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Research article

Complex-valued controlled rectangular metric type spaces and application to linear systems

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Abstract: Fixed point theory can be generalized to cover multidisciplinary areas such as computer science; it can also be used for image authentication to ensure secure communication and detect any malicious modifications. In this article, we introduce the notion of complex-valued controlled rectangular metric-type spaces, where we prove fixed point theorems for self-mappings in such spaces. Furthermore, we present several examples and give two applications of our main results: solving linear systems of equations and finding a unique solution for an equation of the form $f(x) = 0$.

Keywords: complex valued controlled rectangular metric type spaces; fixed point; linear system of equations
Mathematics Subject Classification: 47H10, 54H25

1. Introduction and motivation

The fixed point techniques represent a very useful tool in proving the existence and uniqueness of the solution of different type of equations as: integral equations, differential equations, fractional differential equations. Fixed point theory, which was introduced by Banach in 1922 [1], is growing fast, having applications in mathematical sciences, discrete dynamics, and more recently in super fractals. In fact, fixed point theory is considered the primary tool in studying nonlinear analysis, furthermore, fixed point theory for nonlinear operators have many applications in nonlinear equations and many other subjects (c.f. [2–4]). In 1969, Nadler [5] studied fixed point theorems for several-valued mappings, hence generalizing Banach's fixed point theorem, and this opened a new direction of research in fixed point theorems. Ghaler [6] extended the metric space and presented 2-metric space, which prompted a large number of publications discussing various metric space extensions, including fuzzy metric spaces, Intuitionistic fuzzy metric spaces, Branciari metric spaces, cone metric spaces, D-metric spaces, and modular metric spaces. Mathematicians have long found studying new spaces and their characteristics to be fascinating subjects. In this direction, Bakhtin [7] developed the concept of b-metric space in 1989, and later more work was done by Czerwik [8] by establishing a weaker postulate than the traditional triangle inequality. Lately, the concept of b-metric spaces has undergone considerable generalizations; consult for example, [9, 10]. See [11] for an overview of fixed point theory on b-metric spaces. Shatanawi et al. thereafter obtained fixed point results on extended b-metric spaces [12]. Then, George et al. [13] introduced the notion of rectangular b-metric space, which was later developed into controlled rectangular b-metric space by Mlaiki et al. [14]. Along those same lines, the notion of controlled metric type spaces was introduced in [15], which was then enhanced to include double controlled metric type spaces [16]. Recently, many articles appeared dealing with fixed point theorems on various controlled metric type spaces under different contraction mappings, see for example, [17–23].

Azam et al. [24] were the first to propose complex-valued metric spaces, which are more general than real-valued metric spaces. They came up with fixed point theorems for mappings that adhere to generalized contraction criteria. In 1989, Rao et al. [25] initiated a new type of metric space which is a generalized form of complex-valued metric spaces, and obtained the fixed point theorem on such spaces. Ullah et al. [26] accomplished fixed point results on complex valued extended b-metric space. The concept of complex valued rectangular metric space was introduced by Abbas et al. [27]. Ullah et al. have further researched fixed point results on complex-valued rectangular extended b-metric spaces [28], while Mlaiki et al. [29] introduced complex valued triple controlled metric types spaces and obtained fixed point theorem, and Aslam et al. [30] studied fixed point results on complex-valued controlled metric spaces.

In this article, motivated by the work of Mlaiki et al. [14], we introduce the concept of complex-valued controlled rectangular metric type space and establish fixed point theorems. Moreover, we present several examples, and finally, in Section 4, we give two applications of our results.

2. Preliminaries

We begin our preliminaries by providing the notations that were first proposed by Azam et al. [24], who first discussed complex valued metric spaces in 2011.

Let $\mathbb{C}$ denote the set of complex numbers, for $z_1, z_2 \in \mathbb{C}$, we present a partial ordering $\preceq$ on $\mathbb{C}$ as follows: $z_1 \preceq z_2 \iff Re(z_1) \leq Re(z_2)$ and $Im(z_1) \leq Im(z_2)$. From this we can deduce that $z_1 \preceq z_2$, if any of the following situations exist:
$(C1)$ $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$.
$(C2)$ $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$.
$(C3)$ $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$.
$(C4)$ $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.
Observe that if $z_1 \preceq z_2 \Longrightarrow |z_1| \leq |z_2|$.

Definition 2.1. *[24] Let $\mathbb{Y} \neq \phi$. If the mapping $\mathfrak{L} : \mathbb{Y}^2 \to \mathbb{C}$ satisfies all the following conditions:*

(L_1) $\mathfrak{x} = \mathfrak{y} \iff \mathfrak{L}(\mathfrak{x}, \mathfrak{y}) = 0$;
(L_2) $\mathfrak{L}(\mathfrak{x}, \mathfrak{y}) = \mathfrak{L}(\mathfrak{y}, \mathfrak{x})$;
(L_3) $\mathfrak{L}(\mathfrak{x}, \mathfrak{y}) \preceq \mathfrak{L}(\mathfrak{x}, \mathfrak{z}) + \mathfrak{L}(\mathfrak{z}, \mathfrak{y})$,
for all $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathbb{Y}$. Then, $(\mathbb{Y}, \mathfrak{L})$ is called a complex-valued metric space.

Abbas et al. [27] presented the notion of complex valued rectangular metric space, as follows:

Definition 2.2. *[27] Let $\mathbb{Y} \neq \phi$. If the mapping $\mathfrak{L} : \mathbb{Y}^2 \to \mathbb{C}$ satisfies all the following conditions:*

(L_1) $\mathfrak{x} = \mathfrak{y} \Longleftrightarrow \mathfrak{L}(\mathfrak{x}, \mathfrak{y}) = 0$;
(L_2) $\mathfrak{L}(\mathfrak{x}, \mathfrak{y}) = \mathfrak{L}(\mathfrak{y}, \mathfrak{x})$;
(L_3) $\mathfrak{L}(\mathfrak{x}, \mathfrak{y}) \preceq \mathfrak{L}(\mathfrak{x}, a) + \mathfrak{L}(a, b) + \mathfrak{L}(b, \mathfrak{y})$,
for all $\mathfrak{x}, \mathfrak{y} \in \mathbb{Y}$ and all distinct $a, b \in \mathbb{Y}$, each one is different from $\mathfrak{y}$ and $\mathfrak{x}$. Then, $(\mathbb{Y}, \mathfrak{L})$ is called a complex-valued rectangular metric space.

Ullah et al. [26] developed the notion of complex-valued extended b-metric spaces. Later, they initiated the notion of complex-valued rectangular extended metric spaces [28].

Definition 2.3. *[28] Let $\mathbb{Y} \neq \phi$, and let $\xi : \mathbb{Y}^2 \to [1, \infty)$ be a function. Define the mapping $\mathfrak{L} : \mathbb{Y}^2 \to \mathbb{C}$ for all distinct $\mathfrak{x}, \mathfrak{y}, a, b \in \mathbb{Y}$, as follows:*

(L_1) $\mathfrak{x} = \mathfrak{y} \Longleftrightarrow \mathfrak{L}(\mathfrak{x}, \mathfrak{y}) = 0$;
(L_2) $\mathfrak{L}(\mathfrak{x}, \mathfrak{y}) = \mathfrak{L}(\mathfrak{y}, \mathfrak{x})$;
(L_3) $\mathfrak{L}(\mathfrak{x}, \mathfrak{y}) \preceq \xi(\mathfrak{x}, \mathfrak{y})[\mathfrak{L}(\mathfrak{x}, a) + \mathfrak{L}(a, b) + \mathfrak{L}(b, \mathfrak{y})]$.
Then, $(\mathbb{Y}, \mathfrak{L})$ is called a complex-valued rectangular extended b-metric space.

Being inspired, by Definition 2.3 and motivated by the notion of controlled rectangular b-metric spaces, which was initiated in [14] by Mlaiki et al., we introduce the definition of complex-valued controlled rectangular metric type spaces.

Definition 2.4. *Let $\mathbb{X} \neq \phi$, and let $\Upsilon : \mathbb{X}^4 \to [1, \infty)$ be a function. Consider the mapping $\mathfrak{L} : \mathbb{X}^2 \to \mathbb{C}$ meeting the criteria, for all distinct $u, v, a, b \in \mathbb{X}$:*

1) $\mathfrak{L}(u, v) = 0 \Longleftrightarrow u = v$;
2) $\mathfrak{L}(u, v) = \mathfrak{L}(v, u)$;
3) $\mathfrak{L}(u, v) \preceq \Upsilon(u, v, a, b)[\mathfrak{L}(u, a) + \mathfrak{L}(a, b) + \mathfrak{L}(b, v)]$.

Then $(\mathbb{X}, \mathfrak{L})$ is called a complex-valued controlled rectangular metric type space.

Throughout the rest of this manuscript we denote complex-valued controlled rectangular metric type space by complex-valued $CRMTS$.

Note that every complex-valued rectangular metric space is a complex-valued $CRMTS$, but the converse is invalid, as the example below illustrates.

Example 2.1. *Let $\mathbb{X} = \mathbb{R}$. Define $\mathfrak{L} : \mathbb{X}^2 \to \mathbb{C}$ by*

$$\mathfrak{L}(x, y) = |x - y|^2 + i|x - y|^2.$$

For all $x, y \in \mathbb{X}$. Define $\Upsilon : \mathbb{X}^4 \to [1, \infty)$ by $\Upsilon(x, y, a, b) = \max\{x, y, a, b\} + 2$. Then, it can be easily shown that $(\mathbb{X}, \mathfrak{L})$ is a complex-valued $CRMTS$. But it is not complex-valued rectangular metric space, for example consider

$$\begin{aligned} \mathfrak{L}(3, 1/2) &> \mathfrak{L}(3, 1) + \mathfrak{L}(1, 1/3) + \mathfrak{L}(1/3, 1/2) \\ 6.25 + 6.25i &> (4 + 4i) + (4/9 + 4/9i) + (1/36 + 1/36i) = 161/36 + 161/36i. \end{aligned}$$

Below are some examples of our defined complex-valued $CRMTS$.

Example 2.2. *Let $\mathbb{X} = [0, 1]$. Define $\mathfrak{L} : \mathbb{X}^2 \to \mathbb{C}$ by*

$$\mathfrak{L}(x, y) = e^{ik}|x - y|.$$

For all $x, y \in \mathbb{X}$, and some $k \in \mathbb{R}^+$. Define $\Upsilon : \mathbb{X}^4 \to [1, \infty)$ by $\Upsilon(x, y, a, b) = \max\{x, y, a, b\} + 1$. Then $(\mathbb{X}, \mathfrak{L})$ is a complex-valued $CRMTS$.

Example 2.3. *Let* $\mathbb{X} = \{1, 2, 3, 4\}$. *Define* $\mathfrak{L} : \mathbb{X}^2 \to \mathbb{C}$ *by*
$\mathfrak{L}(x, y) = 0$ *if and only if* $x = y$,
$\mathfrak{L}(1, 2) = \mathfrak{L}(2, 1) = 3i$, *and* $\mathfrak{L}(2, 3) = \mathfrak{L}(3, 2) = \mathfrak{L}(1, 3) = \mathfrak{L}(3, 1) = 1i$, *also* $\mathfrak{L}(1, 4) = \mathfrak{L}(4, 1) = \mathfrak{L}(4, 2) = \mathfrak{L}(2, 4) = \mathfrak{L}(3, 4) = \mathfrak{L}(4, 3) = 4i$.

Define $\Upsilon : \mathbb{X}^4 \to [1, \infty)$ by $\Upsilon(x, y, a, b) = \max\{x, y, a, b\}$. Clearly properties (1) and (2) of Definition 2.4 are easily verified. We only verify (3).

$$\mathfrak{L}(1, 3) = 1i \preceq \Upsilon(1, 3, 2, 4)[\mathfrak{L}(1, 2) + \mathfrak{L}(2, 4) + \mathfrak{L}(4, 3)] = 44i.$$

$$\mathfrak{L}(1, 2) = 3i \preceq \Upsilon(1, 2, 3, 4)[\mathfrak{L}(1, 3) + \mathfrak{L}(3, 4) + \mathfrak{L}(4, 2)] = 36i.$$

$$\mathfrak{L}(3, 4) = 4i \preceq \Upsilon(3, 4, 1, 2)[\mathfrak{L}(3, 1) + \mathfrak{L}(1, 2) + \mathfrak{L}(2, 4)] = 32i.$$

$$\mathfrak{L}(1, 4) = 4i \preceq \Upsilon(1, 4, 2, 3)[\mathfrak{L}(1, 2) + \mathfrak{L}(2, 3) + \mathfrak{L}(3, 4)] = 32i.$$

Thus, $(\mathbb{X}, \mathfrak{L})$ is a complex-valued $CRMTS$.

Before defining the convergence of the sequences, the Cauchy sequence and the open ball in the complex-valued $CRMTS$. We state the following lemma.

Lemma 2.1. *[24] Let* $(\mathbb{X}, \mathfrak{L})$ *be a complex-valued* $CRMTS$, *and let* $\{\chi_l\}$ *be a sequence in* $\mathbb{X}$. *Then*

- The sequence $\{\chi_l\}$ converges to $\chi \iff |\mathfrak{L}(\chi_l, \chi)| \to 0$ as $l \to \infty$.
- The sequence $\{\chi_l\}$ is Cauchy sequence $\iff |\mathfrak{L}(\chi_l, \chi_k)| \to 0$ as $l, k \to \infty$.

Definition 2.5. *Let* $(\mathbb{X}, \mathfrak{L})$ *be a complex-valued* $CRMTS$. *Then the convergence of a sequence and the open ball is defined as follows:*

1) We say a sequence $\{\chi_l\}$ in $(\mathbb{X}, \mathfrak{L})$ is convergent, if there exists $\nu \in \mathbb{X}$, such that $\lim_{l\to\infty} |\mathfrak{L}(\chi_l, \nu)| = 0$.
2) A sequence $\{\chi_l\}$ is Cauchy $\iff \lim_{l,m\to\infty} |\mathfrak{L}(\chi_l, \chi_m)| = 0$.
3) If every Cauchy sequence in $\mathbb{X}$ is convergent, then we say$(\mathbb{X}, \mathfrak{L})$ is complete.
4) Let $a \in \mathbb{X}$, then an open ball with center a and radius $0 \prec \eta \in \mathbb{C}$, in $(\mathbb{X}, \mathfrak{L})$ is defined by

$$B_{\mathfrak{L}}(a, \eta) = \{x \in \mathbb{X} : \mathfrak{L}(a, x) \prec \eta\}.$$

3. Main results

This is a presentation of our first main finding.

Theorem 3.1. *Let* $(\mathbb{X}, \mathfrak{L})$ *be a complete complex-valued* $CRMTS$, *and let* $T : \mathbb{X} \to \mathbb{X}$ *be a mapping satisfying* $\mathfrak{L}(T\chi, T\tilde{\eta}) \preceq k\mathfrak{L}(\chi, \tilde{\eta})$, *for some* $k \in (0, 1)$. *Suppose there exists* $\chi_0 \in \mathbb{X}$, *so that the sequence* $\{\chi_l\}$, *defined by* $\chi_l = T^l\chi_0$, *satisfies the following:*

$$\sup_{m>1} \lim_{l\to\infty} \Upsilon(\chi_l, \chi_{l+1}, \chi_{l+2}, \chi_m) \le \frac{1}{k}. \tag{3.1}$$

Then, T admits a unique fixed point in $\mathbb{X}$.

Proof. Let $\chi_0 \in \mathbb{X}$ and consider the sequence $\{\chi_l\}$, where for each $l \ge 1$, we have $\chi_l = T^l\chi_0$. By the theorem's assumption, we have

$$\mathfrak{L}(\chi_l, \chi_{l+1}) \le k\mathfrak{L}(\chi_{l-1}, \chi_l) \le k^2\mathfrak{L}(\chi_{l-2}, \chi_{l-1}) \le \cdots \le k^l\mathfrak{L}(\chi_0, \chi_1). \tag{3.2}$$

Denote by $L_0 = \mathfrak{L}(\chi_0, \chi_1)$. Thus

$$|\mathfrak{L}(\chi_l, \chi_{l+1})| \le k^l|\mathfrak{L}(\chi_0, \chi_1)| = k^l|L_0|, \tag{3.3}$$

by taking the limit, we obtain

$$|\mathfrak{L}(\chi_l, \chi_{l+1})| \to 0, \text{ as } l \to \infty. \tag{3.4}$$

For all $l \ge 1$, we have 2 cases.

Case 1: There exists at least an integer $l \ne m$, assume $\chi_l = \chi_m$. So, in case $m > l$ then $T^{m-l}(\chi_l) = \chi_l$. Select $\chi = \chi_l$ and $p = m - l$. Then $T^p\chi = \chi$, and that is, χ is a periodic point of the contraction T. Thus, $\mathfrak{L}(\chi, T\chi) = \mathfrak{L}(T^p\chi, T^{p+1}\chi) \le k^p\mathfrak{L}(\chi, T\chi)$. As $k \in (0, 1)$, we get $|\mathfrak{L}(\chi, T\chi)| = 0$, so $\chi = T\chi$, hence T admits a fixed point χ.

Case 2: Assume for all integers $l \ne m$, then $T^lx \ne T^mx$. Let $l < m \in N$, to illustrate that $\{\chi_l\}$ is a $\mathfrak{L}-$Cauchy sequence, we took into account two sub-cases:

Sub-case 1: Suppose that $m = l + 2p + 1$. Due to property (3) of Definition 2.4 we have,

$$\begin{aligned}
\mathfrak{L}(\chi_l, \chi_{l+2p+1}) &\le \Upsilon(\chi_l, \chi_{l+1}, \chi_{l+2}, \chi_{l+2p+1})[\mathfrak{L}(\chi_l, \chi_{l+1}) + \mathfrak{L}(\chi_{l+1}, \chi_{l+2}) + \mathfrak{L}(\chi_{l+2}, \chi_{l+2p+1})].\\
&\le \Upsilon(\chi_l, \chi_{l+1}, \chi_{l+2}, \chi_{l+2p+1})\mathfrak{L}(\chi_l, \chi_{l+1}) + \Upsilon(\chi_l, \chi_{l+1}, \chi_{l+2}, \chi_{l+2p+1})\mathfrak{L}(\chi_{l+1}, \chi_{l+2})\\
&+ \Upsilon(\chi_l, \chi_{l+1}, \chi_{l+2}, \chi_{l+2p+1})\Upsilon(\chi_{l+2}, \chi_{l+3}, \chi_{l+4}, \chi_{l+2p+1})[\mathfrak{L}(\chi_{l+2}, \chi_{l+3})\\
&+ \mathfrak{L}(\chi_{l+3}, \chi_{l+4}) + \mathfrak{L}(\chi_{l+4}, \chi_{l+2p+1})].\\
&\le \Upsilon(\chi_l, \chi_{l+1}, \chi_{l+2}, \chi_{l+2p+1})\mathfrak{L}(\chi_l, \chi_{l+1}) + \Upsilon(\chi_l, \chi_{l+1}, \chi_{l+2}, \chi_{l+2p+1})\mathfrak{L}(\chi_{l+1}, \chi_{l+2})\\
&+ \Upsilon(\chi_l, \chi_{l+1}, \chi_{l+2}, \chi_{l+2p+1})\Upsilon(\chi_{l+2}, \chi_{l+3}, \chi_{l+4}, \chi_{l+2p+1})\mathfrak{L}(\chi_{l+2}, \chi_{l+3})\\
&+ \Upsilon(\chi_l, \chi_{l+1}, \chi_{l+2}, \chi_{l+2p+1})\Upsilon(\chi_{l+2}, \chi_{l+3}, \chi_{l+4}, \chi_{l+2p+1})\mathfrak{L}(\chi_{l+3}, \chi_{l+4})\\
&+ \Upsilon(\chi_l, \chi_{l+1}, \chi_{l+2}, \chi_{l+2p+1})\Upsilon(\chi_{l+2}, \chi_{l+3}, \chi_{l+4}, \chi_{l+2p+1})\mathfrak{L}(\chi_{l+4}, \chi_{l+2p+1})\\
&\cdots\\
&\le \Upsilon(\chi_l, \chi_{l+1}, \chi_{l+2}, \chi_{l+2p+1})\mathfrak{L}(\chi_l, \chi_{l+1}) + \Upsilon(\chi_l, \chi_{l+1}, \chi_{l+2}, \chi_{l+2p+1})\mathfrak{L}(\chi_{l+1}, \chi_{l+2})\\
&+ \Upsilon(\chi_l, \chi_{l+1}, \chi_{l+2}, \chi_{l+2p+1})\Upsilon(\chi_{l+2}, \chi_{l+3}, \chi_{l+4}, \chi_{l+2p+1})\mathfrak{L}(\chi_{l+2}, \chi_{l+3})\\
&+ \Upsilon(\chi_l, \chi_{l+1}, \chi_{l+2}, \chi_{l+2p+1})\Upsilon(\chi_{l+2}, \chi_{l+3}, \chi_{l+4}, \chi_{l+2p+1})\mathfrak{L}(\chi_{l+3}, \chi_{l+4})\\
&+ \cdots + \Upsilon(\chi_l, \chi_{l+1}, \chi_{l+2}, \chi_{l+2p+1})\Upsilon(\chi_{l+2}, \chi_{l+3}, \chi_{l+4}, \chi_{l+2p+1})\\
&\cdots \Upsilon(\chi_{l+2p-2}, \chi_{l+2p-1}, \chi_{l+2p}, \chi_{l+2p+1})\mathfrak{L}(\chi_{l+2p}, \chi_{l+2p+1}).
\end{aligned}$$

Rearranging the terms and then taking the absolute value and using Eq (3.3), we obtain

$$|\mathfrak{L}(\chi_l, \chi_{l+2p+1})| \le \Upsilon(\chi_l, \chi_{l+1}, \chi_{l+2}, \chi_{l+2p+1})[(k^l + k^{l+1})|L_0|].$$

$$+\Upsilon(\chi_l,\chi_{l+1},\chi_{l+2},\chi_{l+2p+1})\Upsilon(\chi_{l+2},\chi_{l+3},\chi_{l+4},\chi_{l+2p+1})[(k^{l+2}+k^{l+3})|L_0|].$$
$$+\cdots+\Upsilon(\chi_l,\chi_{l+1},\chi_{l+2},\chi_{l+2p+1})\Upsilon(\chi_{l+2},\chi_{l+3},\chi_{l+4},\chi_{l+2p+1})\times\cdots$$
$$\times\cdots\Upsilon(\chi_{l+2p-2},\chi_{l+2p-1},\chi_{l+2p},\chi_{l+2p+1})[(k^{l+2p-2}+k^{l+2p-1})|L_0|].$$
$$=\sum_{r=0}^{p-1}\prod_{i=0}^{r}\Upsilon(\chi_{l+2i},\chi_{l+2i+1},\chi_{l+2i+2},\chi_{l+2p+1})[k^{l+2r}+k^{l+2r+1}]|L_0|.$$
$$=\sum_{r=0}^{p-1}\prod_{i=0}^{r}\Upsilon(\chi_{l+2i},\chi_{l+2i+1},\chi_{l+2i+2},\chi_{l+2p+1})[1+k]k^{l+2r}|L_0|.$$

As $k<1$, the following is implied by the above inequalities:

$$|\mathfrak{L}(\chi_l,\chi_{l+2p+1})|<\sum_{r=0}^{p-1}\prod_{i=0}^{r}\Upsilon(\chi_{l+2i},\chi_{l+2i+1},\chi_{l+2i+2},\chi_{l+2p+1})2k^{l+2r}|L_0|.$$

By Eqs (3.1) and (3.4), and then taking the limits as both l and p tends to infinity, we deduce,

$$\begin{aligned}|\mathfrak{L}(\chi_l,\chi_{l+2p+1})| &< \sum_{r=0}^{\infty}\prod_{i=0}^{r}\Upsilon(\chi_{l+2i},\chi_{l+2i+1},\chi_{l+2i+2},\chi_{l+2p+1})2k^{l+2r}|L_0|.\\ &\leq \sum_{r=0}^{\infty}\frac{1}{k^{r+1}}2k^{l+2r}|L_0|.\\ &\leq \sum_{r=0}^{\infty}2k^{l+r-1}|L_0|.\end{aligned}$$

This series $\sum_{r=0}^{\infty}2k^{l+r-1}|L_0|$ converges, by utilizing the ratio test, we have $|\mathfrak{L}(\chi_l,\chi_{l+2p+1})|\to 0$, as l,p tends to ∞.

Sub-case 2: $m=l+2p$

$$\mathfrak{L}(\chi_l,\chi_{l+2})\leq k\mathfrak{L}(\chi_{l-1},\chi_{l+1})\leq k^2\mathfrak{L}(\chi_{l-2},\chi_l)\leq\cdots\leq k^l\mathfrak{L}(\chi_0,\chi_1), \tag{3.5}$$

by taking absolute value $|\mathfrak{L}(\chi_l,\chi_{l+2}|\leq k^l|L_0|$, and then taking the limit, we obtain

$$|\mathfrak{L}(\chi_l,\chi_{l+2})|\to 0,\ \text{as } l\to\infty. \tag{3.6}$$

Repeating similar process to Sub-case 1, we deduce

$$\begin{aligned}\mathfrak{L}(\chi_l,\chi_{l+2p}) &\leq \Upsilon(\chi_l,\chi_{l+1},\chi_{l+2},\chi_{l+2p})[\mathfrak{L}(\chi_l,\chi_{l+1})+\mathfrak{L}(\chi_{l+1},\chi_{l+2})+\mathfrak{L}(\chi_{l+2},\chi_{l+2p})].\\ &\leq \Upsilon(\chi_l,\chi_{l+1},\chi_{l+2},\chi_{l+2p})\mathfrak{L}(\chi_l,\chi_{l+1})+\Upsilon(\chi_l,\chi_{l+1},\chi_{l+2},\chi_{l+2p})\mathfrak{L}(\chi_{l+1},\chi_{l+2})\\ &+\Upsilon(\chi_l,\chi_{l+1},\chi_{l+2},\chi_{l+2p})\Upsilon(\chi_{l+2},\chi_{l+3},\chi_{l+4},\chi_{l+2p})[\mathfrak{L}(\chi_{l+2},\chi_{l+3})\\ &+\mathfrak{L}(\chi_{l+3},\chi_{l+4})+\mathfrak{L}(\chi_{l+4},\chi_{l+2p})].\end{aligned}$$

After several steps, it becomes

$$\mathfrak{L}(\chi_l,\chi_{l+2p})\leq\Upsilon(\chi_l,\chi_{l+1},\chi_{l+2},\chi_{l+2p}))\mathfrak{L}(\chi_l,\chi_{l+1})+\Upsilon(\chi_l,\chi_{l+1},\chi_{l+2},\chi_{l+2p})\mathfrak{L}(\chi_{l+1},\chi_{l+2})$$

$$
\begin{aligned}
&+ \Upsilon(\chi_l,\chi_{l+1},\chi_{l+2},\chi_{l+2p})\Upsilon(\chi_{l+2},\chi_{l+3},\chi_{l+4},\chi_{l+2p})\mathfrak{L}(\chi_{l+2},\chi_{l+3})\\
&+ \Upsilon(\chi_l,\chi_{l+1},\chi_{l+2},\chi_{l+2p})\Upsilon(\chi_{l+2},\chi_{l+3},\chi_{l+4},\chi_{l+2p})\mathfrak{L}(\chi_{l+3},\chi_{l+4})\\
&+ \cdots\\
&+ \Upsilon(\chi_l,\chi_{l+1},\chi_{l+2},\chi_{l+2p})\Upsilon(\chi_{l+2},\chi_{l+3},\chi_{l+4},\chi_{l+2p}) \times \cdots\\
&\times \cdots \Upsilon(\chi_{l+2p-3},\chi_{l+2p-2},\chi_{l+2p-1},\chi_{l+2p})\mathfrak{L}(\chi_{l+2p},\chi_{l+2p+1})\\
&+ \prod_{i=0}^{2p-2} \Upsilon(\chi_{l+2i},\chi_{l+2i+1},\chi_{l+2i+1},\chi_{l+2p})\mathfrak{L}(\chi_{l+2p-2},\chi_{l+2p}).
\end{aligned}
$$

By Eq (3.5) and taking the absolute value we get

$$
\begin{aligned}
|\mathfrak{L}(\chi_l,\chi_{l+2p})| \leq& \sum_{r=0}^{p-1}\prod_{i=0}^{r} \Upsilon(\chi_{l+2i},\chi_{l+2i+1},\chi_{l+2i+2},\chi_{l+2p+1})[k^{l+2r}+k^{l+2r+1}]|L_0|\\
&+ \prod_{i=0}^{2p-2} \Upsilon(\chi_{l+2i},\chi_{l+2i+1},\chi_{l+2i+1},\chi_{l+2p})k^{l+2p-2}|L_0|.\\
\leq& \sum_{r=0}^{p-1}\prod_{i=0}^{r} \Upsilon(\chi_{l+2i},\chi_{l+2i+1},\chi_{l+2i+2},\chi_{l+2p+1})[1+k]k^{l+2r}|L_0|\\
&+ \prod_{i=0}^{2p-2} \Upsilon(\chi_{l+2i},\chi_{l+2i+1},\chi_{l+2i+1},\chi_{l+2p})k^{l+2p-2}|L_0|.
\end{aligned}
$$

Since, $\sup_{m>1}\lim_{l\to\infty}\Upsilon(\chi_l,\chi_{l+1},\chi_{l+2},\chi_m) \leq \frac{1}{k}$, we deduce,

$$
\begin{aligned}
|\mathfrak{L}(\chi_l,\chi_{l+2p})| \leq& \sum_{r=0}^{p-1}\frac{1}{k^{l+1}}[1+k]k^{l+2r}|L_0| + k^{-2p+1}k^{l+2p-2}|L_0|.\\
=& \sum_{r=0}^{p-1}[1+k]k^{l+r-1}|L_0| + k^{l-1}|L_0|.
\end{aligned}
$$

We have $|\mathfrak{L}(\chi_l,\chi_{l+2p})| \leq [\sum_{r=0}^{p-1}[1+k]k^{l+r-1}+k^{l-1}]|L_0|$. Since $k<1$, so $\lim_{m\to\infty}k^m = 0$, and using the ratio test on the series, it tends to zero for any value of p, we deduce that

$$|\mathfrak{L}(\chi_l,\chi_{l+2p})| \to 0, l, p \to \infty.$$

Hence, $\mathfrak{L}(\chi_l,\chi_{l+2p})$ converges as l, p tends to ∞.

Therefore, it is demonstrated by Sub-cases 1 and 2 that the sequence $\{\chi_l\}$ is a Cauchy sequence. Given that $(\mathbb{X},\mathfrak{L})$ is a complete complex-valued $CRMTS$, we deduce that $\{\chi_l\}$ converges to $\nu \in \mathbb{X}$.

Now, to show ν is fixed by T. Without loss of generality, we assume that for all l, we have $\chi_l \notin \{\nu, T\nu\}$. Hence

$$
\begin{aligned}
\mathfrak{L}(\nu,T\nu) &\leq \Upsilon(\nu,\chi_l,\chi_{l+1},T\nu)[\mathfrak{L}(\nu,\chi_l)+\mathfrak{L}(\chi_l,\chi_{l+1})+\mathfrak{L}(\chi_{l+1},T\nu)].\\
&\leq \Upsilon(\nu,\chi_l,\chi_{l+1},T\nu)[\mathfrak{L}(\nu,\chi_l)+\mathfrak{L}(\chi_l,\chi_{l+1})+\mathfrak{L}(T\chi_l,T\nu)].\\
&\leq \Upsilon(\nu,\chi_l,\chi_{l+1},T\nu)[\mathfrak{L}(\nu,\chi_l)+\mathfrak{L}(\chi_l,\chi_{l+1})+k\mathfrak{L}(\chi_l,\nu)].
\end{aligned}
$$

Thus,

$$|\mathfrak{L}(\nu, T\nu)| \leq \Upsilon(\nu, \chi_l, \chi_{l+1}, T\nu))[|\mathfrak{L}(\nu, \chi_l)| + |\mathfrak{L}(\chi_l, \chi_{l+1})| + k|\mathfrak{L}(\chi_l, \nu)|].$$

By taking the limit as l tends to ∞, and using the fact that the sequence $\{\chi_l\}$ converges to ν, we obtain $|\mathfrak{L}(\nu, T\nu)| = 0$, i.e., $\mathfrak{L}(\nu, T\nu) = 0$, which implies that $T\nu = \nu$, thus ν is a fixed point of T.
Last step: Suppose that T has two fixed points, say ν and μ, such that $\nu \neq \mu$. The contractive property of T allows us to state:

$$\mathfrak{L}(\nu, \mu) = \mathfrak{L}(T\nu, T\mu) \leq k\mathfrak{L}(\nu, \mu),$$

which gives

$$|\mathfrak{L}(\nu, \mu)| = |\mathfrak{L}(T\nu, T\mu)| \leq k|\mathfrak{L}(\nu, \mu)| < |\mathfrak{L}(\nu, \mu)|,$$

resulting in a contradiction. Hence, T has a unique fixed point. □

Example 3.1. *Let* $\mathbb{X} = [0, 1]$ *and* $\mathfrak{L} : \mathbb{X} \times \mathbb{X} \to \mathbb{C}$ *be defined as*

$$\mathfrak{L}(u, v) = |u - v|e^{\frac{i}{7}}.$$

Next, we define $\Upsilon\colon X^4 \to [1, \infty)$ by

$$\Upsilon(u, v, a, b) = \max(u, v, a, b) + 1.5.$$

We only show the extended quadrilateral inequality,

$$\mathfrak{L}(u, v) = |u - v|e^{\frac{i}{7}} \leq (|u - a| + |a - b| + |b - v|)e^{\frac{i}{7}} \leq \Upsilon(u, v, a, b)[\mathfrak{L}(u, a) + \mathfrak{L}(a, b) + \mathfrak{L}(b, v)].$$

Thus, $(\mathbb{X}, \mathfrak{L})$ is a complete complex-valued $CRMTS$.

Let $T\colon \mathbb{X} \to \mathbb{X}$ be the contraction mapping defined as $T(x) = \frac{x}{7}$ and take $k = \frac{1}{3} \in (0, 1)$, then we get

$$\begin{aligned}\mathfrak{L}(Tx, Ty) = |Tx - Ty|e^{\frac{i}{7}} &= \frac{1}{7}|x - y|e^{\frac{i}{7}}.\\ &\leq \frac{1}{3}|x - y|e^{\frac{i}{7}} = k|x - y|e^{\frac{i}{7}} = k\mathfrak{L}(x, y).\end{aligned}$$

Let $x_1 = 1$, then we form the sequence, $x_2 = T(1) = \frac{1}{7}$, hence $x_n = T^{n-1}(1) = (\frac{1}{7^{n-1}})$. Then clearly

$$sup_m \lim_{n\to\infty} \Upsilon(x_n, x_{n+1}, x_{n+2}, x_m) \leq 3.$$

As a result, Theorem 3.1 requirements are all met. Thus, T has a unique fixed point $x = 0$.
Our next major finding.

Theorem 3.2. *Let* $(\mathbb{X}, \mathfrak{L})$ *be a complete complex-valued* $CRMTS$, *and let the mapping* $T : \mathbb{X} \longrightarrow \mathbb{X}$ *satisfies the following; for all* $\chi, \tilde{\eta} \in \mathbb{X}$ *you can find* $0 < k < \frac{1}{2}$ *such that*

$$\mathfrak{L}(T\chi, T\tilde{\eta}) \leq k[\mathfrak{L}(\chi, T\chi) + \mathfrak{L}(\tilde{\eta}, T\tilde{\eta})]. \tag{3.7}$$

Moreover,

$$\sup_{m>1} \lim_{l\to\infty} \Upsilon(\chi_l, \chi_{l+1}, \chi_{l+2}, \chi_m) \leq \frac{1}{k}, \tag{3.8}$$

and for all $u, v \in \mathbb{X}$, this holds:

$$\lim_{l\to\infty} \Upsilon(u, v, \chi_l, \chi_{l+1}) \leq 1. \tag{3.9}$$

Then T admits a fixed point in $\mathbb{X}$ which is unique.

Proof. The sequence $\{\chi_l\}$ is defined as follows,

$$\chi_1 = T\chi_0, \chi_2 = T\chi_1 = T^2\chi_0, \cdots, \chi_l = T^l\chi_0, \cdots, \text{ for some } \chi_0 \in \mathbb{X}.$$

For all $l \geq 1$ by Eq (3.7), we have,

$$\begin{aligned} \mathfrak{L}(\chi_l, \chi_{l+1}) &= \mathfrak{L}(T\chi_{l-1}, T\chi_l) \leq k[\mathfrak{L}(\chi_{l-1}, \chi_l) + \mathfrak{L}(\chi_l, \chi_{l+1})]. \\ &\Rightarrow (1-k)\mathfrak{L}(\chi_l, \chi_{l+1}) \leq k\mathfrak{L}(\chi_{l-1}, \chi_l). \\ &\Rightarrow \mathfrak{L}(\chi_l, \chi_{l+1}) \leq \frac{k}{1-k}\mathfrak{L}(\chi_{l-1}, \chi_l). \end{aligned}$$

Since $0 < k < \frac{1}{2}$, one can easily deduce that $0 < \frac{k}{1-k} < 1$. Therefore, let $\alpha = \frac{k}{1-k}$.
Hence,

$$\begin{aligned} \mathfrak{L}(\chi_l, \chi_{l+1}) &\leq \alpha\mathfrak{L}(\chi_{l-1}, \chi_l). \\ &\leq \alpha^2\mathfrak{L}(\chi_{l-2}, \chi_{l-1}). \\ &\leq \cdots \\ &\leq \alpha^l\mathfrak{L}(\chi_0, \chi_1). \end{aligned}$$

Denote by $L_0 = \mathfrak{L}(\chi_0, \chi_1)$, therefore,

$$|\mathfrak{L}(\chi_l, \chi_{l+1})| \leq \alpha^l|\mathfrak{L}(\chi_0, \chi_1)| = \alpha^l|L_0|.$$

Hence,

$$|\mathfrak{L}(\chi_l, \chi_{l+1})| \to 0 \text{as} l \to \infty. \tag{3.10}$$

Also, for all $l \geq 1$ we have

$$\mathfrak{L}(\chi_l, \chi_{l+2}) \leq k[\mathfrak{L}(\chi_{l-1}, \chi_l) + \mathfrak{L}(\chi_{l+1}, \chi_{l+2})],$$

which becomes

$$|\mathfrak{L}(\chi_l, \chi_{l+2})| \leq k[|\mathfrak{L}(\chi_{l-1}, \chi_l)| + |\mathfrak{L}(\chi_{l+1}, \chi_{l+2})|].$$

By using Eq (3.10), we deduce that

$$|\mathfrak{L}(\chi_l, \chi_{l+2})| \to 0 \text{ as } l \to \infty.$$

Following similar methods as of Case 1 and Case 2 of Theorem 3.1, we show that the sequence $\{\chi_l\}$ is a Cauchy sequence. Since $(\mathbb{X}, \mathfrak{L})$ is a complete complex-valued $CRMTS$, we conclude that $\{\chi_l\}$ converges to some $\nu \in \mathbb{X}$.

Next, we prove that T fixes ν. Hence, without loss of generality we may suppose that for all l, we have $\chi_l \notin \{\nu, T\nu\}$. Hence

$$\begin{aligned}
\mathfrak{L}(\nu, T\nu) &\leq \Upsilon(\nu, \chi_l, \chi_{l+1}, T\nu)[\mathfrak{L}(\nu, \chi_l) + \mathfrak{L}(\chi_l, \chi_{l+1}) + \mathfrak{L}(\chi_{l+1}, T\nu)]. \\
&\leq \Upsilon(\nu, \chi_l, \chi_{l+1}, T\nu)[\mathfrak{L}(\nu, \chi_l) + \mathfrak{L}(\chi_l, \chi_{l+1}) + \mathfrak{L}(T\chi_l, T\nu)]. \\
&\leq \Upsilon(\nu, \chi_l, \chi_{l+1}, T\nu)[\mathfrak{L}(\nu, \chi_l) + \mathfrak{L}(\chi_l, \chi_{l+1}) + k\mathfrak{L}(\chi_l, T\chi_l) + k\mathfrak{L}(\nu, T\nu)]. \\
&\leq \Upsilon(\nu, \chi_l, \chi_{l+1}, T\nu)[\mathfrak{L}(\nu, \chi_l) + \mathfrak{L}(\chi_l, \chi_{l+1}) + k\mathfrak{L}(\chi_l, \chi_{l+1}) + k\mathfrak{L}(\nu, T\nu)].
\end{aligned}$$

Thus,

$$|\mathfrak{L}(\nu, T\nu)| \leq |\mathfrak{L}(\nu, \chi_l)| + (k+1)|\mathfrak{L}(\chi_l, \chi_{l+1})| + k|\mathfrak{L}(\nu, T\nu)|.$$

The sequence $\{\chi_l\}$ converges to ν, as l tends to ∞, hence we obtain

$$|\mathfrak{L}(\nu, T\nu)| \leq k|\mathfrak{L}(\nu, T\nu)| < |\mathfrak{L}(\nu, T\nu)|,$$

from this we deduce that $|\mathfrak{L}(\nu, T\nu)| = 0$, that is $\mathfrak{L}(\nu, T\nu) = 0$, which implies that $T\nu = \nu$ and T fixes point ν.

By assuming two fixed points ν and μ of T such that $\nu \neq \mu$, we demonstrate the fixed point's uniqueness as follows:

$$\mathfrak{L}(\nu, \mu) = \mathfrak{L}(T\nu, T\mu) \leq k[\mathfrak{L}(\nu, T\nu) + \mathfrak{L}(\mu, T\mu)] = k[\mathfrak{L}(\nu, \nu) + \mathfrak{L}(\mu, \mu)] = 0.$$

Thus, by taking the absolute value we get $|\mathfrak{L}(\nu, \mu)| = 0$, that is $\mathfrak{L}(\nu, \mu) = 0$ which implies $\nu = \mu$. Thus, T has a unique fixed point as desired. □

4. Applications

In the sequel we will present two applications of our results.

Let $\mathbb{X} = \mathbb{C}^n$ where $\mathbb{C}$ is the set of complex numbers and n a positive integer. Consider the complete complex-valued $CRMTS$ $(\mathbb{X}, \mathfrak{L})$ defined by

$$\mathfrak{L}(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|.$$

where the function $\Upsilon : \mathbb{X}^4 \to [1, \infty)$, is given by $\Upsilon(x, y, z, t) = 2$, for all $x, y, z, t \in \mathbb{X}$.

Theorem 4.1. *Consider the following system:*

$$\begin{cases}
s_{11}x_1 + s_{12}x_2 + s_{13}x_3 + s_{1n}x_n = r_1, \\
s_{21}x_1 + s_{22}x_2 + s_{23}x_3 + s_{2n}x_n = r_2, \\
\vdots \\
s_{n1}x_1 + s_{n2}x_2 + s_{n3}x_3 + s_{nn}x_n = r_n,
\end{cases}$$

if $\sum_{j=1}^{n} |s_{ij} + s_{ii}| < \frac{1}{3}$, for all $i = 1, \cdots, n$, then the above linear system has a unique solution.

Proof. Consider the map $T : \mathbb{X} \to \mathbb{X}$ given by $Tx = (B + I_n)x - r$, where I_n is an $n \times n$ identity matrix, and

$$B = \begin{pmatrix} s_{11} & s_{12} & \cdots & s_{1n} \\ s_{21} & s_{22} & \cdots & s_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n1} & s_{n2} & \cdots & s_{nn} \end{pmatrix}$$

with $x = (x_1, x_2, \cdots, x_n),\ \ r = (r_1, r_2, \cdots, r_n) \in \mathbb{C}^n$.

$$\begin{aligned}
\mathfrak{L}(Tx, Ty) &= \max_{1\le i\le n} |\sum_{j=1}^{n} (s_{ij} + s_{ii})(x_i - y_i)|. \\
&\le \max_{1\le i\le n} \sum_{j=1}^{n} |s_{ij} + s_{ii}||x_i - y_i|. \\
&\le \max_{1\le i\le n} |x_i - y_i| \sum_{j=1}^{n} |s_{ij} + s_{ii}|. \\
&\le \max_{1\le i\le n} |x_i - y_i| \max_{1\le i\le n} \sum_{j=1}^{n} |s_{ij} + s_{ii}|. \\
&= \max_{1\le i\le n} \sum_{j=1}^{n} |s_{ij} + s_{ii}| \mathfrak{L}(x, y). \\
&\le \frac{1}{3} \mathfrak{L}(x, y).
\end{aligned}$$

All the assumptions of Theorem 3.1 are satisfied. Thus, T fixes a unique point and in view of this, the aforementioned linear system has a unique solution. □

Our next application guarantees a unique solution for an equation of the form $f(x) = 0$. We start first by stating our next theorem.

Theorem 4.2. *For any $n \in \mathbb{N}$, the equation*

$$(x + 1)^n + 1 = (n^2 2^n + 1)x(x + 1)^n + n^2 2^n x, \tag{4.1}$$

has a unique solution in the interval $[0, 1]$.

Proof. Define the mapping $T : [0, 1] \to [0, 1]$ by

$$Tx = \frac{(x + 1)^n + 1}{(n^2 2^n + 1)(x + 1)^n + n^2 2^n}, \ \text{ for some } n \in \mathbb{N}. \tag{4.2}$$

Note that x is a fixed point of T if and only if x is a solution of the Eq (4.1). Hence, we will show T has a unique fixed point in $[0, 1]$, by utilizing Theorem 3.1.

Define $\mathfrak{L} : [0,1] \times [0,1] \to \mathbb{C}$ by

$$\mathfrak{L}(x,y) = e^{i\tau}|x-y|, \tag{4.3}$$

for some $\tau > 0$. Let the function $\Upsilon : [0,1]^4 \to [1,\infty)$, be given by

$$\Upsilon(x,y,a,b) = \sum_{i=1}^{n}(x+1)^{n-i}(y+1)^{i-1}. \tag{4.4}$$

Then, $([0,1],\mathfrak{L})$ is a complete complex-valued $CRMTS$.
Observe that

$$\mathfrak{L}(Tx,Ty) \preceq k\,\mathfrak{L}(x,y),\ \text{for}\, x,y\in[0,1],\ \text{ and } k = \frac{n2^{n-1}}{2n^2 2^n+1} \in (0,1).$$

Indeed, by Eq (4.2) we have

$$\begin{aligned}
\mathfrak{L}(Tx,Ty) &= \mathfrak{L}\Big(\frac{(x+1)^n+1}{(n^2 2^n+1)(x+1)^n+n^2 2^n}, \frac{(y+1)^n+1}{(n^2 2^n+1)(y+1)^n+n^2 2^n}\Big).\\
&= e^{i\tau}\Big|\frac{(x+1)^n+1}{(n^2 2^n+1)(x+1)^n+n^2 2^n} - \frac{(y+1)^n+1}{(n^2 2^n+1)(y+1)^n+n^2 2^n}\Big|.\\
&\preceq \frac{e^{i\tau}}{(2n^2 2^n+1)^2}\,|(x+1)^n-(y+1)^n|.\\
&= \frac{e^{i\tau}}{(2n^2 2^n+1)^2}\sum_{i=1}^{n}(x+1)^{n-i}(y+1)^{i-1}\,|x-y|.\\
&\preceq \frac{n2^{n-1}}{(2n^2 2^n+1)^2}\,e^{i\tau}\,|x-y|, \quad \text{by (4.3)}.\\
&= \frac{n2^{n-1}}{(2n^2 2^n+1)^2}\,\mathfrak{L}(x,y) = k\,\mathfrak{L}(x,y).
\end{aligned}$$

Next, if we pick any $x \in [0,1]$, then $Tx \in [0,1]$, hence starting with any $x_0 \in [0,1]$, we can form the sequence $\{x_l\}$, by $x_l = T^l x_0 \in [0,1]$, for all $l \in \mathbb{N}$. By utilizing Eq (4.4), one can show easily this holds

$$\sup_{m>1}\lim_{l\to\infty}\Upsilon(x_l,x_{l+1},x_{l+2},x_m) \le n2^{n-1} \le \frac{1}{k}. \tag{4.5}$$

Thus, all the assumptions of Theorem 3.1 are satisfied. So, T has a unique fixed point in $[0,1]$. Hence, Eq (4.1) has a unique solution in the interval $[0,1]$. □

Example 4.1. *The equation*

$$(x+1)^2(17x-1)+16x-1 = 0, \tag{4.6}$$

has a unique solution in the interval $[0,1]$.

Proof. Note that the equation $(x+1)^2(17x-1)+16x-1 = 0$ is equivalent to

$$(x+1)^2+1 = 17x(x+1)^2+16x.$$

Therefore the result follows from Theorem 4.2 by taking $n = 2$. □

5. Conclusions

In this article, we have introduced the notion of complex-valued controlled rectangular metric type spaces. We have proved the existence and uniqueness of a fixed point for self-mapping in such a space. Moreover, we presented several examples and two applications of our results, which includes solving systems of linear equations and finding a unique solution for an equation of the form $f(x) = 0$. In closing, we would like to bring to the reader's attention that one can introduce complex-valued metric type spaces endowed with a graph.

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Conflict of interest

The authors declare no conflicts of interest.

References

1. S. Banach, Sur les operations dans les ensembles et leur application aux equation sitegrales, *Fund. Math.*, **3** (1922), 133–181. https://doi.org/10.4064/FM-3-1-133-181
2. J. A. Gatica, W. A. Kirk, A fixed point theorem for k-set-contractions defined in a cone, *Pac. J. Math.*, **53** (1974), 131–136.
3. Z. Zhang, K. Wang, On fixed point theorems of mixed monotone operators and applications, *Nonlinear Anal. Theor.*, **70** (2009), 3279–3284. https://doi.org/10.1016/j.na.2008.04.032
4. J. Brzdek, M. Piszczek, Fixed points of some nonlinear operators in spaces of multifunctions and the Ulam stability, *J. Fixed Point Theory Appl.*, **19** (2017), 2441–2448. https://doi.org/10.1007/s11784-017-0441-1
5. S. Nadler, Multi-valued contraction mappings, *Pac. J. Math.*, **30** (1969), 475–488.
6. S. Ghaler, 2-metrishche raume und ihre topologische strukture, *Math. Nachr.*, **26** (1963), 115–148. https://doi.org/10.1002/mana.19630260109
7. I. A. Bakhtin, The contraction mapping principle in almost metric spaces, *Func. Anal. Gos. Ped. Inst. Unianowsk*, **30** (1989), 26–37.
8. S. Czerwik, Contraction mappings in *b*-metric spaces, *Acta Mathematica et Informatica Universitatis Ostraviensis*, **1** (1993), 5–11.
9. M. A. Kutbi, E. Karapınar, J. Ahmad, A. Azam, Some fixed point results for multi-valued mappings in b-metric spaces, *J. Inequal. Appl.*, **2014** (2014), 126. https://doi.org/10.1186/1029-242X-2014-126
10. T. Kamran, M. Samreen, Q. U. Ain, A generalization of *b*-metric space and some fixed point theorems, *Mathematics*, **5** (2017), 19. https://doi.org/10.3390/math5020019

11. E. Karapinar, A short survey on the recent fixed point results on b-metric spaces, *Constructive Mathematical Analysis*, **1** (2018), 15–44. https://doi.org/10.33205/cma.453034

12. W. Shatanawi, K. Abodayeh, A. Mukheimer, Some fixed point theorems in extended b-metric spaces, *UPB Scientific Bulletin, Series A: Applied Mathematics and Physics*, **80** (2018), 71–78.

13. R. George, S. Radenovic, P. K. Reshma, S. Shukla, Rectangular b-metric space and contraction principles, *J. Nonlinear Sci. Appl.*, **8** (2015), 1005–1013. https://doi.org/10.22436/JNSA.008.06.11

14. N. Mlaiki, M. Hajji, T. Abdeljawad, A new extension of the rectangular-metric spaces, *Adv. Math. Phys.*, **2020** (2020), 8319584. https://doi.org/10.1155/2020/8319584

15. N. Mlaiki, H. Aydi, N. Souayah, T. Abdeljawad, Controlled metric type spaces and the related contraction principle, *Mathematics*, **6** (2018), 194. https://doi.org/10.3390/math6100194

16. T. Abdeljawad, N. Mlaiki, H. Aydi, N. Souayah, Double controlled metric type spaces and some fixed point results, *Mathematics*, **6** (2018), 320. https://doi.org/10.3390/math6120320

17. W. Shatanawi, Common fixed points for mappings under contractive conditions of (α,β,ψ)-admissibility type, *Mathematics*, **6** (2018), 261. https://doi.org/10.3390/math6110261

18. N. Y. Özgür, N. Mlaiki, N. Taş, N. Souayah, A new generalization of metric spaces: rectangular M-metric spaces, *Math. Sci.*, **12** (2018), 223–233. https://doi.org/10.1007/s40096-018-0262-4

19. M. Zhou, N. Saleem, X. L. Liu, N. Ozgur, On two new contractions and discontinuity on fixed points, *AIMS Mathematics*, **7** (2022), 1628–1663. https://doi.org/10.3934/math.2022095

20. A. Latif, R. F. A. Subaie, M. O. Ansari, Fixed points of generalized multi-valued contractive mappings in metric type spaces, *J. Nonlinear Var. Anal.*, **6** (2022), 123–138. https://doi.org/10.23952/jnva.6.2022.1.07

21. E. Karapınar, R. P. Agarwal, S. S. Yesilkaya, C. Wang, Fixed-point results for Meir-Keeler type contractions in partial metric spaces: a survey, *Mathematics*, **10** (2022), 3109. https://doi.org/10.3390/math10173109

22. F. M. Azmi, New fixed point results in double controlled metric type spaces with applications, *AIMS Mathematics*, **8** (2023), 1592–1609. https://doi.org/10.3934/math.2023080

23. F. M. Azmi, New contractive mappings and solutions to boundary-value problems in triple controlled metric type spaces, *Symmetry*, **14** (2022), 2270. https://doi.org/10.3390/sym14112270

24. A. Azam, B. Fisher, M. Khan, Common fixed point theorems in complex-valued metric spaces, *Numer. Func. Anal. Opt.*, **32** (2011), 243–253.

25. K. Rao, P. Swamy, J. Prasad, A common fixed point theorem in complex-valued b-metric spaces, *Bulletin of Mathematics and Statistics Research*, **1** (2013), 1–8.

26. N. Ullah, M. S. Shagari, A. Azam, Fixed point theorems in complex-valued extended b-metric spaces, *Journal of Pure and Applied Analysis*, **5** (2019), 140–163. https://doi.org/10.2478/mjpaa-2019-0011

27. M. Abbas, V. Raji, T. Nazir, S. Radenovi, Common fixed point of mappings satisfying rational inequalities in ordered complex valued generalized metric spaces, *Afr. Mat.*, **26** (2015), 17–30. https://doi.org/10.1007/s13370-013-0185-z

28. N. Ullah, M. S. Shagari, Fixed point results in complex-valued rectangular extended b-metric spaces with applications, *Mathematical Analysis and Complex Optimization*, **1** (2020), 107–120. https://doi.org/10.29252/maco.1.2.11

29. N. Mlaiki, T. Abdeljawad, W. Shatanawi, H. Aydi, Y. Gaba, On complex-valued triple controlled metric spaces and applications, *J. Funct. Space.*, **2021** (2021), 5563456. https://doi.org/10.1155/2021/5563456

30. M. S. Aslam, M. S. R. Chowdhury, L. Guran, M. A. Alqudah, T. Abdeljawad, Fixed point theory in complex valued controlled metric spaces with an application, *AIMS Mathematics*, **7** (2022), 11879–11904. https://doi.org/10.3934/math.2022663

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