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Research article

Existence and stability results for a coupled system of impulsive fractional differential equations with Hadamard fractional derivatives

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Abstract: The purpose of this study is to give some findings on the existence, uniqueness, and Hyers-Ulam stability of the solution of an implicit coupled system of impulsive fractional differential equations possessing a fractional derivative of the Hadamard type. The existence and uniqueness findings are obtained using a fixed point theorem of the type of Kransnoselskii. In keeping with this, many forms of Hyers-Ulam stability are examined. Ultimately, to support main results, an example is provided.

Keywords: impulsive effect; existence solution; fixed point technique; Hadamard fractional derivative; Hyers-Ulam stability
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1. Introduction and building system

In last years, it was noted that several real-world phenomena cannot be modeled by partial or ordinary differential equations or classical difference equations defined using the standard integrals and derivatives. These problems required the concept of fractional calculus (fractional integrals and derivatives), where the classical calculus was insufficient. Differential equations of fractional order are considered to be interesting tools in the modeling of several problems in different fields of engineering and science, as electrochemistry, control, electromagnetic, porous media, viscoelasticity. See for example [1–7]. On the other hand, in the recent years impulsive differential equations have become essential as mathematical models of problems in social and physical sciences. There was a great development in impulsive theory in particular in the field of impulsive differential equations with fixed moments. For instance, see the works of Samoilenko and Perestyuk [8], Benchohra et al. [9], Lakshmikantham et al. [10], etc. Further works for differential equations at variable moments of impulse have been appeared. For example, we cite the papers of Frigon and O'Regan [11, 12], Graef and Ouahab [13], Bajo and Liz [14], etc.

It is also observed that fixed point theory is an important mathematical tool to ensure the existence and uniqueness of many problems intervening nonlinear relations. As a consequence, existence and uniqueness problems of fractional differential equations have been resolved using fixed point techniques. This theory has been developed in many directions and has several applications. Moreover, we could apply it in different types of spaces, like metric spaces, abstract spaces, and Sobolev spaces. This use of fixed point theory makes very easier the resolution of many problems modeled by fractional ordinary, partial differential and difference equations. For instance, see [15–20].

The theory for impulsive fractional differential equations in Banach spaces have been sufficiently developed by Feckan et al. [21] by using fixed point techniques. In the real world, many phenomena are subject to transient external effects as they develop. In comparison to the entire duration of the phenomenon being observed, the durations of these external effects are incredibly brief. The logical conclusion is that these external forces are real impulses. Impulsive differential equations are now a major component of the modeling of physical real-world issues in order to study these abrupt shifts. Biological systems including heartbeat, blood flow, and impulse rate have been discussed in relation to many applications of this kind of impulsive differential equations. For more details, see, [22–27].

On the other hand, in last years the study of Hyers-Ulam (HU) stability analysis for nonlinear fractional differential equations has attracted the attention of several researchers. Note that HU stability is considered as an exact solution near the approximate solution for these equations with minimal error. The following works [28–32] deal with such a stability analysis. For Hyers-Ulam (HU) stabilities, there are generalized Hyers-Ulam (GHU), Hyers-Ulam-Rassias (HUR), and generalized Hyers-Ulam-Rassias (GHUR) stabilities.

Much of the work on the topic of fractional differential equations deals with the governing equations involving Riemann-Liouville and Caputo-type fractional derivatives. Another kind of fractional derivative is the Hadamard type [33], which was introduced in 1892. This derivative differs significantly from both the Riemann-Liouville type and the Caputo type in the sense that the kernel of the integral in the definition of the Hadamard derivative contains a logarithmic function of arbitrary exponent. It seems that the abstract fractional differential equations involving Hadamard fractional derivatives and Hilfer-Hadamard fractional derivatives have not been fully explored so far. Several applications of where the Hadamard derivative and the Hadamard integral arise can be found in the papers by Butzer, Kilbas and Trujillo [34–36]. Other important results dealing with Hadamard fractional calculus and Hadamard differential equations can be found in [37, 38]. The presence of the δ-differential operator ($\delta = x\frac{d}{dx}$) in the definition of Hadamard fractional derivatives could make their study uninteresting and less applicable than Riemann-Liouville and Caputo fractional derivatives. Moreover, this operator appears outside the integral in the definition of the Hadamard derivatives just like the usual derivative $D = \frac{d}{dx}$ is located outside the integral in the case of Riemann-Liouville, which makes the fractional derivative of a constant of these two types not equal to zero in general. Hadamard [33] proposed a fractional power of the form $(x\frac{d}{dx})^{\alpha}$. This fractional derivative is invariant with respect to dilation on the whole axis.

The existence and HU stability of the following implicit FDEs involving Hadamard derivatives were investigated in [39] as follows:

$$\begin{cases} {}^{H}D^{\varpi}z(\upsilon) = \phi\left(\upsilon, z(\upsilon), {}^{H}D^{\varpi}z(\upsilon)\right), \ \varpi \in (0,1), \\ z(1) = z_1, \ \ z_1 \in \mathbb{R}, \end{cases}$$

where $\upsilon \in [1, G]$, $G > 1$, ${}^{H}D^{\varpi}$ refers to the Hadamard fractional (HF) derivative of order ϖ.

The following coupled system containing the Caputo derivative was examined in [40] for its existence, uniqueness, and several types of Hyers-Ulam stability:

$$\begin{cases} {}^{C}D^{\varpi}z(\upsilon) = \phi\left(\upsilon, s(\upsilon), {}^{C}D^{\varpi}z(\upsilon)\right), \ \upsilon \in U, \\ {}^{C}D^{\theta}s(\upsilon) = \psi\left(\upsilon, z(\upsilon), {}^{C}D^{\theta}s(\upsilon)\right), \ \upsilon \in U, \\ z'(G) = z''(0) = 0, \ z(1) = \varrho z(\eta) \ \ \varrho, \eta \in (0,1), \\ s'(G) = s''(0) = 0, \ s(1) = \varrho s(\eta) \ \ \varrho, \eta \in (0,1), \end{cases}$$

where $\upsilon \in U = [0, 1]$, $\varpi, \theta \in (2, 3]$ and $\phi, \psi : U \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions.

For the following coupled system containing the Riemann-Liouville derivative, the authors of [41] demonstrated the existence, uniqueness, and several types of Hyers-Ulam stability:

$$\begin{cases} D^{\varpi}z(\upsilon) = \phi\left(\upsilon, s(\upsilon), D^{\varpi}z(\upsilon)\right), \ \ \upsilon \in U, \\ D^{\theta}s(\upsilon) = \psi\left(\upsilon, z(\upsilon), D^{\theta}s(\upsilon)\right), \ \ \upsilon \in U, \\ D^{\varpi-2}z(0^+) = \pi_1 D^{\varpi-2}z(G^-), \ \ D^{\varpi-2}z(0^+) = \ell_1 D^{\varpi-1}z(G^-), \\ D^{\varpi-2}s(0^+) = \pi_2 D^{\varpi-2}s(G^-), \ \ D^{\varpi-2}s(0^+) = \ell_2 D^{\varpi-1}s(G^-), \end{cases}$$

where $\upsilon \in U = [0, G]$, $G > 0$, $\varpi, \theta \in (1, 2]$ and $\pi_1, \pi_2, \ell_1, \ell_2 \neq 1$, D^{ϖ}, D^{θ} are Riemann-Liouville derivatives of fractional orders ϖ, θ respectively and $\phi, \psi : U \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions.

Inspired by the previous work, we investigate the coupled impulsive implicit FDEs (CII-FDEs) incorporating Hadamard derivatives as follows:

$$\begin{cases} {}^{H}D^{\varpi}z(\upsilon) = \phi\left(\upsilon, {}^{H}D^{\varpi}z(\upsilon), {}^{H}D^{\theta}s(\upsilon)\right), \ \upsilon \in U, \ \upsilon \neq \upsilon_i, \ i = 1, 2, ...k, \\ {}^{H}D^{\theta}s(\upsilon) = \psi\left(\upsilon, {}^{H}D^{\theta}s(\upsilon), {}^{H}D^{\varpi}z(\upsilon)\right), \ \upsilon \in U, \ \upsilon \neq \upsilon_j, \ j = 1, 2, ...m, \\ \Delta z(\upsilon_i) = I_i z(\upsilon_i), \ \ \Delta z'(\upsilon_i) = \widetilde{I}_i z(\upsilon_i), \ \ i = 1, 2, ...k, \\ \Delta s(\upsilon_j) = I_j s(\upsilon_j), \ \ \Delta s'(\upsilon_j) = \widetilde{I}_j s(\upsilon_j), \ \ j = 1, 2, ...m, \\ z(G) = \frac{1}{\Gamma(\varpi)} \int_1^G \ln\left(\frac{G}{\eta}\right)^{\varpi-1} B(\eta, z(\eta))\frac{d\eta}{\eta}, \ z'(G) = B^*(z), \\ s(G) = \frac{1}{\Gamma(\theta)} \int_1^G \ln\left(\frac{G}{\eta}\right)^{\theta-1} B(\eta, s(\eta))\frac{d\eta}{\eta}, \ s'(G) = B^*(s), \end{cases} \tag{1.1}$$

where $\varpi, \theta \in (1,2]$, $\phi, \psi : U\times\mathbb{R}^2 \to \mathbb{R}$, $B : U\times C(U,\mathbb{R}) \to \mathbb{R}$ and $B^* : U \to \mathbb{R}$ are continuous functions and

$$\begin{aligned}\Delta z(\upsilon_i) &= z(\upsilon_i^+) - z(\upsilon_i^-), \ \ \Delta z'(\upsilon_i) = z'(\upsilon_i^+) - z'(\upsilon_i^-),\\ \Delta s(\upsilon_i) &= s(\upsilon_i^+) - s(\upsilon_i^-), \ \ \Delta s'(\upsilon_i) = s'(\upsilon_i^+) - s'(\upsilon_i^-).\end{aligned}$$

The derivatives ${}^H D^{\varpi}, {}^H D^{\theta}$ are the Hadamard derivative operators of order ϖ and θ, respectively; $z(\upsilon_i^+), s(\upsilon_i^+)$ are right limits and $z(\upsilon_i^-), s(\upsilon_i^-)$ are left limits; $I_i, I_j, \widetilde{I_i}, \widetilde{I_j} : \mathbb{R} \to \mathbb{R}$ are continuous functions. The system (1.1) is used to describe certain features of applied mathematics and physics such as blood flow problems, chemical engineering, thermoelasticity, underground water flow, and population dynamics. For more details, we refer the readers to see the monograph [42].

Using the Banach contraction and Kransnoselskii FP theorems, we establish necessary and sufficient criteria for the existence and uniqueness of a positive solution for the problem (1.1). Additionally, we analyze other Hyers-Ulam (HU) stabilities such as generalized Hyers-Ulam (GHU), Hyers-Ulam-Rassias (HUR), and generalized Hyers-Ulam-Rassias (GHUR) stabilities.

2. Basic facts

In this part, we present certain key terms and lemmas that are utilized throughout the rest of this paper, for more information, see [42, 43].

Assume that $PC(U,\mathbb{R}_+)$ equipped with the norms $\|z\| = \max\{|z(\upsilon)| : \upsilon \in U\}$, $\|s\| = \max\{|s(\upsilon)| : \upsilon \in U\}$ is a Banach space (shortly, BS), then the products of these norms are also a BS under the norm $\|(z + s)\| = \|z\| + \|s\|$.

Assume that $\mathfrak{I}_1$ and $\mathfrak{I}_2$ represent the piecewise continuous function spaces described as

$$\begin{aligned}\mathfrak{I}_1 &= PC_{2-\varpi,\ln}(U,\mathbb{R}_+) = \{z : U \to \mathbb{R}_+ \text{ so that } z(\upsilon_i^+), z'(\upsilon_i^+) \text{ and } z(\upsilon_i^-), z'(\upsilon_i^-) \text{ exist }, i = 1, 2, ...k\},\\ \mathfrak{I}_2 &= PC_{2-\theta,\ln}(U,\mathbb{R}_+) = \{s : U \to \mathbb{R}_+ \text{ so that } s(\upsilon_j^+), s'(\upsilon_j^+) \text{ and } s(\upsilon_j^-), s'(\upsilon_j^-) \text{ exist }, j = 1, 2, ...m\},\end{aligned}$$

with norms

$$\|z\|_{\mathfrak{I}_1} = \sup\left\{\left|z(\upsilon)\ln(\upsilon)^{2-\varpi}\right|, \ \upsilon \in U\right\} \text{ and } \|s\|_{\mathfrak{I}_2} = \sup\left\{\left|s(\upsilon)\ln(\upsilon)^{2-\theta}\right|, \ \upsilon \in U\right\},$$

respectively. Clearly, the product $\mathfrak{I} = \mathfrak{I}_1 \times \mathfrak{I}_2$ is a BS endowed with $\|(z + s)\|_{\mathfrak{I}} = \|z\|_{\mathfrak{I}_1} + \|s\|_{\mathfrak{I}_2}$.

The following definitions are recalled from [44].

Definition 2.1. For the function $z(\upsilon)$, the Hadamard fractional (HF) integral of order ϖ is described as

$${}^H I^{\varpi} z(\upsilon) = \frac{1}{\Gamma(\varpi)}\int_1^{\upsilon} \ln\left(\frac{\upsilon}{\eta}\right)^{\varpi-1} z(\eta)\frac{d\eta}{\eta}, \ \upsilon \in (1, G]$$

where $\Gamma(.)$ is the Gamma function.

Definition 2.2. For the function $z(\upsilon)$, the HF derivative of order $\varpi \in [a - 1, a)$, $a \in \mathbb{Z}_+$ is described as

$${}^H D^{\varpi} z(\upsilon) = \frac{1}{\Gamma(a - \varpi)}\left(\upsilon\frac{d}{d\upsilon}\right)^a \int_x^{\upsilon} \ln\left(\frac{\upsilon}{\eta}\right)^{a-\varpi+1} z(\eta)\frac{d\eta}{\eta}, \ \upsilon \in (x, G].$$

Lemma 2.3. *[45] Assume that $\varpi > 0$ and z is any function, then the derivative equation ${}^{H}D^{\varpi}z(\upsilon) = 0$ has solutions below:*

$$z(\upsilon) = r_1 (\ln \upsilon)^{\varpi-1} + r_2 (\ln \upsilon)^{\varpi-2} + r_3 (\ln \upsilon)^{\varpi-3} + ... + r_a (\ln \upsilon)^{\varpi-a},$$

and the formula

$${}^{H}I^{\varpi H}D^{\varpi}z(\upsilon) = z(\upsilon) + r_1 (\ln \upsilon)^{\varpi-1} + r_2 (\ln \upsilon)^{\varpi-2} + r_3 (\ln \upsilon)^{\varpi-3} + ... + r_a (\ln \upsilon)^{\varpi-a},$$

is satisfied, where $r_i \in \mathbb{R}$, $i = 1, 2, ..., a$ and $\varpi \in (a-1, a)$.

Theorem 2.4. *[46] Assume that Ξ is a non-empty, convex and closed subset of a BS $\Im$. Let E and $\widetilde{E}$ be operators so that*

(1) for $z, s \in \Xi$, $E(z, s) + \widetilde{E}(z, s) \in \Xi$;
(2) the operator $\widetilde{E}$ is completely continuous;
(3) the operator Ξ is contractive.

Then there is a solution $(z, s) \in \Xi$ for the operator equation $E(z, s) + \widetilde{E}(z, s) = (z, s)$.

3. Definitions of HU stability

The definitions and observations below are taken from [47, 48].

Definition 3.1. The coupled problem (1.1) is called HU stable if there are $\Lambda_{\varpi,\theta} = \max\{\Lambda_{\varpi}, \Lambda_{\theta}\} > 0$ so that, for $\varphi = \max\{\varphi_{\varpi}, \varphi_{\theta}\}$ and for each solution $(z, s) \in \Im$ to inequalities

$$\begin{cases} \left|{}^{H}D^{\varpi}z(\upsilon) - \phi\left(\upsilon, {}^{H}D^{\varpi}z(\upsilon), {}^{H}D^{\theta}s(\upsilon)\right)\right| \le \varphi_{\varpi}, \ \upsilon \in U, \\ |\Delta z(\upsilon_i) - I_i z(\upsilon_i)| \le \varphi_{\varpi}, \ \left|\Delta z'(\upsilon_i) - \overline{I}_i z(\upsilon_i)\right| \le \varphi_{\varpi}, \ i = 1, 2, ...k, \\ \left|{}^{H}D^{\theta}s(\upsilon) - \phi\left(\upsilon, {}^{H}D^{\theta}s(\upsilon), {}^{H}D^{\varpi}z(\upsilon)\right)\right| \le \varphi_{\theta}, \ \upsilon \in U, \\ \left|\Delta s(\upsilon_j) - I_j s(\upsilon_j)\right| \le \varphi_{\theta}, \ \left|\Delta s'(\upsilon_j) - \overline{I}_j s(\upsilon_j)\right| \le \varphi_{\theta}, \ j = 1, 2, ...m, \end{cases} \tag{3.1}$$

there is a unique solution $(\widetilde{z}, \widetilde{s}) \in \Im$ with

$$\|(z, s) - (\widetilde{z}, \widetilde{s})\|_{\Im} \le \Lambda_{\varpi,\theta}\varphi, \ \upsilon \in U.$$

Definition 3.2. The coupled problem (1.1) is called GHU stable if there is $\Phi \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $\xi(0) = 0$, so that, for any solution $(z, s) \in \Im$ of (3.1), there is a unique solution $(\widetilde{z}, \widetilde{s}) \in \Im$ of with of (1.1) fulfilling

$$\|(z, s) - (\widetilde{z}, \widetilde{s})\|_{\Im} \le \Phi(\varphi), \ \upsilon \in U.$$

Set $\mho_{\varpi,\theta} = \max\{\mho_{\varpi}, \mho_{\theta}\} \in C(U, \mathbb{R})$ and $\Lambda_{\mho_{\varpi},\mho_{\theta}} = \max\{\Lambda_{\mho_{\varpi}}, \Lambda_{\mho_{\theta}}\} > 0$.

Definition 3.3. The coupled problem (1.1) is called HUR stable with respect to $\mho_{\varpi,\theta}$ if there is a constant $\Lambda_{\mho_{\varpi},\mho_{\theta}}$ so that, for any solution $(z, s) \in \Im$ for the inequalities below

$$\begin{cases} \left|{}^{H}D^{\varpi}z(\upsilon) - \phi\left(\upsilon, {}^{H}D^{\varpi}z(\upsilon), {}^{H}D^{\theta}s(\upsilon)\right)\right| \le \mho_{\varpi}(\upsilon)\varphi_{\varpi}, \ \upsilon \in U, \\ \left|{}^{H}D^{\theta}s(\upsilon) - \phi\left(\upsilon, {}^{H}D^{\theta}s(\upsilon), {}^{H}D^{\varpi}z(\upsilon)\right)\right| \le \mho_{\theta}(\upsilon)\varphi_{\theta}, \ \upsilon \in U, \end{cases} \tag{3.2}$$

there is a unique solution $(\widetilde{z}, \widetilde{s}) \in \Im$ with

$$\|(z, s) - (\widetilde{z}, \widetilde{s})\|_{\Im} \le \Lambda_{\mho_{\varpi},\mho_{\theta}}\mho_{\varpi,\theta}\varphi, \ \upsilon \in U. \tag{3.3}$$

Definition 3.4. The coupled problem (1.1) is called GHUR stable with respect to $\mho_{\varpi,\theta}$ if there is a constant $\Lambda_{\mho_{\varpi},\mho_{\theta}}$ so that, for any a proximate solution $(z,s) \in \mathfrak{I}$ of (3.2), there is a unique solution $(\widetilde{z},\widetilde{s}) \in \mathfrak{I}$ of with of (1.1) fulfilling

$$\|(z,s)-(\widetilde{z},\widetilde{s})\|_{\mathfrak{I}} \leq \Lambda_{\mho_{\varpi},\mho_{\theta}}\mho_{\varpi,\theta}(\upsilon),\ \upsilon \in U.$$

Remark 3.5. If there are functions $\mathfrak{R}_{\phi}, \mathfrak{R}_{\psi} \in C(U,\mathbb{R})$ depending upon z, s, respectively, so that

(R_1) $\left|\mathfrak{R}_{\phi}(\upsilon)\right| \leq \varphi_{\varpi}, \left|\mathfrak{R}_{\psi}(\upsilon)\right| \leq \varphi_{\theta}, \upsilon \in U$;

(R_2)

$$\begin{cases} {}^{H}D^{\varpi}z(\upsilon) = \phi\left(\upsilon,{}^{H}D^{\varpi}z(\upsilon),{}^{H}D^{\theta}s(\upsilon)\right) + \mathfrak{R}_{\phi}(\upsilon), \\ \qquad \Delta z(\upsilon_i) = I_i(z(\upsilon_i)) + \mathfrak{R}_{\phi_i},\ \ \Delta z'(\upsilon_i) = \widetilde{I}_i(z(\upsilon_i)) + \mathfrak{R}_{\phi_i}, \\ {}^{H}D^{\theta}s(\upsilon) = \phi\left(\upsilon,{}^{H}D^{\theta}s(\upsilon),{}^{H}D^{\varpi}z(\upsilon)\right) + \mathfrak{R}_{\psi}(\upsilon), \\ \qquad \Delta s(\upsilon_j) = I_j\left(s(\upsilon_j)\right) + \mathfrak{R}_{\psi_j},\ \ \Delta s'(\upsilon_j) = \widetilde{I}_j\left(s(\upsilon_j)\right) + \mathfrak{R}_{\psi_j}. \end{cases}$$

Then, $(z,s) \in \mathfrak{I}$ is a solution of the system of inequalities (3.1).

4. Existence consequences

In the following part, we establish requirements for the existence and uniqueness of solutions to the suggested system (1.1)

Theorem 4.1. *For the function w, the solutions of the following subsequent linear impulsive BVP*

$$\begin{cases} {}^{H}D^{\varpi}z(\upsilon) = w(\upsilon),\ \upsilon \in U,\ \upsilon \neq \upsilon_i,\ i = 1,2,...k, \\ \Delta z(\upsilon_i) = I_i(z(\upsilon_i)),\ \ \Delta z'(\upsilon_i) = \widetilde{I}_i(z(\upsilon_i)),\ \ \upsilon \neq \upsilon_i,\ i = 1,2,...k, \\ z(G) = \frac{1}{\Gamma(\varpi)}\int_1^G \ln\left(\frac{G}{\eta}\right)^{\varpi-1} B(\eta,z(\eta))\frac{d\eta}{\eta},\ z'(G) = B^*(z), \end{cases}$$

takes the form

$$\begin{aligned} z(\upsilon) \ =\ & GD_0(\varpi)B^*(z)(\ln\upsilon)^{\varpi-2} + \sum_{i=1}^{u} D_{1i}(\varpi)(\ln\upsilon)^{\varpi-2} I_i z(\upsilon_i) + \sum_{i=1}^{u} D_{2i}(\varpi)(\ln\upsilon)^{\varpi-2}\widetilde{I}_i z(\upsilon_i) \\ & + \frac{D_3(\varpi)(\ln\upsilon)^{\varpi-2}}{\Gamma(\varpi)}\int_1^G \ln\left(\frac{G}{\eta}\right)^{\varpi-1} B(\eta,z(\eta))\frac{d\eta}{\eta} + \frac{D_0(\varpi)(\ln\upsilon)^{\varpi-2}}{\Gamma(\varpi-1)}\int_{\upsilon_u}^G \ln\left(\frac{G}{\eta}\right)^{\varpi-2} w(\eta)\frac{d\eta}{\eta} \\ & + \frac{D_4(\varpi)(\ln\upsilon)^{\varpi-2}}{\Gamma(\varpi)}\int_{\upsilon_u}^G \ln\left(\frac{G}{\eta}\right)^{\varpi-1} w(\eta)\frac{d\eta}{\eta} + \sum_{i=1}^{u}\frac{D_{5i}(\varpi)(\ln\upsilon)^{\varpi-2}}{\Gamma(\varpi)}\int_{\upsilon_{i-1}}^{\upsilon_i} \ln\left(\frac{\upsilon_i}{\eta}\right)^{\varpi-1} w(\eta)\frac{d\eta}{\eta} \\ & + \sum_{i=1}^{u}\frac{\ln\upsilon^{3-\varpi}(\log_{\upsilon_i}\upsilon)^{\varpi-2} D_{5i}(\varpi)(\ln\upsilon)^{\varpi-2}}{\Gamma(\varpi-1)}\int_{\upsilon_{i-1}}^{\upsilon_i} \ln\left(\frac{\upsilon_i}{\eta}\right)^{\varpi-2} w(\eta)\frac{d\eta}{\eta} \\ & + \frac{1}{\Gamma(\varpi)}\int_{\upsilon_u}^{\upsilon} \ln\left(\frac{\upsilon}{\eta}\right)^{\varpi-1} w(\eta)\frac{d\eta}{\eta}, \end{aligned} \tag{4.1}$$

where $u = 1,2,...,k$ *and*

$$D_0(\varpi) \ =\ \ln\left(\frac{\upsilon}{G}\right)\ln(G)^{2-\varpi},$$

$$
\begin{aligned}
D_{1i}(\varpi) &= (\varpi-1)(\ln\upsilon-\varpi+2)(\ln\upsilon_i)^{3-\varpi}-\frac{(\varpi-2)\left(\ln\upsilon^2-\varpi+1\right)(\ln\upsilon_i)^{2-\varpi}}{\ln\upsilon_i},\\
D_{2i}(\varpi) &= \ln\upsilon^{\upsilon_i(3-\varpi)}(\ln\upsilon_i)^{2-\varpi},\\
D_3(\varpi) &= \left(\varpi-1-\log_G\upsilon^{\varpi-2}\right)(\ln\upsilon)^{2-\varpi},\\
D_4(\varpi) &= \log_G\frac{\upsilon}{G^{\varpi-1}}(\ln G)^{2-\varpi},\\
D_{5i}(\varpi) &= \left(\ln\frac{\upsilon^{\varpi-1}}{G^{\varpi-2}}+\log_{\upsilon_i}\left(\frac{G\upsilon_i}{\upsilon^2}\right)^{\varpi-2}\right)(\ln\upsilon_i)^{2-\varpi}.
\end{aligned}
$$

Proof. Assume that

$$
{}^{H}D^{\varpi}z(\upsilon)=w(\upsilon),\ \varpi\in(1,2],\ \upsilon\in U. \tag{4.2}
$$

Using Lemma 2.3, for $\upsilon\in(1,\upsilon_1]$, we have

$$
\begin{aligned}
z(\upsilon) &= r_1(\ln\upsilon)^{\varpi-1}+r_2(\ln\upsilon)^{\varpi-2}+\frac{1}{\Gamma(\varpi)}\int_1^{\upsilon}\ln\left(\frac{\upsilon}{\eta}\right)^{\varpi-1}w(\eta)\frac{d\eta}{\eta}, \qquad (4.3)\\
z'(\upsilon) &= \frac{r_1(\varpi-1)}{\upsilon}(\ln\upsilon)^{\varpi-2}+\frac{r_2(\varpi-2)}{\upsilon}(\ln\upsilon)^{\varpi-3}+\frac{1}{\Gamma(\varpi-1)}\int_1^{\upsilon}\frac{1}{\upsilon}\ln\left(\frac{\upsilon}{\eta}\right)^{\varpi-2}w(\eta)\frac{d\eta}{\eta}.
\end{aligned}
$$

Again, applying Lemma 2.3, for $\upsilon\in(\upsilon_1,\upsilon_2]$, we get

$$
\begin{aligned}
z(\upsilon) &= l_1(\ln\upsilon)^{\varpi-1}+l_2(\ln\upsilon)^{\varpi-2}+\frac{1}{\Gamma(\varpi)}\int_{\upsilon_1}^{\upsilon}\ln\left(\frac{\upsilon}{\eta}\right)^{\varpi-1}w(\eta)\frac{d\eta}{\eta}, \qquad (4.4)\\
z'(\upsilon) &= \frac{l_1(\varpi-1)}{\upsilon}(\ln\upsilon)^{\varpi-2}+\frac{l_2(\varpi-2)}{\upsilon}(\ln\upsilon)^{\varpi-3}+\frac{1}{\Gamma(\varpi-1)}\int_{\upsilon_1}^{\upsilon}\frac{1}{\upsilon}\ln\left(\frac{\upsilon}{\eta}\right)^{\varpi-2}w(\eta)\frac{d\eta}{\eta}.
\end{aligned}
$$

Using initial impulses

$$
\begin{aligned}
l_1 &= r_1-(\varpi-2)(\ln\upsilon_1)^{1-\varpi}I_1(z(\upsilon_1))+\upsilon_1(\ln\upsilon_1)^{2-\varpi}\widetilde{I_1}(z(\upsilon_1))\\
&\quad+\frac{(\ln\upsilon_1)^{2-\varpi}}{\Gamma(\varpi-1)}\int_1^{\upsilon_1}\ln\left(\frac{\upsilon_1}{\eta}\right)^{\varpi-2}w(\eta)\frac{d\eta}{\eta}-\frac{(\varpi-2)(\ln\upsilon_1)^{1-\varpi}}{\Gamma(\varpi)}\int_1^{\upsilon_1}\ln\left(\frac{\upsilon_1}{\eta}\right)^{\varpi-1}w(\eta)\frac{d\eta}{\eta},\\
l_2 &= r_2+(\varpi-1)(\ln\upsilon_1)^{2-\varpi}I_1(z(\upsilon_1))-\upsilon_1(\ln\upsilon_1)^{3-\varpi}\widetilde{I_1}(z(\upsilon_1))\\
&\quad-\frac{(\ln\upsilon_1)^{3-\varpi}}{\Gamma(\varpi-1)}\int_1^{\upsilon_1}\ln\left(\frac{\upsilon_1}{\eta}\right)^{\varpi-2}w(\eta)\frac{d\eta}{\eta}+\frac{(\varpi-1)(\ln\upsilon_1)^{2-\varpi}}{\Gamma(\varpi)}\int_1^{\upsilon_1}\ln\left(\frac{\upsilon_1}{\eta}\right)^{\varpi-1}w(\eta)\frac{d\eta}{\eta}.
\end{aligned}
$$

From l_1 and l_2 on (4.4), one has

$$
\begin{aligned}
z(\upsilon) &= r_1(\ln\upsilon)^{\varpi-1}-r_2(\ln\upsilon)^{\varpi-2}+\left((\varpi-1)-(\varpi-2)(\log_{\upsilon_1}\upsilon)\right)(\log_{\upsilon_1}\upsilon)^{\varpi-2}I_1(z(\upsilon_1))\\
&\quad+\upsilon_1(\ln\upsilon-\ln\upsilon_1)(\log_{\upsilon_1}\upsilon)^{\varpi-2}\widetilde{I_1}(z(\upsilon_1))\\
&\quad+\frac{(\ln\upsilon-\ln\upsilon_1)(\log_{\upsilon_1}\upsilon)^{\varpi-2}}{\Gamma(\varpi-1)}\int_1^{\upsilon_1}\ln\left(\frac{\upsilon_1}{\eta}\right)^{\varpi-2}w(\eta)\frac{d\eta}{\eta}\\
&\quad+\frac{\left((\varpi-1)-(\varpi-2)(\log_{\upsilon_1}\upsilon)\right)(\log_{\upsilon_1}\upsilon)^{\varpi-2}}{\Gamma(\varpi)}\int_1^{\upsilon_1}\ln\left(\frac{\upsilon_1}{\eta}\right)^{\varpi-2}w(\eta)\frac{d\eta}{\eta}
\end{aligned}
$$

$$+\frac{1}{\Gamma(\varpi)}\int_{\upsilon_1}^{\upsilon}\ln\left(\frac{\upsilon}{\eta}\right)^{\varpi-1}w(\eta)\frac{d\eta}{\eta}.$$

Analogously for $\upsilon \in (\upsilon_u, G)$, we have

$$\begin{aligned} z(\upsilon) &= r_1(\ln\upsilon)^{\varpi-1} + r_2(\ln\upsilon)^{\varpi-2} + \sum_{i=1}^{u}\Big((\varpi-1)-(\varpi-2)(\log_{\upsilon_i}\upsilon)\Big)(\log_{\upsilon_i}\upsilon)^{\varpi-2}I_i(z(\upsilon_i)) \\ &+\sum_{i=1}^{u}\upsilon_i(\ln\upsilon-\ln\upsilon_i)(\log_{\upsilon_i}\upsilon)^{\varpi-2}\widetilde{I_i}(z(\upsilon_i)) \\ &+\sum_{i=1}^{u}\frac{(\ln\upsilon-\ln\upsilon_i)(\log_{\upsilon_i}\upsilon)^{\varpi-2}}{\Gamma(\varpi-1)}\int_{\upsilon_{i-1}}^{\upsilon_i}\ln\left(\frac{\upsilon_i}{\eta}\right)^{\varpi-2}w(\eta)\frac{d\eta}{\eta} \\ &+\sum_{i=1}^{u}\frac{\Big((\varpi-1)-(\varpi-2)(\log_{\upsilon_i}\upsilon)\Big)(\log_{\upsilon_i}\upsilon)^{\varpi-2}}{\Gamma(\varpi)}\int_{\upsilon_{i-1}}^{\upsilon_i}\ln\left(\frac{\upsilon_i}{\eta}\right)^{\varpi-2}w(\eta)\frac{d\eta}{\eta} \\ &+\frac{1}{\Gamma(\varpi)}\int_{\upsilon_u}^{\upsilon}\ln\left(\frac{\upsilon}{\eta}\right)^{\varpi-1}w(\eta)\frac{d\eta}{\eta}, \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} z'(\upsilon) &= \frac{(\varpi-1)r_1}{\upsilon}(\ln\upsilon)^{\varpi-2} + \frac{(\varpi-1)r_2}{\upsilon}(\ln\upsilon)^{\varpi-3} \\ &+\sum_{i=1}^{u}\frac{(\varpi-1)(\varpi-2)}{\upsilon}(\log_{\upsilon}e-\log_{e}\upsilon_i)(\log_{\upsilon_i}\upsilon)^{\varpi-2}I_i(z(\upsilon_i)) \\ &+\sum_{i=1}^{u}\frac{\upsilon_i}{\upsilon}\left[(\varpi-1)-(\varpi-2)\log_{\upsilon}\upsilon_i\right](\log_{\upsilon_i}\upsilon)^{\varpi-2}\widetilde{I_i}(z(\upsilon_i)) \\ &+\frac{1}{\upsilon\Gamma(\varpi-1)}\int_{\upsilon_u}^{\upsilon}\ln\left(\frac{\upsilon}{\eta}\right)^{\varpi-2}w(\eta)\frac{d\eta}{\eta}, \\ &+\sum_{i=1}^{u}\frac{((\varpi-1)-(\varpi-2)\log_{\upsilon}\upsilon_i)(\log_{\upsilon_i}\upsilon)^{\varpi-2}}{\upsilon\Gamma(\varpi-1)}\int_{\upsilon_{i-1}}^{\upsilon_i}\ln\left(\frac{\upsilon_i}{\eta}\right)^{\varpi-2}w(\eta)\frac{d\eta}{\eta} \\ &+\sum_{i=1}^{u}\frac{(\varpi-1)(\varpi-2)(\log_{\upsilon}e-\log_{e}\upsilon_i)(\log_{\upsilon_i}\upsilon)^{\varpi-2}}{\upsilon\Gamma(\varpi)}\int_{\upsilon_{i-1}}^{\upsilon_i}\ln\left(\frac{\upsilon_i}{\eta}\right)^{\varpi-2}w(\eta)\frac{d\eta}{\eta}. \end{aligned} \tag{4.6}$$

Applying the boundary stipulations $z(G) = \frac{1}{\Gamma(\varpi)}\int_1^G \ln\left(\frac{G}{\eta}\right)^{\varpi-1}B(\eta, z(\eta))\frac{d\eta}{\eta}$ and $z'(G) = B^*(z)$, we obtain that

$$\begin{aligned} r_1 &= GB^*(z)\ln(G)^{2-\varpi} - \frac{(\ln G)^{1-\varpi}(\varpi-2)}{\Gamma(\varpi)}\int_1^G\ln\left(\frac{G}{\eta}\right)^{\varpi-1}B(\eta,z(\eta))\frac{d\eta}{\eta} \\ &+\frac{(\ln G)^{1-\varpi}}{\Gamma(\varpi)}\int_{\upsilon_u}^{G}\ln\left(\frac{G}{\eta}\right)^{\varpi-1}w(\eta)\frac{d\eta}{\eta} \\ &+\sum_{i=1}^{u}\left(\ln\upsilon_i^{\varpi-1}-\frac{\varpi-2}{\ln\upsilon_i}\right)(\ln\upsilon_i)^{2-\varpi}I_i(z(\upsilon_i)) - (\varpi-2)\sum_{i=1}^{u}\upsilon_i(\ln\upsilon_i)^{\varpi-1}\widetilde{I_i}(z(\upsilon_i)) \end{aligned}$$

$$-\frac{(\varpi-2)}{\Gamma(\varpi-1)}\sum_{i=1}^{u}(\ln\upsilon_i)^{2-\varpi}\int_{\upsilon_{i-1}}^{\upsilon_i}\ln\left(\frac{\upsilon_i}{\eta}\right)^{\varpi-2}w(\eta)\frac{d\eta}{\eta}-\frac{(\ln G)^{2-\varpi}}{\Gamma(\varpi-1)}\int_{\upsilon_u}^{G}\ln\left(\frac{G}{\eta}\right)^{\varpi-2}w(\eta)\frac{d\eta}{\eta}$$
$$+\frac{1}{\Gamma(\varpi)}\sum_{i=1}^{u}\left(\ln\upsilon_i^{\varpi-1}-\frac{\varpi-2}{\ln\upsilon_i}\right)(\ln\upsilon_i)^{2-\varpi}\int_{\upsilon_{i-1}}^{\upsilon_i}\ln\left(\frac{\upsilon_i}{\eta}\right)^{\varpi-1}w(\eta)\frac{d\eta}{\eta},$$

and

$$\begin{aligned} r_2 \;=\;& \frac{(\ln G)^{2-\varpi}}{\Gamma(\varpi-1)}\int_1^G\ln\left(\frac{G}{\eta}\right)^{\varpi-1}B(\eta,z(\eta))\frac{d\eta}{\eta}-GB^*(z)\ln(G)^{3-\varpi}+\sum_{i=1}^{u}\upsilon_i(\ln\upsilon_i)^{3-\varpi}\widetilde{I_i}(z(\upsilon_i))\\ &+(\varpi-1)\sum_{i=1}^{u}\left(\ln G^{(\varpi-2)(\log_{\upsilon_i}e-\log_e\upsilon_i)}-1\right)(\ln\upsilon_i)^{2-\varpi}I_i(z(\upsilon_i))+\frac{(\ln G)^{3-\varpi}}{\Gamma(\varpi-1)}\int_{\upsilon_u}^{\upsilon}\ln\left(\frac{G}{\eta}\right)^{\varpi-2}w(\eta)\frac{d\eta}{\eta}\\ &+\frac{1}{\Gamma(\varpi-1)}\sum_{i=1}^{u}\left(\ln G^{(\varpi-2)(\log_{\upsilon_i}e-\log_e\upsilon_i)}-1\right)(\ln\upsilon_i)^{2-\varpi}\int_{\upsilon_{i-1}}^{\upsilon_i}\ln\left(\frac{\upsilon_i}{\eta}\right)^{\varpi-1}w(\eta)\frac{d\eta}{\eta}\\ &+\frac{1}{\Gamma(\varpi-1)}\sum_{i=1}^{u}(\ln\upsilon_i)^{3-\varpi}\int_{\upsilon_{i-1}}^{\upsilon_i}\ln\left(\frac{\upsilon_i}{\eta}\right)^{\varpi-2}w(\eta)\frac{d\eta}{\eta}-\frac{(\ln G)^{2-\varpi}}{\Gamma(\varpi-1)}\int_{\upsilon_i}^{G}\ln\left(\frac{G}{\eta}\right)^{\varpi-1}w(\eta)\frac{d\eta}{\eta}, \end{aligned}$$

for $u = 1, 2, ..., k$. Substituting r_1 and r_2 in (4.5), we have (4.1). □

Corollary 4.2. *Theorem 2.4 provides the following solution for our coupled problem (1.1):*

$$\begin{aligned} z(\upsilon) \;=\;& GD_0(\varpi)B^*(z)(\ln\upsilon)^{\varpi-2}+\sum_{i=1}^{u}D_{1i}(\varpi)(\ln\upsilon)^{\varpi-2}I_i(z_i)+\sum_{i=1}^{u}D_{2i}(\varpi)(\ln\upsilon)^{\varpi-2}\widetilde{I_i}(z_i)\\ &+\frac{D_3(\varpi)(\ln\upsilon)^{\varpi-2}}{\Gamma(\varpi)}\int_1^G\ln\left(\frac{G}{\eta}\right)^{\varpi-1}B(\eta,z(\eta))\frac{d\eta}{\eta}\\ &+\frac{D_0(\varpi)(\ln\upsilon)^{\varpi-2}}{\Gamma(\varpi-1)}\int_{\upsilon_u}^{G}\ln\left(\frac{G}{\eta}\right)^{\varpi-2}\phi\left(\eta,{}^H D^{\varpi}z(\eta),{}^H D^{\theta}s(\eta)\right)\frac{d\eta}{\eta}\\ &+\frac{D_4(\varpi)(\ln\upsilon)^{\varpi-2}}{\Gamma(\varpi)}\int_{\upsilon_u}^{G}\ln\left(\frac{G}{\eta}\right)^{\varpi-1}\phi\left(\eta,{}^H D^{\varpi}z(\eta),{}^H D^{\theta}s(\eta)\right)\frac{d\eta}{\eta}\\ &+\sum_{i=1}^{u}\frac{D_{5i}(\varpi)(\ln\upsilon)^{\varpi-2}}{\Gamma(\varpi)}\int_{\upsilon_{i-1}}^{\upsilon_i}\ln\left(\frac{\upsilon_i}{\eta}\right)^{\varpi-1}\phi\left(\eta,{}^H D^{\varpi}z(\eta),{}^H D^{\theta}s(\eta)\right)\frac{d\eta}{\eta}\\ &+\sum_{i=1}^{u}\frac{\ln\upsilon^{3-\varpi}(\log_{\upsilon_i}\upsilon)^{\varpi-2}D_{5i}(\varpi)(\ln\upsilon)^{\varpi-2}}{\Gamma(\varpi-1)}\int_{\upsilon_{i-1}}^{\upsilon_i}\ln\left(\frac{\upsilon_i}{\eta}\right)^{\varpi-2}\phi\left(\eta,{}^H D^{\varpi}z(\eta),{}^H D^{\theta}s(\eta)\right)\frac{d\eta}{\eta}\\ &+\frac{1}{\Gamma(\varpi)}\int_{\upsilon_u}^{\upsilon}\ln\left(\frac{\upsilon}{\eta}\right)^{\varpi-1}\phi\left(\eta,{}^H D^{\varpi}z(\eta),{}^H D^{\theta}s(\eta)\right)\frac{d\eta}{\eta}, \end{aligned} \tag{4.7}$$

where $u = 1, 2, ..., k$ and

$$s(\upsilon) \;=\; GD_0(\theta)B^*(s)(\ln\upsilon)^{\theta-2}+\sum_{j=1}^{u}D_{1j}(\theta)(\ln\upsilon)^{\theta-2}I_j(s_j)+\sum_{j=1}^{u}D_{2j}(\theta)(\ln\upsilon)^{\theta-2}\widetilde{I_j}(s_j)$$

$$+\frac{D_3(\theta)(\ln\upsilon)^{\theta-2}}{\Gamma(\theta)}\int_1^G \ln\left(\frac{G}{\eta}\right)^{\theta-1} B(\eta,s(\eta))\frac{d\eta}{\eta}$$
$$+\frac{D_0(\theta)(\ln\upsilon)^{\theta-2}}{\Gamma(\theta-1)}\int_{\upsilon_u}^G \ln\left(\frac{G}{\eta}\right)^{\theta-2} \psi\left(\eta,{}^H D^{\theta}s(\eta),{}^H D^{\varpi}z(\eta)\right)\frac{d\eta}{\eta}$$
$$+\frac{D_4(\theta)(\ln\upsilon)^{\theta-2}}{\Gamma(\theta)}\int_{\upsilon_u}^G \ln\left(\frac{G}{\eta}\right)^{\theta-1} \psi\left(\eta,{}^H D^{\theta}s(\eta),{}^H D^{\varpi}z(\eta)\right)\frac{d\eta}{\eta}$$
$$+\sum_{j=1}^{u}\frac{D_{5i}(\theta)(\ln\upsilon)^{\theta-2}}{\Gamma(\theta)}\int_{\upsilon_{j-1}}^{\upsilon_j} \ln\left(\frac{\upsilon_j}{\eta}\right)^{\theta-1} \psi\left(\eta,{}^H D^{\theta}s(\eta),{}^H D^{\varpi}z(\eta)\right)\frac{d\eta}{\eta}$$
$$+\sum_{j=1}^{u}\frac{\ln\upsilon^{3-\theta}(\log_{\upsilon_j}\upsilon)^{\theta-2}D_{5j}(\theta)(\ln\upsilon)^{\theta-2}}{\Gamma(\theta-1)}\int_{\upsilon_{j-1}}^{\upsilon_j} \ln\left(\frac{\upsilon_i}{\eta}\right)^{\theta-2} \psi\left(\eta,{}^H D^{\theta}s(\eta),{}^H D^{\varpi}z(\eta)\right)\frac{d\eta}{\eta},$$
$$+\frac{1}{\Gamma(\theta)}\int_{\upsilon_u}^{\upsilon} \ln\left(\frac{\upsilon}{\eta}\right)^{\theta-1} \psi\left(\eta,{}^H D^{\theta}s(\eta),{}^H D^{\varpi}z(\eta)\right)\frac{d\eta}{\eta}, \tag{4.8}$$

where $u = 1, 2, ..., m$.

For convenience, we use the notations below:

$$p(\upsilon) = \phi(\upsilon, a_1(\upsilon), a_2(\upsilon)) \le \phi(\upsilon, z(\upsilon), a(\upsilon)) \text{ and } a(\upsilon) = \psi(\upsilon, p_1(\upsilon), p_2(\upsilon)) \le \psi(\upsilon, s(\upsilon), p(\upsilon)).$$

Hence, for $\upsilon \in U$, Eqs (4.7) and (4.8) can be written as

$$\begin{aligned}
&z(\upsilon)\\
=\;& GD_0(\varpi)B^*(z)(\ln\upsilon)^{\varpi-2} + \sum_{i=1}^{u} D_{1i}(\varpi)(\ln\upsilon)^{\varpi-2} I_i(z_i) + \sum_{i=1}^{u} D_{2i}(\varpi)(\ln\upsilon)^{\varpi-2}\widetilde{I}_i(z_i)\\
&+\frac{D_3(\varpi)(\ln\upsilon)^{\varpi-2}}{\Gamma(\varpi)}\int_1^G \ln\left(\frac{G}{\eta}\right)^{\varpi-1} B(\eta,z(\eta))\frac{d\eta}{\eta} + \frac{D_0(\varpi)(\ln\upsilon)^{\varpi-2}}{\Gamma(\varpi-1)}\int_{\upsilon_u}^G \ln\left(\frac{G}{\eta}\right)^{\varpi-2} p(\eta)\frac{d\eta}{\eta}\\
&+\frac{D_4(\varpi)(\ln\upsilon)^{\varpi-2}}{\Gamma(\varpi)}\int_{\upsilon_u}^G \ln\left(\frac{G}{\eta}\right)^{\varpi-1} p(\eta)\frac{d\eta}{\eta} + \sum_{i=1}^{u}\frac{D_{5i}(\varpi)(\ln\upsilon)^{\varpi-2}}{\Gamma(\varpi)}\int_{\upsilon_{i-1}}^{\upsilon_i} \ln\left(\frac{\upsilon_i}{\eta}\right)^{\varpi-1} p(\eta)\frac{d\eta}{\eta}\\
&+\sum_{i=1}^{u}\frac{\ln\upsilon^{3-\varpi}(\log_{\upsilon_i}\upsilon)^{\varpi-2}D_{5i}(\varpi)(\ln\upsilon)^{\varpi-2}}{\Gamma(\varpi-1)}\int_{\upsilon_{i-1}}^{\upsilon_i} \ln\left(\frac{\upsilon_i}{\eta}\right)^{\varpi-2} p(\eta)\frac{d\eta}{\eta} + \frac{1}{\Gamma(\varpi)}\int_{\upsilon_u}^{\upsilon} \ln\left(\frac{\upsilon}{\eta}\right)^{\varpi-1} p(\eta)\frac{d\eta}{\eta},
\end{aligned}$$

for $u = 1, 2, ..., k$ and

$$\begin{aligned}
&s(\upsilon)\\
=\;& GD_0(\theta)B^*(s)(\ln\upsilon)^{\theta-2} + \sum_{j=1}^{u} D_{1j}(\theta)(\ln\upsilon)^{\theta-2} I_j(s_j) + \sum_{j=1}^{u} D_{2j}(\theta)(\ln\upsilon)^{\theta-2}\widetilde{I}_j(s_j)\\
&+\frac{D_3(\theta)(\ln\upsilon)^{\theta-2}}{\Gamma(\theta)}\int_1^G \ln\left(\frac{G}{\eta}\right)^{\theta-1} B(\eta,s(\eta))\frac{d\eta}{\eta} + \frac{D_0(\theta)(\ln\upsilon)^{\theta-2}}{\Gamma(\theta-1)}\int_{\upsilon_u}^G \ln\left(\frac{G}{\eta}\right)^{\theta-2} a(\eta)\frac{d\eta}{\eta}\\
&+\frac{D_4(\theta)(\ln\upsilon)^{\theta-2}}{\Gamma(\theta)}\int_{\upsilon_u}^G \ln\left(\frac{G}{\eta}\right)^{\theta-1} a(\eta)\frac{d\eta}{\eta} + \sum_{j=1}^{u}\frac{D_{5i}(\theta)(\ln\upsilon)^{\theta-2}}{\Gamma(\theta)}\int_{\upsilon_{j-1}}^{\upsilon_j} \ln\left(\frac{\upsilon_j}{\eta}\right)^{\theta-1} a(\eta)\frac{d\eta}{\eta}
\end{aligned}$$

$$+\sum_{j=1}^{u}\frac{\ln\upsilon^{3-\theta}(\log_{\upsilon_j}\upsilon)^{\theta-2}D_{5j}(\theta)(\ln\upsilon)^{\theta-2}}{\Gamma(\theta-1)}\int_{\upsilon_{j-1}}^{\upsilon_j}\ln\left(\frac{\upsilon_i}{\eta}\right)^{\theta-2}a(\eta)\frac{d\eta}{\eta}+\frac{1}{\Gamma(\theta)}\int_{\upsilon_u}^{\upsilon}\ln\left(\frac{\upsilon}{\eta}\right)^{\theta-1}a(\eta)\frac{d\eta}{\eta},$$

for $u = 1, 2, ..., m$.

If z and s are solutions to the CII-FDEs (1.1), then for $\upsilon \in U$, we can write

$$\begin{aligned}
z(\upsilon) &= GD_0(\varpi)B^*(z)(\ln\upsilon)^{\varpi-2}+\sum_{i=1}^{u}D_{1i}(\varpi)(\ln\upsilon)^{\varpi-2}I_i(z_i)+\sum_{i=1}^{u}D_{2i}(\varpi)(\ln\upsilon)^{\varpi-2}\widetilde{I_i}(z_i)\\
&+\frac{D_3(\varpi)(\ln\upsilon)^{\varpi-2}}{\Gamma(\varpi)}\int_1^G\ln\left(\frac{G}{\eta}\right)^{\varpi-1}B(\eta,z(\eta))\frac{d\eta}{\eta}\\
&+\frac{D_0(\varpi)(\ln\upsilon)^{\varpi-2}}{\Gamma(\varpi-1)}\int_{\upsilon_u}^G\ln\left(\frac{G}{\eta}\right)^{\varpi-2}\phi(\eta,a_1(\eta),a_2(\eta))\frac{d\eta}{\eta}\\
&+\frac{D_4(\varpi)(\ln\upsilon)^{\varpi-2}}{\Gamma(\varpi)}\int_{\upsilon_u}^G\ln\left(\frac{G}{\eta}\right)^{\varpi-1}\phi(\eta,a_1(\eta),a_2(\eta))\frac{d\eta}{\eta}\\
&+\sum_{i=1}^{u}\frac{D_{5i}(\varpi)(\ln\upsilon)^{\varpi-2}}{\Gamma(\varpi)}\int_{\upsilon_{i-1}}^{\upsilon_i}\ln\left(\frac{\upsilon_i}{\eta}\right)^{\varpi-1}\phi(\eta,a_1(\eta),a_2(\eta))\frac{d\eta}{\eta}\\
&+\sum_{i=1}^{u}\frac{\ln\upsilon^{3-\varpi}(\log_{\upsilon_i}\upsilon)^{\varpi-2}D_{5i}(\varpi)(\ln\upsilon)^{\varpi-2}}{\Gamma(\varpi-1)}\int_{\upsilon_{i-1}}^{\upsilon_i}\ln\left(\frac{\upsilon_i}{\eta}\right)^{\varpi-2}\phi(\eta,a_1(\eta),a_2(\eta))\frac{d\eta}{\eta}\\
&+\frac{1}{\Gamma(\varpi)}\int_{\upsilon_u}^{\upsilon}\ln\left(\frac{\upsilon}{\eta}\right)^{\varpi-1}\phi(\eta,a_1(\eta),a_2(\eta))\frac{d\eta}{\eta},
\end{aligned}$$

for $u = 1, 2, ..., k$ and

$$\begin{aligned}
s(\upsilon) &= GD_0(\theta)B^*(s)(\ln\upsilon)^{\theta-2}+\sum_{j=1}^{u}D_{1j}(\theta)(\ln\upsilon)^{\theta-2}I_j(s_j)+\sum_{j=1}^{u}D_{2j}(\theta)(\ln\upsilon)^{\theta-2}\widetilde{I_j}(s_j)\\
&+\frac{D_3(\theta)(\ln\upsilon)^{\theta-2}}{\Gamma(\theta)}\int_1^G\ln\left(\frac{G}{\eta}\right)^{\theta-1}B(\eta,s(\eta))\frac{d\eta}{\eta}\\
&+\frac{D_0(\theta)(\ln\upsilon)^{\theta-2}}{\Gamma(\theta-1)}\int_{\upsilon_u}^G\ln\left(\frac{G}{\eta}\right)^{\theta-2}\psi(\eta,p_1(\eta),p_2(\eta))\frac{d\eta}{\eta}\\
&+\frac{D_4(\theta)(\ln\upsilon)^{\theta-2}}{\Gamma(\theta)}\int_{\upsilon_u}^G\ln\left(\frac{G}{\eta}\right)^{\theta-1}\psi(\eta,p_1(\eta),p_2(\eta))\frac{d\eta}{\eta}\\
&+\sum_{j=1}^{u}\frac{D_{5i}(\theta)(\ln\upsilon)^{\theta-2}}{\Gamma(\theta)}\int_{\upsilon_{j-1}}^{\upsilon_j}\ln\left(\frac{\upsilon_j}{\eta}\right)^{\theta-1}\psi(\eta,p_1(\eta),p_2(\eta))\frac{d\eta}{\eta}\\
&+\sum_{j=1}^{u}\frac{\ln\upsilon^{3-\theta}(\log_{\upsilon_j}\upsilon)^{\theta-2}D_{5j}(\theta)(\ln\upsilon)^{\theta-2}}{\Gamma(\theta-1)}\int_{\upsilon_{j-1}}^{\upsilon_j}\ln\left(\frac{\upsilon_i}{\eta}\right)^{\theta-2}\psi(\eta,p_1(\eta),p_2(\eta))\frac{d\eta}{\eta}\\
&+\frac{1}{\Gamma(\theta)}\int_{\upsilon_u}^{\upsilon}\ln\left(\frac{\upsilon}{\eta}\right)^{\theta-1}\psi(\eta,p_1(\eta),p_2(\eta))\frac{d\eta}{\eta},
\end{aligned}$$

for $u = 1, 2, ..., m$.

Our next step is to convert the considered system (1.1) into a FP problem. Give the definition of the operators $E, \widetilde{E} : \mathfrak{I} \to \mathfrak{I}$ as

$$E(z,s)(\upsilon) = (E_1 z(\upsilon), E_2 z(\upsilon)) \text{ and } \widetilde{E}(z,s)(\upsilon) = (E_1(z,s)(\upsilon), E_2(s,z)(\upsilon)),$$

where

$$\begin{cases} E_1(z(\upsilon)) = GD_0(\varpi)B^*(z)(\ln\upsilon)^{\varpi-2} + \sum_{i=1}^{u} D_{1i}(\varpi)(\ln\upsilon)^{\varpi-2} I_i(z_i) + \sum_{i=1}^{u} D_{2i}(\varpi)(\ln\upsilon)^{\varpi-2}\widetilde{I}_i(z_i) \\ \qquad + \frac{D_3(\varpi)(\ln\upsilon)^{\varpi-2}}{\Gamma(\varpi)} \int_1^G \ln\left(\frac{G}{\eta}\right)^{\varpi-1} B(\eta, z(\eta))\frac{d\eta}{\eta}, \quad u = 1,2,...,k, \\ E_2(s(\upsilon)) = GD_0(\theta)B^*(s)(\ln\upsilon)^{\theta-2} + \sum_{j=1}^{u} D_{1j}(\theta)(\ln\upsilon)^{\theta-2} I_j(s_j) + \sum_{j=1}^{u} D_{2j}(\theta)(\ln\upsilon)^{\theta-2}\widetilde{I}_j(s_j) \\ \qquad + \frac{D_3(\theta)(\ln\upsilon)^{\theta-2}}{\Gamma(\theta)} \int_1^G \ln\left(\frac{G}{\eta}\right)^{\theta-1} B(\eta, s(\eta))\frac{d\eta}{\eta}, \quad u = 1,2,...,m, \end{cases} \tag{4.9}$$

and

$$\begin{cases} E_1(z,s)(\upsilon) = \frac{D_0(\varpi)(\ln\upsilon)^{\varpi-2}}{\Gamma(\varpi-1)} \int_{\upsilon_u}^G \ln\left(\frac{G}{\eta}\right)^{\varpi-2} \phi\left(\eta, {}^H D^{\varpi} z(\eta), {}^H D^{\theta} s(\eta)\right)\frac{d\eta}{\eta} \\ + \frac{D_4(\varpi)(\ln\upsilon)^{\varpi-2}}{\Gamma(\varpi)} \int_{\upsilon_u}^G \ln\left(\frac{G}{\eta}\right)^{\varpi-1} \phi\left(\eta, {}^H D^{\varpi} z(\eta), {}^H D^{\theta} s(\eta)\right)\frac{d\eta}{\eta} \\ + \sum_{i=1}^{u} \frac{D_{5i}(\varpi)(\ln\upsilon)^{\varpi-2}}{\Gamma(\varpi)} \int_{\upsilon_{i-1}}^{\upsilon_i} \ln\left(\frac{\upsilon_i}{\eta}\right)^{\varpi-1} \phi\left(\eta, {}^H D^{\varpi} z(\eta), {}^H D^{\theta} s(\eta)\right)\frac{d\eta}{\eta} \\ + \sum_{i=1}^{u} \frac{\ln\upsilon^{3-\varpi}(\log_{\upsilon_i}\upsilon)^{\varpi-2}}{\Gamma(\varpi-1)} \int_{\upsilon_{i-1}}^{\upsilon_i} \ln\left(\frac{\upsilon_i}{\eta}\right)^{\varpi-2} \phi\left(\eta, {}^H D^{\varpi} z(\eta), {}^H D^{\theta} s(\eta)\right)\frac{d\eta}{\eta} \\ + \frac{1}{\Gamma(\varpi)} \int_{\upsilon_u}^{\upsilon} \ln\left(\frac{\upsilon}{\eta}\right)^{\varpi-1} \phi(\eta, a_1(\eta), a_2(\eta))\frac{d\eta}{\eta}, \quad u = 1,2,...,k, \\ E_2(s,z)(\upsilon) = \frac{D_0(\theta)(\ln\upsilon)^{\theta-2}}{\Gamma(\theta-1)} \int_{\upsilon_u}^G \ln\left(\frac{G}{\eta}\right)^{\theta-2} \psi\left(\eta, {}^H D^{\theta} s(\eta), {}^H D^{\varpi} z(\eta)\right)\frac{d\eta}{\eta} \\ + \frac{D_4(\theta)(\ln\upsilon)^{\theta-2}}{\Gamma(\theta)} \int_{\upsilon_u}^G \ln\left(\frac{G}{\eta}\right)^{\theta-1} \psi\left(\eta, {}^H D^{\theta} s(\eta), {}^H D^{\varpi} z(\eta)\right)\frac{d\eta}{\eta} \\ + \sum_{j=1}^{u} \frac{D_{5i}(\theta)(\ln\upsilon)^{\theta-2}}{\Gamma(\theta)} \int_{\upsilon_{j-1}}^{\upsilon_j} \ln\left(\frac{\upsilon_j}{\eta}\right)^{\theta-1} \psi\left(\eta, {}^H D^{\theta} s(\eta), {}^H D^{\varpi} z(\eta)\right)\frac{d\eta}{\eta} \\ + \sum_{j=1}^{u} \frac{\ln\upsilon^{3-\theta}(\log_{\upsilon_j}\upsilon)^{\theta-2}}{\Gamma(\theta-1)} \int_{\upsilon_{j-1}}^{\upsilon_j} \ln\left(\frac{\upsilon_i}{\eta}\right)^{\theta-2} \psi\left(\eta, {}^H D^{\theta} s(\eta), {}^H D^{\varpi} z(\eta)\right)\frac{d\eta}{\eta} \\ + \frac{1}{\Gamma(\theta)} \int_{\upsilon_u}^{\upsilon} \ln\left(\frac{\upsilon}{\eta}\right)^{\theta-1} \psi\left(\eta, {}^H D^{\theta} s(\eta), {}^H D^{\varpi} z(\eta)\right)\frac{d\eta}{\eta}, \quad u = 1,2,...,m. \end{cases} \tag{4.10}$$

The preceding assertions must be true in order to conduct further analysis:

(A_1) For $\upsilon \in U$ and $a_1, a_2, p_1, p_2 \in \mathbb{R}$, there exist $\ell_0, \ell_1, \ell_2, \rho_0, \rho_1, \rho_2 \in C(U, \mathbb{R}_+)$, so that

$$\begin{aligned} |\phi(\upsilon, a_1(\upsilon), a_2(\upsilon))| &\leq \ell_0(\upsilon) + \ell_1(\upsilon)|a_1(\upsilon)| + \ell_2(\upsilon)|a_2(\upsilon)|, \\ |\psi(\upsilon, p_1(\upsilon), p_2(\upsilon))| &\leq \rho_0(\upsilon) + \rho_1(\upsilon)|p_1(\upsilon)| + \rho_2(\upsilon)|p_2(\upsilon)|, \end{aligned}$$

with $\widetilde{\ell}_0 = \sup_{\upsilon\in U}\ell_0(\upsilon)$, $\widetilde{\ell}_1 = \sup_{\upsilon\in U}\ell_1(\upsilon)$, $\widetilde{\ell}_2 = \sup_{\upsilon\in U}\ell_2(\upsilon)$, $\widetilde{\rho}_0 = \sup_{\upsilon\in U}\rho_0(\upsilon)$, $\widetilde{\rho}_1 = \sup_{\upsilon\in U}\rho_1(\upsilon)$, and $\widetilde{\rho}_2 = \sup_{\upsilon\in U}\rho_2(\upsilon) < 1$.

(A_2) For the continuous functions $B^*, I_u, \widetilde{I}_u : \mathbb{R} \to \mathbb{R}$ there are positive constants $O_B, O_I, O_{\widetilde{I}}, O_I'', O_{\widetilde{I}}'', \widetilde{O}_B, \widetilde{O}_I, \widetilde{O}_{\widetilde{I}}, \widetilde{O}_I'', \widetilde{O}_{\widetilde{I}}''$ so that for any $(z,s) \in \mathfrak{I}$

$$\begin{aligned} |B^*(z)| &\leq O_{B^*}, \quad |I_u(z(\upsilon))| \leq O_I|z| + O_I'', \quad \left|\widetilde{I}_u(z(\upsilon))\right| \leq O_{\widetilde{I}}|z| + O_{\widetilde{I}}'', \\ |B^*(s)| &\leq \widetilde{O}_{B^*}, \quad |I_u(s(\upsilon))| \leq \widetilde{O}_I|s| + \widetilde{O}_I'', \quad \left|\widetilde{I}_u(s(\upsilon))\right| \leq \widetilde{O}_{\widetilde{I}}|s| + \widetilde{O}_{\widetilde{I}}'', \end{aligned}$$

where $u = \{0, 1, 2, ..., k\}$.

(A_3) For all $\upsilon \in U$ and $s, z \in \mathbb{R}$, there are $\varrho_1, \delta_1, \varrho_2, \delta_2 \in C(U, \mathbb{R}_+)$, so that

$$|B(\upsilon, z(\upsilon))| \leq \varrho_1(\upsilon) + \delta_1 |z(\upsilon)| \text{ and } |B(\upsilon, s(\upsilon))| \leq \varrho_2(\upsilon) + \delta_1 |s(\upsilon)|,$$

with $\varrho_1^* = \sup_{\upsilon\in U} \varrho_1(\upsilon)$, $\delta_1^* = \sup_{\upsilon\in U} \delta_1(\upsilon)$, $\varrho_2^* = \sup_{\upsilon\in U} \varrho_2(\upsilon)$, $\delta_2^* = \sup_{\upsilon\in U} \delta_2(\upsilon) < 1$.

(A_4) For each $a_1, a_2, \widetilde{a}_1, \widetilde{a}_2, p_1, p_2, \widetilde{p}_1, \widetilde{p}_2 \in \mathbb{R}$, and for all $\upsilon \in U$, there are constants $L_\phi, L_\psi > 0$, and $\widetilde{L}_\phi, \widetilde{L}_\psi \in (0, 1)$ so that

$$\begin{aligned}
|\phi(\upsilon, a_1(\upsilon), a_2(\upsilon)) - \phi(\upsilon, \widetilde{a}_1(\upsilon), \widetilde{a}_2(\upsilon))| &\leq L_\phi |a_1 - \widetilde{a}_1| + \widetilde{L}_\phi |a_2 - \widetilde{a}_2|,\\
|\psi(\upsilon, p_1(\upsilon), p_2(\upsilon)) - \psi(\upsilon, \widetilde{p}_1(\upsilon), \widetilde{p}_2(\upsilon))| &\leq L_\psi |p_1 - \widetilde{p}_1| + \widetilde{L}_\psi |p_2 - \widetilde{p}_2|.
\end{aligned}$$

(A_5) For the continuous functions $I_u, \widetilde{I}_u : \mathbb{R} \to \mathbb{R}$, there are positive constants $L_I, L_{\widetilde{I}}, \widetilde{L}_I, \widetilde{L}_{\widetilde{I}}$ so for any $(z, s), (\widetilde{z}, \widetilde{s}) \in \mathfrak{I}$

$$\begin{aligned}
|I_u(z(\upsilon)) - I_u(\widetilde{z}(\upsilon))| \leq L_I |z - \widetilde{z}|, &\quad |I_u(s(\upsilon)) - I_u(\widetilde{s}(\upsilon))| \leq \widetilde{L}_I |s - \widetilde{s}|,\\
\left|\widetilde{I}_u(z(\upsilon)) - \widetilde{I}_u(\widetilde{z}(\upsilon))\right| \leq L_{\widetilde{I}} |z - \widetilde{z}| &\quad \left|\widetilde{I}_u(s(\upsilon)) - \widetilde{I}_u(\widetilde{s}(\upsilon))\right| \leq \widetilde{L}_{\widetilde{I}} |s - \widetilde{s}|.
\end{aligned}$$

(A_6) For each $s, z, \widetilde{s}, \widetilde{z} \in \mathbb{R}$ and for all $\upsilon \in U$, there are $L_B, L_{B^*}, \widetilde{L}_B, \widetilde{L}_{B^*} > 0$, so that

$$\begin{aligned}
|B(\upsilon, z(\upsilon)) - B(\upsilon, \widetilde{z}(\upsilon))| &\leq L_B |z - \widetilde{z}|, \quad |B^*(z) - B^*(\widetilde{z})| \leq L_{B^*} |z - \widetilde{z}|,\\
|B(\upsilon, s(\upsilon)) - B(\upsilon, \widetilde{s}(\upsilon))| &\leq \widetilde{L}_B |s - \widetilde{s}|, \quad |B^*(s) - B^*(\widetilde{s})| \leq \widetilde{L}_{B^*} |z - \widetilde{z}|.
\end{aligned}$$

Here, we demonstrate that the operator $E + \widetilde{E}$ has at least one FP using Kransnoselskii's FP theorem. For this, we choose a closed ball

$$\mathfrak{I}_x = \left\{(z, s) \in \mathfrak{I} : \|(z, s)\| \leq y, \ \|z\| \leq \frac{y}{2} \text{ and } \|s\| \leq \frac{y}{2}\right\} \subset \mathfrak{I},$$

where

$$x \geq \frac{M_1^* + M_1^{**} + \frac{(\widetilde{\ell}_0 + \widetilde{\ell}_2 \widetilde{\rho}_0) M_3^* + (\widetilde{\rho}_0 + \widetilde{\rho}_2 \widetilde{\ell}_0) M_3^{**}}{\widetilde{\ell}_2 \widetilde{\rho}_2 - 1}}{1 - M_2^* - M_2^{**} - \frac{Y_1^* M_2^* + Y_2^* M_2^{**}}{\widetilde{\ell}_2 \widetilde{\rho}_2 - 1}}.$$

Theorem 4.3. *There exists at least one solution to the CII-FDEs (1.1) provided that the assertions* (A_1) *and* (A_2) *are true.*

Proof. For any $(z, s) \in \mathfrak{I}_y$, we get

$$\left\|E(z, s)(\upsilon) + \widetilde{E}(z, s)\right\|_{\mathfrak{I}} \leq \|E_1(z)\|_{\mathfrak{I}_1} + \|E_2(s)\|_{\mathfrak{I}_2} + \left\|\widetilde{E}_1(z, s)\right\|_{\mathfrak{I}_1} + \left\|\widetilde{E}_1(z, s)\right\|_{\mathfrak{I}_2}. \tag{4.11}$$

From (4.9), we have

$$\begin{aligned}
\left|E_1 z(\upsilon)(\ln \upsilon)^{2-\varpi}\right| \leq\ & G |D_0(\varpi)| |B^*(z)| + \sum_{i=1}^{u} |D_{1i}(\varpi)| |I_i(z(\upsilon_i))| + \sum_{i=1}^{u} |D_{2i}(\varpi)| \left|\widetilde{I}_i(z(\upsilon_i))\right|\\
& + \frac{|D_3(\varpi)|}{\Gamma(\varpi)} \int_1^G \left|\ln\left(\frac{G}{\eta}\right)^{\varpi-1}\right| |B(\eta, z(\eta))| \frac{d\eta}{\eta},
\end{aligned}$$

for $u = 1, 2, ..., k$. This leads to

$$
\begin{aligned}
\|E_1(z)\|_{\Im_1} &\leq GO_{B^*}|D_0(\varpi)| + u|D_1(\varpi)|(O_I\|z\| + O_I'') + u|D_2(\varpi)|\left(O_{\widetilde{I}}\|z\| + O_{\widetilde{I}}''\right) \\
&\quad -\frac{|D_3(\varpi)|\left(\varrho_1^*(\upsilon) + \delta_1^*\|z\|\right)}{\varpi\Gamma(\varpi)}|\ln(G)^{\varpi}| \\
&= GO_{B^*}|D_0(\varpi)| + uO_I''|D_1(\varpi)| + uO_{\widetilde{I}}''|D_2(\varpi)| + uO_I|D_1(\varpi)|\|z\| \\
&\quad +uO_{\widetilde{I}}|D_2(\varpi)|\|z\| - \frac{|D_3(\varpi)|\left(\varrho_1^*(\upsilon) + \delta_1^*\|z\|\right)}{\Gamma(\varpi+1)}|\ln(G)^{\varpi}| \\
&\leq M_1^* + M_2^*\|z\|. \qquad (4.12)
\end{aligned}
$$

Analogously, one can write

$$
\|E_2(z)\|_{\Im_2} \leq M_1^{**} + M_2^{**}\|s\|, \qquad (4.13)
$$

where

$$
\begin{aligned}
M_1^* &= GO_{B^*}|D_0(\varpi)| + uO_I''|D_1(\varpi)| + uO_{\widetilde{I}}''|D_2(\varpi)| - \frac{|D_3(\varpi)|\varrho_1^*(\upsilon)}{\Gamma(\varpi+1)}|\ln(G)^{\varpi}|, \ u = 1, 2, ..., k, \\
M_2^* &= uO_I|D_1(\varpi)| + uO_{\widetilde{I}}|D_2(\varpi)| - \frac{\delta_1^*|D_3(\varpi)|}{\Gamma(\varpi+1)}|\ln(G)^{\varpi}|, \quad u = 1, 2, ..., k, \\
M_1^{**} &= G\widetilde{O}_{B^*}|D_0(\theta)| + u\widetilde{O}_I''|D_1(\theta)| + u\widetilde{O}_{\widetilde{I}}''|D_2(\theta)| - \frac{|D_3(\theta)|\varrho_2^*(\upsilon)}{\Gamma(\theta+1)}\left|\ln(G)^{\theta}\right|, \ u = 1, 2, ..., m, \\
M_2^{**} &= u\widetilde{O}_I|D_1(\theta)| + u\widetilde{O}_{\widetilde{I}}|D_2(\theta)| - \frac{\delta_2^*|D_3(\theta)|}{\Gamma(\theta+1)}\left|\ln(G)^{\theta}\right|, \quad u = 1, 2, ..., m.
\end{aligned}
$$

Further, we obtain for $u = 1, 2, ..., k$, that

$$
\begin{aligned}
&\left|\widetilde{E}_1(z,s)(\upsilon)(\ln\upsilon)^{2-\varpi}\right| \\
\leq\ & \frac{|D_0(\varpi)|}{\Gamma(\varpi-1)}\int_{\upsilon_u}^{G}\left|\ln\left(\frac{G}{\eta}\right)^{\varpi-2}\right||p(\eta)|\frac{d\eta}{\eta} + \frac{|D_4(\varpi)|}{\Gamma(\varpi)}\int_{\upsilon_u}^{G}\left|\ln\left(\frac{G}{\eta}\right)^{\varpi-1}\right||p(\eta)|\frac{d\eta}{\eta} \\
&+\sum_{i=1}^{u}\frac{|D_{5i}(\varpi)|}{\Gamma(\varpi)}\int_{\upsilon_{i-1}}^{\upsilon_i}\left|\ln\left(\frac{\upsilon_i}{\eta}\right)^{\varpi-1}\right||p(\eta)|\frac{d\eta}{\eta} + \frac{\left|(\ln\upsilon)^{2-\varpi}\right|}{\Gamma(\varpi)}\int_{\upsilon_u}^{\upsilon}\left|\ln\left(\frac{\upsilon}{\eta}\right)^{\varpi-1}\right||p(\eta)|\frac{d\eta}{\eta} \\
&+\sum_{i=1}^{u}\frac{\left|\ln\upsilon^{3-\varpi}(\ln\upsilon_i)^{2-\varpi}\right|}{\Gamma(\varpi-1)}\int_{\upsilon_{i-1}}^{\upsilon_i}\ln\left(\frac{\upsilon_i}{\eta}\right)^{\varpi-2}|p(\eta)|\frac{d\eta}{\eta}. \qquad (4.14)
\end{aligned}
$$

From assertion (A_1), we can write

$$
\begin{aligned}
|p(\upsilon)| &= |\phi(\upsilon, a_1(\upsilon), a_2(\upsilon))| \leq \phi(\upsilon, z(\upsilon), a(\upsilon)) \leq \\
&\quad \ell_0(\upsilon) + \ell_1(\upsilon)|z(\upsilon)| + \ell_2(\upsilon)|a(\upsilon)| \\
&= \ell_0(\upsilon) + \ell_1(\upsilon)|z(\upsilon)| + \ell_2(\upsilon)|\psi(\upsilon, p_1(\upsilon), p_2(\upsilon))| \\
&\leq \ell_0(\upsilon) + \ell_1(\upsilon)|z(\upsilon)| + \ell_2(\upsilon)|\psi(\upsilon, s(\upsilon), p(\upsilon))| \\
&\leq \ell_0(\upsilon) + \ell_1(\upsilon)|z(\upsilon)| + \ell_2(\upsilon)\left[\rho_0(\upsilon) + \rho_1(\upsilon)|s(\upsilon)| + \rho_2(\upsilon)|p(\upsilon)|\right] \\
&\leq \frac{\ell_0(\upsilon) + \ell_2(\upsilon)\rho_0(\upsilon)}{1 - \ell_2(\upsilon)\rho_2(\upsilon)} + \frac{\ell_1(\upsilon)|z(\upsilon)| + \ell_2(\upsilon)\rho_1(\upsilon)|s(\upsilon)|}{1 - \ell_2(\upsilon)\rho_2(\upsilon)},
\end{aligned}
$$

which implies that

$$\|p\| \leq \frac{\widetilde{\ell}_0 + \widetilde{\ell}_2\widetilde{\rho}_0}{1 - \widetilde{\ell}_2\widetilde{\rho}_2} + \frac{\widetilde{\ell}_1 \|z\| + \widetilde{\ell}_2\widetilde{\rho}_1 \|s\|}{1 - \widetilde{\ell}_2\widetilde{\rho}_2}. \tag{4.15}$$

Taking $\sup_{\upsilon \in U}$ on (4.14) and using (4.15), one has

$$\begin{aligned}
\left\|\widetilde{E}_1(z,s)\right\|_{\mathfrak{I}_1} &\leq \left(\frac{\widetilde{\ell}_0 + \widetilde{\ell}_2\widetilde{\rho}_0}{\widetilde{\ell}_2\widetilde{\rho}_2 - 1} + \frac{\widetilde{\ell}_1 \|z\| + \widetilde{\ell}_2\widetilde{\rho}_1 \|s\|}{\widetilde{\ell}_2\widetilde{\rho}_2 - 1}\right) \\
&\quad \times \left(\frac{|D_0(\varpi)| \left|\ln\left(\frac{G}{\upsilon_u}\right)^{\varpi-1}\right|}{\Gamma(\varpi)} + \frac{|D_4(\varpi)| \left|\ln\left(\frac{G}{\upsilon_u}\right)^{\varpi}\right|}{\Gamma(\varpi+1)} + \frac{u\,|D_5(\varpi)| \left|\left(\ln\frac{\upsilon_i}{\upsilon_{i-1}}\right)^{\varpi}\right|}{\Gamma(\varpi+1)}\right. \\
&\quad \left. + \frac{\left|(\ln\upsilon)^{2-\varpi}\right| \left|\left(\ln\frac{\upsilon}{\upsilon_u}\right)^{\varpi}\right|}{\Gamma(\varpi+1)} + \frac{u\left|\ln\upsilon^{3-\varpi}(\ln\upsilon_i)^{2-\varpi}\right| \left|\left(\ln\frac{\upsilon_i}{\upsilon_{i-1}}\right)^{\varpi-1}\right|}{\Gamma(\varpi)}\right) \\
&\leq \frac{\left(\widetilde{\ell}_0 + \widetilde{\ell}_2\widetilde{\rho}_0\right) M_3^*}{\widetilde{\ell}_2\widetilde{\rho}_2 - 1} + \frac{\left(\widetilde{\ell}_1 \|z\| + \widetilde{\ell}_2\widetilde{\rho}_1 \|s\|\right) M_3^*}{\widetilde{\ell}_2\widetilde{\rho}_2 - 1} \\
&\leq \frac{\left(\widetilde{\ell}_0 + \widetilde{\ell}_2\widetilde{\rho}_0\right) M_3^*}{\widetilde{\ell}_2\widetilde{\rho}_2 - 1} + \frac{Y_1^* M_3^*}{\widetilde{\ell}_2\widetilde{\rho}_2 - 1} \|(z,s)\|.
\end{aligned} \tag{4.16}$$

In the same scenario, we get

$$\left\|\widetilde{E}_2(z,s)\right\|_{\mathfrak{I}_2} \leq \frac{\left(\widetilde{\rho}_0 + \widetilde{\rho}_2\widetilde{\ell}_0\right) M_3^{**}}{\widetilde{\ell}_2\widetilde{\rho}_2 - 1} + \frac{Y_2^* M_3^{**}}{\widetilde{\ell}_2\widetilde{\rho}_2 - 1} \|(z,s)\|, \tag{4.17}$$

where

$$\begin{aligned}
M_3^* &= \left(\begin{array}{c} \frac{|D_0(\varpi)|\left|\ln\left(\frac{G}{\upsilon_u}\right)^{\varpi-1}\right|}{\Gamma(\varpi)} + \frac{|D_4(\varpi)|\left|\ln\left(\frac{G}{\upsilon_u}\right)^{\varpi}\right|}{\Gamma(\varpi+1)} + \frac{u|D_5(\varpi)|\left|\left(\ln\frac{\upsilon_i}{\upsilon_{i-1}}\right)^{\varpi}\right|}{\Gamma(\varpi+1)} \\ + \frac{\left|(\ln\upsilon)^{2-\varpi}\right|\left|\left(\ln\frac{\upsilon}{\upsilon_u}\right)^{\varpi}\right|}{\Gamma(\varpi+1)} + \frac{u\left|\ln\upsilon^{3-\varpi}(\ln\upsilon_i)^{2-\varpi}\right|\left|\left(\ln\frac{\upsilon_i}{\upsilon_{i-1}}\right)^{\varpi-1}\right|}{\Gamma(\varpi)} \end{array}\right), \ u = 1,2,...,k, \\
M_3^{**} &= \left(\begin{array}{c} \frac{|D_0(\theta)|\left|\ln\left(\frac{G}{\upsilon_u}\right)^{\theta-1}\right|}{\Gamma(\theta)} + \frac{|D_4(\theta)|\left|\ln\left(\frac{G}{\upsilon_u}\right)^{\theta}\right|}{\Gamma(\theta+1)} + \frac{u|D_5(\theta)|\left|\left(\ln\frac{\upsilon_i}{\upsilon_{i-1}}\right)^{\theta}\right|}{\Gamma(\theta+1)} \\ + \frac{\left|(\ln\upsilon)^{2-\theta}\right|\left|\left(\ln\frac{\upsilon}{\upsilon_u}\right)^{\theta}\right|}{\Gamma(\theta+1)} + \frac{u\left|\ln\upsilon^{3-\theta}(\ln\upsilon_i)^{2-\theta}\right|\left|\left(\ln\frac{\upsilon_i}{\upsilon_{i-1}}\right)^{\theta-1}\right|}{\Gamma(\theta)} \end{array}\right), \ u = 1,2,...,m, \\
Y_1^* &= \max\left\{\widetilde{\ell}_1, \widetilde{\ell}_2\widetilde{\rho}_1\right\}, \qquad Y_2^* = \max\left\{\widetilde{\rho}_2\widetilde{\ell}_1, \widetilde{\rho}_1\right\}.
\end{aligned}$$

Applying (4.12), (4.13), (4.16) and (4.17) in (4.11), we have

$$\begin{aligned}
\left\|E(z,s) + \widetilde{E}(z,s)\right\|_{\mathfrak{I}} &\leq M_1^* + M_1^{**} + \frac{\left(\widetilde{\ell}_0 + \widetilde{\ell}_2\widetilde{\rho}_0\right) M_3^* + \left(\widetilde{\rho}_0 + \widetilde{\rho}_2\widetilde{\ell}_0\right) M_3^{**}}{\widetilde{\ell}_2\widetilde{\rho}_2 - 1} \\
&\quad + \frac{Y_1^* M_3^* + Y_2^* M_3^{**}}{\widetilde{\ell}_2\widetilde{\rho}_2 - 1} \|(z,s)\| + M_2^* \|z\| + M_2^{**} \|s\|
\end{aligned}$$

$$
\begin{aligned}
&\leq M_1^* + M_1^{**} + \frac{\left(\widetilde{\ell}_0 + \widetilde{\ell}_2\widetilde{\rho}_0\right) M_3^* + \left(\widetilde{\rho}_0 + \widetilde{\rho}_2\widetilde{\ell}_0\right) M_3^{**}}{\widetilde{\ell}_2\widetilde{\rho}_2 - 1} \\
&\quad + \left(M_2^* + M_2^{**} + \frac{Y_1^* M_3^* + Y_2^* M_3^{**}}{\widetilde{\ell}_2\widetilde{\rho}_2 - 1} \right) \|(z,s)\| \\
&\leq x,
\end{aligned}
$$

which implies that $E(z, s)(\upsilon) + \widetilde{E}(z, s) \in \Im_x$. After that, for any $\upsilon \in U$ and $s, z, \widetilde{s}, \widetilde{z} \in \Im$, one writes

$$
\begin{aligned}
&\|E(z,s) - E(\widetilde{z},\widetilde{s})\|_{\Im} \\
\leq\ & \|E_1(z) - E_1(\widetilde{z})\|_{\Im_1} + \|E_2(s) - E_2(\widetilde{s})\|_{\Im_2} \\
\leq\ & G\,|D_0(\varpi)|\,|B^*(z) - B^*(\widetilde{z})| + \sum_{i=1}^{u} |D_{1i}(\varpi)|\,|I_i(z_i) - I_i(\widetilde{z}_i)| + \sum_{i=1}^{u} |D_{2i}(\varpi)| \left|\widetilde{I}_i(z_i) - \widetilde{I}_i(\widetilde{z}_i)\right| \\
& + \frac{|D_3(\varpi)|}{\Gamma(\varpi)} \int_1^G \left| \ln\left(\frac{G}{\eta}\right)^{\varpi-1} \right| |B(\eta, z(\eta)) - B(\eta, \widetilde{z}(\eta))| \frac{d\eta}{\eta} \\
& + G\,|D_0(\theta)|\,|B^*(s) - B^*(\widetilde{s})| + \sum_{j=1}^{u} D_{1j}(\theta) \left|I_j(s_j) - I_j(\widetilde{s}_j)\right| + \sum_{j=1}^{u} D_{2j}(\theta) \left|\widetilde{I}_j(s_j) - \widetilde{I}_j(\widetilde{s}_j)\right| \\
& + \frac{|D_3(\theta)|}{\Gamma(\theta)} \int_1^G \left| \ln\left(\frac{G}{\eta}\right)^{\theta-1} \right| |B(\eta, s(\eta)) - B(\eta, \widetilde{s}(\eta))| \frac{d\eta}{\eta}.
\end{aligned}
$$

Applying (A_5) and (A_6), one has

$$
\begin{aligned}
&\|E(z,s) - E(\widetilde{z},\widetilde{s})\|_{\Im} \\
\leq\ & \left[G L_{B^*} |D_0(\varpi)| + u L_I |D_1(\varpi)| + u L_{\widetilde{I}} |D_2(\varpi)| - \frac{L_B |D_3(\varpi)|\,|(\ln G)^{\varpi}|}{\Gamma(\varpi+1)} \right] \|z - \widetilde{z}\| \\
& + \left[G \widetilde{L}_{B^*} |D_0(\theta)| + u \widetilde{L}_I |D_1(\theta)| + u \widetilde{L}_{\widetilde{I}} |D_2(\theta)| - \frac{\widetilde{L}_B |D_3(\theta)| \left|(\ln G)^{\theta}\right|}{\Gamma(\theta+1)} \right] \|s - \widetilde{s}\| \\
\leq\ & L(\Delta_1 + \Delta_2) \|(z - \widetilde{z}, s - \widetilde{s})\|,
\end{aligned}
$$

where

$$
L = \max\left\{ L_{B^*}, L_I, L_{\widetilde{I}}, \widetilde{L}_{B^*}, \widetilde{L}_I, \widetilde{L}_{\widetilde{I}}, L_B, \widetilde{L}_B \right\},
$$

and

$$
\begin{aligned}
\Delta_1 &= G\,|D_0(\varpi)| + u\,|D_1(\varpi)| + u\,|D_2(\varpi)| - \frac{|D_3(\varpi)|\,|(\ln G)^{\varpi}|}{\Gamma(\varpi+1)},\ u = 1,2,...,k, \\
\Delta_2 &= G\,|D_0(\theta)| + u\,|D_1(\theta)| + u\,|D_2(\theta)| - \frac{|D_3(\theta)| \left|(\ln G)^{\theta}\right|}{\Gamma(\theta+1)},\ u = 1,2,...,m.
\end{aligned}
$$

Hence, E is a contraction mapping. Now, we claim that $\widetilde{E}$ is continuous and compact. For this, we build a sequence $G_n = (z_n, s_n)$ in $\Im$ so that $\lim_{n\to\infty}(z_n, s_n) = (z, s) \in \Im_x$. Hence, we obtain

$$
\left\|\widetilde{E}(z,s) - \widetilde{E}(z_n,s_n)\right\|_{\Im} \leq \left\|\widetilde{E}_1(z_n,s_n) - \widetilde{E}_1(z,s)\right\|_{\Im_1} + \left\|\widetilde{E}_2(z_n,s_n) - \widetilde{E}_2(z,s)\right\|_{\Im_2}. \tag{4.18}
$$

Since

$$\begin{aligned}
&\left\|\widetilde{E_1}(z_n,s_n)-\widetilde{E}_1(z,s)\right\|_{\mathfrak{I}_1}\\
\leq\ &\left(\frac{|D_0(\varpi)|\left|\ln\left(\frac{G}{\upsilon_u}\right)^{\varpi-1}\right|}{\Gamma(\varpi)}+\frac{|D_4(\varpi)|\left|\ln\left(\frac{G}{\upsilon_u}\right)^{\varpi}\right|}{\Gamma(\varpi+1)}+\frac{u\,|D_5(\varpi)|\left|\left(\ln\frac{\upsilon_u}{\upsilon_{u-1}}\right)^{\varpi}\right|}{\Gamma(\varpi+1)}\right.\\
&\left.+\frac{\left|(\ln\upsilon)^{2-\varpi}\right|\left|\left(\ln\frac{\upsilon}{\upsilon_u}\right)^{\varpi}\right|}{\Gamma(\varpi+1)}+\frac{u\left|\ln\upsilon^{3-\varpi}(\ln\upsilon_u)^{2-\varpi}\right|\left|\left(\ln\frac{\upsilon_u}{\upsilon_{u-1}}\right)^{\varpi-1}\right|}{\Gamma(\varpi)}\right)\left(\frac{L_\phi\,\|z_n-z\|+\widetilde{L}_\phi L_\psi\,\|s_n-s\|}{\widetilde{L}_\phi\widetilde{L}_\psi-1}\right)\\
\leq\ &M_3^*\left(\frac{L_\phi\,\|z_n-z\|+\widetilde{L}_\phi L_\psi\,\|s_n-s\|}{\widetilde{L}_\phi\widetilde{L}_\psi-1}\right),
\end{aligned}\tag{4.19}$$

and

$$\begin{aligned}
&\left\|\widetilde{E}_2(z_n,s_n)-\widetilde{E}_2(z,s)\right\|_{\mathfrak{I}_2}\\
\leq\ &\left(\frac{|D_0(\theta)|\left|\ln\left(\frac{G}{\upsilon_u}\right)^{\theta-1}\right|}{\Gamma(\theta)}+\frac{|D_4(\theta)|\left|\ln\left(\frac{G}{\upsilon_u}\right)^{\theta}\right|}{\Gamma(\theta+1)}+\frac{u\,|D_5(\theta)|\left|\left(\ln\frac{\upsilon_u}{\upsilon_{u-1}}\right)^{\theta}\right|}{\Gamma(\theta+1)}\right.\\
&\left.+\frac{\left|(\ln\upsilon)^{2-\theta}\right|\left|\left(\ln\frac{\upsilon}{\upsilon_u}\right)^{\theta}\right|}{\Gamma(\theta+1)}+\frac{u\left|\ln\upsilon^{3-\theta}(\ln\upsilon_u)^{2-\theta}\right|\left|\left(\ln\frac{\upsilon_u}{\upsilon_{i-1}}\right)^{\theta-1}\right|}{\Gamma(\theta)}\right)\left(\frac{L_\phi\widetilde{L}_\psi\,\|z_n-z\|+L_\psi\,\|s_n-s\|}{\widetilde{L}_\phi\widetilde{L}_\psi-1}\right)\\
\leq\ &M_3^{**}\left(\frac{L_\phi\widetilde{L}_\psi\,\|z_n-z\|+L_\psi\,\|s_n-s\|}{\widetilde{L}_\phi\widetilde{L}_\psi-1}\right).
\end{aligned}\tag{4.20}$$

Applying (4.19) and (4.20) in (4.18), we conclude that

$$\left\|\widetilde{E}(z,s)-\widetilde{E}(z_n,s_n)\right\|_{\mathfrak{I}}\leq M_3^*\left(\frac{L_\phi\,\|z_n-z\|+\widetilde{L}_\phi L_\psi\,\|s_n-s\|}{\widetilde{L}_\phi\widetilde{L}_\psi-1}\right)+M_3^{**}\left(\frac{L_\phi\widetilde{L}_\psi\,\|z_n-z\|+L_\psi\,\|s_n-s\|}{\widetilde{L}_\phi\widetilde{L}_\psi-1}\right),$$

which yields $\left\|\widetilde{E}(z,s)-\widetilde{E}(z_n,s_n)\right\|_{\mathfrak{I}}\to 0$ as $n\to\infty$, this proves the continuity of $\widetilde{E}$. Next, using (4.16) and (4.17), we get

$$\begin{aligned}
\left\|\widetilde{E}\,(z,s)\,(\upsilon)\right\|_{\mathfrak{I}}\ &\leq\ \left\|\widetilde{E_1}\,(z,s)\,(\upsilon)\right\|_{\mathfrak{I}_1}+\left\|\widetilde{E}_2\,(z,s)\right\|_{\mathfrak{I}_2}\\
&\leq\ \frac{\left(\widetilde{\ell}_0+\widetilde{\ell}_2\widetilde{\rho}_0\right)M_3^*}{\widetilde{\ell}_2\widetilde{\rho}_2-1}+\frac{\left(\widetilde{\rho}_0+\widetilde{\rho}_2\widetilde{\ell}_0\right)M_3^{**}}{\widetilde{\ell}_2\widetilde{\rho}_2-1}+\left(\frac{Y_1^*M_3^*}{\widetilde{\ell}_2\widetilde{\rho}_2-1}+\frac{Y_2^*M_3^{**}}{\widetilde{\ell}_2\widetilde{\rho}_2-1}\right)\|(z,s)\|\\
&\leq\ x.
\end{aligned}$$

Therefore, $\widetilde{E}$ is uniformly bounded on $\mathfrak{I}_x$. Finally, we show that $\widetilde{E}$ is equicontinuous. To get this result, take $\upsilon_1,\upsilon_2\in U$ with $\upsilon_1<\upsilon_2$ and for any $(z,s)\in\mathfrak{I}_x\subset\mathfrak{I}$ (clearly $\mathfrak{I}_x$ is bounded), we obtain

$$\begin{aligned}
&\left\|\widetilde{E}_1(z,s)\,(\upsilon_1)-\widetilde{E}_1(z,s)\,(\upsilon_2)\right\|_{\mathfrak{I}_1}\\
=\ &\max\left\{\left|\left[\widetilde{E}_1(z,s)\,(\upsilon_1)-\widetilde{E}_1(z,s)\,(\upsilon_2)\right](\ln\upsilon)^{2-\varpi}\right|\right\}
\end{aligned}$$

$$\leq \left[\left(\frac{|D_0(\varpi)| \left| \ln \left(\frac{G}{\upsilon_u} \right)^{\varpi-1} \right|}{\Gamma(\varpi)} + \frac{|D_4(\varpi)| \left| \ln \left(\frac{G}{\upsilon_u} \right)^{\varpi} \right|}{\Gamma(\varpi+1)} + \frac{u\,|D_5(\varpi)| \left| \left(\ln \frac{\upsilon_u}{\upsilon_{u-1}} \right)^{\varpi} \right|}{\Gamma(\varpi+1)} \right) \right.$$

$$\times \left| (\ln \upsilon)^{2-\varpi} \right| \left| (\ln \upsilon_1)^{\varpi-2} - (\ln \upsilon_2)^{\varpi-2} \right|$$

$$\left. + \frac{u \left| (\ln \upsilon)^{2-\varpi} \right| \left| \left(\ln \frac{\upsilon}{\upsilon_u} \right)^{\varpi} \right| \left| \ln \upsilon_1^{3-\varpi} \left(\log_{\upsilon_u} \upsilon_1 \right)^{\varpi-2} - \ln \upsilon_2^{3-\varpi} \left(\log_{\upsilon_u} \upsilon_2 \right)^{\varpi-2} \right|}{\Gamma(\varpi)} \right]$$

$$\times \left(\frac{\widetilde{\ell}_0 + \widetilde{\ell}_2 \widetilde{\rho}_0}{1 - \widetilde{\ell}_2 \widetilde{\rho}_2} + \frac{\widetilde{\ell}_1 \|z\| + \widetilde{\ell}_2 \widetilde{\rho}_1 \|s\|}{1 - \widetilde{\ell}_2 \widetilde{\rho}_2} \right) + \frac{\left| (\ln \upsilon)^{2-\varpi} \right|}{\Gamma(\varpi)} \left| \int_{\upsilon_u}^{\upsilon_1} \ln \left(\frac{\upsilon_1}{\eta} \right)^{\varpi-1} \phi \left(\upsilon, {}^H D^{\varpi} z(\upsilon), {}^H D^{\theta} s(\upsilon) \right) \frac{d\eta}{\eta} \right.$$

$$\left. - \int_{\upsilon_u}^{\upsilon_2} \ln \left(\frac{\upsilon_2}{\eta} \right)^{\varpi-1} \phi \left(\upsilon, {}^H D^{\varpi} z(\upsilon), {}^H D^{\theta} s(\upsilon) \right) \frac{d\eta}{\eta} \right|,$$

which yields that

$$\left\| \widetilde{E}_1(z,s)(\upsilon_1) - \widetilde{E}_1(z,s)(\upsilon_2) \right\|_{\mathfrak{I}_1} \to 0, \text{ as } \upsilon_1 \to \upsilon_2.$$

Similarly, we get

$$\left\| \widetilde{E}_2(z,s)(\upsilon_1) - \widetilde{E}_2(z,s)(\upsilon_2) \right\|_{\mathfrak{I}_2} \to 0, \text{ as } \upsilon_1 \to \upsilon_2.$$

Hence

$$\left\| \widetilde{E}(z,s)(\upsilon_1) - \widetilde{E}(z,s)(\upsilon_2) \right\|_{\mathfrak{I}} \to 0, \text{ as } \upsilon_1 \to \upsilon_2.$$

Therefore $\widetilde{E}$ is a relatively compact on $\mathfrak{I}_x$. Thanks to the theorem of Arzelà-Ascoli, $\widetilde{E}$ is compact. Thus, it is completely continuous. So, the CII-FDEs (1.1) admits at least one solution. This finishes the proof. □

Theorem 4.4. *Assume that* (A_4)–(A_6) *are fulfilled with*

$$\mho_1 + \mho_3 + \frac{\mho_2 \left(L_\phi + \widetilde{L}_\phi L_\psi \right) + \mho_4 \left(L_\phi \widetilde{L}_\psi + L_\psi \right)}{\widetilde{L}_\phi \widetilde{L}_\psi - 1} < 1, \tag{4.21}$$

then the CII-FDEs (1.1) possesses a unique solution.

Proof. Let $\aleph = (\aleph_1, \aleph_1) : \mathfrak{I} \to \mathfrak{I}$ be an operator defined by $\aleph(z,s)(\upsilon) = (\aleph_1(z,s), \aleph_2(z,s))(\upsilon)$, where

$$\begin{aligned} \aleph_1(z,s) &= G D_0(\varpi) B^*(z) (\ln \upsilon)^{\varpi-2} + \sum_{i=1}^{u} D_{1i}(\varpi) (\ln \upsilon)^{\varpi-2} I_i(z(\upsilon_i)) \\ &\quad + \sum_{i=1}^{u} D_{2i}(\varpi) (\ln \upsilon)^{\varpi-2} \widetilde{I}_i(z(\upsilon_i)) + \frac{D_3(\varpi) (\ln \upsilon)^{\varpi-2}}{\Gamma(\varpi)} \int_1^G \ln \left(\frac{G}{\eta} \right)^{\varpi-1} B(\eta, z(\eta)) \frac{d\eta}{\eta} \\ &\quad + \frac{D_0(\varpi) (\ln \upsilon)^{\varpi-2}}{\Gamma(\varpi-1)} \int_{\upsilon_u}^G \ln \left(\frac{G}{\eta} \right)^{\varpi-2} \phi \left(\eta, {}^H D^{\varpi} z(\eta), {}^H D^{\theta} s(\eta) \right) \frac{d\eta}{\eta} \\ &\quad + \frac{D_4(\varpi) (\ln \upsilon)^{\varpi-2}}{\Gamma(\varpi)} \int_{\upsilon_u}^G \ln \left(\frac{G}{\eta} \right)^{\varpi-1} \phi \left(\eta, {}^H D^{\varpi} z(\eta), {}^H D^{\theta} s(\eta) \right) \frac{d\eta}{\eta} \end{aligned}$$

$$+\sum_{i=1}^{u}\frac{D_{5i}(\varpi)(\ln\upsilon)^{\varpi-2}}{\Gamma(\varpi)}\int_{\upsilon_{i-1}}^{\upsilon_i}\ln\left(\frac{\upsilon_i}{\eta}\right)^{\varpi-1}\phi\left(\eta,{}^{H}D^{\varpi}z(\eta),{}^{H}D^{\theta}s(\eta)\right)\frac{d\eta}{\eta}$$
$$+\sum_{i=1}^{u}\frac{\ln\upsilon^{3-\varpi}(\log_{\upsilon_i}\upsilon)^{\varpi-2}}{\Gamma(\varpi-1)}\int_{\upsilon_{i-1}}^{\upsilon_i}\ln\left(\frac{\upsilon_i}{\eta}\right)^{\varpi-2}\phi\left(\eta,{}^{H}D^{\varpi}z(\eta),{}^{H}D^{\theta}s(\eta)\right)\frac{d\eta}{\eta}$$
$$+\frac{1}{\Gamma(\varpi)}\int_{\upsilon_u}^{\upsilon}\ln\left(\frac{\upsilon}{\eta}\right)^{\varpi-1}\phi\left(\eta,{}^{H}D^{\varpi}z(\eta),{}^{H}D^{\theta}s(\eta)\right)\frac{d\eta}{\eta},$$

for $u = 1, 2, ..., k$ and

$$\begin{aligned}\aleph_2(z,s) &= GD_0(\theta)B^*(s)(\ln\upsilon)^{\theta-2}+\sum_{j=1}^{u}D_{1j}(\theta)(\ln\upsilon)^{\theta-2}I_j(s_j)+\sum_{j=1}^{u}D_{2j}(\theta)(\ln\upsilon)^{\theta-2}\widetilde{I}_j(s_j)\\
&+\frac{D_3(\theta)(\ln\upsilon)^{\theta-2}}{\Gamma(\theta)}\int_1^G\ln\left(\frac{G}{\eta}\right)^{\theta-1}B(\eta,s(\eta))\frac{d\eta}{\eta}\\
&+\frac{D_0(\theta)(\ln\upsilon)^{\theta-2}}{\Gamma(\theta-1)}\int_{\upsilon_u}^G\ln\left(\frac{G}{\eta}\right)^{\theta-2}\psi(\eta,p_1(\eta),p_2(\eta))\frac{d\eta}{\eta}\\
&+\frac{D_4(\theta)(\ln\upsilon)^{\theta-2}}{\Gamma(\theta)}\int_{\upsilon_u}^G\ln\left(\frac{G}{\eta}\right)^{\theta-1}\psi(\eta,p_1(\eta),p_2(\eta))\frac{d\eta}{\eta}\\
&+\sum_{j=1}^{u}\frac{D_{5i}(\theta)(\ln\upsilon)^{\theta-2}}{\Gamma(\theta)}\int_{\upsilon_{j-1}}^{\upsilon_j}\ln\left(\frac{\upsilon_j}{\eta}\right)^{\theta-1}\psi(\eta,p_1(\eta),p_2(\eta))\frac{d\eta}{\eta}\\
&+\sum_{j=1}^{u}\frac{\ln\upsilon^{3-\theta}(\log_{\upsilon_j}\upsilon)^{\theta-2}}{\Gamma(\theta-1)}\int_{\upsilon_{j-1}}^{\upsilon_j}\ln\left(\frac{\upsilon_i}{\eta}\right)^{\theta-2}\psi(\eta,p_1(\eta),p_2(\eta))\frac{d\eta}{\eta}\\
&+\frac{1}{\Gamma(\theta)}\int_{\upsilon_u}^{\upsilon}\ln\left(\frac{\upsilon}{\eta}\right)^{\theta-1}\psi(\eta,p_1(\eta),p_2(\eta))\frac{d\eta}{\eta},\end{aligned}$$

for $u = 1, 2, ..., m$. In light of Theorem 4.3, one can obtain

$$\begin{aligned}&\left|(\aleph_1(z,s)-\aleph_1(\widetilde{z},\widetilde{s}))\left((\ln\upsilon)^{\varpi-2}\right)\right|\\
\leq\ &\Bigg[GL_{B^*}|D_0(\varpi)|+uL_I|D_1(\varpi)|+uL_{\widetilde{I}}|D_2(\varpi)|-\frac{L_B|D_3(\varpi)||(\ln G)^{\varpi}|}{\Gamma(\varpi+1)}\\
&+\Bigg(\frac{|D_0(\varpi)|\left|\ln\left(\frac{G}{\upsilon_u}\right)^{\varpi-1}\right|}{\Gamma(\varpi)}+\frac{|D_4(\varpi)|\left|\ln\left(\frac{G}{\upsilon_u}\right)^{\varpi}\right|}{\Gamma(\varpi+1)}+\frac{u|D_5(\varpi)|\left|\left(\ln\frac{\upsilon_u}{\upsilon_{u-1}}\right)^{\varpi}\right|}{\Gamma(\varpi+1)}\\
&+\frac{\left|(\ln\upsilon)^{2-\varpi}\right|\left|\left(\ln\frac{\upsilon}{\upsilon_u}\right)^{\varpi}\right|}{\Gamma(\varpi+1)}+\frac{u\left|\ln\upsilon^{3-\varpi}(\ln\upsilon_u)^{2-\varpi}\right|\left|\left(\ln\frac{\upsilon_u}{\upsilon_{u-1}}\right)^{\varpi-1}\right|}{\Gamma(\varpi)}\Bigg)\left(\frac{L_\phi}{\widetilde{L}_\phi\widetilde{L}_\psi-1}\right)\Bigg]|z-\widetilde{z}|\\
&+\Bigg(\frac{|D_0(\varpi)|\left|\ln\left(\frac{G}{\upsilon_u}\right)^{\varpi-1}\right|}{\Gamma(\varpi)}+\frac{|D_4(\varpi)|\left|\ln\left(\frac{G}{\upsilon_u}\right)^{\varpi}\right|}{\Gamma(\varpi+1)}+\frac{u|D_5(\varpi)|\left|\left(\ln\frac{\upsilon_u}{\upsilon_{u-1}}\right)^{\varpi}\right|}{\Gamma(\varpi+1)}\end{aligned}$$

$$+\frac{\left|(\ln\upsilon)^{2-\varpi}\right|\left|\left(\ln\frac{\upsilon}{\upsilon_u}\right)^{\varpi}\right|}{\Gamma(\varpi+1)}+\frac{u\left|\ln\upsilon^{3-\varpi}(\ln\upsilon_u)^{2-\varpi}\right|\left|\left(\ln\frac{\upsilon_u}{\upsilon_{u-1}}\right)^{\varpi-1}\right|}{\Gamma(\varpi)}\Bigg)\left(\frac{\widetilde{L}_\phi L_\psi\,|s-\widetilde{s}|}{\widetilde{L}_\phi\widetilde{L}_\psi-1}\right),$$

for $u=1,2,...,k$. Passing $\sup_{\upsilon\in U}$, we have

$$\|\aleph_1(z,s)-\aleph_1(\widetilde{z},\widetilde{s})\|_{\mathfrak{I}_1}\leq\left(\mho_1+\frac{\mho_2\left(L_\phi+\widetilde{L}_\phi L_\psi\right)}{\widetilde{L}_\phi\widetilde{L}_\psi-1}\right)\|(z,s)-(\widetilde{z},\widetilde{s})\|,\ u=1,2,...,k,$$

where

$$\begin{aligned}\mho_1 &= GL_{B^*}\,|D_0(\varpi)|+uL_I\,|D_1(\varpi)|+uL_{\widetilde{I}}|D_2(\varpi)|-\frac{L_B\,|D_3(\varpi)|\,|(\ln G)^{\varpi}|}{\Gamma(\varpi+1)},\\ \mho_2 &= \frac{|D_0(\varpi)|\left|\ln\left(\frac{G}{\upsilon_u}\right)^{\varpi-1}\right|}{\Gamma(\varpi)}+\frac{|D_4(\varpi)|\left|\ln\left(\frac{G}{\upsilon_u}\right)^{\varpi}\right|}{\Gamma(\varpi+1)}+\frac{u\,|D_5(\varpi)|\left|\left(\ln\frac{\upsilon_u}{\upsilon_{u-1}}\right)^{\varpi}\right|}{\Gamma(\varpi+1)}\\ &\quad+\frac{\left|(\ln\upsilon)^{2-\varpi}\right|\left|\left(\ln\frac{\upsilon}{\upsilon_u}\right)^{\varpi}\right|}{\Gamma(\varpi+1)}+\frac{u\left|\ln\upsilon^{3-\varpi}(\ln\upsilon_u)^{2-\varpi}\right|\left|\left(\ln\frac{\upsilon_u}{\upsilon_{u-1}}\right)^{\varpi-1}\right|}{\Gamma(\varpi)}.\end{aligned}$$

Analogously,

$$\|\aleph_2(z,s)-\aleph_2(\widetilde{z},\widetilde{s})\|_{\mathfrak{I}_2}\leq\left(\mho_3+\frac{\mho_4\left(L_\psi+L_\phi\widetilde{L}_\psi\right)}{\widetilde{L}_\phi\widetilde{L}_\psi-1}\right)\|(z,s)-(\widetilde{z},\widetilde{s})\|,\ u=1,2,...,m,$$

where

$$\begin{aligned}\mho_3 &= G\widetilde{L}_{B^*}\,|D_0(\theta)|+u\widetilde{L}_I\,|D_1(\theta)|+u\widetilde{L}_{\widetilde{I}}|D_2(\theta)|-\frac{\widetilde{L}_B\,|D_3(\theta)|\left|(\ln G)^{\theta}\right|}{\Gamma(\theta+1)},\\ \mho_4 &= \frac{|D_0(\theta)|\left|\ln\left(\frac{G}{\upsilon_u}\right)^{\theta-1}\right|}{\Gamma(\theta)}+\frac{|D_4(\theta)|\left|\ln\left(\frac{G}{\upsilon_u}\right)^{\theta}\right|}{\Gamma(\theta+1)}+\frac{u\,|D_5(\theta)|\left|\left(\ln\frac{\upsilon_u}{\upsilon_{u-1}}\right)^{\theta}\right|}{\Gamma(\theta+1)}\\ &\quad+\frac{\left|(\ln\upsilon)^{2-\theta}\right|\left|\left(\ln\frac{\upsilon}{\upsilon_u}\right)^{\theta}\right|}{\Gamma(\theta+1)}+\frac{u\left|\ln\upsilon^{3-\theta}(\ln\upsilon_u)^{2-\theta}\right|\left|\left(\ln\frac{\upsilon_u}{\upsilon_{u-1}}\right)^{\theta-1}\right|}{\Gamma(\theta)}.\end{aligned}$$

Hence

$$\|\aleph(z,s)-\aleph(\widetilde{z},\widetilde{s})\|_{\mathfrak{I}}\leq\left(\mho_1+\mho_3+\frac{\mho_2\left(L_\phi+\widetilde{L}_\phi L_\psi\right)+\mho_4\left(L_\psi+L_\phi\widetilde{L}_\psi\right)}{\widetilde{L}_\phi\widetilde{L}_\psi-1}\right)\|(z,s)-(\widetilde{z},\widetilde{s})\|.$$

This suggests that $\aleph$ is a contraction. Consequently, the CII-FDEs (1.1) has a unique solution. □

5. Analyzing stabilities

In this section, we examine various stability types for the suggested system, including the HU, GHU, HUR, and GHUR stability.

Theorem 5.1. *If the assertions* (A_1)–(A_3) *and the condition (4.21) are true and*

$$\beth = 1 - \frac{L_\phi \widetilde{L}_\phi L_\psi \widetilde{L}_\psi \mho_2 \mho_4}{\left[\left(\widetilde{L}_\phi \widetilde{L}_\psi - 1\right)\left((\ln \upsilon)^{\varpi-2} - \mho_1\right) - \mho_2 L_\phi\right]\left[\left(\widetilde{L}_\phi \widetilde{L}_\psi - 1\right)\left((\ln \upsilon)^{\theta-2} - \mho_3\right) - \mho_4 L_\psi\right]} > 0,$$

then the unique solution of CII-FDEs (1.1) is HU stable and as a result, GHU stable.

Proof. Take into account that $(z, s) \in \Im$ is an approximate solution of (3.1) and consider $\widehat{(z, s)} \in \Im$ is a solution of the coupled problem shown below

$$\begin{cases} {}^H D^{\varpi} \widehat{z}(\upsilon) = \phi\left(\upsilon, {}^H D^{\varpi} \widehat{z}(\upsilon), {}^H D^{\theta} \widehat{s}(\upsilon)\right), \ \upsilon \in U, \ \upsilon \neq \upsilon_i, \ i = 1, 2, ...k, \\ {}^H D^{\theta} \widehat{s}(\upsilon) = \psi\left(\upsilon, {}^H D^{\theta} \widehat{s}(\upsilon), {}^H D^{\varpi} \widehat{z}(\upsilon)\right), \ \upsilon \in U, \ \upsilon \neq \upsilon_j, \ j = 1, 2, ...m, \\ \Delta z(\widehat{\upsilon_i}) = I_i \widehat{z}(\upsilon_i), \quad \Delta \widehat{z}'(\upsilon_i) = \widetilde{I_i} \widehat{z}(\upsilon_i), \quad i = 1, 2, ...k, \\ \Delta \widehat{s}(\upsilon_j) = I_j \widehat{s}(\upsilon_j), \quad \Delta \widehat{s}'(\upsilon_j) = \widetilde{I_j} \widehat{s}(\upsilon_j), \quad j = 1, 2, ...m, \\ \widehat{z}(G) = \frac{1}{\Gamma(\varpi)} \int_1^G \ln\left(\frac{G}{\eta}\right)^{\varpi-1} B(\eta, \widehat{z}(\eta)) \frac{d\eta}{\eta}, \ \widehat{z}'(G) = B^*(\widehat{z}), \\ \widehat{s}(G) = \frac{1}{\Gamma(\theta)} \int_1^G \ln\left(\frac{G}{\eta}\right)^{\theta-1} B(\eta, \widehat{s}(\eta)) \frac{d\eta}{\eta}, \ \widehat{s}'(G) = B^*(\widehat{s}). \end{cases} \tag{5.1}$$

From Remark 3.5, we get

$$\begin{cases} {}^H D^{\varpi} z(\upsilon) = \phi\left(\upsilon, {}^H D^{\varpi} z(\upsilon), {}^H D^{\theta} s(\upsilon)\right) + \Re_\phi(\upsilon), \ \upsilon \in U, \ \upsilon \neq \upsilon_i, \ i = 1, 2, ...k, \\ \Delta z(\upsilon_i) = I_i(z(\upsilon_i)) + \Re_{\phi_i}, \ \ \Delta z'(\upsilon_i) = \widetilde{I}_i(z(\upsilon_i)) + \Re_{\phi_i}, \ i = 1, 2, ...k, \\ {}^H D^{\theta} s(\upsilon) = \phi\left(\upsilon, {}^H D^{\theta} s(\upsilon), {}^H D^{\varpi} z(\upsilon)\right) + \Re_\psi(\upsilon), \ \upsilon \in U, \ \upsilon \neq \upsilon_j, \ j = 1, 2, ...m, \\ \Delta s(\upsilon_j) = I_j\left(s(\upsilon_j)\right) + \Re_{\psi_j}, \ \ \Delta s'(\upsilon_j) = \widetilde{I}_j\left(s(\upsilon_j)\right) + \Re_{\psi_j}, \ j = 1, 2, ..., m. \end{cases} \tag{5.2}$$

It follows from Corollary 4.2 that the solution of system (5.2) is □

$$\begin{aligned} z(\upsilon) \ = \ & G D_0(\varpi) B^*(z) (\ln \upsilon)^{\varpi-2} + \sum_{i=1}^{u} D_{1i}(\varpi) (\ln \upsilon)^{\varpi-2} \left(I_i(z_i) + \Re_{\phi_i}\right) \\ & + \sum_{i=1}^{u} D_{2i}(\varpi) (\ln \upsilon)^{\varpi-2} \left(\widetilde{I}_i(z_i) + \Re_{\phi_i}\right) \\ & + \frac{D_0(\varpi) (\ln \upsilon)^{\varpi-2}}{\Gamma(\varpi - 1)} \int_{\upsilon_u}^{G} \ln\left(\frac{G}{\eta}\right)^{\varpi-2} \left[\phi\left(\eta, {}^H D^{\varpi} z(\eta), {}^H D^{\theta} s(\eta)\right) + \Re_\phi(\upsilon)\right] \frac{d\eta}{\eta} \\ & + \frac{D_4(\varpi) (\ln \upsilon)^{\varpi-2}}{\Gamma(\varpi)} \int_{\upsilon_u}^{G} \ln\left(\frac{G}{\eta}\right)^{\varpi-1} \left[\phi\left(\eta, {}^H D^{\varpi} z(\eta), {}^H D^{\theta} s(\eta)\right) + \Re_\phi(\upsilon)\right] \frac{d\eta}{\eta} \\ & + \sum_{i=1}^{u} \frac{D_{5i}(\varpi) (\ln \upsilon)^{\varpi-2}}{\Gamma(\varpi)} \int_{\upsilon_{i-1}}^{\upsilon_i} \ln\left(\frac{\upsilon_i}{\eta}\right)^{\varpi-1} \left[\phi\left(\eta, {}^H D^{\varpi} z(\eta), {}^H D^{\theta} s(\eta)\right) + \Re_\phi(\upsilon)\right] \frac{d\eta}{\eta} \\ & + \sum_{i=1}^{u} \frac{\ln \upsilon^{3-\varpi} (\log_{\upsilon_i} \upsilon)^{\varpi-2}}{\Gamma(\varpi - 1)} \int_{\upsilon_{i-1}}^{\upsilon_i} \ln\left(\frac{\upsilon_i}{\eta}\right)^{\varpi-2} \left[\phi\left(\eta, {}^H D^{\varpi} z(\eta), {}^H D^{\theta} s(\eta)\right) + \Re_\phi(\upsilon)\right] \frac{d\eta}{\eta} \\ & + \frac{D_3(\varpi) (\ln \upsilon)^{\varpi-2}}{\Gamma(\varpi)} \int_{1}^{G} \ln\left(\frac{G}{\eta}\right)^{\varpi-1} B(\eta, z(\eta)) \frac{d\eta}{\eta} \end{aligned}$$

$$+\frac{1}{\Gamma(\varpi)}\int_{\upsilon_u}^{\upsilon}\ln\left(\frac{\upsilon}{\eta}\right)^{\varpi-1}\left[\phi\left(\eta,^{H}D^{\varpi}z(\eta),^{H}D^{\theta}s(\eta)\right)+\Re_{\phi}(\upsilon)\right]\frac{d\eta}{\eta}, \tag{5.3}$$

for $u = 1, 2, ..., k$ and

$$\begin{aligned} s(\upsilon) &= GD_0(\theta)B^*(s)(\ln\upsilon)^{\theta-2}+\sum_{j=1}^{u}D_{1j}(\theta)(\ln\upsilon)^{\theta-2}\left(I_j(s_j)+\Re_{\psi_j}\right) \\ &+\sum_{j=1}^{u}D_{2j}(\theta)(\ln\upsilon)^{\theta-2}\left(I_j(z_j)+\Re_{\psi_j}\right) \\ &+\frac{D_0(\theta)(\ln\upsilon)^{\theta-2}}{\Gamma(\theta-1)}\int_{\upsilon_u}^{G}\ln\left(\frac{G}{\eta}\right)^{\theta-2}\left[\phi\left(\upsilon,^{H}D^{\theta}s(\upsilon),^{H}D^{\varpi}z(\upsilon)\right)+\Re_{\psi}(\upsilon)\right]\frac{d\eta}{\eta} \\ &+\frac{D_4(\theta)(\ln\upsilon)^{\theta-2}}{\Gamma(\theta)}\int_{\upsilon_u}^{G}\ln\left(\frac{G}{\eta}\right)^{\theta-1}\left[\phi\left(\upsilon,^{H}D^{\theta}s(\upsilon),^{H}D^{\varpi}z(\upsilon)\right)+\Re_{\psi}(\upsilon)\right]\frac{d\eta}{\eta} \\ &+\sum_{j=1}^{u}\frac{D_{5i}(\theta)(\ln\upsilon)^{\theta-2}}{\Gamma(\theta)}\int_{\upsilon_{j-1}}^{\upsilon_j}\ln\left(\frac{\upsilon_j}{\eta}\right)^{\theta-1}\left[\phi\left(\upsilon,^{H}D^{\theta}s(\upsilon),^{H}D^{\varpi}z(\upsilon)\right)+\Re_{\psi}(\upsilon)\right]\frac{d\eta}{\eta} \\ &+\sum_{j=1}^{u}\frac{\ln\upsilon^{3-\theta}(\log_{\upsilon_j}\upsilon)^{\theta-2}}{\Gamma(\theta-1)}\int_{\upsilon_{j-1}}^{\upsilon_j}\ln\left(\frac{\upsilon_i}{\eta}\right)^{\theta-2}\left[\phi\left(\upsilon,^{H}D^{\theta}s(\upsilon),^{H}D^{\varpi}z(\upsilon)\right)+\Re_{\psi}(\upsilon)\right]\frac{d\eta}{\eta} \\ &+\frac{D_3(\theta)(\ln\upsilon)^{\theta-2}}{\Gamma(\theta)}\int_{1}^{G}\ln\left(\frac{G}{\eta}\right)^{\theta-1}B(\eta,s(\eta))\frac{d\eta}{\eta} \\ &+\frac{1}{\Gamma(\theta)}\int_{\upsilon_u}^{\upsilon}\ln\left(\frac{\upsilon}{\eta}\right)^{\theta-1}\left[\phi\left(\upsilon,^{H}D^{\theta}s(\upsilon),^{H}D^{\varpi}z(\upsilon)\right)+\Re_{\psi}(\upsilon)\right]\frac{d\eta}{\eta}, \end{aligned} \tag{5.4}$$

for $u = 1, 2, ..., m$. Consider

$$\begin{aligned} &\left|(z(\upsilon)-\widehat{z}(\upsilon))(\ln\upsilon)^{2-\theta}\right| \\ \leq\ & G\left|D_0(\varpi)\right|\left|B^*(z)-B^*(\widehat{z})\right|+\sum_{i=1}^{u}\left|D_{1i}(\varpi)\right|\left|I_i(z_i)-I_i(\widehat{z_i})\right|+\sum_{i=1}^{u}\left|D_{2i}(\varpi)\right|\left|\widetilde{I_i}(z_i)-\widetilde{I_i}(\widehat{z_i})\right| \\ &+\frac{|D_0(\varpi)|}{\Gamma(\varpi-1)}\int_{\upsilon_u}^{G}\left|\ln\left(\frac{G}{\eta}\right)^{\varpi-2}\right|\left|\phi\left(\eta,^{H}D^{\varpi}z(\eta),^{H}D^{\theta}s(\eta)\right)-\phi\left(\eta,^{H}D^{\varpi}\widehat{z}(\eta),^{H}D^{\theta}\widehat{s}(\eta)\right)\right|\frac{d\eta}{\eta} \\ &+\frac{|D_4(\varpi)|}{\Gamma(\varpi)}\int_{\upsilon_u}^{G}\left|\ln\left(\frac{G}{\eta}\right)^{\varpi-1}\right|\left|\phi\left(\eta,^{H}D^{\varpi}z(\eta),^{H}D^{\theta}s(\eta)\right)-\phi\left(\eta,^{H}D^{\varpi}\widehat{z}(\eta),^{H}D^{\theta}\widehat{s}(\eta)\right)\right|\frac{d\eta}{\eta} \\ &+\sum_{i=1}^{u}\frac{|D_{5i}(\varpi|)}{\Gamma(\varpi)}\int_{\upsilon_{i-1}}^{\upsilon_i}\left|\ln\left(\frac{\upsilon_i}{\eta}\right)^{\varpi-1}\right|\left|\phi\left(\eta,^{H}D^{\varpi}z(\eta),^{H}D^{\theta}s(\eta)\right)-\phi\left(\eta,^{H}D^{\varpi}\widehat{z}(\eta),^{H}D^{\theta}\widehat{s}(\eta)\right)\right|\frac{d\eta}{\eta} \\ &+\sum_{i=1}^{u}\frac{\left|\ln\upsilon^{3-\varpi}\right|\left|(\ln\upsilon_i)^{\varpi-2}\right|}{\Gamma(\varpi-1)}\int_{\upsilon_{i-1}}^{\upsilon_i}\left|\ln\left(\frac{\upsilon_i}{\eta}\right)^{\varpi-2}\right| \\ &\times\left|\phi\left(\eta,^{H}D^{\varpi}z(\eta),^{H}D^{\theta}s(\eta)\right)-\phi\left(\eta,^{H}D^{\varpi}\widehat{z}(\eta),^{H}D^{\theta}\widehat{s}(\eta)\right)\right|\frac{d\eta}{\eta} \end{aligned}$$

$$+\frac{\left|(\ln \upsilon)^{2-\theta}\right|}{\Gamma(\varpi)}\int_{\upsilon_u}^{\upsilon}\left|\ln\left(\frac{\upsilon}{\eta}\right)^{\varpi-1}\right|\left|\phi\left(\eta,{}^{H}D^{\varpi}z(\eta),{}^{H}D^{\theta}s(\eta)\right)-\phi\left(\eta,{}^{H}D^{\varpi}\widehat{z}(\eta),{}^{H}D^{\theta}\widehat{s}(\eta)\right)\right|\frac{d\eta}{\eta}$$
$$+\frac{|D_3(\varpi)|}{\Gamma(\varpi)}\int_1^G\left|\ln\left(\frac{G}{\eta}\right)^{\varpi-1}\right|\left|B(\eta,z(\eta))-B(\eta,\widehat{z}(\eta))\right|\frac{d\eta}{\eta}$$
$$+\sum_{i=1}^{u}|D_{2i}(\varpi)|\left|\Re_{\phi_i}\right|+\sum_{i=1}^{u}|D_{1i}(\varpi)|\left|\Re_{\phi_i}\right|+\frac{|D_0(\varpi)|}{\Gamma(\varpi-1)}\int_{\upsilon_u}^{G}\left|\ln\left(\frac{G}{\eta}\right)^{\varpi-2}\right|\left|\Re_{\phi}(\upsilon)\right|\frac{d\eta}{\eta}$$
$$+\frac{|D_4(\varpi)|}{\Gamma(\varpi)}\int_{\upsilon_u}^{G}\left|\ln\left(\frac{G}{\eta}\right)^{\varpi-1}\right|\left|\Re_{\phi}(\upsilon)\right|\frac{d\eta}{\eta}+\sum_{i=1}^{u}\frac{|D_{5i}(\varpi)|}{\Gamma(\varpi)}\int_{\upsilon_{i-1}}^{\upsilon_i}\left|\ln\left(\frac{\upsilon_i}{\eta}\right)^{\varpi-1}\right|\left|\Re_{\phi}(\upsilon)\right|\frac{d\eta}{\eta}$$
$$+\sum_{i=1}^{u}\frac{\left|\ln\upsilon^{3-\varpi}\right|\left|(\ln\upsilon_i)^{\varpi-2}\right|}{\Gamma(\varpi-1)}\int_{\upsilon_{i-1}}^{\upsilon_i}\left|\ln\left(\frac{\upsilon_i}{\eta}\right)^{\varpi-2}\right|\left|\Re_{\phi}(\upsilon)\right|\frac{d\eta}{\eta}$$
$$+\frac{\left|(\ln \upsilon)^{2-\theta}\right|}{\Gamma(\varpi)}\int_{\upsilon_u}^{\upsilon}\left|\ln\left(\frac{\upsilon}{\eta}\right)^{\varpi-1}\right|\left|\Re_{\phi}(\upsilon)\right|\frac{d\eta}{\eta}.$$

As in Theorem 4.4, one has

$$\begin{aligned}\left\|z-\widehat{z}\right\|_{\mathfrak{I}_1} &\le \left(\mho_1+\frac{\mho_2 L_\phi}{\widetilde{L}_\phi\widetilde{L}_\psi-1}\right)(\ln\upsilon)^{2-\varpi}\left\|z-\widehat{z}\right\|_{\mathfrak{I}_1}+\frac{\mho_2\widetilde{L}_\phi L_\psi}{\widetilde{L}_\phi\widetilde{L}_\psi-1}(\ln\upsilon)^{2-\varpi}\left\|s-\widehat{s}\right\|_{\mathfrak{I}_1}\\ &\quad+(\mho_2+u|D_1(\varpi)|+u|D_2(\varpi)|)\varphi_\varpi,\end{aligned}\tag{5.5}$$

for $u=1,2,...,k$ and

$$\begin{aligned}\left\|s-\widehat{s}\right\|_{\mathfrak{I}_2} &\le \left(\frac{\mho_4 L_\phi\widetilde{L}_\psi}{\widetilde{L}_\phi\widetilde{L}_\psi-1}\right)(\ln\upsilon)^{2-\theta}\left\|z-\widehat{z}\right\|_{\mathfrak{I}_2}+\left(\mho_3+\frac{\mho_4 L_\psi}{\widetilde{L}_\phi\widetilde{L}_\psi-1}\right)(\ln\upsilon)^{2-\theta}\left\|s-\widehat{s}\right\|_{\mathfrak{I}_2}\\ &\quad+(\mho_4+u|D_1(\theta)|+u|D_2(\theta)|)\varphi_\theta.\end{aligned}\tag{5.6}$$

Arranging (5.5) and (5.6), we get

$$\left\|z-\widehat{z}\right\|_{\mathfrak{I}_1}-\frac{\mho_2\widetilde{L}_\phi L_\psi}{\left(\widetilde{L}_\phi\widetilde{L}_\psi-1\right)\left((\ln\upsilon)^{\varpi-2}-\mho_1\right)-\mho_2 L_\phi}\left\|s-\widehat{s}\right\|_{\mathfrak{I}_1}\le\frac{(\mho_2+u|D_1(\varpi)|+u|D_2(\varpi)|)}{1-\left(\mho_1+\frac{\mho_2 L_\phi}{\widetilde{L}_\phi\widetilde{L}_\psi-1}\right)(\ln\upsilon)^{2-\varpi}}\varphi_\varpi,\tag{5.7}$$

and

$$\left\|s-\widehat{s}\right\|_{\mathfrak{I}_2}-\frac{\mho_4 L_\phi\widetilde{L}_\psi}{\left(\widetilde{L}_\phi\widetilde{L}_\psi-1\right)\left((\ln\upsilon)^{\theta-2}-\mho_3\right)-\mho_4 L_\psi}\left\|z-\widehat{z}\right\|_{\mathfrak{I}_2}\le\frac{(\mho_4+u|D_1(\theta)|+u|D_2(\theta)|)}{1-\left(\mho_3+\frac{\mho_4 L_\psi}{\widetilde{L}_\phi\widetilde{L}_\psi-1}\right)(\ln\upsilon)^{2-\theta}}\varphi_\theta,\tag{5.8}$$

respectively. Assume that $\eth_\varpi=1-\left(\mho_1+\frac{\mho_2 L_\phi}{\widetilde{L}_\phi\widetilde{L}_\psi-1}\right)(\ln\upsilon)^{2-\varpi}$ and $\eth_\theta=1-\left(\mho_3+\frac{\mho_4 L_\psi}{\widetilde{L}_\phi\widetilde{L}_\psi-1}\right)(\ln\upsilon)^{2-\theta}$. Then (5.7) and (5.8) can be written as

$$\begin{bmatrix}1 & -\frac{\mho_2\widetilde{L}_\phi L_\psi}{\left(\widetilde{L}_\phi\widetilde{L}_\psi-1\right)\left((\ln\upsilon)^{\varpi-2}-\mho_1\right)-\mho_2 L_\phi}\\ -\frac{\mho_4 L_\phi\widetilde{L}_\psi}{\left(\widetilde{L}_\phi\widetilde{L}_\psi-1\right)\left((\ln\upsilon)^{\theta-2}-\mho_3\right)-\mho_4 L_\psi} & 1\end{bmatrix}\begin{bmatrix}\left\|z-\widehat{z}\right\|_{\mathfrak{I}_1}\\ \left\|s-\widehat{s}\right\|_{\mathfrak{I}_1}\end{bmatrix}\le\begin{bmatrix}\eth_\varpi\varphi_\varpi\\ \eth_\theta\varphi_\theta\end{bmatrix}.$$

Hence

$$\begin{bmatrix} \left\|z-\widehat{z}\right\|_{\mathfrak{I}_1} \\ \left\|s-\widehat{s}\right\|_{\mathfrak{I}_2} \end{bmatrix} \leq \begin{bmatrix} \frac{1}{\beth} & \frac{\mho_2 \widetilde{L}_\phi L_\psi}{(\widetilde{L}_\phi \widetilde{L}_\psi - 1)((\ln \upsilon)^{\varpi-2} - \mho_1) - \mho_2 L_\phi} \frac{1}{\beth} \\ \frac{\mho_4 L_\phi \widetilde{L}_\psi}{(\widetilde{L}_\phi \widetilde{L}_\psi - 1)((\ln \upsilon)^{\theta-2} - \mho_3) - \mho_4 L_\psi} \frac{1}{\beth} & \frac{1}{\beth} \end{bmatrix} \begin{bmatrix} \eth_\varpi \varphi_\varpi \\ \eth_\theta \varphi_\theta \end{bmatrix}, \tag{5.9}$$

where

$$\beth = 1 - \frac{L_\phi \widetilde{L}_\phi L_\psi \widetilde{L}_\psi \mho_2 \mho_4}{\left[\left(\widetilde{L}_\phi \widetilde{L}_\psi - 1\right)\left((\ln \upsilon)^{\varpi-2} - \mho_1\right) - \mho_2 L_\phi\right]\left[\left(\widetilde{L}_\phi \widetilde{L}_\psi - 1\right)\left((\ln \upsilon)^{\theta-2} - \mho_3\right) - \mho_4 L_\psi\right]} > 0.$$

From system (5.9), we observe that

$$\begin{aligned} \left\|z-\widehat{z}\right\|_{\mathfrak{I}_1} &= \frac{\eth_\varpi \varphi_\varpi}{\beth} + \frac{\mho_2 \widetilde{L}_\phi L_\psi \eth_\theta \varphi_\theta}{\left(\widetilde{L}_\phi \widetilde{L}_\psi - 1\right)\left((\ln \upsilon)^{\varpi-2} - \mho_1\right) - \mho_2 L_\phi} \frac{1}{\beth}, \\ \left\|s-\widehat{s}\right\|_{\mathfrak{I}_2} &= \frac{\mho_4 L_\phi \widetilde{L}_\psi \eth_\varpi \varphi_\varpi}{\left(\widetilde{L}_\phi \widetilde{L}_\psi - 1\right)\left((\ln \upsilon)^{\theta-2} - \mho_3\right) - \mho_4 L_\psi} \frac{1}{\beth} + \frac{\eth_\theta \varphi_\theta}{\beth}, \end{aligned}$$

which yields that

$$\begin{aligned} \left\|z-\widehat{z}\right\|_{\mathfrak{I}_1} + \left\|s-\widehat{s}\right\|_{\mathfrak{I}_2} \leq{}& \frac{\eth_\varpi \varphi_\varpi}{\beth} + \frac{\eth_\theta \varphi_\theta}{\beth} + \frac{\mho_2 \widetilde{L}_\phi L_\psi \eth_\theta \varphi_\theta}{\left(\widetilde{L}_\phi \widetilde{L}_\psi - 1\right)\left((\ln \upsilon)^{\varpi-2} - \mho_1\right) - \mho_2 L_\phi} \frac{1}{\beth} \\ &+ \frac{\mho_4 L_\phi \widetilde{L}_\psi \eth_\varpi \varphi_\varpi}{\left(\widetilde{L}_\phi \widetilde{L}_\psi - 1\right)\left((\ln \upsilon)^{\theta-2} - \mho_3\right) - \mho_4 L_\psi} \frac{1}{\beth}. \end{aligned}$$

Let us consider $\varphi = \max\{\varphi_\theta, \varphi_\varpi\}$ and

$$\begin{aligned} \eth_{\varpi,\theta} ={}& \frac{\eth_\varpi}{\beth} + \frac{\eth_\theta}{\beth} + \frac{\mho_2 \widetilde{L}_\phi L_\psi \eth_\theta}{\left(\widetilde{L}_\phi \widetilde{L}_\psi - 1\right)\left((\ln \upsilon)^{\varpi-2} - \mho_1\right) - \mho_2 L_\phi} \frac{1}{\beth} \\ &+ \frac{\mho_4 L_\phi \widetilde{L}_\psi \eth_\varpi}{\left(\widetilde{L}_\phi \widetilde{L}_\psi - 1\right)\left((\ln \upsilon)^{\theta-2} - \mho_3\right) - \mho_4 L_\psi} \frac{1}{\beth}. \end{aligned}$$

Then, we can write

$$\left\|(z,s) - (\widehat{z}, \widehat{s})\right\|_{\mathfrak{I}} \leq \eth_{\varpi,\theta} \varphi,$$

which leads to the supposed coupled problem (1.1) is HU stable. Further, if

$$\left\|(z,s) - (\widehat{z}, \widehat{s})\right\|_{\mathfrak{I}} \leq \eth_{\varpi,\theta} \Phi(\varphi),\ \Phi(0) = 0.$$

Then the suggested coupled problem (1.1) is GHU stable.

For the final result, we suppose the following assertion:

(A_7) There are nondecreasing functions $\daleth_\varpi, \daleth_\theta \in C(U, \mathbb{R}_+)$ so that

$$^{H}D^{\varpi} \daleth_\varpi(\upsilon) \leq L_\varpi \daleth_\varpi(\upsilon) \text{ and } ^{H}D^{\theta} \daleth_\theta(\upsilon) \leq L_\theta \daleth_\theta(\upsilon),\ \text{for } L_\varpi, L_\theta > 0.$$

Theorem 5.2. *If the assertions* (A_1)–(A_3) *and* (A_7) *and the condition (4.21) are fulfilled and*

$$\beth = 1 - \frac{L_\phi \widetilde{L}_\phi L_\psi \widetilde{L}_\psi \mho_2 \mho_4}{\left[\left(\widetilde{L}_\phi \widetilde{L}_\psi - 1\right)\left((\ln \upsilon)^{\varpi-2} - \mho_1\right) - \mho_2 L_\phi\right]\left[\left(\widetilde{L}_\phi \widetilde{L}_\psi - 1\right)\left((\ln \upsilon)^{\theta-2} - \mho_3\right) - \mho_4 L_\psi\right]} > 0,$$

then the unique solution of CII-FDEs (1.1) is HUR stable and consequently GHUR stable.

Proof. According to Definitions 3.3 and 3.4, we can get our conclusion by following the same procedures as in Theorem 5.1. □

6. Illustrative example

Example 6.1. Consider

$$\begin{cases} {}^H D^{\frac{6}{5}} z(\upsilon) = \frac{2 + {}^H D^{\frac{6}{5}} z(\upsilon) + {}^H D^{\frac{5}{4}} s(\upsilon)}{70 e^{20+\upsilon}\left(1 + {}^H D^{\frac{6}{5}} z(\upsilon) + {}^H D^{\frac{5}{4}} s(\upsilon)\right)}, \ \upsilon \neq 1.5, \\ {}^H D^{\frac{5}{4}} s(\upsilon) = \frac{1}{50}\left(\upsilon \cos z(\upsilon) - s(\upsilon)\sin(\upsilon)\right) + \frac{{}^H D^{\frac{6}{5}} z(\upsilon) + {}^H D^{\frac{5}{4}} s(\upsilon)}{25 + {}^H D^{\frac{6}{5}} z(\upsilon) + {}^H D^{\frac{5}{4}} s(\upsilon)}, \ \upsilon \neq 1.5, \\ \Delta z(1.5) = I_1 z(1.5) = \frac{|z(1.5)|}{2+|z(1.5)|}, \ \Delta z'(1.5) = \widetilde{I}_1 z(1.5) = \frac{|z(1.5)|}{25+|z(1.5)|}, \\ \Delta s(1.5) = I_1 s(1.5) = \frac{|s(1.5)|}{2+|s(1.5)|}, \ \Delta s'(1.5) = \widetilde{I}_1 s(1.5) = \frac{|s(1.5)|}{25+|s(1.5)|}, \ \upsilon_1 = 1.5, \\ z(e) = \frac{1}{\Gamma(\frac{6}{5})}\int_1^e \ln\left(\frac{e}{\eta}\right)^{\frac{1}{5}} \frac{\eta^2 + z(\eta)}{60}\frac{d\eta}{\eta}, \ z'(e) = \sum_{u=1}^{10} \frac{1}{B_u^*}\left|z(\zeta_u)\right|, \quad 1 < \zeta_u < 2B_u^*, \\ s(e) = \frac{1}{\Gamma(\frac{6}{5})}\int_1^e \ln\left(\frac{e}{\eta}\right)^{\frac{1}{5}} \frac{\eta^2 + s(\eta)}{60}\frac{d\eta}{\eta}, \ s'(e) = \sum_{u=1}^{10} \frac{1}{B_u^*}\left|s(\zeta_u)\right|, \quad 1 < \zeta_u < 2B_u^*, \end{cases} \tag{6.1}$$

where $\sum_{u=1}^{10} \frac{1}{B_u^*} < 0.5$ for $\upsilon \in [1, e]$. In view of problem (6.1), we observe that $\varpi = \frac{6}{5}$, $\theta = \frac{5}{4}$, $G = e$, $k = 1$ and $\upsilon_1 = 1.5$. Further, it's simple to locate $L_{B^*} = \widetilde{L}_{B^*} = 0.5$, $L_B = \widetilde{L}_B = \frac{1}{60}$, $L_I = L_{\widetilde{I}} = 0.5$, $\widetilde{L}_I = \widetilde{L}_{\widetilde{I}} = 0.04$, $L_\phi = \widetilde{L}_\phi = \frac{1}{70e^{20}}$ and $L_\psi = \widetilde{L}_\psi = 0.04$. Based on Theorem 4.4, we find that

$$\mho_1 + \mho_3 + \frac{\mho_2\left(L_\phi + \widetilde{L}_\phi L_\psi\right) + \mho_4\left(L_\phi \widetilde{L}_\psi + L_\psi\right)}{\widetilde{L}_\phi \widetilde{L}_\psi - 1} \simeq 0.537.$$

Therefore problem (6.1) has a unique solution. Further

$$\begin{aligned} \beth &= 1 - \frac{L_\phi \widetilde{L}_\phi L_\psi \widetilde{L}_\psi \mho_2 \mho_4}{\left[\left(\widetilde{L}_\phi \widetilde{L}_\psi - 1\right)\left((\ln \upsilon)^{\varpi-2} - \mho_1\right) - \mho_2 L_\phi\right]\left[\left(\widetilde{L}_\phi \widetilde{L}_\psi - 1\right)\left((\ln \upsilon)^{\theta-2} - \mho_3\right) - \mho_4 L_\psi\right]} \\ &= 0.023 > 0. \end{aligned}$$

Therefore, according to Theorem 5.1, the coupled system (6.1) is HU stable and consequently GHU stable. Similarly, we can confirm that Theorems 4.3 and 5.2 are true.

7. Conclusions

In this manuscript, we used fixed point results of Banach and Kransnoselskii to give necessary and sufficient conditions for the existence of a unique positive solution for a system of impulsive fractional differential equations intervening a fractional derivative of the Hadamard type. We also studied some Hyers-Ulam (HU) stabilities such as generalized Hyers-Ulam (GHU), Hyers-Ulam-Rassias (HUR), and generalized Hyers-Ulam-Rassias (GHUR) stabilities. At the end, we provided a concrete example making effective the obtained results.

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Conflict of interest

The authors declare that they have no competing interests.

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