{aligned} \tag{7.3.15}$$

In order to recover the convergence rate of $\mu - \upsilon$, let $k_{\mathrm{I}} = \gamma$ and choose k_{I} such that

$$k_{\mathrm{I}} = \theta^* k_{\mathrm{p}},$$

where $\theta^* := \min(\theta_1, \theta_2, \theta_3, \theta_4)$ and let

$$k_{\mathrm{p},2}^* := \max\left(\kappa_1, \kappa_2, \frac{4\rho_2(\mu - \upsilon)}{a\lambda_{\min}(X^2)}, \frac{4\rho_2(\theta^*)^2(\mu - \upsilon)}{\lambda_{\min}(Y^\top Y)}\right).$$

Then for any $k_{\mathrm{p}} \geq k_{\mathrm{p},2}^*$, it follows from the construction that

$$2(\mu - \upsilon) \leq \frac{a}{2\rho_2}k_{\mathrm{p}}\lambda_{\min}(X^2) \tag{7.3.16}$$

$$2(\mu - \upsilon) \leq \frac{1}{2\rho_2}\frac{2\gamma - k_{\rm I}}{k_{\rm p}} k_{\rm I}\lambda_{\min}(Y^\top Y) = \frac{1}{2\rho_2}\frac{k_{\rm p}}{(\theta^*)^2}\lambda_{\min}(Y^\top Y) \tag{7.3.17}$$

Additionally, it can be checked that

$$k_{\rm p} \geq k_{\rm p,1}^* = \max\left(\kappa_1, \kappa_2, \frac{\theta^*}{\theta_1}k_{\rm p}, \frac{\theta^*}{\theta_2}k_{\rm p}, \frac{\theta^*}{\theta_3}k_{\rm p}, \frac{\theta^*}{\theta_4}k_{\rm p}\right).$$

Hence, it follows from (7.3.15), (7.3.16) and (7.3.17) that the convergence rate becomes

$$\min\left(2\left(\mu - \upsilon\right), \frac{a}{2\rho_2}k_{\rm p}\lambda_{\min}(X^2), \frac{1}{2\rho_2}\frac{2\gamma - k_{\rm I}}{k_{\rm p}}k_{\rm I}\lambda_{\min}(Y^\top Y)\right) = 2(\mu - \upsilon).$$

In conclusion, we obtain

$$\dot{V} \leq -2(\mu - \upsilon)V,$$

which means that the solution satisfies

$$\left\| \begin{bmatrix} \delta(t) \\ \tilde{w}(t) \\ \delta\tilde{\xi}(t) \end{bmatrix} \right\| \leq \sqrt{\frac{\rho_2}{\rho_1}} e^{(-\mu+\upsilon)t} \cdot \left\| \begin{bmatrix} \delta(0) \\ \tilde{w}(0) \\ \delta\tilde{\xi}(0) \end{bmatrix} \right\|, \tag{7.3.18}$$

i.e., the solution is exponentially stable with the rate $\mu - \upsilon$. □

From Theorem 7.3.3, it holds that the convergence rate can be decomposed into 3 components: 1) convergence rate of the blended dynamics, 2) convergence rate related to proportional feedback, and 3) convergence rate related to integral feedback. Most importantly, with sufficiently large gains, the convergence rate follows the convergence rate of the blended dynamics. For instance, if the blended dynamics is the heavy-ball method, then by setting $\upsilon = \mu/2$, the propose dynamics has the convergence rate proportional to $\sqrt{\alpha}$, exhibiting a faster convergence rate than the gradient descent based methods.

Additionally, convergence rate that is arbitrarily close to the convergence rate of the blended dynamics can be obtained by choosing υ sufficiently small. (However, this requires the corresponding coupling gain $k_{\rm p}$ to be even larger.)

Remark 7.3.3. In order to recover the convergence rate, $k_{\rm I} = \gamma = \theta^* k_{\rm p}$ should be used with sufficiently large k_{p}. For the simple cases, the value of θ^* can be easily computed. Suppose that $\sigma_{\min}(E) = \sigma_{\max}(E) = 1$, the input (7.2.2) is used, $\Lambda_{\mathrm{p}} = \Lambda_{\mathrm{I}}$ and that $a = 1$, $b = 2$ are used. Then, for θ_1, noting the fact that $Y\Lambda_{\mathrm{I}}^{-1}Y^\top = Y\Lambda_{\mathrm{p}}^{-1}Y^\top = X$, applying Schur's complement to V_2 implies we may take $\theta_1 = \sqrt{2}$ instead. For θ_2, it follows from the proof that $\theta_2 = 2$. For θ_4, since $Y(I_N \otimes EW)X = X^2$, it can be checked that taking $\theta_4 = 1/2$ leads to positivity of the term 2. Finally, one may ignore θ_3 when choosing θ^* since it is only used in obtaining the constant ρ_1, which does not affect the convergence rate. (In this case, the only difference is that the gain in (7.3.18) may be larger than the estimation given in the Theorem 7.3.3.) Thus, we obtain $\theta^* = \min(\sqrt{2}, 2, 1/2) = 1/2$. This means that k_{I} (and γ) can be simply chosen as $k_{\mathrm{I}} = \gamma = k_{\mathrm{p}}/2$. ♢

Remark 7.3.4. The convergence of the proposed algorithm may resemble the results of [LS20b, Thm. 3]. In particular, define state of each agent can be augmented with the state of the integrator ξ_i as $\chi_i := [x_i; \xi_i]$ and write it as

$$\dot{\chi}_i = F_i(\chi_i) + \begin{bmatrix} k_{\mathrm{p}} I_n & k_{\mathrm{I}} I_n \\ -\gamma I_n & 0 \end{bmatrix} \sum_{j\in\mathcal{N}_i} (\chi_j - \chi_i) =: F_i(\chi_i) + B \sum_{j\in\mathcal{N}_i} (\chi_j - \chi_i).$$

Then, it can be checked $\mathrm{Re}\,(\lambda_i(B)) > 0$. Therefore, one may apply the transformation to diagonalize B such that the coupling matrix becomes a symmetric positive definite matrix as studied in [LS20b]. Nonetheless, there are some key differences that prohibit the usage of results from [LS20b] directly. First, the coupling matrix is not in form of kB, where k is some common coupling gain. Specifically, each entry of B has different gains, e.g., $k_{\mathrm{p}}, k_{\mathrm{I}}$, and γ that can be controlled independently. More importantly, the classical definition of blended dynamics (which may be obtained assuming $k_{\mathrm{p}} = k_{\mathrm{I}} = \gamma = k$ and require high integral gain) will be given by

$$\begin{bmatrix} \dot{s}_x \\ \dot{s}_\xi \end{bmatrix} = \begin{bmatrix} \frac{1}{N}\sum_{i=1}^N h_i(s_x) \\ 0 \end{bmatrix}.$$

Hence, it does not contain a compact invariant set. The set is stable for any value of s_ξ, making it unbounded. This limits the usage of [LS20b] due to technical reasons. Finally, the work of [LS20b] only provides general sufficient conditions for asymptotic convergence and did not propose a systematic design method to satisfy such assumptions. In this sense, it can be seen that we have explicitly constructed an appropriate controller to achieve asymptotic consensus and characterize its trajectories. $\Diamond$

7.3.3 Convergence under Fast Switching

Previous section studied the convergence when communication topology is time-invariant. In this section, convergence is established for the switching network. For this, suppose that the dynamics of each agent is given by a time-varying function $h_i(t, x_i)$ and the communication network is described by

$$\mathfrak{L}_{\mathrm{p}}(t) := \sum_{l=1}^{L_{\mathrm{p}}} \mathfrak{L}^l(t) \otimes (C_{\mathrm{p}}^l)^\top C_{\mathrm{p}}^l,$$

while $\mathfrak{D}_{\mathrm{I}}(t)$ is defined similarly (for simplicity, we suppose that the input (7.2.2) is used). Then, the overall dynamics (7.2.4) can be written as

$$\begin{aligned} \dot{x} &= h(t/\theta, x) - k_{\mathrm{p}}(I_N \otimes E)^\top \mathfrak{L}_{\mathrm{p}}(t/\theta)(I_N \otimes E)x - k_{\mathrm{I}}(I_N \otimes E)^\top \mathfrak{D}_{\mathrm{I}}(t/\theta)\xi \\ \dot{\xi} &= \gamma(I_N \otimes P)\mathfrak{L}_{\mathrm{p}}(t/\theta)(I_N \otimes E)x, \end{aligned} \tag{7.3.19}$$

where $\theta > 0$ is a small parameter representing the fast switching, i.e., the dynamics is switched more frequently as θ approaches 0. Note that (7.3.19) is exactly the (7.2.4) with the additional properties that h_i and graphs are time-varying. We make following assumptions regarding the time-varying nature of the system.

Assumption 7.3.3. The time-varying functions $h_i(t, x_i)$ and $\mathfrak{L}^l(t)$ are piecewise continuous[3] and periodic with period $\delta > 0$. In addition, $h_i(t, x_i)$ is uniformly globally Lipschitz. $\Diamond$

Since time-varying functions are assumed to be periodic, define the time average of $\mathfrak{L}_{\mathrm{P}}(t), \mathfrak{D}_{\mathrm{I}}(t)$ and $h_i(t, x_i)$ as $\bar{\mathfrak{L}}_{\mathrm{P}} := (1/\delta) \int_0^\delta \sum_{l=1}^{L_{\mathrm{P}}} \mathfrak{L}_{\mathrm{P}}^l(\tau) \otimes (C_{\mathrm{P}}^l)^\top C_{\mathrm{P}}^l d\tau$, $\bar{\mathfrak{D}}_{\mathrm{I}} := (1/\delta) \int_0^\delta \sum_{l=1}^{L_{\mathrm{I}}} \mathfrak{L}_{\mathrm{I}}^l(\tau) \otimes (C_{\mathrm{I}}^l)^\top C_{\mathrm{I}}^l d\tau$, and $\bar{h}_i(x_i) := (1/\delta) \int_0^\delta h_i(\tau, x_i) d\tau$.

Then the following assumptions similar to Assumptions 7.2.1 and 7.2.2 are made for the averaged system.

Assumption 7.3.4. Suppose that $\lambda_{q+1}\left(\bar{\mathfrak{L}}_{\mathrm{P}}\right) > 0$ and $\lambda_{q+1}\left(\bar{\mathfrak{D}}_{\mathrm{I}}\right) > 0$. $\Diamond$

Assumption 7.3.5. The system given by

$$\begin{aligned} \dot{z} &= (I_N \otimes Z)^\top \bar{h}\Big((I_N \otimes Z)z + (1_N \otimes W)\bar{w}\Big), \\ \dot{\bar{w}} &= \frac{1}{N}(1_N \otimes W)^\top \bar{h}\Big((I_N \otimes Z)z + (1_N \otimes W)\bar{w}\Big), \end{aligned} \tag{7.3.20}$$

where $\bar{h}(x) := [\bar{h}_1(x_1); \cdots ; \bar{h}_N(x_N)]$, has a unique globally exponentially stable equilibrium point $(z^*, \bar{w}^*)$. Also, there exists a Lyapunov function $\bar{V}(\delta)$ such that $c_1|\delta|^2 \le \bar{V}(\delta) \le c_2|\delta|^2$,

$$\frac{\partial \bar{V}}{\partial \delta} \cdot \begin{bmatrix} (I_N \otimes Z)^\top \bar{h}\Big((I_N \otimes Z)z + (1_N \otimes W)\bar{w}\Big) \\ \frac{1}{N}(1_N \otimes W)^\top \bar{h}\Big((I_N \otimes Z)z + (1_N \otimes W)\bar{w}\Big) \end{bmatrix} \le -\nu|\delta|^2,$$

for some $\nu > 0$, and $\partial\bar{V}/\partial\delta$ is globally Lipschitz. Finally, $h(t, x^*) = 0$ for all $t \ge 0$ where $x^* = (I_N \otimes Z)z^* + (1_N \otimes W)\bar{w}^*$. $\Diamond$

Remark 7.3.5. Assumption 7.3.5 supposes global exponential stability of the blended dynamics of the averaged system, which is analogous to Assumption 7.2.2 ans its exponential stability. In addition to the stability, Assumption 7.3.5 requires two additional property: 1) x^* to be an equilibrium point of the time-varying system and 2) existence of the Lyapunov function and Requiring x^* to be

[3] We say a function $f(t)$ is piecewise continuous on an interval $[a, b]$ if there exists a finite number of points t_i satisfying $a = t_0 < t_1 < \cdots < t_n = b$ so that 1) $f(t)$ is continuous on each subinterval (t_{i-1}, t_i) for all $i = 0, \ldots, n$ and 2) $\lim_{t\to t_0^+} f(t)$, $\lim_{t\to t_n^-} f(t)$, $\lim_{t\to t_i^-} f(t)$ and $\lim_{t\to t_i^+} f(t)$ are finite for all $i = 1, \ldots, n-1$. If $f(t)$ is periodic in addition to this, then $f(t)$ has a finite dwell time.

an equilibrium point for the time-varying system is typical assumption used for the analysis of the averaged system and it is needed to establish asymptotic convergence under switching topology. Existence of the Lyapunov function satisfying all the conditions is a restrictive assumption. One may use the converse Lyapunov theorem to obtain the Lyapunov function with quadratic upper/under bounds and negativity along the dynamics, but such Lyapunov function does not guarantee $\partial \bar{V}/\partial \delta$ to be globally Lipschitz. (In fact, typical converse Lyapunov theorem only guarantees $|\partial \bar{V}/\partial \delta| \le c_4 |\delta|$.) Hence, one either need to find a Lyapunov function where $\partial \bar{V}/\partial \delta$ is globally Lipschitz, or impose additional assumptions such as boundedness of partial derivatives of dynamics up to second order which is used in [AP99, Thm. 3]. Unfortunately, approach of [AP99] only results in semi-global convergence (even when the required conditions hold globally) which hinders the development of optimization algorithms. $\Diamond$

Then the following proposition stating the convergence under fast switching easily follows from [AP99].

Proposition 7.3.4. Consider the time-varying dynamics (7.3.19). Suppose that Assumptions 7.3.3, 7.3.4 and 7.3.5 hold. Then, there exists $k_{\rm p}^* > 0$ and $\theta^* > 0$ such that for all $k_{\rm p} > k_{\rm p}^*$ and $0 < \theta < \theta^*$, (7.3.19) converges exponentially. $\Diamond$

Proof. For the proof, we follow the arguments of [AP99]. Let the Lyapunov function be

$$V(e) := \eta \bar{V}(\delta) + \frac{1}{2} \begin{bmatrix} \tilde{w} \\ \delta\tilde{\xi} \end{bmatrix}^\top \begin{bmatrix} aX & \epsilon Y \\ \epsilon Y^\top & b\Lambda_{\rm I} \end{bmatrix} \begin{bmatrix} \tilde{w} \\ \delta\tilde{\xi} \end{bmatrix}$$

where $\bar{V}$ is from the assumption. Also recall that this Lyapunov function is identical to the one used in Theorem 7.3.2.

Let $p := [\delta z, \delta \bar{w}; \tilde{w}; \delta\tilde{\xi}]$ and ψ such that $\dot{p} = \psi(t,p)$. Then, we claim that there exists $\nu > 0$ such that the Lyapunov function satisfies

$$\frac{\partial V}{\partial p}(p) \cdot \int_{t^*}^{t^*+\delta} \psi(\tau, p)\, d\tau \le -\nu |p|^2 \tag{7.3.21}$$

for all $t^* \geq 0$ and $p \in \mathbb{R}^{Nn-q}$. Since we have

$$\int_{t^*}^{t^*+\delta} \psi\,(\tau, p)\, d\tau = \delta \cdot \bar{\psi}(p),$$

it follows from the proof of Theorem 7.3.1 that (7.3.21) holds with $\nu := \delta\lambda_{\min}(\Xi)$. Hence, applying [AP99, Thm. 2], there exists θ^* such that origin of (7.3.19) is exponentially stable.

For the convergence rate, we obtain from [AP99, Equation (28)] and the proof of Theorem 7.3.3 that

$$V(n\theta\delta) - V((n-1)\theta\delta) \leq -r\theta\delta V((n-1)\theta\delta) - \theta\delta\lambda_{\min}(\Xi)|p|^2 + \theta^2 L(\theta)|p|^2$$

for all $n = 1, 2, \ldots$, where $L(\theta)$ is defined as

$$L(\theta) := 2d_1 e^{d_2\delta\theta}\delta^2 + d_3 e^{2d_4\theta\delta}\theta\delta^3.$$

for some positive constants $d_i > 0$ related to Lipschitz constants of dynamics and $\partial V/\partial p$. Hence, θ^* can be found such that

$$V(n\theta\delta) \leq (1 - r\theta\delta)\, V((n-1)\theta\delta) \tag{7.3.22}$$

for all $0 < \theta < \theta^*$, where $1 - r\theta^*\delta < 1$.

On the other hand, since the overall dynamics is globally Lipschitz, we have

$$V(t) \leq \bar{c}_2|p(t)|^2 \leq \bar{c}_2 e^{2L'\theta\delta}|p(0)|^2, \quad \forall t \in [0, \theta\delta]. \tag{7.3.23}$$

where L' is the Lipschitz constant of the overall dynamics and $\bar{c}_2$ is such that $V \leq \bar{c}_2|p|^2$. Then, combining (7.3.22) and (7.3.23) and referring to Fig. 7.1, we obtain

$$V(t) \leq \bar{c}_2 e^{2L'\theta\delta} \cdot \frac{1}{1 - r\theta\delta}|p(0)|^2 \cdot e^{-r't}$$

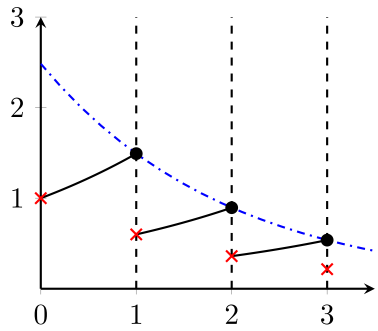


Figure 7.1: Graph of the values of the Lyapunov function $V(t)$. Red marks the value of V given by (7.3.22) and black circle denotes the value of $V(t)$ given by (7.3.23). Blue dash dotted line gives the upper bound of the $V(t)$ over all time.

where the rate $r' > 0$ is defined as

$$e^{-r'\theta\delta} = 1 - r\theta\delta \implies r' = -\frac{\ln(1 - r\theta\delta)}{\theta\delta}.$$

It can be verified that

$$-\frac{\ln(1 - r\theta\delta)}{\theta\delta} \geq \frac{r^2}{2}\theta\delta + r \geq r.$$

Hence, we obtain the convergence rate of r for V when θ is sufficiently small. $\square$

Remark 7.3.6. For the estimate of θ^*, let K_V and K_ψ be the Lipschitz constant of $\partial V(e)/\partial e$ and $\psi(t, e)$ respectively, where K_V is independent of k_p. Then, it follows from [AP99, Remark 4] that θ must satisfy

$$\delta\theta e^{K_\psi T\theta} \leq \frac{1}{K_\psi}\left(-1 + \sqrt{1 + \frac{\lambda_{\min}(\Xi)}{K_V K_\psi \delta}}\right) =: \alpha. \tag{7.3.24}$$

Taking logarithms on (7.3.24) and using the fact that $\ln(x) \geq 1 - (1/x)$, it can be shown that (7.3.24) holds if

$$\theta < \frac{\ln(\alpha/\delta) - 1 + \sqrt{(\ln(\delta/\alpha) + 1)^2 + 4K_\psi\delta}}{2K_\psi\delta} =: \bar{\theta}^*.$$

Hence, $\bar{\theta}^*$ is a conservative estimate of θ^*. ◊

Remark 7.3.7. Let $r'(r, \theta\delta) = -\ln(1 - r\theta\delta)/\theta\delta$ be the convergence rate of the switching system. Then for fixed r,

$$\lim_{\theta \cdot \delta \to 0} r'(r, \theta\delta) = r,$$

which means that it recovers the convergence rate of the averaged system. Similar to the time-invariant system, the rate r can be made arbitrarily close to the convergence rate of the blended dynamics of the time-averaged system. However, this means $\lambda_{\min}(\Xi)$ is smaller, leading to even smaller $\theta\delta$ for the convergence. Nonetheless, the desired convergence rate is still obtained. ◊

7.4 Construction of Distributed Algorithms

In this section, we present how results from Section 7.3 can be used to obtain distributed algorithms to solve (7.1.1) and propose novel distributed algorithms using the heavy-ball method. For this, suppose throughout this section that the communication network is given by a connected graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ whose Laplacian matrix is given by $\mathfrak{L}$ and $F(x)$ is α-strongly convex.

7.4.1 Distributed Gradient Descent Method

First, recall that the typical PI algorithm is given by

$$\begin{aligned} \dot{x}_i &= -\nabla f_i(x_i) + k_{\mathrm{P}} \sum_{j \in \mathcal{N}_i} (x_j - x_i) + k_{\mathrm{I}} \sum_{j \in \mathcal{N}_i} (\xi_j - \xi_i) \\ \dot{\xi}_i &= -\gamma \sum_{j \in \mathcal{N}_i} (x_j - x_i). \end{aligned} \tag{7.4.1}$$

In particular, it can be seen that $\zeta_i = x_i$ and hence it follows that $E = I_n$, $W = I_n$ and Z is null. Then, it can be verified that the blended dynamics of (7.4.1) becomes

$$\dot{\bar{x}} = -\frac{1}{N} \sum_{i=1}^{N} \nabla f_i(\bar{x}). \tag{7.4.2}$$

Notice that (7.4.2) is the centralized gradient descent algorithm to (7.1.1). Therefore, if $F(x) := (1/N)\sum_{i=1}^{N} f_i(x)$ is strongly convex, then the blended dynamics has an exponentially stable equilibrium point at w^*. In fact, it can be checked that $x_i \to w^*$ follows from Theorem 7.3.2 (with sufficiently high gain k_p). Note that convexity of f_i or incremental passivity of h_i is not needed, and only the property of its average is used. The convergence result is stated below.

Corollary 7.4.1. (Distributed Gradient Algorithm) Consider the distributed PI algorithm based on the gradient descent algorithm (7.4.1) and $0 < \upsilon < \alpha$. Suppose that the gains are designed as in Theorem 7.3.3 with k_p being sufficiently large. Then there exists $M > 0$ and $r > 0$ such that

$$\lim_{t\to\infty} |x_i(t) - w^*| \leq Me^{-(\alpha-\upsilon)t}.$$

For instance, the rate becomes $\alpha/2$ if $\upsilon = \alpha/2$. ◊

Proof. Let Lyapunov function for the blended dynamics (7.4.2) be defined as

$$\bar{V}(\bar{x}) = \frac{1}{2}(\bar{x} - w^*)^\top(\bar{x} - w^*).$$

Using $\bar{V}$ and following the proof of Theorem 7.3.3, it follows that

$$\dot{V} \leq -rV,$$

where $r > 0$ is given by

$$r = \min\left(2\left(\mu - \upsilon\right), \frac{a}{2\rho_2}k_\mathrm{p}\lambda_{\min}(\Lambda_\mathrm{p})^2\sigma_{\min}(E)^4, \frac{1}{2\rho_2}(\theta^*)^2k_\mathrm{p}\lambda_{\min}(\Lambda_\mathrm{I})^2\sigma_{\min}(E)^2\right).$$

Hence, with sufficiently large k_p, convergence rate becomes $\mu - \upsilon = \alpha - \upsilon$ since the blended dynamics (i.e., the gradient descent algorithm) converges with the rate α, which follows from Lemma 2.3.4.

To compute the constant M, let R_i be the i-th row of R. Then,

$$\begin{aligned}|x_i(t) - w^*|^2 = |\bar{x} + R_i\tilde{x} - w^*|^2 &\leq 2|\bar{x} - w^*|^2 + 2|R_i|^2|\tilde{x}|^2 \\ &\leq 2\left(|\bar{x} - w^*|^2 + |\tilde{x}|^2\right)\end{aligned}$$

$$\le \frac{2}{c}V(t)$$
$$\le \frac{2}{c}V(0)e^{-rt},$$

where $c > 0$ is such that

$$c\left(|\bar{x} - w^*|^2 + |\tilde{w}|^2 + |\delta\tilde{\xi}|^2\right) \le V.$$

Hence, M is given by

$$M := \sqrt{\frac{2}{c}V(0)}. \qquad \square$$

7.4.2 Distributed Heavy-ball Method

The argument used to obtain the convergence of (7.4.1) can be done in reverse to obtain the distributed algorithm from a centralized algorithm. By setting the blended dynamics as the desired centralized algorithm, Theorem 7.3.2 provides a constructive method to obtain a distributed algorithm. This approach is not limited to the gradient descent method. Suppose that we desire a heavy-ball method to solve (7.1.1) [Sie19], which is given by

$$\dot{\bar{w}} = -z, \quad \dot{z} = -2\sqrt{\alpha}z - \frac{1}{N}\sum_{i=1}^{N}\nabla f_i(\bar{w}). \tag{7.4.3}$$

Since $z = (1/N)\sum_{i=1}^{N} z$, it follows that (7.4.3) is the blended dynamics of

$$\begin{bmatrix}\dot{w}_i\\ \dot{z}_i\end{bmatrix} = \begin{bmatrix}-z_i\\ -2\sqrt{\alpha}z_i - \nabla f_i(w_i)\end{bmatrix} + k_{\mathrm{p}}\sum_{j\in\mathcal{N}_i}\left(\begin{bmatrix}w_j\\ z_j\end{bmatrix} - \begin{bmatrix}w_i\\ z_i\end{bmatrix}\right)$$
$$+ k_{\mathrm{I}}\sum_{j\in\mathcal{N}_i}(\xi_j - \xi_i) \tag{7.4.4a}$$

$$\dot{\xi}_i = -\gamma\sum_{j\in\mathcal{N}_i}\left(\begin{bmatrix}w_j\\ z_j\end{bmatrix} - \begin{bmatrix}w_i\\ z_i\end{bmatrix}\right) \tag{7.4.4b}$$

with $E = I_{2n}$. Then using the similar argument as in the gradient descent method, it can be shown that $w_i(t)$ and $z_i(t)$ of (7.4.4) converges to the equilibrium point of (7.4.3), i.e., $w_i(t) \to w^*$ and $z_i(t) \to 0$. In particular, $V(\bar{w}, z) = F(\bar{w}) - F(w^*) + (1/2)|\sqrt{\alpha}(\bar{w} - w^*) + z|^2$ is a Lyapunov function for (7.4.3) and satisfies $\dot{V} \leq -\sqrt{\alpha}V$ [Sie19]. This implies that the convergence rate of the distributed algorithm (7.4.4) is also proportional to $\sqrt{\alpha}$ (instead of being proportional to α as in (7.4.1)). The convergence result is stated below.

Corollary 7.4.2. (Distributed Heavy-ball Algorithm with State Communication) Consider the distributed algorithm given by (7.4.4) and $0 < \upsilon < \sqrt{\alpha}/2$. Suppose that the gains are designed as in Theorem 7.3.3 with k_{p} being sufficiently large. Then there exists $M > 0$ and $r > 0$ such that

$$\lim_{t\to\infty} |x_i(t) - w^*| \leq M e^{-(\frac{\sqrt{\alpha}}{2} - \upsilon)t}.$$

For instance, if $\upsilon = \sqrt{\alpha}/4$, then the rate becomes $\sqrt{\alpha}/4$. ♢

Proof. Similar to the proof of Corollary 7.4.1, recall from Theorem 2.3.6 that the blended dynamics is exponentially stable with rate $\sqrt{\alpha}/2$. Thus, we obtain

$$\dot{V} \leq -rV,$$

where $r > 0$ is given by

$$\begin{aligned} r &= \min\left(2\left(\mu - \upsilon\right), \frac{a}{2\rho_2} k_{\mathrm{p}} \lambda_{\min}(\Lambda_{\mathrm{p}})^2 \sigma_{\min}(E)^4, \frac{1}{2\rho_2}(\theta^*)^2 k_{\mathrm{p}} \lambda_{\min}(\Lambda_{\mathrm{I}})^2 \sigma_{\min}(E)^2\right) \\ &= 2(\mu - \upsilon), \end{aligned}$$

where the last equality holds for suitably chosen gains $k_{\mathrm{p}}, k_{\mathrm{I}}$ and γ. Also recall that the blended dynamics is a centralized heavy-ball method. Hence, $\mu = \sqrt{\alpha}/2$ follows from Theorem 2.3.6. Constant M can be computed using similar arguments as in the proof of Corollary 7.4.1. □

The algorithm (7.4.4) achieves state consensus of $x_i = [w_i; z_i]$ since $E = I_{2n}$. However, since z_i is auxiliary variable added for the performance, communication of z_i may not be necessary. Motivated by this, let $E = [I_n \; 0]$ which results in

$$\begin{bmatrix} \dot{w}_i \\ \dot{z}_i \end{bmatrix} = \begin{bmatrix} -z_i \\ -2\sqrt{\alpha} z_i - \nabla f_i(w_i) \end{bmatrix} + k_{\mathrm{p}} \begin{bmatrix} I_n \\ 0 \end{bmatrix} \sum_{j \in \mathcal{N}_i} (w_j - w_i) + k_{\mathrm{I}} \begin{bmatrix} I_n \\ 0 \end{bmatrix} \sum_{j \in \mathcal{N}_i} (\xi_j - \xi_i) \tag{7.4.5a}$$

$$\dot{\xi}_i = -\gamma \sum_{j \in \mathcal{N}_i} (w_j - w_i) . \tag{7.4.5b}$$

The algorithm (7.4.5) only communicates $\zeta_i = w_i$ (and its integral) and applies control input only to w_i. It can be checked that the blended dynamics of (7.4.5) becomes

$$\dot{\bar{w}} = -\frac{1}{N} \sum_{i=1}^{N} z_i, \quad \dot{z}_i = -2\sqrt{\alpha} z_i - \nabla f_i(\bar{w}) \tag{7.4.6}$$

for all $i \in \mathcal{N}$, which is a $(N+1)n$-dimensional system. Equilibrium point of (7.4.6) is given by

$$(w^*, -\tfrac{1}{2\sqrt{\alpha}} \nabla f_1(w^*), \dots, -\tfrac{1}{2\sqrt{\alpha}} \nabla f_N(w^*))$$

and exponential convergence of the equilibrium point as well as the convergence rate proportional to $\sqrt{\alpha}$ is shown below.

Lemma 7.4.3. (Exponential Convergence of the Reduced System with Specific Parameters) Consider the system given by

$$\begin{aligned} \dot{z}_i &= -2\sqrt{\alpha} z_i - \nabla f_i(w), \\ \dot{w} &= \frac{1}{N} \sum_{i=1}^{N} z_i, \end{aligned} \tag{7.4.7}$$

where $\bar{f}(w) := (1/N) \sum_{i=1}^{N} f_i(w)$ is α–strongly convex and $f_i(w)$ is L–smooth for all $i = 1, \dots, N$. Then the solution of (7.4.7) satisfies

$$\lim_{t \to \infty} z_i(t) = -\frac{1}{2\sqrt{\alpha}} \nabla f_i(w^*), \quad \lim_{t \to \infty} w(t) = w^*,$$

and the convergence is exponentially fast. ◊

Proof. Let $\psi := [w; z_1; \cdots ; z_N]$. Similar to the Theorem 2.3.6, consider the function

$$V(\psi) = \frac{1}{N}\sum_{i=1}^{N}\Big(f_i(w) - f_i(w^*)\Big) + \frac{1}{2}\left|\sqrt{\alpha}(w - w^*) + \frac{1}{N}\sum_{i=1}^{N} z_i\right|^2.$$

Let $\bar{z} := (1/N)\sum_{i=1}^{N} z_i$. Then the time-derivative of $V(\psi)$ along (7.4.7) becomes

$$\begin{aligned}\dot{V} &= \frac{1}{N}\sum_{i=1}^{N}\nabla f_i(w)^\top \cdot \bar{z} + \big(\sqrt{\alpha}(w - w^*) + \bar{z}\big)^\top \cdot \big(\sqrt{\alpha}\cdot\bar{z} - 2\sqrt{\alpha}\bar{z} - \nabla\bar{f}(w)\big)\\
&= \nabla\bar{f}(w)^\top \cdot \bar{z} - \alpha(w - w^*)^\top\bar{z} - \sqrt{\alpha}(w - w^*)^\top\nabla\bar{f}(w) - \sqrt{\alpha}\bar{z}^\top\bar{z} - \bar{z}^\top\nabla\bar{f}(w)\\
&= -\alpha(w - w^*)^\top\bar{z} - \sqrt{\alpha}(w - w^*)^\top\nabla\bar{f}(w) - \sqrt{\alpha}\bar{z}^\top\bar{z}.\end{aligned}$$

Here we use the definition of strong convexity (2.3.2) with $y = w^*$ to obtain

$$-\sqrt{\alpha}\nabla\bar{f}(w)^\top(w - w^*) \le -\sqrt{\alpha}\cdot\Big(\bar{f}(w) - \bar{f}(w^*) + \frac{\alpha}{2}|w - w^*|^2\Big).$$

Substituting this, it follows that

$$\begin{aligned}\dot{V} &\le -\alpha(w - w^*)^\top\bar{z} - \sqrt{\alpha}\cdot\Big(\bar{f}(w) - \bar{f}(w^*) + \frac{\alpha}{2}|w - w^*|^2\Big) - \sqrt{\alpha}\bar{z}^\top\bar{z}\\
&= -\sqrt{\alpha}\Big(\bar{f}(w) - \bar{f}(w^*) + \frac{\alpha}{2}|w - w^*|^2 + \sqrt{\alpha}(w - w^*)^\top\bar{z} + \bar{z}^\top\bar{z}\Big)\\
&= -\sqrt{\alpha}\left(\bar{f}(w) - \bar{f}(w^*) + \frac{1}{2}\big|\sqrt{\alpha}(w - w^*) + \bar{z}\big|^2 + \frac{1}{2}\bar{z}^\top\bar{z}\right)\\
&= -\sqrt{\alpha}V(\psi) - \frac{\sqrt{\alpha}}{2}\bar{z}^\top\bar{z}.\end{aligned}$$

Therefore, we obtain

$$V(\psi(t)) \le e^{-\sqrt{\alpha}t}V(\psi(0)).$$

In fact, regarding $V(\psi)$ as $V(w, \bar{z})$, its structure is identical to the one used for the proof of Theorem 2.3.6. Hence, there exists $M > 0$ such that

$$\left|\begin{bmatrix} w(t) - w^* \\ \bar{z}(t)\end{bmatrix}\right|^2 \le Me^{-\sqrt{\alpha}t}\cdot\left|\begin{bmatrix} w(0) - w^* \\ \bar{z}(0)\end{bmatrix}\right|^2 \tag{7.4.8}$$

Rest of the proof now shows similar convergence property holds for $\psi - \psi^* := [w - w^*; z_1 - z_1^*; \cdots ; z_N - z_N^*]$.

First note that

$$\begin{aligned} |\bar{z}|^2 = \frac{1}{N^2}\left|\sum_{i=1}^N z_i - z_i^* + z_i^*\right|^2 &= \frac{1}{N^2}\left|\sum_{i=1}^N z_i - z_i^* + \sum_{i=1}^N z_i^*\right|^2 \\ &\leq \frac{1}{N^2} \cdot N \cdot \sum_{i=1}^N |z_i - z_i^*|^2 \\ &= \frac{1}{N}\sum_{i=1}^N |z_i - z_i^*|^2 \end{aligned}$$

where we used the fact that $\sum_{i=1}^N z_i^* = -1/(2\sqrt{\alpha}) \cdot \sum_{i=1}^N \nabla f_i(w^*) = 0$ and $|\sum z_i - z_i^*|^2 \leq N \sum |z_i - z_i^*|^2$. Therefore, we have

$$\left|\begin{bmatrix} w - w^* \\ \bar{z} \end{bmatrix}\right|^2 \leq \left|\begin{bmatrix} w - w^* \\ \frac{1}{\sqrt{N}}(z_1 - z_1^*) \\ \vdots \\ \frac{1}{\sqrt{N}}(z_N - z_N^*) \end{bmatrix}\right|^2 \leq \left|\begin{bmatrix} w - w^* \\ z_1 - z_1^* \\ \vdots \\ z_N - z_N^* \end{bmatrix}\right|^2, \tag{7.4.9}$$

where the second inequality holds since $N \geq 1$. Combining (7.4.8) with (7.4.9),

$$|w(t) - w^*|^2 \leq \left|\begin{bmatrix} w(t) - w^* \\ \bar{z}(t) \end{bmatrix}\right|^2 \leq M e^{-\sqrt{\alpha}t} \cdot |\psi(0)|^2.$$

Thus, $|w(t) - w^*|$ converges exponentially fast with the rate $-\sqrt{\alpha}/2$, while its coefficient depending on $w(0)$ and $z_i(0)$.

Now to show z_i converges exponentially, let $e_i := z_i - z_i^*$. Then, its dynamics is given by

$$\begin{aligned} \dot{e}_i &= -2\sqrt{\alpha} z_i - \nabla f_i(w) \\ &= -2\sqrt{\alpha} e - 2\sqrt{\alpha} z_i^* - \nabla f_i(w) \\ &= -2\sqrt{\alpha} e + \nabla f_i(w^*) - \nabla f_i(w). \end{aligned}$$

Since f_i is L–smooth, it holds that

$$|\nabla f_i(w(t)) - \nabla f_i(w^*)| \leq L|w(t) - w^*| \leq L\sqrt{M}e^{-\sqrt{\alpha}/2t}|\psi(0)|.$$

Since dynamics of e_i is Hurwitz with exponentially decaying input $u(t)$, it follows from Lemma A.1.2 that $e(t)$ also decays exponentially fast, i.e., z_i converges to z_i^* exponentially fast with rate $\sqrt{\alpha}/2$. □

Remark 7.4.1. Similar to that of Theorem 2.3.6, the algorithm given by (7.4.7) achieves convergence rate proportional to $\sqrt{\alpha}$. This implies that the rate is improved over the traditional centralized gradient descent method when α is small.◊

Since the blended dynamics is exponentially stable, it follows from Theorem 7.3.2 that $w_i \to w^*$ holds for (7.4.5). Note that (7.4.5) achieves same convergence rate as (7.4.4), while only communicating w_i and its integral. Its convergence is stated below.

Corollary 7.4.4. (Distributed Heavy-ball Algorithm with Output Communication) Consider the distributed algorithm given by (7.4.5) and $0 < \upsilon < \sqrt{\alpha}/2$. Suppose that the gains are designed as in Theorem 7.3.3 with k_p being sufficiently large. Then there exists $M > 0$ and $r > 0$ such that

$$\lim_{t\to\infty} |w_i(t) - w^*| \leq Me^{-(\frac{\sqrt{\alpha}}{2} - \upsilon)t}.$$

For instance, if $\upsilon = \sqrt{\alpha}/4$, then the convergence rate becomes $\sqrt{\alpha}/4$. ◊

Proof. Since the blended dynamics of (7.4.5) is exponentially stable with convergence rate $\sqrt{\alpha}/2$ as shown in Lemma 7.4.3, we may follow the proof of Theorem 7.3.3 to obtain

$$\dot{V} \leq -rV,$$

where $r > 0$ is given by

$$r = \min\left(2\left(\mu - \upsilon\right), \frac{a}{2\rho_2}k_\mathrm{p}\lambda_{\min}(\Lambda_\mathrm{p})^2\sigma_{\min}(E)^4, \frac{1}{2\rho_2}(\theta^*)^2k_\mathrm{p}\lambda_{\min}(\Lambda_\mathrm{I})^2\sigma_{\min}(E)^2\right).$$

Therefore, we obtain $\sqrt{\alpha}/2 - \upsilon$ with suitably chosen gains. $\square$

Finally, recall that the algorithm (7.4.5) can be also implemented without communicating ξ_i if the specific initialization is used if (7.2.3) is used. In particular, the algorithm given by

$$\begin{bmatrix} \dot{w}_i \\ \dot{z}_i \end{bmatrix} = \begin{bmatrix} -z_i \\ -2\sqrt{\alpha} z_i - \nabla f_i(w_i) \end{bmatrix} + k_{\mathrm{p}} \begin{bmatrix} I_n \\ 0 \end{bmatrix} \sum_{j \in \mathcal{N}_i} (w_j - w_i) - k_{\mathrm{I}} \begin{bmatrix} I_n \\ 0 \end{bmatrix} \xi_i \quad (7.4.10a)$$

$$\dot{\xi}_i = -\gamma \sum_{j \in \mathcal{N}_i} (w_j - w_i) \,. \quad (7.4.10b)$$

If the initial condition satisfies $\sum_{i=1}^{N} \xi_i(0) = 0$, (7.4.10) achieves the same convergence rate of $\sqrt{\alpha}/2 - \upsilon$ while communicating n-dimensional information.

7.4.3 Distributed Heavy-ball Method with Cyclic Coordinate Descent

Communication load of (7.4.4) can also be reduced by using the cyclic coordinate descent method [Nes12, ST13]. Recall that the centralized cyclic coordinate descent methods cycle through each coordinate axis update the corresponding value. This technique can be also applied to distributed algorithms which can be analyzed using Proposition 7.3.4 and switching network.

For illustration, consider a period $\delta > 0$ let $\delta^l > 0$ be such that $\sum_{l=1}^{L} \delta^l = \delta$. Define the periodic switching signal $\sigma(t) : \mathbb{R} \to \mathcal{L}$ as $\sigma(t) := \pi\left(t - \lfloor t/\delta \rfloor \cdot \delta\right)$, where $\lfloor \cdot \rfloor$ is the floor function and $\pi(t) : [0, \delta) \to \mathcal{L}$ is defined as $\pi(t) := q$ for $q \in \{1, \dots, L\}$ such that $\sum_{l=1}^{q-1} \delta^l \le t < \sum_{l=1}^{q} \delta^l$. In particular, $\sigma(t)$ cycles through 1 to L, spending δ^l amount of time on each mode. Consequently, construct a time-varying multilayer graph $\mathcal{G}(t)$ with L layers, where Laplacian matrices are given by

$$\mathfrak{L}^l(t) := \begin{cases} \mathfrak{L} & \text{if } \pi(t) = l, \\ 0_{N \times N} & \text{otherwise.} \end{cases}$$

The corresponding output matrices $C^l \in \mathbb{R}^{n^l \times n}$ represents coordinate of $[w_i; z_i]$

and satisfies $(1/\delta)\int_0^\delta (C^{\sigma(t)})^\top C^{\sigma(t)} = I_{2n}$. Specifically, we suppose that $(C^l)^\top(C^l)$ is a diagonal matrix with positive diagonal entries representing the subset of coordinate to be updated.

With the communication network constructed as $\mathcal{G}(t)$, we propose modification of (7.4.5) as

$$\begin{aligned}
\begin{bmatrix}\dot{w}_i\\ \dot{z}_i\end{bmatrix} &= (C^{\sigma(t/\theta)})^\top C^{\sigma(t/\theta)}\begin{bmatrix} z_i \\ -2\sqrt{\alpha}z_i - \nabla f_i(w_i)\end{bmatrix}\\
&\quad + k_{\mathrm{p}}\sum_{l\in\mathcal{L}}(C^l)^\top\sum_{j\in\mathcal{N}_i(t/\theta)} C^l\left(\begin{bmatrix}w_j\\ z_j\end{bmatrix} - \begin{bmatrix}w_i\\ z_i\end{bmatrix}\right)\\
&\quad + k_{\mathrm{I}}\sum_{l\in\mathcal{L}}(C^l)^\top\sum_{j\in\mathcal{N}_i(t/\theta)} C^l(\xi_j - \xi_i) \qquad (7.4.11a)\\
\dot{\xi}_i &= -\gamma\left[\sum_{l=1}^{L}(C^l)^\top\sum_{j\in\mathcal{N}_i^l(t)}\left(C^l\begin{bmatrix}w_j\\ z_j\end{bmatrix} - C^l\begin{bmatrix}w_i\\ z_i\end{bmatrix}\right)\right], \qquad (7.4.11b)
\end{aligned}$$

where $\mathcal{N}_i(t)$ is the set of neighbors of agent i at time t and $\theta > 0$ is a constant.

The blended dynamics (7.3.20) of (7.4.11) is identical to (7.4.6) and it can be easily verified that Assumptions 7.3.3 to 7.3.5 hold. Therefore, it follows from Proposition 7.3.4 that the solution of 7.3.4 satisfies $w_i(t) \to w^*$ and $z_i(t) \to 0$ with sufficiently large k_{p} and sufficiently small θ.

Corollary 7.4.5. (Distributed Heavy-ball Algorithm with State Communication and Cyclic Coordinate Descent) Consider the distributed algorithm given by (7.4.11). Suppose that the gains are designed as Theorem 7.3.3. Then, there exists θ^* such that for all $0 < \theta < \theta^*$, it holds that

$$\lim_{t\to\infty} |w_i(t) - w^*| \le Me^{-(r/2)t},$$

where $M(w(0), z(0), \xi(0), \alpha, k_{\mathrm{p}})$ and r is proportional to $\sqrt{\alpha}$. ◊

Proof. Proof follows directly from Proposition 7.3.4. □

The proposed algorithm (7.4.11) implements the heavy-ball method as well as cyclic coordinate descent for each agent and does not require any specific initial condition. For comparison, typical distributed algorithms are compared in

Study	Type	Require convexity of f_i	Arbitrary initialization	Required communication	Convergence rate (strongly convex)
[WE10]	Continuous	Yes	✓	$2n$	Asymptotic
[KCM15]	Continuous	Yes	✗	n	Exponential[4]
[YLW17]	Continuous	Yes	✗	n	Asymptotic
[HCIL18]	Continuous	Yes	✓	$2n$	Asymptotic
[XK19]	Discrete	Yes	✗	$2n$	Exponential[5] $(\mathcal{O}([1 - C(\frac{\alpha}{L})^{5/7}]^t))$
[QL20]	Discrete	Yes	✗	$3n$	Exponential[6]
(7.4.5)	Continuous	No	✓	$2n$	Exponential $(\mathcal{O}(\sqrt{\alpha}))$
(7.4.10)	Continuous	No	✗	n	Exponential $(\mathcal{O}(\sqrt{\alpha}))$

Table 7.1: Table comparing various distributed algorithms and the required assumptions.

[4] Relation between convergence rate with strongly convexity of the function (i.e., α) is not shown explicitly. However, a simple example is presented in [KCM15] which shows the rate of $\mathcal{O}(\alpha)$ with sufficiently high gains.

[5] Here, $C > 0$ is a constant, L is Lipschitz constant of $\nabla F(x)$ and t is the discrete time-step. Also note that an appropriate step size is chosen to achieve the given convergence rate.

[6] Although the exponential convergence is proved, relation between the convergence rate and the condition number is not shown explicitly.

Table 7.1. The classical PI algorithms (e.g., [KCM15, YLW17]) communicate n-dimensional information to its neighbors while requiring a specific initial condition. Algorithms are proposed which does not require a specific initial condition (e.g., [HCIL18, WE10]), but these communicate $2n$-dimensional information. Additionally, most works only prove asymptotic convergence. Authors of [KCM15] prove exponential convergence but accelerated methods are not used. For discrete-time algorithms, the distributed Nesterov method studied in [QL20] communicates $3n$-dimensional information and requires initialization. The distributed heavy-ball method proposed in [XK19] communicates $2n$-dimensional information but still requires a specific initial condition. Additionally, convergence rates of discrete-time algorithms did not match the rate of the corresponding centralized algorithms.

The proposed algorithm (7.4.5) implements the distributed heavy-ball method while only communicating $2n$-dimensional information and converges from any arbitrary initial condition. We also recover the convergence rate of the centralized heavy-ball method, i.e., the accelerated rate of$\sqrt{\alpha}/2$. Furthermore, (7.4.10) is also proposed which achieves the accelerated rate of convergence while only communicating n-dimensional information with the cost of arbitrary initialization. Finally, (7.4.11) implements the heavy-ball method as well as the cyclic coordinate descent. At each time instant, (7.4.11) cycles through different layers and only communicates $2n^l$-dimensional information. The n^l is a design parameter and in extreme case, it can be set to 1 by choosing $C^l = e_l^\top \in \mathbb{R}^{1\times 2n}$ for $l = 1, \ldots, 2n$, where e_l is an elementary vector whose l-th element is 1 and 0 otherwise. This leads to an algorithm only communicating 2-dimensional information.

Although the discussion of this section employed the heavy-ball method, the main results provide a general framework for the construction and analysis of distributed algorithms. In particular, any continuous-time algorithm satisfying the required assumptions (e.g., Assumption 7.3.5) may be used to obtain a similar result. For example, algorithms like accelerated triple momentum algorithm [SGK] can be employed to obtain new distributed algorithm.

Remark 7.4.2. Coordinate descent algorithm based on (7.4.5) can be also obtained in a similar manner. However, for the analysis, we must show Assumption

7.3.5 hold. Although the exponential stability is shown in Lemma 7.4.3, it did not construct the Lyapunov function. Hence, Proposition 7.3.4 cannot be applied. Nonetheless, such design is simulated in Section 7.5 and shown to converge to the optimal solution. $\Diamond$

7.5 Numerical Experiments

For the numerical simulation, consider the distributed quadratic problem with $N = 12$ agents where cost function of each agent is given by $f_i(x) = x^\top A_i x + b_i^\top x$ where $A_i \in \mathbb{R}^{6\times 6}$ is a symmetric matrix and $b_i \in \mathbb{R}^6$. For instance, $f_i(x)$ may represent squared losses for the linear regression problem. It is supposed that $\sum_{i=1}^N A_i$ is positive definite while A_i may be indefinite. The condition number κ of $\sum_{i=1}^N f_i(x)$ is defined as the ratio of the maximum to the minimum eigenvalue of $\sum_{i=1}^N A_i$. For communication network, a random (connected) graph is generated using the Erdős-Rényi model with each edge having a probability of 0.2.

7.5.1 Distributed PI Algorithm

The classical PI algorithm given by

$$\begin{aligned}\dot{x}_i &= -\nabla f_i(x_i) + k_{\mathrm{p}} \sum_{j\in\mathcal{N}_i} (x_j - x_i) + k_{\mathrm{I}} \sum_{j\in\mathcal{N}_i} (\xi_j - \xi_i)\\ \dot{\xi}_i &= -k_{\mathrm{I}} \sum_{j\in\mathcal{N}_i} (x_j - x_i)\end{aligned}$$

is implemented to illustrate the impact of gains on the performance. Simulation result is shown in Fig. 7.2 when $\kappa = 1$. Solid lines denote the performance with $k_{\mathrm{p}} = 1$ and dotted lines denote the performance with $k_{\mathrm{I}} = 1$. The performance of the centralized gradient descent is also plotted with a red dashed line. With $k_{\mathrm{p}} = k_{\mathrm{I}} = 1$ as a basis, observe that as k_{p} is increased, the performance actually decreases and solution converges slowly. This is consistent with the convergence rate obtained in Theorem 7.3.3. On the other hand, if k_{I} is increased instead, we see that the performance is indeed improved. Moreover, performance recovers that of the blended dynamics (i.e., the CGD) when $k_{\mathrm{I}} = 2$.

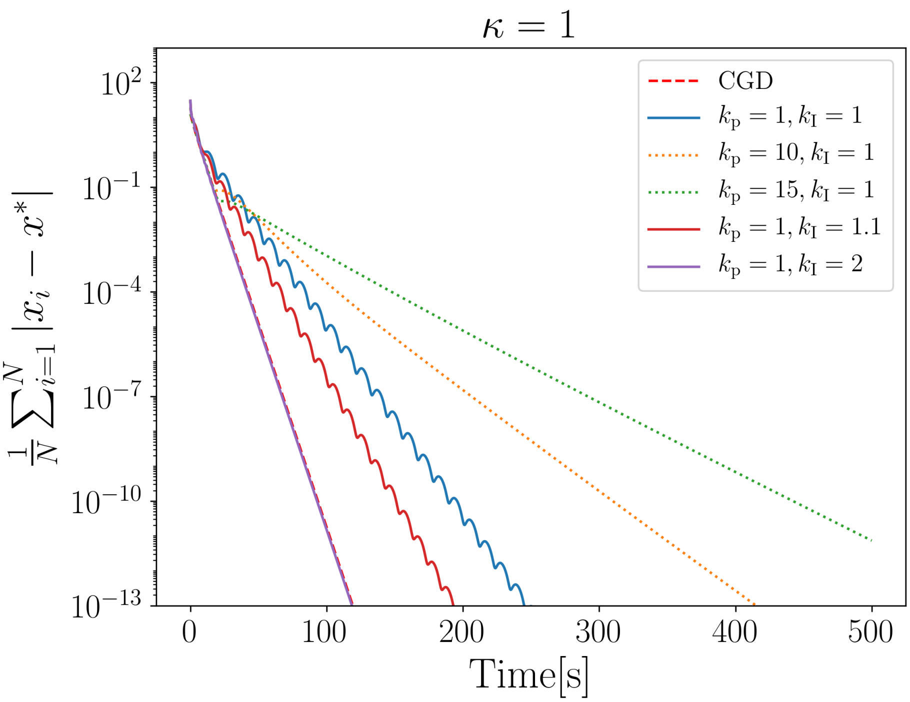


Figure 7.2: Simulation result for PI algorithm with varying gains.

In more detail, it can be seen that performance closely follows that of CGD at first (and stays closer longer for higher k_p) but slows down (and slows down more for higher k_p). This behavior can be explained by the work of [LS20b]. Specifically, it is known that the trajectory of each agent follows that of blended dynamics up to finite time for high k_p. However, fast convergence to the solution of blended dynamics degrades the performance in the long run because the integrator must catch up for asymptotic convergence. For asymptotic convergence, the state of the integrator must converge to a fixed value, which takes longer now since the consensus error is relatively small when k_p is high. As k_I is increased, it speeds up the convergence rate of the integrators which results in faster convergence for the overall system. Such behavior is encoded in the convergence rate derived in Theorem 7.3.3 (although the exact order might be improved).

7.5.2 Distributed Heavy-ball Algorithm

For distributed heavy-ball algorithm, construct the switching multilayer graph as described in Section 7.4.3 with $L = 2$ and C^l given by

$$C^1 = 2\begin{bmatrix} I_3 & 0_{3\times 3} & I_3 & 0_{3\times 3} \end{bmatrix}, \quad C^2 = 2\begin{bmatrix} 0_{3\times 3} & I_3 & 0_{3\times 3} & I_3 \end{bmatrix},$$

with $\delta^l = 2$ or 10 for all $l \in \mathcal{L}$. The dynamics (7.4.11) is implemented with $k_\mathrm{p} = 1$ and $k_\mathrm{I} = 0.1$ or 0.01. Specifically, with communication network and switching matrices defined as above, we have a cyclic coordinate algorithm. Finally, $2\sqrt{\alpha} = 0.01$ is used for the heavy-ball methods. Note that this is not the exact value of the minimum eigenvalue of $\sum_{i=1}^{N} A_i$. Exact value is approximately $2\sqrt{1/\kappa} = 2\sqrt{1/\kappa} \approx 0.07$. However, finding the exact value is challenging in the practice and hence we use the value within the same order of magnitude.

Coordinate descent version of (7.4.5) is also implemented whose dynamics is given by

$$\begin{bmatrix} \dot{w}_i \\ \dot{z}_i \end{bmatrix} = \begin{bmatrix} (C_o^{\sigma(t/\theta)})^\top C_o^{\sigma(t/\theta)} z_i \\ (C_o^{\sigma(t/\theta)})^\top C_o^{\sigma(t/\theta)}(-2\sqrt{\alpha} z_i - \nabla f_i(w_i)) \end{bmatrix} + k_\mathrm{p} \begin{bmatrix} I_n \\ 0 \end{bmatrix} \sum_{l\in\mathcal{L}} (C_o^l)^\top \sum_{j\in\mathcal{N}_i(t/\theta)} C_o^l\,(w_j - w_i) + k_\mathrm{I} \begin{bmatrix} I_n \\ 0 \end{bmatrix} \sum_{j\in\mathcal{L}} (C_o^l)^\top \sum_{j\in\mathcal{N}_i(t/\theta)} C_o^l(\xi_j - \xi_i) \tag{7.5.1a}$$

$$\dot{\xi}_i = -\gamma \left[\sum_{l=1}^{L} (C_o^l)^\top \sum_{j\in\mathcal{N}_i^l(t)} \left(C_o^l w_j - C_o^l w_i \right) \right], \tag{7.5.1b}$$

where C_o^l is the corresponding output matrices.

Results are compared with the following continuous-time algorithms:

1. Centralized gradient descent (CGD)
2. Centralized momentum method (CMM)
3. PI algorithm (PI) [HCIL18]
4. Heavy-ball method with full state communication given in (7.4.4) (HB-

State)

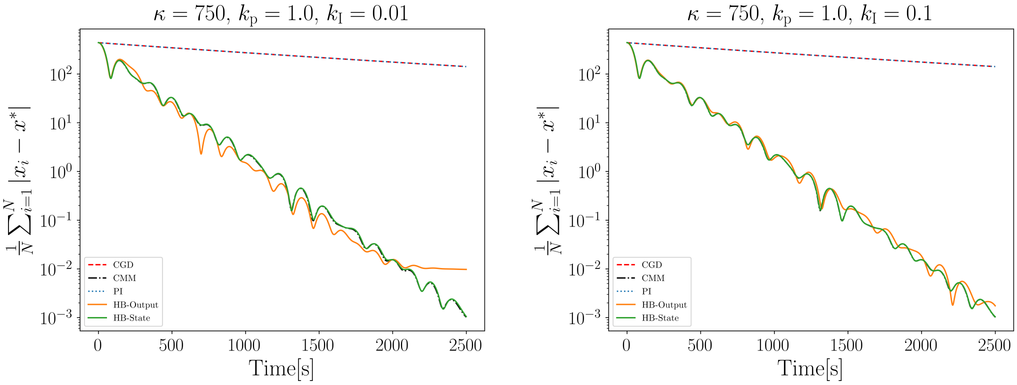
5. Heavy-ball method with full state and switching communication (HB-State (Switching)) given in (7.4.11)

6. Heavy-ball method with output communication given in (7.4.5) (HB-Output)

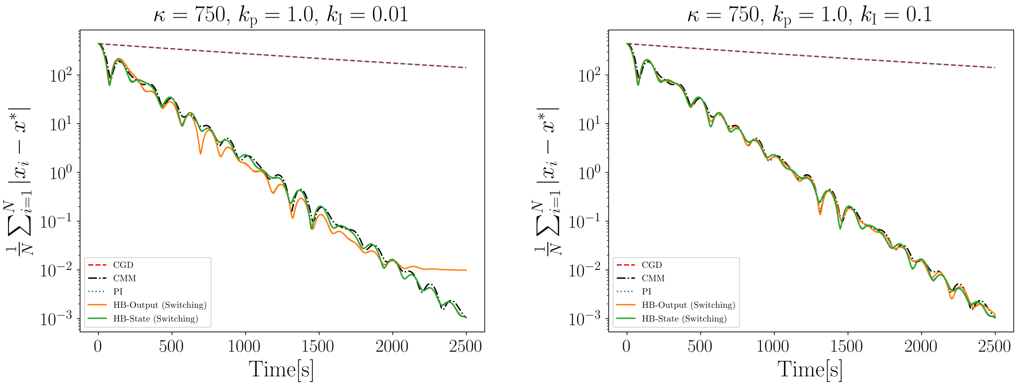
7. Heavy-ball method with output and switching communication (HB-Output (Switching)) given in (7.5.1)

Simulation results are shown in Fig. 7.3 with varying parameters, where distance to the optimal value is plotted in vertical axis (i.e., $(1/N)\sum_{i=1}^{N}|x_i(t)-x^*|$ for distributed algorithms and $|x(t)-x^*|$ for centralized algorithms) and horizontal axis is the time t. Simulation is done in the case when the condition number κ is high to see the effectiveness of the momentum. It can be seen in all cases that the momentum methods outperforms gradient based methods as expected. The proposed algorithm (7.4.5) (in solid orange line) even outperforms the centralized gradient descent algorithm (red dashed line) when $\kappa = 750$. This is because the distributed algorithm follows the trajectory of the centralized momentum method with sufficiently large gains, and hence achieve similar performance of the centralized heavy-ball method. In particular, it can be seen from Fig. 7.3(a) that with sufficiently high $k_{\rm p}$ and $k_{\rm I}$, both HB-Output and HB-State achieves similar performance as the centralized momentum method. Additionally, by analyzing the blended dynamics of respective algorithm, it can be verified that the $x_i(t)$ (the trajectory of the solution) will be identical, which is also reflected on the simulation results.

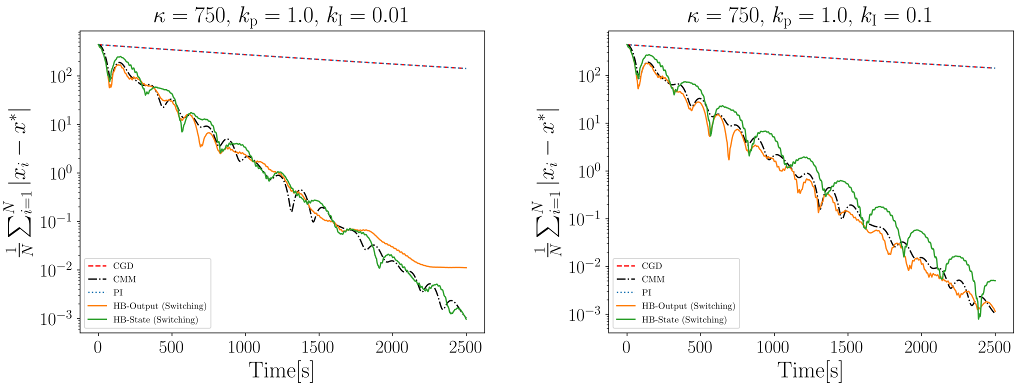
For the coordinate descent algorithms shown in Fig. 7.3(b) and Fig. 7.3(c), only 6-dimensional information (i.e., $C^l w_i \in \mathbb{R}^3$ and $C^l \xi_i \in \mathbb{R}^3$) is communicated at each time instant and only the half elements in vectors w_i and z_i are updated. Nonetheless, Fig. 7.3(b) shows that with sufficiently fast switching time, coordinate descent algorithms and regular algorithms have similar performances. In fact, since the coordinate descent algorithms only require half of the computation and communicational load when compared with (7.4.5), it can operate at twice as fast in theory. This means that the coordinate descent algorithms may achieve



(a) Static network



(b) Switching network with $\delta^l = 2$



(c) Switching network with $\delta^l = 10$

Figure 7.3: Performance of the algorithms with various parameters when $\kappa = 750$.

better performance in practice. Finally, Fig. 7.3(c) shows the simulation results when graph is switched more slowly. We see that the trajectories deviates from the centralized methods, but still achieve comparable performances.

7.6 Remark on the Study of Continuous-time Algorithms

We end this chapter with a brief discussion on the importance of continuous-time optimization algorithms. Traditionally, construction and analysis of the optimization algorithms are done in the discrete-time domain as algorithms must be implemented into computing devices. Centralized algorithms such as gradient descent, proximal operators, Nesterov gradient methods [Ber16, Nes04] are well-studied and have a long history stemming back to the 1960s. Consequently, distributed algorithms are also mainly studied in the discrete-time domain. Meanwhile, analysis of the optimization algorithms in the continuous-time domain has been done (e.g., see [BBB89] and references therein) as well. Such an approach has a long history, but it is recently gaining increased attention with the work of [WSC16] and followed with works such as [WRJ16, MJ19, SDSJ19, SDJS18]. These works mainly studied accelerated gradient methods such as Nesterov's gradient method to provide further insights into the optimization algorithms that are otherwise too hard to express in the discrete-time domain. Moreover, improvements in the continuous-time domain along with the proper discretization of such continuous-time algorithms resulted in various versions of the discrete-time algorithm that can be used [SDSJ19].

In the same philosophy, we believe that the studies of continuous-time distributed algorithms will lead to a similar conclusion, i.e., a better understanding of algorithms, development of improved algorithms, and simpler theoretical analysis of discrete-time algorithms. For example, stochastic discrete-time algorithms can be analyzed by studying its deterministic continuous-time counterpart, which is often easier to analyze (e.g., see [Bor08]) or work such as [CGC17], which analyzed continuous-time saddle-point dynamics using rich theory from the analysis of (continuous-time) nonlinear system.

Another aspect that the analysis of the continuous-time distributed algorithm can also contribute is to provide a common framework to categorize and analyze various discrete-time algorithms. A plethora of discrete-time distributed algorithms is mainly developed independently based on intuition and a combination of different optimization and consensus algorithms. (See [YYW$^+$19] for a survey of various algorithms.) Recently, there were attempts to unify these algorithms such as [Jak19, AS20] (using primal-dual interpretation as discussed in Remark 7.2.2) or [XTSS20] (which uses similar approach along with the operator splitting). However, since these works approached distributed algorithms from a discrete-time perspective, there are limited to the class of algorithms these frameworks can express. These limitations come from the basic update structures these work suppose, which may not apply to all discrete-time algorithms (e.g., see [AS20] which discusses combine-then-adapt and adapt-then-combine schemes). However, using continuous-time algorithms, discrete-time algorithms may be expressed as a combination of a continuous-time dynamical system with a particular discretization method. In this sense, we believe that a unified framework for various distributed algorithms can be developed. Moreover, discretization schemes other than Euler's method with different properties and strengths can be employed to generate new discrete-time algorithms. For example, backward discretization might be used to promote stability, symplectic discretization developed for Hamiltonian systems for accuracy, or higher-order methods can be used to develop different versions of the same continuous-time algorithm. The result of this dissertation does not explore these possibilities. However, these may give more value to the analysis of the continuous-time algorithms provided in this dissertation.

Chapter 8

Conclusions and Further Issues

The study of the multi-agent system has been done extensively for the past few decades, each focusing and extending different aspects of the multi-agent systems. Numerous studies can be categorized based on i) complexity of dynamics of each agent, ii) structure of the communication network and iii) details on the information exchanged between agents. This dissertation builds upon the study of the consensus problem and mainly investigates the novel aspects of information exchange. Specifically, we study the case when different information is exchanged over multiple different communication networks. From hereafter, we summarize the main contributions for each chapter.

The most important result of this dissertation is presented in Chapter 3. We have formulated the consensus problem of multi-agent systems over a multilayer network and studied various properties of the problem. First, motivated by the classical definition of undetectable subspace and by identifying invariant subspace for the consensus problem, we have proposed a novel necessary condition for state consensus problem over undirected multilayer network. A different interpretation of the proposed condition is given such as observability aspect, geometric interpretation, and finally the algebraic condition. Each interpretation extends the previous result for the consensus problem over the single-layer network differently yet it is shown that these conditions are equivalent. Perhaps most importantly (and obviously in hindsight), the proposed condition combines both graph theoretical concepts and system theoretical concepts. Connectivity of the graph, as well as the detectability, are captured together and the interplay between two aspects of the consensus problem is elegantly summarized into a single statement.

Furthermore, another main contribution of Chapter 3 is designing a novel dynamic controller to achieve state consensus using the proposed condition. The proposed design is motivated and generalizes existing designs for the single-layer network. It utilizes a partial observer for each layer and a full-order observer to combine partial information from each layer to obtain relative state information, whose convergence is guaranteed by the proposed necessary condition. In conclusion, we have shown that the proposed condition is necessary and sufficient for the consensusability of the multi-agent systems.

In Chapter 4, results from Chapter 3 is extended to the output consensus problem over directed multilayer networks. Extensions of various aspects of the necessary condition are proposed. Unfortunately, the output consensus over a directed multilayer network is a much more challenging problem. For example, proposed extensions are no longer equivalent and clearly not sufficient as illustrated through various counter-examples. Nonetheless, an appropriate assumption on the dynamics of the agent is proposed to recover equivalence for output consensus problem over an undirected network. Specifically, we have proposed an assumption that essentially requires the plant to have different *modes*. With the assumption, sufficient condition and controller design is proposed for a directed multilayer network. The sufficient condition also embodies a nice physical representation which can be summarized as *each mode of the plant must have a rooted spanning tree*. For an undirected multilayer network, an additional assumption is also investigated which is motivated by the physical interpretation of the corresponding necessary condition obtained from the state consensus problem. The same dynamic controller designed for a directed network is shown to work under different assumptions. Regardless of which assumption to use, equivalence is recovered for undirected output consensus problems.

From Chapter 5 to Chapter 7, various applications of the consensus over multilayer network are presented. In Chapter 5, the distributed state estimation problem is formulated into the consensus problem over a multilayer network. It is shown that the proposed necessary condition generalizes previous conditions reported for the estimation problem. Applying a multilayer network also led to a novel design for a distributed observer with a reduced communication burden. Specifically, a static controller is designed and a notion of the minimal communication burden is proposed and analyzed. Results are further extended to switching networks and estimation is achieved when the communication network as well as the output measurements are switching.

In Chapter 6, the formation control problem is viewed as a consensus problem over a multilayer network. Specifically, we consider a scenario where the desired formation is given by a combination of bearing and relative positional constraints. The dynamic controller proposed in Chapter 3 can be directly applied to achieve the desired formation. The proposed controller is especially useful when the formation wants to scale its overall size and numerical simulations are presented to illustrate the efficacy.

Chapter 7 investigates a slightly different application of the multilayer network. We consider applying the general concept of a multilayer network to the distributed optimization problem. In particular, it is shown that the switching multilayer network may represent the coordinate descent algorithms in the continuous-time domain and related results are presented. This chapter also develops proportional-integral control applied to nonlinear heterogeneous agents using the approach of the blended dynamics, which provides a systematical framework for designing a distributed algorithm from a centralized algorithm. In particular, a novel distributed optimization algorithm based on the centralized heavy-ball method with cyclic coordinate descent is proposed which recovers the convergence rate of $\sqrt{\alpha}$ (when the cost function is α-strongly convex) under appropriate designs.

So far we have summarized a number of novel concepts and designs for the consensus problem over the multilayer network as well as various applications. Surely this does not mean closure, and in fact, opens up many questions for future research.

First and foremost, further study is certainly needed to fully characterize the output consensus problem over a directed network. Finding the general statement without requiring the additional assumptions as well as finding similar intuitive understanding seems to be a interesting topic. For this, it is natural that we need innovations from both the system theoretical side and graph theoretical side of the problem. From the system theoretical aspect, it is expected that the classical theory developed for the linear system will be critical for further development. Innovation from graph theoretical aspect seems more challenging. In order to fully understand the directed multilayer network, I believe the extension (and examination) of results such as [LWHF14] may be fruitful. Secondly, the proposed dynamic controller requires each agent to communicate the state of the controller on the projection graph. Although a similar requirement was necessary for the designs under a single-layer communication network, the work of [SSB09] removed such requirement by using the low-gain approach. The design proposed in [SSB09] is inherently different compared to the observer-based controller that is used in this dissertation. Hence, extending such ideas to a multilayer network seems to be a natural next step (but challenging!). Application to the distributed estimation problem also calls for further investigation. Specifically, the development of a static feedback controller for a more general system (other than marginally stable systems) will directly yield a distributed observer design with reduced communication for a wider range of systems which is more practical than marginally stable systems. In particular, an extension to include integrators is an important open problem.

Additionally, various other aspects of the problem can be improved upon as the control community did for the consensus problem over a single-layer network. For instance, frameworks such as event-triggered systems (e.g., see the survey paper [DHGZ18]), discrete-time systems, or sampled-data systems [YG14] can be applied to a consensus over a multilayer network. Nonlinear systems or heterogeneous agents can also be studied and results of [LS20b, KYS+16] may provide some insights and guidance. Other considerations including but not limited to communication delays, disturbances rejection, cooperative tracking, or optimal control problems are all viable extensions to study the consensus problem over a multilayer network.

Finally, blended dynamics approach to the distributed optimization problem studied in Chapter 7 also has interesting topics to work on. For instance, a systematical analysis of the stability of the blended dynamics needs more work especially when agents are coupled via output. In particular, further analysis may lead to convergence proof for distributed heavy-ball algorithm with output coupling using coordinate descent method. Extension of the proposed approach to the larger class of system is also of interest such that the distributed algorithms and the convergence rates of the corresponding algorithms can be characterized for convex (but not necessarily strongly convex) problems. In this direction, one of the most interesting question is whether the approach can describe the Nesterov gradient method and whether we can develop a distributed Nesterov method.

APPENDIX

A.1 Technical Lemmas

Technical lemmas used throughout the dissertation is presented in this section. First lemma summarizes properties of a symmetric matrix.

Lemma A.1.1. (Property of Positive Semidefinite Matrix) Let $M \in \mathbb{R}^{n\times n}$ be a symmetric matrix. Then the followings are equivalent.

1. $M \geq 0$.

2. $x^\top M x > 0, \quad \forall x \notin \ker M$.

3. $x^\top M x > 0, \quad \forall x \in (\ker M)^\perp$.

4. $\lambda_{d+1}(M) > 0$, where $d := \dim \ker M$. $\Diamond$

Proof. Proof follows easily from the basic definitions. For more details, see [RAH19]. $\square$

It is well known that for stable linear system with exponentially decaying input is stable. For instance, such concept is studied in [Kha02, Chapter 4.9] as input-to-state stability. Following result reiterates some of these findings but provides explicit bound for linear systems.

Lemma A.1.2. (Linear System with Vanishing Input) Consider the linear system given by

$$\dot{x} = Ax + Bu(t)$$

where $u(t)$ is exponentially decaying, i.e., there exists constants $M_u, \lambda_u > 0$ such that

$$|u(t)| \leq M_u e^{-\lambda_u t} |u(0)|$$

and that A is a Hurwitz matrix. Then, there exists $M \in \mathbb{R}_{>0}$ such that

$$|x(t)| \leq M e^{-\alpha t},$$

where $\alpha = (1/2) \min(\lambda_u, \lambda_{\min}(A))$. ◊

Proof. Proof can be found in [Wie10, Lem. B. 1.]. Specifically, the statement is a special case of [Wie10, Lem. B. 1.] when A is Hurwitz. □

Next lemma constructs a dynamics which estimates the relative state difference.

Lemma A.1.3. (Estimating Relative State Difference) Consider a multi-agent system given by

$$\dot{z}_i = A z_i + B u_i, \quad y_i = C z_i, \quad \forall i \in \mathcal{N},$$

where (C, A) is detectable and the communication network consists of $c \geq 1$ independently strongly connected component (iSCC) [Wie10]. Suppose that the dynamics of $\hat{z}_i$ is given by

$$\dot{\hat{z}}_i = A \hat{z}_i + B u_i + G \left[\sum_{j \in \mathcal{N}_i} a_{ij} C (\hat{z}_j - \hat{z}_i) - a_{ij} (y_j - y_i) \right] + \varepsilon_i(t)$$

for all $i \in \mathcal{N}$ where $\varepsilon_i(t)$ is an exponentially decaying signal. Then there exists a gain G such that

$$\lim_{t \to \infty} \sum_{j \in \mathcal{N}_i} \alpha_{ij} (\hat{z}_j(t) - \hat{z}_i(t)) - \sum_{j \in \mathcal{N}_i} \alpha_{ij} (z_j(t) - z_i(t)) = 0,$$

for all $i \in \mathcal{N}$. ◊

Proof. Let $\mathfrak{L} \in \mathbb{R}^{N \times N}$ be the Laplacian matrix of the communication network. Define the columns of $V_{\bar{e}} \in \mathbb{C}^{N \times c}$ and $V_{\tilde{e}} \in \mathbb{C}^{N \times (N-c)}$ as the eigenvectors of $\mathfrak{L}$

such that

$$\mathfrak{L}[V_{\bar{e}}\ V_{\tilde{e}}] = [V_{\bar{e}}\ V_{\tilde{e}}]\mathrm{diag}(0_c, \Lambda),$$

where $\Lambda := \mathrm{diag}(\lambda_{c+1}(\mathfrak{L}), \ldots, \lambda_N(\mathfrak{L})) \in \mathbb{C}^{(N-c)\times(N-c)}$. Specifically, columns of $V_{\bar{e}}$ are eigenvectors with eigenvalue of 0. Let $e_i := \hat{z}_i - z_i$, $e := [e_1; \cdots ; e_N]$ and $\varepsilon(t) := [\varepsilon_1(t); \cdots ; \varepsilon_N(t)]$. Then, the dynamics of e become

$$\dot{e} = [(I_N \otimes A) - (\mathfrak{L} \otimes GC)]\, e + \varepsilon(t).$$

Now, apply the transformation given by

$$\begin{bmatrix} \bar{e} \\ \tilde{e} \end{bmatrix} = \left(\begin{bmatrix} V_{\bar{e}} & V_{\tilde{e}} \end{bmatrix}^{-1} \otimes I_n \right) e =: \left(\begin{bmatrix} W_{\bar{e}} \\ W_{\tilde{e}} \end{bmatrix} \otimes I_n \right) e.$$

Then, we obtain

$$\begin{aligned} \dot{\bar{e}} &= (I_c \otimes A)\bar{e} + W_{\bar{e}}\varepsilon(t) \\ \dot{\tilde{e}} &= [(I_{N-c} \otimes A) - \Lambda \otimes GC]\, \tilde{e} + W_{\tilde{e}}\varepsilon(t) \end{aligned}$$

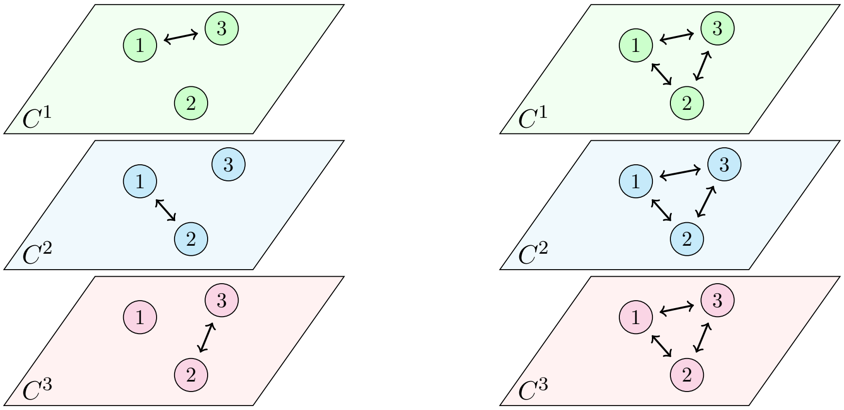
It follows from [Tun08] that G can be found such that $I_{N-c} \otimes A - \Lambda \otimes GC$ is Hurwitz. Since $\varepsilon(t) \to 0$, this implies $\tilde{e} \to 0$. Finally, note that

$$(\mathfrak{L} \otimes I_n)e = (\mathfrak{L}V_{\bar{e}} \otimes I_n)\bar{e} + (\mathfrak{L}V_{\tilde{e}} \otimes I_n)\tilde{e} = (\mathfrak{L}V_{\tilde{e}} \otimes I_n)\tilde{e} \to 0.$$

This completes the proof. □

A.2 Comparisons with Existing Consensus Problems

In this section, the consensus problem over multilayer network is compared with other frameworks. Specifically, it is shown that the classical consensus problem over multilayer network as well as the consensus problem of heterogeneous agents over single-layer network cannot represent the consensus problem over multilayer network (even when the output matrices are modified). A concept of the matrix weighted network, which is proposed recently in [Tun17], is shown to be equivalent to the consensus problem over multilayer network.



(a) Multilayer graph $\mathcal{G}$ for Example A.2.1.

(b) Multilayer graph $\mathcal{G}_{sm}$ representing the single-layer network.

Figure A.1: Multilayer graphs for different formulations.

Throughout this section, we will use the following example for the multilayer network.

Example A.2.1. Consider the consensus problem of 3 agents over multilayer graph $\mathcal{G}$ with 3 layers, whose dynamics is given by

$$
\begin{aligned}
\dot{x}_i &= Ax_i + Bu_i, \\
y_i^l &= C^l x_i, \quad \forall i \in \mathcal{N}, l \in \mathcal{L}.
\end{aligned}
$$

The multilayer graph $\mathcal{G}$ is defined as in Fig. A.1(a). $\Diamond$

In the following sections, we will investigate whether the system considered in Example A.2.1 can be represented using the different frameworks.

A.2.1 Consensus Problem of Homogeneous Agents over Single-layer Network

We will first study whether the classical consensus problem over single-layer network can represent the consensus problem over multilayer network. For this, consider the consensus of 3 agents whose dynamics is given by

$$\begin{aligned} \dot{x}_i &= Ax_i + Bu_i \\ y_i &= Ex_i \end{aligned} \tag{A.2.1}$$

where E is the common output matrix with the single-layer network G_{s}. In particular, suppose that G_{s} is the all-to-all network.

In an attempt to represent the multilayer graph using (A.2.1), let

$$E := \begin{bmatrix} C^1 \\ C^2 \\ C^3 \end{bmatrix}.$$

Then the resulting consensus problem can be represented using the multilayer graph $\mathcal{G}_{sm}$ given by Fig. A.1(b). Let α_{ij}^l be the entries of the adjacency matrices of $\mathcal{G}_{sm}$ and α_{ij} be the entries of the adjacency matrix of G_s. Then, since the output matrix E is defined as a stack of C^l, it can be observed that

$$\begin{aligned} \alpha_{ij}^1 = \alpha_{ij}^2 = \alpha_{ij}^3 = 1 &\iff \alpha_{ij} = 1, \\ \alpha_{ij}^1 = \alpha_{ij}^2 = \alpha_{ij}^3 = 0 &\iff \alpha_{ij} = 0. \end{aligned}$$

Therefore, using single-layer network lacks the flexibility compared with the general multilayer network, which may have different values for α_{ij}^l, e.g., $\alpha_{12}^1 =$ but $\alpha_{13}^2 = \alpha_{13}^3 = 0$ as in Fig. A.1(a).

One may try to change the definition of E, to perhaps only be $E = [C^1; C^2]$. However, corresponding multilayer graph can be also obtained, and it has an effect of removing a layer. In conclusion, single-layer network with different definition of E (including the one using the lifting) cannot represent the multilayer network.

Fundamental limitation of the single-layer network is that the *common* information is transmitted to all neighbors of agent i. On the other hand, agent may transmit *different* information to *different* neighbors, which cannot be described using a single-layer network.

A.2.2 Consensus Problem of Heterogeneous Agents over Single-layer Network

As seen from the previous discussion, if a framework can describe multilayer network using a single-layer network, one must be able to transmit different information. Another consensus problem studied in the literature that may achieve this is to use heterogeneous agents [WSA11]. Consider the following consensus problem of 3 agents with the same dynamics but with different output matrices as given by

$$\begin{aligned} \dot{x}_i &= Ax_i + Bu_i \\ y_i &= E_i x_i, \end{aligned}$$

where E_i is the output matrix of agent i. In this problem, we suppose that each agent only use the relative output information $E_i x_i - E_j x_j$ to compute the control input and that the communication network is given by G_s.

In order to see whether the heterogeneous agents can represent the multilayer network, consider the agent 3 on both heterogeneous single-layer network and multilayer network. On the heterogeneous single-layer network, information available to agent 3 can be written as

$$\delta_{3,\mathrm{s}} = \begin{bmatrix} E_1 x_1 - E_3 x_3 \\ E_2 x_2 - E_3 x_3 \end{bmatrix}, \tag{A.2.2}$$

while the information available to agent 3 over multilayer network is given by

$$\delta_{3,\mathrm{m}} = \begin{bmatrix} C^1(x_1 - x_3) \\ C^2(x_2 - x_3) \end{bmatrix}. \tag{A.2.3}$$

Therefore, the question now becomes whether there exists E_1 and E_3 such that (A.2.2) is equivalent to (A.2.3), i.e., $\delta_{3,\mathrm{s}} = \delta_{3,\mathrm{m}}$.

As a first attempt, let $E_1 = E_3 = C^1$ such that $E_1 x_1 - E_3 x_3 = C^1(x_1 - x_3)$. However, this means that $E_2 x_2 - C^1 x_3 \neq C^2 x_2 - C^2 x_3$. One may try other options for E_i, such as $E_1 = [C^1; C^2]$ and $E_3 = [C^1; 0]$ which results in

$$E_1 x_1 - E_3 x_3 = \begin{bmatrix} C^1 x_1 - C^1 x_3 \\ C^2 x_1 - 0 \end{bmatrix} \not\Leftrightarrow C^1(x_1 - x_3).$$

Moreover, information between agent 3 and agent 2 cannot be matched.

To be more precise, the information available to all agents through single-layer network can be written as

$$\delta_{\mathrm{s}} := \begin{bmatrix} \delta_{1,\mathrm{s}} \\ \delta_{2,\mathrm{s}} \\ \delta_{3,\mathrm{s}} \end{bmatrix} = \left[\begin{array}{ccc} E_1 & 0 & -E_3 \\ E_1 & -E_2 & 0 \\ \hline -E_1 & E_2 & 0 \\ 0 & E_2 & -E_3 \\ \hline -E_1 & 0 & E_3 \\ 0 & -E_2 & E_3 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{A.2.4}$$

while the information available to all agents through multilayer network is given by

$$\delta_{\mathrm{m}} := \begin{bmatrix} \delta_{1,\mathrm{m}} \\ \delta_{2,\mathrm{m}} \\ \delta_{3,\mathrm{m}} \end{bmatrix} = \left[\begin{array}{ccc} C^1 & 0 & -C^1 \\ C^2 & -C^2 & 0 \\ \hline -C^2 & C^2 & 0 \\ 0 & C^3 & -C^3 \\ \hline -C^1 & 0 & C^1 \\ 0 & -C^3 & C^3 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}. \tag{A.2.5}$$

The matrices E_i is $q \times n$ matrices, where the size q is larger than q^l. If the size of q is different from q^l, then δ_{m} may be padded with zero rows such that the dimension of δ_{s} is same as δ_{m}. Finally, by comparing the first columns of (A.2.4) with (A.2.5) that no E_1 exists such that $\delta_{\mathrm{s}} = \delta_{\mathrm{m}}$.

In the discussions so far, we have implicitly assumed that relative information is available per agent basis, i.e., $y_j - y_i$ instead of $\sum_{j \in \mathcal{N}_i}(y_j - y_i)$. If consider the summation of relative output information, $\delta_{\rm s}$ becomes

$$\delta_{\rm s} = \begin{bmatrix} 2E_1 & -E_2 & -E_3 \\ -E_1 & 2E_2 & -E_3 \\ -E_1 & -E_2 & 2E_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Comparing the above with (A.2.5), it can be easily checked that such E_i does not exists.

In conclusion, even if we use heterogeneous output over single-layer network, the corresponding system cannot represent the information exchange over the multilayer network. The same challenge as studied in earlier section still remains. Namely, each agent transmit same information to all of its neighbors, while in the multilayer network agent may transmit different information to different neighbors.

Remark A.2.1. Multilayer also cannot represent heterogeneous output matrices. In particular, one needs to find C^l such that

$$E_i x_i - E_j x_j = C^l (x_i - x_j), \quad \forall i, j \in \mathcal{N},$$

which does not hold in general. ◊

A.2.3 Consensus Problem over Matrix-weighted Network

A fundamental limitation for single-layer network studied so far is that each agent cannot transmit different information to each of its neighbors. However, consider the same dynamics with communication network G_s and the output structure given by

$$\begin{aligned} \dot{x}_i &= A x_i + B u_i \\ y_{ij} &= C_{ij}(x_i - x_j), \end{aligned} \tag{A.2.6}$$

where C_{ij} is the matrix associated with the edge (i, j) and that each agent uses y_{ij} for all $j \in \mathcal{N}_i$. This is the problem studied in [Tun17]. A graphical representation is shown in Fig. A.2. In particular, (A.2.6) can easily represent the multilayer

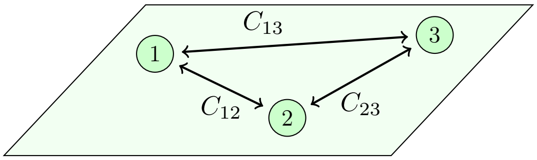


Figure A.2: Graphical representation of the communication structure for the system (A.2.6).

network considered in (A.2.1). For this suppose $C_{ij} = C_{ji}$ and define output matrices as

$$C_{12} = \begin{bmatrix} 0 \\ C^2 \\ 0 \end{bmatrix}, \quad C_{13} = \begin{bmatrix} C^1 \\ 0 \\ 0 \end{bmatrix}, \quad C_{23} = \begin{bmatrix} 0 \\ 0 \\ C^3 \end{bmatrix}.$$

In general, a multilayer graph $\mathcal{G}$ with the corresponding output matrices C^l can be represented using a matrix weighted network. Specifically, let the single-layer graph $G_s = \text{proj}(\mathcal{G})$ and let

$$C_{ij} = \begin{bmatrix} \alpha_{ij}^1 C^1 \\ \vdots \\ \alpha_{ij}^L C^L \end{bmatrix}.$$

Then, it can be easily checked that the corresponding system describes the identical information structure given by the multilayer network.

A.3 Detectability Interpretation of the Necessary Conditions

In this section, we present a detectability interpretation of the necessary condition (3.2.2). Consider the multi-agent system with multilayer network given by

$$\begin{aligned} \dot{x}_i &= Ax_i, \\ y_i^l &= C^l x_i, \\ \zeta_i &= Rx_i, \quad \forall i \in \mathcal{N}, l \in \mathcal{L}, \end{aligned} \tag{A.3.1}$$

where $x_i \in \mathbb{R}^n$, $y_i^l \in \mathbb{R}^{q^l}$ and $\zeta_i \in \mathbb{R}^q$ is the desired output.

We can prove the following results which resembles the observability (or detectability), which extend the ideas proposed in [Tun17] to output consensus problem over directed graphs.

Definition A.3.1. The multi-agent system over the multilayer network $\mathcal{G}$ is *detectable* if

$$y_j^l(t) - y_i^l(t) \equiv 0 \quad \forall t \geq 0,\ \forall (i,j) \in \mathcal{E}^l \implies \lim_{t\to\infty} |\zeta_j(t) - \zeta_i(t)| = 0, \quad \forall i,j \in \mathcal{N},$$

and *observable* if the later holds for all $t \geq 0$. ◊

Theorem A.3.1. The system (A.3.1) is detectable if and only if

$$\bar{\mathcal{K}} \subseteq \ker \Pi \otimes Z_R^\top. \qquad \Diamond$$

Proof. ($\Longleftarrow$) Suppose $y_j^l(t) - y_i^l(t) \equiv 0$ for all $(i,j) \in \mathcal{E}^l$ and $t \geq 0$. By taking derivative, it holds that

$$\begin{bmatrix} C^l \\ C^l A \\ \vdots \\ C^l A^{n-1} \end{bmatrix} (x_j(0) - x_i(0)) = 0.$$

This implies $x_j(0) - x_i(0) \in \langle \ker C^l \mid A \rangle$. Now decompose each state to stable and unstable part as $x_i(0) = x_{i,s}(0) + x_{i,u}(0)$, where $x_{i,s}(0) \in \mathcal{X}^s(A)$ and $x_{i,u}(0) \in \mathcal{X}^u(A)$. Then we have

$$\begin{aligned} &(x_{j,s}(0) - x_{i,s}(0)) + (x_{j,u}(0) - x_{i,u}(0)) \\ &\in \langle \ker C^l \mid A \rangle \\ &= \langle \ker C^l \mid A \rangle \cap \mathcal{X}^s(A) \oplus \langle \ker C^l \mid A \rangle \cap \mathcal{X}^u(A). \end{aligned}$$

Thus,

$$(x_{j,s}(0) - x_{i,s}(0)) \in \langle \ker C^l \mid A \rangle \cap \mathcal{X}^s(A).$$

By stacking $x_{j,s}$ and since graphs are undirected, we obtain

$$x_u(0) \in \bar{\mathcal{K}} \subseteq \ker \Pi \otimes {Z_R}^\top.$$

Since $\bar{\mathcal{K}}$ is invariant, using the property of the linear system, we get

$$\zeta_j(t) - \zeta_i(t) = R(x_j^u(t) - x_i^u(t)) + R(x_j^s(t) - x_i^s(t)).$$

Unstable terms are zero due to assumption and second term decays to zero, which completes the proof.

($\Longrightarrow$) Proof is similar to the previous case. First break down $x(0)$ into stable and unstable parts and use property of the linear system to have output in from of

$$\zeta_j(t) - \zeta_i(t) = R(x_j^u(t) - x_i^u(t)) + R(x_j^s(t) - x_i^s(t)) \to 0.$$

Notice that stable parts decays to zero, and to have the relative output to converge to zero, unstable part must satisfy

$$\begin{bmatrix} R \\ RA \\ \vdots \\ RA^{n-1} \end{bmatrix} (x_j^u(0) - x_i^u(0)) = 0.$$

This implies $x_j(0) \in \ker \Pi \otimes {Z_R}^\top$. Since $x_j(0) \in \bar{\mathcal{K}}$, this completes the proof. $\square$

Result of Theorem A.3.1 provides a detectability interpretation of the proposed necessary condition. Also recall that the necessary condition, i.e., the *detectability* of the MAS over multilayer network, is sufficient for state consensus problem over undirected graphs. However, as discussed in the Chapter 4, detectability of the MAS is not sufficient to design controller for the output consensus problem over directed graphs.

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국문초록

Consensus of Linear Time Invariant Multi-agent Systems over Multilayer Network

다층레이어 네트워크 구조를 가지는 선형 시불변 다개체 시스템의 상태일치

전통적으로 다 개체 시스템의 상태 일치 문제는 한 가지의 네트워크 상에서 한 가지의 정보를 주고받는 경우에 대해서 주로 연구가 되었다. 하지만 최근에는 이러한 가정은 보다 복잡한 상호작용을 나타내는 데 한계가 있기 때문에 새로운 접근법이 필요한 상황이다. 본 논문에서는 각 개체가 서로 다른 정보를 서로 다른 네트워크 상에서 주고받는 경우를 고려한다. 이러한 관계를 표현하기 위해 다층레이어 네트워크 (multilayer network)라는 개념을 도입하였다. 이때 동적인 제어기로 방향성이 없는 네트워크에서 상태 일치를 이루는 새로운 필요충분조건을 제시한다. 특히 제시한 조건은 그래프 이론적인 조건과 시스템 이론적인 조건을 결합하였으며, 통신 네트워크와 주고받는 정보의 상호작용을 강조한다. 더 나아가 제시한 조건을 사용하여 방향성이 없는 네트워크상에서 상태 일치를 이루는 관측기 기반 동적 제어기를 제시한다.

주요 결과는 방향성이 있는 네트워크 상에서 출력 일치를 이루는 문제로 확장한다. 아쉽게도 이 상황에서는 제시한 조건은 더 이상 필요충분조건이 되지 못하며 이런 어려움들을 다양한 예제를 통해서 설명한다. 그럼에도 불구하고, 개체의 동역학에 추가적인 조건을 가함으로써 방향성이 없는 네트워크에서 필요충분조건을 회복한다. 또한 방향성이 있는 네트워크에서 출력 일치 문제를 푸는 충분조건을 제시하고 이를 이루는 제어기를 제안한다.

본 논문의 효용성은 여러 가지 적용 예제를 통해 보인다. 첫 번째로 분산 관측 문제를 다층 레이어 네트워크 상의 상태 일치 문제로 표현한다. 제시된 방법을 사용하면 주변 개체와의 통신량을 기존 결과들 보다 줄이는 새로운 분산 관측기를 제시한다. 두 번째로 논문의 결과를 사용해 편대 제어 문제를 푼다. 특히, 원하는 편대의 모양이 개체의 상대적인 위치와 상대적인 각도로 주어진 경우를 고려한다. 제시한 방법을 사용하여 원하는 편대를 이루는 동적 제어기를 제시하였고, 편대의 크기를 유기적으로 조절하는 알고리즘을 제시한다. 마지막으로 다층 레이어 네트워크를 분산 최적화 문제에 적용을 한다. 이를 통해 매시간 결정 변수의 일부분만을 통신하는 통신적으로 더 효율적인 알고리즘을 제시한다.

주요어: 동기화, 선형 다 개체 시스템, 다층레이어 네트워크, 편대 제어, 분산 추정, 분산 최적화
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