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Fixed Point Approaches for Multi-Valued Prešić Multi-Step Iterative Mappings with Applications

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Abstract: The purpose of this paper is to present some fixed point approaches for multi-valued Prešić k-step iterative-type mappings on a metric space. Furthermore, some corollaries are obtained to unify and extend many symmetrical results in the literature. Moreover, two examples are provided to support the main result. Ultimately, as potential applications, some contributions of integral type are investigated and the existence of a solution to the second-order boundary value problem (BVP) is presented.

Keywords: Prešić-type contraction; Φ-contraction; existence solution; boundary value problem

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1. Introduction

In nonlinear analysis, fixed point (FP) theory is regarded as one of the most potent and practical tools. FP theory is a thriving area of nonlinear analysis with numerous potential future developments. It is a field that is significant in both pure and applied mathematics. Due to the ease and smoothness of the FP method, as well as its numerous and fascinating applications in fields such as economics, biology, chemistry, game theory, engineering, physics, etc., it has now become the standard for nonlinear analysis, following the publication of a large number of valuable papers that have used it effectively.

The strength of FPs appears clearly when applied to contraction mappings in complete metric spaces (MSs). From here, many writers headed in this direction, either by generalizing space or by generalizing contractions. Then, theoretical results were applied in many applications, such as studying the existence and uniqueness of the solution to differential, integral, matrix, and functional equations. For more details, see [1–9].

To generalize the above results, Prešić [10] introduced mappings under mild conditions on a finite product space and introduced some FP results for such mappings. Many authors have been interested in this idea and have discovered new fixed points with new applications. For more information on this trend, see [11–20].

On the other hand, multi-valued mappings are important in a variety of mathematical sciences, including economics, optimization theory, and problems involving optimal control. With the help of the FP approach, it is possible to examine the existence and uniqueness of the solution to fractional differential and integral equations [21–24]. Additionally, this subject has been thoroughly researched, with some noteworthy findings reported in [25–29].

Nadler [30] extended BCPs to multi-valued contraction mappings and obtained important results as a continuation of this approach. Choudhury et al. [31] generalized Nadler's FP theorem by using the notion of α-admissible contractions tin multi-valued contraction mappings, and presented nice results on fixed point theorems in this line. For further generalizations in this regard, see [32–36].

In this paper, some FP results for multi-valued Prešić type Φ-contraction mappings are introduced in MSs. Furthermore, some results were related to previous contributions obtained as corollaries. Moreover, two examples are presented to support the first main result. Ultimately, as applications, some contributions of integral type are obtained and the existence of a solution to the second-order boundary value problem (BVP) is discussed.

2. Preliminaries

In this part, we provide some basic definitions and concepts that help us in our desired goal and also facilitate the reader to understand our manuscript.

Let $\eth$ be any non-empty set and $\daleth : \eth \to \eth$ be a given mapping. A functional equation $\daleth(\varkappa) = \varkappa$ is known as fixed point (FP) equation and its solution is called a FP of $\daleth$. The existence of solutions to such equations depends on the nature of the mapping $\daleth$ and the distance or topological structure of the set $\eth$. If for any $\varkappa, y \in \eth$ there exists some real number $\mathfrak{A} \in [0,1)$ such that the following condition holds:

$$\Xi(\daleth\varkappa, \daleth y) \leq \mathfrak{A}\Xi(\varkappa, y).$$

Then, $\daleth$ is called a contraction mapping. The BCP [1] is stated as follows:

Theorem 1. *Let $(\eth, \Xi)$ be a complete MS and $\daleth : \eth \to \eth$ be a self mapping which is also a contraction. Then, the FP equation involving the mapping $\daleth$ has a solution $\eth$ which is unique. Furthermore, for every $\varkappa_0 \in \eth$, the iterative sequence defined by $\varkappa_{n+1} = \daleth(\varkappa_n)$ converges to a FP $\varkappa^*$.*

Here, consider the function $\Phi : (0,\infty) \to (1,\infty)$ such that the properties below hold:
(Φ_1) Φ is non-decreasing;
(Φ_2) If $\{\mathfrak{y}_n\}$ is a sequence in $(0,\infty)$, then $\lim\limits_{n\to\infty} \Phi(\mathfrak{y}_n) = 1$ if $\lim_{n\to\infty} \mathfrak{y}_n = 0$;
(Φ_3) There exists $0 < r < 1$ and $0 \leq h < \infty$ such that

$$\lim_{t\to 0} \frac{\Phi(t) - 1}{\mathfrak{y}^r} = h;$$

(Φ_4) For all $\beta \in (0,1)$ and $\mathfrak{y} > 0$, $\Phi(\mathfrak{y}) \leq [\Phi(\frac{\mathfrak{y}}{\beta})]^{\sqrt{\beta}}$.

Furthermore, let Θ be the set of all functions which satisfies (Φ_1–Φ_3), whereas Θ^* is the set of all functions that fulfils (Φ_1–Φ_4).

For examples of the above functions, define $\daleth, g : (0,\infty) \to (1,\infty)$ by

$$\daleth(\mathfrak{y}) = e^{\sqrt{\mathfrak{y}}}$$

and

$$g(\mathfrak{y}) = e^{\sqrt{\mathfrak{y}e^{\mathfrak{y}}}}.$$

Clearly, both $\daleth$ and g belong to Θ and Θ^*, respectively.

Theorem 2 ([8])**.** *Let $(\eth, \Xi)$ be a complete MS and $\daleth : \eth \to \eth$ be a self-mapping. If there exists $\Phi \in \Theta$ and $\mathfrak{A} \in (0,1)$ such that*

$$\Phi(\Xi(\daleth\varkappa, \daleth y)) \leq [\Phi(\Xi(\varkappa, y))]^{\mathfrak{A}}$$

holds, for all $\varkappa, y \in \eth$ with $\Xi(\varkappa, y) > 0$, then $\daleth$ has a unique FP in $\eth$.

It is obvious that if $\Phi(\mathfrak{y}) = e^{\sqrt{\mathfrak{y}}}$, then

$$e^{\sqrt{\Xi(f\varkappa, fy)}} \leq [e^{\sqrt{\Xi(\varkappa,y)}}]^{\mathfrak{A}} = [e^{\mathfrak{A}\sqrt{\Xi(\varkappa,y)}}],$$

which implies that

$$\Xi(\daleth\varkappa, \daleth y)) \leq \mathfrak{A}^2 \Xi(\varkappa, y),$$

and hence $\daleth$ is a contraction mapping.

Theorem 3 ([10]). *Let $(\eth, \Xi)$ be a complete MS and $\daleth : \eth^k \to \eth$ be a mapping. If there exists constants $r_1, r_2, \ldots, r_k \geq 0$ such that for all $\varkappa_1, \varkappa_2, \ldots, \varkappa_{k+1} \in \eth$ we have*

$$\begin{aligned} & \Xi(\daleth(\varkappa_1, \varkappa_2, \ldots, \varkappa_k), \daleth(\varkappa_2, \varkappa_3, \ldots \varkappa_{k+1})) \\ \leq \ & r_1 \Xi(\varkappa_1, \varkappa_2) + r_2 \Xi(\varkappa_2, \varkappa_3) + \ldots + r_k \Xi(\varkappa_k, \varkappa_{k+1}), \end{aligned}$$

then there exists a point $\varkappa \in \eth$ such that

$$\daleth(\varkappa, \varkappa, \ldots., \varkappa) = \varkappa$$

provided that $r_1 + r_2 + \ldots + r_k < 1$. Moreover, for any $\varkappa_1, \varkappa_2, \ldots \varkappa_k \in \eth$, the sequence

$$\varkappa_{n+k} = \daleth(\varkappa_n, \varkappa_{n+1}, \ldots, \varkappa_{n+k-1}), \quad n = 0, 1, 2, 3, \ldots$$

converges to $\varkappa$.

If we take $k = 1$ in the above result, one obtains the BCP. For the sake of simplicity, a point $\varkappa$ in $\eth$ is called a Prešić FP of $\daleth$ if

$$\daleth(\varkappa, \varkappa, \ldots., \varkappa) = \varkappa.$$

In addition, we refer to the sequence $\{\varkappa_{n+k}\}$ given in the above theorem as a Prešić–Picard iterative sequence starting from $\varkappa_1, \varkappa_2, \ldots \varkappa_k \in \eth$.

Ćirić and Prešić [12] extended Theorem 3 in the following theorem.

Theorem 4 ([12]). *Suppose that $(\eth, \Xi)$ is a complete MS and $\daleth : \eth^k \to \eth$. If there exists a real number $0 \leq r < 1$ such that for all $\varkappa_1, \varkappa_2, \ldots, \varkappa_{k+1} \in \eth$,*

$$\Xi(\daleth(\varkappa_1, \varkappa_2, \ldots, \varkappa_k), \daleth(\varkappa_2, \varkappa_3, \ldots \varkappa_{k+1})) \leq r \max\{\Xi(\varkappa_1, \varkappa_2), \Xi(\varkappa_2, \varkappa_3), \ldots, \Xi(\varkappa_k, \varkappa_{k+1})\},$$

then there exists a Prešić FP $\varkappa$ of $\daleth$. Moreover, the Prešić–Picard iterative sequence starting from $\varkappa_1, \varkappa_2, \ldots \varkappa_k \in \eth$ converges to $\varkappa$. In addition, if

$$\Xi(\daleth(\varkappa_1, \varkappa_1, \ldots, \varkappa_1), \daleth(\varkappa_2, \varkappa_2, \ldots, \varkappa_2)) < \Xi(\varkappa_1, \varkappa_2)$$

holds for all $\varkappa_1, \varkappa_2 \in \eth$, with $\varkappa_1 \neq \varkappa_2$, then $\varkappa$ is a unique Prešić FP of $\daleth$.

Recently, Altun et al. [13] obtained some Prešić FP consequences under the mapping $\daleth : \eth^k \to \eth$ as shown in the following section.

Definition 1 ([13]). *Let $(\eth, \Xi)$ be a MS and $\Phi \in \Theta$. A mapping $\daleth : \eth^k \to \eth$ is known as a Ćirić–Prešić type Φ-contraction if there exists $\mathfrak{A} \in (0, 1)$ such that for all $\varkappa_1, \varkappa_2, \ldots, \varkappa_{k+1} \in \eth$ and $\Xi(\daleth(\varkappa_1, \varkappa_2, \ldots, \varkappa_k), \daleth(\varkappa_2, \varkappa_3, \ldots, \varkappa_{k+1})) > 0$, we have*

$$\Phi(\Xi(\daleth(\varkappa_1, \varkappa_2, \ldots, \varkappa_k), \daleth(\varkappa_2, \varkappa_3, \ldots, \varkappa_{k+1}))) \leq \Phi(\max\{\Xi(\varkappa_i, \varkappa_{i+1})\})^{\sqrt{\mathfrak{A}}},$$

where $\Phi \in \Theta^*$ and $1 \leq i \leq k$.

Theorem 5 ([13]). *Let $(\eth, \Xi)$ be a complete MS and $\daleth : \eth^k \to \eth$ be a mapping which is a Ćirić–Prešić type Φ-contraction with $\Phi \in \Theta^*$. Then, there exists a point $\varkappa \in \eth$ such that $\varkappa$ is a Prešić

FP of $\urcorner$*. Moreover, the Prešić–Picard iterative sequence starting from* $\varkappa_1, \varkappa_2, \ldots \varkappa_k \in \eth$ *converges to* $\varkappa$*. In addition, if for all* $\mathfrak{y}, \varkappa^* \in \eth$ *with* $\mathfrak{y} \neq \varkappa^*$*,*

$$\Xi(\urcorner(\mathfrak{y}, \mathfrak{y}, \ldots, \mathfrak{y}), \urcorner(\varkappa^*, \varkappa^*, \ldots, \varkappa^*)) < \Xi(\mathfrak{y}, \varkappa^*),$$

then $\varkappa$ *is a unique Prešić FP of* $\urcorner$*.*

Theorems 3 and 4 are crucial for understanding the issue of global asymptotic stability of an equilibrium for the nonlinear difference equation

$$\varkappa_{n+k} = f(\varkappa_n, \varkappa_{n+1}, \ldots, \varkappa_{n+k-1}), \quad n = 0, 1, 2, 3, \ldots,$$

which was considered in [14,15].

To study the FPs of set-valued Prešić-type contraction mappings in the setup of MSs, we need the following concepts.

For an MS $(\eth, \Xi)$, we set $\mathcal{N}(\eth), \mathcal{B}(\eth), \mathcal{CB}(\eth)$, and $\mathcal{C}(\eth)$ as the collection of all non-empty, non-empty bounded, non-empty closed bounded, and non-empty compact subsets of $\eth$, respectively.

The distance $\Xi(\varkappa, \Omega)$ of a point $\varkappa \in \eth$ from $\Omega \in \mathcal{N}(\eth)$ is given by

$$\Xi(\varkappa, \Omega) = \inf\{\Xi(\varkappa, z) : z \in \Omega\}.$$

For $\Omega, \mho \in \mathcal{N}(\eth)$, we define

$$\delta(\Omega, \mho) = \sup\{\Xi(\varkappa, \mho) : \varkappa \in \Omega\},$$

and

$$\mathcal{H}(\Omega, \mho) = \max\{\delta(\Omega, \mho), \delta(\mho, \Omega)\}.$$

Then, $\mathcal{H}$ is known as Pompeiu–Hausdorff metric on $\mathcal{CB}(\eth)$. Furthermore, $(\mathcal{CB}(\eth), \mathcal{H})$ is a complete MS if $(\eth, \Xi)$ is a complete MS.

Nadler [30] extended the BCP to multi-valued contraction mappings by introducing result of the following theorem.

Theorem 6 ([30])**.** *Let* $(\eth, \Xi)$ *be a complete MS and* $\urcorner : \eth \to \mathcal{CB}(\eth)$*. If for any* $\varkappa, y \in \eth$*, the following holds:*

$$\mathcal{H}(\urcorner\varkappa, \urcorner y) \leq \mathfrak{A}\Xi(\varkappa, y),$$

where $0 \leq \mathfrak{A} < 1$*, then there exists* u *in* $\eth$ *such that* $u \in \urcorner(u)$*.*

The following lemmas, which are obtained from [30], are very important in the sequel.

Lemma 1. *If* $\Omega, \mho \in \mathcal{CB}(\eth)$*,* $h > 1$*, and* $\varkappa \in \Omega$*., then there exists* $y \in \mho$ *such that*

$$\Xi(\varkappa, y) \leq h\mathcal{H}(\Omega, \mho).$$

Lemma 2. *If* $\Omega, \mho \in \mathcal{CB}(\eth)$*,* $h > 0$*, and* $\varkappa \in \Omega$*, then there exists* $y \in \mho$ *such that*

$$\Xi(\varkappa, y) \leq \mathcal{H}(\Omega, \mho) + h.$$

Lemma 3. *If* $\Omega, \mho \in \mathcal{CB}(\eth)$*, then for any* $a \in \Omega$

$$\Xi(a, \mho) \leq \mathcal{H}(\Omega, \mho).$$

Lemma 4. *If* $\Omega, \mho \in \mathcal{C}(\eth)$*, and* $a \in \Omega$*, then there exists* $b \in \mho$ *such that*

$$\Xi(a, b) \leq \mathcal{H}(\Omega, \mho)$$

holds.

Indeed, $\Xi(a,\mho) = \inf\{\Xi(a,y) : y \in \mho\}$. Since $\mho$ is compact, there exists b in $\mho$ such that $\Xi(a,\mho) = \Xi(a,b)$.

Recently, Shulka et al. [35] and Abbas et al. [36] introduced the notion of the set-valued Prešić-type contraction mapping in product spaces as the following:

Definition 2. *An MV mapping $\daleth : \eth^k \to \mathcal{CB}(\eth)$ is called a set-valued Prešić-type contraction if*

$$\mathcal{H}(\daleth(\varkappa_1,\varkappa_2,\ldots,\varkappa_k),\daleth(\varkappa_2,\varkappa_3,\ldots,\varkappa_{k+1})) \le \sum_{i=1}^{k} \alpha_i \Xi(\varkappa_i,\varkappa_{i+1})$$

holds for all $(\varkappa_1,\varkappa_2,\ldots,\varkappa_{k+1}) \in \eth^{k+1}$, where $\alpha_i \ge 0$ and $\sum_{i=1}^{k} \alpha_i < 1$.

For an MV mapping $\daleth : \eth^k \to \mathcal{N}(\eth)$, a point $\varkappa \in \eth$ is called a FP of $\daleth$ if $\varkappa \in \daleth(\varkappa,\varkappa,\ldots,\varkappa)$. The collection of all fixed points of $\daleth$ is denoted by $Fix(\daleth)$. A point $\varkappa \in \eth$ is called an end point of $\daleth$ if $\daleth(\varkappa,\varkappa,\ldots,\varkappa) = \{\varkappa\}$.

3. Main Results

This section is devoted to presenting some FP results for a Ćirić–Prešić multi-valued Φ-contraction type mapping in the setting of complete MSs. We begin with the following theorem:

Theorem 7. *Let $(\eth,\Xi)$ be a complete MS, $\daleth : \eth^k \to \mathcal{C}(\eth)$ and $\Phi \in \Theta^*$. If there exists $\mathfrak{A} \in (0,1)$ such that for all $\varkappa_1,\varkappa_2,\ldots,\varkappa_{k+1} \in \eth$ with*

$$\mathcal{H}(\daleth(\varkappa_1,\varkappa_2,\ldots,\varkappa_k),\daleth(\varkappa_2,\varkappa_3,\ldots,\varkappa_{k+1})) > 0,$$

the following condition holds

$$\Phi(\mathcal{H}(\daleth(\varkappa_1,\varkappa_2,\ldots,\varkappa_k),\daleth(\varkappa_2,\varkappa_3,\ldots,\varkappa_{k+1}))) \le \Phi(\max\{\Xi(\varkappa_i,\varkappa_{i+1})\})^{\sqrt{\mathfrak{A}}}, \tag{1}$$

where $1 \le i \le k$, then there exists a point $\varkappa \in \eth$ such that $\varkappa \in \daleth(\varkappa,\varkappa,\ldots,\varkappa)$. Moreover, if $\varkappa_1,\varkappa_2,\ldots,\varkappa_k$ are arbitrary points in $\eth$ and

$$\varkappa_{\wp+k} \in \daleth(\varkappa_\wp,\varkappa_{\wp+1},\ldots,\varkappa_{\wp+k-1}) \text{ for each } \wp,$$

then the sequence $\{\varkappa_\wp\}$ converges to $\varkappa$.

Proof. Let us denote

$$F_{\varkappa_i}^{\varkappa_{i+j-1}} = (\varkappa_i,\varkappa_{i+1},\ldots,\varkappa_k,\varkappa_{j+i-1}) \in \eth^k,$$

and

$$F_{\varkappa}^{\varkappa} = (\varkappa,\varkappa,\ldots,\varkappa) \in \eth^k.$$

Consider $\varkappa_1,\varkappa_2,\ldots,\varkappa_k,\varkappa_{k+1}$ as arbitrary points in $\eth$ such that $\varkappa_{k+1} \in \daleth(F_{\varkappa_1}^{\varkappa_k})$. By Lemma 4, there exists $\varkappa_{k+2} \in \daleth(F_{\varkappa_2}^{\varkappa_{k+1}})$ such that

$$\Xi(\varkappa_{k+1},\varkappa_{k+2}) \le \mathcal{H}(\daleth(F_{\varkappa_1}^{\varkappa_k}),\daleth(F_{\varkappa_2}^{\varkappa_{k+1}})).$$

Since $\daleth(F_{\varkappa_2}^{\varkappa_{k+1}}),\daleth(F_{\varkappa_3}^{\varkappa_{k+2}}) \in \mathcal{C}(\eth)$, by Lemma 4, there exists $\varkappa_{k+3} \in \daleth(F_{\varkappa_3}^{\varkappa_{k+2}})$ such that

$$\Xi(\varkappa_{k+2},\varkappa_{k+3}) \le \mathcal{H}(\daleth(F_{\varkappa_2}^{\varkappa_{k+1}}),\daleth(F_{\varkappa_3}^{\varkappa_{k+2}})).$$

Continuing this way, for every natural number $\wp$ and for $\varkappa_{k+\wp-1} \in \daleth(F^{\varkappa_{\wp+k-2}}_{\varkappa_{\wp-1}})$, there exists $\varkappa_{k+\wp} \in \daleth(F^{\varkappa_{\wp+k-1}}_{\varkappa_{\wp}})$ such that

$$\Xi(\varkappa_{k+\wp-1}, \varkappa_{k+\wp}) \leq \mathcal{H}(\daleth(F^{\varkappa_{\wp+k-2}}_{\varkappa_{\wp-1}}), \daleth(F^{\varkappa_{\wp+k-1}}_{\varkappa_{\wp}})).$$

Hence, we have a sequence $\varkappa_{\wp}$ in $\eth$ described as

$$\varkappa_{\wp+k} \in \daleth(F^{\varkappa_{\wp+k-1}}_{\varkappa_{\wp}}); \quad \wp = 1, 2, \ldots \tag{2}$$

which satisfies

$$\Xi(\varkappa_{k+\wp-1}, \varkappa_{k+\wp}) \leq \mathcal{H}(\daleth(F^{\varkappa_{\wp+k-2}}_{\varkappa_{\wp-1}}), \daleth(F^{\varkappa_{\wp+k-1}}_{\varkappa_{\wp}})).$$

Using the condition (Φ_1), we obtain that

$$\Phi(\Xi(\varkappa_{\wp+k}, \varkappa_{\wp+k+1})) \leq \Phi(\mathcal{H}(F^{\varkappa_{\wp+k-1}}_{\varkappa_{\wp}}), \daleth(F^{\varkappa_{\wp+k}}_{\varkappa_{\wp+1}}))).$$

If we assume $\varkappa_i = \varkappa_{i+1}$ for all $i = \wp, \wp+1, \ldots, \wp+k-1$, then from (2), we obtain

$$\varkappa_i \in \daleth(F^{\varkappa_i}_{\varkappa_i}),$$

that is, $\varkappa_i$ is an FP of $\daleth$. Therefore, assuming that $\varkappa_i \neq \varkappa_{i+1}$ for some $i = \wp, \wp+1, \ldots, \wp+k-1$, we shall show that for every $\wp$, the following inequalities hold

$$\Phi(\Xi(\varkappa_{\wp}, \varkappa_{\wp+1})) \leq [\Phi(\max_{1\leq i\leq k}\{\frac{1}{\mathfrak{A}^{\frac{i}{k}}}\Xi(\varkappa_i, \varkappa_{i+1})\})]^{\sqrt[2k]{\mathfrak{A}^{\wp}}}, \quad \forall \wp \in N,$$

or

$$\Phi(\Xi_{\wp}) \leq [\Phi(\Bbbk)]^{\sqrt[2k]{\mathfrak{A}^{\wp}}}, \quad \forall \wp \in N, \tag{3}$$

where $\Xi_{\wp} = \Xi(\varkappa_{\wp}, \varkappa_{\wp+1})$ and $\Bbbk = \max_{1\leq i\leq k}\{\frac{1}{\mathfrak{A}^{\frac{i}{k}}}\Xi(\varkappa_i, \varkappa_{i+1})\}$.

If $i \in \{1, 2, \ldots, k\}$, then by (Φ_4), we obtain

$$\Phi(\Xi_i) \leq [\Phi(\Xi_i/\mathfrak{A}^{i/k})]^{\sqrt{\mathfrak{A}^{i/k}}} \leq \Phi(\max_{1\leq i\leq k}\{\frac{1}{\mathfrak{A}^{\frac{i}{k}}}\Xi(\varkappa_i, \varkappa_{i+1})\})^{\sqrt{\mathfrak{A}^{\frac{i}{k}}}} = [\Phi(\Bbbk)]^{\sqrt{\mathfrak{A}^{\frac{i}{k}}}},$$

and the inequality (3) holds for $\wp = 1, 2, 3, \ldots, k$. Let the following inequalities be true

$$\Phi(\Xi_{\wp+i-1}) = \Phi(\Xi(\varkappa_{\wp+i-1}, \varkappa_{\wp+i})) \leq [\Phi(\Bbbk)]^{\sqrt[2k]{\mathfrak{A}^{\wp+i-1}}},$$

for $i \in \{1, 2, \ldots, k\}$. Then, we have

$$\begin{aligned}
\Phi(\Xi_{\wp+k}) &= \Phi(\Xi(\varkappa_{\wp+k}, \varkappa_{\wp+k+1})) \\
&\leq \Phi(\mathcal{H}(\daleth(F^{\varkappa_{\wp+k-1}}_{\varkappa_{\wp}}), \daleth(F^{\varkappa_{\wp+k}}_{\varkappa_{\wp+1}})) \\
&\leq \Phi(\max_{\wp\leq i\leq \wp+k-1}\{\Xi(\varkappa_i, \varkappa_{i+1})\})^{\sqrt{\mathfrak{A}}} \\
&\leq (\max_{1\leq i\leq k}\{[\Phi(\Bbbk)]^{\sqrt[2k]{\mathfrak{A}^{\wp+i-1}}}\})^{\sqrt{\mathfrak{A}}} \\
&\leq ([\Phi(\Bbbk)]^{\sqrt[2k]{\mathfrak{A}^{\wp}}})^{\sqrt{\mathfrak{A}}} \\
&= [\Phi(\Bbbk)]^{\sqrt[2k]{\mathfrak{A}^{\wp+k}}}.
\end{aligned}$$

Therefore, by induction, the inequality (3) holds for all $\wp \in N$. Hence,

$$\Phi(\Xi_{\wp+k}) \leq [\Phi(\Bbbk)]^{\sqrt[2k]{\mathfrak{A}^{\wp+k}}}, \quad \forall \wp \in N.$$

If $\wp \to \infty$, then

$$\lim_{\wp\to\infty}\Phi(\Xi_{k+\wp}) = 1,$$

and hence by (Φ_2), we have

$$\lim_{\wp\to\infty}\Xi_{k+\wp}=0.$$

Now, by (Φ_3),

$$\lim_{\wp\to 0^+}\frac{\Phi(\Xi_{\wp+k})-1}{[\Xi_{\wp+k}]^r}=h,$$

and there exists $\wp_0$ such that

$$\Xi_{k+\wp}\leq\frac{1}{\wp^{1/r}}\quad\text{for all}\quad\wp\geq\wp_0.$$

Now, we prove that $\{\varkappa_\wp\}$ is a Cauchy sequence. If $m\geq\wp\geq\wp_0$, then we obtain

$$\begin{aligned}\Xi(\varkappa_{\wp+2k},\varkappa_{m+2k})&\leq\mathcal{H}(\urcorner(F^{\varkappa_{\wp+2k-1}}_{\varkappa_{\wp+k}}),\urcorner(F^{\varkappa_{m+2k-1}}_{\varkappa_{m+k}}))\\&\leq\mathcal{H}(\urcorner(F^{\varkappa_{\wp+2k-1}}_{\varkappa_{\wp+k}}),\urcorner(F^{\varkappa_{\wp+2k}}_{\varkappa_{\wp+k+1}}))+\mathcal{H}(\urcorner(F^{\varkappa_{\wp+2k}}_{\varkappa_{\wp+k+1}}),\urcorner(F^{\varkappa_{\wp+2k+1}}_{\varkappa_{\wp+k+2}}))\\&+\ldots+\mathcal{H}(\urcorner(F^{\varkappa_{m+2k-2}}_{\varkappa_{m+k-1}}),\urcorner(F^{\varkappa_{m+2k-1}}_{\varkappa_{m+k}}))\\&\leq(\max_{1\leq i\leq k}\{\Xi_{(\wp+i-1)+k}\})^{\sqrt{\mathfrak{A}}}+(\max_{1\leq i\leq k}\{\Xi_{(\wp+i)+k}\})^{\sqrt{\mathfrak{A}}}\\&+\ldots+(\max_{1\leq i\leq k}\{\Xi_{(m+i-2)+k}\})^{\sqrt{\mathfrak{A}}}\\&\leq(\max_{1\leq i\leq k}\{\frac{1}{(\wp+i-1)^{1/r}}\})^{\sqrt{\mathfrak{A}}}+(\max_{1\leq i\leq k}\{\frac{1}{(\wp+i)^{1/r}}\})^{\sqrt{\mathfrak{A}}}\\&+\ldots+(\max_{1\leq i\leq k}\{\frac{1}{(m+i-2)^{1/r}}\})^{\sqrt{\mathfrak{A}}}.\end{aligned}$$

Thus,

$$\begin{aligned}\Xi(\varkappa_{\wp+2k},\varkappa_{m+2k})&\leq\frac{1}{\wp^{\sqrt{\mathfrak{A}}/r}}+\frac{1}{(\wp+1)^{\sqrt{\mathfrak{A}}/r}}+,\ldots,+\frac{1}{(m-1)^{\sqrt{\mathfrak{A}}/r}}\\&=\sum_{i=\wp}^{m-1}\frac{1}{i^{\sqrt{\mathfrak{A}}/r}}\leq\sum_{i=\wp}^{m-1}\frac{1}{i^{1/r}}\to 0\ \text{as}\ \wp,m\to\infty.\end{aligned}$$

Therefore, $\{\varkappa_\wp\}$ is a Cauchy sequence in $(\eth,\Xi)$. The completeness of $(\eth,\Xi)$ implies that there exists $\varkappa\in\eth$ such that

$$\lim_{\wp\to\infty}\Xi(\varkappa_\wp,\varkappa)=0.$$

Now, for any $\wp\in N$, we obtain

$$\begin{aligned}\Xi(\varkappa,\urcorner(F^{\varkappa}_{\varkappa}))&\leq\Xi(\varkappa,\varkappa_{\wp+k})+\Xi(\varkappa_{\wp+k},\urcorner(F^{\varkappa}_{\varkappa}))\\&\leq\Xi(\varkappa,\varkappa_{\wp+k})+\mathcal{H}(\urcorner(F^{\varkappa_{\wp+k-1}}_{\varkappa_\wp}),\urcorner(F^{\varkappa}_{\varkappa}))\\&\leq\Xi(\varkappa,\varkappa_{\wp+k})+\mathcal{H}(\urcorner(F^{\varkappa}_{\varkappa}),\urcorner(\varkappa,\varkappa,\ldots,\varkappa_\wp))\\&+\ldots+\mathcal{H}(\urcorner(\varkappa,\varkappa_\wp,\ldots,\varkappa_{\wp+k-2}),\urcorner(F^{\varkappa_{\wp+k-1}}_{\varkappa_\wp}))\\&\leq\Xi(\varkappa,\varkappa_{\wp+k})+\Xi(\varkappa,\varkappa_\wp)^{\sqrt{\mathfrak{A}}}+\max\{\Xi(\varkappa,\varkappa_\wp),\Xi(\varkappa_\wp,\varkappa_{\wp+1})\}^{\sqrt{\mathfrak{A}}}+\ldots\\&+\max\{\Xi(\varkappa,\varkappa_\wp),\Xi(\varkappa_\wp,\varkappa_{\wp+1}),\Xi(\varkappa_{\wp+1},\varkappa_{\wp+2}),\ldots,\Xi(\varkappa_{\wp+k-2},\varkappa_{\wp+k-1})\}^{\sqrt{\mathfrak{A}}}\end{aligned}$$

Taking the limit as $\wp\to\infty$, we have

$$\Xi(\varkappa,\urcorner(F^{\varkappa}_{\varkappa}))=0\quad\Rightarrow\quad\varkappa\in\urcorner(F^{\varkappa}_{\varkappa}).$$

□

The following corollaries give generalized results to some of the previous literature:

Corollary 1. *Let (∂, Ξ) be a complete MS, $\urcorner : \partial^2 \to \mathcal{C}(\partial)$ and $\Phi \in \Theta^*$. If there exists $\mathfrak{A} \in (0,1)$ such that for all $\varkappa_1, \varkappa_2, \varkappa_3 \in \partial$ with $\mathcal{H}(\urcorner(\varkappa_1, \varkappa_2), \urcorner(\varkappa_2, \varkappa_3)) > 0$, the following condition holds:*

$$\Phi(\mathcal{H}(\urcorner(\varkappa_1, \varkappa_2), \urcorner(\varkappa_2, \varkappa_3))) \leq \Phi(\max\{\Xi(\varkappa_1, \varkappa_2), \Xi(\varkappa_2, \varkappa_3))^{\sqrt{\mathfrak{A}}}.$$

Then, there exists a point $\varkappa \in \partial$ such that $\varkappa \in \urcorner(\varkappa, \varkappa)$. Moreover, if $\varkappa_1$ and $\varkappa_2$ are arbitrary points in ∂ such that for $\wp \in N$, $\varkappa_{\wp+2} \in \urcorner(\varkappa_\wp, \varkappa_{\wp+1})$, then the sequence $\{\varkappa_\wp\}$ converges to $\varkappa$.

Proof. Follow the proof of Theorem 7 for $k = 2$. □

Corollary 2. *Let (∂, Ξ) be a complete MS and $\urcorner : \partial^2 \to \mathcal{C}(\partial)$. If there exists $\mathfrak{A} \in (0,1)$ such that*

$$\frac{\mathcal{H}(\urcorner(\varkappa, y), \urcorner(y, z))}{\max\{\Xi(\varkappa, y), \Xi(y, z)\}} \exp\{\mathcal{H}(\urcorner(\varkappa, y), \urcorner(y, z)) - \max\{\Xi(\varkappa, y), \Xi(y, z)\}\} \leq \mathfrak{A}$$

holds for all $\varkappa, y, z \in \partial$ with $\mathcal{H}(\urcorner(\varkappa, y), \urcorner(y, z)) > 0$, then there exists a point $u \in \partial$ such that $u \in \urcorner(u, u)$. Moreover, if $\varkappa_1$ and $\varkappa_2$ are arbitrary points in ∂ such that for $\wp \in N$, we take

$$\varkappa_{\wp+2} \in \urcorner(\varkappa_\wp, \varkappa_{\wp+1}),$$

then the sequence $\{\varkappa_\wp\}$ converges to u.

Proof. The results follows immediately by taking $k = 2$ and $\Phi(t) = e^{\sqrt{te^t}}$ in Theorem 5. □

Corollary 3. *Let (∂, Ξ) be a complete MS and $\urcorner : \partial^2 \to \mathcal{C}(\partial)$. If there exists $\mathfrak{A} \in (0,1)$ such that*

$$\mathcal{H}(\urcorner(\varkappa, y), \urcorner(y, z)) \leq \mathfrak{A} \max\{\Xi(\varkappa, y), \Xi(y, z)\}$$

holds for all $\varkappa, y, z \in \partial$ with $\mathcal{H}(\urcorner(\varkappa, y), \urcorner(y, z)) > 0$, then there exists a point $u \in \partial$ such that $u \in \urcorner(u, u)$. Moreover, if $\varkappa_1$ and $\varkappa_2$ are arbitrary points in ∂ such that for $\wp \in N$, we take

$$\varkappa_{\wp+2} \in \urcorner(\varkappa_\wp, \varkappa_{\wp+1}),$$

then the sequence $\{\varkappa_\wp\}$ converges to u.

Proof. The result follows by taking $k = 2$ and $\Phi(t) = e^{\sqrt{t}}$ in Theorem 7. □

Now, for a fixed point of mapping $F : \partial^k \to [\mathcal{C}(\partial)]^\wp$, we present the following theorem:

Theorem 8. *Let (∂, Ξ) be a complete MS and $\Phi \in \Theta^*$. $\urcorner : \partial^k \to \mathcal{C}(\partial)$ satisfies (1) for some $\mathfrak{A} \in (0,1)$. For any $\varkappa_1, \varkappa_2, \ldots, \varkappa_k \in \partial$ and $\wp \in N$, define $F : \partial^k \to [\mathcal{C}(\partial)]^\wp$ by*

$$F(\varkappa_1, \varkappa_2, \ldots, \varkappa_k) = (\urcorner(\varkappa_1, \varkappa_2, \ldots, \varkappa_k), \ldots, \urcorner(\varkappa_1, \varkappa_2, \ldots, \varkappa_k)).$$

Then, there exists a point $(\varkappa, \varkappa, \ldots, \varkappa) \in \partial^k$ such that

$$(\varkappa, \varkappa, \ldots, \varkappa) \in F(\varkappa, \varkappa, \ldots, \varkappa).$$

Proof. $\urcorner$ fulfils the relation (1); therefore, by Theorem 7, there exists $\varkappa \in \partial$ such that $\varkappa \in \urcorner(\varkappa, \varkappa, \ldots \varkappa)$ and hence $(\varkappa, \varkappa, \ldots, \varkappa) \in F(\varkappa, \varkappa, \ldots, \varkappa)$. □

To obtain an FP for the mapping $\urcorner : \partial^k \to \mathcal{CB}(\partial)$, we consider a subclass $\Theta^{**}$of Θ^*, which consists of the elements $\Phi \in \Theta^*$ and satisfies the following condition:

(Φ_5) $\Phi(\inf A) \geq \inf \Phi(A)$, for any subset A of $(0, \infty)$ with $\inf A > 0$.

Theorem 9. *Let (∂, Ξ) be a complete MS, $\urcorner : \partial^k \to \mathcal{CB}(\partial)$ and $\Phi \in \Theta^{**}$. If there exists $\mathfrak{A} \in (0,1)$ such that for all $\varkappa_1, \varkappa_2, \dots, \varkappa_{k+1} \in \partial$, with $\mathcal{H}(\urcorner(\varkappa_1, \varkappa_2, \dots, \varkappa_k), \urcorner(\varkappa_2, \varkappa_3, \dots, \varkappa_{k+1})) > 0$,*

$$\Phi(\mathcal{H}(\urcorner(\varkappa_1, \varkappa_2, \dots, \varkappa_k), \urcorner(\varkappa_2, \varkappa_3, \dots, \varkappa_{k+1}))) \leq \Phi(\max_{1 \leq i \leq k}\{\Xi(\varkappa_i, \varkappa_{i+1})\})^{\sqrt{\mathfrak{A}}}, \tag{4}$$

then there exists a point $\varkappa \in \partial$ such that $\varkappa \in \urcorner(\varkappa, \varkappa, \dots, \varkappa)$. Moreover, if $\varkappa_1, \varkappa_2, \dots, \varkappa_k$ are arbitrary points in ∂, and for $\wp \in N$, we take

$$\varkappa_{\wp+k} \in \urcorner(\varkappa_\wp, \varkappa_{\wp+1}, \dots, \varkappa_{\wp+k-1}),$$

then the sequence $\{\varkappa_\wp\}$ converges to $\varkappa$.

Proof. Let $\varkappa_1, \varkappa_2, \varkappa_3, \dots, \varkappa_{k+1} \in \partial$ be such that $\varkappa_{k+1} \in \urcorner(\varkappa_1, \varkappa_2, \dots, \varkappa_k)$.

Now, for $\urcorner(\varkappa_1, \varkappa_2, \dots, \varkappa_k), \urcorner(\varkappa_2, \varkappa_3, \dots, \varkappa_{k+1}) \in \mathcal{CB}(\partial)$, we have

$$\urcorner(\varkappa_1, \varkappa_2, \dots, \varkappa_k) \neq \urcorner(\varkappa_2, \varkappa_3, \dots, \varkappa_{k+1}).$$

$\mathcal{H}(\urcorner(\varkappa_1, \varkappa_2, \dots, \varkappa_k), \urcorner(\varkappa_2, \varkappa_3, \dots, \varkappa_{k+1})) > 0$, thus it follows from Lemma 3 that

$$\Phi(\Xi(\varkappa_{k+1}, \urcorner(\varkappa_2, \varkappa_3, \dots, \varkappa_{k+1}))) \leq \Phi(\mathcal{H}(\urcorner(\varkappa_1, \varkappa_2, \dots, \varkappa_k), \urcorner(\varkappa_2, \varkappa_3, \dots, \varkappa_{k+1}))).$$

By condition Φ_5, we have

$$\inf_{y \in \urcorner(\varkappa_2, \varkappa_3, \dots, \varkappa_{k+1})} \Phi(\Xi(\varkappa_{k+1}, y)) \leq \Phi(\mathcal{H}(\urcorner(\varkappa_1, \varkappa_2, \dots, \varkappa_k), \urcorner(\varkappa_2, \varkappa_3, \dots, \varkappa_{k+1})))$$

and hence there exists $\varkappa_{k+2} \in \urcorner(\varkappa_2, \varkappa_3, \dots, \varkappa_{k+1})$ such that

$$\Phi(\Xi(\varkappa_{k+1}, \varkappa_{k+2})) \leq \Phi(\mathcal{H}(\urcorner(\varkappa_1, \varkappa_2, \dots, \varkappa_k), \urcorner(\varkappa_2, \varkappa_3, \dots, \varkappa_{k+1}))).$$

Furthermore, for $\urcorner(\varkappa_2, \varkappa_3, \dots, \varkappa_{k+1}), \urcorner(\varkappa_3, \varkappa_4, \dots, \varkappa_{k+2}) \in \mathcal{CB}(\partial)$, we have

$$\urcorner(\varkappa_2, \varkappa_3, \dots, \varkappa_{k+1}) \neq \urcorner(\varkappa_3, \varkappa_4, \dots, \varkappa_{k+2}).$$

As $\mathcal{H}(\urcorner(\varkappa_2, \varkappa_3, \dots, \varkappa_{k+1}), \urcorner(\varkappa_3, \varkappa_4, \dots, \varkappa_{k+2})) > 0$, by Lemma 3, we have

$$\Phi(\Xi(\varkappa_{k+2}, \urcorner(\varkappa_3, \varkappa_4, \dots, \varkappa_{k+2}))) \leq \Phi(\mathcal{H}(\urcorner(\varkappa_2, \varkappa_3, \dots, \varkappa_{k+1}), \urcorner(\varkappa_3, \varkappa_4, \dots, \varkappa_{k+2}))).$$

By condition Φ_5, we obtain

$$\inf_{y \in \urcorner(\varkappa_3, \varkappa_4, \dots, \varkappa_{k+2})} \Phi(\Xi(\varkappa_{k+2}, y)) \leq \Phi(\mathcal{H}(\urcorner(\varkappa_2, \varkappa_3, \dots, \varkappa_{k+1}), \urcorner(\varkappa_3, \varkappa_4, \dots, \varkappa_{k+2})))$$

and there exists $\varkappa_{k+3} \in \urcorner(\varkappa_3, \varkappa_4, \dots, \varkappa_{k+2})$ such that

$$\Phi(\Xi(\varkappa_{k+2}, \varkappa_{k+3})) \leq \Phi(\mathcal{H}(\urcorner(\varkappa_2, \varkappa_3, \dots, \varkappa_{k+1}), \urcorner(\varkappa_3, \varkappa_4, \dots, \varkappa_{k+2}))).$$

Continuing this way, we obtain a sequence $\{\varkappa_\wp\}$ such that

$$\varkappa_{\wp+k} \in \urcorner(\varkappa_\wp, \varkappa_{\wp+1}, \dots, \varkappa_{\wp+k-1}), \quad \wp = 1, 2, 3, \dots$$

which satisfies the following relation for all $\wp \in N$

$$\Phi(\Xi(\varkappa_{\wp+k}, \varkappa_{\wp+k+1})) \leq \mathcal{H}(\urcorner(\varkappa_\wp, \varkappa_{\wp+1}, \varkappa_{\wp+2}, \dots, \varkappa_{\wp+k-1}), \urcorner(\varkappa_{\wp+1}, \varkappa_{\wp+2}, \varkappa_{\wp+3}, \dots, \varkappa_{\wp+k})).$$

Using (4), one obtains

$$\Phi(\Xi(\varkappa_{\wp+k},\varkappa_{\wp+k+1}))\leq\Phi(\max_{1\leq i\leq k}\{\Xi(\varkappa_i,\varkappa_{i+1})\})^{\sqrt{\mathfrak{A}}}.$$

By induction and the properties of $\Phi\in\Theta^{**}$, one can write

$$\Xi_{k+\wp}=\Xi(\varkappa_{\wp+k},\varkappa_{\wp+k+1})\leq\frac{1}{\wp^{1/r}}\qquad\forall\wp\geq\wp_0,\ \text{where}\ r\in(0,1).$$

This proves that $\{\varkappa_\wp\}$ is a Cauchy sequence in $(\eth,\Xi)$ and there exists $\varkappa\in\eth$ such that $\lim_{\wp\to\infty}\Xi(\varkappa_\wp,\varkappa)=0$, and

$$\Xi(\varkappa,\daleth(\varkappa,\varkappa,\ldots,\varkappa))=0\Rightarrow\varkappa\in\daleth(\varkappa,\varkappa,\varkappa,\ldots,\varkappa).$$

□

The following examples support Theorem 7.

Example 1. *Let $\eth=[0,k]$, where $k\geq 2$ is a natural number and $\Xi:\eth\times\eth\to\mathcal{C}(\eth)$ is a usual metric defined as follows:*

$$\Xi(\varkappa,y)=|\varkappa-y|,\qquad\forall\varkappa,y\in\eth.$$

A multivalued mapping $\daleth:\eth^k\to C(\eth)$ is defined for all $(\varkappa_1,\varkappa_2,\varkappa_3,\cdots,\varkappa_k)\in\eth^k$ as follows:

$$\daleth(\varkappa_1,\varkappa_2,\varkappa_3,\cdots,\varkappa_k)=\left[0,\frac{\max\{\varkappa_1,\varkappa_2,\varkappa_3,\cdots,\varkappa_k\}}{2k^2}\right].$$

Let us denote

$$F_{\varkappa_i}^{\varkappa_{i+j-1}}=(\varkappa_i,\varkappa_{i+1},\ldots,\varkappa_k,\varkappa_{j+i-1})\in\eth^k.$$

Then, for any $\varkappa_1,\varkappa_2,\varkappa_3,\cdots,\varkappa_{k+1}\in\eth$, we have

$$\begin{aligned}\Phi(\mathcal{H}(\daleth(F_{\varkappa_1}^{\varkappa_k}),\daleth(F_{\varkappa_2}^{\varkappa_{k+1}})))&=e^{\sqrt{\mathcal{H}(\daleth(\varkappa_1,\varkappa_2,\varkappa_3,\cdots,\varkappa_k),\daleth(\varkappa_2,\varkappa_3,\varkappa_4,\cdots,\varkappa_{k+1}))}},\\&=e^{\sqrt{\mathcal{H}\left(\left[0,\frac{\max\{\varkappa_1,\varkappa_2,\varkappa_3,\cdots,\varkappa_k\}}{2k^2}\right],\left[0,\frac{\max\{\varkappa_2,\varkappa_3,\varkappa_4\cdots,\varkappa_{k+1}\}}{2k^2}\right]\right)}},\\&\leq e^{\sqrt{\frac{1}{2k}|\varkappa_1-\varkappa_{k+1}|}},\\&\leq e^{\sqrt{\frac{1}{2}\max\{|\varkappa_i-\varkappa_{i+1}|:1\leq i\leq k\}}},\\&=(e^{\sqrt{\max\{|\varkappa_i-\varkappa_{i+1}|:1\leq i\leq k\}}})^{\sqrt{1/2}},\\&=\Phi(\max\{|\varkappa_i-\varkappa_{i+1}|:1\leq i\leq k\})^{\sqrt{1/2}},\\&=\Phi(\max\{\Xi(\varkappa_i,\varkappa_{i+1}):1\leq i\leq k\})^{\sqrt{\mathfrak{A}}}.\end{aligned}$$

Hence,

$$\Phi(\mathcal{H}(\daleth(F_{\varkappa_1}^{\varkappa_k}),\daleth(F_{\varkappa_2}^{\varkappa_{k+1}})))\leq\Phi(\max\{\Xi(\varkappa_i,\varkappa_{i+1}):1\leq i\leq k\})^{\sqrt{\mathfrak{A}}}$$

and thus $\daleth$ satisfies all the conditions of Theorem 7 for $\mathfrak{A}=1/2$ and $\Phi(t)=e^{\sqrt{t}}$. Hence, $\daleth$ has a fixed point which is $0\in\daleth(0,0,0,\cdots,0)$.

Example 2. *Let $\eth=\{0,1,2,3,\ldots\}$ and the metric Ξ on $\eth$ be defined as follows:*

$$\Xi(\varkappa,\mathfrak{z})=\begin{cases}0 & ;\ \varkappa=\mathfrak{z},\\ \varkappa+\mathfrak{z} & ;\ \varkappa\neq\mathfrak{z}.\end{cases}$$

Since every finite subset of $\eth$ is compact, we can describe the mapping $\urcorner : \eth^2 \to \mathcal{C}(\eth)$ as

$$\urcorner(\varkappa,\mathfrak{z}) = \begin{cases} \{0\} & ; \ \varkappa = \mathfrak{z}, \\ \{0,1,2,3,\ldots,\max\{\varkappa,\mathfrak{z}\}-1\} & ; \ \varkappa \neq \mathfrak{z}. \end{cases}$$

This mapping does not satisfy the condition

$$\mathcal{H}(\urcorner(\varkappa,\mathfrak{z}),\urcorner(\mathfrak{z},\rho)) \leq \mathfrak{A}\max\{\Xi(\varkappa,\mathfrak{z}),\Xi(\mathfrak{z},\rho)\},$$

but it satisfies the required condition of Theorem 7 for $\Phi(\mathfrak{y}) = e^{\sqrt{\mathfrak{y}e^{\mathfrak{y}}}}$.

Indeed, for any $\rho > 1$, $\varkappa = 0$, and $\mathfrak{z} = 1$, one has

$$\mathcal{H}(\urcorner(0,1),\urcorner(1,\rho)) = \mathcal{H}(\{0\},\{0,1,2,\ldots,\rho-1\}) = \Xi(0,\rho-1) = \rho-1$$

and

$$\max\{\Xi(0,1),\Xi(1,\rho)\} = \max\{1,1+\rho\} = 1+\rho.$$

Since $\sup_{\rho\in\eth}\frac{\rho-1}{\rho+1} = 1$, then there does not exist any $\mathfrak{A}$ such that the above relation holds for all $\varkappa,\mathfrak{z},\rho \in \eth$. However, we now show our main result is applicable here for $\mathfrak{A} = e^{-1}$.
If $\varkappa < \mathfrak{z} < \rho$,

$$\begin{aligned} A &= \frac{\mathcal{H}(\urcorner(\varkappa,\mathfrak{z}),\urcorner(\mathfrak{z},\rho))e^{\mathcal{H}(\urcorner(\varkappa,\mathfrak{z}),\urcorner(\mathfrak{z},\rho))}}{\max\{\Xi(\varkappa,\mathfrak{z}),\Xi(\mathfrak{z},\rho)\}e^{\max\{\Xi(\varkappa,\mathfrak{z}),\Xi(\mathfrak{z},\rho)\}}} \\ &= \frac{\mathcal{H}(\{0,1,2,\ldots,\mathfrak{z}-1\},\{0,1,2,\ldots,\rho-1\})e^{\mathcal{H}(\{0,1,2,\ldots,\mathfrak{z}-1\},\{0,1,2,\ldots,\rho-1\})}}{\max\{\varkappa+\mathfrak{z},\mathfrak{z}+\rho\}e^{\max\{\varkappa+\mathfrak{z},\mathfrak{z}+\rho\}}} \\ &= \frac{\max\{\Xi(\mathfrak{z}-1,0),\Xi(0,\rho-1)\}e^{\max\{\Xi(\mathfrak{z}-1,0),\Xi(0,\rho-1)\}}}{(\mathfrak{z}+\rho)e^{\mathfrak{z}+\rho}} \\ &= \frac{(\rho-1)e^{\rho-1}}{(\mathfrak{z}+\rho)e^{\mathfrak{z}+\rho}} = \frac{(\rho-1)e^{-1-\mathfrak{z}}}{(\mathfrak{z}+\rho)} \leq e^{-1}. \end{aligned}$$

If $\varkappa = \mathfrak{z} < \rho$,

$$\begin{aligned} \frac{\mathcal{H}(\urcorner(\varkappa,\mathfrak{z}),\urcorner(\mathfrak{z},\rho))e^{\mathcal{H}(\urcorner(\varkappa,\mathfrak{z}),\urcorner(\mathfrak{z},\rho))}}{\max\{\Xi(\varkappa,\mathfrak{z}),\Xi(\mathfrak{z},\rho)\}e^{\max\{\Xi(\varkappa,\mathfrak{z}),\Xi(\mathfrak{z},\rho)\}}} &= \frac{\mathcal{H}(\{0\},\{0,1,2,\ldots,\rho-1\})e^{\mathcal{H}(\{0\},\{0,1,2,\ldots,\rho-1\})}}{\max\{\varkappa+\mathfrak{z},\mathfrak{z}+\rho\}e^{\max\{\varkappa+\mathfrak{z},\mathfrak{z}+\rho\}}} \\ &= \frac{\max(0,\Xi(0,\rho-1))e^{\max(\Xi(0,\rho-1),0)}}{(\mathfrak{z}+\rho)e^{\mathfrak{z}+\rho}} \\ &= \frac{(\rho-1)e^{-1-\mathfrak{z}}}{(\mathfrak{z}+\rho)} \leq e^{-1}. \end{aligned}$$

Similarly, if $\varkappa < \mathfrak{z} = \rho$, one obtains

$$\frac{\mathcal{H}(\urcorner(\varkappa,\mathfrak{z}),\urcorner(\mathfrak{z},\rho))e^{\mathcal{H}(\urcorner(\varkappa,\mathfrak{z}),\urcorner(\mathfrak{z},\rho))}}{\max\{\Xi(\varkappa,\mathfrak{z}),\Xi(\mathfrak{z},\rho)\}e^{\max\{\Xi(\varkappa,\mathfrak{z}),\Xi(\mathfrak{z},\rho)\}}} \leq e^{-1}.$$

Thus, for any $\varkappa,\mathfrak{z},\rho \in \eth$, we have

$$\mathcal{H}(\urcorner(\varkappa,\mathfrak{z}),\urcorner(\mathfrak{z},\rho))e^{\mathcal{H}(\urcorner(\varkappa,\mathfrak{z}),\urcorner(\mathfrak{z},\rho))} \leq e^{-1}\max\{\Xi(\varkappa,\mathfrak{z}),\Xi(\mathfrak{z},\rho)\}e^{\max\{\Xi(\varkappa,\mathfrak{z}),\Xi(\mathfrak{z},\rho)\}},$$

or

$$\sqrt{\mathcal{H}(\urcorner(\varkappa,\mathfrak{z}),\urcorner(\mathfrak{z},\rho))e^{\mathcal{H}(\urcorner(\varkappa,\mathfrak{z}),\urcorner(\mathfrak{z},\rho))}} \leq \sqrt{e^{-1}\max\{\Xi(\varkappa,\mathfrak{z}),\Xi(\mathfrak{z},\rho)\}e^{\max\{\Xi(\varkappa,\mathfrak{z}),\Xi(\mathfrak{z},\rho)\}}},$$

or

$$e^{\sqrt{\mathcal{H}(\urcorner(\varkappa,\mathfrak{z}),\urcorner(\mathfrak{z},\rho))e^{\mathcal{H}(\urcorner(\varkappa,\mathfrak{z}),\urcorner(\mathfrak{z},\rho))}}} \leq e^{\sqrt{e^{-1}\max\{\Xi(\varkappa,\mathfrak{z}),\Xi(\mathfrak{z},\rho)\}e^{\max\{\Xi(\varkappa,\mathfrak{z}),\Xi(\mathfrak{z},\rho)\}}}}$$

$$\Xi(\mathcal{H}(\urcorner(\varkappa,\mathfrak{z}),\urcorner(\mathfrak{z},\rho))) \leq \Xi(\max\{\Xi(\varkappa,\mathfrak{z}),\Xi(\mathfrak{z},\rho)\})^{\sqrt{\mathfrak{A}}}.$$

Therefore, by Theorem 7, $\urcorner$ has an FP, which is $0 \in \urcorner(0,0)$.

4. Applications

This part is considered as the mainstay of this paper because it indicates the applications that contribute to solving some nonlinear integral systems that attract many readers and researchers and show the importance of fixed point theory in many areas.

4.1. Some Contributions of Integral Type

Let Ω be class of functions $\varpi : [0,+\infty) \to [0,+\infty)$ that fulfils the following postulates:
(1) For each compact subset of $[0,+\infty)$, ϖ is a positive Lebesgue integrable mapping;
(2) $\int_0^{\epsilon} \varpi(\ell)d\ell > 0$ for all $\epsilon > 0$.

Corollary 4. *Replace the condition (1) of Theorem 7 by the condition*

$$\mathcal{H}(\urcorner(\varkappa_1,\varkappa_2,\ldots,\varkappa_k),\urcorner(\varkappa_2,\varkappa_3,\ldots,\varkappa_{k+1})) > 0,$$

and

$$\int_0^{\Phi(\mathcal{H}(\urcorner(\varkappa_1,\varkappa_2,\ldots,\varkappa_k),\urcorner(\varkappa_2,\varkappa_3,\ldots,\varkappa_{k+1})))} \varpi(\ell)d\ell \leq \int_0^{\Phi(\max\{\Xi(\varkappa_i,\varkappa_{i+1})\})^{\sqrt{\mathfrak{A}}}} \varpi(\ell)d\ell \tag{5}$$

If the remaining conditions of Theorem 7 are true, then the sequence $\{\varkappa_\wp\}$ converges to $\varkappa$.

Proof. Assume the function $\Upsilon(\wp) = \int_0^{\wp} \varpi(\ell)d\ell$, then (5) becomes

$$\Upsilon(\Phi(\mathcal{H}(\urcorner(\varkappa_1,\varkappa_2,\ldots,\varkappa_k),\urcorner(\varkappa_2,\varkappa_3,\ldots,\varkappa_{k+1})))) \leq \Upsilon\Big(\Phi(\max\{\Xi(\varkappa_i,\varkappa_{i+1})\})^{\sqrt{\mathfrak{A}}}\Big).$$

Letting $\Upsilon(\ell) = \ell$ and since $\Upsilon(\ell) \geq 0$, then the proof is quickly completed from Theorem 7. □

By the same line in [37], let a fixed number $p \in \mathbb{N}$. Suppose that $\{\varpi_j\}_{1\leq j\leq p}$ is a collection of p functions which belong to Ω. For each $\ell \geq 0$, we define

$$\begin{aligned}
J_1(\ell) &= \int_0^{\ell} \varpi_1(\rho)d\rho, \\
J_2(\ell) &= \int_0^{J_1(\ell)} \varpi_2(\rho)d\rho = \int_0^{\int_0^{\ell}\varpi_1(\rho)d\rho,} \varpi_2(\rho)d\rho, \\
J_3(\ell) &= \int_0^{J_2(\ell)} \varpi_3(\rho)d\rho = \int_0^{\int_0^{\int_0^{\ell}\varpi_1(\rho)d\rho,}\varpi_2(\rho)d\rho} \varpi_3(\rho)d\rho, \\
&\cdots \\
J_p(\ell) &= \int_0^{J_{(p-1)}(\ell)} \varpi_p(\rho)d\rho.
\end{aligned}$$

We have the following result:

Corollary 5. *Replace the inequality (1) of Theorem 7 by the the following assumption: there is $\varpi \in \Omega$ such that*

$$J_p(\Phi(\mathcal{H}(\urcorner(\varkappa_1, \varkappa_2, \ldots, \varkappa_k), \urcorner(\varkappa_2, \varkappa_3, \ldots, \varkappa_{k+1})))) \le J_p(\Phi(\max\{\Xi(\varkappa_i, \varkappa_{i+1})\})^{\sqrt{\mathfrak{A}}}). \quad (6)$$

If the remaining conditions of Theorem 7 hold, then the sequence $\{\varkappa_\wp\}$ converges to $\varkappa$.

Proof. Specify $J_p(\varkappa) = \varkappa$, then the inequality (6) takes the form

$$\Phi(\mathcal{H}(\urcorner(\varkappa_1, \varkappa_2, \ldots, \varkappa_k), \urcorner(\varkappa_2, \varkappa_3, \ldots, \varkappa_{k+1}))) \le \Phi(\max\{\Xi(\varkappa_i, \varkappa_{i+1})\})^{\sqrt{\mathfrak{A}}}.$$

Applying Theorem 7, we obtain the desired result. □

4.2. Solve a Three Point Boundary Value Problem

An ordinary differential equation, partial differential equation, or a differential equation with a well-posed issue should have a single solution that changes over time depending on the sources. The operator that converts the data into the solution for linear equations is often a linear integral operator [38] and Green's function is the kernel. For various scenarios and problems, ref. [39] contains a set of formula for such Green's functions. According to [40], the best method for solving a boundary value problem (BVP) is to calculate its Green's function. By using the integral expression, it is also possible to obtain some additional qualitative information about the solutions of the problem under consideration, such as their sign, oscillation properties, a priori bounds, or their stability. In this part, we discuss an application of our results by examining the existence of solutions to the following three point BVP:

$$\begin{cases} \frac{d^2\mathfrak{u}}{d\rho^2} + \Upsilon(\rho, \mathfrak{u}(\rho)) = 0 & , \quad \rho \in [r_1, r_2], \\ \mathfrak{u}(r_2) = 0, \quad \mathfrak{u}(r_1) = \alpha\mathfrak{u}(\eta), \end{cases} \quad (7)$$

where Υ is a real-valued continuous function defined on the interval $[r_1, r_2]$ and η is a real number lying between r_1 and r_2 such that $\alpha(r_2 - \eta) \ne r_2 - r_1$. Let us consider the following Green's function [41]:

$$\mathcal{G}(\rho, \varrho) = \mathcal{K}(\rho, \varrho) + \frac{\alpha(r_2 - 1)}{r_2 - r_1 - \alpha(r_2 - \eta)}\mathcal{K}(\eta, \varrho),$$

where

$$\mathcal{K}(\rho, \varrho) = \begin{cases} \frac{(\varrho - r_1)(r_2 - \rho)}{r_2 - r_1} & , \quad r_1 \le \varrho \le \rho \le r_2, \\ \frac{(\rho - r_1)(r_2 - \varrho)}{r_2 - r_1} & , \quad r_1 \le \rho \le \varrho \le r_2. \end{cases}$$

Clearly, the problem (7) is equivalent to the following integral equation:

$$\mathfrak{u}(\rho) = \int_{r_1}^{r_2} \mathcal{G}(\rho, \varrho)\Upsilon(\varrho, \mathfrak{u}(\varrho))d\varrho \quad ; \quad \rho \in [r_1, r_2]. \quad (8)$$

Therefore, $\mathfrak{u}$ is a solution of (7) if and only if it is a solution of (8).
Assume that

$$\partial = \bar{\mathbf{C}}[r_1, r_2]$$

is a class of all real continuous valued functions defined on $[r_1, r_2]$, equipped with the norm

$$||\mathfrak{u}||_\infty = \sup\{|\mathfrak{u}(\rho)| : \rho \in [r_1, r_2]\}.$$

Obviously, the space $(\partial, ||.||_\infty)$ is a complete MS.
Now, our main theorems in this part are as follows:

Theorem 10. *The problem (7) has a solution provided that the following assertions hold:*
(i) There exists two continuous functions $\Upsilon, \hbar : [r_1, r_2] \times R \to R$ *such that*

$$f(\rho, \mathfrak{u}) = \Upsilon(\rho, \mathfrak{u}) + \hbar(\rho, \mathfrak{u})$$

and a continuous function $p : [r_1, r_2] \to [0, \infty)$ *such that*

$$|\Upsilon(\rho, \mathfrak{u}(\rho)) + \hbar(\rho, v(\rho)) - \Upsilon(\rho, v(\rho)) - \hbar(\rho, w(\rho))| \leq Ap(\rho)$$

for all $\mathfrak{u}, v, w \in R$*. Here,* $A = \max\{||\mathfrak{u} - v||_\infty, ||v - w||_\infty\}$*.*
(ii) There exists $\mathfrak{A} < 1$ *such that*

$$e^{\sqrt{\int_{r_1}^{r_2} \mathcal{G}(\rho,\varrho)p(\varrho)d\varrho}} \leq e^{\sqrt{\mathfrak{A}}}.$$

Proof. Let $\eth = \bar{\mathbf{C}}[r_1, r_2]$. Define the MV mapping $\daleth : \eth^2 \to C(\eth)$ by

$$\daleth(\mathfrak{u}(\rho), v(\rho)) = \{F(\mathfrak{u}, v)\}, \quad where \quad F(\mathfrak{u}, v) = \int_{r_1}^{r_2} \mathcal{G}(, \rho)\{\Upsilon(\rho, \mathfrak{u}(\rho)) + \hbar(\rho, v(\rho))\}dt.$$

Then, for any $\mathfrak{u}, v, w \in \eth$ and $\Phi(\rho) = e^{\sqrt{\rho}}$, we have

$$\begin{aligned}
\Phi(\mathcal{H}(\daleth(\mathfrak{u}, v), \daleth(v, w))) &= \Phi(\mathcal{H}(\{F(\mathfrak{u}, v)\}, \{F(v, w)\}\} \\
&= \Phi(|\int_{r_1}^{r_2} \mathcal{G}(\rho, \varrho)(\Upsilon(\varrho, \mathfrak{u}(\varrho)) + h(\varrho, v(\varrho)))d\varrho \\
&- \int_{r_1}^{r_2} \mathcal{G}(\rho, \varrho)(\Upsilon(\varrho, v(\varrho)) + \hbar(\varrho, w(\varrho)))d\varrho|) \\
&\leq \Phi(\int_{r_1}^{r_2} \mathcal{G}(\rho, \varrho)|\Upsilon(\varrho, \mathfrak{u}(\varrho)) + \hbar(\varrho, v(\varrho)) - \Upsilon(\varrho, v(\varrho)) - \hbar(\varrho, w(\varrho))|d\varrho) \\
&\leq \Phi(A \int_{r_1}^{r_1} \mathcal{G}(\rho, \varrho)p(\varrho)d\varrho) \\
&= e^{\sqrt{A \int_{r_1}^{r_2} \mathcal{G}(\rho,\varrho)p(\varrho)d\varrho}} \\
&\leq (e^{\sqrt{A}})^{\sqrt{\mathfrak{A}}} \\
&= \Phi(A)^{\sqrt{\mathfrak{A}}}.
\end{aligned}$$

Hence, it satisfies (1), so by Theorem 5, there exists $\mathfrak{u} \in \eth$ such that $\mathfrak{u} \in \daleth(\mathfrak{u}, \mathfrak{u}) = \{F(\mathfrak{u}, \mathfrak{u})\}$. Therefore,

$$\mathfrak{u}(\rho) = \int_{r_1}^{r_2} \mathcal{G}(\rho, \varrho)(\Upsilon(\varrho, \mathfrak{u}(\varrho)) + \hbar(\varrho, \mathfrak{u}(\varrho)))d\varrho = \int_{r_1}^{r_2} \mathcal{G}(\rho, \varrho)f(\varrho, \mathfrak{u}(\varrho))d\varrho.$$

This illustrates that the BVP (7) has a solution on $\eth$. □

Remark 1. *The existence of a solution to the BVP (7) can be obtained also if we replace condition (i) of Theorem 10 with the following condition:*
(iii) There exist two continuous functions $\Upsilon, \hbar : [r_1, r_2] \times R \to R$ *such that*

$$f(\rho, \mathfrak{u}(\rho)) = \Upsilon(\rho, \mathfrak{u}(\rho))\hbar(\rho, \mathfrak{u}(\rho))$$

and a continuous function $p : [r_1, r_2] \to [0, \infty)$ *such that the inequality below is true*

$$|\Upsilon(\rho, \mathfrak{u}(\rho))\hbar(\rho, v(\rho)) - \Upsilon(\rho, v(\rho))\hbar(\rho, w(\rho))| \leq Ap(\rho)$$

for all $\mathfrak{u}, v, w \in R$. *In addition, taking*

$$F(\mathfrak{u}, v) = \int_{r_1}^{r_2} \mathcal{G}(\rho, \varrho)\Upsilon(\varrho, \mathfrak{u}(\varrho))\hbar(\varrho, v(\varrho))d\varrho.$$

It follows that for any $\mathfrak{u}, v, w \in \eth$ *and* $\Phi(\rho) = e^{\sqrt{\rho}}$, *we have*

$$\begin{aligned}\Phi(\mathcal{H}(\daleth(\mathfrak{u}, v), \daleth(v, w))) &= \Phi(\mathcal{H}(\{F(\mathfrak{u}, v)\}, \{F(v, w)\}) \\ &= \Phi(|\int_{r_1}^{r_2} \mathcal{G}(\rho, \varrho)\Upsilon(\varrho, \mathfrak{u}(\varrho))\hbar(\varrho, v(\varrho))d\varrho \\ &- \int_{r_1}^{r_2} \mathcal{G}(\rho, \varrho)(\Upsilon(\varrho, v(\varrho))\hbar(\varrho, w(\varrho)))d\varrho|) \\ &\leq \Phi(\int_{r_1}^{r_2} \mathcal{G}(\rho, \varrho)|\Upsilon(\varrho, \mathfrak{u}(\varrho))\hbar(\varrho, v(\varrho)) - \Upsilon(\varrho, v(\varrho))\hbar(\varrho, w(\varrho))|d\varrho) \\ &\leq \Phi(A\int_{r_1}^{r_2} \mathcal{G}(\rho, \varrho)p(\varrho)d\varrho) \\ &= e^{\sqrt{A\int_a^b \mathcal{G}(\rho,\varrho)p(\varrho)d\varrho}} \\ &\leq (e^{\sqrt{A}})^{\sqrt{\mathfrak{A}}} \\ &= \Phi(A)^{\sqrt{\mathfrak{A}}}.\end{aligned}$$

Therefore, we the end of the proof is the same as Theorem 10. Therefore, the BVP (7) has a solution on $\eth$.

5. Conclusions

The analytical solution of BVPs by using multi-valued contractive mappings is an important application in fixed point theory, which has attracted the interest of many authors in academic research. Continuing in this direction, this paper discusses some FP results for multi-valued Prešić-type Φ-contraction mappings in MSs. Furthermore, some results were related to previous contributions obtained as corollaries. Moreover, two examples are presented to support the first main result. Ultimately, the applications of some contributions of integral type are discussed and the existence of a solution to the second-order BVP is investigated.

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References

1. Banach, S. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fund. Math.* **1922**, *3*, 133–181. [CrossRef]
2. Kannan, R. Some results on fixed point. *Proc. Am. Math. Soc.* **1973**, *38*, 111–118. [CrossRef]
3. Boyd, D.W.; Wong, J.S.W. On nonlinear contractions. *Proc. Am. Math. Soc.* **1969**, *20*, 458–464. [CrossRef]
4. Rhoades, B.E. Some theorems on weakly contractive maps. *Nonlinear Anal.* **2001**, *47*, 2683–2693. [CrossRef]

5. Berinde, V.; Păcurar, M. Fixed points theorems for unsaturated and saturated classes of contractive mappings in Banach spaces. *Symmetry* **2021**, *13*, 713. [CrossRef]
6. Dutta, P.N.; Choudhury, B.S. A generalization of contraction principle in metric spaces. *Fixed Point Theory Appl.* **2008**, *2008*, 1–8. [CrossRef]
7. Wang, M.; Saleem, N.; Liu, X.; Ansari, A.H.; Zhou, M. Fixed Point of (α, β)-Admissible Generalized Geraghty F-Contraction with Application. *Symmetry* **2016**, *14*, 1016. [CrossRef]
8. Jleli, M.; Samet, B. A new generalization of the Banach contraction principle. *J. Ineq. Appl.* **2014**, *38*, 1–8. [CrossRef]
9. Pant, R.; Patel, P.; Shukla, R.; De la Sen, M. Fixed point theorems for nonexpansive type mappings in Banach spaces. *Symmetry* **2021**, *13*, 585. [CrossRef]
10. Prešić, S.B. Sur une classe d'inéquations aux différences finies et sur la convergence de certaines suites. *Publ. Inst. Math.* **1965**, *5*, 75–78.
11. George, R.; Reshma, K.P.; Rajagopalan, R. A generalised fixed point theorem of Presic type in cone metric spaces and application to Markov process. *Fixed Point Theory Appl.* **2011**, *2011*, 85. [CrossRef]
12. Ćirić, L.B.; Prešić, S.B. On Prešić type generalization of the Banach contraction mapping principle. *Acta Math. Univ. Comen.* **2007**, *76*, 143–147.
13. Altun, I.; Qasim, M.; Olgun, M. A new result of Preši ć type theorems with applications to second order boundary value problems. *Filomat* **2021**, *35*, 2257–2266. [CrossRef]
14. Abbas, M.; Illic, D.; Nazir, T. Iterative approximation of fixed points of generalized weak Prešić type k-step iterative method for a class of operators. *Filomat* **2015**, *29*, 713–724. [CrossRef]
15. Chen, Y.Z. A Prešić type contractive condition and its applications. *Nonlinar Anal.* **2009**, *71*, 2012–2017. [CrossRef]
16. Păcurar, M. A multi-step iterative method for approximating fixed points of Prešić-Kannan operators. *Acta Math. Univ. Comen.* **2010**, *79*, 77–88.
17. Berinde, V.; Păcurar, M. An iterative method for approximating fixed points of Prešić nonexpansive mappings. *Rev. Anal. Numer. Theor. Approx.* **2009**, *38*, 144–153.
18. Berinde, V.; Păcurar, M. Two elementary applications of some Prešić type fixed point theorems. *Creat. Math. Inform.* **2011**, *20*, 32–42. [CrossRef]
19. Khan, M.S.; Berzig, M.; Samet, B. Some convergence results for iterative sequences of Prešić type and applications. *Adv. Differ. Equ.* **2012**, *2012*, 1–12. [CrossRef]
20. Rao, K.P.R.; Sadik, S.K.; Manro, S. Prešić type fixed point theorem for four maps in MSs. *J. Math.* **2016**, *2016*, 2121906. [CrossRef]
21. Hammad, H.A.; De la Sen, M. Tripled fixed point techniques for solving system of tripled fractional differential equations. *AIMS Math.* **2020**, *6*, 2330–2343. [CrossRef]
22. Hammad, H.A.; Aydi, H.; De la Sen, M. Solutions of fractional differential type equations by fixed point techniques for multi-valued contractions. *Complexity* **2021**, *2021*, 5730853. [CrossRef]
23. Fabiano, N.; Nikolič, N.; Thenmozhi, S.; Radenović, S.; Čıtaković, N. Tenth order boundary value problem solution existence by fixed point theorem. *J. Inequal. Appl.* **2020**, *2020*, 166. [CrossRef]
24. Afshari, H.; Kalantari, S.; Karapınar, E. Solution of fractional differential equations via coupled fixed point. *Electron. J. Differ. Equ.* **2015**, *2015*, 286.
25. Bhaskar, T.G.; Lakshmikantham, V. Fixed point theorems in partially ordered metric spaces and applications. *Nonlinear Anal.* **2006**, *65*, 1379–1393. [CrossRef]
26. Lakshmikantham, V.; Cirić, L. Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces. *Nonlinear Anal.* **2009**, *70*, 4341–4349. [CrossRef]
27. Samet, B.; Vetro, C. Coupled fixed point theorems for multi-valued nonlinear contraction mappings in partially ordered metric spaces. *Nonlinear Anal.* **2011**, *74*, 4260–4268. [CrossRef]
28. Hammad, H.A.; Albaqeri, D.M.; Rashwan, R.A. Coupled coincidence point technique and its application for solving nonlinear integral equations in RPOCbML spaces. *J. Egypt. Math. Soc.* **2020**, *28*, 1–17. [CrossRef]
29. Hammad, H.A.; De la Sen, M.; Aydi, H. Analytical solution for differential and nonlinear integral equations via F_{ϖ_e}-Suzuki contractions in modified ϖ_e-metric-like spaces. *J. Funct. Spaces* **2021**, *2021*, 6128586.
30. Nadler, S.B., Jr. Multivalued contraction mapping. *Pac. J. Math.* **1969**, 30, 475–488.
31. Choudhury, B.S.; Metiya, N.; Bandyopadhyay, C. Fixed points of multivalued α-admissible mappings and stability of fixed point sets in metric spaces. *Rend. Circ. Mat. Palermo* **2015**, *64*, 43–55. [CrossRef]
32. Latif, A.; Beg, I. Geometric fixed points for single and multivalued mappings. *Demonstr. Math.* **1997**, *30*, 791–800. [CrossRef]
33. Rasham, T.; Shoaib, A.; Hussain, N.; Alamri, B.A.; Arshad, M. Multivalued fixed point results in dislocated b-metric spaces with application to the system of nonlinear integral equations. *Symmetry* **2019**, *11*, 40. [CrossRef]
34. Abbas, M.; Anjum, R.; Berinde, V. Enriched Multivalued Contractions with Applications to Differential Inclusions and Dynamic Programming. *Symmetry* **2021**, *13*, 1350. [CrossRef]
35. Shukla, S.; Radojević, S.; Veljković, Z.A.; Radenovi ć, S. Some coincidence and common fixed point theorems for ordered Pre šić-Reich type contractions. *J. Inequal. Appl.* **2013**, *2013*, 520. [CrossRef]
36. Latif, A.; Nazir, T.; Abbas, M. Fixed Point Results for Multivalued Prešić Type Weakly Contractive Mappings. *Mathematics* **2019**, *7*, 601. [CrossRef]

37. Nashine, H.K.; Samet, B. Fixed point results for mappings satisfying $(\psi, \phi)-$weakly contractive condition in partially ordered metric spaces. *Nonlinear Anal.* **2011**, *74*, 2201–2209. [CrossRef]
38. Kythe, P.K. *Green's Functions and Linear Differential Equations*; Chapman & Hall/CRC Applied Mathematics and Nonlinear Science Series; CRC Press: Boca Raton, FL, USA, 2011.
39. Duffy, D.G. *Green's Functions with Applications*; Chapman & Hall/CRC: Boca Raton, FL, USA, 2001.
40. Cabada, A.; Cid, J.Á.; Máquez-Villamarín, B. Computation of Green's functions for boundary value problems with Mathematica. *Appl. Math. Comput.* **2012**, 219, 1919–1936. [CrossRef]
41. Zhao, Z. Solutions and Green's functions for some linear second-order three-point boundary value problems. *Comp. Math. Appl.* **2008**, *56*, 104–113. [CrossRef]