Higher Order Evolution Equations and Dynamic Boundary Value Problems

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Ti-Jun Xiao
aus Sichuan, China

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1. Berichterstatter:	Professor Dr. R. Nagel
2. Berichterstatter:	Professor Dr. J. A. Goldstein

Zusammenfassung in deutscher Sprache

In dieser Arbeit untersuchen wir Cauchyprobleme höherer Ordnung der Form (ACP_n)

$$\begin{cases} u^{(n)}(t) + \sum\limits_{i=0}^{n-1} A_i u^{(i)}(t) = 0, \quad t \geq 0, \\ u^{(k)}(0) = u_k, \quad 0 \leq k \leq n-1, \end{cases}$$

wobei A_0, A_1, $\cdots$, A_{n-1} lineare Operatoren auf einem Banachraum X sind . Dazu führen wir in Kapitel 1 Operatoren, sogenannte Existenzfamilien, ein, die einen weiteren Banachraum Y in X abbilden. Damit erhalten wir eine grosse Flexibilität und können Existenz und stetige Abhängigkeit der Lösungen von (ACP_n) und seiner inhomogenen Version beweisen. Analog werden Eindeutigkeitsfamilien definiert zur Charakterisierung der Eindeutigkeit der Lösungen. Die Verbindung dieser beiden Konzepte gestattet die Verallgemeinerung aller bisher bekannten Resultate zur Lösung von (ACP_n).

In Kapitel 2 werden dann multiplikative und additive Störungsresultate vom Desch-Schappacher-Typ für (ACP_n) bewiesen und angewandt.

Im zweiten Teil der Arbeit untersuchen wir dynamische Randbedingungen für Cauchyprobleme erster und zweiter Ordnung. Dynamische Randbedingungen kommen in verschiedenen konkreten Problemen vor, zum Beispiel in Modellen von dynamischen Vibrationen von linearen viscoelastischen Stäben mit Spitze-Masse (tip masses) auf ihren bewegenden Enden. Die mathematische Untersuchung von Evolutionsgleichungen mit dynamische Randbedingungen geht auf 1961 zurück, als J. L. Lions solche Gleichungen behandelte und schwache Lösungen mit Hilfe von Variationsmethoden gab.

Kapitel 3 presentiert eine Lösung für ein Problem, das A. Favini, G. R. Goldstein, J. A. Goldstein and S. Romanelli [34] gestellt haben bezüglich des gemischten Problems für Wellengleichungen mit verallgemeinerten Wentzell Randbedingungen.

Im vierten Kapitel wird der zugehörige nichtautonome Fall betrachtet. Hier erhalten wir nicht nur Existenz- und Eindeutigkeitsresultate sondern auch präzise Aussagen zur Regularität der Lösungen.

Schliesslich enthalten Kapitel 5 und 6 eine einheitliche Behandlung gemischter

Probleme (Anfangs-Randwert Probleme) mit dynamischen Randbedingungen für parabolische und hyperbolische oder allgemeine Gleichungen zweiter Ordnung. Wir beschäftigen uns direkt mit Problemen zweiter Ordnung, ohne sie auf erste Ordnung zu reduzieren. Es stellt sich heraus, daß diese direkte Methoden starke Lösungen von erwünschter Regularität liefern und sogar allgemeine Theoreme ermöglichen. Eine Reihe von ganz neuen Resultaten werden bewiesen. Die Ergebnisse werden dann auf konkrete partielle Differentialgleichungen angewandt.

Contents

Introduction

Higher order evolution equations

A very interesting and important area of modern mathematical study are evolution equations. The reason for this stems from the fact that many problems in partial differential equations arising from mechanics, physics, engineering, control theory, etc., can be translated into the form of initial value or initial boundary value problems for evolution equations in appropriate infinite dimensional spaces.

A considerable effort has been devoted since the well-known Hille-Yosida theorem came out in 1948 for the investigation of the Cauchy problem for first order evolution equations

$$\begin{cases} u'(t) = Au(t), \quad t \geq 0, \\ u(0) = u_0, \end{cases} \tag{ACP_1}$$

(A being a linear operator in an infinite dimensional space) and related equations. The general theory and basic results for first order abstract Cauchy problems and operator semigroups are available in the monographs of Arendt, Batty, Hieber and Neubrander [5], Davies [15], deLaubenfels [19], Engel and Nagel [26], Fattorini [30, 31], Goldstein [38], Hille [41], Hille and Phillips [42], Lions and Magenes [60], Pazy [67], Reed and Simon [71], Xiao and Liang [84] and others.

On the other hand, since the pioneer work of Lions [57] in 1957, the Cauchy problem for higher order ($n \geq 2$) evolution equations

$$\begin{cases} u^{(n)}(t) + \sum\limits_{i=0}^{n-1} A_i u^{(i)}(t) = 0, \quad t \geq 0, \\ u^{(k)}(0) = u_k, \quad 0 \leq k \leq n-1, \end{cases} \tag{ACP_n}$$

where A_0, A_1, $\cdots$, A_{n-1} are linear operators in an infinite dimensional space, has been extensively explored (see, e.g., Engel and Nagel [26], Fattorini [31], Goldstein [38], Krein [51], Xiao and Liang [84]). However, this theory is far from being perfect, as compared with that of first order abstract Cauchy problems. Many interesting problems connected closely to (ACP_n) still remain open.

In Chapter 1, we introduce a new operator family of bounded linear operators from another Banach space Y to X, called an existence family for (ACP_n), to study the existence and continuous dependence on initial data of the solutions of (ACP_n)

and its inhomogeneous version ($IACP_n$), and obtain some basic results in a quite general setting. A sufficient and necessary condition ensuring (ACP_n) to possess an exponentially bounded existence family, in terms of Laplace transforms, is presented. As a partner of the existence family, we define, for (ACP_n), a uniqueness family of bounded linear operators on X to guarantee the uniqueness of the solutions. These two operator families are generalizations of strongly continuous semigroups and sine operator functions, C-regularized semigroups and sine operator functions, existence and uniqueness families for (ACP_1), and C-propagation families for (ACP_n). They have a special function in treating those illposed (ACP_n) and ($IACP_n$) whose coefficient operators lack commutativity.

Chapter 2 is intended to establish Desch-Schappacher type multiplicative and additive perturbation theorems for existence families for (ACP_n) (with $A_1 = \cdots = A_{n-1} = 0$). As a consequence, perturbation results for regularized semigroups and regularized cosine operator functions are obtained generalizing the previous ones. An example is also given to illustrate possible applications.

Dynamic boundary value problems

Dynamic boundary conditions occur in diverse practical problems, for instance, in those modelling the dynamic vibrations of linear viscoelastic rods and beams with tip masses attached at their free ends (see, e.g., [6]). The study of evolution equations with dynamic boundary conditions from the mathematical point of view dates back to 1961, when J. L. Lions [59, p. 117, 118] treated such equations and gave weak solutions by means of the variational method. Since then, this issue has been investigated to a large extent (see, e.g., [8, 9, 25, 27, 32–35, 37, 43, 50, 53, 59, 74] and references therein). I would like to mention that A. Favini, G. R. Goldstein, J. A. Goldstein and S. Romanelli have recently done a systematic study and established a series of very interesting and significant theorems for parabolic problems of first order in time with (generalized) Wentzell boundary conditions (see, e.g., [32–35] and references therein). Most recently, K. -J. Engel, R. Nagel *et al* made also very nice contributions to this field (see, e.g., [9, 25, 50]). While most of the previous research concerns the case of first order in time, there have been few results regarding the second order (in time) case, for which there seems to be a lack of general theory of wellposedness. In this dissertation, following an investigation of wave equations and heat equations in the space $C[0, 1]$ of continuous functions, we consider second order dynamic boundary value problems of both *parabolic* and *hyperbolic* type in the setting of general Banach spaces, and deal with them in a direct way without reduction to first order systems. One will see that the direct approach will yield strong solutions with desirable regularity, as well as build up theorems of a general nature.

Chapter 3 presents a solution to an open problem put forward by A. Favini, G. R. Goldstein, J. A. Goldstein and S. Romanelli [34], concerning the mixed problem for wave equations with generalized Wentzell boundary conditions.

The subsequent chapter concerns the nonautonomous heat equation with generalized Wentzell boundary conditions. It is shown, under appropriate assumptions, that there exists a unique evolution family for this problem and that the family satisfies various regularity properties. This enables us to obtain, for the corresponding inhomogeneous problem, classical and strict solutions having optimal regularity.

In Chapter 5, we exhibit a unified treatment of the mixed initial boundary value problem for second order (in time) parabolic linear differential equations in Banach spaces whose boundary conditions are of a dynamical nature. Results regarding existence, uniqueness, continuous dependence (on initial data) and regularity of classical and strict solutions are established. Moreover, two examples are given as samples for possible applications.

In the final Chapter 6, we continue to deal with the mixed initial boundary value problem for complete second order (in time) linear differential equations in Banach spaces, in which time-derivatives occur in the boundary conditions. General well-posedness theorems are obtained (for the first time) which are used to solve the corresponding inhomogeneous problems. Examples of applications to initial boundary value problems for partial differential equations are also presented.

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Chapter 1

Existence and uniqueness families for higher order abstract Cauchy problems

1.1 Summary

Of concern are the higher order abstract Cauchy problem (ACP_n) in a Banach space X and its inhomogeneous version $(IACP_n)$. We introduce a new operator family of bounded linear operators from another Banach space Y to X, called an existence family for (ACP_n), to study the existence and continuous dependence on initial data of the solutions of (ACP_n) and $(IACP_n)$, and obtain some basic results in a quite general setting. A sufficient and necessary condition ensuring (ACP_n) to possess an exponentially bounded existence family, in terms of Laplace transforms, is presented. As a partner of the existence family, we define, for (ACP_n), a uniqueness family of bounded linear operators on X to guarantee the uniqueness of the solutions. These two operator families are generalizations of the strongly continuous semigroups and sine operator functions, the C-regularized semigroups and sine operator functions, the existence and uniqueness families for (ACP_1), and the C-propagation families for (ACP_n). They have a special function in treating those illposed (ACP_n) and $(IACP_n)$ whose coefficient operators lack commutativity.

1.2 Introduction

Let $A_0, \ldots, A_{n-1}$ be linear operators on a Banach space X. Of concern are the abstract Cauchy problem for higher order linear differential equations

$$\begin{cases} u^{(n)}(t) + \sum_{i=0}^{n-1} A_i u^{(i)}(t) = 0, & t \geq 0, \\ u^{(k)}(0) = u_k, & 0 \leq k \leq n-1, \end{cases} \qquad (ACP_n)$$

and its inhomogeneous version $(IACP_n)$ in X (see the beginning of Section 3). As indicated in Fattorini [31, Preface, p. v - vi], Favini and Obrecht [36], Pazy [67, p. 253], and Xiao and Liang [82] and [84, Preface, p. vii - viii], there are many advantages to treat (ACP_n) directly instead to reduce it to first order systems in a suitable phase space and then use the theory of operator semigroups. Although it is usually hard to deal with (ACP_n) directly (cf., e.g., [31, 36, 82, 84]), we will obtain quite general results. As one will see, it seems to be impractical to deduce these results using the theory of first order systems.

Let $C \in \mathcal{L}(X)$ be injective. Based on Lions [58] and the paper [12], Da Prato [13] introduced C-regularized semigroups on X in 1966. Since Davies and Pang ([16]) rediscovered it in 1987, these semigroups have been investigated extensively (cf., e.g., [19, 21, 39, 44, 62, 78, 84]) and have been applied to deal with many ill-posed (in classical sense) abstract Cauchy problems for which strongly continuous semigroups are not applicable. Following these works about C regularized semigroups, we introduced in [84, Section 3.5] a strong C-propagation family $\{S_0, \ldots, S_{n-1}\}$ on X to govern (ACP_n) for both wellposed and illposed problems (see also [83] regarding (ACP_2)). Here C serves as a regularizing operator which is injective and commutes with each of coefficient operators $A_0, \ldots, A_{n-1}$ (when $C = I$, the operator family $S_0, \ldots, S_{n-1}$ controls the wellposed problems in the classical sense (cf. [29, 31] and [83, Chapter 2]). Here we are concerned with another important problem:

> *How to treat those (ACP_n) for which it is impossible or difficult to find a regularizing operator commuting with the coefficient operators?*

We will define an operator family $\{E(t)\}_{t\geq 0}$ of bounded linear operators from a Banach space Y (may be different from X) to X, called an existence family for (ACP_n) (Definition 1.3.1 (1)), as a new tool for handling (ACP_n). The family is associated with a regularizing operator $E_0 := E(0)$ which may not be injective and may not commute with A_i $(0 \le i \le n-1)$ even if $Y = X$. It will be shown that if (ACP_n) has an existence family $\{E(t)\}_{t\ge 0}$, then it admits a solution which can be represented explicitly by $\{E(t)\}_{t\ge 0}$ and depends continuously on initial data in some sense (Theorem 1.3.4 (1) and Theorem 1.3.7 (b)). We will also exhibit how an existence family of (ACP_n) gives the solutions of $(IACP_n)$ (Theorem 1.4.1). A sufficient and necessary condition ensuring (ACP_n) to possess an exponentially bounded existence family, in terms of Laplace transforms, will be presented (Theorem 1.3.7 (a)). Moreover we will define, as a companion of $\{E(t)\}_{t\ge 0}$, a uniqueness family $\{U(t)\}_{t\ge 0}$ (Definition 1.3.1 (2)) of bounded linear operators on X which guarantees the uniqueness of solutions to (ACP_n) (see Theorem 1.3.4 (2) and Theorem 1.3.9). An example will be given to show that the flexibility of Y, sometimes produces a larger set of initial data for which the solutions exist (Example 1.3.5). Also two concrete initial value problems for partial differential equations will be investigated as samples of possible applications (Examples 1.5.1 and 1.5.2).

We mention here that the study of the existence families and uniqueness families for (ACP_1), which are more general than the regularized semigroups as well as the classical strongly continuous semigroups (cf., e.g., [15, 18, 19, 26, 38, 67, 71]), was initiated by deLaubenfels ([18]). Moreover, deLaubenfels introduced in [17] the semi-closed (Y, X) semigroup where Y is continuously embedded in X. It will be seen that the existence families and uniqueness families for (ACP_n) given below are extensions of the operator families in [17, 18] as well as the classical sine operator functions, the C regularized sine operator functions, and the C-propagation families for (ACP_n). In addition, specializations of our theorems to the case $n = 1$ extend some related results in [17–20] (see Remarks 1.3.2, 1.3.6, 1.3.8 and 1.4.3).

In this chapter, X, Y are Banach spaces, $\mathcal{L}(Y,\ X)$ is the space of all bounded linear operators from Y into X, and $\mathcal{L}(X,\ X)$ is abbreviated to $\mathcal{L}(X)$.

For a linear operator A in X, $\mathcal{D}(A)$, $\mathcal{R}(A)$, and $\rho(A)$ stand for its domain, range, and resolvent set, respectively. By $[\mathcal{D}(A)]$ we mean the normed space $\mathcal{D}(A)$ with the graph norm

$$\|x\|_{[\mathcal{D}(A)]} := \|x\| + \|Ax\| \quad (x \in \mathcal{D}(A)).$$

When $C \in \mathcal{L}(Y,\ X)$, $[\mathcal{R}(C)]$ denotes the Banach space $\mathcal{R}(C)$ with the norm

$$\|x\|_{[\mathcal{R}(C)]} := \inf\{\|y\|;\ Cy = x\}.$$

By $C^i(R^+, X)$, $i \in N$, we denote the space of all i-times continuously differentiable

X-valued functions on $R^+ := [0, \infty)$, and by $C(R^+, X)$ the space of all continuous X-valued functions on R^+.

For a function $F : (\omega,\ \infty) \longrightarrow \mathcal{L}(Y,\ X)$, we write $F \in LT_\omega - \mathcal{L}(Y,\ X)$ (or $LT - \mathcal{L}(Y,\ X)$) to mean that there exists a strongly continuous mapping $H : [0,\ \infty) \longrightarrow \mathcal{L}(Y,\ X)$ satisfying

$$\|H\| \leq Me^{\omega t} \quad (t \geq 0) \quad \text{for some constants } M > 0 \text{ and } \omega \in R$$

(with this estimate it is called an $O(e^{\omega t})$ mapping) such that

$$F(\lambda)y = \int_0^\infty e^{-\lambda t} H(t) y dt \quad (y \in Y), \quad \text{for } \lambda > \omega.$$

We refer the reader to [4, 5] and [84, Section 1.2] for the Widder-type theorems which characterize the space $LT_\omega - \mathcal{L}(Y,\ X)$.

Definition 1.2.1. By a solution of (ACP_n), we mean a function $u(\cdot) \in C^n(R^+, X)$ such that $u^{(i)}(t) \in \mathcal{D}(A_i)$ $(t \geq 0,\ 0 \leq i \leq n-1)$, $A_i u^{(i)}(\cdot) \in C(R^+, X)$, and (ACP_n) is satisfied.

We write, for $\lambda \in \mathbf{C}$,

$$P_\lambda := \lambda^n + \sum_{i=0}^{n-1} \lambda^i A_i \quad \text{and} \quad R_\lambda := P_\lambda^{-1}$$

if the inverse exists.

1.3 Existence and uniqueness families for (ACP_n) and wellposedness of (ACP_n)

Let $E_0 \in \mathcal{L}(Y, X)$, and let $U_0 \in \mathcal{L}(X)$ be injective. We first give the definitions of the existence and uniqueness families for (ACP_n).

Definition 1.3.1. (1) A strongly continuous family of operators $\{E(t)\}_{t\geq 0} \subset \mathcal{L}(Y, X)$ is called an E_0-*existence family* for (ACP_n) if $E(\cdot)y \in C^{n-1}(R^+, X)$,

$E^{(i-1)}(t)y \in \mathcal{D}(A_i)$, $A_iE^{(i-1)}(\cdot)y \in C(R^+, X)$ (for all $y \in Y$, $t \geq 0$, $0 \leq i \leq n-1$), and

$$E(t)y + \sum_{i=0}^{n-1} A_i \int_0^t \frac{(t-s)^{n-i-1}}{(n-i-1)!} E(s)yds = \frac{t^{n-1}}{(n-1)!} E_0 y. \tag{1.3.1}$$

Here and in the sequel, for $s \geq 0$, $y \in Y$, we write

$$E^{(j)}(s)y := \frac{d^j}{ds^j}\left(E(s)y\right), \quad j \in N \cup \{0\},$$

$$E^{(-j)}(s)y := \int_0^t \frac{(t-s)^{j-1}}{(j-1)!} E(s)yds, \quad j \in N. \tag{1.3.2}$$

We also say that (ACP_n) *has an E_0-existence family* $\{E(t)\}_{t\geq 0}$.

(2) A strongly continuous family of operators $\{U(t)\}_{t\geq 0} \subset \mathcal{L}(X)$ is called a U_0-*uniqueness family* for (ACP_n) if for all $x \in \cap_{i=0}^{n-1}\mathcal{D}(A_i)$, $t \geq 0$,

$$U(t)x + \sum_{i=0}^{n-1} \int_0^t \frac{(t-s)^{n-i-1}}{(n-i-1)!} U(s)A_i x ds = \frac{t^{n-1}}{(n-1)!} U_0 x. \tag{1.3.3}$$

We also say that (ACP_n) *has a U_0-uniqueness family* $\{U(t)\}_{t\geq 0}$.

Remark 1.3.2. When $n = 1$ and $Y = X$, the E_0-existence family $\{E(t)\}_{t\geq 0}$ in Definition 1.3.1 is just the mild E_0-existence family for $-A_0$ in [19, p. 8] denoted by $\{W(t)\}_{t\geq 0}$ there (see also [18, 20]). Let $n = 1$, Y be continuously embedded in X, and $E_0 = I$. Then $\{E(t)\}_{t\geq 0}$ reduces to the semi-closed (Y, X)-semigroup introduced by deLaubenfels [17].

It is known from [18–20] that for each $u_0 \in \mathcal{R}(E_0)$, $u(\cdot) := E(\cdot)v_0$ ($E_0 v_0 = u_0$) is a mild solution of (ACP_1), i.e.,

$$u(t) + A_0 \int_0^t u(s)ds = u_0, \quad t \geq 0.$$

Moreover, for all $u_0 \in E_0(\mathcal{D}(A_0))$, $u(\cdot)$ is a (strict) solution of (ACP_1) provided A_0 is closed and

$$E(\cdot)x \in C(R^+, [\mathcal{D}(A_0)]), \quad x \in \mathcal{D}(A_0). \tag{1.3.4}$$

Condition (1.3.4) is automatically satisfied when $E(\cdot)$ is a C-regularized semigroup for $-A_0$ (which implies that $E(t)A_0 \subset A_0E(t)$ for all $t \geq 0$). We will show (Theorem

1.3.4 (1) for $n = 1$) that, without condition (1.3.4), $\{E(t)\}_{t\geq 0}$ also yields (strict) solutions of (ACP_1) for a set $\mathbf{D}_0$ of initial data (see Definition 1.3.3 below) which is larger than $E_0(\mathcal{D}(A_0))$ in many cases (cf. Example 1.3.5 and Remark 1.3.6).

When $Y = X$ and E_0 commutes with A_i $(0 \leq i \leq n-1)$, $E(\cdot)$ becomes, under some conditions, the $S_{n-1}(\cdot)$ of the strong E_0-propagation family $\{S_0, \ldots, S_{n-1}\}$ for (ACP_n) in ([84, p. 115]) where S_0, …, S_{n-2} are determined by S_{n-1} (cf. [84, p. 116]). Moreover, if $n = 2$ and $A_1 = 0$, then $S_{n-1}(= S_1)$ is a E_0-regularized sine operator function.

For $n = 1$, $\{U(t)\}_{t\geq 0}$ in Definition 1.3.1 coincides with the uniqueness family in [20].

Next, we define a class of sets, which will be used as spaces of initial data for solutions of (ACP_n).

Definition 1.3.3. For $0 \leq k \leq n-1$,

$$\mathbf{D}_k := \left\{ x \in \bigcap_{j=0}^{k} \mathcal{D}(A_j);\ A_j x \in \mathcal{R}(E_0) \ \text{ for all } \ 0 \leq j \leq k \right\}.$$

Theorem 1.3.4. (1) *Assume that A_i $(0 \leq i \leq n-1)$ are closed. If there is an E_0-existence family $\{E(t)\}_{t\geq 0}$ for (ACP_n), then for $u_0 \in \mathbf{D}_0$, …, $u_{n-1} \in \mathbf{D}_{n-1}$, (ACP_n) admits a solution given by*

$$u(t) := \sum_{i=0}^{n-1} \left[\frac{t^i}{i!} u_i - \sum_{j=0}^{i} \int_0^t \frac{(t-s)^{i-j}}{(i-j)!} E(s) v_{i,j} ds \right], \quad t \geq 0, \tag{1.3.5}$$

where $v_{i,j} \in Y$ such that

$$A_j u_i = E_0 v_{i,j}, \quad 0 \leq j \leq i,\ 0 \leq i \leq n-1. \tag{1.3.6}$$

The solution satisfies

$$\|u^{(n)}(t)\|, \|u^{(k)}(t)\|_{[\mathcal{D}(A_k)]} \ \leq \ M(t) \sum_{i=0}^{n-1} \left(\|u_i\| + \sum_{j=0}^{i} \|A_j u_i\|_{[\mathcal{R}(E_0)]} \right), \quad t \geq 0,\ 0 \leq k \leq n-1, \tag{1.3.7}$$

for some locally bounded positive function $M(\cdot)$ on R^+.

(2) *If there is a U_0-uniqueness family $\{U(t)\}_{t\geq 0}$ for (ACP_n), then all solutions of (ACP_n) are unique.*

Proof. (1) Let $u_0 \in \mathbf{D}_0$, ..., $u_{n-1} \in \mathbf{D}_{n-1}$, and let $v_{i,j}$ be as in (1.3.6). We claim that $u(\cdot)$ given by (1.3.5) is a solution of (ACP_n). In fact, noting (1.3.2) we have

$$\begin{aligned} u^{(n)}(t) &= -\sum_{i=0}^{n-1}\sum_{j=0}^{i} E^{(n-i+j-1)}(t)v_{i,j}, \\ u^{(l)}(t) &= \sum_{i=l}^{n-1} \frac{t^{i-l}}{(i-l)!}u_i - \sum_{i=0}^{n-1}\sum_{j=0}^{i} E^{(l-i+j-1)}(t)v_{i,j}, \quad 0 \leq l \leq n-1. \end{aligned}$$

Furthermore, from (1.3.1) and the closedness of A_i $(0 \leq i \leq n-1)$ we obtain for all $t \geq 0$, $y \in Y$,

$$\begin{cases} E^{(n-k-1)}(t)y + \displaystyle\sum_{l=0}^{n-1} A_l E^{(l-k-1)}(t)y = \frac{t^k}{k!}E_0 y, \quad 0 \leq k \leq n-1, \\ E^{(j)}(0)y = 0, \quad 0 \leq j \leq n-2. \end{cases} \tag{1.3.8}$$

Accordingly, we deduce that

$$u^{(j)}(0) = u_j, \quad 0 \leq j \leq n-1,$$

and that

$$\begin{aligned} &u^{(n)}(t) + \sum_{l=0}^{n-1} A_l u^{(l)}(t) \\ = &\sum_{l=0}^{n-1}\sum_{i=l}^{n-1} \frac{t^{i-l}}{(i-l)!}A_l u_i - \sum_{i=0}^{n-1}\sum_{j=0}^{i} E^{(n-i+j-1)}(t)v_{i,j} \\ &\qquad -\sum_{l=0}^{n-1}\sum_{i=0}^{n-1}\sum_{j=0}^{i} A_l E^{(l-i+j-1)}(t)v_{i,j} \\ = &\sum_{i=0}^{n-1}\left\{\sum_{l=0}^{i} \frac{t^{i-l}}{(i-l)!}A_l u_i - \sum_{j=0}^{i}\left[E^{(n-i+j-1)}(t)v_{i,j} + \sum_{l=0}^{n-1} A_l E^{(l-i+j-1)}(t)v_{i,j}\right]\right\} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{n-1}\left\{\sum_{l=0}^{i}\frac{t^{i-l}}{(i-l)!}A_l u_i - \sum_{j=0}^{i}\frac{t^{i-j}}{(i-j)!}E_0 v_{i,j}\right\} \\
&= 0
\end{aligned}
$$

by (1.3.6). Observe that $v_{i,j}$ in (1.3.6) is arbitrary, that for any $0 \le j \le i$, $0 \le i \le n-1$,

$$\inf\{\|v_{i,j}\|;\ A_j u_i = E_0 v_{i,j}\} = \|A_j u_i\|_{[\mathcal{R}(E_0)]},$$

$$\|A_j u_i\| \le \|E_0\|\|A_j u_i\|_{[\mathcal{R}(E_0)]},$$

and that each of $\|E^{(n-1)}(t)\|$ and $\|A_i E^{(i-1)}(t)\|$ $(0 \le i \le n-1)$ is locally bounded on R^+ by the Banach-Steinhaus theorem. So (1.3.7) follows from (1.3.5) immediately.

(2) For every $x \in \cap_{i=0}^{n-1}\mathcal{D}(A_i)$ and $t \ge 0$, we define

$$
\begin{aligned}
V_0(t)x &= U_0 x - \int_0^t U(s)A_0 x ds, \\
V_j(t)x &= \int_0^t (V_{j-1}(s)x - U(s)A_j x)\, ds, \quad 1 \le j \le n-1, \qquad \text{if } n \ge 2.
\end{aligned}
$$

Thus for x and t as above,

$$U(t)x = V_{n-1}(t)x \tag{1.3.9}$$

by (1.3.3). Let now $\widetilde{u}(\cdot)$ be a solution of (ACP_n) with $\widetilde{u}^{(j)}(0) = 0$, $0 \le j \le n-1$. Clearly, $\widetilde{u}(t) \in \cap_{i=0}^{n-1}\mathcal{D}(A_i)$ for $t \ge 0$ and $A_i\widetilde{u}(\cdot) \in C(R^+, X)$, $0 \le i \le n-1$. For $t \ge 0$, put

$$w(t) = \begin{cases} \widetilde{u}(t) & \text{if } \ n = 1, \\ \displaystyle\int_0^t \frac{(t-s)^{n-2}}{(n-2)!}\widetilde{u}(s)ds & \text{if } \ n \ge 2. \end{cases}$$

It is easy to see that $w(t)$ is also a solution of (ACP_n) with $w^{(j)}(0) = 0$, $0 \le j \le n-1$, and furthermore for all $0 \le j \le n-1$,

$$w^{(j)}(t) \in \bigcap_{i=0}^{n-1}\mathcal{D}(A_i), \quad t \ge 0.$$

According to this and (1.3.9), we have for $t \geq s \geq 0$,

$$\begin{aligned}
\frac{d}{ds}\left(\sum_{j=0}^{n-1} V_j(t-s)w^{(j)}(s)\right) &= \sum_{j=0}^{n-1} V_j(t-s)w^{(j+1)}(s) + \sum_{j=0}^{n-1} U(t-s)A_j w^{(j)}(s) \\
&\quad - \begin{cases} 0 & \text{if} \quad n=1 \\ \sum_{j=1}^{n-1} V_{j-1}(t-s)w^{(j)}(s) & \text{if} \quad n \geq 2 \end{cases} \\
&= 0.
\end{aligned}$$

Therefore

$$U_0 w(t) = V_0(0)w(t) = \sum_{j=0}^{n-1} V_j(t)w^{(j)}(0) = 0, \quad t \geq 0,$$

which implies that $w(t) \equiv 0$ by the injectivity of U_0. So $\widetilde{u}(t) \equiv 0$. This ends the proof.

□

The following is an example indicating that the choice of a Banach space Y different from X produces a larger set of initial data.

Example 1.3.5. Look at the (ACP_1) in the space $C_0(R)$, where A_0 is the linear operator defined by

$$(A_0 f)(x) := xf(x) \quad (x \in R)$$

for all

$$f \in \mathcal{D}(A_0) := \{f \in C_0(R);\ xf(x) \in C_0(R)\}.$$

It is easy to verify that this (ACP_1) has an E_0-existence family $\{E(t)\}_{t\geq 0}$ of operators in $\mathcal{L}(C_b(R), C_0(R))$, as well as a U_0-uniqueness family $\{U(t)\}_{t\geq 0}$ of operators in $\mathcal{L}(C_0(R))$. Here for $x \in R$, $t \geq 0$,

$$\begin{cases} (E(t)f)(x) := e^{-x^2} e^{-tx} f(x), \quad f \in C_b(R), \\ E_0 = E(0), \end{cases}$$

$$U(t) := E(t)\Big|_{C_0(R)}, \quad U_0 = U(0). \tag{1.3.10}$$

Therefore, by virtue of Theorem 1.3.4 we know that this (ACP_1) admits a unique solution whenever the initial value is in

$$\mathbf{D}_0 = \{f \in C_0(R);\ xe^{x^2}f(x) \text{ is bounded on } R\}.$$

On the other hand, the U_0-uniqueness family $\{U(t)\}_{t\geq 0}$ in (1.3.10) is also a U_0-existence family, which is in $\mathcal{L}(C_0(R))$, for (ACP_1). This then yields that (ACP_1) admits a unique solution for every initial data in $U_0(\mathcal{D}(A_0))$. But clearly,

$$\begin{aligned} & U_0(\mathcal{D}(A_0)) \\ = \ & \{x \in \mathcal{D}(A_0);\ A_0 x \in \mathcal{R}(U_0)\} \\ = \ & \{f \in C_0(R);\ xe^{x^2}f(x) \in C_0(R)\} \end{aligned}$$

is smaller than $\mathbf{D}_0$.

Remark 1.3.6. Even in the case of $X = Y$ (whereby $E_0\mathcal{D}(A_0)$ makes sense) and $E_0A_0 \subset A_0E_0$, it is also possible that $\mathbf{D}_0$ is larger than $E_0\mathcal{D}(A_0)$. Actually, for this case, it is not difficult to see that

$$E_0\mathcal{D}(A_0) \subset \mathbf{D}_0 \cap \mathcal{R}(E_0).$$

The opposite inclusion holds true if and only if $\{x;\ A_0E_0x \in \mathcal{R}(E_0)\} \subset \mathcal{D}(A_0)$. Moreover, if A_0 is a one to one mapping of $\mathcal{D}(A_0)$ onto X, then

$$E_0\mathcal{D}(A_0) = \mathbf{D}_0. \tag{1.3.11}$$

However, the following two counterexamples indicate that the equality (1.3.11) may fail if A_0 is not injective or not surjective.

(a) Let X be an infinite-dimensional Banach space and A_0 a linear operator in X which is surjective but not injective. Let $E_0 \equiv 0$ in X. Then

$$E_0A_0 \subset A_0E_0, \quad E_0\mathcal{D}(A_0) = \{0\},$$

$$\mathbf{D}_0 = \{x \in X;\ A_0x = 0\} \neq \{0\}.$$

(b) Let $X = l^2$ and let A_0, E_0 be the operators defined by

$$A_0\{u_m\} = \{\widetilde{u}_m\} \quad \text{for all } \{u_m\} \in X,$$

$$E_0\{u_m\} = \{\overline{u}_m\} \quad \text{for all } \{u_m\} \in X,$$

where $\widetilde{u}_1 = u_1$, $\overline{u}_1 = 0$ and

$$\widetilde{u}_m = \overline{u}_m = \begin{cases} 0 & \text{if } m \text{ is even} \\ u_{i+1} & \text{if } m = 2i+1 \ (i = 1, 2, \cdots). \end{cases}$$

Clearly $A_0,\ E_0 \in \mathcal{L}(X)$, A_0 is injective but not surjective, $E_0A_0 = A_0E_0$, and

$$\mathbf{D}_0 = \{\{v_m\} \in X; \quad v_1 = 0\},$$

$$E_0\mathcal{D}(A_0) = \{\{v_m\} \in X; \quad v_1 = v_m = 0 \text{ for any even } m\}.$$

Next, we present a sufficient and necessary condition ensuring (ACP_n) to possess an $O(e^{\omega t})$ E_0-existence families in terms of Laplace transforms.

Theorem 1.3.7. *Suppose that A_i $(0 \le i \le n-1)$ are closed and P_λ is injective for $\lambda > \omega$. Then the following holds.*

(a) *(ACP_n) has an E_0-existence family $\{E(t)\}_{t\ge 0} \subset \mathcal{L}(Y, X)$ satisfying*

$$\left\|E^{(n-1)}(t)\right\|,\ \left\|A_iE^{(i-1)}(t)\right\| \le Me^{\omega t} \quad (0 \le i \le n-1,\ t \ge 0) \tag{1.3.12}$$

if and only if $\mathcal{R}(E_0) \subset \mathcal{R}(P_\lambda)$ (for $\lambda > \omega$) and

$$\lambda^{n-1}R_\lambda E_0,\ \lambda^{i-1}A_iR_\lambda E_0 \in LT_\omega - \mathcal{L}(Y, X)\ (1 \le i \le n-1). \tag{1.3.13}$$

In this case, $M(t)$ in (1.3.7) can be taken as $Me^{\omega t}$.

(b) *Let (1.3.13) hold. In the case of $n \ge 2$ assume, in addition, that $\mathcal{R}(A_iE_0) \subset \mathcal{R}(P_\lambda)$ $(1 \le i \le n-1,\ \lambda > \omega)$ and*

$$\lambda^{i-1}R_\lambda A_iE_0y = \int_0^\infty e^{-\lambda t}S_i(t)y dt, \quad y \in \mathcal{D}(A_iE_0),\ \lambda > \omega, \tag{1.3.14}$$

for some strongly continuous family of operators $\{S_i(t)\}_{t\ge 0} \subset \mathcal{L}(Y, X)$ with $\|S_i(t)\| \le Me^{\omega t}$ $(t \ge 0)$. Then for $u_0 \in \mathbf{D}_0 \cap \mathcal{R}(E_0), \ldots, u_{n-1} \in \mathbf{D}_{n-1} \cap \mathcal{R}(E_0)$, (ACP_n) admits a solution $u(t)$ satisfying

$$\|u(t)\| \le Me^{\omega t} \sum_{i=0}^{n-1} \|u_i\|_{[\mathcal{R}(E_0)]}, \quad t \ge 0. \tag{1.3.15}$$

Proof. (a) The "only if" part. Taking Laplace transforms in (1.3.1), we obtain from the closedness of A_i $(0 \le i \le n-1)$,

$$\left(I + \sum_{i=0}^{n-1} \lambda^{i-n} A_i \right) \int_0^\infty e^{-\lambda t} E(t) y dt = \lambda^{-n} E_0 y, \quad y \in Y, \ \lambda > \omega.$$

Hence $\mathcal{R}(E_0) \subset \mathcal{R}(P_\lambda)$ for $\lambda > \omega$ and

$$R_\lambda E_0 y = \int_0^\infty e^{-\lambda t} E(t) y dt, \quad y \in Y, \ \lambda > \omega.$$

This in conjunction with (1.3.12) gives that for $y \in Y$, $\lambda > \omega$,

$$\lambda^i R_\lambda E_0 y = \int_0^\infty e^{-\lambda t} E^{(i)}(t) y dt, \quad 0 \le i \le n-1, \tag{1.3.16}$$

$$\lambda^{i-1} A_i R_\lambda E_0 y = \int_0^\infty e^{-\lambda t} A_i E^{(i-1)}(t) y dt, \quad 0 \le i \le n-1.$$

Hence (1.3.13) is satisfied.

The "if" part. By hypothesis we have $R_\lambda E_0 \in LT_\omega - \mathcal{L}(Y, X)$. So there exists a strongly continuous family of operators $\{\widetilde{E}(t)\}_{t \ge 0} \subset \mathcal{L}(Y, X)$ with $\|\widetilde{E}(t)\| \le M e^{\omega t}$ $(t \ge 0)$ such that

$$R_\lambda E_0 y = \int_0^\infty e^{-\lambda t} \widetilde{E}(t) y dt, \quad y \in Y, \ \lambda > \omega. \tag{1.3.17}$$

In view of Theorem 1.1.9 of [84], (1.3.17) combined with the assumption $\lambda^{n-1} R_\lambda E_0 \in LT_\omega - \mathcal{L}(Y, X)$ indicates that

$$\widetilde{E}(\cdot) y \in C^{n-1}(R^+, X) \quad \text{and} \quad \left\| \widetilde{E}^{(n-1)}(t) \right\| \le M e^{\omega t} \ \ (\text{for all} \ \ y \in Y, \ t \ge 0).$$

On the other hand, we observe by (1.3.13) that

$$\lambda^{i-1} A_i R_\lambda E_0 \in LT_\omega - \mathcal{L}(Y, X)$$

for all $0 \le i \le n-1$ since

$$\lambda^{-1} A_0 R_\lambda E_0 = \lambda^{-1} E_0 - \lambda^{n-1} R_\lambda E_0 - \sum_{i=1}^{n-1} \lambda^{i-1} A_i R_\lambda E_0.$$

This gives, by Theorem 1.1.10 of [84], that for any $y \in Y$, $t \ge 0$, $0 \le i \le n-1$,

$$\widetilde{E}^{(i-1)}(t) y \in \mathcal{D}(A_i) \quad \text{and} \quad \left\| A_i \widetilde{E}^{(i-1)}(t) \right\| \le M e^{\omega t}.$$

Here for $t \geq 0$, $y \in Y$,

$$\widetilde{E}^{(i-1)}(t)y := \begin{cases} \dfrac{d^{i-1}}{dt^{i-1}}\left(\widetilde{E}(t)y\right) & \text{if} \quad 1 \leq i \leq n, \\ \displaystyle\int_0^t \widetilde{E}(s)yds & \text{if} \quad i = 0. \end{cases}$$

Consequently, we obtain for every $y \in Y$,

$$\begin{aligned} & \int_0^\infty e^{-\lambda t}\frac{t^{n-1}}{(n-1)!}E_0ydt \\ = \ & \lambda^{-n}E_0y \\ = \ & R_\lambda E_0 y + \sum_{i=0}^{n-1}\lambda^{i-n}A_iR_\lambda E_0y \\ = \ & \int_0^\infty e^{-\lambda t}\left(\widetilde{E}(t)y + \sum_{i=0}^{n-1}A_i\int_0^t \frac{(t-s)^{n-i-1}}{(n-i-1)!}\widetilde{E}(s)yds\right)dt, \quad \lambda > \omega. \end{aligned}$$

Thus an application of the uniqueness theorem for Laplace transforms yields the desired result.

(b) Let $u_0 \in \mathbf{D}_0 \cap \mathcal{R}(E_0)$, $\ldots$, $u_{n-1} \in \mathbf{D}_{n-1} \cap \mathcal{R}(E_0)$. By Theorem 1.3.4 (1), (ACP_n) admits a solution $u(t)$ given by (1.3.5). Using (1.3.16) and (1.3.6), we have for $\lambda > \omega$,

$$\begin{aligned} \int_0^\infty e^{-\lambda t}u(t)dt &= \sum_{i=0}^{n-1}\left(\lambda^{-i-1}u_i - \sum_{j=0}^{i}\lambda^{j-i-1}R_\lambda E_0 v_{i,j}\right) \\ &= \sum_{i=0}^{n-1}\lambda^{-i-1}\left(\lambda^n R_\lambda + \sum_{j=0}^{n-1}\lambda^j R_\lambda A_j - \sum_{j=0}^{i}\lambda^j R_\lambda A_j\right)u_i \\ &= \sum_{i=0}^{n-1}\left(\lambda^{n-i-1}R_\lambda u_i + \begin{cases} 0 & \text{if} \quad n = 1 \\ \displaystyle\sum_{j=i+1}^{n-1}\lambda^{j-i-1}R_\lambda A_j u_i & \text{if} \quad n \geq 2 \end{cases}\right). \end{aligned} \tag{1.3.18}$$

Take $w_i \in Y$ such that $E_0 w_i = u_i$ $(0 \leq i \leq n-1)$. We thus obtain from (1.3.14), (1.3.16) and (1.3.18),

$$\int_0^\infty e^{-\lambda t}u(t)dt = \int_0^\infty e^{-\lambda t}\left(E^{(n-1)}(t)w_0 + w_1(t) + w_2(t)\right)dt, \quad \lambda > \omega,$$

where

$$w_1(t) := \begin{cases} 0 & \text{if} \quad n = 1, \\ \sum_{j=1}^{n-1} S_j(t) w_0 & \text{if} \quad n \geq 2, \end{cases}$$

$$w_2(t) := \begin{cases} 0 & \text{if} \quad n = 1, \\ E(t) w_1 & \text{if} \quad n = 2, \\ \sum_{i=1}^{n-1} E^{(n-i-1)}(t) w_i + \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \int_0^t \frac{(t-s)^{i-1}}{(i-1)!} S_j(s) w_i ds & \text{if} \quad n \geq 3. \end{cases}$$

It follows that

$$u(t) = E^{(n-1)}(t) w_0 + w_1(t) + w_2(t), \quad t \geq 0,$$

according to the uniqueness theorem for Laplace transforms. Now, (1.3.15) follows by the arbitrariness of w_i for each $0 \leq i \leq n-1$. The proof is then complete.

□

Remark 1.3.8. Let $n = 1$ and $Y = X$. Then Theorem 1.3.7 (a) reduces to the result in [20, p. 1489] where [4, Theorem 1] was used. If $n = 1$, E_0 is injective and there exists $\mu_0 \in \rho(A_0)$ such that

$$(\mu_0 - A_0)^{-1} \mathcal{R}(E_0) \subset \mathcal{R}(E_0),$$

then Theorem 1.3.7 (b) is Theorem 12 in [17].

Theorem 1.3.9. *Suppose that $\{U(t)\}_{t \geq 0}$ is an exponentially bounded strongly continuous family of operators in $\mathcal{L}(X)$. Then $\{U(t)\}_{t \geq 0}$ is a U_0-uniqueness family for (ACP_n) if and only if*

$$U_0 x = \int_0^\infty e^{-\lambda t} U(t) P_\lambda x dt, \quad \text{for } \; x \in \cap_{i=0}^{n-1} \mathcal{D}(A_i), \;\; \lambda > \omega. \tag{1.3.19}$$

Proof. The "only if" part. Taking Laplace transforms in (1.3.2), we obtain for $x \in \cap_{i=0}^{n-1} \mathcal{D}(A_i)$, $\lambda > \omega$,

$$\int_0^\infty e^{-\lambda t} U(t) \left(x + \sum_{i=0}^{n-1} \lambda^{i-n} A_i x \right) dt = \lambda^{-n} U_0 x,$$

and so (1.3.19) follows.

The "if" part. Reverse the process in the "only if" part and make use of the uniqueness theorem for Laplace transforms.

□

1.4 Inhomogeneous Cauchy problems

In this section, we focus on investigating the inhomogeneous version of ACP_n).

Let $T > 0$. Assume that $A_0, \ldots, A_{n-1}$ are closed linear operators in X, $E_0 \in \mathcal{L}(Y, X)$, and that $f \in C([0,T], X)$, $g \in C([0,T], Y)$ such that

$$E_0 g(t) = f(t), \quad t \in [0,T].$$

We consider

$$\begin{cases} v^{(n)}(t) + \sum_{i=0}^{n-1} A_i v^{(i)}(t) = f(t), & t \in [0,T], \\ v^{(k)}(0) = u_k, & 0 \le k \le n-1. \end{cases} \qquad (IACP_n)$$

By a solution of $(IACP_n)$, we mean a function $v(\cdot) \in C^n([0,T], X)$ such that

$$v^{(i)}(t) \in \mathcal{D}(A_i) \ \ (t \in [0,T],\ 0 \le i \le n-1), \quad A_i v^{(i)}(\cdot) \in C([0,T], X)$$

and $(IACP_n)$ is satisfied.

Theorem 1.4.1. *Suppose that (ACP_n) has an E_0-existence family $\{E(t)\}_{t \ge 0}$ in $\mathcal{L}(Y, X)$ and a U_0-uniqueness family in $\mathcal{L}(X)$, and $\mathbf{D}_0, \cdots, \mathbf{D}_{n-1}$ from Definition* 1.3.3. *If either*

(i) *$g \in C^1([0,T], X)$, or*

(ii) *there are $h_i \in L^1([0,T], Y)$ such that $A_i f(t) = E_0 h_i(t)$ $(0 \le i \le n-1$, $t \in [0,T])$,*

then for every $u_0 \in \mathbf{D}_0, \cdots, u_{n-1} \in \mathbf{D}_{n-1}$, $(IACP_n)$ admits a unique solution given by

$$v(t) = u(t) + \int_0^t E(t-s) g(s) ds, \quad t \in [0,T], \qquad (1.4.1)$$

where $u(t)$ is the solution of (ACP_n).

Proof. Put

$$v_*(t) := \int_0^t E(t-\sigma)g(\sigma)d\sigma, \quad t \in [0,T].$$

By Definition 1.3.1 (1) and (1.3.8), we have $v_* \in C^{n-1}([0,T],X)$ and

$$v_*^{(k)}(t) = \int_0^t E^{(k)}(t-\sigma)g(\sigma)d\sigma, \quad t \in [0,T],\ k = 0,\dots,n-1. \tag{1.4.2}$$

Using Fubini's theorem gives

$$\begin{aligned}\int_0^t v_*(\tau)d\tau &= \int_0^t \left[\int_0^\tau E(\tau-\sigma)g(\sigma)d\sigma\right]d\tau \\ &= \int_0^t \left[\int_0^{t-\sigma} E(\tau)g(\sigma)d\tau\right]d\sigma, \quad \text{for } t \in [0,T].\end{aligned}$$

By (1.3.8),

$$\begin{aligned}A_0\int_0^{t-\sigma} E(\tau)g(\sigma)d\tau &= E_0g(\sigma) - E^{(n-1)}(t-\sigma)g(\sigma) - \sum_{i=1}^{n-1} A_iE^{(i-1)}(t-\sigma)g(\sigma), \\ &\qquad 0 \le \sigma \le t,\ t \in [0,T].\end{aligned}$$

It follows that

$$\int_0^t v_*(\tau)d\tau \in \mathcal{D}(A_0) \quad \text{for each } t \in [0,T],$$

and

$$\begin{aligned}A_0\int_0^t v_*(\tau)d\tau &= \int_0^t \left[A_0\int_0^{t-\sigma} E(\tau)g(\sigma)d\tau\right]d\sigma \\ &= \int_0^t f(\sigma)d\sigma - \int_0^t E^{(n-1)}(t-\sigma)g(\sigma)d\sigma \\ &\quad -\sum_{i=1}^{n-1}\int_0^t A_iE^{(i-1)}(t-\sigma)g(\sigma)d\sigma \quad \text{for each } t \in [0,T].\end{aligned}$$

So

$$A_0\int_0^t v_*(\tau)d\tau = \int_0^t f(\sigma)d\sigma - v_*^{(n-1)}(t) - \sum_{i=1}^{n-1} A_iv_*^{(i-1)}(t), \quad t \in [0,T] \tag{1.4.3}$$

by (1.4.2) and the closedness of A_i $(1 \le i \le n-1)$.

If $g \in C^1([0,T],X)$, then by (1.4.2),

$$t \mapsto v_*^{(n-1)}(t) = \int_0^t E^{(n-1)}(\sigma)g(t-\sigma)d\sigma \quad \in \quad C^1([0,T],X),$$

$$t \mapsto A_i v_*^{(i-1)}(t) = \int_0^t A_i E^{(i-1)}(\sigma)g(t-\sigma)d\sigma \quad \in \quad C^1([0,T],X), \quad 1 \le i \le n-1.$$

Therefore, we see from (1.4.3) that

$$t \mapsto A_0 \int_0^t v_*(s)ds \in C^1([0,T],X)$$

and

$$A_0 v_*(t) = f(t) - v_*^{(n)}(t) - \sum_{i=1}^{n-1} A_i v_*^{(i)}(t), \quad t \in [0,T], \tag{1.4.4}$$

due to A_0 and A_i $(1 \le i \le n-1)$ being closed.

Let now hypothesis (ii) be satisfied. Set

$$\begin{aligned} r_\sigma(s) \quad := \quad & E(s)g(\sigma) - \frac{s^{n-1}}{(n-1)!}f(\sigma) \\ & + \sum_{j=0}^{n-1} \int_0^s \frac{(s-\tau)^{n-j-1}}{(n-j-1)!}E(\tau)h_j(\sigma)d\tau, \quad s,\ \sigma \in [0,T]. \end{aligned}$$

It is not difficult to verify by (1.3.8) that

$$\begin{cases} r_\sigma^{(n-1)}(s) + \sum_{i=0}^{n-1} A_i r_\sigma^{(i-1)}(s) = 0, \quad s,\ \sigma \in [0,T], \\ r_\sigma^{(k)}(0) = 0, \quad 0 \le k \le n-1. \end{cases}$$

Then, arguing similarly as in the proof of Theorem 1.3.4 (2), we obtain

$$r_\sigma(s) \equiv 0 \quad \text{for all } s,\ \sigma \in [0,T].$$

Therefore

$$E(t-\sigma)g(\sigma) \quad = \quad \frac{(t-\sigma)^{n-1}}{(n-1)!}f(\sigma) + \sum_{j=0}^{n-1} \int_0^{t-\sigma} \frac{(t-\sigma-\tau)^{n-j-1}}{(n-j-1)!}E(\tau)h_j(\sigma)d\tau,$$

$$0 \le \sigma \le t,\ t \in [0,T].$$

From this we see that for $0 \le \sigma \le t$, $t \in [0,T]$ and $0 \le i \le n-1$,

$$E^{(i)}(t-\sigma)g(\sigma) \in \mathcal{D}(A_i)$$

and

$$A_i E^{(i)}(t-\sigma)g(\sigma) = \frac{(t-\sigma)^{n-i-1}}{(n-i-1)!}E_0 h_i(\sigma) + \sum_{j=0}^{n-1} A_i E^{(j-n+i)}(t-\sigma)h_j(\sigma).$$

Thus,

$$t \mapsto \int_0^t A_i E^{(i)}(t-\sigma)g(\sigma)d\sigma \in C([0,T],X), \quad 0 \le i \le n-1,$$

and hence

$$A_i v_*^{(i-1)}(\cdot) \in C^1([0,T],X), \quad 0 \le i \le n-1.$$

So (1.4.4) holds too in this case.

Consequently, in both of the cases (i) and (ii), $v(t)$ given in (1.4.1) is the unique solution of $(IACP_n)$ by an application of Theorem 1.3.4. This completes the proof.

□

Corollary 1.4.2. *Suppose that* $\{E(t)\}_{t\ge 0}$ *is an* E_0*-existence family for* (ACP_1)*. If* $g \in C^1([0,T],Y)$ *and* $t \in [0,T]$*, then* $\int_0^t E(s)g(s)ds \in \mathcal{D}(A_0)$ *and*

$$A_0 \int_0^t E(s)g(s)ds = -E(t)g(t) + E_0 g(0) + \int_0^t E(s)g'(s)ds. \tag{1.4.5}$$

Proof. Fix $t \in [0,T]$ and define

$$h(s) = \begin{cases} g(t-s), & s \in [0,t], \\ 2g(0) - g(s-t), & s \in (t,T]. \end{cases}$$

Then $h \in C^1([0,T],Y)$. Making use of Theorem 1.4.1 with $n=1$ and h in place of g, we obtain for $t \in [0,T]$,

$$\int_0^t E(t-s)h(s)ds \in \mathcal{D}(A_0)$$

and

$$\frac{d}{dt}\int_0^t E(t-s)h(s)ds + A_0 \int_0^t E(t-s)h(s)ds = E_0 h(t).$$

Therefore

$$\begin{aligned} A_0 \int_0^t E(t-s)h(s)ds &= -\frac{d}{dt}\int_0^t E(s)h(t-s)ds + E_0 h(t) \\ &= -\int_0^t E(s)h'(t-s)ds - E(t)h(0) + E_0 h(t). \end{aligned}$$

Accordingly

$$\int_0^t E(s)g(s)ds = \int_0^t E(s)h(t-s)ds = \int_0^t E(t-s)h(s)ds \in \mathcal{D}(A_0)$$

and

$$\begin{aligned} A_0 \int_0^t E(s)g(s)ds &= A_0 \int_0^t E(t-s)h(s)ds \\ &= \int_0^t E(s)g'(s)ds - E(t)g(t) + E_0 g(0). \end{aligned}$$

The proof is then complete.

□

Remark 1.4.3. Corollary 1.4.2 presents a formula for existence families. The formula was already proved by deLaubenfels [19, Theorem 3.4 (c)] for regularized semigroups. His proof could not be adapted to the case of general existence families because it is based on the semigroup property of regularized semigroups. We used here a different approach and obtained the formula for general existence families.

1.5 Applications

Example 1.5.1. We consider the Cauchy problem for a modified Klein-Gordon equation (cf., e.g., [64, 72]) in $L^p(R)$ $(1 \le p \le \infty)$:

$$\begin{cases} \dfrac{\partial^2 u(t,\ x)}{\partial t^2} + a\dfrac{\partial^2 u(t,\ x)}{\partial x \partial t} - r\dfrac{\partial^2 u(t,\ x)}{\partial x^2} + \gamma(x)u(t,\ x) = f(t,x), \quad (t,\ x) \in R^+ \times R, \\ u(0,\ x) = \phi(x),\ u_t(0,\ x) = \psi(x), \quad x \in R, \end{cases} \tag{1.5.1}$$

where $a \in R$, $r > 0$, $\gamma(\cdot) \in W^{1,\infty}(R)$, and $f \in C^1\left(R^+, W^{1,p}(R)\right)$.

Set

$$X = L^p(R),$$

$$A_1 := a\frac{d}{dx} \quad \text{with } \mathcal{D}(A_1) = W^{1,p}(R),$$

$$A_0 = A_{01} + A_{02} \quad \text{with } \mathcal{D}(A_0) = W^{2,p}(R),$$

where $A_{01} := -r\dfrac{d^2}{dx^2}$ and A_{02} is the multiplication operator by $\gamma(\cdot)$.

Write

$$P_{0\lambda} := \lambda^2 + \lambda A_1 + A_{01}, \quad \lambda \in R.$$

Then by Theorem 3.5.6 (2) and Theorem 3.5.4 in [84], we have

$$\lambda P_{0\lambda}^{-1}\left(I - \frac{d^2}{dx^2}\right)^{-\frac{1}{2}}, \quad A_1 P_{0\lambda}^{-1}\left(I - \frac{d^2}{dx^2}\right)^{-\frac{1}{2}} \in LT_\omega - \mathcal{L}(X). \tag{1.5.2}$$

Take $\mu_0 \in \rho(A_1)$. Using the equalities

$$\lambda P_{0\lambda}^{-1}(\mu_0 - A_1)^{-1} = \lambda P_{0\lambda}^{-1}\left(I - \frac{d^2}{dx^2}\right)^{-\frac{1}{2}}\left[\left(I - \frac{d^2}{dx^2}\right)^{\frac{1}{2}}(\mu_0 - A_1)^{-1}\right],$$

$$\begin{aligned} P_{0\lambda}^{-1} &= \left[\mu_0 P_{0\lambda}^{-1}\left(I - \frac{d^2}{dx^2}\right)^{-\frac{1}{2}} - A_1 P_{0\lambda}^{-1}\left(I - \frac{d^2}{dx^2}\right)^{-\frac{1}{2}}\right] \\ &\quad \cdot \left[\left(I - \frac{d^2}{dx^2}\right)^{\frac{1}{2}}(\mu_0 - A_1)^{-1}\right] \end{aligned}$$

and noting the boundedness of the operator $\left(I - \frac{d^2}{dx^2}\right)^{\frac{1}{2}}(\mu_0 - A_1)^{-1}$, we deduce from (1.5.2) that

$$P_{0\lambda}^{-1} \in LT_\omega - \mathcal{L}(X), \tag{1.5.3}$$

$$\lambda P_{0\lambda}^{-1}(\lambda_0 - A_1)^{-1} \in LT_\omega - \mathcal{L}(X). \tag{1.5.4}$$

From (6.4.3) it follows that

$$A_{02}P_{0\lambda}^{-1} \in LT_\omega - \mathcal{L}(X), \quad A_{02}P_{0\lambda}^{-1} \in LT_\omega - \mathcal{L}\left([\mathcal{D}(A_1)]\right),$$

because $A_{02} \in \mathcal{L}(X)$ and $A_{02}(\mathcal{D}(A_1)) \subset \mathcal{D}(A_1)$. Accordingly, we obtain in view of Theorem 1.1.11 of [84],

$$\left(I + A_{02}P_{0\lambda}^{-1}\right)^{-1} - I \in LT_\omega - \mathcal{L}(X), \tag{1.5.5}$$

$$\left(I + A_{02}P_{0\lambda}^{-1}\right)^{-1} - I \in LT_\omega - \mathcal{L}([\mathcal{D}(A_1)]). \tag{1.5.6}$$

Combining (1.5.3) and (1.5.5) yields that for λ large enough, $P_\lambda := \lambda^2 + \lambda A_1 + A_0$ is invertible and

$$R_\lambda := P_\lambda^{-1} = P_{0\lambda}^{-1} + P_{0\lambda}^{-1}\left[\left(I + A_{02}P_{0\lambda}^{-1}\right)^{-1} - I\right] \in LT_\omega - \mathcal{L}(X). \tag{1.5.7}$$

This means that in view of Theorem 1.3.9, there exists an I-uniqueness family for (ACP_2).

On the other hand, we observe that for λ large enough

$$\begin{aligned} R_\lambda(\mu_0 - A_1)^{-1} &= P_{0\lambda}^{-1}(\mu_0 - A_1)^{-1} + \left(P_{0\lambda}^{-1}(\mu_0 - A_1)^{-1}\right) \\ &\quad \times (\mu_0 - A_1)\left[\left(I + A_{02}P_{0\lambda}^{-1}\right)^{-1} - I\right](\mu_0 - A_1)^{-1}. \end{aligned}$$

Hence,

$$\lambda R_\lambda(\mu_0 - A_1)^{-1},\ A_1 R_\lambda(\mu_0 - A_1)^{-1} \in LT_\omega - \mathcal{L}(X)$$

by (1.5.4) and (1.5.6). Moreover, (1.5.7) implies that

$$R_\lambda A_1(\mu_0 - A_1)^{-1} \in LT_\omega - \mathcal{L}(X).$$

Thus we infer, by Theorem 1.3.7 (a), that (1.5.1) has a $(\mu_0 - A_1)^{-1}$-existence family. Set $g = (\mu_0 - A_1)f$. Then $g \in C^1(R^+, X)$ by hypothesis. Applying now Theorem 1.4.1 and (1.3.15) we conclude that for every $\phi \in W^{3,p}(R)$, $\psi \in W^{3,p}(R)$, (1.5.1) has a unique solution $u \in C^2\left(R^+, L^p(R)\right) \cap C^1\left(R^+, W^{1,p}(R)\right)$ and

$$\|u\|_{L^p(R)} \le M e^{\omega t}\left(\|\phi\|_{W^{1,p}(R)} + \|\psi\|_{W^{1,p}(R)} + \int_0^t \|f(s,\cdot)\|_{W^{1,p}(R)} ds\right), \quad t \in R^+.$$

Example 1.5.2. Let $1 \le p \le \infty$, $\rho_1 \in R$, $\rho_2 > 0$, $c \in \mathbf{C}$, and let $a \in W^{3,\infty}(R)$, $f \in C^1\left(R^+, W^{3,p}(R)\right)$. Consider the following initial value problem in $L^p(R)$:

$$\begin{cases} \dfrac{\partial^2 u(t,\ x)}{\partial t^2} + \left(\rho_1 \dfrac{\partial^3}{\partial x^3} - \rho_2 \dfrac{\partial^2}{\partial x^2}\right)\dfrac{\partial u(t,\ x)}{\partial t} \\ \qquad + \left(c\dfrac{\partial^2}{\partial x^2} + a(x)\right)u(t,\ x) = f(t,x), \quad (t,\ x) \in R^+ \times R, \\ u(0,\ x) = \phi(x),\ u_t(0,\ x) = \psi(x), \quad x \in R. \end{cases} \tag{1.5.8}$$

Set

$$X = L^p(R),$$

$$A_1 = \rho_1 \frac{\partial^3}{\partial x^3} - \rho_2 \frac{\partial^2}{\partial x^2} \quad \text{with } \mathcal{D}(A_1) = W^{3,p}(R),$$

$$A_0 = A_{01} + A_{02} \quad \text{with } \mathcal{D}(A_0) = W^{2,p}(R),$$

where $A_{01} := c\dfrac{\partial^2}{\partial x^2}$ and A_{02} is the multiplication operator by $a(\cdot)$. It is known from Theorem 1.5.9 of [84] that $-A_1$ generates a once integrated semigroup.

As in Example 1.5.1, we write $P_{0\lambda} = \lambda^2 + \lambda A_1 + A_{01}$ ($\lambda \in R$). Then by the equality (1.3.10) on page 95 of [84], we see that for λ large enough $P_{0\lambda}$ is invertible and

$$P_{0\lambda}^{-1} \in LT_\omega - \mathcal{L}(X), \quad \lambda P_{0\lambda}^{-1}(\mu_0 - A_1)^{-1} \in LT_\omega - \mathcal{L}(X),$$

where $\mu_0 \in \rho(A_1)$. Thus the same reasoning as in Example 1.5.1 gives that (1.5.8) has a $(\mu_0 - A_1)^{-1}$-existence family and for every $\phi \in W^{5,p}(R)$, $\psi \in W^{6,p}(R)$, (1.5.8) has a unique solution $u \in C^2(R^+, L^p(R)) \cap C^1(R^+, W^{3,p}(R))$ which satisfies

$$\|u\|_{L^p(R)} \le Me^{\omega t}\left(\|\phi\|_{W^{3,p}(R)} + \|\psi\|_{W^{3,p}(R)} + \int_0^t \|f(s,\cdot)\|_{W^{3,p}(R)} ds\right), \quad t \in R^+.$$

Chapter 2

Perturbations of existence families for higher order abstract Cauchy problems

2.1 Summary

In this chapter, we establish Desch-Schappacher type multiplicative and additive perturbation theorems for existence families for arbitrary order abstract Cauchy problems in a Banach space

$$\begin{cases} u^{(n)}(t) = Au(t) \quad (t \geq 0), \\ u^{(j)}(0) = x_j \quad (0 \leq j \leq n-1). \end{cases}$$

As a consequence, we obtain perturbation results for regularized semigroups and regularized cosine operator functions. An example is also given to illustrate possible applications.

2.2 Introduction

We consider the abstract Cauchy problem:

$$\begin{cases} u^{(n)}(t) = Au(t) \quad (t \geq 0), \\ u^{(j)}(0) = x_j \quad (0 \leq j \leq n-1). \end{cases} \tag{2.2.1}$$

where $n \in N$, and A is a closed linear operator in a Banach space X.

Definition 2.2.1. (compare with Definition 1.3.1). The strongly continuous family of operators $\{S(t)\}_{t\geq 0} \subset \mathcal{L}(X)$ is called a *C-existence family* for (2.2.1) if for all $x \in X$, $t \geq 0$,

$$S(\cdot)x \in C^{n-1}(R^+, X), \quad A\int_0^t S(s)xds \in C(R^+, X),$$

and

$$S(t)x = \frac{t^{n-1}}{(n-1)!}Cx + A\int_0^t \frac{(t-s)^{n-1}}{(n-1)!}S(s)xds. \tag{2.2.2}$$

We also say that (2.2.1) *has a C-existence family* $\{S(t)\}_{t\geq 0}$.

It is known from [19, Chapter III] that the C-existence family reduces to C-regularized semigroup when $n = 1$ and $S(t)A \subset AS(t)$ $(t \geq 0)$. Moreover, taking $n = 2$ and $S(t)A \subset AS(t)$ $(t \geq 0)$ in Definition 2.2.1 gives the C-regularized cosine operator function $\{S'(t)\}_{t\geq 0}$.

The Desch-Schappacher perturbations were first studied in [22] for strongly continuous semigroups in 1989. In recent years, this type of perturbations has drawn many researchers' attention, and the related theory has been developed (cf., e.g., Engel and Nagel [26, Section III.3], [10, 21, 23, 46, 66, 69, 70] and references therein). In [23], Diekmann, Gyllenberg and Thieme showed a new view at the perturbations of Desch-Schappacher type by solving Stieltjes' renewal equations with the basic assumption on the behaviour of semivariation of the step response function (see also [69]). In [46], Jung investigated how certain properties, e.g., analyticity, norm continuity, of the original semigroup are inherited by the perturbed semigroup. In [21, Section V] by deLaubenfels and Yao, nonlinear additive perturbations of this type for C-regularized semigroups were discussed and a local existence and uniqueness theorem on the classical solutions of the Cauchy problem for the associated perturbed equation was given. Moreover, in [10, 66, 70], one can see such results for perturbations of strongly continuous cosine operator functions, solution families or n-times integrated solution families of linear Volterra equations.

In this chapter, we will present Desch-Schappacher type multiplicative and additive perturbation theorems for the general existence family given by Definition 2.2.1, and show the uniqueness of solutions for the corresponding perturbed problem

(2.2.1) (Theorems 2.3.1 and 2.3.2). As a consequence, we obtain Desch-Schappacher type perturbation theorems for regularized semigroups and regularized cosine operator functions (Corollaries 2.4.1, 2.4.2 and 2.4.4) which recover the corresponding results in [18, 19, 22, 69, 70] (see Remarks 2.4.3 and 2.4.5]). With a new observation on the ranges of perturbation operators, we exhibit in Theorem 2.4.6 two classes of perturbation operators satisfying the conditions of Theorem 2.3.1 or Theorem 2.3.2. Finally, an example (Example 2.4.7) is given to illustrate possible applications. This example also reflects the feature of Theorem 2.4.6 (see Remark 2.4.8).

The following result on exponentially bounded existence families (shown in Theorem 1.3.7) will be used in the sequel.

Proposition 2.2.2. *Let $\lambda^n - A$ be injective for $\lambda > \omega$. Then* (2.2.1) *has a C-existence family $\{S(t)\}_{t\geq 0}$ on X with*

$$\left\|S^{(n-1)}(t)\right\| \leq Me^{\omega t}, \quad t \geq 0,$$

if and only if

$$\begin{cases} \mathcal{R}(C) \subset \mathcal{R}(\lambda^n - A) \quad \text{for } \lambda > \omega, \\ \text{the function } \lambda \mapsto \lambda^{n-1} \left(\lambda^n - A\right)^{-1} C \in LT - \mathcal{L}(X). \end{cases}$$

In this case, for $x_j \in \mathcal{D}(A)$ with $Ax_j \in \mathcal{R}(C)$ $(0 \leq j \leq n-1)$, (2.2.1) *admits a solution $u(\cdot)$ satisfying*

$$\left\|u^{(n)}(t)\right\|, \ \|u(t)\|_{[\mathcal{D}(A)]} \leq Me^{\omega t} \sum_{i=0}^{n-1} \left(\|u_i\| + \left\|C^{-1}Au_i\right\|\right), \quad t \geq 0,$$

and

$$\lambda^{n-1} \left(\lambda^n - A\right)^{-1} Cx = \int_0^\infty e^{-\lambda t} S^{(n-1)}(t) x dt, \quad x \in X, \ \lambda > \omega. \tag{2.2.3}$$

2.3 Perturbations of existence families for (ACP_n)

We first give a Desch-Schappacher type mixed (right) multiplicative and additive perturbation theorem.

Theorem 2.3.1. *Let A and $\{S(t)\}_{t\geq 0}$ be as in Proposition* 2.2.2*, and let α, $\beta \in \mathbf{C}$. Suppose $B \in \mathcal{L}(X)$ and $\mathcal{R}(B) \subset \mathcal{R}(C)$. If for every $f \in C(R^+, X)$ and $t \geq 0$*

$$\left\| A \int_0^t S(t-s)C^{-1}Bf(s)ds \right\| \leq M \int_0^t e^{\omega(t-s)} \|f(s)\| ds, \tag{2.3.1}$$

then

(i) *the Cauchy problem*

$$\begin{cases} u^{(n)}(t) = (A(I+\alpha B) + \beta B)u(t), \quad t \geq 0, \\ u^{(j)}(0) = x_j, \quad 0 \leq j \leq n-1 \end{cases} \tag{2.3.2}$$

has a C-existence family $\{U(t)\}_{t\geq 0}$ on X and $\left\|U^{(n-1)}(\cdot)\right\|$ is exponentially bounded;

(ii) *all solutions of* (2.3.2) *are unique provided $CA \subset AC$.*

Proof. Fixing $f \in C(R^+, X)$, by (2.3.1) and (2.2.2) we see that for $0 \leq t_2 \leq t_1 < \infty$

$$\begin{aligned} & \left\| A \int_0^{t_1} S(t_1 - s)C^{-1}Bf(s)ds - A \int_0^{t_2} S(t_2 - s)C^{-1}Bf(s)ds \right\| \\ \leq & \left\| A \int_0^{t_1} S(t_1 - s)C^{-1}B\left[f(s) - f(s - t_1 + t_2)\right] ds \right\| \\ & + \left\| A \int_0^{t_1 - t_2} S(t_1 - s)C^{-1}Bf(s - t_1 + t_2)ds \right\| \\ \leq & Me^{\omega t_1} \max_{0\leq s\leq t_1} \|f(s) - f(s - t_1 + t_2)\| + \left\| A \int_{t_2}^{t_1} S(s)C^{-1}Bf(0)ds \right\| \\ \leq & Me^{\omega t_1} \max_{0\leq s\leq t_1} \|f(s) - f(s - t_1 + t_2)\| + \|[S^{(n-1)}(t_1) - S^{(n-1)}(t_2)]C^{-1}Bf(0)\|, \end{aligned}$$

where $f(-s) := f(0)$ for $s > 0$. This implies that

$$t \mapsto A \int_0^t S(t-s)C^{-1}Bf(s)ds \in C(R^+, X), \quad f \in C(R^+, X).$$

We set

$$W_0(t) := S^{(n-1)}(t), \quad t \geq 0,$$

and define $W_n(t)$ inductively by

$$W_n(t)x = (\beta + \alpha A)\int_0^t S(t-s)C^{-1}BW_{n-1}(s)xds, \quad x \in X,\ t \geq 0,\ n \in N. \quad (2.3.3)$$

Clearly, $\{W_n(t)\}_{t\geq 0}$ is a strongly continuous family of bounded linear operators on X for each $n \in N$. We know by hypothesis that $S(\cdot)$ and $W_0(\cdot)$ are exponentially bounded. So using (2.3.1) we infer by induction that

$$\|W_n(t)\| \leq M_1^{n+1}e^{\omega_1 t}\frac{t^n}{n!}, \quad t \geq 0,\ n \in N \cup \{0\},$$

for certain constants $M_1 > M$, $\omega_1 > \omega$. Define

$$W(t) := \sum_{n=0}^{\infty} W_n(t), \quad t \geq 0.$$

We see by the above arguments that the series converges in the uniform operator topology, uniformly on bounded intervals of R^+ with

$$\|W(t)\| \leq M_1 e^{(\omega_1+M_1)t}, \quad t \geq 0.$$

Hence $\{W(t)\}_{t\geq 0} \subset \mathcal{L}(X)$ is a strongly continuous family. Thus, by (2.3.3), we have

$$W(t)x = S^{(n-1)}(t)x + (\beta + \alpha A)\int_0^t S(t-s)C^{-1}BW(s)xds, \quad x \in X,\ t \geq 0.$$

Taking Laplace transforms we obtain by (2.2.3) that for λ large enough and $x \in X$,

$$\begin{aligned}&\int_0^\infty e^{-\lambda t}W(t)xdt\\ = \;& \lambda^{n-1}(\lambda^n - A)^{-1}Cx + (\beta + \alpha A)(\lambda^n - A)^{-1}B\int_0^\infty e^{-\lambda t}W(t)xdt.\end{aligned}$$

Therefore for such λ we have

$$(\lambda^n - A(I + \alpha B) - \beta B)\int_0^\infty e^{-\lambda t}W(t)xdt = \lambda^{n-1}Cx, \quad x \in X, \quad (2.3.4)$$

by the equalities

$$\begin{aligned}&(\lambda^n - A)\left[I - (\beta + \alpha A)(\lambda^n - A)^{-1}B\right]\\ = \;& (\lambda^n - A)\left[I + \alpha B - \alpha\lambda^n(\lambda^n - A)^{-1}B - \beta(\lambda^n - A)^{-1}B\right] \quad (2.3.5)\\ = \;& \lambda^n - A(I + \alpha B) - \beta B.\end{aligned}$$

Finally, we show that $I-(\beta+\alpha A)(\lambda^n-A)^{-1}B$ is invertible for large λ. In order to do this, we observe by (2.3.1) and (2.2.3) that for each $x \in X$, $t \geq 0$,

$$\begin{aligned}
&\left\|(\beta+\alpha A)\int_0^t S(t-s)C^{-1}Bxds\right\| \\
\leq\ & M_2\int_0^t e^{\omega_2(t-s)}\|x\|ds \\
\leq\ & \frac{M_2}{\omega_2}\left(e^{\omega_2 t}-1\right)\|x\|,
\end{aligned}$$

and

$$\begin{aligned}
&(\beta+\alpha A)(\lambda^n-A)^{-1}Bx \\
=\ & \lambda\int_0^\infty e^{-\lambda t}\left[(\beta+\alpha A)\int_0^t S(t-s)C^{-1}Bxds\right]dt, \quad \lambda>\omega_2,
\end{aligned}$$

where M_2 and ω_2 are positive constants. So for $\lambda>\omega_2$ and $x \in X$,

$$\begin{aligned}
&\left\|(\beta+\alpha A)(\lambda^n-A)^{-1}Bx\right\| \\
\leq\ & \frac{M_2\lambda}{\omega_2}\int_0^\infty e^{-\lambda t}\left(e^{\omega_2 t}-1\right)\|x\|dt \\
=\ & \frac{M_2\|x\|}{\lambda(\lambda-\omega_2)}.
\end{aligned}$$

Thus for $\lambda>2M_2+\omega_2+1$,

$$\left\|(\beta+\alpha A)(\lambda^n-A)^{-1}B\right\|<\frac{1}{2},$$

so that $I-(\beta+\alpha A)(\lambda^n-A)^{-1}B$ is invertible. This together with (2.3.5) yields that for $\lambda>2M_2+\omega_2+\omega+1$, $\lambda^n-A(I+\alpha B)-\beta B$ is injective since λ^n-A is injective for $\lambda>\omega$. In conclusion, we obtain from (2.3.4) that for λ sufficiently large,

$$\lambda^{n-1}(\lambda^n-A(I+\alpha B)-\beta B)^{-1}Cx=\int_0^\infty e^{-\lambda t}W(t)xdt, \quad x \in X.$$

Set, for $t \geq 0$ and $x \in X$,

$$U(t)x:=\begin{cases} W(t)x & \text{if} \quad n=1, \\ \displaystyle\int_0^t \frac{(t-s)^{n-2}}{(n-2)!}W(s)xds & \text{if} \quad n\geq 2. \end{cases}$$

Then an application of Proposition 2.2.2 gives assertion (i).

In order to verify assertion (ii), we let $v(\cdot)$ be a solution of (2.3.2) with initial data $x_j = 0$ $(0 \le j \le n-1)$. Evidently

$$v(t) = (A(I+\alpha B)+\beta B)\int_0^t \frac{(t-\sigma)^{n-1}}{(n-1)!}v(\sigma)d\sigma, \quad t \ge 0. \tag{2.3.6}$$

The assumption $CA \subset AC$ implies

$$S(t)C = CS(t)\ (t \ge 0), \quad S(t)Ax = AS(t)x\ (x \in \mathcal{D}(A),\ t \ge 0), \tag{2.3.7}$$

according to (2.2.3) and the uniqueness theorem for Laplace transforms. So (2.2.2) yields

$$S^{(n)}(t)x = S(t)Ax, \quad x \in \mathcal{D}(A),\ t \ge 0. \tag{2.3.8}$$

Thus, by (2.3.6) – (2.3.8) we obtain that for $t \ge s \ge 0$,

$$\begin{aligned}
&\frac{d}{ds}\left[\sum_{i=0}^{n-1} S^{(i)}(t-s)(I+\alpha B)\int_0^s \frac{(s-\sigma)^i}{i!}v(\sigma)d\sigma\right] \\
=\ & S(t-s)(I+\alpha B)v(s) + \sum_{i=1}^{n-1} S^{(i)}(t-s)(I+\alpha B)\int_0^s \frac{(s-\sigma)^{i-1}}{(i-1)!}v(\sigma)d\sigma \\
& -\sum_{i=0}^{n-1} S^{(i+1)}(t-s)(I+\alpha B)\int_0^s \frac{(s-\sigma)^{i}}{(i-1)!}v(\sigma)d\sigma \\
=\ & S(t-s)(I+\alpha B)v(s) - S^{(n)}(t-s)(I+\alpha B)\int_0^s \frac{(s-\sigma)^{n-1}}{(n-1)!}v(\sigma)d\sigma \\
=\ & CS(t-s)\left[\alpha C^{-1}Bv(s) + \beta C^{-1}B\int_0^s \frac{(s-\sigma)^{n-1}}{(n-1)!}v(\sigma)d\sigma\right].
\end{aligned}$$

Noting

$$S^{(n-1)}(0) = I, \quad S^{(i)}(0) = 0\ (0 \le i \le n-2) \tag{2.3.9}$$

from (2.2.2), we then infer that for $t \ge 0$,

$$\begin{aligned}
&C(I+\alpha B)\int_0^t \frac{(t-\sigma)^{n-1}}{(n-1)!}v(\sigma)d\sigma \\
=\ & C\int_0^t S(t-s)\left[\alpha C^{-1}Bv(s) + \beta C^{-1}B\int_0^s \frac{(s-\sigma)^{n-1}}{(n-1)!}v(\sigma)d\sigma\right]ds.
\end{aligned}$$

Since C is injective, it follows from (2.3.6) that for $t \geq 0$,

$$v(t) = \beta B \int_0^t \frac{(t-\sigma)^{n-1}}{(n-1)!} v(\sigma) d\sigma + \alpha A \int_0^t S(t-\sigma) C^{-1} B v(\sigma) d\sigma$$

$$+\beta A \int_0^t S(t-\sigma) C^{-1} B \left(\int_0^\sigma \frac{(\sigma-\tau)^{n-1}}{(n-1)!} v(\tau) d\tau \right) d\sigma.$$

Fix $T > 0$. Then by (2.3.1) there exists a constant $M_0 > 0$ such that for each $t \in [0, T]$,

$$\max_{0 \leq s \leq t} \|v(s)\| \leq M_0 \int_0^t \max_{0 \leq \tau \leq \sigma} \|v(\tau)\| d\sigma.$$

So Gronwall-Bellman's inequality shows that $v(t) = 0$ for $t \in [0, T]$. Because T was arbitrary, $v(t) \equiv 0$ for $t \geq 0$. This ends the proof.

□

The following is a Desch-Schappacher type additive perturbation theorem which can also be regarded as a (left) multiplicative perturbation theorem.

Theorem 2.3.2. *Let A and $\{S(t)\}_{t \geq 0}$ be as in Proposition* 2.2.2. *Suppose* $\mathbf{B}$ *is a closed linear operator in X such that $\mathcal{D}(\mathbf{B}) \supset \mathcal{D}(A)$ and $\mathcal{R}(\mathbf{B}) \subset \mathcal{R}(C)$. If for each $f \in C(R^+, [\mathcal{D}(A)])$ and $t \geq 0$,*

$$\left\| A \int_0^t S(t-s) C^{-1} \mathbf{B} f(s) ds \right\| \leq M \int_0^t e^{\omega(t-s)} \|f(s)\|_{[\mathcal{D}(A)]} ds, \qquad (2.3.10)$$

then

(i) *the Cauchy problem*

$$\begin{cases} u^{(n)}(t) = (A + \mathbf{B}) u(t), \quad t \geq 0, \\ u^{(j)}(0) = x_j, \quad 0 \leq j \leq n-1 \end{cases} \qquad (2.3.11)$$

has a C-existence family $\{V(t)\}_{t \geq 0}$ on $[\mathcal{D}(A)]$ and $\left\| V^{(n-1)}(\cdot) \right\|_{\mathcal{L}([\mathcal{D}(A)])}$ is exponentially bounded;

(ii) *for any $x_j \in C(\mathcal{D}(A))$ $(0 \leq j \leq n-1)$, the function*

$$\sum_{j=0}^{n-1} V^{(n-1-j)}(\cdot) C^{-1} x_j$$

is a solution of (2.3.11).

(iii) *all solutions of* (2.3.11) *are unique, provided* $CA \subset AC$.

Proof. Define

$$Y(t) = \sum_{n=0}^{\infty} Y_n(t), \quad t \geq 0,$$

where

$$\begin{cases} Y_0(t) = S^{(n-1)}(t), \quad t \geq 0, \\ Y_n(t)x = \int_0^t S(t-s)C^{-1}\mathbf{B}Y_{n-1}(s)xds, \quad t \geq 0, \ x \in \mathcal{D}(A), \ n \in N. \end{cases}$$

Arguing similarly as in the proof of Theorem 2.3.1, we deduce that $\{Y(t)\}_{t\geq 0}$ is an exponentially bounded, strongly continuous family of bounded linear operators on $[\mathcal{D}(A)]$. For λ large enough,

$$(\lambda^n - (A+\mathbf{B}))\int_0^{\infty} e^{-\lambda t}Y(t)xdt = \lambda^{n-1}Cx, \quad x \in \mathcal{D}(A),$$

and

$$\left\|(\lambda^n - A)^{-1}\mathbf{B}\right\|_{\mathcal{L}([\mathcal{D}(A)])} < \frac{1}{2}, \tag{2.3.12}$$

so that $\lambda^n - (A+\mathbf{B})$ is injective and

$$\lambda^{n-1}(\lambda^n - (A+\mathbf{B}))^{-1}Cx = \int_0^{\infty} e^{-\lambda t}Y(t)xdt, \quad x \in \mathcal{D}(A). \tag{2.3.13}$$

Therefore, (2.3.11) has a C-existence family $\{V(t)\}_{t\geq 0}$ on $[\mathcal{D}(A)]$ given by

$$V(t)x := \begin{cases} Y(t)x & \text{if} \quad n = 1, \\ \int_0^t \frac{(t-s)^{n-2}}{(n-2)!}Y(s)xds & \text{if} \quad n \geq 2, \end{cases}$$

in view of Proposition 2.2.2. This completes the proof of part (i).

Next we have

$$\begin{aligned} V(t)x - \frac{t^{n-1}}{(n-1)!}Cx &= (A+\mathbf{B})\int_0^t \frac{(t-s)^{n-1}}{(n-1)!}V(s)xds \\ &= \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}(A+\mathbf{B})V(s)xds, \quad x \in \mathcal{D}(A), \ t \geq 0, \end{aligned}$$

since $\{V(t)\}_{t\geq 0}$ is a C-existence family on $[\mathcal{D}(A)]$. This leads to part (ii) immediately.

To prove part (iii) we let $w(\cdot)$ be a solution of (2.3.11) with $x_j = 0$ for all $0 \leq j \leq n-1$. By (2.3.7) and (2.3.8) we deduce

$$\frac{d}{ds}\left[\sum_{i=0}^{n-1} S^{(i)}(t-s)w^{(n-1-i)}(s)\right] = CS(t-s)C^{-1}\mathbf{B}w(s), \quad t \geq s \geq 0,$$

so that

$$w(t) = \int_0^t S(t-s)C^{-1}\mathbf{B}w(s)ds, \quad t \geq 0$$

by (2.3.9). Thus from (2.3.10) we have

$$\left\|e^{-\omega t}w(t)\right\|_{[\mathcal{D}(A)]} \leq M' \int_0^t \left\|e^{-\omega t}w(t)\right\|_{[\mathcal{D}(A)]} ds, \quad t \geq 0,$$

for some constant $M' > 0$. It follows that $w(t) \equiv 0$ for $t \geq 0$ by using Gronwall-Bellman's inequality. The proof is then complete.

□

2.4 Perturbations of regularized semigroups and regularized cosine operator functions

In what follows, we give multiplicative and additive perturbation theorems with regard to exponentially bounded regularized semigroups and regularized cosine operator functions, as consequences of Theorems 2.3.1 and 2.3.2. Let A and $\{S(t)\}_{t\geq 0}$ be as in Proposition 2.2.2. If $n = 1$ (resp. $n = 2$) and $CA \subset AC$, then $S(\cdot)$ (resp. $C(\cdot) := S'(\cdot)$) is an exponentially bounded C-regularized semigroup (resp. cosine operator function) with $C^{-1}AC$ as its generator. In this case, A is called a subgenerator of $S(\cdot)$ (resp. $C(\cdot)$), or in other words, A subgenerates $S(\cdot)$ (resp. $C(\cdot)$). For more information on regularized semigroups and regularized cosine operator functions, we refer to, e.g., [19, 52, 84] and references therein.

Theorem 2.4.1. *Assume that A subgenerates an exponentially bounded C-regularized semigroup $\{S(t)\}_{t\geq 0}$ (resp. cosine operator function $\{C(t)\}_{t\geq 0}$) on X.*

Let α, $\beta \in \mathbf{C}$, and $B \in \mathcal{L}(X)$ with $\mathcal{R}(B) \subset \mathcal{R}(C)$, and let $C_1 \in \mathcal{L}(X)$ be injective such that $\mathcal{R}(C_1) \subset \mathcal{R}(C)$ and

$$C_1[A(I+\alpha B)+\beta B] \subset [A(I+\alpha B)+\beta B]C_1.$$

If (2.3.1) *holds (in the case of the cosine operator function, $S(t)x := \int_0^t C(s)xds$), then $A(I+\alpha B)+\beta B$ subgenerates an exponentially bounded C_1-regularized semigroup (resp. cosine operator function) on X.*

Proof. Apply Theorem 2.3.1. Then $U(t)C^{-1}C_1$ (resp. $U'(t)C^{-1}C_1$) is the C_1-regularized semigroup (resp. cosine operator function) as claimed.

□

Theorem 2.4.2. *Assume that A subgenerates an exponentially bounded C-regularized semigroup $\{S(t)\}_{t\geq 0}$ (resp. cosine operator function $\{C(t)\}_{t\geq 0}$) on X. Let $\mathbf{B}$ be a closed linear operator in X such that $\mathcal{D}(\mathbf{B}) \supset \mathcal{D}(A)$ and $\mathcal{R}(\mathbf{B}) \subset \mathcal{R}(C)$. Let $C_1 \in \mathcal{L}(X)$ be injective such that*

$$\mathcal{R}(C_1) \subset \mathcal{R}(C), \quad C^{-1}C_1 : \mathcal{D}(A) \to \mathcal{D}(A), \quad C_1(A+\mathbf{B}) \subset (A+\mathbf{B})C_1.$$

If (2.3.10) *holds (in the case of the cosine operator function, $S(t)x := \int_0^t C(s)xds$), then $A+\mathbf{B}$ subgenerates an exponentially bounded C_1-regularized semigroup (resp. cosine operator function) on X provided that $\rho(A)$ contains a sequence of real numbers tending to $+\infty$.*

Proof. From (2.3.12) we see that there exists a $\mu_0 \in \rho(A)$ such that

$$\|(\mu_0 - A)^{-1}\mathbf{B}\|_{\mathcal{L}([\mathcal{D}(A))]} < \frac{1}{2},$$

and therefore

$$\mu_0 - (A+\mathbf{B}) = (\mu_0 - A)\left(I - (\mu_0 - A)^{-1}\mathbf{B}\right)$$

is invertible on X. Letting $Y(\cdot)$ be as in (2.3.13) with $n = 1$ (resp. $n = 2$), we put

$$\widetilde{Y}(t) := [\mu_0 - (A+\mathbf{B})]\, Y(t)C^{-1}C_1 \,[\mu_0 - (A+\mathbf{B})]^{-1}, \quad t \geq 0.$$

Then $\{\widetilde{Y}(t)\}_{t\geq 0}$ is a strongly continuous family of operators in $\mathcal{L}(X)$, and for λ large enough,

$$\begin{aligned}&\int_0^\infty e^{-\lambda t}\widetilde{Y}(t)xdt\\ =\ &\lambda^{n-1}\left[\mu_0-(A+\mathbf{B})\right]\left[\lambda^n-(A+\mathbf{B})\right]^{-1}C_1\left[\mu_0-(A+\mathbf{B})\right]^{-1}\\ =\ &\lambda^{n-1}\left[\lambda^n-(A+\mathbf{B})\right]^{-1}C_1x,\quad t\geq 0,\ x\in X,\end{aligned}$$

with $n=1$ (resp. $n=2$).

□

Remark 2.4.3. For the case when $C=C_1=I$, $\alpha=1$, $\beta=0$, and A generates a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ (resp. strongly continuous cosine operator function $\{C(t)\}_{t\geq 0}$) on X, Corollaries 2.4.1 and 2.4.2 can be found in [22, 69, 70]. In this case, $\{S(t)\}_{t\geq 0}$ (resp. $\{C(t)\}_{t\geq 0}$) is exponentially bounded and $\rho(A)$ contains a right half plane, automatically.

It is evident that (2.3.1) holds for

$$B\in\mathcal{L}(X)\quad \text{with}\ \ \mathcal{R}(B)\subset\mathcal{D}(AC^{-1}),\tag{2.4.1}$$

and that (2.3.10) holds for any closed linear operator $\mathbf{B}$ in X with $\mathcal{D}(\mathbf{B})\supset\mathcal{D}(A)$ and $\mathcal{R}(\mathbf{B})\subset\mathcal{D}(AC^{-1})$. Specifically we have the following result.

Corollary 2.4.4. *Suppose that A subgenerates an exponentially bounded C-regularized semigroup on X, and that $B_1\in\mathcal{L}(X)$ and $\mathcal{R}(B_1)\subset\mathcal{R}(C)$. Then*

(i) *the Cauchy problem*

$$(*)\qquad \begin{cases} u'(t)=(A+B_1)u(t), & t\geq 0,\\ u(0)=x\end{cases}$$

has an exponentially bounded C-existence family on X.

(ii) *all solutions of $(*)$ are unique provided $CA\subset AC$.*

(iii) *$A+B_1$ subgenerates an exponentially bounded C_1-regularized semigroup on X, whenever $C_1\in\mathcal{L}(X)$ is injective, $\mathcal{R}(C_1)\subset\mathcal{R}(C)$ and $C_1(A+B_1)\subset(A+B_1)C_1$.*

Proof. Take $\alpha = 0$ and $\beta = 1$ in Theorem 2.3.1 and Theorem 2.4.1.

□

Remark 2.4.5. Conclusion (i) of Corollary 2.4.4 appeared in [18, 19]. Generally speaking, a C-existence family for a first order Cauchy problem ensures uniqueness of the exponentially bounded solutions, but not all solutions (see [19, Proposition 2.9]). This indicates the significance of the assertion (ii). Conclusion (iii) is due to [78].

Let A_0 be a linear operator in X satisfying

$$(\omega, \infty) \subset \rho(A_0), \quad \sup_{\lambda > \omega} \left\| \lambda(\lambda - A_0)^{-1} \right\| < \infty. \tag{2.4.2}$$

We set

$$F_{A_0} := \left\{ x \in X;\ \overline{\lim}_{\lambda \to +\infty} \left\| \lambda A_0 (\lambda - A_0)^{-1} x \right\| < \infty \right\}.$$

It is easy to verify that F_{A_0} endowed with the norm

$$\|x\|_{F_{A_0}} := \|x\| + \overline{\lim}_{\lambda \to +\infty} \left\| \lambda A_0 (\lambda - A_0)^{-1} x \right\|$$

is a Banach space. When A_0 is the generator of a strongly continuous semigroup $\{T(t)\}_{t \geq 0}$, F_{A_0} coincides with the Favard class of $T(t)$, cf. [26, Proposition 5.12, p. 130].

Theorem 2.4.6. *Let A and $\{S(t)\}_{t \geq 0}$ be as in Proposition 2.2.2, and let $C_0 \in \mathcal{L}(X)$ with $CC_0 = C_0 C$ and $C_0 A \subset AC_0$. Suppose A_0 is a densely defined linear operator in X satisfying (2.4.2), such that $\mathcal{D}(A_0) \subset \mathcal{D}(AC_0)$, $CA_0 \subset A_0 C$, and $(\lambda - A_0)^{-1} A \subset A(\lambda - A_0)^{-1}$ for $\lambda > \omega$. Then*

(i) *(2.3.1) is valid for $B = CC_0 B_0$ if $B_0 \in \mathcal{L}(X)$ and $\mathcal{R}(B_0) \subset F_{A_0}$.*

(ii) *(2.3.10) is valid for $\mathbf{B} = CC_0 \mathbf{B}_0$ if $\mathbf{B}_0$ is a closed linear operator in X, $\mathcal{D}(\mathbf{B}_0) \supset \mathcal{D}(A)$, and $\mathcal{R}(\mathbf{B}_0) \subset F_{A_0}$.*

Proof. Using the density of $\mathcal{D}(A_0)$ and (2.4.2), we have

$$\lim_{\lambda \to +\infty} \lambda(\lambda - A_0)^{-1} x = x, \quad x \in X. \tag{2.4.3}$$

Moreover, by hypothesis,

$$A_0(\lambda - A_0)^{-1}(\mu^n - A)^{-1}C = (\mu^n - A)^{-1}CA_0(\lambda - A_0)^{-1}, \quad \lambda,\ \mu > \omega.$$

In combination with (2.2.3), this shows that for $\lambda > \omega$, $t \geq 0$,

$$A_0(\lambda - A_0)^{-1}S(t) = S(t)A_0(\lambda - A_0)^{-1}, \tag{2.4.4}$$

by the uniqueness theorem for Laplace transforms. Likewise,

$$C_0S(t) = S(t)C_0, \quad t \geq 0. \tag{2.4.5}$$

Let B_0 and B be as in (i). For each $f \in C(R^+, X)$ and $t > 0$, we take a sequence $\{f_m\}_{m\in N} \subset C^1([0,t], X)$ such that

$$\max_{s\in[0,t]} \|f_m(s) - f(s)\| \to 0 \quad \text{as } m \to \infty. \tag{2.4.6}$$

From Theorem 1.4.1, we know

$$\int_0^t S(t-s)C_0B_0f_m(s)ds \in \mathcal{D}(A), \quad m \in N.$$

Therefore, noting $\mathcal{D}(AC_0) \supset \mathcal{D}(A_0)$ and using (2.4.3) – (2.4.5) we obtain

$$\begin{aligned}
&\left\|A\int_0^t S(t-s)C^{-1}Bf_m(s)ds\right\| \\
= \ & \left\|AC_0\int_0^t S(t-s)B_0f_m(s)ds\right\| \\
\leq \ & \left\|AC_0(\omega + 1 - A_0)^{-1}\right\| \\
& \qquad \times \lim_{\lambda\to+\infty}\left\|\int_0^t S(t-s)\left[\lambda(\omega+1-A_0)(\lambda - A_0)^{-1}\right]B_0f_m(s)ds\right\| \\
\leq \ & \widetilde{M}\int_0^t e^{\omega(t-s)}\|B_0f_m(s)\|_{F_{A_0}}ds \\
\leq \ & \widetilde{M}\|B_0\|_{\mathcal{L}(X,F_{A_0})}\int_0^t e^{\omega(t-s)}\|f_m(s)\|ds, \quad m \in N,
\end{aligned}$$

where $\widetilde{M}$ is a constant independent of m and t. This proves part (i) by (2.4.6) and the closedness of A. The same type of argument gives part (ii).

□

Example 2.4.7. Let $X = UC_b(R)$ the space of uniformly continuous and bounded functions,

$$A = i\frac{d^2}{d\xi^2} \quad \text{with} \quad \mathcal{D}(A) = \{f \in C^2(R);\ f \text{ is bounded and } \ f'' \in X\},$$

$$A_0 = \frac{d^2}{d\xi^2} \quad \text{with} \quad \mathcal{D}(A_0) = \mathcal{D}(A).$$

It is known that the operator $D := \frac{d^2}{d\xi^2}$ with domain $W^{2,1}(R)$ generates a strongly continuous semigroup on $L^1(R)$, and A_0 is the generator of its sun dual semigroup on X. Thus F_{A_0} coincides with the domain of the adjoint operator of D (cf. [26, Proposition 5.19, p. 135]). Hence it is not hard to see that

$$F_{A_0} = \{f \in C^1(R);\ f \text{ is bounded and } f' \text{ is Lipschitz continuous}\}.$$

From [49], we see that A generates an exponentially bounded once integrated semigroup, and so generates an exponentially bounded C-regularized semigroup (cf. [19, Theorem 18.3]) for $C := (1 - A_0)^{-1}$. Moreover, define B_0 by

$$(B_0 f)(\xi) = ig(\xi)\int_a^b f(\sigma)d\sigma, \quad f \in X,$$

where $g(\xi) \in F_{A_0}$, and $a,\, b \in R$. Then $B_0 \in \mathcal{L}(X)$ and $\mathcal{R}(B_0) \subset F_{A_0}$. Taking

$$\alpha = 1, \quad \beta = -i, \quad C_0 = I, \quad B = CC_0B_0,$$

we have

$$A(\alpha I + B) + \beta B = A - iB_0.$$

Applying Theorem 2.3.1, Theorem 2.4.6 and Proposition 2.2.2, we conclude that for each

$$\phi \in UC_b(R) \cap C^4(R) \ \text{ with } \ \phi^{(4)} \in UC_b(R)$$

the Cauchy problem

$$\begin{cases} \dfrac{\partial u(t,\xi)}{\partial t} = i\dfrac{\partial^2 u(t,\xi)}{\partial \xi^2} + g(\xi)\displaystyle\int_a^b u(t,\sigma)d\sigma, \quad t \geq 0,\ \xi \in R, \\ u(0,\xi) = \phi(\xi), \quad \xi \in R, \end{cases}$$

has a unique solution in $C^1(R^+, UC_b(R))$.

Remark 2.4.8. Given an exponentially bounded C-regularized semigroup $\{S(t)\}_{t\geq 0}$, one can define the Favard class of $\{S(t)\}_{t\geq 0}$ similarly as in the case of strongly continuous semigroups (see, e.g., [22], [26, Section III.3]) by

$$Fav(S(t)) := \left\{ x \in X;\ \sup_{t>0} \left\| \frac{1}{t}(S(t)x - x) \right\| < \infty \right\}.$$

Then one proves that (2.3.1) holds for $B \in \mathcal{L}(X)$ with

$$\mathcal{R}(B) \subset C^2(Fav(S(t)), \tag{2.4.7}$$

and that (2.3.10) holds for a closed linear operator $\mathbf{B}$ in X with

$$\mathcal{D}(\mathbf{B}) \supset \mathcal{D}(A) \quad \text{and} \quad \mathcal{R}(\mathbf{B}) \subset C^2(Fav(S(t)).$$

When $\{S(t)\}_{t\geq 0}$ is a strongly continuous semigroup, this result is essentially Theorem 2.4.6.

In Example 2.4.7, the permissible space for $\mathcal{R}(B)$ can be large as

$$C(F_{A_0}) = \{f \in C^3(R);\ f \ \text{ is bounded and } f''' \text{ is Lipschitz continuous}\}.$$

However, if either (2.4.7) or (2.4.1) were used, the range of B would be restricted to a set which is smaller than or equal to

$$\mathcal{R}(C^2) = \{f \in C^4(R);\ f \text{ is bounded and } f^{(4)} \in UC_b(R)\}.$$

It is clear that in this case $C(F_{A_0})$ strictly contains $\mathcal{R}(C^2)$. This reflects the feature of Theorem 2.4.6 on which Example 2.4.7 was based.

Chapter 3

Wave equations with generalized Wentzell boundary conditions

3.1 Summary

In this chapter we solve an open problem put forward by A. Favini, G. R. Goldstein, J. A. Goldstein and S. Romanelli [34], concerning the mixed problem for wave equations with generalized Wentzell boundary conditions. As a consequence, we also develop the previous wellposedness result regarding the mixed problem for heat equations with generalized Wentzell boundary conditions.

3.2 Introduction

Of concern is the following wave equation with generalized Wentzell boundary conditions on $[0,1]$.

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le 1,\ t \in R, \\ u(x,0) = f(x),\ \dfrac{\partial u}{\partial t}(x,0) = g(x), \quad 0 \le x \le 1, \\ \dfrac{\partial^2 u}{\partial x^2}(j,t) + \beta_j \dfrac{\partial u}{\partial x}(j,t) + \gamma_j u(j,t) = 0, \quad j = 0,1,\ t \in R, \end{cases} \tag{3.2.1}$$

where $c > 0$, and β_j, γ_j $(j = 0,1)$ are scalar coefficients. For the parabolic problem of first order in time involving (generalized) Wentzell boundary conditions, A. Favini, G. R. Goldstein, J. A. Goldstein and S. Romanelli have recently made a systematic study and established a series of significant theorems (see [33–35] and references therein). However, just as pointed out by them ([34]), the case of (generalized) Wentzell boundary conditions for the wave equation is much trickier. So far, only for the case of $\beta_j = \gamma_j = 0$ ($j = 0, 1$), problem (3.2.1) has been shown to be wellposed in $C[0,1]$ (see [34, Theorem 2.1]). On the other hand, one knows that the corresponding first order problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le 1, \ t \ge 0, \\ u(x,0) = f(x), \quad 0 \le x \le 1, \\ \dfrac{\partial^2 u}{\partial x^2}(j,t) + \beta_j \dfrac{\partial u}{\partial x}(j,t) + \gamma_j u(j,t) = 0, \quad j = 0, 1, \ t \ge 0, \end{cases} \tag{3.2.2}$$

is wellposed in $C[0,1]$ whenever

$$\gamma_0, \gamma_1 \ge 0, \quad \beta_1 > 0 > \beta_0 \tag{3.2.3}$$

(see [33, Theorem 1.1]). In [34], A. Favini, G. R. Goldstein, J. A. Goldstein and S. Romanelli put forward an attractive problem:

Is the mixed problem (3.2.1) wellposed in $C[0,1]$ in the case of (3.2.3)?

It is a challenging question since so little is known about wave equations with generalized Wentzell boundary conditions, and the methods in [33] and [34] appear to be no longer applicable to the new situation. The importance of the problem in theory and application makes it worthwhile to study it.

The present chapter aims at solving this open problem. In fact, we shall prove the wellposedness of (3.2.1) without any restrictions on the *complex numbers* β_0, β_1, γ_0 and γ_1. As a byproduct, we develop the previous wellposedness result about (3.2.2) and show that the semigroup governing (3.2.2) is analytic in the *right half plane*. Our approach depends on a delicate analysis of certain operator matrices. Such types of operator matrices have been considered before for problems of first order in time (see, e.g., [3, 9, 35]).

We write $C^i[0,1]$, $i = 0, 1, 2$, for the space of all i-times continuously differentiable complex valued functions on $[0,1]$ endowed with the norm

$$\|f\|_{C^i[0,1]} = \sum_{k=0}^{i} \max_{x \in [0,1]} \|f^{(k)}(x)\|,$$

and $C[0,1] := C^0[0,1]$.

We define a linear operator A in $C[0,1]$ by

$$\begin{cases} (Af)(x) = c^2 f''(x) \quad 0 \le x \le 1, \\ \mathcal{D}(A) := \{f \in C^2[0,1]; \quad f''(j) + \beta_j f'(j) + \gamma_j f(j) = 0 \ \text{ at } \ j = 0,1\}. \end{cases} \tag{3.2.4}$$

Then (3.2.1) and (3.2.2) are realized in the space $C[0,1]$ as, respectively,

$$\begin{cases} u''(t) = Au(t), \quad t \in R, \\ u(0) = f, \ u'(0) = g, \end{cases} \tag{ACP_2}$$

and

$$\begin{cases} u'(t) = Au(t), \quad t \ge 0, \\ u(0) = f. \end{cases} \tag{ACP_1}$$

For an arbitrary linear operator A in a Banach space X, we recall the following definitions (cf., e.g., [26, 28, 31, 38, 68, 84]).

Definition 3.2.1. (ACP_1) is called *wellposed* in X if $\mathcal{D}(A)$ is dense in X, (ACP_1) has a unique solution for each $f \in \mathcal{D}(A)$, and there exists a locally bounded positive function $M(t)$ satisfying

$$\|u(t)\| \le M(t)\|u(0)\|, \quad t \ge 0,$$

for any solution $u(t)$ of (ACP_1).

Definition 3.2.2. (ACP_2) is called *wellposed* in X if $\mathcal{D}(A)$ is dense in X, (ACP_2) has a unique solution for every $f,\ g \in \mathcal{D}(A)$, and there exists a locally bounded positive function $M(t)$ satisfying

$$\|u(t)\| \le M(t)(\|u(0)\| + \|u'(0)\|), \quad t \in R,$$

for any solution $u(t)$ of (ACP_2).

Definition 3.2.3. A family of operators $\{C(t)\}_{t\in R} \subset \mathcal{L}(X)$ is called a *strongly continuous cosine function* on X if

(i) $C(0) = I$,

(ii) $C(t+s) + C(t-s) = 2C(t)C(s)$ for all $t,\ s \in R$,

(iii) $C(\cdot)f$ is continuous on R for each $f \in X$.

The operator A defined by

$$\mathcal{D}(A) := \left\{x \in X;\ \lim_{t\to 0^+} \frac{2}{t^2}(C(t)-I)x \text{ exists}\right\},$$

$$Ax := \lim_{t\to 0^+} \frac{2}{t^2}(C(t)-I)x \quad \text{for } x \in \mathcal{D}(A)$$

is called the *generator* of the strongly continuous cosine function $\{C(t)\}_{t\in R}$.

3.3 Wellposedness of (ACP_2) and (ACP_1)

Throughout this section, we assume that $c > 0$, β_j, $\gamma_j \in \mathbf{C}$ $(j = 0, 1)$, and the operator A is as in (3.2.4).

Theorem 3.3.1. *(ACP_2) is wellposed in $C[0,1]$.*

Proof. Set

$$A_{02} = c^2 \frac{d^2}{dx^2}, \quad \mathcal{D}(A_{02}) = H^2(0,1) \cap H_0^1(0,1),$$

$$A_{0c} = c^2 \frac{d^2}{dx^2}, \quad \mathcal{D}(A_{0c}) = \{u \in C^2[0,1],\ u(0) = u(1) = 0\}.$$

It is known from d'Alembert's formula that the operator family $\{C_{02}(t)\}_{t\in R}$ given by

$$(C_{02}(t)f)(x) = \frac{1}{2}\left[\widetilde{f}(x+ct) + \widetilde{f}(x-ct)\right], \quad f \in L^2(0,1),$$

is the strongly continuous cosine function on $L^2(0,1)$ generated by A_{02}. Here and in the sequel we always define

$$\widetilde{f}(\xi) := \begin{cases} f(\xi), & \xi \in [0,1), \\ -f(-\xi), & \xi \in [-1,0), \end{cases}$$

$$\widetilde{f}(\xi \pm 2n) := \widetilde{f}(\xi), \quad \xi \in [-1,1),\ n \in N.$$

Then $\{S_{02}(t)\}_{t\in R}$ given by

$$(S_{02}(t)f)(x) = \frac{1}{2c}\int_{x-ct}^{x+ct} \widetilde{f}(\sigma)d\sigma, \quad f \in L^2(0,1), \tag{3.3.1}$$

is the corresponding sine function, and so for $\lambda > 0$,

$$\left(\lambda^2 - A_{02}\right)^{-1} f = \int_0^\infty e^{-\lambda t} S_{02}(t) f dt, \quad f \in L^2(0,1). \tag{3.3.2}$$

Clearly the restrictions

$$S_{0c}(t) := S_{02}(t)\big|_{C[0,1]}, \quad t \in R, \tag{3.3.3}$$

leave $C[0,1]$ invariant and form a strongly continuous $\mathcal{L}(C[0,1])$-valued function satisfying

$$\|S_{0c}(t)\|_{\mathcal{L}(C[0,1])} \le |t|, \quad t \in R. \tag{3.3.4}$$

Therefore we have by (3.3.2)

$$\left(\lambda^2 - A_{0c}\right)^{-1} f = \int_0^\infty e^{-\lambda t} S_{0c}(t) f dt \quad \text{for } \lambda > 0 \text{ and } f \in C[0,1]. \tag{3.3.5}$$

Next we define linear operators $A_c : C^2[0,1] \subset C[0,1] \to C[0,1]$ and $P : C[0,1] \to \mathbf{C}^2$ by

$$(A_c f)(x) := c^2 f''(x), \quad Pf := \begin{pmatrix} f(0) \\ f(1) \end{pmatrix}.$$

It is easy to see that

$$P(\mathcal{D}(A_c)) = P(C^2[0,1]) = \mathbf{C}^2. \tag{3.3.6}$$

Noting that $A_c \in \mathcal{L}(C^2[0,1], C[0,1])$ and $P\big|_{C^2[0,1]} \in \mathcal{L}(C^2[0,1], \mathbf{C}^2)$, we deduce as in [40, Lemma 1.2] that for any $\lambda \in \rho(A_{0c})$, the restriction $P\big|_{\ker(\lambda - A_c)}$ is an isomorphism of $\left(\ker(\lambda - A_c),\ \|\cdot\|_{C^2[0,1]}\right)$ onto $\mathbf{C}^2$ because of (3.3.6). Write

$$D_\lambda = \left(P\big|_{\ker(\lambda - A_c)}\right)^{-1}, \quad \lambda \in \rho(A_{0c}),$$

which is called the *Dirichlet operator*. We then have

$$D_\lambda \in \mathcal{L}(\mathbf{C}^2, C^2[0,1]). \tag{3.3.7}$$

Moreover, it follows as in [40, Lemma 1.3] that

$$D_\lambda = D_\mu - (\lambda - \mu)\,(\lambda - A_{0c})^{-1}\,D_\mu, \quad \lambda,\ \mu \in \rho(A_{0c}). \tag{3.3.8}$$

Let us now consider the operator matrix

$$\mathbb{A}_0 := \begin{pmatrix} A_c & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{D}\left(\mathbb{A}_0\right) := \left\{ \begin{pmatrix} f \\ z \end{pmatrix} \in C^2[0,1] \times \mathbf{C}^2; \quad Pf = z \right\}$$

on the product space $\mathcal{E} := C[0,1] \times \mathbf{C}^2$. As in [9, Lemma 2.6 and Theorem 2.7 (i)], one has $\rho(A_{0c}) \setminus \{0\} \subset \rho(\mathbb{A}_0)$, and for $\lambda > 0$,

$$\lambda^2 - \mathbb{A}_0 = \begin{pmatrix} \lambda^2 - A_{0c} & 0 \\ 0 & \lambda^2 \end{pmatrix} \begin{pmatrix} I & -D_{\lambda^2} \\ 0 & I \end{pmatrix},$$

$$\begin{aligned} \left(\lambda^2 - \mathbb{A}_0\right)^{-1} &= \begin{pmatrix} \left(\lambda^2 - A_{0c}\right)^{-1} & \lambda^{-2} D_{\lambda^2} \\ 0 & \lambda^{-2} \end{pmatrix} \\ &= \begin{pmatrix} \left(\lambda^2 - A_{0c}\right)^{-1} & \lambda^{-2} D_0 - \left(\lambda^2 - A_{0c}\right)^{-1} D_0 \\ 0 & \lambda^{-2} \end{pmatrix} \end{aligned} \tag{3.3.9}$$

by (3.3.8). So by (3.3.5)

$$\left(\lambda^2 - \mathbb{A}_0\right)^{-1} \begin{pmatrix} f \\ z \end{pmatrix} = \int_0^\infty e^{-\lambda t} \mathbb{S}_0(t) \begin{pmatrix} f \\ z \end{pmatrix} dt \quad \text{for } \lambda > 0,\ \begin{pmatrix} f \\ z \end{pmatrix} \in \mathcal{E}, \tag{3.3.10}$$

where

$$\mathbb{S}_0(t) := \begin{pmatrix} S_{0c}(t) & tD_0 - S_{0c}(t) D_0 \\ 0 & t \end{pmatrix}, \quad t \in R. \tag{3.3.11}$$

Obviously, there is a constant $b_0 > 0$ such that

$$\|\mathbb{S}_0(t)\|_{\mathcal{L}(\mathcal{E})} \le b_0 |t|, \quad t \in R, \tag{3.3.12}$$

by (3.3.4). Define

$$Gf := -c^2 \begin{pmatrix} \beta_0 f'(0) + \gamma_0 f(0) \\ \beta_1 f'(1) + \gamma_1 f(1) \end{pmatrix}, \quad f \in \mathcal{D}(G) := C^1[0,1],$$

$$\mathbb{G} := \begin{pmatrix} 0 & 0 \\ G & 0 \end{pmatrix}, \quad \mathcal{D}(\mathbb{G}) := C^1[0,1] \times \mathbf{C}^2.$$

We then have by (3.3.9)

$$\mathbb{G}\left(\lambda^2 - \mathbb{A}_0\right)^{-1} = \begin{pmatrix} 0 & 0 \\ G\left(\lambda^2 - A_{0c}\right)^{-1} & \lambda^{-2} G D_0 - G\left(\lambda^2 - A_{0c}\right)^{-1} D_0 \end{pmatrix} \quad \text{for } \lambda > 0. \tag{3.3.13}$$

Since $G \in \mathcal{L}\left(C^1[0,1], \mathbf{C}^2\right)$ we have $G A_{0c}^{-1} \in \mathcal{L}\left(C[0,1], \mathbf{C}^2\right)$. It follows from (3.3.5) that for $f \in \mathcal{D}(A_{0c})$, $\lambda > 0$,

$$\begin{aligned} G\left(\lambda^2 - A_{0c}\right)^{-1} f &= G \int_0^\infty e^{-\lambda t} S_{0c}(t) f dt \\ &= \left(G A_{0c}^{-1}\right) \int_0^\infty e^{-\lambda t} S_{0c}(t) A_{0c} f dt \\ &= \int_0^\infty \left(G A_{0c}^{-1}\right) e^{-\lambda t} S_{0c}(t) A_{0c} f dt \\ &= \int_0^\infty e^{-\lambda t} G S_{0c}(t) f dt. \end{aligned} \tag{3.3.14}$$

Using (3.3.1) and (3.3.3), we obtain

$$G S_{0c}(t) f = -c \begin{pmatrix} \beta_0 \widetilde{f}(ct) \\ \beta_1 \widetilde{f}(1+ct) \end{pmatrix}, \quad f \in \mathcal{D}(A_{0c}),\ t \in R.$$

Writing

$$H(t) f := -c \begin{pmatrix} \beta_0 \widetilde{f}(ct) \\ \beta_1 \widetilde{f}(1+ct) \end{pmatrix}, \quad f \in C[0,1],\ t \in R, \tag{3.3.15}$$

we see from (3.3.14) that

$$G\left(\lambda^2 - A_{0c}\right)^{-1} f = \int_0^\infty e^{-\lambda t} H(t) f dt, \quad f \in \mathcal{D}(A_{0c}),\ \lambda > 0. \tag{3.3.16}$$

Given $f \in C[0,1]$, there exists a sequence $\{f_n\}_{n \in N} \subset \mathcal{D}(A_{0c})$ such that

$$\sup_{n \in N} \|f_n\|_{C[0,1]} < \infty,$$

and

$$\|f_n - f\|_{L^2(0,1)} \to 0, \quad f_n \to f \ \text{ a.e. in } \ [0,1]$$

as $n \to \infty$. Hence we deduce that for $\lambda > 0$,

$$\begin{aligned}
\lim_{n\to\infty} G\left(\lambda^2 - A_{0c}\right)^{-1} f_n &= \lim_{n\to\infty} G\left(\lambda^2 - A_{02}\right)^{-1} f_n \\
&= G\left(\lambda^2 - A_{02}\right)^{-1} f \qquad (3.3.17)\\
&= G\left(\lambda^2 - A_{0c}\right)^{-1} f,
\end{aligned}$$

where we observe that $G \in \mathcal{L}\left(C^1[0,1], \mathbf{C}^2\right)$ and $\left(\lambda^2 - A_{02}\right)^{-1} \in \mathcal{L}\left(L^2[0,1], C^1[0,1]\right)$. The latter can be derived from the fact

$$\left(\lambda^2 - A_{02}\right)^{-1} : L^2[0,1] \to H^2(0,1) \subset C^1[0,1]$$

with the aid of the closed graph theorem. On the other hand, we find from (3.3.15) that

$$\|H(t)(f_n - f)\|_{\mathbf{C}^2} \leq \text{const} \quad \text{for all } t \geq 0 \text{ and } n \in N,$$

and

$$\lim_{n\to\infty} H(t)(f_n - f) = 0 \quad \text{for almost all } t \in [0,1].$$

According to the dominated convergence theorem, this yields

$$\lim_{n\to\infty} \int_0^\infty e^{-\lambda t} H(t)(f_n - f)dt = 0, \quad \lambda > 0.$$

This, together with (3.3.17), shows that (3.3.16) holds for all $f \in C[0,1]$. Thus, in view of (3.3.13) we obtain

$$\begin{aligned}
\mathbb{G}\left(\lambda^2 - \mathbb{A}_0\right)^{-1} \begin{pmatrix} f \\ z \end{pmatrix} &= \int_0^\infty e^{-\lambda t} \begin{pmatrix} 0 & 0 \\ H(t) & tGD_0 - H(t)D_0 \end{pmatrix} \begin{pmatrix} f \\ z \end{pmatrix} dt \qquad (3.3.18)\\
&\quad \text{for } \lambda > 0 \text{ and } \begin{pmatrix} f \\ z \end{pmatrix} \in \mathcal{E}.
\end{aligned}$$

Clearly

$$\mathbb{H}_0(t) := \begin{pmatrix} 0 & 0 \\ H(t) & tGD_0 - H(t)D_0 \end{pmatrix} \qquad (3.3.19)$$

by (3.3.15) is strongly continuous in $t \in R \setminus \{0, \pm c^{-1}, \pm 2c^{-1}, \ldots\}$, and there is a constant $b_1 > 0$ such that

$$\|\mathbb{H}_0(t)\|_{\mathcal{L}(\mathcal{E})} \leq b_1(|t| + 1), \quad t \in R.$$

Thus we have

$$\|[\mathbb{H}_0(t)]^{*m}\|_{\mathcal{L}(\mathcal{E})} \leq b_1^m e^t \frac{t^{m-1}}{(m-1)!}, \quad t \geq 0,\ m \in N,$$

where $*m$ indicates the mth convolution power. This means that

$$\mathbb{H}(t) := \sum_{m=1}^{\infty} [\mathbb{H}_0(t)]^{*m}, \quad t \geq 0,$$

defines a strongly continuous $\mathcal{L}(\mathcal{E})$-valued function on $(0,\infty) \setminus \{c^{-1}, 2c^{-1}, \dots\}$ satisfying

$$\|\mathbb{H}(t)\| \leq \sum_{m=1}^{\infty} b_1^m e^t \frac{t^{m-1}}{(m-1)!} = b_1 e^{(b_1+1)t}, \quad t \geq 0. \tag{3.3.20}$$

We hence infer from (3.3.18) that for $\begin{pmatrix} f \\ z \end{pmatrix} \in \mathcal{E}$, $\lambda > b_1 + 1$,

$$\begin{aligned}
\left[I - \mathbb{G}\left(\lambda^2 - \mathbb{A}_0\right)^{-1}\right]^{-1} \begin{pmatrix} f \\ z \end{pmatrix} - \begin{pmatrix} f \\ z \end{pmatrix} &= \sum_{m=1}^{\infty} \left[\mathbb{G}\left(\lambda^2 - \mathbb{A}_0\right)^{-1}\right]^m \begin{pmatrix} f \\ z \end{pmatrix} \\
&= \sum_{m=1}^{\infty} \int_0^{\infty} e^{-\lambda t} [\mathbb{H}_0(t)]^{*m} \begin{pmatrix} f \\ z \end{pmatrix} dt \\
&= \int_0^{\infty} e^{-\lambda t} \mathbb{H}(t) \begin{pmatrix} f \\ z \end{pmatrix} dt.
\end{aligned}$$

In combination with (3.3.10), this yields that $\lambda^2 \in \rho(\mathbb{A}_0 + \mathbb{G})$ for $\lambda > b_1 + 1$ and

$$\begin{aligned}
&\left(\lambda^2 - \mathbb{A}_0 - \mathbb{G}\right)^{-1} \begin{pmatrix} f \\ z \end{pmatrix} \\
=\ & \left(\lambda^2 - \mathbb{A}_0\right)^{-1} \begin{pmatrix} f \\ z \end{pmatrix} + \left(\lambda^2 - \mathbb{A}_0\right)^{-1} \left\{\left[I - \mathbb{G}\left(\lambda^2 - \mathbb{A}_0\right)^{-1}\right]^{-1} - I\right\} \begin{pmatrix} f \\ z \end{pmatrix} \\
=\ & \int_0^{\infty} e^{-\lambda t} \mathbb{S}_0(t) \begin{pmatrix} f \\ z \end{pmatrix} dt + \int_0^{\infty} e^{-\lambda t} (\mathbb{S}_0 * \mathbb{H})(t) \begin{pmatrix} f \\ z \end{pmatrix} dt
\end{aligned}$$

for $\lambda > b_1 + 1$ and $\begin{pmatrix} f \\ z \end{pmatrix} \in \mathcal{E}$.

We know from (3.3.1) and (3.3.3) that

$$\|S_{0c}(t) - S_{0c}(s)\|_{\mathcal{L}(C[0,1])} \leq |t-s|, \quad t,\ s \geq 0,$$

which implies, by (3.3.11), the existence of a constant b_2 such that

$$\|\mathbb{S}_0(t) - \mathbb{S}_0(s)\|_{\mathcal{L}(\mathcal{E})} \leq b_2|t-s|, \quad t,\ s \geq 0.$$

Therefore,

$$\begin{aligned}
&\left\| [(\mathbb{S}_0 * \mathbb{H})(t) - (\mathbb{S}_0 * \mathbb{H})(s)] \begin{pmatrix} f \\ z \end{pmatrix} \right\|_{\mathcal{E}} \\
\leq\ & \left\| \int_0^s [\mathbb{S}_0(t-\tau) - \mathbb{S}_0(s-\tau)]\mathbb{H}(\tau) \begin{pmatrix} f \\ z \end{pmatrix} d\tau \right\|_{\mathcal{E}} + \left\| \int_s^t \mathbb{S}_0(t-\tau)\mathbb{H}(\tau) \begin{pmatrix} f \\ z \end{pmatrix} d\tau \right\|_{\mathcal{E}} \\
\leq\ & b_1(b_0+b_2)(t-s)e^{(b_1+2)t} \left\| \begin{pmatrix} f \\ z \end{pmatrix} \right\|_{\mathcal{E}}, \quad \begin{pmatrix} f \\ z \end{pmatrix} \in \mathcal{E},\ t \geq s \geq 0,
\end{aligned}$$

by the use of (3.3.12) and (3.3.20). Defining $\mathbb{S}_1(t) := \mathbb{S}_0(t) + (\mathbb{S}_0 * \mathbb{H})(t)$ $(t \geq 0)$ and summing up the arguments above, we obtain $\lambda^2 \in \rho(\mathbb{A}_0 + \mathbb{G})$ for $\lambda > b_1 + 3$ and

$$\left(\lambda^2 - \mathbb{A}_0 - \mathbb{G}\right)^{-1} \begin{pmatrix} f \\ z \end{pmatrix} = \int_0^\infty e^{-\lambda t}\mathbb{S}_1(t) \begin{pmatrix} f \\ z \end{pmatrix} dt \quad \text{for } \lambda > b_1 + 3 \text{ and } \begin{pmatrix} f \\ z \end{pmatrix} \in \mathcal{E}. \tag{3.3.21}$$

In addition, $\mathbb{S}_1(\cdot)$ is an $\mathcal{L}(\mathcal{E})$-valued function, strongly continuous on $[0,\infty)$, satisfying

$$\|\mathbb{S}_1(t) - \mathbb{S}_1(s)\|_{\mathcal{L}(\mathcal{E})} \leq b(t-s)e^{(b_1+2)t}, \quad t \geq s \geq 0, \tag{3.3.22}$$

for a constant $b > 0$. We now denote by $\mathbb{A}$ the part of $\mathbb{A}_0 + \mathbb{G}$ in the closure

$$\mathcal{E}_1 := \overline{\mathcal{D}(\mathbb{A}_0 + \mathbb{G})} = \overline{\mathcal{D}(\mathbb{A}_0)}.$$

It is not hard to see that

$$\mathcal{E}_1 = \left\{ \begin{pmatrix} f \\ z \end{pmatrix} \in C[0,1] \times \mathbf{C}^2;\ Pf = z \right\},$$

and that, by (3.3.21), $((b_1+3)^2, \infty) \in \rho(\mathbb{A})$ and all operators

$$\mathbb{S}(t) := \mathbb{S}_1(t)\big|_{\mathcal{E}_1}, \quad t \geq 0,$$

leave $\mathcal{E}_1$ invariant. Hence, it follows from (3.3.21) and (3.3.22) that $\mathbb{S}(t)$ is strongly continuously differentiable in $t \geq 0$, and $\mathbb{C}(t) := \mathbb{S}'(t)$ satisfies

$$\|\mathbb{C}(t)\|_{\mathcal{L}(\mathcal{E}_1)} \leq be^{(b_1+2)t}, \quad t \geq 0, \tag{3.3.23}$$

$$\lambda \left(\lambda^2 - \mathbb{A}\right)^{-1} \begin{pmatrix} f \\ z \end{pmatrix} = \int_0^\infty e^{-\lambda t}\mathbb{C}(t) \begin{pmatrix} f \\ z \end{pmatrix} dt, \quad \begin{pmatrix} f \\ z \end{pmatrix} \in \mathcal{E}_1,\ \lambda > b_1 + 3. \tag{3.3.24}$$

Here for the strong differentiability of $\mathbb{S}(t)$ we used first the equality

$$\mathbb{S}(t) \begin{pmatrix} f \\ z \end{pmatrix} = t \begin{pmatrix} f \\ z \end{pmatrix} + \int_0^t (t-s)\mathbb{S}(s)(\mathbb{A}_0 + \mathbb{G}) \begin{pmatrix} f \\ z \end{pmatrix} ds, \quad t \geq 0,\ \begin{pmatrix} f \\ z \end{pmatrix} \in \mathcal{D}(\mathbb{A}_0),$$

derived from (3.3.21) by the uniqueness theorem of Laplace transforms, to ensure the differentiability of $\mathbb{S}(t) \begin{pmatrix} f \\ z \end{pmatrix}$ for $\begin{pmatrix} f \\ z \end{pmatrix} \in \mathcal{D}(\mathbb{A}_0)$, and then relied on a density argument according to the estimate

$$\left\| \mathbb{S}'(t) \begin{pmatrix} f \\ z \end{pmatrix} \right\|_{\mathcal{E}_1} \leq be^{(b_1+2)t} \left\| \begin{pmatrix} f \\ z \end{pmatrix} \right\|_{\mathcal{E}_1}, \quad t \geq 0,\ \begin{pmatrix} f \\ z \end{pmatrix} \in \mathcal{D}(\mathbb{A}_0)$$

deduced from (3.3.22). Finally, we look at the operator A defined in (3.2.4) and observe that $f \in \mathcal{D}(A)$ if and only if

$$f \in C^2[0,1], \quad \text{and} \quad PA_c f = Gf$$

if and only if

$$f \in C^2[0,1], \quad \text{and} \quad (\mathbb{A}_0 + \mathbb{G}) \begin{pmatrix} f \\ Pf \end{pmatrix} \in \mathcal{E}_1$$

if and only if $\begin{pmatrix} f \\ Pf \end{pmatrix} \in \mathcal{D}(\mathbb{A})$. Therefore, $f \in \mathcal{D}(A)$ and $(\lambda^2 - A)f = g$ if and only if $\begin{pmatrix} f \\ Pf \end{pmatrix} \in \mathcal{D}(\mathbb{A})$, and

$$(\lambda^2 - \mathbb{A}) \begin{pmatrix} f \\ Pf \end{pmatrix} = \begin{pmatrix} (\lambda^2 - A_c)f \\ -Gf + \lambda^2 Pf \end{pmatrix} = \begin{pmatrix} g \\ Pg \end{pmatrix}.$$

Accordingly it follows from (3.3.24) that $\lambda^2 \in \rho(A)$ for $\lambda > b_1 + 3$ and

$$\begin{aligned} \lambda(\lambda^2 - A)^{-1}g &= \pi_1\left(\lambda(\lambda^2 - \mathbb{A})^{-1}\begin{pmatrix} g \\ Pg \end{pmatrix}\right) \\ &= \pi_1\left(\int_0^\infty e^{-\lambda t}\mathbb{C}(t)\begin{pmatrix} g \\ Pg \end{pmatrix} dt\right) \\ &= \int_0^\infty e^{-\lambda t}C(t)g dt, \quad \text{for } \lambda > b_1 + 3 \text{ and } \ g \in C[0,1], \end{aligned} \tag{3.3.25}$$

where π_1 is the projection from $C[0,1] \times \mathbf{C}^2$ onto $C[0,1]$ and

$$C(t)g := \pi_1\left(\mathbb{C}(t)\begin{pmatrix} g \\ Pg \end{pmatrix}\right), \quad t \geq 0, \ g \in C[0,1].$$

Clearly $C(\cdot)$ is a strongly continuous $\mathcal{L}(C[0,1])$-valued function satisfying

$$\|C(t)\|_{C[0,1]} \leq 2be^{(b_1+2)t}, \quad t \geq 0,$$

by (3.3.23). This and (3.3.25) enable us to conclude that $\{C(t)\}_{t\in R}$ with $C(t) := C(-t)$ for $t < 0$ is a strongly continuous cosine function on $C[0,1]$ generated by A (cf. [84, Lemma 4.2, p. 181]). So the wellposedness of (ACP_2) follows immediately in view of [28, Theorem 5.9] (see also, e.g., [31, Chapter II] and [38, Theorem 8.2, p. 118]). The proof is then complete.

□

Corollary 3.3.2. *The operator A in* (3.2.4) *is the generator of a strongly continuous cosine function on $C[0,1]$.*

Proof. This has been established in the final part of the proof of Theorem 3.3.1.

□

Corollary 3.3.3. *The operator A in* (3.2.4) *is the generator of a strongly continuous analytic semigroup on $C[0,1]$ of angle $\frac{\pi}{2}$.*

Proof. Using Corollary 3.3.2 and Romanov's formula ([73]; see also, e.g., [38, Theorem 8.7, p. 120]), we obtain the result.

Corollary 3.3.4. *(ACP_1) is wellposed in $C[0,1]$ for the above operator A .*

Proof. It follows from a direct application of a classical theorem due to Hille [11] and Phillips [12] (see also, e.g., [26, Corollary 6.9, p. 151]), [31, Chapter I]) and [38, Theorem 1.2, p. 83])) since by Corollary 3.3.3 A generates a strongly continuous semigroup on $C[0,1]$.

□

Remark 3.3.5. By a perturbation argument and a similarity transformation as in [26, Chapter VI, Section 4b], we could see that the conclusions of Corollaries 3.3.3 and 3.3.4 are also true for A being a general nondegenerate second order differential operator with generalized Wentzell boundary conditions. This fact is also an immediate consequence of the main result in [56] where the authors treat directly nonautonomous heat equations with generalized Wentzell boundary conditions.

Chapter 4

The mixed problem for time dependent heat equations with generalized Wentzell boundary conditions

4.1 Summary

In this chapter, we study the nonautonomous heat equation in $C[0,1]$ with generalized Wentzell boundary conditions. It is shown, under appropriate assumptions, that there exists a unique evolution family for this problem and that the family satisfies various regularity properties. This enables us to obtain, for the corresponding inhomogeneous problem, classical and strict solutions having optimal regularity.

4.2 Introduction

For second order differential operators

$$\mathcal{A}(x,t) = a(x,t)\frac{d^2}{dx^2} + q(x,t)\frac{d}{dx} + r(x,t), \quad x \in [0,1],\ t \in [0,T],$$

where $a(x,t) > 0$ ($x \in [0,1]$, $t \in [0,T]$), we consider the following time dependent heat equation with a generalized Wentzell boundary condition.

$$\begin{cases} \dfrac{\partial u}{\partial t} = \mathcal{A}(x,t)u, \quad 0 \le x \le 1,\ 0 \le s < t \le T, \\ u(x,s) = f(x), \quad 0 \le x \le 1, \\ \mathcal{A}(j,t)u(j,t) + \beta_j(t)\dfrac{\partial u}{\partial x}(j,t) + \gamma_j(t)u(j,t) = 0, \quad j = 0,1,\ 0 \le s < t \le T. \end{cases} \tag{4.2.1}$$

To the equation we associate the nonautonomous abstract Cauchy problem

$$\begin{cases} u'(t) = A(t)u(t), \quad 0 \le s < t \le T, \\ u(s) = f, \end{cases} \tag{$NACP$}$$

in the Banach space $C[0,1]$, where the operators $A(t)$ are defined by

$$\begin{cases} (A(t)f)(x) = \mathcal{A}(x,t)f(x), \quad 0 \le x \le 1, \\ \mathcal{D}(A(t)) := \{f \in C^2[0,1]; \quad \mathcal{A}(j,t)f(j) + \beta_j(t)f'(j) + \gamma_j(t)f(j) = 0 \ \text{ at } \ j = 0,1\}. \end{cases} \tag{4.2.2}$$

Moreover, we assume for the coefficients that

$$\begin{gathered} a,\ q,\ r \in C^\alpha([0,T];C[0,1]), \quad \beta_j,\ \gamma_j \in C^\alpha([0,T];\mathbf{C}) \\ \text{for some } \alpha \in (0,1),\ j = 0,1, \end{gathered} \tag{4.2.3}$$

where $C^\alpha([0,T];X)$ (for a Banach space X) is the Banach space of Hölder continuous functions on $q : [0,T] \to X$ with exponent α and norm given by

$$\sup_{0 \le t \le T} \|q(t)\|_X + \sup_{0 \le s < t \le T} (t-s)^{-\alpha}\|q(t) - q(s)\|_X.$$

With these assumptions we will show the wellposedness of $(NACP)$ and (4.2.1).

The first result about the wellposedness of problem (4.2.1) with Robin boundary conditions (i.e. with $\mathcal{A}$ in the third line of (4.2.1) replaced by zero) was established in 1956 by T. Kato using C_0-semigroups ([47]). Later, T. Kato and H. Tanabe [48] sharpened this result using analytic semigroups. Recently, the wellposedness for the autonomous version of (4.2.1) and $(NACP)$ has been studied by A. Favini, G. R. Goldstein, J. A. Goldstein and S. Romanelli ([33] and [35]), and most recently, the analyticity of the corresponding semigroup was also shown (see [25], [80], [86]). Using the known facts about the autonomous problem, one is faced in the nonautonomous case with difficulties caused by the variable domains $\mathcal{D}(A(t))$. So we will use suitable operator matrices on a product space to avoid this problem. Such matrices appeared before in abstract form as operator matrices with non-diagonal domain in [63] or as one-sided coupled operator matrices in [24]. Such operator matrices were also used in [3], [9], [35] and [86].

We now define, for $t \in [0,T]$, linear operators $A_c(t) : C^2[0,1] \subset C[0,1] \to C[0,1]$ by

$$(A_c(t)f)(x) := \mathcal{A}(x,t)f(x),$$

and linear operators $Q(t) : C^1[0,1] \subset C[0,1] \to \mathbf{C}^2$ by

$$Q(t)f := -\begin{pmatrix} \beta_0(t)f'(0) + \gamma_0(t)f(0) \\ \beta_1(t)f'(1) + \gamma_1(t)f(1) \end{pmatrix}.$$

The restriction of $A_c(t)$ to the subspace $\{f \in C^2[0,1]; \quad f(0) = f(1) = 0\}$ is

$$A_{0c}(t) := A_c(t)\big|_{C_0[0,1]\cap C^2[0,1]}, \quad t \in [0,T].$$

4.3 Preliminary results

Lemma 4.3.1. *Fix $t \in [0,T]$. For each $\theta \in (\frac{\pi}{2}, \pi)$, there exist constants M_θ, $\omega_\theta > 0$ such that*

$$\Sigma(\theta,\omega_\theta) := \{z \in \mathbf{C}; \quad z \neq \omega_\theta,\ |arg(z-\omega_\theta)| < \theta\} \subset \rho(A_{0c}(t))$$

and

$$\left\|(\lambda - A_{0c}(t))^{-1}\right\|_{\mathcal{L}(C[0,1])} \leq M_\theta |\lambda|^{-1}, \tag{4.3.1}$$

$$\left\|(\lambda - A_{0c}(t))^{-1}\right\|_{\mathcal{L}(C[0,1],C^2[0,1])} \leq M_\theta \tag{4.3.2}$$

for $\lambda \in \Sigma(\theta,\omega_\theta)$.

Proof. The estimate (4.3.1) comes from [26, Chapter VI, Section 4b and the corresponding notes]. Pick $\mu \in \rho(A_{0c}(t))$. It is clear that $(\mu - A_{0c}(t))^{-1} \in \mathcal{L}(C[0,1], C^2[0,1])$. Hence, for $\lambda \in \Sigma(\theta, \omega_\theta)$, we obtain

$$\begin{aligned}
& \left\|(\lambda - A_{0c}(t))^{-1}\right\|_{\mathcal{L}(C[0,1],C^2[0,1])} \\
\leq\ & \left\|(\mu - A_{0c}(t))^{-1}\right\|_{\mathcal{L}(C[0,1],C^2[0,1])} \left\|(\mu - A_{0c}(t))\,(\lambda - A_{0c}(t))^{-1}\right\|_{\mathcal{L}(C[0,1])} \\
\leq\ & \text{const } \left\|\mu\,(\lambda - A_{0c}(t))^{-1} - \lambda\,(\lambda - A_{0c}(t))^{-1} + I\right\|_{\mathcal{L}(C[0,1])} \\
\leq\ & \text{const}, \quad \text{by (4.3.1)}.
\end{aligned}$$

□

We now consider the product space $\mathcal{E} := C[0,1] \times \mathbf{C}^2$ and operators thereon

$$\mathbb{A}(t) := \begin{pmatrix} A_c(t) & 0 \\ Q(t) & 0 \end{pmatrix}, \quad \mathcal{D}\left(\mathbb{A}(t)\right) := \left\{ \begin{pmatrix} f \\ y \end{pmatrix} \in C^2[0,1] \times \mathbf{C}^2; \;\; Pf = y \right\},$$

where $Pf := \begin{pmatrix} f(0) \\ f(1) \end{pmatrix}$ for $f \in C[0,1]$. For these operators we have an estimate analogous to (4.3.1).

Lemma 4.3.2. *Let $t \in [0,T]$. For each $\theta \in (\frac{\pi}{2}, \pi)$, there exist constants M'_θ, $\omega'_\theta > 0$ such that $\Sigma(\theta, \omega'_\theta) \subset \rho(A_{0c}(t))$ and*

$$\left\|(\lambda - \mathbb{A}(t))^{-1}\right\|_{\mathcal{L}(\mathcal{E})} \leq M'_\theta |\lambda|^{-1} \tag{4.3.3}$$

for $\lambda \in \Sigma(\theta, \omega'_\theta)$.

Proof. Fix $t \in [0,T]$, $\theta \in (\frac{\pi}{2}, \pi)$, and $\mu \in \rho(A_{0c}(t))$. In order to use perturbation arguments we write $\mathbb{A}(t) := \mathbb{A}_0(t) + \mathbb{Q}(t)$ with

$$\mathbb{A}_0(t) := \begin{pmatrix} A_c(t) & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{with } \mathcal{D}\left(\mathbb{A}_0(t)\right) := \mathcal{D}\left(\mathbb{A}(t)\right).$$

Since $P(\mathcal{D}(A_c(t))) = P(C^2[0,1]) = \mathbf{C}^2$, $A_c(t) \in \mathcal{L}(C^2[0,1], C[0,1])$, and $P\big|_{C^2[0,1]} \in \mathcal{L}(C^2[0,1], \mathbf{C}^2)$, we can define, similarly to [40, Lemmas 1.2 and 1.3] (see also [9, Lemma 2.2]), the Dirichlet operators with respect to $A_c(t)$ by

$$D_\lambda(t) := \left(P\big|_{\ker(\lambda - A_c(t))}\right)^{-1}, \quad \lambda \in \rho(A_{0c}(t)),$$

such that

$$D_\lambda(t) \in \mathcal{L}(\mathbf{C}^2, C^2[0,1]) \ \text{ and } \ D_\lambda(t) = D_\mu(t) - (\lambda-\mu)\,(\lambda - A_{0c}(t))^{-1}\, D_\mu(t) \quad (4.3.4)$$

for λ, $\mu \in \rho(A_{0c}(t))$. Thus we have

$$\lambda - \mathbb{A}_0(t) \;=\; \begin{pmatrix} \lambda - A_{0c}(t) & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} I & -D_\lambda(t) \\ 0 & I \end{pmatrix}$$

for $\lambda \in \rho(A_{0c}(t))$ and $t \in [0,T]$. Therefore, $\rho(A_{0c}(t)) \setminus \{0\} \subset \rho(\mathbb{A}_0(t))$ and

$$(\lambda - \mathbb{A}_0(t))^{-1} \;=\; \begin{pmatrix} (\lambda - A_{0c}(t))^{-1} & \lambda^{-1} D_\lambda(t) \\ 0 & \lambda^{-1} \end{pmatrix}, \quad (4.3.5)$$

$$\mathbb{Q}(t)\,(\lambda - \mathbb{A}_0(t))^{-1} = \begin{pmatrix} 0 & 0 \\ Q(t)\,(\lambda - A_{0c}(t))^{-1} & \lambda^{-1} Q(t) D_\lambda(t) \end{pmatrix} \quad (4.3.6)$$

for $\lambda \in \rho(A_{0c}(t)) \setminus \{0\}$.

From (4.3.1), (4.3.4) and (4.3.5), we see that

$$\left\|(\lambda - \mathbb{A}_0(t))^{-1}\right\|_{\mathcal{L}(\mathcal{E})} \le \text{const } |\lambda|^{-1}, \quad \lambda \in \Sigma(\theta, \omega_\theta). \quad (4.3.7)$$

We now estimate $\left\|\mathbb{Q}(t)\,(\lambda - \mathbb{A}_0(t))^{-1}\right\|_{\mathcal{L}(\mathcal{E})}$. To this purpose we use the fact (cf., e.g., [26, (2.2), p. 170]) that for each $\varepsilon > 0$ there exists b_ε such that

$$\|f'\|_{C[0,1]} \le \varepsilon \|f''\|_{C[0,1]} + b_\varepsilon \|f\|_{C[0,1]}, \quad f \in C^2[0,1].$$

Since $Q(t) \in \mathcal{L}(C^1[0,1], \mathbf{C}^2)$, we then deduce by (4.3.1) and (4.3.2) that

$$\begin{aligned} &\left\|Q(t)\left(\lambda - A_{0c}(t))^{-1} f\right)\right\|_{\mathbf{C}^2} \\ \le\ & \|Q(t)\|_{\mathcal{L}(C^1[0,1],\mathbf{C}^2)} \left\{\varepsilon \left\|(\lambda - A_{0c}(t))^{-1} f\right\|_{C^2[0,1]} + b_\varepsilon \left\|(\lambda - A_{0c}(t))^{-1} f\right\|_{C[0,1]}\right\} \\ \le\ & \|Q(t)\|_{\mathcal{L}(C^1[0,1],\mathbf{C}^2)} \left(\varepsilon + b_\varepsilon |\lambda|^{-1}\right) M_\theta \|f\|_{C[0,1]} \end{aligned}$$

for all $\lambda \in \Sigma(\theta, \omega_\theta)$, $f \in C[0,1]$.

Choose ε small enough such that $\varepsilon\|Q(t)\|_{\mathcal{L}(C^1[0,1],\mathbf{C}^2)}M_\theta \leq \frac{1}{4}$ and then choose $\omega'_\theta > \omega_\theta$ such that

$$\|Q(t)\|_{\mathcal{L}(C^1[0,1],\mathbf{C}^2)} b_\varepsilon M_\theta |\lambda|^{-1} \leq \frac{1}{4} \quad \text{for } \lambda \in \mathbf{C} \text{ with } |\lambda| > \omega'_\theta.$$

Hence

$$\left\|Q(t)\left(\lambda - A_{0c}(t)\right)^{-1}\right\|_{\mathcal{L}(C[0,1],\mathbf{C}^2)} \leq \frac{1}{2}, \quad \lambda \in \Sigma(\theta, \omega'_\theta), \tag{4.3.8}$$

and, by (4.3.1), (4.3.2) and (4.3.4),

$$\left\|\lambda^{-1} Q(t) D_\lambda(t)\right\|_{\mathcal{L}(\mathbf{C}^2)} \leq \frac{1}{2}, \quad \lambda \in \Sigma(\theta, \omega'_\theta).$$

This combined with (4.3.6) and (4.3.8) yields that

$$\left\|\mathbb{Q}(t)\left(\lambda - \mathbb{A}_0(t)\right)^{-1}\right\|_{\mathcal{L}(\mathcal{E})} \leq \frac{1}{2}, \quad \lambda \in \Sigma(\theta, \omega'_\theta).$$

So the operator $\lambda - \mathbb{A}(t) = \left[I - \mathbb{Q}(t)\left(\lambda - \mathbb{A}_0(t)\right)^{-1}\right]\left(\lambda - \mathbb{A}_0(t)\right)$ is invertible with

$$\left(\lambda - \mathbb{A}(t)\right)^{-1} = \left(\lambda - \mathbb{A}_0(t)\right)^{-1}\left[I - \mathbb{Q}(t)\left(\lambda - \mathbb{A}_0(t)\right)^{-1}\right]^{-1}, \quad \lambda \in \Sigma(\theta, \omega'_\theta).$$

Thus we obtain (4.3.3) by recalling (4.3.7).

□

We now investigate the continuity of the map $t \mapsto \mathbb{A}(t)$. If we define

$$\mathcal{D} := \left\{ \begin{pmatrix} f \\ y \end{pmatrix} \in C^2[0,1] \times \mathbf{C}^2; \quad Pf = y \right\}$$

endowed with the norm

$$\left\| \begin{pmatrix} f \\ y \end{pmatrix} \right\|_{\mathcal{D}} := \|f\|_{C^2[0,1]},$$

we obtain the following.

Lemma 4.3.3. *Under our assumptions, the map* $t \mapsto \mathbb{A}(t)$ *belongs to* $C^\alpha([0,T]; \mathcal{L}(\mathcal{D}, \mathcal{E}))$.

Proof. For $\begin{pmatrix} f \\ y \end{pmatrix} \in \mathcal{D}$ and $t,\ s \in [0,T]$, we estimate by (4.2.3) that

$$\begin{aligned}
&\left\| (\mathbb{A}(t) - \mathbb{A}(s)) \begin{pmatrix} f \\ y \end{pmatrix} \right\|_{\mathcal{E}} \\
= &\left\| \begin{pmatrix} (A_c(t) - A_c(s))f \\ (Q(t) - Q(s))f \end{pmatrix} \right\|_{\mathcal{E}} \\
= &\|(A_c(t) - A_c(s))f\|_{C[0,1]} + \|(Q(t) - Q(s))f\|_{\mathbf{C}^2} \\
\leq &\|a(\cdot,t) - a(\cdot,s)\|_{C[0,1]}\|f''\|_{C[0,1]} + \|q(\cdot,t) - q(\cdot,s)\|_{C[0,1]}\|f'\|_{C[0,1]} \\
&\qquad + \|r(\cdot,t) - r(\cdot,s)\|_{C[0,1]}\|f\|_{C[0,1]} \\
&\qquad + \sum_{j=0,1} (|\beta_j(t) - \beta_j(s)||f'(j)| + |\gamma_j(t) - \gamma_j(s)||f(j)|) \\
\leq &\text{ const } |t-s|^\alpha \|f\|_{C^2[0,1]} \\
\leq &\text{ const } |t-s|^\alpha \left\| \begin{pmatrix} f \\ y \end{pmatrix} \right\|_{\mathcal{D}}.
\end{aligned}$$

□

Lemma 4.3.4. *For each $t \in [0,T]$, the Banach spaces $\mathcal{D}$ and $[\mathcal{D}(\mathbb{A}(t))]$ are isomorphic, and the constants M_θ, $\omega_\theta > 0$ in Lemma 4.3.2 can be chosen to be independent of $t \in [0,T]$.*

Proof. An isomorphism is easy to find. The independence of the constants is implied by Lemma 4.3.3 (cf. [14, Appendix]).

□

The following result covers the corresponding ones in [25, 80, 86] with a different approach.

Proposition 4.3.5. *If $A(t)$ is as in (4.2.2), then it generates a strongly continuous analytic semigroup of angle $\frac{\pi}{2}$ satisfying*

$$\left\|e^{zA(t)}\right\| \leq M_\varphi e^{\omega_\varphi |z|}, \quad z \in \Sigma(\varphi, 0),\ t \in [0,T],$$

where M_φ, $\omega_\varphi > 0$ are constants dependent on $\varphi \in (0, \frac{\pi}{2})$ but independent of $t \in [0,T]$.

Proof. By Lemma 4.3.2, we infer (cf. [75]) that each $\mathbb{A}(t)$ generates an analytic semigroup $\{e^{s\mathbb{A}(t)}\}_{s\geq 0}$ on $\mathcal{E}$, and the restrictions of $e^{s\mathbb{A}(t)}$ to $\mathcal{E}_1 := \overline{\mathcal{D}(\mathbb{A}(t))}$ leave $\mathcal{E}_1$ invariant and become a strongly continuous analytic semigroup on $\mathcal{E}_1$, generated by the part $\mathbb{A}_1(t)$ of $\mathbb{A}(t)$ in $\mathcal{E}_1$. As a consequence, $\mathcal{D}(\mathbb{A}_1(t))$ is dense in $\mathcal{E}_1$. Clearly

$$\mathcal{E}_1 = \left\{ \begin{pmatrix} f \\ y \end{pmatrix} \in C[0,1] \times \mathbf{C}^2; \quad Pf = y \right\}.$$

It is not hard to see that

$$f \in \mathcal{D}(A(t)) \ \text{ if and only if } \ \begin{pmatrix} f \\ Pf \end{pmatrix} \in \mathcal{D}(\mathbb{A}_1(t)) \tag{4.3.9}$$

and that

$$(\lambda - A(t))^{-1} f = \pi_1 \left((\lambda - \mathbb{A}_1(t))^{-1} \begin{pmatrix} f \\ Pf \end{pmatrix} \right) \quad \text{for } \lambda \in \rho(\mathbb{A}_1(t)) \text{ and } \ f \in C[0,1], \tag{4.3.10}$$

where π_1 is the canonical projection from $C[0,1] \times \mathbf{C}^2$ onto $C[0,1]$. From (4.3.9) we know that $\mathcal{D}(A(t))$ is dense in $C[0,1]$ since $\mathcal{D}(\mathbb{A}_1(t))$ is dense in $\mathcal{E}_1$. Combining (4.3.9) and Lemma 4.3.4 yields that for each $\theta \in (\frac{\pi}{2}, \pi)$ there exist constants M_θ, $\omega_\theta > 0$ (independent of $t \in [0,T]$) such that

$$\begin{aligned} \left\|(\lambda - A(t))^{-1} f\right\|_{C[0,1]} &\leq \left\| (\lambda - \mathbb{A}_1(t))^{-1} \begin{pmatrix} f \\ Pf \end{pmatrix} \right\|_{\mathcal{E}_1} \\ &\leq M_\theta |\lambda|^{-1} \left(\|f\|_{C[0,1]} + \|Pf\|_{\mathbf{C}^2} \right) \\ &\leq 3M_\theta |\lambda|^{-1} \|f\|_{C[0,1]}, \quad \lambda \in \Sigma(\theta, \omega_\theta),\ f \in C[0,1]. \end{aligned}$$

This estimate implies the assertion.

□

4.4 Existence of evolution family for $(NACP)$ and regularity for $(NACP)$ and $(INACP)$

We now return to $(NACP)$ as well as to its inhomogeneous version

$$\begin{cases} u'(t) = A(t)u(t) + F(t), & 0 \le s < t \le T, \\ u(s) = f, \end{cases} \tag{$INACP$}$$

where $F(\cdot)$ is a given function from $[0,T]$ to $C[0,1]$.

Before stating the main result we briefly recall the basic concepts for nonautonomous abstract Cauchy problems (compare [61, Definition 6.0.1] or [26, Chapter VI, Definition 9.2]). We do so for arbitrary linear operators $A(t)$ $(t \in [0,T])$ in a Banach space X.

Definition 4.4.1. A family of linear operators $\{U(t,s)\}_{0\le s\le t\le T} \subset \mathcal{L}(X)$ is called an *evolution family* for $(NACP)$ if

(I) $U(s,s) = I$ for $0 \le s \le T$,

(II) $U(t,s)U(s,r) = U(t,r)$ for $0 \le r \le s \le t \le T$,

(III) $(t,s) \to U(t,s)$ is strongly continuous for $0 \le s \le t \le T$,

(IV) $t \mapsto U(t,s)$ is strongly continuously differentiable in $(s,T]$ and

$$\frac{\partial}{\partial t}U(t,s) = A(t)U(t,s), \quad 0 \le s < t \le T.$$

Definition 4.4.2. (i) Let $F(\cdot) \in C((s,T];X)$. A function $u(\cdot)$ is called a *classical solution* of $(INACP)$ if $u(\cdot) \in C^1((s,T];X) \cap C([s,T];X)$ and $(INACP)$ is satisfied.

(ii) Let $F(\cdot) \in C([s,T];X)$. A function $u(\cdot)$ is called a *strict solution* of $(INACP)$ if $u(\cdot) \in C^1([s,T];X)$ and $(INACP)$ is satisfied.

We now prove our main results.

Theorem 4.4.3. *Let $A(t)$ be as in (4.2.2). Then there exists a unique evolution family $\{U(t,s)\}_{0\le s\le t\le T} \subset \mathcal{L}(C[0,1])$ for $(NACP)$ with the following properties.*

(i) $\|A(t)U(t,s)\|_{\mathcal{L}(C[0,1])} \le$ *const* $(t-s)^{-1}$ *and*

$$\|A(t)U(t,s)\|_{\mathcal{L}(C^2[0,1],C[0,1])} \le const$$

for $0 \le s < t \le T$.

(ii) $U(\cdot,s)f \in C([s,T];C[0,1]) \cap C^{1+\alpha}([s+\varepsilon,T];C[0,1]) \cap C^{\alpha}([s+\varepsilon,T];C^2[0,1])$ *for* $f \in C[0,1]$, $s \in [0,T)$, $\varepsilon \in (0,T-s)$.

(iii) $U(\cdot,s)f \in C^1([s,T];C[0,1])$ *for* $f \in \mathcal{D}(A(s))$, $s \in [0,T)$.

(iv) $U(t,\cdot)f \in C([0,t];C[0,1])$ *for* $f \in C[0,1]$, $t \in (0,T]$.

(v) $U(t,\cdot)f \in C^1([0,t];C[0,1]) \cap C([0,t];C^2[0,1])$ *for* $f \in \mathcal{D}(A(t))$, $t \in (0,T]$.

Proof. By Lemmas 4.3.2 and 4.3.3 there exists, in view of [1, 2] or [61, Sections 6.1 and 6.2], a family of linear operators $\{\mathbb{U}(t,s)\}_{0\le s\le t\le T} \subset \mathcal{L}(\mathcal{E})$ with the following properties.

(a) For $\begin{pmatrix} f \\ Pf \end{pmatrix} \in \mathcal{E}_1$, the $\mathcal{E}$-valued function $\mathcal{U}(\cdot) := \mathbb{U}(\cdot,s)\begin{pmatrix} f \\ Pf \end{pmatrix}$ is the unique classical solution of the problem

$$\begin{cases} \mathcal{U}'(t) = \mathbb{A}(t)\mathcal{U}(t), \quad s < t \le T, \\ \mathcal{U}(s) = \begin{pmatrix} f \\ Pf \end{pmatrix} \end{cases} \tag{4.4.1}$$

and belongs to $C([s,T];\mathcal{E}) \cap C^{1+\alpha}([s+\varepsilon,T];\mathcal{E}) \cap C^{\alpha}([s+\varepsilon,T];\mathcal{D})$ for each $\varepsilon \in (0,T-s)$.

(b) For $\begin{pmatrix} f \\ Pf \end{pmatrix} \in \mathcal{D}(\mathbb{A}_1(s))$, $\mathbb{U}(\cdot,s)\begin{pmatrix} f \\ Pf \end{pmatrix}$ belongs to $C^1([s,T];\mathcal{E})$.

(c) For $0 \le s < t \le T$,

$$\|\mathbb{U}(t,s)\|_{\mathcal{L}(\mathcal{E})} \le \text{const}, \tag{4.4.2}$$

$$\|\mathbb{A}(t)\mathbb{U}(t,s)\|_{\mathcal{L}(\mathcal{E})} \le \text{const } (t-s)^{-1}, \quad \|\mathbb{U}(t,s)\|_{\mathcal{L}(\mathcal{D},\mathcal{E})} \le \text{const}.$$

(d) For $t \in (0,T]$,

$$\mathbb{U}(t,\cdot)\begin{pmatrix} f \\ Pf \end{pmatrix} \in C([0,t];\mathcal{E}), \quad \begin{pmatrix} f \\ Pf \end{pmatrix} \in \mathcal{E}_1,$$

$$\mathbb{U}(t,\cdot)\begin{pmatrix} f \\ Pf \end{pmatrix} \in C^1([0,t];\mathcal{E}) \cap C([0,t];\mathcal{D}), \quad \begin{pmatrix} f \\ Pf \end{pmatrix} \in \mathcal{D}(\mathbb{A}_1(t)).$$

We now take the first coordinate of $\mathbb{U}(t,s)\begin{pmatrix} f \\ Pf \end{pmatrix}$ and define

$$U(t,s)f := \pi_1\left(\mathbb{U}(t,s)\begin{pmatrix} f \\ Pf \end{pmatrix}\right) \qquad \text{for } 0 \le s \le t \le T \text{ and } \ f \in C[0,1].$$

Observe that $u(\cdot)$ is a classical solution of $(NACP)$ if and only if $\mathcal{U}(\cdot) = \begin{pmatrix} u(\cdot) \\ Pu(\cdot) \end{pmatrix}$ is a classical solution of (4.4.1), and

$$f \in C[0,1] \ \text{ if and only if } \ \begin{pmatrix} f \\ Pf \end{pmatrix} \in \mathcal{E}_1,$$

$$f \in C^2[0,1] \ \text{ if and only if } \ \begin{pmatrix} f \\ Pf \end{pmatrix} \in \mathcal{D},$$

$$f \in \mathcal{D}(A(t)) \ \text{ if and only if } \ \begin{pmatrix} f \\ Pf \end{pmatrix} \in \mathcal{D}(\mathbb{A}_1(t)).$$

Accordingly, we obtain assertions (i) - (v), as well as (I) and (IV) in Definition 4.4.1, by the corresponding properties of $\mathbb{U}(t,s)$ listed above. Furthermore, we know that for each $f \in C[0,1]$, $(NACP)$ has a unique classical solution since the classical solution of (4.4.1) is unique.

Next, we show properties (II) and (I) in Definition 4.4.1. To this end, we let $f_1 \in C[0,1]$ and $0 \le r \le s \le t \le T$. By (ii),

$$U(\cdot, r)f_1, \quad U(\cdot, s)U(s, r)f_1 \in C([s,T]; C[0,1]).$$

This combined with (IV) in Definition 4.4.1 yields that $t \mapsto U(t,r)f_1$ and $t \mapsto U(t,s)U(s,r)f_1$ are classical solutions of $(NACP)$ with the same initial datum $U(s,r)f_1$ at $t = s$. Therefore (II) in Definition 4.4.1 is satisfied. The assertion (III) in Definition 4.4.1 follows from (II), (ii), (iv) and the uniform boundedness of $\|U(t,s)\|_{\mathcal{L}(C[0,1])}$ for $0 \le s < t \le T$ (derived from (4.4.2)).

Finally, the uniqueness of the classical solution of $(NACP)$ implies the uniqueness of the evolution family for $(NACP)$.

□

The inhomogeneous problem can be solved as follows.

Theorem 4.4.4. *Let $\beta \in (0, \alpha]$ and $F \in C^\beta([s,T]; C[0,1])$.*

1) *If $f \in C[0,1]$, then $(INACP)$ has a unique classical solution $u(\cdot) \in C^{1+\beta}([s+\varepsilon, T]; C[0,1]) \cap C^\beta([s+\varepsilon, T]; C^2[0,1])$ for every $\varepsilon \in (0, T-s)$ and is given by*

$$u(t) = U(t,s)f + \int_s^t U(t,\sigma)F(\sigma)d\sigma, \quad s \le t \le T.$$

2) *If $f \in \mathcal{D}(A(s))$, the above $u(\cdot)$ is a strict solution of $(INACP)$.*

Proof. Observe that $u(\cdot)$ is a classical (resp. strict) solution of $(INACP)$ if and only if $\mathcal{U}(\cdot) \begin{pmatrix} u(\cdot) \\ Pu(\cdot) \end{pmatrix}$ is a classical (resp. strict) solution of the following inhomogeneous nonautonomous abstract Cauchy problem

$$\begin{cases} \mathcal{U}'(t) = \mathbb{A}(t)\mathcal{U}(t) + \begin{pmatrix} F(t) \\ PF(t) \end{pmatrix}, \quad s < t \le T, \\ \mathcal{U}(s) = \begin{pmatrix} f \\ Pf \end{pmatrix}. \end{cases} \tag{4.4.3}$$

Therefore, we obtain the desired conclusions from the corresponding results for (4.4.3) available because of Lemmas 4.3.2 and 4.3.3 (see the papers [1, 2] or the book [61, Corollary 6.1.6 (i) and (iii) and Corollary 6.2.4] stemming from the classical Sobolevskii-Tanabe work [76, 77] for abstract nonautonomous parabolic equations).

□

Remark 4.4.5. In the same way, we can derive other properties of the evolution family $\{U(t,s)\}_{0\leq s\leq t\leq T}$ (of the solutions of $(INACP)$, resp.) from the corresponding ones of $\{\mathbb{U}(t,s)\}_{0\leq s\leq t\leq T}$ (of (4.4.3), resp.).

Chapter 5

Second order abstract parabolic equations with dynamic boundary conditions

5.1 Summary

In this chapter, we exhibit a unified treatment of the mixed initial boundary value problem for second order (in time) parabolic linear differential equations in Banach spaces whose boundary conditions are of a dynamical nature. Results regarding existence, uniqueness, continuous dependence (on initial data) and regularity of classical and strict solutions are established. Moreover, two examples are given as samples for possible applications.

5.2 Introduction

Of concern is the inhomogeneneous complete second order differential equation

$$u''(t) + Au(t) + Bu'(t) = f(t), \quad t > 0, \tag{5.2.1}$$

in a Banach spaces E, where A and B are linear operators in E, and f an E-valued function. The Cauchy problem for (5.2.1) has been extensively studied since the end of 1950s (see H. O. Fattorini [30, 31] and T. J. Xiao and J. Liang [84, 87] for surveys).

In this chapter, we consider a mixed initial boundary value problem for (5.2.1), in which besides the usual initial condition

$$u(0) = u_0, \quad u'(0) = u_1, \tag{5.2.2}$$

there is also a boundary condition given by

$$x''(t) + A_1 x(t) + B_1 x'(t) = G_0 u(t) + G_1 u'(t) + g(t), \quad t > 0. \tag{5.2.3}$$

Here $x(\cdot)$ stands for the boundary value of the state function $u(\cdot)$, these two functions being connected by a linear boundary operator P (from $\mathcal{D}(A)$ to another Banach space X)

$$x(t) := Pu(t), \quad t > 0. \tag{5.2.4}$$

Moreover, A_1 and B_1 are linear operators in X, g an X-valued function, and G_i $(i = 0, 1)$ are linear operators (*feedback operators*) from $\mathcal{D}(G_i) \subset E$ to X. The boundary condition (5.2.3) is of a dynamic nature, for which we initially have

$$x(0) = x_0, \quad x'(0) = x_1. \tag{5.2.5}$$

The study of evolution equations with dynamic boundary conditions from the mathematical point of view dates back to 1961, when J. L. Lions [59, p. 117, 118] treated such equations and gave weak solutions by means of the variational method. Since then, this issue has been investigated to a large extent (see, e.g., [8, 9, 27, 32, 33, 35, 37, 43, 53, 59, 74] and references therein). While most of the previous research concerns the case of first order in time, there has been few regarding the second order (in time) case. In the present chapter, we shall consider the second order problem (5.2.1) - (5.2.5) and deal with it in a direct way, without reduction. This approach will yield strong solutions with desirable regularity, as well as build up theorems of a general nature.

To begin with, define operators on $\mathbf{E} := E \times X$ by

$$\mathbb{A} := \begin{pmatrix} A & 0 \\ -G_0 & A_1 \end{pmatrix}, \quad \mathcal{D}(\mathbb{A}) := \left\{ \begin{pmatrix} u \\ x \end{pmatrix} \in (\mathcal{D}(A) \cap \mathcal{D}(G_0)) \times \mathcal{D}(A_1); \quad x = Pu \right\},$$

$$\mathbb{B} := \begin{pmatrix} B & 0 \\ -G_1 & B_1 \end{pmatrix}, \quad \mathcal{D}(\mathbb{B}) := (\mathcal{D}(B) \cap \mathcal{D}(G_1)) \times \mathcal{D}(B_1).$$

By setting

$$y(t) := \begin{pmatrix} u(t) \\ x(t) \end{pmatrix}, \quad h(t) := \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}, \quad y_0 := \begin{pmatrix} u_0 \\ x_0 \end{pmatrix}, \quad y_1 := \begin{pmatrix} u_1 \\ x_1 \end{pmatrix},$$

problem (5.2.1) - (5.2.5) is converted into an abstract Cauchy problem in $\mathbf{E}$ of the following form.

$$\begin{cases} y''(t) + \mathbb{A}y(t) + \mathbb{B}y'(t) = h(t), \quad t > 0, \\ y(0) = y_0, \ y'(0) = y_1. \end{cases}$$

How can one deal with this problem involving two operator matrices? We shall present some ideas about it. This chapter is confined to equations of parabolic type, and those of general case will be considered in the next chapter.

In order to carry out our strategy, we still need to introduce another boundary operator P_1 as a linear operator from $\mathcal{D}(B)$ to the quotient space X/X_0 (X_0 is a closed linear subspace of X to be kept fixed in the following). The P_1 can be chosen flexibly in applications (see Examples 5.5.1 and 5.5.3) such that the relation

$$x'(t) \in P_1 u'(t), \quad t > 0, \tag{5.2.6}$$

is implied by (5.2.1), (5.2.3) and (5.2.4). The simplest P_1 is in the case of $X_0 = X$.

For the two operators A and B in the state space E, we define

$$A_0 := A\Big|_{\ker P}, \quad B_0 := B\Big|_{\ker P_1}. \tag{5.2.7}$$

Then A_0 and B_0 have zero boundary values in some sense. A condition of parabolic type will be given on the operator pair (A_0, B_0) (also on (A_1, B_1)), holding quite often in concrete situations. Moreover, for equations (5.2.1) and (5.2.3), we regard A, B, A_1, and B_1 as principal operators to which G_0 and G_1 are subordinated. For a wider applicability, we shall include four more perturbing (linear) operators into our consideration:

$$\widetilde{A} : \mathcal{D}(\widetilde{A}) \subset E \to E, \quad \widetilde{B} : \mathcal{D}(\widetilde{B}) \subset E \to E,$$

$$\widetilde{A}_1 : \mathcal{D}(\widetilde{A}_1) \subset X \to X, \quad \widetilde{B}_1 : \mathcal{D}(\widetilde{B}_1) \subset X \to X.$$

Thus, we shall actually study

$$\begin{cases} y''(t) + \left(\mathbf{A} + \widetilde{\mathbf{A}}\right) y(t) + \left(\mathbf{B} + \widetilde{\mathbf{B}}\right) y'(t) = h(t), \quad t > 0, \\ y(0) = y_0, \ y'(0) = y_1 \end{cases} \tag{5.2.8}$$

in space $\mathbf{E}$, with the main operator matrices $\mathbf{A}$, $\mathbf{B}$ and the perturbing operators $\widetilde{\mathbf{A}}$, $\widetilde{\mathbf{B}}$ defined as follows:

$$\mathbf{A} := \begin{pmatrix} A & 0 \\ 0 & A_1 \end{pmatrix}, \quad \mathcal{D}(\mathbf{A}) := \left\{ \begin{pmatrix} u \\ x \end{pmatrix} \in \mathcal{D}(A) \times \mathcal{D}(A_1); \quad x = Pu \right\},$$

$$\mathbf{B} := \begin{pmatrix} B & 0 \\ 0 & B_1 \end{pmatrix}, \quad \mathcal{D}(\mathbf{B}) := \left\{ \begin{pmatrix} u \\ x \end{pmatrix} \in \mathcal{D}(B) \times \mathcal{D}(B_1); \quad x \in P_1 u \right\}.$$

$$\widetilde{\mathbf{A}} := \begin{pmatrix} \widetilde{A} & 0 \\ -G_0 & \widetilde{A}_1 \end{pmatrix}, \quad \widetilde{\mathbf{B}} := \begin{pmatrix} \widetilde{B} & 0 \\ -G_1 & \widetilde{B}_1 \end{pmatrix}.$$

In Section 2, we shall show under suitable conditions that the operator pair $\big(\mathbf{A}+\widetilde{\mathbf{A}}, \mathbf{B}+\widetilde{\mathbf{B}}\big)$ possesses certain parabolicity (Theorem 5.3.3), and then construct an operator function $\widetilde{\mathbf{S}}(\cdot)$ (a fundamental solution operator of (5.2.8)) having a holomorphic extension to a sector Σ_θ ($\theta \in (0, \frac{\pi}{2}]$) and satisfying various nice properties (Theorem 5.3.4). Making use of this, we will formulate and prove, in Section 3, our main theorem (Theorem 5.4.3) with regard to the existence and uniqueness of classical and strict solutions for (5.2.8), to continuous dependence (on initial data) and regularity of the solutions. Finally, in Section 4 we shall discuss two applications of our theorems to platelike equations and damped beam equations with dynamic boundary conditions.

Notation: Write

$$\Sigma_\theta := \{\lambda \in \mathbf{C}; \quad \lambda \neq 0, \ |\arg \lambda| < \theta\}, \quad \theta \in (0, \pi],$$
$$R_i(\lambda) := (\lambda^2 + A_i + \lambda B_i)^{-1}, \quad i = 0, 1, \quad \lambda \in \mathbf{C}, \tag{5.2.9}$$
$$\widetilde{\mathbf{R}}(\lambda) := \left(\lambda^2 + \left(\mathbf{A} + \widetilde{\mathbf{A}}\right) + \lambda \left(\mathbf{B} + \widetilde{\mathbf{B}}\right)\right)^{-1}, \quad \lambda \in \mathbf{C},$$

if the inverse operators exist, and

$$\rho(A_0, B_0) := \{\lambda \in \mathbf{C}; \ R_0(\lambda) \text{ exists and belongs to } \mathcal{L}(E)\}.$$

By $[\mathcal{D}(A)]_P$ we denote the space $\mathcal{D}(A)$ equipped with the norm

$$\|u\|_{A,P} := \|u\| + \|Au\| + \|Pu\|,$$

$[\mathcal{D}(B)]_{P_1}$ the space $\mathcal{D}(B)$ with the norm

$$\|u\|_{B,P_1} := \|u\| + \|Bu\| + \|P_1 u\|_{X/X_0},$$

$[\mathcal{D}(A) \cap \mathcal{D}(B)]$ the space $\mathcal{D}(A) \cap \mathcal{D}(B)$ with the norm

$$\|u\|_{A,B} := \|u\| + \|Au\| + \|Bu\|,$$

and $[\mathcal{D}(A) \cap \mathcal{D}(B)]_P$ the space $\mathcal{D}(A) \cap \mathcal{D}(B)$ with the norm

$$\|u\|_{A,B,P} := \|u\| + \|Au\| + \|Bu\| + \|Pu\|.$$

5.3 Parabolicity

We first give some basic properties of the operators A, B and P.

Lemma 5.3.1. *Suppose that the following* (H_1) *is satisfied.*

(H_1) $[\mathcal{D}(A)]_P$ *and* $[\mathcal{D}(B)]_{P_1}$ *are complete,* $P(\mathcal{D}(A) \cap \mathcal{D}(B)) = X$*, and* $Pu \in P_1 u$ *for any* $u \in \mathcal{D}(A) \cap \mathcal{D}(B)$.

Then

(1) *The space* $[\mathcal{D}(A) \cap \mathcal{D}(B)]_P$ *is complete.*

(2) *If* $\lambda \in \rho(A_0, B_0)$, $\lambda \neq 0$*, then* $P\Big|_{\ker(\lambda^2+A+\lambda B)}$ *is a bijection of* $\ker(\lambda^2 + A + \lambda B)$ *onto* X*, and*

$$D_\lambda := \left(P\Big|_{\ker(\lambda^2+A+\lambda B)} \right)^{-1}$$

is bounded from X *to* $(\ker(\lambda^2 + A + \lambda B), \|\cdot\|_{A,B,P})$.

(3) *For every* $\lambda, \mu \in \rho(A_0, B_0)$ *with* λ, $\mu \neq 0$,

$$D_\lambda := D_\mu + (\mu - \lambda) R_0(\lambda)(\mu + \lambda + B) D_\mu. \tag{5.3.1}$$

Proof. (1) Suppose that $\{u_n\}_{n\in N}$ is a Cauchy sequence in $[\mathcal{D}(A)\cap\mathcal{D}(B)]_P$. Then it is easy to see that $\{u_n\}_{n\in N}$ is a Cauchy sequence in $[\mathcal{D}(A)]_P$. So there exists $u\in\mathcal{D}(A)$ such that

$$u_n \to u, \quad Au_n \to Au, \quad Pu_n \to Pu \text{ as } \ n\to\infty. \tag{5.3.2}$$

Moreover, $\{u_n\}_{n\in N}$ is also a Cauchy sequence in $[\mathcal{D}(B)]_{P_1}$ because of

$$\|P_1 u_n\|_{X/X_0} \le \|Pu_n\|$$

by (H_1). Therefore there is $v\in\mathcal{D}(B)$ such that

$$\lim_{n\to\infty} u_n = v, \quad \lim_{n\to\infty} Bu_n = Bv. \tag{5.3.3}$$

Combining (5.3.2) and (5.3.3) shows that $u=v$, and so

$$u\in\mathcal{D}(B), \quad \lim_{n\to\infty} Bu_n = Bu.$$

This verifies the completeness of $[\mathcal{D}(A)\cap\mathcal{D}(B)]_P$.

(2) Assume that $u,v\in\ker(\lambda^2+A+\lambda B)$ with $Pu=Pv$. Then

$$(\lambda^2+A+\lambda B)(u-v)=0 \quad\text{and}\quad P(u-v)=0$$

which implies $P_1(u-v)=0$. Therefore

$$u-v\in\mathcal{D}(A_0)\cap\mathcal{D}(B_0)$$

by the definitions of A_0 and B_0. Thus we have

$$(\lambda^2+A_0+\lambda B_0)(u-v)=0.$$

This yields that $u-v=0$ since $\lambda\in\rho(A_0,B_0)$. Hence $P\Big|_{\ker(\lambda^2+A+\lambda B)}$ is injective. Next take $x\in X$. Then there is $u\in\mathcal{D}(A)\cap\mathcal{D}(B)$ such that $Pu=x$, by (H_1). Put

$$v_1 := R_0(\lambda)(\lambda^2+A+\lambda B)u, \quad v_2 := u-v_1.$$

We see easily that $v_1\in\mathcal{D}(A_0)$ and $(\lambda^2+A+\lambda B)v_2=0$. So

$$Pv_1=0, \quad Pv_2=Pu-Pv_1=x,$$

and $v_2 \in \ker(\lambda^2 + A + \lambda B)$. This indicates that $P\Big|_{\ker(\lambda^2+A+\lambda B)}$ is surjective. Finally, we observe that $(\ker(\lambda^2 + A + \lambda B), \|u\|_{A,B,P})$ is a Banach space in view of (1), and $P\Big|_{\ker(\lambda^2+A+\lambda B)}$ is a bounded linear operator from $(\ker(\lambda^2 + A + \lambda B), \|\cdot\|_{A,B,P})$ onto X. So the open mapping theorem gives the boundedness of D_λ.

(3) Write

$$Q := [I + (\mu - \lambda)R_0(\lambda)(\mu + \lambda + B)]D_\mu.$$

Then for each $x \in X$,

$$\begin{aligned}(\lambda^2 + A + \lambda B)Qx &= [(\lambda^2 + A + \lambda B) + \mu^2 - \lambda^2 + (\mu - \lambda)B]D_\mu x \\ &= (\mu^2 + A + \mu B)D_\mu x = 0\end{aligned}$$

since $D_\mu x \in \ker(\mu^2 + A + \mu B)$. Thus we see that the range of Q is contained in $\ker(\lambda^2 + A + \lambda B)$. Moreover, we have $PQ = PD_\mu = I$, noting $PR_0(\lambda) = 0$. Therefore, we deduce $Q = D_\lambda$ as claimed. The proof is then complete.

□

The following are the hypotheses of parabolic type on A_0,B_0 (see (5.2.7)) and on A_1, B_1.

(H$_2$) The operators A_0 and B_0 are closed, and for each $\varphi \in (0, \theta)$ ($\theta \in (0, \frac{\pi}{2}]$), there exist M_φ, $\omega_\varphi > 0$ such that

$$\|\lambda R_0(\lambda)\|,\ \left\|\lambda^{-1}A_0R_0(\lambda)\right\| \le M_\varphi|\lambda|^{-1}, \quad \lambda \in \omega_\varphi + \Sigma_{\frac{\pi}{2}+\varphi}.$$

(H$_3$) The operators A_1 and B_1 are closed, and for each $\varphi \in (0, \theta)$ ($\theta \in (0, \frac{\pi}{2}]$), there exist M_φ, $\omega_\varphi > 0$ such that

$$\|\lambda R_1(\lambda)\|,\ \left\|\lambda^{-1}A_1R_1(\lambda)\right\| \le M_\varphi|\lambda|^{-1}, \quad \lambda \in \omega_\varphi + \Sigma_{\frac{\pi}{2}+\varphi}.$$

Remark 5.3.2. In concrete problems, it happens quite often that A_1 and B_1 are bounded operators on X. In this situation, (H$_3$) holds automatically.

Prior to Theorem 5.3.3 below concerning (among others) parabolicity of $\left(\mathbf{A} + \widetilde{\mathbf{A}}, \mathbf{B} + \widetilde{\mathbf{B}}\right)$, we recall the following notion (cf., e.g., [26, p. 169]):

A linear operator $\mathbb{B}$ in a Banach space Y is called $\mathbb{A}$-*bounded*, for a linear operator $\mathbb{A}$ in Y, if $\mathcal{D}(\mathbb{A}) \subset \mathcal{D}(\mathbb{B})$ and there exist constants a, $b > 0$ such that

$$\|\mathbb{B}y\| \le a\|\mathbb{A}y\| + b\|y\| \tag{5.3.4}$$

for all $y \in \mathcal{D}(\mathbb{A})$. The $\mathbb{A}$-*bound* of $\mathbb{B}$ is

$$\inf\{a > 0;\ \text{there is } b > 0 \text{ such that (5.3.4) holds}\}.$$

Theorem 5.3.3. *Let $\theta \in (0, \frac{\pi}{2}]$. Suppose that* (H_1) – (H_3) *hold. Let*

$$\widetilde{A} \in \mathcal{L}\left([\mathcal{D}(A)]_P, E\right), \quad \widetilde{B} \in \mathcal{L}\left([\mathcal{D}(B)]_{P_1}, E\right), \tag{5.3.5}$$

$$G_0 \in \mathcal{L}\left([\mathcal{D}(A)]_P, X\right), \quad G_1 \in \mathcal{L}\left([\mathcal{D}(B)]_{P_1}, X\right), \tag{5.3.6}$$

$$\widetilde{A}_1 \in \mathcal{L}\left([\mathcal{D}(A_1)], X\right), \quad \widetilde{B}_1 \in \mathcal{L}\left([\mathcal{D}(B_1)] X\right), \tag{5.3.7}$$

be such that $\widetilde{A}$, G_0 are A_0-bounded with A_0-bound zero, $\widetilde{B}$, G_1 are B_0-bounded with B_0-bound zero, $\widetilde{A}_1$ is A_1-bounded with A_1-bound zero, and $\widetilde{B}_1$ is B_1-bounded with B_1-bound zero. Then

(1) $\mathbf{A}$ *and* $\mathbf{B}$ *are closed, and*

$$\widetilde{\mathbf{A}} \in \mathcal{L}([\mathcal{D}(\mathbf{A})], \mathbf{E}), \quad \widetilde{\mathbf{B}} \in \mathcal{L}([\mathcal{D}(\mathbf{B})], \mathbf{E}). \tag{5.3.8}$$

(2) *There exist $M'_\varphi > M_\varphi$, $\omega'_\varphi > \omega_\varphi$ such that*

$$\|\lambda\widetilde{\mathbf{R}}(\lambda)\|, \quad \left\|\lambda^{-1}\mathbf{A}\widetilde{\mathbf{R}}(\lambda)\right\|, \quad \left\|\mathbf{B}\widetilde{\mathbf{R}}(\lambda)\right\| \le M'_\varphi|\lambda|^{-1}, \quad \lambda \in \omega'_\varphi + \Sigma_{\frac{\pi}{2}+\varphi}. \tag{5.3.9}$$

Proof. We take $\begin{pmatrix} u_n \\ x_n \end{pmatrix}_{n\in N} \subset \mathcal{D}(\mathbf{B})$ and let

$$\lim_{n\to\infty} \begin{pmatrix} u_n \\ x_n \end{pmatrix} = \begin{pmatrix} u \\ x \end{pmatrix} \quad \text{and} \quad \lim_{n\to\infty} \mathbf{B} \begin{pmatrix} u_n \\ x_n \end{pmatrix} = \begin{pmatrix} v \\ y \end{pmatrix}.$$

Then

$$\begin{cases} \lim\limits_{n\to\infty} x_n = x \\ \lim\limits_{n\to\infty} B_1 x_n = y \end{cases}, \quad \begin{cases} \lim\limits_{n\to\infty} u_n = u \\ \lim\limits_{n\to\infty} B u_n = v \end{cases},$$

and $\{P_1 u_n\}_{n\in N}$ is a Cauchy sequence in X/X_0 since

$$x_n \in P_1 u_n \quad \text{and} \quad \|P_1(u_n - u_m)\| \le \|x_n - x_m\|, \quad m, n \in N.$$

This combined with the closedness of B_1 and the completeness of $[\mathcal{D}(B)]_{P_1}$ implies that

$$x \in \mathcal{D}(B_1), \quad u \in \mathcal{D}(B), \quad B_1 x = y, \quad Bu = v, \quad \lim_{n\to\infty} P_1 u_n = P_1 u.$$

We observe that

$$\operatorname{dist}(x_n,\ P_1 u) = \|P_1 u_n - P_1 u\|_{X/X_1}$$

because of $x_n \in P_1 u_n$. It follows that

$$\operatorname{dist}(x,\ P_1 u) = \lim_{n\to\infty} \operatorname{dist}(x_n,\ P_1 u) = 0,$$

and therefore $x \in P_1 u$. Thus we know that $\mathbf{B}$ is closed. A similar and simpler argument shows the closedness of $\mathbf{A}$.

Next, we observe that

$$\left\| \begin{pmatrix} u \\ x \end{pmatrix} \right\|_{[\mathcal{D}(\mathbf{A})]} = \|u\| + \|x\| + \|Au\| + \|A_1 x\| \quad \text{for} \quad \begin{pmatrix} u \\ x \end{pmatrix} \in \mathcal{D}(\mathbf{A}),$$

$$\left\| \begin{pmatrix} u \\ x \end{pmatrix} \right\|_{[\mathcal{D}(\mathbf{B})]} = \|u\| + \|x\| + \|Bu\| + \|B_1 x\| \quad \text{for} \quad \begin{pmatrix} u \\ x \end{pmatrix} \in \mathcal{D}(\mathbf{B}),$$

and that

$$\begin{pmatrix} u \\ x \end{pmatrix} \in \mathcal{D}(\mathbf{A}) \quad (\text{resp.} \quad \begin{pmatrix} u \\ x \end{pmatrix} \in \mathcal{D}(\mathbf{B}))$$

implies

$$x = Pu \quad (\text{resp.}\ \ x \in P_1 u).$$

From this and (5.3.5) – (5.3.7), we see easily that (5.3.8) is true.

Now, fix $\varphi \in (0, \theta)$ and let $\lambda \in \omega_\varphi + \Sigma_{\frac{\pi}{2}+\varphi}$. If $\begin{pmatrix} u \\ x \end{pmatrix} \in \mathcal{D}(\mathbf{A}) \cap \mathcal{D}(\mathbf{B})$, then by Lemma 5.3.1 (2),

$$(\lambda^2 + A + \lambda B) D_\lambda x = 0,$$

$$u - D_\lambda x \in (\ker P) \cap \mathcal{D}(A) \cap \mathcal{D}(B) = \mathcal{D}(A_0) \cap \mathcal{D}(B_0).$$

So we obtain

$$
\begin{aligned}
(\lambda^2 + \mathbf{A} + \lambda \mathbf{B}) \begin{pmatrix} u \\ x \end{pmatrix} &= \begin{pmatrix} \lambda^2 + A + \lambda B & 0 \\ 0 & \lambda^2 + A_1 + \lambda B_1) \end{pmatrix} \begin{pmatrix} u \\ x \end{pmatrix} \\
&= \begin{pmatrix} \lambda^2 + A + \lambda B & 0 \\ 0 & \lambda^2 + A_1 + \lambda B_1 \end{pmatrix} \begin{pmatrix} u - D_\lambda x \\ x \end{pmatrix} \\
&= \begin{pmatrix} \lambda^2 + A_0 + \lambda B_0 & 0 \\ 0 & \lambda^2 + A_1 + \lambda B_1 \end{pmatrix} \begin{pmatrix} u - D_\lambda x \\ x \end{pmatrix} \\
&= \begin{pmatrix} \lambda^2 + A_0 + \lambda B_0 & 0 \\ 0 & \lambda^2 + A_1 + \lambda B_1 \end{pmatrix} \begin{pmatrix} I & -D_\lambda \\ 0 & I \end{pmatrix} \begin{pmatrix} u \\ x \end{pmatrix}.
\end{aligned}
$$

We then have

$$
\lambda^2 + \mathbf{A} + \lambda \mathbf{B} = \begin{pmatrix} \lambda^2 + A_0 + \lambda B_0 & 0 \\ 0 & \lambda^2 + A_1 + \lambda B_1 \end{pmatrix} \begin{pmatrix} I & -D_\lambda \\ 0 & I \end{pmatrix},
$$

noting that

$$
\begin{pmatrix} I & -D_\lambda \\ 0 & I \end{pmatrix} \begin{pmatrix} u \\ x \end{pmatrix} \in (\mathcal{D}(A_0) \cap \mathcal{D}(B_0)) \times (\mathcal{D}(A_1) \cap \mathcal{D}(B_1))
$$

implies $\begin{pmatrix} u \\ x \end{pmatrix} \in \mathcal{D}(\mathbf{A}) \cap \mathcal{D}(\mathbf{B})$. It follows that $\lambda^2 + \mathbf{A} + \lambda \mathbf{B}$ is invertible and

$$
\begin{aligned}
\mathbf{R}(\lambda) &:= (\lambda^2 + \mathbf{A} + \lambda \mathbf{B})^{-1} \\
&= \begin{pmatrix} I & D_\lambda \\ 0 & I \end{pmatrix} \begin{pmatrix} R_0(\lambda) & 0 \\ 0 & R_1(\lambda) \end{pmatrix} \\
&= \begin{pmatrix} R_0(\lambda) & D_\lambda R_1(\lambda) \\ 0 & R_1(\lambda) \end{pmatrix},
\end{aligned} \tag{5.3.10}
$$

$$
\mathbf{A}\mathbf{R}(\lambda) = \begin{pmatrix} A_0 R_0(\lambda) & A D_\lambda R_1(\lambda) \\ 0 & A_1 R_1(\lambda) \end{pmatrix}. \tag{5.3.11}
$$

Take $\mu \in \omega_\varphi + \Sigma_{\frac{\pi}{2}+\varphi}$. Then

$$
AD_\mu, \quad BD_\mu \in \mathcal{L}(X, E), \tag{5.3.12}
$$

by Lemma 5.3.1 (2). Using (5.3.12) and (H_2), we obtain from (5.3.1)

$$\sup\left\{\|D_\lambda\| + \left\|\lambda^{-2}AD_\lambda\right\|;\ \lambda \in \omega_\varphi + \Sigma_{\frac{\pi}{2}+\varphi}\right\} < \infty.$$

This combined with (H_2) and (H_3) yields that

$$\|\lambda\mathbf{R}(\lambda)\|,\ \left\|\lambda^{-1}\mathbf{A}\mathbf{R}(\lambda)\right\| \le M'|\lambda|^{-1}, \quad \lambda \in \omega_\varphi + \Sigma_{\frac{\pi}{2}+\varphi}, \tag{5.3.13}$$

for some constant $M' > M_\varphi$. From (5.3.10) we have

$$\widetilde{\mathbf{A}}\mathbf{R}(\lambda) = \begin{pmatrix} \widetilde{A}R_0(\lambda) & \widetilde{A}D_\lambda R_1(\lambda) \\ -G_0R_0(\lambda) & -G_0D_\lambda R_1(\lambda) + \widetilde{A}_1R_1(\lambda) \end{pmatrix}, \tag{5.3.14}$$

$$\widetilde{\mathbf{B}}\mathbf{R}(\lambda) = \begin{pmatrix} \widetilde{B}R_0(\lambda) & \widetilde{B}D_\lambda R_1(\lambda) \\ -G_1R_0(\lambda) & -G_1D_\lambda R_1(\lambda) + \widetilde{B}_1R_1(\lambda) \end{pmatrix}. \tag{5.3.15}$$

Since $\widetilde{A}$ (resp. $\widetilde{B}$) has A_0-bound (resp. B_0-bound) zero, there exists $a(\delta) > 0$, for each $\delta > 0$, such that for $\lambda \in \omega_\varphi + \Sigma_{\frac{\pi}{2}+\varphi}$,

$$\begin{aligned} \left\|\widetilde{A}R_0(\lambda)\right\| &\le \delta\|A_0R_0(\lambda)\| + a(\delta)\|R_0(\lambda)\| \\ &\le \delta\sup\left\{\|A_0R_0(\lambda)\|;\quad \lambda \in \omega_\varphi + \Sigma_{\frac{\pi}{2}+\varphi}\right\} \\ &\qquad +a(\delta)\sup\left\{\|\lambda^2R_0(\lambda)\|;\quad \lambda \in \omega_\varphi + \Sigma_{\frac{\pi}{2}+\varphi}\right\}|\lambda|^{-2}, \\ \left\|\lambda\widetilde{B}R_0(\lambda)\right\| &\le \delta\|\lambda B_0R_0(\lambda)\| + a(\delta)\|\lambda R_0(\lambda)\| \\ &\le \delta\sup\left\{\|\lambda B_0R_0(\lambda)\|;\quad \lambda \in \omega_\varphi + \Sigma_{\frac{\pi}{2}+\varphi}\right\} \\ &\qquad +a(\delta)\sup\left\{\|\lambda^2R_0(\lambda)\|;\quad \lambda \in \omega_\varphi + \Sigma_{\frac{\pi}{2}+\varphi}\right\}|\lambda|^{-1}. \end{aligned}$$

Recalling (H_2), which implies

$$\|B_0R_0(\lambda)\| \le (1 + 2M_\varphi)|\lambda|^{-1}, \quad \lambda \in \omega_\varphi + \Sigma_{\frac{\pi}{2}+\varphi},$$

we see that the above suprema are all finite. Hence, for each $\varepsilon > 0$, there exists $\beta(\varepsilon) > 0$ such that for $\lambda \in \omega_\varphi + \Sigma_{\frac{\pi}{2}+\varphi}$,

$$\left\|\widetilde{A}R_0(\lambda)\right\|, \quad \left\|\lambda\widetilde{B}R_0(\lambda)\right\| \le \varepsilon + \beta(\varepsilon)|\lambda|^{-1}.$$

The same is true of each of $\|G_0R_0(\lambda)\|$, $\|\lambda G_1R_0(\lambda)\|$, $\left\|\widetilde{A}_1R_1(\lambda)\right\|$, $\left\|\lambda\widetilde{B}_1R_1(\lambda)\right\|$.

Note that

$$\widetilde{A}D_\mu, \quad \widetilde{B}D_\mu \in \mathcal{L}(X,E), \tag{5.3.16}$$

by (5.3.5) and Lemma 5.3.1 (2). We deduce from (5.3.1), (5.3.12) and (5.3.16) that for $\lambda \in \omega_\varphi + \Sigma_{\frac{\pi}{2}+\varphi}$,

$$\begin{aligned}\left\|\widetilde{A}D_\lambda R_1(\lambda)\right\| &\leq \left\|\widetilde{A}D_\mu\right\| \|R_1(\lambda)\| + \left\|\widetilde{A}R_0(\lambda)\right\| \|D_\mu\| \,\|(\mu^2-\lambda^2)R_1(\lambda)\| \\ &\quad + \left\|\widetilde{A}R_0(\lambda)\right\| \|BD_\mu\| \|(\mu-\lambda)R_1(\lambda)\|, \\ \left\|\lambda\widetilde{B}D_\lambda R_1(\lambda)\right\| &\leq \left\|\widetilde{B}D_\mu\right\| \|\lambda R_1(\lambda)\| + \left\|\lambda\widetilde{B}R_0(\lambda)\right\| \|D_\mu\| \,\|(\mu^2-\lambda^2)R_1(\lambda)\| \\ &\quad + \left\|\lambda\widetilde{B}R_0(\lambda)\right\| \|BD_\mu\| \|(\mu-\lambda)R_1(\lambda)\|.\end{aligned}$$

Then, by (H_3) there is a constant $C_0 > 0$ such that for $\lambda \in \omega_\varphi + \Sigma_{\frac{\pi}{2}+\varphi}$,

$$\left\|\widetilde{A}D_\lambda R_1(\lambda)\right\| \leq C_0\left(|\lambda|^{-2} + \left\|\widetilde{A}R_0(\lambda)\right\|\right),$$

$$\left\|\lambda\widetilde{B}D_\lambda R_1(\lambda)\right\| \leq C_0\left(|\lambda|^{-1} + \left\|\lambda\widetilde{B}R_0(\lambda)\right\|\right).$$

Similarly, we have

$$\|G_0 D_\lambda R_1(\lambda)\| \leq C_1\left(|\lambda|^{-2} + \|G_0 R_0(\lambda)\|\right), \quad \lambda \in \omega_\varphi + \Sigma_{\frac{\pi}{2}+\varphi},$$

$$\|\lambda G_1 D_\lambda R_1(\lambda)\| \leq C_1\left(|\lambda|^{-1} + \|\lambda G_1 R_0(\lambda)\|\right) \quad \lambda \in \omega_\varphi + \Sigma_{\frac{\pi}{2}+\varphi},$$

for some constant $C_1 > 0$.

The above arguments imply the existence of a constant $\omega'_\varphi > \omega_\varphi$ such that

$$\left\|\widetilde{\mathbf{A}}\mathbf{R}(\lambda)\right\| + \left\|\lambda\widetilde{\mathbf{B}}\mathbf{R}(\lambda)\right\| \leq \frac{1}{2}, \quad \lambda \in \omega'_\varphi + \Sigma_{\frac{\pi}{2}+\varphi},$$

by the use of (5.3.14) and (5.3.15). Accordingly, we see that for $\lambda \in \omega'_\varphi + \Sigma_{\frac{\pi}{2}+\varphi}$,

$$\lambda^2 + \left(\mathbf{A} + \widetilde{\mathbf{A}}\right) + \lambda\left(\mathbf{B} + \widetilde{\mathbf{B}}\right) = \left[I + \widetilde{\mathbf{A}}\mathbf{R}(\lambda) + \lambda\widetilde{\mathbf{B}}\mathbf{R}(\lambda)\right](\lambda^2 + \mathbf{A} + \lambda\mathbf{B})$$

is invertible, and

$$\widetilde{\mathbf{R}}(\lambda) = \mathbf{R}(\lambda)\left[I + \widetilde{\mathbf{A}}\mathbf{R}(\lambda) + \lambda\widetilde{\mathbf{B}}\mathbf{R}(\lambda)\right]^{-1}.$$

This, together with (5.3.13), yields that for $\lambda \in \omega'_\varphi + \Sigma_{\frac{\pi}{2}+\varphi}$,

$$\left\|\lambda\widetilde{\mathbf{R}}(\lambda)\right\|, \left\|\lambda^{-1}\widetilde{\mathbf{A}}\widetilde{\mathbf{R}}(\lambda)\right\| \le 2M'|\lambda|^{-1},$$

$$\left\|\widetilde{\mathbf{B}}\widetilde{\mathbf{R}}(\lambda)\right\| \le \left\|\lambda^{-1} - \lambda\widetilde{\mathbf{R}}(\lambda) - \lambda^{-1}\widetilde{\mathbf{A}}\widetilde{\mathbf{R}}(\lambda)\right\| \le (1+2M')|\lambda|^{-1}.$$

The proof is now complete.

□

By virtue of Theorem 5.3.3, we can obtain a fundamental solution operator of (5.2.8) as below.

Theorem 5.3.4. *Assume that the conditions of Theorem* 5.3.3 *hold. Define*

$$\widetilde{\mathbf{S}}(0) = 0, \quad \widetilde{\mathbf{S}}(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\widetilde{\mathbf{R}}(\lambda)d\lambda \quad (t>0), \tag{5.3.17}$$

where Γ *is any piecewise smooth curve in* $\omega'_\varphi + \Sigma_{\frac{\pi}{2}+\varphi}$ $(\varphi \in (0,\theta))$ *going from* $\omega'_\varphi + \infty e^{-i\delta}$ *to* $\omega'_\varphi + \infty e^{i\delta}$ *(for some* $\delta \in \left(\frac{\pi}{2}, \frac{\pi}{2}+\varphi\right)$*), and leaving* ω'_φ *to its left. Then the following holds.*

(1) *The operator function* $\widetilde{\mathbf{S}}(\cdot)$ *can be extended analytically to* Σ_θ *such that*

$$\widetilde{\mathbf{S}}(z)y \in \mathcal{D}(\mathbf{A}) \cap \mathcal{D}(\mathbf{B}) \quad \text{for } y \in Y,\ z \in \Sigma_\theta,$$

and $\mathbf{A}\widetilde{\mathbf{S}}(\cdot)$, $\mathbf{B}\widetilde{\mathbf{S}}(\cdot)$ *are analytic in* Σ_θ.

(2) *For any* $\varphi \in (0,\theta)$, $\widetilde{\mathbf{S}}(\cdot)$ *is strongly continuous in* $\overline{\Sigma}_\varphi$.

(3) *For each* $y \in \overline{\mathcal{D}(\mathbf{A}) \cap \mathcal{D}(\mathbf{B})}$,

$$\lim_{t\to 0^+}\widetilde{\mathbf{S}}'(t)y = y, \lim_{t\to 0^+}\ \mathbf{B}\widetilde{\mathbf{S}}(t)y = 0,\ \lim_{t\to 0^+}\mathbf{A}\int_0^t \widetilde{\mathbf{S}}(s)yds = 0. \tag{5.3.18}$$

(4) *For each* $\varphi \in (0,\theta)$, *there exists* $\breve{M}_\varphi > 0$ *such that*

$$\left\|\widetilde{\mathbf{S}}'(z)\right\|,\ \left\|\mathbf{B}\widetilde{\mathbf{S}}(z)\right\|,\ \left\|\mathbf{A}\int_0^z \widetilde{\mathbf{S}}(\tau)d\tau\right\| \le \breve{M}_\varphi e^{\omega'_\varphi Re z}, \quad \text{for } z \in \Sigma_\varphi. \tag{5.3.19}$$

(5) *For any* $k \in \{0,1,2,3,4\}$, *there exist* M, $\omega > 0$ *such that*

$$\left\|\widetilde{\mathbf{S}}^{(k)}(t)\right\|, \ \left\|\mathbf{B}\widetilde{\mathbf{S}}^{(k-1)}(t)\right\|, \ \left\|\mathbf{A}\widetilde{\mathbf{S}}^{(k-2)}(t)\right\| \leq Mt^{-(k-1)}e^{\omega t}, \quad t>0, \tag{5.3.20}$$

where

$$\widetilde{\mathbf{S}}^{(-i)}(t) := \int_0^t (t-s)^{i-1}\widetilde{\mathbf{S}}(s)ds, \quad i=1,2.$$

(6) *For every* $z \in \Sigma_\theta$,

$$\widetilde{\mathbf{S}}''(z) + \left(\mathbf{B}+\widetilde{\mathbf{B}}\right)\widetilde{\mathbf{S}}'(z) + \left(\mathbf{A}+\widetilde{\mathbf{A}}\right)\widetilde{\mathbf{S}}(z) = 0, \tag{5.3.21}$$

$$\widetilde{\mathbf{S}}''(z)y + \widetilde{\mathbf{S}}'(z)\left(\mathbf{B}+\widetilde{\mathbf{B}}\right)y + \widetilde{\mathbf{S}}(z)\left(\mathbf{A}+\widetilde{\mathbf{A}}\right)y = 0, \quad y \in \mathcal{D}(\mathbf{A})\cap\mathcal{D}(\mathbf{B}). \tag{5.3.22}$$

Proof. By means of Theorem 5.3.3, the arguments similar to those in the proof of the implication (ii) $\Longrightarrow$ (i) of [84, Theorem 1.1, Section 4.1] justify assertions (1) - (4) and (6). In order to show assertion (5), we choose $\Gamma = \omega_\varphi + \Gamma_1$ with

$$\Gamma_1 := \left\{\rho e^{\pm i\frac{\pi+\varphi}{2}}; \quad \rho \geq 1\right\} \cup \left\{e^{i\theta}; |\theta| \leq \delta\right\}.$$

From (5.3.17) we have

$$\begin{aligned}
\widetilde{\mathbf{S}}^{(k)}(t) &= \frac{1}{2\pi i}\int_\Gamma \lambda^k e^{\lambda t}\widetilde{\mathbf{R}}(\lambda)d\lambda \\
&= \frac{t^{-1}e^{\omega_\varphi t}}{2\pi i}\int_{t\Gamma_1} \left(t^{-1}\mu+\omega_\varphi\right)^k e^{\mu}\widetilde{\mathbf{R}}(t^{-1}\mu+\omega_\varphi)d\mu \\
&= \frac{t^{-1}e^{\omega_\varphi t}}{2\pi i}\int_{\Gamma_1} \left(t^{-1}\mu+\omega_\varphi\right)^k e^{\mu}\widetilde{\mathbf{R}}(t^{-1}\mu+\omega_\varphi)d\mu, \\
\mathbf{B}\widetilde{\mathbf{S}}^{(k-1)}(t) &= \frac{1}{2\pi i}\int_\Gamma \lambda^{k-1} e^{\lambda t}\mathbf{B}\widetilde{\mathbf{R}}(\lambda)d\lambda \\
&= \frac{t^{-1}e^{\omega_\varphi t}}{2\pi i}\int_{\Gamma_1} \left(t^{-1}\mu+\omega_\varphi\right)^{k-1} e^{\mu}\mathbf{B}\widetilde{\mathbf{R}}(t^{-1}\mu+\omega_\varphi)d\mu, \\
\mathbf{A}\widetilde{\mathbf{S}}^{(k-2)}(t) &= \frac{1}{2\pi i}\int_\Gamma \lambda^{k-2} e^{\lambda t}\mathbf{A}\widetilde{\mathbf{R}}(\lambda)d\lambda \\
&= \frac{t^{-1}e^{\omega_\varphi t}}{2\pi i}\int_{\Gamma_1} \left(t^{-1}\mu+\omega_\varphi\right)^{k-2} e^{\mu}\mathbf{A}\widetilde{\mathbf{R}}(t^{-1}\mu+\omega_\varphi)d\mu.
\end{aligned}$$

Therefore, using Theorem 5.3.3 yields that for $t > 0$,

$$\begin{aligned}
&\left\|\widetilde{\mathbf{S}}^{(k)}(t)\right\|, \ \left\|\mathbf{B}\widetilde{\mathbf{S}}^{(k-1)}(t)\right\|, \ \left\|\mathbf{A}\widetilde{\mathbf{S}}^{(k-2)}(t)\right\| \\
\leq\ & \frac{1}{2\pi}t^{-(k-1)}e^{\omega_\varphi t}\int_\Gamma \left|\mu + t^2\right|^{k-2} e^{\mathrm{Re}\mu}|d\mu| \\
\leq\ & \text{const } t^{-(k-1)}\left(1+t^2\right)e^{\omega_\varphi t}.
\end{aligned}$$

The proof is then complete.

5.4 The main theorem for problem (5.2.8)

Definition 5.4.1. Assume that $\mathbf{A}$, $\mathbf{B}$ are closed, and $\widetilde{\mathbf{A}}$, $\widetilde{\mathbf{B}}$ satisfy (5.3.8). Let $h \in C([0,T];\mathbf{E})$.

(i) A function $y(\cdot)$ is called a *classical solution* of (5.2.8) if $y(\cdot) \in C^2((0,T];\mathbf{E}) \cap C^1([0,T];\mathbf{E})$,

$$y(\cdot) \in C((0,T];[\mathcal{D}(\mathbf{A})]), \quad \int_0^\cdot y(\sigma)d\sigma \in C([0,T];[\mathcal{D}(\mathbf{A})]),$$

$$y'(\cdot) \in C((0,T];[\mathcal{D}(\mathbf{B})]), \quad y(\cdot) - y(0) \in C([0,T];[\mathcal{D}(\mathbf{B})]),$$

and (5.2.8) is satisfied.

(ii) A function $y(\cdot)$ is called a *strict solution* of (5.2.8) if $y(\cdot) \in C^2([0,T];\mathbf{E}) \cap C([0,T];[\mathcal{D}(\mathbf{A})])$, $y'(\cdot) \in C([0,T];[\mathcal{D}(\mathbf{B})])$, and (5.2.8) is satisfied.

Remark 5.4.2. It can be seen from (5.3.8) that

(1) if $y(\cdot)$ is a classical solution of (5.2.8), then

$$\widetilde{\mathbf{B}}y'(\cdot), \quad \widetilde{\mathbf{A}}y(\cdot) \in C((0,T];\mathbf{E}),$$

$$\widetilde{\mathbf{B}}(y(\cdot) - y(0)), \quad \widetilde{\mathbf{A}}\int_0^\cdot y(\sigma)d\sigma \in C([0,T];\mathbf{E});$$

(2) if $y(\cdot)$ is a strict solution of (5.2.8), then

$$\widetilde{\mathbf{B}}y'(\cdot), \quad \widetilde{\mathbf{A}}y(\cdot) \in C([0,T];\mathbf{E}).$$

We here introduce a subset $\boldsymbol{\Upsilon}$ of $\mathbf{E}$, which is closely related to the Brézis-Fraenkel condition in [7] (see also [36, Appendix], [65]). Put

$$\boldsymbol{\Upsilon} := \left\{ y \in \mathcal{D}(\mathbf{B}); \quad \lim_{t\to 0^+} \Psi(t,y) = 0 \right\}, \tag{5.4.1}$$

where

$$\Psi(t,y) := \inf_{v\in\mathcal{D}(\mathbf{A})\cap\mathcal{D}(\mathbf{B})} \left(t\|v\|_{[\mathcal{D}(\mathbf{A})\cap\mathcal{D}(\mathbf{B})]} + \|y-v\|_{[\mathcal{D}(\mathbf{B})]} + t^{-1}\|y-v\| \right), \quad t \in (0,T], \ y \in \mathcal{D}(\mathbf{B}). \tag{5.4.2}$$

It is not difficult to see that

$$\mathcal{D}(\mathbf{A}) \cap \mathcal{D}(\mathbf{B}) \subset \boldsymbol{\Upsilon} \subset \overline{\mathcal{D}(\mathbf{A}) \cap \mathcal{D}(\mathbf{B})}.$$

We are now in a position to present our main theorem.

Theorem 5.4.3. *Let the hypotheses of Theorem* 5.3.3 *hold,* $h \in C^\alpha([0,T];\mathbf{E})$ $(\alpha \in (0,1))$*,* $y_0 \in \mathcal{D}(\mathbf{A}) \cup \boldsymbol{\Upsilon}$*, and* $y_1 \in \overline{\mathcal{D}(\mathbf{A}) \cap \mathcal{D}(\mathbf{B})}$*. Then*

(1) *problem* (5.2.8) *has a unique classical solution* $y(\cdot)$*, given by*

$$y(t) = \widetilde{\mathbf{C}}(t)y_0 + \widetilde{\mathbf{S}}(t)y_1 + \int_0^t \widetilde{\mathbf{S}}(t-s)h(s)ds, \quad t \in [0,T], \tag{5.4.3}$$

where for $t \in [0,T]$*,*

$$\widetilde{\mathbf{C}}(t)y_0 := \begin{cases} y_0 - \displaystyle\int_0^t \widetilde{\mathbf{S}}(s)(\mathbf{A}+\widetilde{\mathbf{A}})y_0 ds & if \quad y_0 \in \mathcal{D}(\mathbf{A}), \\ \left(\widetilde{\mathbf{S}}'(t) + \widetilde{\mathbf{S}}(t)(\mathbf{B}+\widetilde{\mathbf{B}})\right) y_0 & if \quad y_0 \in \boldsymbol{\Upsilon}. \end{cases} \tag{5.4.4}$$

(2) *the function* $y(\cdot)$ *satisfies the following regularity property and estimates:*

$$y''(\cdot), \ \mathbf{B}y'(\cdot), \mathbf{A}y(\cdot) \in C^\alpha([\varepsilon,T];\mathbf{E}), \quad \varepsilon \in (0,T); \tag{5.4.5}$$

$$\|y(t)\| \;\le\; const\,\Big(\|h\|_{C([0,T];\mathbf{E})} + \|y_0\|_{[\mathcal{D}(\mathbf{B})]} + \|y_1\|\Big) \qquad \text{if } \; y_0 \in \boldsymbol{\Upsilon},\; t\in[0,T]; \tag{5.4.6}$$

$$\|y'(t)\| \;\le\; const\,\Big(\|h\|_{C([0,T];\mathbf{E})} + \|y_0\|_{[\mathcal{D}(\mathbf{A})]} + \|y_1\|\Big) \qquad \text{if } \; y_0 \in \mathcal{D}(\mathbf{A}),\; t\in[0,T]; \tag{5.4.7}$$

$$\begin{aligned} &\|y''(t)\| + \|y'(t)\|_{[\mathcal{D}(\mathbf{B})]} + \|y(t)\|_{[\mathcal{D}(\mathbf{A})]} \\ \le\;& const\,\Big(\|h\|_{C^\alpha([0,T];\mathbf{E})} + \|y_0\|_{[\mathcal{D}(\mathbf{A})]} + \|y_1\|_{[\mathcal{D}(\mathbf{A})\cap\mathcal{D}(\mathbf{B})]}\Big) \\ &\qquad \text{if } \; y_0 \in \mathcal{D}(\mathbf{A}),\; y_1\in\mathcal{D}(\mathbf{A})\cap\mathcal{D}(\mathbf{B}),\; t\in(0,T]. \end{aligned} \tag{5.4.8}$$

(3) *the function* $y(t)$ *is a strict solution of* (5.2.8) *provided* $y_0 \in \mathcal{D}(\mathbf{A})$, $y_1 \in \boldsymbol{\Upsilon}$, *and*

$$(\mathbf{A}+\widetilde{\mathbf{A}})y_0 + (\mathbf{B}+\widetilde{\mathbf{B}})y_1 - h(0) \in \overline{\mathcal{D}(\mathbf{A})\cap\mathcal{D}(\mathbf{B})}. \tag{5.4.9}$$

Proof. We will use freely the closedness of $\mathbf{A}$, $\mathbf{B}$ and the fact (5.3.8) concerning $\widetilde{\mathbf{A}}$ and $\widetilde{\mathbf{B}}$. Put

$$y_*(t) := \int_0^t \widetilde{\mathbf{S}}(t-s)h(s)ds, \quad t\in[0,T].$$

We then have (noting $\widetilde{\mathbf{S}}(0)=0$)

$$y_*(t) = \int_0^t \widetilde{\mathbf{S}}(\sigma)h(t)d\sigma + \int_0^t \widetilde{\mathbf{S}}(t-\sigma)(h(\sigma)-h(t))d\sigma, \quad t\in[0,T], \tag{5.4.10}$$

$$y_*'(t) = \widetilde{\mathbf{S}}(t)h(t) + \int_0^t \widetilde{\mathbf{S}}'(t-\sigma)(h(\sigma)-h(t))d\sigma, \quad t\in[0,T], \tag{5.4.11}$$

$$y_*''(t) = \widetilde{\mathbf{S}}'(t)h(t) + \int_0^t \widetilde{\mathbf{S}}''(t-\sigma)(h(\sigma)-h(t))d\sigma, \quad t\in(0,T], \tag{5.4.12}$$

in view of the estimates

$$\|h(\sigma)-h(t)\| \le \text{const}\;(t-\sigma)^\alpha, \quad 0\le\sigma\le t\le T, \tag{5.4.13}$$

and (5.3.20). Thus, we infer by (5.4.13), (5.3.20), (5.3.21) and Theorem 5.3.4 (1) and (2) that

$$y_*''(\cdot), \quad \mathbf{B}y_*'(\cdot), \quad \mathbf{A}y_*(\sigma) \in C((0,T];\mathbf{E}), \tag{5.4.14}$$

$$y_*'(\cdot),\quad \mathbf{B}y_*(\cdot),\quad \mathbf{A}\int_0^{\cdot} y_*(\sigma)d\sigma \in C([0,T];\mathbf{E}), \tag{5.4.15}$$

$$y_*''(t) + \left(\mathbf{B} + \widetilde{\mathbf{B}}\right) y_*'(t) + \left(\mathbf{A} + \widetilde{\mathbf{A}}\right) y_*(t) = h(t), \quad t \in (0,T]. \tag{5.4.16}$$

Clearly

$$y_*(0) = 0, \quad y_*'(0) = 0, \tag{5.4.17}$$

by (5.4.10), (5.4.11) and (5.3.17). Next, we fix $\varepsilon \in (0,T)$. Using (5.4.12), (5.4.13) and (5.3.20) yields that for $\varepsilon \le s < t \le T$,

$$\begin{aligned}
&\|y_*''(t) - y_*''(s)\| \\
\le\ & \left\|\widetilde{\mathbf{S}}'(t)\right\| \|h(t) - h(s)\| + \left\|\int_s^t \widetilde{\mathbf{S}}''(\sigma)d\sigma\right\| \|h(s)\| \\
&\quad + \int_0^s \left\|\widetilde{\mathbf{S}}''(t-\sigma) - \widetilde{\mathbf{S}}''(s-\sigma)\right\| \|h(\sigma) - h(s)\| d\sigma \\
&\quad + \left\|\int_0^s \widetilde{\mathbf{S}}''(t-\sigma)d\sigma\right\| \|h(s) - h(t)\| \\
&\quad + \int_s^t \left\|\widetilde{\mathbf{S}}''(t-\sigma)\right\| \|h(\sigma) - h(t)\| d\sigma \\
\le\ & \text{const}\left[(t-s)^\alpha + \int_s^t \sigma^{-1}d\sigma + \int_0^s \left|\int_{s-\sigma}^{t-\sigma} \tau^{-2}d\tau\right| (s-\sigma)^\alpha d\sigma \right. \\
&\quad \left. + \left\|\widetilde{\mathbf{S}}(t-s) - \widetilde{\mathbf{S}}(t)\right\| (t-s)^\alpha + \int_s^t (t-\sigma)^{\alpha-1}d\sigma\right] \\
\le\ & \text{const}\left[(t-s)^\alpha + \varepsilon^{-1}(t-s) + (t-s)\int_0^s (t-\sigma)^{-1}(s-\sigma)^{\alpha-1}d\sigma\right] \\
\le\ & \text{const}\ (t-s)^\alpha.
\end{aligned}$$

In a similar way, we obtain from (5.4.10) and (5.4.11)

$$\|\mathbf{B}y_*'(t) - \mathbf{B}y_*'(s)\|,\ \|\mathbf{A}y_*(t) - \mathbf{A}y_*(s)\| \le \text{const}\ (t-s)^\alpha, \quad \varepsilon \le s < t \le T.$$

Therefore

$$y_*''(\cdot),\ \mathbf{B}y_*'(\cdot), \mathbf{A}y_*(\cdot) \in C^\alpha([\varepsilon,T];\mathbf{E}), \quad \varepsilon \in (0,T). \tag{5.4.18}$$

We now take care of $\widetilde{\mathbf{C}}(\cdot)y_0$ and $\widetilde{\mathbf{S}}(\cdot)y_1$. By (5.4.4) and the related properties of $\widetilde{\mathbf{S}}(\cdot)$ (see Theorem 5.3.4), we get

$$\widetilde{\mathbf{C}}(0)y_0 = y_0,\quad \widetilde{\mathbf{S}}(0)y_1 = 0,\quad \widetilde{\mathbf{C}}'(0)y_0 = 0,\quad \widetilde{\mathbf{S}}'(0)y_1 = y_1, \tag{5.4.19}$$

$$\widetilde{\mathbf{C}}''(\cdot)y_0,\ \mathbf{B}\widetilde{\mathbf{C}}'(\cdot)y_0,\ \mathbf{A}\widetilde{\mathbf{C}}(\cdot)y_0 \in C((0,T];\mathbf{E}), \tag{5.4.20}$$

$$\widetilde{\mathbf{S}}''(\cdot)y_1,\ \mathbf{B}\widetilde{\mathbf{S}}'(\cdot)y_1,\ \mathbf{A}\widetilde{\mathbf{S}}(\cdot)y_1 \in C((0,T];\mathbf{E}), \tag{5.4.21}$$

and

$$\begin{aligned}
&\widetilde{\mathbf{C}}''(t)y_0 + \widetilde{\mathbf{S}}''(t)y_1 + \left(\mathbf{B} + \widetilde{\mathbf{B}}\right)\left(\widetilde{\mathbf{C}}'(t)y_0 + \widetilde{\mathbf{S}}'(t)y_1\right) \\
&\quad + \left(\mathbf{A} + \widetilde{\mathbf{A}}\right)\left(\widetilde{\mathbf{C}}(t)y_0 + \widetilde{\mathbf{S}}(t)y_1\right) \\
&= 0, \quad t \in (0,T].
\end{aligned} \tag{5.4.22}$$

Moreover, using (5.3.20), we see easily that for $\varepsilon \le s < t \le T$,

$$\left.\begin{array}{l}
\left\|\widetilde{\mathbf{C}}''(t)y_0 - \widetilde{\mathbf{C}}''(s)y_0\right\| \\
\left\|\mathbf{B}\widetilde{\mathbf{C}}'(t)y_0 - \mathbf{B}\widetilde{\mathbf{C}}'(s)y_0\right\| \\
\left\|\mathbf{A}\widetilde{\mathbf{C}}(t)y_0 - \mathbf{A}\widetilde{\mathbf{C}}(s)y_0\right\|
\end{array}\right\} \le \text{const } (t-s), \tag{5.4.23}$$

$$\left.\begin{array}{l}
\left\|\widetilde{\mathbf{S}}''(t)y_1 - \widetilde{\mathbf{S}}''(s)y_1\right\| \\
\left\|\mathbf{B}\widetilde{\mathbf{S}}'(t)y_1 - \mathbf{B}\widetilde{\mathbf{S}}'(s)y_1\right\| \\
\left\|\mathbf{A}\widetilde{\mathbf{S}}(t)y_1 - \mathbf{A}\widetilde{\mathbf{S}}(s)y_1\right\|
\end{array}\right\} \le \text{const } (t-s). \tag{5.4.24}$$

In the following, we will show that

$$\widetilde{\mathbf{C}}'(t)y_0,\ \mathbf{B}\left(\widetilde{\mathbf{C}}(t)y_0 - y_0\right),\ \mathbf{A}\int_0^t \widetilde{\mathbf{C}}(\sigma)y_0 d\sigma \longrightarrow 0 \tag{5.4.25}$$

as $t \to 0^+$. When $y_0 \in \mathcal{D}(\mathbf{A})$, (5.4.25) follows immediately from (5.4.4) and Theorem 5.3.4 (2) and (4). Let now $y_0 \in \boldsymbol{\Upsilon}$. Making use of (5.3.20), (5.3.22) and noting $\widetilde{\mathbf{S}}(0) = 0$, $\widetilde{\mathbf{S}}'(0)v = v$ for $v \in \mathcal{D}(\mathbf{A}) \cap (\mathcal{D}(\mathbf{B})$, we obtain

$$\begin{aligned}
&\left\|\widetilde{\mathbf{C}}'(t)y_0\right\| \\
&= \inf_{v \in \mathcal{D}(\mathbf{A}) \cap \mathcal{D}(\mathbf{B})} \left\|\widetilde{\mathbf{C}}'(t)(y_0 - v) - \widetilde{\mathbf{S}}(t)\left(\mathbf{A} + \widetilde{\mathbf{A}}\right)v\right\| \\
&\le \text{const} \inf_{v \in \mathcal{D}(\mathbf{A}) \cap \mathcal{D}(\mathbf{B})} \left(t^{-1}\|y_0 - v\| + \left\|\left(\mathbf{B} + \widetilde{\mathbf{B}}\right)(y_0 - v)\right\| + t\left\|\left(\mathbf{A} + \widetilde{\mathbf{A}}\right)v\right\|\right) \\
&\le \text{const } \Psi(t, y_0), \quad t \in (0,T] \qquad \text{(by (5.4.2))},
\end{aligned}$$

$$
\begin{aligned}
& \|\mathbf{B}\left(\widetilde{\mathbf{C}}(t)y_0 - y_0\right)\| \\
= & \inf_{v\in\mathcal{D}(\mathbf{A})\cap\mathcal{D}(\mathbf{B})} \left\|\mathbf{B}\widetilde{\mathbf{C}}(t)(y_0 - v) - \mathbf{B}\int_0^t \widetilde{\mathbf{S}}(\sigma)(\mathbf{A}+\widetilde{\mathbf{A}})v d\sigma + \mathbf{B}(v - y_0)\right\| \\
\leq & \text{ const } \Psi(t, y_0), \quad t \in (0, T],
\end{aligned}
$$

and

$$
\begin{aligned}
& \left\|\mathbf{A}\int_0^t \widetilde{\mathbf{C}}(\sigma)y_0 d\sigma\right\| \\
= & \inf_{v\in\mathcal{D}(\mathbf{A})\cap\mathcal{D}(\mathbf{B})} \Bigg\|\mathbf{A}\left(\widetilde{\mathbf{S}}(t) + \int_0^t \widetilde{\mathbf{S}}(\sigma)\left(\mathbf{B}+\widetilde{\mathbf{B}}\right)d\sigma\right)(y_0 - v) \\
& \qquad - \mathbf{A}\int_0^t (t-\sigma)\widetilde{\mathbf{S}}(\sigma)\left(\mathbf{A}+\widetilde{\mathbf{A}}\right)v d\sigma + t\mathbf{A}v\Bigg\| \\
\leq & \text{ const } \Psi(t, y_0), \quad t \in (0, T].
\end{aligned}
$$

This leads to (5.4.25) in view of the definition of $\mathbf{\Upsilon}$ (see (5.4.1)). Combining (5.3.18), (5.4.14) – (5.4.17), (5.4.19) – (5.4.22), and (5.4.25) together, we deduce that the function $y(\cdot)$ defined by (5.4.3) is a classical solution of problem (5.2.8).

In order to show the uniqueness, let $v(\cdot)$ be another classical solution of (5.2.8). Then

$$
v'(t) - y'(t) + \left(\mathbf{B}+\widetilde{\mathbf{B}}\right)(v(t) - y(t)) + \left(\mathbf{A}+\widetilde{\mathbf{A}}\right)\int_0^t (v(s) - y(s))ds = 0, \quad t \in [0, T].
$$

So a calculation involving integration by parts shows that for $t \in [0, T]$, λ large enough,

$$
\begin{aligned}
& \left(\lambda + \left(\mathbf{B}+\widetilde{\mathbf{B}}\right) + \lambda^{-1}\left(\mathbf{A}+\widetilde{\mathbf{A}}\right)\right)\int_0^t e^{\lambda(t-s)}(v(s) - y(s))ds \\
= & -v(t) + y(t) + \lambda^{-1}\left(\mathbf{A}+\widetilde{\mathbf{A}}\right)\int_0^t (v(\sigma) - y(\sigma))d\sigma.
\end{aligned}
$$

Hence for $t \in [0, T]$,

$$
\lim_{\lambda\to\infty} e^{-\lambda}\int_0^t e^{\lambda(t-s)}(v(s) - y(s))ds = 0
$$

since $\lim_{\lambda\to\infty} \lambda^2 e^{-\lambda}\widetilde{\mathbf{R}}(\lambda)w = 0$ ($w \in \mathbf{E}$). This yields that $v(t) = y(t)$ for all $t \in [0, T]$, in view of [67, Lemma 1.1, p. 100]. Therefore, assertion (1) is valid. The regularity property (5.4.5) comes from (5.4.18), (5.4.23) and (5.4.24). Based on the expression (5.4.3) of $y(t)$, we derive the estimates (5.4.6) – (5.4.8) by (5.3.19) and (5.3.21).

Finally, assume $y_0 \in \mathcal{D}(\mathbf{A})$ and $y_1 \in \mathbf{\Upsilon}$ satisfying (5.4.9). To prove that $y(\cdot)$ (in this case) is a strict solution, we observe by (5.4.3), (5.3.22) and (5.4.12) that

$$\begin{aligned} y''(t) &= -\widetilde{\mathbf{S}}'(t)\left(\left(\mathbf{A}+\widetilde{\mathbf{A}}\right)y_0 + \left(\mathbf{B}+\widetilde{\mathbf{B}}\right)y_1 - h(0)\right) \\ &\quad +\widetilde{\mathbf{S}}''(t)y_1 + \widetilde{\mathbf{S}}'(t)\left(\mathbf{B}+\widetilde{\mathbf{B}}\right)y_1 + \widetilde{\mathbf{S}}'(t)(h(t)-h(0)) \\ &\quad + \int_0^t \widetilde{\mathbf{S}}''(t-\sigma)(h(\sigma)-h(t))d\sigma, \quad t \in (0, T]. \end{aligned}$$

The same reasoning as for (5.4.25) (in the case of $y_0 \in \mathbf{\Upsilon}$) gives that

$$\lim_{t\to 0^+}\left(\widetilde{\mathbf{S}}''(t)y_1 + \widetilde{\mathbf{S}}'(t)\left(\mathbf{B}+\widetilde{\mathbf{B}}\right)y_1\right) = 0.$$

Therefore

$$\lim_{t\to 0^+} y''(t) = -\left(\mathbf{A}+\widetilde{\mathbf{A}}\right)y_0 - \left(\mathbf{B}+\widetilde{\mathbf{B}}\right)y_1 + h(0), \tag{5.4.26}$$

by (5.3.18) – (5.3.20) and (5.4.13). Analogously, we obtain

$$\lim_{t\to 0^+} \mathbf{B}y'(t) = \mathbf{B}y_1, \quad \lim_{t\to 0^+} \mathbf{A}y(t) = \mathbf{A}y_0. \tag{5.4.27}$$

Thus, (5.4.26) and (5.4.27) together with assertion (1) justify assertion (3). This finishes the proof.

□

5.5 Examples

In this section, we present two examples, which do not aim at generality but indicate how our theorems can be applied to concrete problems.

Example 5.5.1. Let Ω be a bounded domain in R^n with smooth boundary $\partial\Omega$, and let $\rho > 0$. We consider the mixed boundary control problem for a structurally damped platelike equation (cf., e.g., [54, 55]):

$$\begin{cases} \partial_t^2 u + \Delta^2 u - \rho\Delta\partial_t u = 0 & \text{in } [0,T]\times\Omega, \\ \partial_t^2 \left(u\big|_{\partial\Omega}\right) = w & \text{in } [0,T]\times\partial\Omega, \\ \Delta u\big|_{\partial\Omega} = 0 & \text{in } [0,T]\times\partial\Omega, \\ u(0,\cdot) = \varphi_0, \quad \partial_t u(0,\cdot) = \varphi_1 & \text{in } \Omega, \end{cases} \tag{5.5.1}$$

where w is the control force.

The objective is to show that problem (5.5.1) (with a suitable w) is wellposed in $L^p(\Omega)$ $(1 < p < \infty)$.

We consider the case where w is built up by a feedback control law:

$$w = \langle \Delta u, a\rangle b + g$$

with

$$a \in L^q(\Omega) \ \left(\frac{1}{q} + \frac{1}{p} = 1\right), \quad b \in W^{2,p}(\partial\Omega),$$
$$g \in C^\alpha\left([0,T]; W^{2,p}(\partial\Omega)\right) \ (\alpha \in (0,1)).$$

When $a = 0$, (5.5.1) becomes an open loop problem.

In order to apply our theorems, we take

$$E = L^p(\Omega), \quad X = W^{2-\frac{1}{p},p}(\partial\Omega),$$

$$B = -\rho\Delta \quad \text{with} \quad \mathcal{D}(B) = W^{2,p}(\Omega),$$

$$A = \Delta^2 \quad \text{with} \quad \mathcal{D}(A) = \{\varphi \in \mathcal{D}(B^2); \ \ \Delta\varphi\big|_{\partial\Omega} = 0\},$$

$$G_0\varphi = \langle \Delta\varphi, a\rangle b \quad \text{for} \quad \varphi \in \mathcal{D}(G_0) := \mathcal{D}(A),$$
$$P\varphi = \varphi\Big|_{\partial\Omega} \quad \text{for} \quad \varphi \in \mathcal{D}(P) := \mathcal{D}(A), \quad P_1 = P,$$

$$A_1 = 0, \ \ B_1 = 0, \ \ \widetilde{A} = 0, \ \ \widetilde{A}_1 = 0, \ \ \widetilde{B} = 0, \ \ \widetilde{B}_1 = 0, \ \ G_1 = 0.$$

We claim that (H_1) is satisfied. In fact, a trace theorem [79, Section 5.5.2, p. 390, 391] says that

$$\mathcal{P} : \varphi \longmapsto \left(\Delta\varphi, \ \varphi\big|_{\partial\Omega}\right)$$

is an isomorphic mapping from $W^{2,p}(\Omega)$ onto $L^p(\Omega)\times W^{2-\frac{1}{p},p}(\partial\Omega)$. Hence, given $x\in W^{2-\frac{1}{p},p}(\partial\Omega)$, there exist $\varphi\in W^{2,p}(\Omega)$ such that

$$\Delta\varphi=0,\quad \varphi\big|_{\partial\Omega}=x.$$

It follows immediately that

$$\varphi\in\mathcal{D}(A)\quad\text{and}\quad P\varphi=x.$$

So $P(\mathcal{D}(A)\cap\mathcal{D}(B))=X$. Next we show the completeness of $[\mathcal{D}(A)]_P$. To this end, we take a Cauchy sequence $\{\psi_n\}_{n\in N}$ in $[\mathcal{D}(A)]_P$. Then, there exist r, $r_0\in L^p(\Omega)$ and $v\in W^{2-\frac{1}{p},p}(\partial\Omega)$ such that

$$\lim_{n\to\infty}\|\psi_n-r\|_{L^p(\Omega)}=0, \tag{5.5.2}$$

$$\lim_{n\to\infty}\left\|\Delta^2\psi_n-r_0\right\|_{L^p(\Omega)}=0, \tag{5.5.3}$$

$$\lim_{n\to\infty}\left\|\psi_n\big|_{\partial\Omega}-v\right\|_{W^{2-\frac{1}{p},p}(\partial\Omega)}=0, \tag{5.5.4}$$

$$\Delta\psi_n\big|_{\partial\Omega}=0. \tag{5.5.5}$$

According to (5.5.3) and (5.5.5), the isomorphism $\mathcal{P}$ implies the existence of r_1, $r_2\in W^{2,p}(\Omega)$ such that

$$\lim_{n\to\infty}\|\Delta\psi_n-r_1\|_{L^p(\Omega)}=0, \tag{5.5.6}$$

$$\Delta r_1=r_0,\quad r_1\big|_{\partial\Omega}=0. \tag{5.5.7}$$

Using (5.5.2), (5.5.4) and (5.5.6) yields that

$$r\in W^{2,p}(\Omega),\quad \Delta r=r_1,\quad r\big|_{\partial\Omega}=v.$$

From this, (5.5.7) and (5.5.2) – (5.5.5), we deduce that

$$r\in\mathcal{D}(A)\quad\text{and}\quad \lim_{n\to\infty}\|\psi_n-r\|_{A,P}=0.$$

Therefore $[\mathcal{D}(A)]_P$ is complete. The completeness of $[\mathcal{D}(B)]_{P_1}$ can be verified in the same way. Moreover, using the $\mathcal{P}$ again we find that

$$\|\cdot\|_{A,P}\sim\|\cdot\|_{W^{2,p}(\Omega)}+\|\Delta\cdot\|_{W^{2,p}(\Omega)},\quad \|\cdot\|_{B,P_1}\sim\|\cdot\|_{W^{2,p}(\Omega)}. \tag{5.5.8}$$

Clearly $B_0:=B\big|_{\ker P_1}=-\rho\Delta_D$ and $A_0:=A\big|_{\ker P}=\Delta_D^2$ (Δ_D is the Dirichlet Laplacian). By [36, Theorem 3.4], (H_2) holds. The first equivalent relation in (5.5.8) tells us that $G_0 \in \mathcal{L}([\mathcal{D}(A)]_P, X)$. Obviously, G_0 is relatively Δ_D^2-bounded with Δ_D^2-bound zero. Thus the hypotheses of Theorem 5.3.3 are fulfilled. So Theorem 5.3.3 is applicable to this situation, in which $\overline{\mathcal{D}(\mathbf{A}) \cap \mathcal{D}(\mathbf{B})} = \mathbf{E}$,

$$y(t) = \begin{pmatrix} u(t) \\ x(t) \end{pmatrix}, \quad u(t) = u(t,\cdot), \quad x(t) := u(t,\cdot)\big|_{\partial\Omega}, \quad t \in [0,T].$$

Noting (5.5.8), we then obtain the following conclusion:

For every $\varphi_0,\ \varphi_1 \in W^{2,p}(\Omega)$ *with* $\Delta\varphi_0,\ \Delta\varphi_1 \in W^{2,p}(\Omega)$ *and* $\Delta\varphi_0\big|_{\partial\Omega} = \Delta\varphi_1\big|_{\partial\Omega} = 0$*, problem* (5.5.1) *has a unique solution*

$$u \in C^2\left([0,T]; L^p(\Omega)\right) \cap C^1\left([0,T]; W^{2,p}(\Omega)\right); \tag{5.5.9}$$

moreover,

$$\Delta u \in C([0,T]; W^{2,p}(\Omega)),$$

$$\partial_t^2 u \in C^\alpha\left([\varepsilon,T]; L^p(\Omega)\right), \quad \partial_t u,\ \Delta u \in C^\alpha([\varepsilon,T]; W^{2,p}(\Omega)), \qquad \varepsilon \in (0,T),$$

and for $t \in [0,T]$*,*

$$\begin{aligned} &\left\|\partial_t^2 u(t,\cdot)\right\|_{L^p(\Omega)} + \|\partial_t u(t,\cdot)\|_{W^{2,p}(\Omega)} + \|\Delta u(t,\cdot)\|_{W^{2,p}(\Omega)} \\ \leq\ & const \left(\|g\|_{C^\alpha([0,T];W^{2,p}(\partial\Omega))} + \sum_{j=0}^{1} \|\varphi_j\|_{W^{2,p}(\Omega)} + \|\Delta\varphi_j\|_{W^{2,p}(\Omega)} \right). \end{aligned}$$

Here, for obtaining the uniqueness we used the fact that if u is a solution of problem (5.5.1) satisfying (5.5.9), then $(\partial_t u(t,\cdot))\big|_{\partial\Omega} = \partial_t \left(u(t,\cdot)\big|_{\partial\Omega}\right)$, by virtue of the isomorphism $\mathcal{P}$, and therefore

$$x'(t) = P_1 u'(t) \ \ (t \in [0,T]), \quad x(0) = \varphi_0\big|_{\partial\Omega}, \quad x'(0) = \varphi_1\big|_{\partial\Omega}.$$

Remark 5.5.2. To our knowledge, the result in Example 5.5.1(involving the second order dynamic on the boundary) is new even for the case of $p = 2$ and $a = 0$.

Example 5.5.3. Let $\rho > 0$, $\alpha \in (0,1)$, $f \in C^\alpha([0,T]; C[0,1])$,

$$g_j,\ h_j \in C^\alpha([0,T]; \mathbf{C}), \quad j = 0,1.$$

For each i, $j = 0$, 1, let $\mathcal{A}_{ij}(\partial_\xi)$ (resp. $\mathcal{B}_{ij}(\partial_\xi)$) be a linear differential operator in $[0, 1]$ with complex coefficients of the order not exceeding 3 (resp. of order one). We consider a damped Euler-Bernoulli beam equation (cf., e.g., [6, 11, 45]) with dynamic boundary conditions:

$$\begin{cases} \partial_t^2 u + \partial_\xi^4 u - \rho \partial_\xi^2 \partial_t u = f & \text{in } (0, T] \times [0, 1], \\ \partial_t^2 u(t, j) + \mathcal{A}_{0j}(\partial_\xi) u(t, j) + \mathcal{B}_{0j}(\partial_\xi) \partial_t u(t, j) = g_j & \text{in } (0, T] \times \{0, 1\}, \\ \partial_t^2 \partial_\xi^2 u(t, j) + \mathcal{A}_{1j}(\partial_\xi) u(t, j) + \mathcal{B}_{1j}(\partial_\xi) \partial_t u(t, j) = h_j & \text{in } (0, T] \times \{0, 1\}, \\ u(0, \cdot) = \varphi_0, \quad \partial_t u(0, \cdot) = \varphi_1 & \text{in } [0, 1], \\ \partial_t \partial_\xi^2 u(0, j) = \psi_j, & j = 0, 1. \end{cases} \tag{5.5.10}$$

Take

$$E = C[0, 1], \quad X = \mathbf{C}^4,$$

$$A = \frac{d^4}{d\xi^4} \quad \text{with} \quad \mathcal{D}(A) = C^4[0, 1],$$

$$B = -\rho \frac{d^2}{d\xi^2} \quad \text{with} \quad \mathcal{D}(B) = C^2[0, 1],$$

$$G_0 \varphi = - \begin{pmatrix} \mathcal{A}_{00}(\partial_\xi)\varphi(0) \\ \mathcal{A}_{01}(\partial_\xi)\varphi(1) \\ \mathcal{A}_{10}(\partial_\xi)\varphi(0) \\ \mathcal{A}_{11}(\partial_\xi)\varphi(1) \end{pmatrix} \quad \text{for} \quad \varphi \in \mathcal{D}(G_0) := \mathcal{D}(A),$$

$$G_1 \varphi = - \begin{pmatrix} \mathcal{B}_{00}(\partial_\xi)\varphi(0) \\ \mathcal{B}_{01}(\partial_\xi)\varphi(1) \\ \mathcal{B}_{10}(\partial_\xi)\varphi(0) \\ \mathcal{B}_{11}(\partial_\xi)\varphi(1) \end{pmatrix} \quad \text{for} \quad \varphi \in \mathcal{D}(G_1) := \mathcal{D}(B),$$

$$P\varphi = \begin{pmatrix} \varphi(0) \\ \varphi(1) \\ \varphi''(0) \\ \varphi''(1) \end{pmatrix} \quad \text{for} \quad \varphi \in \mathcal{D}(P) := \mathcal{D}(A),$$

$$P_1\varphi = \left\{ \begin{pmatrix} \varphi(0) \\ \varphi(1) \\ z_3 \\ z_4 \end{pmatrix} ; \quad z_3, \ z_4 \in \mathbf{C} \right\} \quad \text{for} \quad \varphi \in \mathcal{D}(P_1) := \mathcal{D}(B),$$

$$A_1 = 0, \quad B_1 = 0, \quad \widetilde{A} = 0, \quad \widetilde{A}_1 = 0, \quad \widetilde{B} = 0, \quad \widetilde{B}_1 = 0.$$

Then we have

$$A_0 = \frac{d^4}{d\xi^4} \ \text{ with } \ \mathcal{D}(A_0) = \{\varphi \in C^4[0,1]; \ \varphi(0) = \varphi(1) = \varphi''(0) = \varphi''(1) = 0\},$$

$$B_0 = -\rho\frac{d^2}{d\xi^2} \ \text{ with } \ \mathcal{D}(B_0) = \{\varphi \in C^2[0,1]; \ \varphi(0) = \varphi(1) = 0\},$$

$$[\mathcal{D}(A)]_P \simeq C^4[0,1], \quad [\mathcal{D}(B)]_{P_1} \simeq C^2[0,1].$$

Obviously (H_1) and (H_3) are satisfied. So is (H_2) by [36, p. 1017, line 4]. Furthermore, we know that G_0 (resp. G_1) is A_0-bounded (resp. B_0-bounded) with A_0-bound (resp. B_0-bound) zero (cf. [26, p. 170]). Thus the hypotheses of Theorem 5.3.3 are all satisfied. Therefore Theorem 5.4.3 is applicable. In this case,

$$y(t) = \begin{pmatrix} u(t) \\ x(t) \end{pmatrix}, \quad u(t) = u(t,\cdot), \quad x(t) := \begin{pmatrix} u(t,0) \\ u(t,1) \\ \partial_\xi^2 u(t,0) \\ \partial_\xi^2 u(t,1) \end{pmatrix}, \quad t \in (0,T],$$

$$\overline{\mathcal{D}(\mathbf{A}) \cap \mathcal{D}(\mathbf{B})} = \left\{ \begin{pmatrix} \varphi \\ x \end{pmatrix} \in C[0,1] \times \mathbf{C}^4; \quad x \in P_1\varphi \right\}, \tag{5.5.11}$$

$$\boldsymbol{\Upsilon} \supset \left\{ \begin{pmatrix} \varphi \\ x \end{pmatrix} \in C^2[0,1] \times \mathbf{C}^4; \quad P\varphi = x \right\}. \tag{5.5.12}$$

It is not hard to verify (5.5.11). For (5.5.12), we exploit the fact (shown in the proof of [36, Theorem 5.1]) that

$$\lim_{t\to 0^+} \inf \left\{ t\|\psi\|_{C^4[0,1]} + \|\varphi-\psi\|_{C^2[0,1]} + t^{-1}\|\varphi-\psi\|_{C[0,1]}; \quad \psi \in \mathcal{D}(A_0) \right\} = 0 \tag{5.5.13}$$

for every

$$\varphi \in \Omega := \left\{ \psi \in C^2[0,1]; \;\; \psi(0) = \psi(1) = \psi''(0) = \psi''(1) = 0 \right\}.$$

Given $\varphi \in C^2[0,1]$, we put

$$\varphi_*(\xi) = \varphi(0) + (\varphi(1)-\varphi(0))\xi + \frac{1}{2}\varphi''(0)\xi^2 + \frac{1}{6}(\varphi''(1)-\varphi''(0))\xi^3, \quad \xi \in [0,1].$$

Then $\varphi - \varphi_* \in \Omega$. This in combination with (5.5.13) yields that

$$\lim_{t\to 0^+} \Psi\left(t, \begin{pmatrix} \varphi \\ P\varphi \end{pmatrix}\right) = 0$$

and so $\begin{pmatrix} \varphi \\ P\varphi \end{pmatrix} \in \boldsymbol{\Upsilon}$. We now use Theorem 5.4.3 to conclude:

(i) *For every $\varphi_0 \in C^2[0,1]$, $\varphi_1 \in C^1[0,1]$, $\psi_j \in \mathbf{C}$ $(j = 0,1)$, problem* (5.5.10) *has a unique solution*

$$u \in \bigcap_{i=0}^{2} C^i\left((0,T]; C^{4-2i}[0,1]\right) \bigcap \left(\bigcap_{k=0}^{1} C^k([0,T]; C^{2-2k}[0,1] \right); \tag{5.5.14}$$

(ii) $\partial_t^i u \in C^\alpha\left([\varepsilon,T]; C^{4-2i}[0,1]\right)$ $(\varepsilon \in (0,T),\ i = 0,\ 1,\ 2)$ *and*

$$\|u(t,\cdot)\|_{C[0,1]} \le const \left[\|f\|_{C([0,T];C[0,1])} + \sum_{j=0}^{1} \Big(\|g_j\|_{C([0,T];\mathbf{C})} \right.$$

$$\left. + \|h_j\|_{C([0,T];\mathbf{C})} + |\psi_j| \Big) + \|\varphi_0\|_{C^2[0,1]} + \|\varphi_1\|_{C[0,1]} \right], \quad t \in [0,T].$$

(iii) *If $\varphi_0 \in C^4[0,1]$, $\varphi_1 \in C^2[0,1]$, $\psi_j = \varphi_1''(j)$ $(j = 0,1)$, and*

$$\varphi_0^{(4)}(j) + \mathcal{A}_{0j}(\partial_\xi)\varphi_0(j) - \rho\varphi_1''(j) + \mathcal{B}_{0j}(\partial_\xi)\varphi_1(j) = f(0,j) - g_j, \quad j = 0,1,$$

then the solution

$$u \in \bigcap_{i=0}^{2} C^i\left([0,T]; C^{4-2i}[0,1]\right).$$

Here, for obtaining the uniqueness, the following fact was taken into account: if u is a solution of problem (5.5.10) satisfying (5.5.14), then

$$x(t) = Pu(t), \quad x'(t) = Pu'(t), \quad t \in (0, T],$$

$$x(0) = \begin{pmatrix} \varphi_0(0) \\ \varphi_0(1) \\ \varphi_0''(0) \\ \varphi_0''(1) \end{pmatrix}, \quad x'(0) = \begin{pmatrix} \varphi_1(0) \\ \varphi_1(1) \\ \psi_0 \\ \psi_1 \end{pmatrix}.$$

Remark 5.5.4. In the case of zero boundary value, i.e., when $\mathcal{A}_{ij}$, $\mathcal{B}_{ij}$, g_i, h_j, $\varphi_i(j)$, and ψ_j (i, $j = 0$, 1) are all zero, Conclusion (i) and a weaker form of conclusion (iii) are due to [36, Theorem 5.1].

Chapter 6

Complete second order abstract differential equations with dynamic boundary conditions

6.1 Summary

In this chapter, we continue to deal with the mixed initial boundary value problem (5.2.1) - (5.2.6) for complete second order (in time) linear differential equations in Banach spaces, in which time-derivatives occur in the boundary conditions. General wellposedness theorems are obtained (for the first time) which are used to solve the corresponding inhomogeneous problems. Examples of applications to initial boundary value problems for partial differential equations are also presented.

6.2 Preliminaries

In this section, we recall the definition of strong wellposedness for a general second order abstract Cauchy problem, introduce the notion of strong quasi-wellposedness and give the corresponding characterization theorems.

Let $\mathbf{X}$ and $\mathbf{Y}$ be two Banach spaces, and $\mathbf{A}$ and $\mathbf{B}$ closed linear operators in $\mathbf{X}$. We shall use the following notations.

$\mathbf{R}(\lambda) := (\lambda^2 + \mathbf{A} + \lambda\mathbf{B})^{-1}$ (if the inverse exists),

$\mathbf{Y} \hookrightarrow \mathbf{X}$: $\mathbf{Y}$ continuously embedded in $\mathbf{X}$,

$\rho(\mathbf{A}, \mathbf{B}) := \{\lambda \in \mathbf{C};\ \ \mathbf{R}(\lambda)$ exists and is in $\mathcal{L}(\mathbf{X})\}$,

$\rho_0(\mathbf{A},\mathbf{B}) := \{\lambda \in \rho(\mathbf{A},\mathbf{B});\ \ \mathbf{R}(\lambda)\mathbf{A}$ is closable$\}$,

$C(R^+;\mathcal{L}_s(\mathbf{X},\mathbf{Y}))$: the space of all strongly continuous $\mathcal{L}(\mathbf{X},\mathbf{Y})$-valued functions on R^+,

$C_{eb}(R^+;\mathcal{L}_s(\mathbf{X},\mathbf{Y})) := \{\mathbf{K} \in C(R^+;\mathcal{L}_s(\mathbf{X},\mathbf{Y}));$ there are constants M, $\omega \geq 0$ such that $\|\mathbf{K}(t)\|_{\mathcal{L}(\mathbf{X},\mathbf{Y})} \leq Me^{\omega t}$ $(t \geq 0)\}$.

Definition 6.2.1. The Cauchy problem

$$\begin{cases} \mathbf{x}''(t) + \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{x}'(t) = 0, \quad t \geq 0, \\ \mathbf{x}(0) = \mathbf{x}_0,\ \mathbf{x}'(0) = \mathbf{x}_1, \end{cases} \tag{$ACP_2;\mathbf{A},\mathbf{B}$}$$

is *wellposed* if

(i) there exist dense subspace $\mathbf{X}_0$, $\mathbf{X}_1$ of $\mathbf{X}$ such that for any $\mathbf{x}_0 \in \mathbf{X}_0$, $\mathbf{x}_1 \in \mathbf{X}_1$, $(ACP_2;\mathbf{A},\mathbf{B})$ has a solution;

(ii) there exists a locally bounded function $M(\cdot) : R^+ \to R^+$ such that

$$\|\mathbf{x}(t)\| \leq M(t)(\|\mathbf{x}_0\| + \|\mathbf{x}_1\|), \quad t \geq 0, \tag{6.2.1}$$

for any solution $\mathbf{x}(t)$ of $(ACP_2;\mathbf{A},\mathbf{B})$.

For $t \geq 0$, $\mathbf{x}_0 \in \mathbf{X}_0$, $\mathbf{x}_1 \in \mathbf{X}_1$, set

$$\mathbf{C}(t)\mathbf{x}_0 := \mathbf{x}_0(t), \quad \mathbf{S}(t)\mathbf{x}_1 := \mathbf{x}_1(t),$$

where $\mathbf{x}_0(\cdot)$ (resp. $\mathbf{x}_1(\cdot)$) is the solution of $(ACP_2;\mathbf{A},\mathbf{B})$ with $\mathbf{x}_0(0) = \mathbf{x}_0$, $\mathbf{x}_0'(0) = 0$ (resp. $\mathbf{x}_1(0) = 0$, $\mathbf{x}_1'(0) = \mathbf{x}_1$). By (6.2.1), $\mathbf{C}(t)$ and $\mathbf{S}(t)$ (for each $t \geq 0$) can be extended to all of $\mathbf{X}$ as bounded linear operators, since $\mathbf{X}_0$ and $\mathbf{X}_1$ are dense in $\mathbf{X}$. We call $\mathbf{C}(\cdot)$, $\mathbf{S}(\cdot)$ the propagators (or solution operators) of $(ACP_2;\mathbf{A},\mathbf{B})$.

Definition 6.2.2. $(ACP_2;\mathbf{A},\mathbf{B})$ is called to be *strongly wellposed* if it is wellposed, and

$$\mathbf{S}(\cdot)\mathbf{x} \in C^1(R^+;\mathbf{X}) \cap C(R^+;[\mathcal{D}(\mathbf{B})]) \quad \text{for every } \mathbf{x} \in \mathbf{X}. \tag{6.2.2}$$

Proposition 6.2.3. ([31, Chapter VIII]) *Suppose that* $(ACP_2;\mathbf{A},\mathbf{B})$ *is strongly wellposed. Then*

(i) $\mathcal{D}(\mathbf{A}) \cap \mathcal{D}(\mathbf{B})$ *is dense in* $\mathbf{X}$*, and* $\mathcal{D}(\mathbf{A}) \subset \mathbf{X}_0$*,* $\mathcal{D}(\mathbf{A}) \cap \mathcal{D}(\mathbf{B}) \subset \mathbf{X}_1$*;*

(ii) *there exists constants* M*,* $\omega > 0$ *such that*

$$\|\mathbf{C}(t)\|,\ \|\mathbf{S}'(t)\|,\ \|\mathbf{B}\mathbf{S}(t)\| \le Me^{\omega t}, \quad t \ge 0; \tag{6.2.3}$$

(iii) $(\omega, \infty) \subset \rho_0(\mathbf{A}, \mathbf{B})$ *and for* $\lambda > \omega$*,*

$$\int_0^\infty e^{-\lambda t}\mathbf{S}(t)\mathbf{x}dt = \mathbf{R}(\lambda)\mathbf{x}, \quad \mathbf{x} \in \mathbf{X}; \tag{6.2.4}$$

(iv)

$$\mathbf{C}(t)\mathbf{x} := \mathbf{x} - \int_0^t \mathbf{S}(s)\mathbf{A}\mathbf{x}ds, \quad \mathbf{x} \in \mathcal{D}(\mathbf{A}). \tag{6.2.5}$$

As an immediate consequence of Proposition 6.2.3 and [82, Theorem 1] (see also [84, Theorem 2.3, p. 57]), we have

Proposition 6.2.4. *$(ACP_2; \mathbf{A}, \mathbf{B})$ is strongly wellposed if and only if $\mathcal{D}(\mathbf{A}) \cap \mathcal{D}(\mathbf{B})$ is dense in $\mathbf{X}$, $(\omega, \infty) \subset \rho_0(\mathbf{A}, \mathbf{B})$ for some $\omega > 0$, and*

$$\lambda \mapsto \lambda\mathbf{R}(\lambda),\ \lambda \mapsto \lambda^{-1}\mathbf{A}\mathbf{R}(\lambda),\ \lambda \mapsto \lambda^{-1}\overline{\mathbf{R}(\lambda)\mathbf{A}} \in LT - \mathcal{L}(\mathbf{X}).$$

Definition 6.2.5. Let $\mathcal{D}(\mathbf{A}) \cap \mathcal{D}(\mathbf{B})$ be dense in $\mathbf{X}$. $(ACP_2; \mathbf{A}, \mathbf{B})$ is called to be *strongly quasi-wellposed* if it has a solution for $\mathbf{x}_0 \in \mathcal{D}(\mathbf{A})$, $\mathbf{x}_1 \in \mathcal{D}(\mathbf{A}) \cap \mathcal{D}(\mathbf{B})$, and there exist two operator functions (propagators)

$$\mathbf{C}(\cdot) \in C(R^+; \mathcal{L}_s([\mathcal{D}(\mathbf{A})])), \quad \mathbf{S}(\cdot) \in C(R^+; \mathcal{L}_s(\mathbf{X})), \tag{6.2.6}$$

satisfying (6.2.2) and (6.2.3) (with $\|\mathbf{C}(t)\|_{\mathcal{L}([\mathcal{D}(\mathbf{A})])}$ instead of $\|\mathbf{C}(t)\|$), such that every solution can be expressed as

$$\mathbf{x}(t) = \mathbf{C}(t)\mathbf{x}_0 + \mathbf{S}(t)\mathbf{x}_1, \quad t \ge 0. \tag{6.2.7}$$

Remark 6.2.6. Clearly, $(ACP_2; \mathbf{A}, \mathbf{B})$ is strongly quasi-wellposed if it is strongly wellposed.

Proposition 6.2.7. *$(ACP_2; \mathbf{A}, \mathbf{B})$ is strongly quasi-wellposed if and only if $\mathcal{D}(\mathbf{A}) \cap \mathcal{D}(\mathbf{B})$ is dense in $\mathbf{X}$, $(\omega, \infty) \subset \rho(\mathbf{A}, \mathbf{B})$ for some $\omega > 0$, and*

$$\lambda \mapsto \lambda \mathbf{R}(\lambda), \ \lambda \mapsto \lambda^{-1} \mathbf{A}\mathbf{R}(\lambda) \in LT - \mathcal{L}(\mathbf{X}).$$

In this case, (6.2.4) and (6.2.5) hold.

Proof. The "only if" part.

It is not difficult to obtain

$$\mathbf{S}'(t)\mathbf{x} + \mathbf{B}\mathbf{S}(t)\mathbf{x} + \mathbf{A} \int_0^t \mathbf{S}(s)\mathbf{x}ds = \mathbf{x}, \quad t \geq 0, \ \mathbf{x} \in \mathbf{X}, \tag{6.2.8}$$

$$\mathbf{S}'(t)\mathbf{x} + \mathbf{S}(t)\mathbf{B}\mathbf{x} + \int_0^t \mathbf{S}(s)\mathbf{A}\mathbf{x}ds = \mathbf{x}, \quad t \geq 0, \ \mathbf{x} \in \mathcal{D}(\mathbf{A}) \cap \mathcal{D}(\mathbf{B}). \tag{6.2.9}$$

Taking Laplace transforms, integrating by parts and using the closedness of $\mathbf{A}$ and $\mathbf{B}$ yields that for $\lambda > \omega$,

$$\left(\lambda + \mathbf{B} + \lambda^{-1}\mathbf{A}\right) \int_0^\infty e^{-\lambda t}\mathbf{S}(t)\mathbf{x}dt = \lambda^{-1}\mathbf{x}, \quad \mathbf{x} \in \mathbf{X},$$

$$\int_0^\infty e^{-\lambda t}\mathbf{S}(t) \left(\lambda + \mathbf{B} + \lambda^{-1}\mathbf{A}\right) \mathbf{x}dt = \lambda^{-1}\mathbf{x}, \quad \mathbf{x} \in \mathcal{D}(\mathbf{A}) \cap \mathcal{D}(\mathbf{B}).$$

So (6.2.4) follows. We see from (6.2.3) and (6.2.4) that

$$\lambda \mapsto \lambda \mathbf{R}(\lambda), \ \lambda \mapsto \mathbf{B}\mathbf{R}(\lambda) \in LT - \mathcal{L}(\mathbf{X}).$$

Therefore

$$\lambda \mapsto \lambda^{-1}\mathbf{A}\mathbf{R}(\lambda) = \lambda^{-1} - \lambda\mathbf{R}(\lambda) - \mathbf{B}\mathbf{R}(\lambda) \in LT - \mathcal{L}(\mathbf{X}).$$

The "if" part.

By hypothesis, there exists $\mathbf{J}_i(\cdot) \in C_{eb}(R^+; \mathcal{L}_s(\mathbf{X}))$ $(i = 1, 2)$ such that for λ large enough,

$$\lambda \mathbf{R}(\lambda)\mathbf{x} = \int_0^\infty e^{-\lambda t}\mathbf{J}_1(t)\mathbf{x}dt, \quad \mathbf{x} \in \mathbf{X},$$

$$\mathbf{B}\mathbf{R}(\lambda)\mathbf{x} = \int_0^\infty e^{-\lambda t}\mathbf{J}_2(t)\mathbf{x}dt, \quad \mathbf{x} \in \mathbf{X}.$$

Define, for $t \geq 0$,

$$\mathbf{S}(t)\mathbf{x} := \int_0^t \mathbf{J}_1(s)\mathbf{x}ds, \quad \mathbf{x} \in \mathbf{X},$$

$$\mathbf{C}(t)\mathbf{x} := \mathbf{x} - \int_0^t \mathbf{S}(s)\mathbf{A}\mathbf{x}ds, \quad \mathbf{x} \in \mathcal{D}(\mathbf{A}).$$

Then we see that

$$\mathbf{BS}(t) = \mathbf{J}_2(t), \quad t \geq 0,$$

$$\mathbf{A}\int_0^t \mathbf{S}(s)\mathbf{x}ds = \mathbf{x} - \mathbf{J}_1(t)\mathbf{x} - \mathbf{J}_2(t)\mathbf{x}, \quad t \geq 0,$$

by the uniqueness theorem for Laplace transforms. Therefore (6.2.2), (6.2.3) and (6.2.6) are true. The same reasoning as in [84, p. 63] gives that

$$\mathbf{x}(\cdot) := \mathbf{C}(\cdot)\mathbf{x}_0 + \mathbf{S}(\cdot)\mathbf{x}_1$$

is a solution of $(ACP_2; \mathbf{A}, \mathbf{B})$ for every $\mathbf{x}_0 \in \mathcal{D}(\mathbf{A})$, $\mathbf{x}_1 \in \mathcal{D}(\mathbf{A}) \cap \mathcal{D}(\mathbf{B})$.

Finally, let $\mathbf{w}(\cdot)$ be a solution of $(ACP_2; \mathbf{A}, \mathbf{B})$. Then obviously, $\mathbf{w}(0) \in \mathcal{D}(\mathbf{A})$. Putting

$$\mathbf{w}_*(t) = \mathbf{w}(t) - \mathbf{C}(t)\mathbf{w}(0), \quad t \geq 0,$$

we see that

$$\mathbf{w}_*''(t) + \mathbf{B}\mathbf{w}_*'(t) + \mathbf{A}\mathbf{w}_*(t) = 0, \quad t \geq 0,$$

$$\mathbf{w}_*(0) = 0, \quad \mathbf{w}_*'(0) = \mathbf{w}'(0).$$

So

$$\mathbf{w}_*'(t) + \mathbf{B}\mathbf{w}_*(t) + \mathbf{A}\int_0^t \mathbf{w}_*(s)ds = \mathbf{w}'(0), \quad t \geq 0.$$

This combined with (6.2.8) implies that

$$\mathbf{v}(\cdot) := \mathbf{w}_*(\cdot) - \mathbf{S}(\cdot)\mathbf{w}'(0)$$

satisfies

$$\mathbf{v}'(t) + \mathbf{B}\mathbf{v}(t) + \mathbf{A}\int_0^t \mathbf{v}(s)ds = 0, \quad t \geq 0,$$

$$\mathbf{v}(0) = \mathbf{v}'(0) = 0.$$

Then arguing similarly as in the proof of [84, Lemma 3.1, p. 67], we obtain $\mathbf{v}(t) \equiv 0$ on $[0, \infty)$. Hence

$$\mathbf{w}(t) = \mathbf{C}(t)\mathbf{w}(0) + \mathbf{S}(t)\mathbf{w}'(0), \quad t \geq 0.$$

The proof is complete.

□

We close this section by stating some assumptions on the operators A_0, B_0, A_1 and B_1 which will be used selectively in our theorems.

(H_4) The operators A_0 and B_0 are closed, with dense $\mathcal{D}(A_0) \cap \mathcal{D}(B_0)$, such that $(\omega, \infty) \subset \rho_0(A_0, B_0)$ for some $\omega > 0$, and

$$\lambda \mapsto \lambda R_0(\lambda),\ \lambda \mapsto \lambda^{-1} A_0 R_0(\lambda),\ \lambda \mapsto \lambda^{-1} \overline{R_0(\lambda) A_0} \in LT - \mathcal{L}(E),$$

where $R_0(\lambda)$ and $R_1(\lambda)$ in (H_5) below are as in (5.2.9).

(H_4') The operators A_0 and B_0 are closed, with dense $\mathcal{D}(A_0) \cap \mathcal{D}(B_0)$, such that $(\omega, \infty) \subset \rho(A_0, B_0)$ for some $\omega > 0$, and

$$\lambda \mapsto \lambda R_0(\lambda),\ \lambda \mapsto \lambda^{-1} A_0 R_0(\lambda) \in LT - \mathcal{L}(E).$$

(H_5) The operators A_1 and B_1 are closed, with dense $\mathcal{D}(A_1) \cap \mathcal{D}(B_1)$, such that $(\omega, \infty) \subset \rho_0(A_1, B_1)$ for some $\omega > 0$, and

$$\lambda \mapsto \lambda R_1(\lambda),\ \lambda \mapsto \lambda^{-1} A_1 R_1(\lambda),\ \lambda \mapsto \lambda^{-1} \overline{R_1(\lambda) A_1} \in LT - \mathcal{L}(X).$$

The two propagators of $(ACP_2; A_i, B_i)$ will be denoted by $C_i(\cdot)$ and $S_i(\cdot)$ $(i = 0, 1)$.

6.3 Strong wellposedness and quasi-wellposedness

In this section, $\mathbb{A}$, $\mathbb{B}$, $y(t)$, y_0, y_1, $\mathbb{Y}$ are as in Section 5.2, except that $\mathcal{D}(\mathbb{B})$ is replaced by

$$\mathcal{D}(\mathbb{B}) := \left\{ \begin{pmatrix} u \\ x \end{pmatrix} \in (\mathcal{D}(B) \cap \mathcal{D}(G_1)) \times \mathcal{D}(B_1); \quad x \in P_1 u \right\}.$$

We look at the abstract Cauchy problem in $\mathbb{Y}$:

$$\begin{cases} y''(t) + \mathbb{A}y(t) + \mathbb{B}y'(t) = 0, \quad t \geq 0, \\ y(0) = y_0, \ y'(0) = y_1. \end{cases} \tag{$ACP_2; \mathbb{A}, \mathbb{B}$}$$

Theorem 6.3.1. *Suppose that* (H_1), (H_4) *and* (H_5) *hold,* $G_0 = G_1 = 0$, *and* $\rho(A_0) \neq \emptyset$. *Take* $\mu \in \rho(A_0, B_0)/\{0\}$ *fixed. Then* $(ACP_2; \mathbb{A}, \mathbb{B})$ *is strongly wellposed if and only if*

$$\begin{cases} K_1(\cdot), \ K_3(\cdot) \in C_{eb}(R^+; \mathcal{L}_s(X, [\mathcal{D}(B_0)])), \\ K_2(\cdot), \ K_4(\cdot) \in C_{eb}(R^+; \mathcal{L}_s(X, [\mathcal{D}(A_0)])), \end{cases} \tag{6.3.1}$$

where for each $x \in X$ *and* $t \geq 0$,

$$\begin{cases} K_1(t)x := \int_0^t S_0'(t-s) D_\mu S_1'(s) x ds, \\ K_2(t)x := \int_0^t S_0(t-s) D_\mu S_1'(s) x ds, \\ K_3(t)x := \int_0^t S_0'(t-s) D_\mu C_1(s) x ds, \\ K_4(t)x := \int_0^t S_0(t-s) D_\mu C_1(s) x ds. \end{cases} \tag{6.3.2}$$

Proof. We first show the denseness of $\mathcal{D}(\mathbb{A}) \cap \mathcal{D}(\mathbb{B})$. It is clear that

$$\mathcal{D}(\mathbb{A}) \cap \mathcal{D}(\mathbb{B}) := \left\{ \begin{pmatrix} u \\ x \end{pmatrix} \in (\mathcal{D}(A) \cap \mathcal{D}(B)) \times (\mathcal{D}(A_1) \cap \mathcal{D}(B_1)); \quad x = Pu \right\}. \tag{6.3.3}$$

Let $\begin{pmatrix} u \\ x \end{pmatrix} \in E \times X$. Because $\mathcal{D}(A_1) \cap \mathcal{D}(B_1)$ is dense in X by (H_5), there exists a sequence $\{x_n\}_{n \in N} \subset \mathcal{D}(A_1) \cap \mathcal{D}(B_1)$ such that $\lim_{n\to\infty} x_n = x$. (H_1) ensures the existence of a sequence $\{u_n\}_{n \in N} \subset \mathcal{D}(A) \cap \mathcal{D}(B)$ such that $Pu_n = x_n$, $n \in N$. Noting $\mathcal{D}(A_0) \cap \mathcal{D}(B_0)$ is dense in E by (H_4), we infer that there exists a sequence $\{v_n\}_{n \in N} \subset \mathcal{D}(A_0) \cap \mathcal{D}(B_0)$ such that $\lim_{n\to\infty}(u_n - v_n) = u$. Accordingly

$$\lim_{n\to\infty} \begin{pmatrix} u_n - v_n \\ x_n \end{pmatrix} = \begin{pmatrix} u \\ x \end{pmatrix}.$$

But for each $n \in N$,

$$u_n - v_n \in \mathcal{D}(A) \cap \mathcal{D}(B), \quad x_n \in \mathcal{D}(A_1) \cap \mathcal{D}(B_1),$$

and

$$P(u_n - v_n) = Pu_n = x_n,$$

i.e., $\begin{pmatrix} u_n - v_n \\ x_n \end{pmatrix} \in \mathcal{D}(\mathbb{A}) \cap \mathcal{D}(\mathbb{B})$. Therefore $\mathcal{D}(\mathbb{A}) \cap \mathcal{D}(\mathbb{B})$ is dense in $\mathbb{E}$.

Moreover, $\mathbb{A}$ is closed by the closedness of A_1 and the completeness of $[\mathcal{D}(A)]_P$; so is $\mathbb{B}$ by the closedness of B_1 and the completeness of $[\mathcal{D}(B)]_{P_1}$.

Let $\lambda \in \rho(A_0, B_0) \cap \rho(A_1, B_1)$, and $\lambda \neq 0$. We obtain

$$\begin{aligned} \lambda^2 + \mathbb{A} + \lambda\mathbb{B} &= \begin{pmatrix} \lambda^2 + A + \lambda B & 0 \\ 0 & \lambda^2 + A_1 + \lambda B_1 \end{pmatrix} \\ &= \begin{pmatrix} \lambda^2 + A_0 + \lambda B_0 & 0 \\ 0 & \lambda^2 + A_1 + \lambda B_1 \end{pmatrix} \begin{pmatrix} I & -D_\lambda \\ 0 & I \end{pmatrix}, \end{aligned}$$

after observing by Lemma 5.3.1 (2) that $(\lambda^2 + A + \lambda B)D_\lambda = 0$, and that

$$u - D_\lambda x \in \mathcal{D}(A_0) \cap \mathcal{D}(B_0) \Longleftrightarrow u \in \mathcal{D}(A) \cap \mathcal{D}(B) \text{ and } Pu = x.$$

From this, we see that $\lambda \in \rho(\mathbb{A}, \mathbb{B})$ and

$$(\lambda^2 + \mathbb{A} + \lambda\mathbb{B})^{-1} = \begin{pmatrix} R_0(\lambda) & D_\lambda R_1(\lambda) \\ 0 & R_1(\lambda) \end{pmatrix}, \tag{6.3.4}$$

noting

$$\begin{pmatrix} I & -D_\lambda \\ 0 & I \end{pmatrix}^{-1} = \begin{pmatrix} I & D_\lambda \\ 0 & I \end{pmatrix}.$$

Now take $\gamma \in \rho(A_0)$. Since $[\mathcal{D}(A)]_P$ is complete, it follows from [40, Lemma 1.2] that the restriction $P\Big|_{\ker(\gamma - A)} : \ker(\gamma - A) \to X$ is invertible and its inverse

$$D := \left(P\Big|_{\ker(\gamma - A)}\right)^{-1} \in \mathcal{L}(X, E).$$

So

$$\gamma - \mathbb{A} = \begin{pmatrix} \gamma - A_0 & 0 \\ 0 & \gamma - A_1 \end{pmatrix} \begin{pmatrix} I & -D \\ 0 & I \end{pmatrix}.$$

Thus, (6.3.4) gives that

$$\mathbb{A}(\lambda^2+\mathbb{A}+\lambda\mathbb{B})^{-1} = \begin{pmatrix} A_0R_0(\lambda) & AD_\lambda R_1(\lambda) \\ 0 & A_1R_1(\lambda) \end{pmatrix}, \tag{6.3.5}$$

$$\begin{aligned} &(\lambda^2+\mathbb{A}+\lambda\mathbb{B})^{-1}(\gamma-\mathbb{A}) \\ = \;& \begin{pmatrix} R_0(\lambda)(\gamma-A_0) & D_\lambda R_1(\lambda)(\gamma-A_1) \\ 0 & R_1(\lambda)(\gamma-A_1) \end{pmatrix}\begin{pmatrix} I & -D \\ 0 & I \end{pmatrix}. \end{aligned} \tag{6.3.6}$$

Observe by (5.3.1) that

$$\begin{aligned} &D_\lambda R_1(\lambda) \\ = \;& D_\mu R_1(\lambda) + (\mu-\lambda)R_0(\lambda)(BD_\mu)R_1(\lambda) + (\mu^2-\lambda^2)R_0(\lambda)D_\mu R_1(\lambda). \end{aligned} \tag{6.3.7}$$

But $AD_\mu,\ BD_\mu \in \mathcal{L}(X,E)$, and

$$\begin{aligned} \lambda^3 R_0(\lambda)D_\mu R_1(\lambda) &= D_\mu(\lambda R_1(\lambda)) - \lambda B_0R_0(\lambda)D_\mu(\lambda R_1(\lambda)) \\ &\quad - A_0R_0(\lambda)D_\mu(\lambda R_1(\lambda)), \end{aligned}$$

$$\begin{aligned} &\lambda R_0(\lambda)D_\mu R_1(\lambda)(\gamma-A_1) \\ = \;& (\gamma R_0(\lambda))D_\mu(\lambda R_1(\lambda)) - D_\mu\left(\lambda^{-1}R_1(\lambda)A_1\right) \\ &+ \lambda B_0R_0(\lambda)D_\mu\left(\lambda^{-1}R_1(\lambda)A_1\right) + A_0R_0(\lambda)D_\mu\left(\lambda^{-1}R_1(\lambda)A_1\right). \end{aligned}$$

We deduce that

$$\begin{cases} \lambda\mapsto \lambda D_\lambda R_1(\lambda),\ \lambda\mapsto\lambda^{-1}AD_\lambda R_1(\lambda) \in LT-\mathcal{L}(X,E), \\ \lambda\mapsto\lambda^{-1}\overline{D_\lambda R_1(\lambda)(\gamma-A_1)} \in LT-\mathcal{L}(E) \end{cases}$$

if and only if

$$\begin{cases} \lambda\mapsto \lambda B_0R_0(\lambda)D_\mu(\lambda R_1(\lambda)),\ \lambda\mapsto A_0R_0(\lambda)D_\mu(\lambda R_1(\lambda)) \in LT-\mathcal{L}(X,E), \\ \lambda\mapsto \lambda B_0R_0(\lambda)D_\mu\left(\lambda^{-1}\overline{R_1(\lambda)A_1}\right) \in LT-\mathcal{L}(X,E), \\ \lambda\mapsto A_0R_0(\lambda)D_\mu\left(\lambda^{-1}\overline{R_1(\lambda)A_1}\right) \in LT-\mathcal{L}(X,E). \end{cases} \tag{6.3.8}$$

This, together with (6.3.4) - (6.3.6), (H_4), (H_5) and Proposition 6.2.4, implies that $(ACP_2;\mathbb{A},\mathbb{B})$ is strongly wellposed in $\mathbb{E}$ if and only if (6.3.8) is valid. On the other hand, for $x\in X$,

$$\begin{cases} \lambda R_0(\lambda)D_\mu(\lambda R_1(\lambda))x = \int_0^\infty e^{-\lambda t}K_1(t)xdt, \\ R_0(\lambda)D_\mu(\lambda R_1(\lambda))x = \int_0^\infty e^{-\lambda t}K_2(t)xdt, \\ \lambda R_0(\lambda)D_\mu\lambda^{-1}(I-\overline{R_1(\lambda)A_1})x = \int_0^\infty e^{-\lambda t}K_3(t)xdt, \\ R_0(\lambda)D_\mu\lambda^{-1}(I-\overline{R_1(\lambda)A_1})x = \int_0^\infty e^{-\lambda t}K_4(t)xdt, \end{cases} \tag{6.3.9}$$

because of (6.2.4) and (6.2.5). Therefore (6.3.8) is valid if and only if (6.3.1) holds. This completes the proof.

□

Theorem 6.3.2. *Let the hypotheses (including (6.3.1)) of Theorem 6.3.1 hold. Then the propagators of $(ACP_2;\mathbb{A},\mathbb{B})$ have the following expressions.*

$$\mathbb{C}(t)\begin{pmatrix} u \\ x \end{pmatrix} = \begin{pmatrix} C_0(t)u - \int_0^t J(s)A_1xds \\ C_1(t)x \end{pmatrix}, \quad t\ge 0,\ u\in E,\ x\in\mathcal{D}(A_1), \tag{6.3.10}$$

$$\mathbb{S}(t)\begin{pmatrix} u \\ x \end{pmatrix} = \begin{pmatrix} S_0(t)u - J(t)x \\ S_1(t)x \end{pmatrix}, \quad t\ge 0,\ u\in E,\ x\in X, \tag{6.3.11}$$

where for $t\ge 0$, $x\in X$,

$$\begin{aligned} J(t)x \ &:=\ D_\mu S_1(t)x + \mu\int_0^t S_0(t-s)(B+\mu)D_\mu S_1(s)xds \\ &\quad -\int_0^t S_0'(t-s)BD_\mu S_1(s)xds - \int_0^t S_0'(t-s)D_\mu S_1'(s)xds. \end{aligned}$$

Proof. For each $x\in X$,

$$\begin{aligned} \int_0^\infty e^{-\lambda t}J(t)xdt \ &:=\ D_\mu R_1(\lambda)x + \mu R_0(\lambda)(B+\mu)D_\mu R_1(\lambda)x \\ &\quad -\lambda R_0(\lambda)(B+\lambda)D_\mu R_1(\lambda)x \\ &=\ D_\lambda R_1(\lambda), \end{aligned}$$

by (6.3.7) . So for $\begin{pmatrix} u \\ x \end{pmatrix} \in \mathbb{E}$, and λ large enough,

$$\int_0^\infty e^{-\lambda t} \begin{pmatrix} S_0(t) & J(t) \\ 0 & S_1(t) \end{pmatrix} \begin{pmatrix} u \\ x \end{pmatrix} dt = (\lambda^2 + \mathbb{A} + \lambda\mathbb{B})^{-1} \begin{pmatrix} u \\ x \end{pmatrix}.$$

This gives (6.3.11) because of (6.2.4). From (6.2.5), we deduce that (6.3.10) is satisfied in the case of $\begin{pmatrix} u \\ x \end{pmatrix} \in \mathcal{D}(\mathbb{A})$. Let now $u \in E$ and $x \in \mathcal{D}(A_1)$. The same reasoning as in the first paragraph of the proof of Theorem 6.3.1 shows the existence of a sequence

$$\begin{pmatrix} u_n \\ x \end{pmatrix}_{n \in N} \in \mathcal{D}(\mathbb{A})$$

such that $u_n \to u$. This justifies (6.3.10) and completes the proof.

□

Corollary 6.3.3. *Suppose that* (H_1) *and* (H_4) *hold,* $\rho(A_0) \neq \emptyset$, *and* $A_1,\ B_1 \in \mathcal{L}(X)$. *Then* $(ACP_2; \mathbb{A}, \mathbb{B})$ *is strongly wellposed.*

Proof. Since $A_1,\ B_1 \in \mathcal{L}(X)$, $(ACP_2; A_1, B_1)$ is automatically strongly wellposed, and so (from Proposition 6.2.3 (i)) both $S_1(\cdot)x$ and $C_1(\cdot)x$ are solutions of $(ACP_2; A_1, B_1)$ for any $x \in X$. This implies that

$$S_1''(\cdot), \quad C_1'(\cdot) \in C_{eb}(R^+; \mathcal{L}(X)). \tag{6.3.12}$$

According to this and from (6.3.1), we get, integrating by parts, for $t \geq 0$ and $x \in X$,

$$\begin{aligned}
K_1(t)x &= S_0(t)D_\mu x + \int_0^t S_0(t-s)D_\mu S_1''(s)xds, \\
K_2(t)x &= \int_0^t S_0(\tau)D_\mu xd\tau + \int_0^t \left(\int_0^{t-s} S_0(\tau)d\tau\right) D_\mu S_1''(s)xds, \\
K_3(t)x &= S_0(t)D_\mu x + \int_0^t S_0(t-s)D_\mu C_1'(s)xds, \\
K_4(t)x &= \int_0^t S_0(\tau)D_\mu xd\tau + \int_0^t \left(\int_0^{t-s} S_0(\tau)d\tau\right) D_\mu C_1'(s)xds.
\end{aligned}$$

On the other hand

$$B_0 S_0(\cdot),\ A_0 \int_0^{\cdot} S_0(\sigma) d\sigma \in C_{eb}(R^+; \mathcal{L}_s(E)).$$

This indicates that (6.3.1) is satisfied. Therefore $(ACP_2; \mathbb{A}, \mathbb{B})$ is strongly wellposed, in view of Theorem 6.3.1. The proof is then complete.

□

In the sequel, E_1 is a Banach space such that

$$[\mathcal{D}(A)]_P \hookrightarrow E_1 \hookrightarrow E, \tag{6.3.13}$$

$$\lambda \mapsto R_0(\lambda) \in LT - \mathcal{L}(E, E_1). \tag{6.3.14}$$

In the case of $[\mathcal{D}(A)]_P \hookrightarrow [\mathcal{D}(B)]_{P_1}$ and under the hypothesis (H_4'), we can take $E_1 = [\mathcal{D}(B)]_{P_1}$, for which (6.3.13) and (6.3.14) are valid.

Theorem 6.3.4. *Suppose that* (H_1) *and* (H_4') *hold. Let*

$$G_0 \in \mathcal{L}(E_1, X), \tag{6.3.15}$$

and let

$$G_1 \in \mathcal{L}(E, X), \quad A_1, B_1 \in \mathcal{L}(X). \tag{6.3.16}$$

Then $(ACP_2; \mathbb{A}, \mathbb{B})$ *is strongly quasi-wellposed, and its second propagator* $\mathbb{S}(\cdot)$ *satisfies*

$$\mathbb{S}(\cdot) \in C_{eb}(R^+; \mathcal{L}_s(\mathbb{E}, E_1 \times X)). \tag{6.3.17}$$

Proof. Using (6.2.4) and (6.3.12) , we obtain

$$\lambda^2 R_1(\lambda)x - x = \int_0^{\infty} e^{-\lambda t} S_1''(t) x dt, \quad x \in X, \tag{6.3.18}$$

for λ sufficiently large. This yields that

$$\lambda \mapsto \lambda D_\lambda R_1(\lambda),\ \lambda \mapsto \lambda^{-1} A D_\lambda R_1(\lambda) \in LT - \mathcal{L}(X, E), \tag{6.3.19}$$

$$\lambda \mapsto D_\lambda R_1(\lambda) \in LT - \mathcal{L}(X, E_1), \tag{6.3.20}$$

by Lemma 5.3.1 (2), (H_4'), (6.3.7), (6.3.13) and (6.3.14) . Write

$$\mathbb{A}_0 := \begin{pmatrix} A & 0 \\ 0 & A_1 \end{pmatrix}, \quad \mathcal{D}(\mathbb{A}_0) := \left\{ \begin{pmatrix} u \\ x \end{pmatrix} \in \mathcal{D}(A) \times X; \quad x = Pu \right\},$$

$$\mathbb{B}_0 := \begin{pmatrix} B & 0 \\ 0 & B_1 \end{pmatrix}, \quad \mathcal{D}(\mathbb{B}_0) := \left\{ \begin{pmatrix} u \\ x \end{pmatrix} \in \mathcal{D}(B) \times X; \quad x \in P_1 u \right\}.$$

Making use of (6.3.4) and (6.3.5) (with $\mathbb{A}_0$, $\mathbb{B}_0$ in place of $\mathbb{A}$, $\mathbb{B}$ there), we infer by (H_4') and (6.3.18) - (6.3.20) that

$$\lambda \mapsto \lambda(\lambda^2 + \mathbb{A}_0 + \lambda\mathbb{B}_0)^{-1},\ \lambda \mapsto \lambda^{-1}\mathbb{A}_0(\lambda^2 + \mathbb{A}_0 + \lambda\mathbb{B}_0)^{-1} \in LT - \mathcal{L}(\mathbb{E}), \quad (6.3.21)$$

$$\lambda \mapsto (\lambda^2 + \mathbb{A}_0 + \lambda\mathbb{B}_0)^{-1} \in LT - \mathcal{L}(\mathbb{E}, E_1 \times X). \quad (6.3.22)$$

Noting that

$$\begin{pmatrix} 0 & 0 \\ G_0 & 0 \end{pmatrix} (\lambda^2 + \mathbb{A}_0 + \lambda\mathbb{B}_0)^{-1} = \begin{pmatrix} 0 & 0 \\ G_0 R_0(\lambda) & G_0 D_\lambda R_1(\lambda) \end{pmatrix}$$

for $\lambda \in \rho(A_0, B_0) \cap \rho(A_1, B_1)$ (cf. (6.3.4)), we obtain

$$\lambda \mapsto \begin{pmatrix} 0 & 0 \\ G_0 & 0 \end{pmatrix} (\lambda^2 + \mathbb{A}_0 + \lambda\mathbb{B}_0)^{-1} \in LT - \mathcal{L}(\mathbb{E}) \quad (6.3.23)$$

by (6.3.14), (6.3.15), and (6.3.20). Also it is clear from (6.3.21) that

$$\lambda \mapsto \lambda \begin{pmatrix} 0 & 0 \\ G_1 & 0 \end{pmatrix} (\lambda^2 + \mathbb{A}_0 + \lambda\mathbb{B}_0)^{-1} \in LT - \mathcal{L}(\mathbb{E}) \quad (6.3.24)$$

since

$$\begin{pmatrix} 0 & 0 \\ G_1 & 0 \end{pmatrix} \in \mathcal{L}(\mathbb{E}).$$

According to (6.3.21) - (6.3.24) , we deduce in view of [84, Theorem 1.10] that for λ large enough,

$$\begin{aligned} \mathbb{R}(\lambda) &:= (\lambda^2 + \mathbb{A} + \lambda\mathbb{B})^{-1} \\ &= (\lambda^2 + \mathbb{A}_0 + \lambda\mathbb{B}_0)^{-1} \Bigg[I - \begin{pmatrix} 0 & 0 \\ G_0 & 0 \end{pmatrix} (\lambda^2 + \mathbb{A}_0 + \lambda\mathbb{B}_0)^{-1} \\ &\qquad - \lambda \begin{pmatrix} 0 & 0 \\ G_1 & 0 \end{pmatrix} (\lambda^2 + \mathbb{A}_0 + \lambda\mathbb{B}_0)^{-1} \Bigg]^{-1} \end{aligned}$$

exists, and

$$\lambda \mapsto \lambda\mathbb{R}(\lambda), \quad \lambda \mapsto \lambda^{-1}\mathbb{A}\mathbb{R}(\lambda) \in LT - \mathcal{L}(\mathbb{E}), \tag{6.3.25}$$

$$\lambda \mapsto \mathbb{R}(\lambda) \in LT - \mathcal{L}(\mathbb{E}, E_1 \times X). \tag{6.3.26}$$

Moreover, $\mathbb{A}$ and $\mathbb{B}$ are closed and densely defined operators in $\mathbb{E}$ by (H_1), (6.3.13), (6.3.15), (6.3.16) and the fact that $\mathcal{D}(\mathbb{A}_0)$ and $\mathcal{D}(\mathbb{B}_0)$ are dense (from the proof of Theorem 6.3.1). This and (6.3.25) together justify the strong quasi-wellposedness of $(ACP_2; \mathbb{A}, \mathbb{B})$, in view of Proposition 6.2.7.

Finally, a combination of (6.2.4) and (6.3.26) leads to (6.3.17) . The proof is complete.

□

Theorem 6.3.5. *Suppose that* (H_1) *and* (H_4') *hold. Let* $G_0 \in \mathcal{L}(E_1, X)$, $G_1 \in \mathcal{L}(E, X)$, $B \in \mathcal{L}(E)$, *and* $A_1, B_1 \in \mathcal{L}(X)$. *Then* $(ACP_2; \mathbb{A}, \mathbb{B})$ *is strongly wellposed, and its second propagator* $\mathbb{S}(\cdot)$ *satisfies* (6.3.17).

Proof. It is easy to see by hypothesis that

$$\mathbb{B} \in \mathcal{L}(\mathbb{E}). \tag{6.3.27}$$

From the proof of Theorem 6.3.4, we know that $(ACP_2; \mathbb{A}, \mathbb{B})$ is strongly quasi-wellposed, and that (6.3.17) and (6.3.25) are satisfied. From (6.3.25) and the identities

$$\mathbb{R}(\lambda)\mathbb{A} = I - \lambda^2\mathbb{R}(\lambda) - \lambda\mathbb{R}(\lambda)\mathbb{B},$$

we see that

$$\lambda \mapsto \lambda^{-1}\overline{\mathbb{R}(\lambda)\mathbb{A}} \in LT - \mathcal{L}(\mathbb{E}),$$

since $\mathbb{B}$ is bounded. Consequently, $(ACP_2; \mathbb{A}, \mathbb{B})$ is strongly wellposed by Proposition 6.2.4.

□

Remark 6.3.6. When $B \in \mathcal{L}(E)$, (H_4') holds if and only if

$$\lambda \mapsto \lambda(\lambda^2 + A_0)^{-1} \in LT - \mathcal{L}(E)$$

if and only if

$-A_0$ generates a strongly continuous cosine operator function on E

(cf., e.g., [81] or [84, Section 1.4 and Theorem 5.1, p. 75]).

Corollary 6.3.7. *Suppose that $[\mathcal{D}(A)]_P$ is complete, $P(\mathcal{D}(A)) = X$, and $-A_0$ generates a strongly continuous cosine operator function on E. Let $G_0 \in \mathcal{L}(E_1, X)$ (with $(\lambda^2 + A_0)^{-1}$ instead of $R_0(\lambda)$ in (6.3.14)), and $A_1 \in \mathcal{L}(X)$. Define $\widetilde{\mathbb{E}} := \mathcal{D}(\mathbb{A})$ endowed with the norm*

$$\left\| \begin{pmatrix} u \\ x \end{pmatrix} \right\|_{\widetilde{\mathbb{E}}} = \|u\|_{[\mathcal{D}(A)]_P}.$$

Denote by $\widetilde{\mathbb{A}}$ the part of $\mathbb{A}$ in $\widetilde{\mathbb{E}}$. Then

(1) *$(ACP_2; \mathbb{A}, 0)$ is strongly wellposed in $\mathbb{E}$, or equivalently, $-\mathbb{A}$ generates a strongly continuous cosine operator function on $\mathbb{E}$;*

(2) *$(ACP_2; \widetilde{\mathbb{A}}, 0)$ is strongly wellposed in $\widetilde{\mathbb{E}}$, or equivalently, $-\widetilde{\mathbb{A}}$ generates a strongly continuous cosine operator function on $\widetilde{\mathbb{E}}$.*

Proof. Clearly, the conditions of Theorem 6.3.5 are satisfied (see Remark 6.3.6). Thus $(ACP_2; \mathbb{A}, 0)$ is strongly wellposed in $\mathbb{E}$. This indicates that

$$\lambda \mapsto \lambda(\lambda^2 + \mathbb{A})^{-1} \in LT - \mathcal{L}(\mathbb{E}).$$

Therefore,

$$\lambda \mapsto \lambda(\lambda^2 + \widetilde{\mathbb{A}})^{-1} \in LT - \mathcal{L}([\mathcal{D}(\mathbb{A})]).$$

But $\|\cdot\|_{[\mathcal{D}(\mathbb{A})]}$ is equivalent to $\|\cdot\|_{\widetilde{\mathbb{E}}}$. So

$$\lambda \mapsto \lambda(\lambda^2 + \widetilde{\mathbb{A}})^{-1} \in LT - \mathcal{L}(\widetilde{\mathbb{E}}). \tag{6.3.28}$$

Thus we infer that $-\widetilde{\mathbb{A}}$ generates a strongly continuous cosine operator function on $\widetilde{\mathbb{E}}$. This finishes the proof.

□

Corollary 6.3.8. *Let the conditions of Corollary 6.3.7 be satisfied. Define an operator $\widetilde{A}$ on $[\mathcal{D}(A)]_P$ by*

$$\widetilde{A}u := Au, \quad \mathcal{D}(\widetilde{A}) := \{u \in \mathcal{D}(A^2); \quad PAu + Gu - A_1 Pu = 0\}.$$

Then $-\widetilde{A}$ generates a strongly continuous cosine operator function on $[\mathcal{D}(A)]_P$.

Proof. It can be seen from (6.3.28) that

$$\lambda \mapsto \lambda(\lambda^2 + \widetilde{A})^{-1} \in LT - \mathcal{L}([\mathcal{D}(A)]_P).$$

This justifies the claim.

□

6.4 Solutions to inhomogeneous problems

In this section, we are concerned with the following inhomogeneous problem:

$$\begin{cases} y''(t) + \mathbb{A}y(t) + \mathbb{B}y'(t) = h(t), \quad t \in [0,T], \\ y(0) = y_0,\ y'(0) = y_1. \end{cases} \tag{6.4.1}$$

Definition 6.4.1. Let $h \in C([0,T];\mathbb{E})$.

(ii) A function $y(\cdot)$ is called a *solution* of $(ACP_2;\mathbb{A},\mathbb{B})$ if $y(\cdot) \in C^2([0,T];\mathbb{E}) \cap C([0,T];[\mathcal{D}(\mathbb{A})])$, $y'(\cdot) \in C([0,T];[\mathcal{D}(\mathbb{B})])$, and $(ACP_2;\mathbb{A},\mathbb{B})$ is satisfied.

Theorem 6.4.2. *Let the hypotheses of either Theorem* 6.3.1 *or Corollary* 6.3.3 *or Theorem* 6.3.5 *hold. Let $h \in C^1([0,T];\mathbb{E})$, $y_0 \in \mathcal{D}(\mathbb{A})$, and $y_1 \in \mathcal{D}(\mathbb{A}) \cap \mathcal{D}(\mathbb{B})$. Then*

(1) *problem* (6.4.1) *has a unique solution $y(\cdot)$, given by*

$$y(t) = \mathbb{C}(t)y_0 + \mathbb{S}(t)y_1 + \int_0^t \mathbb{S}(t-s)h(s)ds, \quad t \in [0,T], \tag{6.4.2}$$

where $\mathbb{C}(\cdot)$ and $\mathbb{S}(\cdot)$ are the two propagators of $(ACP_2;\mathbb{A},\mathbb{B})$;

(2) *the $y(\cdot)$ satisfies*

$$y'(\cdot) \in C([0,T];E_1 \times X), \tag{6.4.3}$$

$$\|y(t)\| \le M\left(\|h\|_{C([0,T];\mathbb{E})} + \|y_0\| + \|y_1\|\right), \quad t \in [0,T], \tag{6.4.4}$$

$$\begin{aligned} &\|y''(t)\| + \|y(t)\|_{[\mathcal{D}(\mathbb{A})]} + \|y'(t)\|_{[\mathcal{D}(\mathbb{B})]} + \|y'(t)\|_{E_1\times X} \\ \le\ & M\Big(\|h\|_{C^1([0,T];\mathbb{E})} + \|y_0\|_{[\mathcal{D}(\mathbb{A})]} + \|y_1\|_{[\mathcal{D}(\mathbb{A})]} + \|y_1\|_{[\mathcal{D}(\mathbb{B})]}\Big), \quad t \in [0,T], \end{aligned} \tag{6.4.5}$$

for some constant $M > 0$.

Proof. By hypothesis, $(ACP_2;\mathbb{A},\mathbb{B})$ is strongly wellposed. Set

$$w(t) := \int_0^t \mathbb{S}(t-s)h(s)ds, \quad t \in [0,T]. \tag{6.4.6}$$

Then, according to Definition 6.2.2, we infer that $w(0) = w'(0) = 0$,

$$w(t) = \int_0^t \mathbb{S}(\sigma)h(0)d\sigma + \int_0^t \left(\int_0^s \mathbb{S}(\sigma)d\sigma\right) h'(t-s)ds, \quad t \in [0,T], \tag{6.4.7}$$

$$w'(t) = \mathbb{S}(t)h(0) + \int_0^t \mathbb{S}(s)h'(t-s)ds, \quad t \in [0,T], \tag{6.4.8}$$

and

$$w''(t) = \mathbb{S}'(t)h(0) + \int_0^t \mathbb{S}'(s)h'(t-s)ds, \quad t \in [0,T]; \tag{6.4.9}$$

therefore

$$w''(t) + \mathbb{A}w(t) + \mathbb{B}w'(t) = h(0) + \int_0^t h'(t-s)ds = h(t), \quad t \in [0,T],$$

by (6.2.9). This means that (6.4.2) gives the unique solution $y(\cdot)$ of problem (6.4.1) by Proposition 6.2.3 (i).

Combining (6.2.5), (6.4.8) and (6.3.17) together, we obtain (6.4.3). The estimate (6.4.4) follows from (6.4.2) immediately. Using (6.2.5), (6.2.9), (6.3.17), and (6.4.7) - (6.4.9) verifies estimate (6.4.5). This completes the proof.

□

Theorem 6.4.3. *Let the hypotheses of Theorem* 6.3.4 *hold. Let* $h \in C^1([0,T];\mathbb{E})$, $y_0 \in \mathcal{D}(\mathbb{A})$, *and* $y_1 \in \mathcal{D}(\mathbb{A}) \cap \mathcal{D}(\mathbb{B})$. *Then the conclusions of Theorem* 6.4.2 *hold, except* (6.4.4).

Proof. Similar to the proof of Theorem 6.4.2.

□

Corollary 6.4.4. *Let the conditions of Corollary* 6.3.7 *be satisfied. Let* $h \in C^1([0,T];\widetilde{\mathbb{E}})$, *and* $y_0, y_1 \in \mathcal{D}(\mathbb{A}^2)$. *Then*

(1) *the conclusions of Theorem* 6.4.2 *hold;*

(2) *the solution* $y(\cdot)$ *is in* $C^2([0,T];\widetilde{\mathbb{E}})$.

Proof. Assertion (1) is obvious.

By hypothesis, $y_0, y_1 \in \mathcal{D}(\widetilde{\mathbb{A}})$. It follows that

$$\mathbb{C}(\cdot)y_0 + \mathbb{S}(\cdot)y_1 \in C^2(R^+; \widetilde{\mathbb{E}}),$$

since $(ACP_2; \widetilde{\mathbb{A}}, 0)$ is strongly wellposed in $\widetilde{\mathbb{E}}$ by Corollary 6.3.7. Moreover

$$\mathbb{S}'(\cdot)\Big|_{\widetilde{\mathbb{E}}} \in C(R^+; \mathcal{L}_s(\widetilde{\mathbb{E}})).$$

Hence, we get assertion (2) by (6.4.2) and (6.4.9).

□

6.5 Examples

Example 6.5.1. Let Ω be a bounded domain in R^n with smooth boundary $\partial\Omega$. Let $A(\xi, \partial_\xi)$ be a second order strongly elliptic operator with smooth coefficients.

We consider the second order hyperbolic equation with a boundary condition of Wentzell type:

$$\begin{cases} \partial_t^2 u = A(\xi, \partial_\xi)\, u, & \text{in } [0,T] \times \Omega, \\ A(\xi, \partial_\xi)\, u\Big|_{\partial\Omega} = Fu, & \text{in } [0,T] \times \partial\Omega, \\ u(0,\cdot) = \varphi_0, \quad \partial_t u(0,\cdot) = \varphi_1, & \text{in } \Omega, \end{cases} \tag{6.5.1}$$

where $F \in \mathcal{L}(L^2(\Omega), L^2(\partial\Omega))$. Obviously, problem (6.5.1) is equivalent to the following one with a dynamical boundary condition:

$$\begin{cases} \partial_t^2 u = A(\xi, \partial_\xi)\, u, & \text{in } [0,T] \times \Omega, \\ \partial_t^2 u\Big|_{\partial\Omega} = Fu, & \text{in } [0,T] \times \partial\Omega, \\ u(0,\cdot) = \varphi_0, \quad \partial_t u(0,\cdot) = \varphi_1, & \text{in } \Omega. \end{cases} \tag{6.5.2}$$

We let

$$E = E_1 = L^2(\Omega), \quad X = L^2(\partial\Omega),$$

$$A\varphi = -A(\xi, \partial_\xi)\varphi \quad \text{for} \quad \varphi \in \mathcal{D}(A) := \{\varphi \in H^{\frac{1}{2}}(\Omega);\ A(\xi, \partial_\xi)\varphi \in L^2(\Omega)\},$$

$$P\varphi = \varphi\Big|_{\partial\Omega} \quad \text{for} \quad \varphi \in \mathcal{D}(P) := \mathcal{D}(A),$$

$$G = F, \quad A_1 = 0.$$

Take $\lambda_0 \geq 0$ such that if

$$(\lambda_0 - A(\xi, \partial_\xi))\varphi = 0$$

for a $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$, then $\varphi = 0$. From [60, Theorem 7.4, p. 188], we know that the mapping

$$\mathcal{P} : \varphi \longmapsto \left((\lambda_0 - A(\xi, \partial_\xi))\varphi,\ \varphi\Big|_{\partial\Omega}\right)$$

is an algebraic and topological isomorphism of $D_A^{\frac{1}{2}}(\Omega)$ onto $\Xi^{-\frac{3}{2}}(\Omega) \times L^2(\partial\Omega)$, where $D_A^{\frac{1}{2}}(\Omega)$ and $\Xi^{-\frac{3}{2}}(\Omega)$ are defined in [60] satisfying

$$D_A^{\frac{1}{2}}(\Omega) \hookrightarrow H^{\frac{1}{2}}(\Omega), \quad L^2(\Omega) \hookrightarrow \Xi^{-\frac{3}{2}}(\Omega),$$

and

$$D_A^{\frac{1}{2}}(\Omega) = \left\{\varphi \in H^{\frac{1}{2}}(\Omega);\ \Delta\varphi \in \Xi^{-\frac{3}{2}}(\Omega)\right\}.$$

Hence, for each $w \in L^2(\partial\Omega)$ there exists $\varphi \in H^{\frac{1}{2}}(\Omega)$ such that

$$(\lambda_0 - A(\xi, \partial_\xi))\varphi = 0, \quad \varphi\Big|_{\partial\Omega} = w.$$

This implies that

$$P(\mathcal{D}(A))) = X.$$

Moreover $[\mathcal{D}(A)]_P$ is complete. In fact, if $\{\varphi_n\}_{n\in N}$ is a Cauchy sequence in $[\mathcal{D}(A)]_P$, then there exist $\psi_i \in L^2(\Omega)$ $(i = 0, 1)$ and $w_0 \in L^2(\partial\Omega)$ such that

$$\lim_{n\to\infty} \|\varphi_n - \psi_0\|_{L^2(\Omega)} = 0,$$

$$\lim_{n\to\infty} \|A(\xi, \partial_\xi)\varphi_n - \psi_1\|_{L^2(\Omega)} = 0,$$

$$\lim_{n\to\infty} \left\|\varphi_n\Big|_{\partial\Omega} - w_0\right\|_{L^2(\partial\Omega)} = 0.$$

The mapping $\mathcal{P}$ tells us that

$$\psi_0 \in H^{\frac{1}{2}}(\Omega), \quad (\lambda_0 - A(\xi, \partial_\xi))\varphi = \lambda_0\psi_0 - \psi_1, \quad \varphi_0\Big|_{\partial\Omega} = w_0.$$

So $\psi_0 \in \mathcal{D}(A)$ and

$$\lim_{n\to\infty} \|\varphi_n - \psi_0\|_{A,P} = 0.$$

Next, put

$$A_D = -A\Big|_{H^2(\Omega)\cap H_0^1(\Omega)}.$$

It is clear that $\lambda_0 \in \rho(A_D)$. Let $\varphi \in \mathcal{D}(A)$ with $\varphi\Big|_{\partial\Omega} = 0$. Then

$$\mathcal{P}\varphi = \mathcal{P}\left((\lambda_0 - A_D)^{-1}((\lambda_0 + A)\varphi)\right) = ((\lambda_0 + A)\varphi,\ 0).$$

Since $\mathcal{P}$ is injective, it follows that

$$\varphi = (\lambda_0 - A_D)^{-1}((\lambda_0 + A)\varphi) \in \mathcal{D}(A_D).$$

Therefore, we obtain

$$A_0 \ \left(:= A\Big|_{\ker P}\right) \ = -A_D.$$

It is known that A_D is the generator of a strongly continuous cosine operator function on $L^2(\Omega)$. Thus the conditions of Corollary 6.3.7 are satisfied. Consequently, the operator

$$\mathbb{A} := \begin{pmatrix} A(\xi, \partial_\xi) & 0 \\ F & 0 \end{pmatrix}, \quad \text{with } \mathcal{D}(\mathbb{A}) := \left\{ \begin{pmatrix} \varphi \\ w \end{pmatrix} \in \mathcal{D}(A) \times L^2(\partial\Omega);\ \ \varphi\Big|_{\partial\Omega} = w \right\}$$

generates a strongly continuous cosine operator function on $L^2(\Omega) \times L^2(\partial\Omega)$. Write $\mathcal{H} := \mathcal{D}(A)$ equipped with the norm

$$\|\varphi\|_{\mathcal{H}} := \|\varphi\|_{L^2(\Omega)} + \|A(\xi, \partial_\xi)\varphi\|_{L^2(\Omega)} + \left\|\varphi\Big|_{\partial\Omega}\right\|_{L^2(\partial\Omega)},$$

and

$$\mathcal{A}\varphi := A(\xi, \partial_\xi)\varphi \quad \text{for } \ \varphi \in \mathcal{D}(\mathcal{A}) := \left\{\varphi \in \mathcal{D}(A^2);\ A(\xi, \partial_\xi)\varphi\Big|_{\partial\Omega} = F\varphi\right\}.$$

Then, we claim by Corollary 6.3.8 that $\mathcal{A}$ generates a strongly continuous cosine operator function on $\mathcal{H}$.

Example 6.5.2. Let Ω be a bounded domain in R^n with smooth boundary $\partial\Omega$. Let $\rho \geq 0$, k, $m \in N$ with $m > k$,

$$f \in C^1([0,T]; L^2(\Omega)), \quad g_i \in C^3([0,T]; H^2(\partial\Omega)) \ \ (i = 1, \ldots, m-1),$$

$v \in L^2(\partial\Omega)$, and $w \in H^2(\partial\Omega)$.

We consider

$$\begin{cases} \partial_t^2 u + (-1)^m \Delta^m u + (-1)^k \rho \Delta^k \partial_t u = f, & \text{in } [0,T] \times \Omega, \\ \partial_t^2 u = \left\langle \dfrac{\partial u}{\partial \nu}, \ v \right\rangle_{L^2(\partial\Omega)} w, & \text{in } [0,T] \times \partial\Omega, \\ \Delta^i u = g_i \ \ (i = 1, \ldots, m-1), & \text{in } [0,T] \times \partial\Omega, \\ u(0,\cdot) = \varphi_0, \ \ \partial_t u(0,\cdot) = \varphi_1, & \text{in } \Omega, \end{cases} \tag{6.5.3}$$

Where $\dfrac{\partial}{\partial \nu}$ is the outward normal derivative on $\partial\Omega$.

Take

$$E = L^2(\Omega), \quad E_1 = H^2(\Omega), \quad X = (H^{\frac{3}{2}}(\partial\Omega))^m,$$

$$A = (-1)^m \Delta_\Omega^m, \quad B = (-1)^k \rho \Delta_\Omega^k$$

(where Δ_Ω is the Laplacian on Ω, with $\mathcal{D}(\Delta_\Omega) := H^2(\Omega)$),

$$P\varphi = \begin{pmatrix} \varphi\Big|_{\partial\Omega} \\ \Delta\varphi\Big|_{\partial\Omega} \\ \cdot \\ \cdot \\ \cdot \\ \Delta^{m-1}\varphi\Big|_{\partial\Omega} \end{pmatrix} \quad \text{for} \quad \varphi \in \mathcal{D}(P) := \mathcal{D}(A),$$

$$P_1\varphi = \left\{ \begin{pmatrix} \varphi\big|_{\partial\Omega} \\ \vdots \\ \Delta^{k-1}\varphi\big|_{\partial\Omega} \\ w_k \\ \vdots \\ w_{m-1} \end{pmatrix} ; \quad w_i \in H^{\frac{3}{2}}(\partial\Omega),\ i = k, \cdots, m-1 \right\}$$

$$\text{for} \quad \varphi \in \mathcal{D}(P_1) := \mathcal{D}(B),$$

$$G_0\varphi = \begin{pmatrix} \left\langle \dfrac{\partial\varphi}{\partial\nu},\ v \right\rangle_{L^2(\partial\Omega)} w \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{for} \quad \varphi \in \mathcal{D}(G_0) := E_1,$$

$$A_1 = 0, \quad B_1 = 0, \quad G_1 = 0.$$

First, we show that

$$P(\mathcal{D}(A) \cap \mathcal{D}(B)) = X. \tag{6.5.4}$$

To this end, we recall (see, e.g., [79, p. 390-391]) that the mapping

$$\mathcal{M} : \varphi \longmapsto \left(\Delta\varphi,\ \varphi\big|_{\partial\Omega} \right)$$

is an algebraic and topological isomorphism of $H^2(\Omega)$ onto $L^2(\Omega) \times H^{\frac{3}{2}}(\partial\Omega)$. Therefore, given $w_0, \cdots, w_{m-1} \in H^{\frac{3}{2}}(\partial\Omega)$, there exist $\psi_1, \cdots, \psi_{m-1}, \varphi \in H^2(\Omega)$ such that

$$\begin{cases} \Delta\psi_1 = 0, \\ \psi_1\big|_{\partial\Omega} = w_{m-1}, \end{cases} \qquad \begin{cases} \Delta\psi_2 = 0, \\ \psi_2\big|_{\partial\Omega} = w_{m-2}, \end{cases} \quad \cdots, \begin{cases} \Delta\varphi = \psi_{m-1}, \\ \varphi\big|_{\partial\Omega} = w_0. \end{cases}$$

It is readily seen that

$$\varphi \in \mathcal{D}(A) \quad \text{and} \quad P\varphi = \begin{pmatrix} w_0 \\ \vdots \\ w_{m-1} \end{pmatrix}.$$

So we obtain (6.5.4) , noting $\mathcal{D}(A) = \mathcal{D}(A) \cap \mathcal{D}(B)$. Let now $\{\varphi_n\}_{n\in N}$ be a Cauchy sequence in $[\mathcal{D}(B)]_{P_1}$. By the definition of $[\mathcal{D}(B)]_{P_1}$ (cf. Section 1 of last chapter), there exist r, $r_0 \in L^2(\Omega)$, $v_0, \cdots, v_{k-1} \in H^{\frac{3}{2}}(\partial\Omega)$ such that

$$\lim_{n\to\infty} \|\varphi_n - r\|_{L^2(\Omega)} = 0,$$

$$\lim_{n\to\infty} \left\|\Delta^k\varphi_n - r_0\right\|_{L^2(\Omega)} = 0,$$

$$\lim_{n\to\infty} \left\|\Delta^i\varphi_n\Big|_{\partial\Omega} - v_i\right\|_{H^{\frac{3}{2}}(\partial\Omega)} = 0, \quad i = 0, 1, \ldots, k-1.$$

Accordingly, the isomorphism $\mathcal{M}$ implies the existence of $r_1, \cdots, r_{k-1} \in H^2(\Omega)$ such that

$$\lim_{n\to\infty} \left\|\Delta^{k-1}\varphi_n - r_1\right\|_{L^2(\Omega)} = 0, \quad \Delta r_1 = r_0, \quad r_1\Big|_{\partial\Omega} = v_{k-1},$$

$$\cdots,$$

$$\lim_{n\to\infty} \|\Delta\varphi_n - r_{k-1}\|_{L^2(\Omega)} = 0, \quad \Delta r_{k-1} = r_{k-2}, \quad r_{k-1}\Big|_{\partial\Omega} = v_1,$$

$$r \in H^2(\Omega), \quad \Delta r = r_{k-1}, \quad r\Big|_{\partial\Omega} = v_0.$$

Hence

$$r \in \mathcal{D}(B) \quad \text{and} \quad \lim_{n\to\infty} \|\varphi_n - r\|_{B,P_1} = 0.$$

Thus we have proved the completeness of $[\mathcal{D}(B)]_{P_1}$. The completeness of $[\mathcal{D}(A)]_P$ can be shown in a similar way.

Making use of the mapping $\mathcal{M}$ again, we deduce that

$$\|\cdot\|_{A,P} \text{ is equivalent to } \sum_{i=0}^{m-1} \|\Delta^i \cdot\|_{H^2(\Omega)},$$

and

$$\|\cdot\|_{B,P_1} \text{ is equivalent to } \sum_{i=0}^{k-1} \|\Delta^i \cdot \|_{H^2(\Omega)}.$$

This implies that

$$[\mathcal{D}(A)]_P \hookrightarrow E_1.$$

Next, denote by Δ_D the Dirichlet Laplacian, i.e., $\Delta_D = \Delta_\Omega\Big|_{H^2(\Omega)\cap H_0^1(\Omega)}$. It is clear that

$$A_0 = (-1)^m \Delta_D^m, \quad B_0 = (-1)^k \rho \Delta_D^k.$$

So (H_4') and (6.3.14) hold (cf., e.g., [84, p. 232]). Thus, the conditions of Theorem 6.3.4 are satisfied, and therefore Theorem 6.4.3 is applicable to this situation. Letting

$$y(t) = \begin{pmatrix} u(t) \\ x(t) \end{pmatrix}, \quad u(t) = u(t,\cdot), \quad x(t) = \begin{pmatrix} u(t,\cdot)\Big|_{\partial\Omega} \\ \vdots \\ \Delta^{m-1} u(t,\cdot)\Big|_{\partial\Omega} \end{pmatrix}, \qquad t \in [0,T],$$

we obtain:

For every $\varphi_0,\ \varphi_1 \in \mathcal{D}(\Delta_\Omega^m)$ with $(\Delta^j \varphi_0)\Big|_{\partial\Omega} = g_j(0,\cdot)$ and $(\Delta^j \varphi_1)\Big|_{\partial\Omega} = \partial_t g_j(0,\cdot)$ $(j = 1, \cdots, m-1)$, problem (6.5.3) *has a unique solution*

$$u \in C^2\left([0,T]; L^2(\Omega)\right) \cap C^1([0,T]; H^2(\Omega)) \tag{6.5.5}$$

with

$$\begin{aligned} &\Delta^i u \in C^1([0,T]; H^2(\Omega)), \quad i = 1, \dots, k-1, \\ &\Delta^i u \in C([0,T]; H^2(\Omega)), \quad i = k, \dots, m-1; \end{aligned} \tag{6.5.6}$$

moreover, u satisfies

$$\begin{aligned} &\left\|\partial_t^2 u(t,\cdot)\right\|_{L^2(\Omega)} + \left\| \sum_{i=0}^{k-1} \Delta^i \partial_t u(t,\cdot) \right\|_{H^2(\Omega)} + \left\| \sum_{i=k}^{m-1} \Delta^i u(t,\cdot) \right\|_{H^2(\Omega)} \\ \leq \;& const \left[\|f\|_{C^1([0,T];L^2(\Omega))} + \sum_{i=1}^{m-1} \left(\|\partial_t^2 g_i\|_{C^1([0,T];H^2(\partial\Omega))} \right.\right. \\ &\left.\left. + \sum_{j=0}^{1} \left(\|\varphi_j\|_{H^2(\Omega)} + \|\Delta^i \varphi_j\|_{H^2(\Omega)} \right) \right) \right], \quad t \in [0,T]. \end{aligned}$$

Here, for getting the uniqueness, we used the fact that if u is a solution of problem (6.5.3) satisfying (6.5.5) and (6.5.6), then $x'(t) \in P_1 u'(t)$ $(t \in [0, T])$,

$$x(0) = \begin{pmatrix} \varphi_0\big|_{\partial\Omega} \\ \Delta\varphi_0\big|_{\partial\Omega} \\ \vdots \\ \Delta^{m-1}\varphi_0\big|_{\partial\Omega} \end{pmatrix}, \quad x'(0) = \begin{pmatrix} \varphi_1\big|_{\partial\Omega} \\ \Delta\varphi_1\big|_{\partial\Omega} \\ \vdots \\ \Delta^{m-1}\varphi_1\big|_{\partial\Omega} \end{pmatrix}.$$

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Lebenslauf

01.07.1964	geboren in Longchang, Sichuan Provinz, China
1971 - 1976	Besuch der Grundschule in Longchang
1976 - 1981	Besuch des Gymnasiums in Longchang
1981 - 1985	Bachelor-Studium der Mathematik an der Universität Sichuan in Chengdu, Sichuan Provinz, China Abschluß mit Bachelor of Science
1985 - 1988	Studium der Mathematik an der Universität Sichuan in Chengdu, Sichuan Provinz, China Abschluß mit Master of Science
Juli 1988	Diplom in Mathematik
1988-1998	Wissenschaftliche Angestellte an der Mathematischen Fakultät der YTU (Yunnan Teachers' University)
seit 1998	Wissenschaftliche Angestellte an der Mathematischen Fakultät der USTC (University of Science and Technology of China)
seit Okt. 2001	Promotion-Studium an der Fakultät für Mathematik und Physik der Eberhard-Karls-Universität Tübingen

Meine akademischen Lehrer

in China:	F. L. Huang, M. K. Luo, D. H. Yu, A. M. Li, A. S. Xu
in Tübingen:	R. Nagel, G. Huisken, U. Schlotterbeck, H. Ruder