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Research article

Analytical solutions for nonlinear systems using Nucci's reduction approach and generalized projective Riccati equations

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Abstract: In this study, the Nucci's reduction approach and the method of generalized projective Riccati equations (GPREs) were utilized to derive novel analytical solutions for the (1+1)-dimensional classical Boussinesq equations, the generalized reaction Duffing model, and the nonlinear Pochhammer-Chree equation. The nonlinear systems mentioned earlier have been solved using analytical methods, which impose certain limitations on the interaction parameters and the coefficients of the guess solutions. However, in the case of the double sub-equation guess solution, analytic solutions were allowed. The soliton solutions that were obtained through this method display real positive values for the wave phase transformation, which is a novel result in the application of the generalized projective Riccati method. In previous applications of this method, the real positive properties of the solutions were not thoroughly investigated.

Keywords: the Nucci's reduction method; the generalized projective Riccati equations method; the (1+1)-dimensional classical Boussinesq equations; the generalized reaction duffing model; the nonlinear Pochhammer-Chree equation; traveling wave solutions
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1. Introduction

In the fields of physics, chemistry, biology, and even social science, nonlinear partial differential equations (NPDEs) are frequently employed to represent a wide range of events. When exact solutions are available, they can be used to help understand nonlinear events. One of the most crucial and vital tasks in nonlinear science is finding and creating exact solutions, especially solitons, to NPDEs. Exact solutions in mathematics and physics are highly valued as they provide insights into the behavior of complex systems. Soliton solutions are a type of exact solution that represents solitary waves that maintain their shape and speed even after collisions with other solitons. Wave solutions, on the other hand, represent periodic or quasi-periodic waves that propagate through space with a certain frequency and amplitude. They are often used to model wave phenomena in various fields, such as fluid dynamics, electromagnetics, and acoustics. Lump solutions are another type of exact solution that represents localized, non-dispersive structures that do not spread out over time. They are often used to model coherent structures in various fields, such as turbulence, plasma physics, and nonlinear optics. The study of these exact solutions is important as they can provide valuable insights into the behavior of physical systems and aid in the development of mathematical models that can accurately predict the behavior of such systems [1–4].

To find exact solutions to NPDEs, numerous strong and effective techniques have recently been developed and enhanced. Among these strategies, we can highlight the homogeneous balance method [5, 6], the tanh-function method [7], the extended tanh-function method [8], the Jacobi elliptic function method [9–11], the auxiliary equation method [12], invariant subspace method [13, 14], and Lie symmetry method [15, 16]. The following techniques have also been used to explore the analytical solutions of NPDEs: $\frac{G'}{G}-$ expansion method [17], the expansion function method and the modified $exp\left(-\Omega\left(\varphi\right)\right)$-expansion function method [18, 19], the double sub-equation method [20, 21], the F-expansion procedure [22] and many others.

In this study, we aim to use two powerful techniques to obtain new exact solutions to three nonlinear differential equations. The first technique we employ is Nucci's reduction method, which yields both generic implicit and explicit solutions for all the equations considered, as well as their reduced equations' first integrals. In addition to Nucci's method, we use the powerful GPREs method to create novel solutions to the equations.

The structure of this paper is organized as follows. Section 2 discusses the descriptions of the implied analytical methods. Section 3 summarizes the main findings of this paper, as well as the suggested strategies and analytical solutions. The final section includes a summary and conclusions.

2. Methodology

In this section, we will briefly introduce the techniques used to find exact solutions to upcoming nonlinear differential equations.

2.1. Nucci's reduction method

To obtain exact solutions and first integrals, we use Nucci's reduction method. This technique was proposed for the first time in [23] to find nonlocal symmetries of differential equations. This method is used in the [24] for group analysis of the mathematical model for thin liquid films. The Nucci's reduction method is used to investigate evolution type differential equations with recently defined Atangana-Baleanu fractional derivatives in [25].

In the current paper, all reduced ODEs are second order nonlinear ODEs. Therefore, we can assume the governing reduced ODE as:

$$U''(\zeta) = \mathcal{F}(U(\zeta), U'(\zeta)). \tag{2.1}$$

Assume we change variables according to the proposed reduction method [16, 26, 27]:

$$\varpi_1(\zeta) = \mathcal{U}(\zeta), \;\; \varpi_2(\zeta) = \mathcal{U}'(\zeta). \tag{2.2}$$

The following system of ODEs can be obtained from the Eq (2.1) using these assumptions:

$$\begin{cases} \varpi_1' = \varpi_2, \\ \varpi_2' = \mathcal{F}(\varpi_1, \varpi_2). \end{cases} \tag{2.3}$$

If we choose ϖ_1 as a new independent variable, then the system (2.3) is converted into:

$$\frac{d\varpi_2(\varpi_1)}{d\varpi_1} = \frac{\mathcal{F}(\varpi_1, \varpi_2(\varpi_1))}{\varpi_2(\varpi_1)}. \tag{2.4}$$

To solve Eq (2.4), we obtain $\varpi_2(\varpi_1)$. Then, we can return the variable ϖ_1 as the dependent variable and substitute it into the first equation of (2.3) to obtain the final first-order ODE. Solving the resultant ODE yields the final solution of the considered model.

2.2. Generalized projective Riccati equations method

In the study of nonlinear phenomena, it is necessary to obtain solutions of NPDEs of the following form:

$$\mathcal{N}(\Theta, \Theta_t, \Theta_z, \Theta_{tt}, \Theta_{zz}, \Theta_{zt}, \ldots) = 0. \tag{2.5}$$

The expression $\mathcal{N}$ involves a polynomial in $\Theta(z,t)$ as well as its partial derivatives, and includes nonlinear terms and higher order derivatives.

Here we present the basic steps of the GPREs method.

Step 1. By applying the wave transformation:

$$\Theta(z,t) \;\; = \Theta(\varphi), \quad \varphi = \varrho z + \nu t, \tag{2.6}$$

where ϱ and ν are constants, the NPDE (2.5) is transformed into the following ODE:

$$\mathcal{H}(\Theta, \Theta', \Theta'', \ldots) = 0, \tag{2.7}$$

where $\mathcal{H}$ is a polynomial in $\Theta(\varphi)$ and its total derivatives, for example $\Theta'(\varphi) = \frac{d\Theta(\varphi)}{d\varphi}$.

Step 2. We assume that the formal solution of the ODE (2.7) exists as [28–31]

$$\Theta(\varphi) \;\; = \;\; b_0 + \sum_{i=1}^{m} \eta^{i-1}(\varphi)\left(b_{i,\kappa}\kappa(\varphi) + b_{i,\eta}\eta(\varphi)\right), \tag{2.8}$$

with b_0, $b_{i,\kappa}$ and $b_{i,\eta}$, are constants to be determined later. The functions $\eta(\varphi)$ and $\kappa(\varphi)$ satisfy the ODEs

$$\eta'(\varphi) = \lambda\eta(\varphi)\kappa(\varphi),\ \kappa'(\varphi) = \Upsilon - \zeta\eta(\varphi) + \lambda\kappa(\varphi)^2, \tag{2.9}$$

which are related between them by the following first integral

$$\kappa(\varphi)^2 = -\lambda\left(\frac{\left(\zeta^2-1\right)\eta(\varphi)^2}{\Upsilon} + \Upsilon - 2\zeta\eta(\varphi)\right), \tag{2.10}$$

for the general case when Υ and ζ are nonzero constants. If $\Upsilon = \zeta = 0$, (2.7) has the formal solution

$$\Theta(\varphi) = \sum_{i=1}^{m} b_{i,\kappa}\kappa^i(\varphi), \tag{2.11}$$

and $\kappa(\varphi)$ satisfy the ODE

$$\kappa'(\varphi) = \kappa(\varphi)^2. \tag{2.12}$$

Step 3. In (2.8), the value of the positive integer number m must be determined by applying the homogeneous balance between the highest order derivatives and the nonlinear terms in (2.7).

It is possible that, for a given nonlinear equation, the balance procedure leads to a non-positive integer number m. If $m = p/q$, the following transformation will be applied:

$$\Theta(\varphi) = \Phi^{q/p}(\varphi), \tag{2.13}$$

and when the substitution into the Eq (2.13) is performed the Eq (2.7) will be transformed to an ODE for the function $\Phi(\varphi)$ for which the balance procedure results into a positive integer number $\mathcal{M}$, see for example [28].

Step 4. The substitution of the Eq (2.8) or (2.11) along with (2.9), (2.10) or (2.12) into (2.7), allow us to collect all terms of the same power order of $\kappa^i(\varphi)\eta^j(\varphi)$ $(j = 0, 1, ..., i = 0, 1)$ (or $\eta^j(\varphi)$ $(j = 0, 1, ...)$. Then each of this coefficients in front of the functions $\kappa^i(\varphi)\eta^j(\varphi)$ will be set equal to zero.

This procedure generates a set of algebraic equations that can be solved to determine the values of b_0, $b_{i,\kappa}$ and $b_{i,\eta}$, ϱ, ν, ζ and Υ.

Step 5. The Eq (2.9) has the following solutions (see for example [28]).

(i) If $\lambda = -1$, $\Upsilon \neq 0$,

$$\eta_1(\varphi) = \frac{\Upsilon\operatorname{sech}\left(\sqrt{\Upsilon}\varphi\right)}{\zeta\operatorname{sech}\left(\sqrt{\Upsilon}\varphi\right)+1}, \qquad \kappa_1(\varphi) = \frac{\sqrt{\Upsilon}\tanh\left(\sqrt{\Upsilon}\varphi\right)}{\zeta\operatorname{sech}\left(\sqrt{\Upsilon}\varphi\right)+1},$$
$$\eta_2(\varphi) = \frac{\Upsilon\operatorname{csch}\left(\sqrt{\Upsilon}\varphi\right)}{\zeta\operatorname{csch}\left(\sqrt{\Upsilon}\varphi\right)+1}, \qquad \kappa_2(\varphi) = \frac{\sqrt{\Upsilon}\coth\left(\sqrt{\Upsilon}\varphi\right)}{\zeta\operatorname{csch}\left(\sqrt{\Upsilon}\varphi\right)+1}. \tag{2.14}$$

(ii) If $\lambda = 1$, $\Upsilon \neq 0$,

$$\eta_3(\varphi) = \frac{\Upsilon\sec\left(\sqrt{\Upsilon}\varphi\right)}{\zeta\sec\left(\sqrt{\Upsilon}\varphi\right)+1}, \qquad \kappa_3(\varphi) = \frac{\sqrt{\Upsilon}\tan\left(\sqrt{\Upsilon}\varphi\right)}{\zeta\sec\left(\sqrt{\Upsilon}\varphi\right)+1},$$

$$\eta_4(\varphi) = \frac{\Upsilon \csc\left(\sqrt{\Upsilon}\varphi\right)}{\zeta \csc\left(\sqrt{\Upsilon}\varphi\right) + 1}, \qquad \kappa_4(\varphi) = \frac{\sqrt{\Upsilon}\cot\left(\sqrt{\Upsilon}\varphi\right)}{\zeta \csc\left(\sqrt{\Upsilon}\varphi\right) + 1}. \tag{2.15}$$

(iii) If $\zeta = 0$, $\Upsilon = 0$

$$\eta_5(\varphi) = \frac{\Xi}{\varphi}, \ \kappa_5(\varphi) = \frac{1}{\lambda\varphi}, \tag{2.16}$$

where Ξ is constant different from zero.

Step 6. The substitution of the coefficients b_0, $b_{i,\kappa}$ and $b_{i,\eta}$, ϱ, ν, ζ and Υ and the functions (2.14)–(2.16) into (2.8) or (2.11), results into the exact solutions of (2.7).

3. Illustrative examples

We apply the Nucci's reduction method and the GPREs method to three nonlinear dynamical models to investigate their effectiveness: The (1+1)-dimensional classical Boussinesq equations, the generalized reaction Duffing model, and the nonlinear Pochhammer-Chree equation.

3.1. The (1+1)-dimensional classical Boussinesq equations

Wu and Zhang in [32] developed the (1+1)-dimensional classical Boussinesq equations to simulate nonlinear and dispersive long gravity waves traveling in two horizontal directions on shallow water of uniform depth. Different types of exact solutions with various analytical techniques such as bilinear method, modified (G'/G^2)-expansion method and extended (G'/G)-expansion method are investigated for this derived equation in [33–35].

We consider the (1+1)-dimensional classical Boussinesq equations in its following formulation:

$$\begin{aligned} &w(z,t)\,\frac{\partial v(z,t)}{\partial z} + \frac{\partial v(z,t)}{\partial t} + (v(z,t)+1)\,\frac{\partial w(z,t)}{\partial z} = -\frac{1}{3}\frac{\partial^3 w(z,t)}{\partial z^3}, \\ &\frac{\partial v(z,t)}{\partial z} + \frac{\partial w(z,t)}{\partial t} + w(z,t)\,\frac{\partial w(z,t)}{\partial z} = 0. \end{aligned} \tag{3.1}$$

In Eq (3.1), the water wave's elevation is represented by v, and its surface velocity along the z-axis is defined by w. The Euler equation served as the direct inspiration for this system. The run-up of ocean waves, such as tsunami waves, on dykes and dams can be studied using this approach. Designing harbors and coastlines benefits from having a solid understanding of precise solutions. Finding more diverse solutions is therefore crucial for fluid dynamics.

Let's now solve (3.1) using the suggested techniques for the original Boussinesq equations. To do this, we will take into consideration the following traveling wave variable transformation (2.6)

$$v\,(z,t) = v\,(\varphi)\,, \quad w\,(z,t) = w\,(\varphi)\,, \quad \varphi = \varrho\, z + \nu t. \tag{3.2}$$

The above transformation permit us to convert the Eq (3.1) into the following ODEs

$$\varrho\, w(\varphi)v'(\varphi) + \varrho(1+v(\varphi))w'(\varphi) + \frac{1}{3}\varrho^3 w^{(3)}(\varphi) + \nu v'(\varphi) = 0,$$

$$\varrho\, v'(\varphi) + \varrho w(\varphi) w'(\varphi) + \nu\, w'(\varphi) = 0.$$

When the aforementioned system is integrated, we get

$$\begin{aligned} &\varrho(1+v(\varphi))w(\varphi) + \frac{1}{3}\varrho^3 w''(\varphi) + \nu v(\varphi) + K_2 = 0,\\ &\varrho\, v(\varphi) + \frac{1}{2}\varrho\, w(\varphi)^2 + \nu w(\varphi) + K_1 = 0, \end{aligned} \tag{3.3}$$

where K_1 and K_2 are arbitrary integration constants [34] and

$$v(\varphi) = -\frac{\varrho\, w(\varphi)^2 + 2K_1 + 2\nu\, w(\varphi)}{2\,\varrho}. \tag{3.4}$$

Thus, the following ODE is obtained by combining Eqs (3.3) and (3.4):

$$K_2 - \frac{K_1 \nu}{\varrho} w(\varphi)\left(-\frac{\nu^2}{\varrho} + \varrho - K_1\right) - \frac{1}{2}\varrho\, w(\varphi)^2 - \frac{3}{2}\nu w(\varphi)^2 + \frac{1}{3}\varrho^3 w''(\varphi) = 0. \tag{3.5}$$

a) Nucci's reduction method

By using the change of variables (2.2), the system of (2.3) can be written as

$$\begin{cases} \varpi_1' = \varpi_2, \\ \varpi_2' = -\dfrac{3}{\varrho^3}\left(K_2 - \dfrac{K_1\nu}{\varrho}\varpi_1\left(-\dfrac{\nu^2}{\varrho} + \varrho - K_1\right) - \dfrac{1}{2}\varrho\,\varpi_1^2 - \dfrac{3}{2}\nu\varpi_1^2\right). \end{cases} \tag{3.6}$$

If ϖ_1 is chosen as a new independent variable, the system (3.6) transforms into

$$\frac{d\varpi_2(\varpi_1)}{d\varpi_1} = -\frac{3}{\varrho^3 \varpi_2(\varpi_1)}\left(K_2 + \lambda\varpi_1 + \mu\varpi_1^2\right),$$

where $\lambda = -\frac{K_1\nu}{\varrho}\left(-\frac{\nu^2}{\varrho} + \varrho - K_1\right)$, and $\mu = -\frac{1}{2}(\varrho + 3\nu)$. Solving this equation concludes

$$\varpi_2(\varpi_1) = \frac{\sqrt{\varrho\left(R_1\,\varrho^3 - 2\mu\,\varpi_1^3 - 3\,\lambda\,\varpi_1^2 - 6K_2\,\varpi_1\right)}}{\varrho^2}, \tag{3.7}$$

where R_1 is an arbitrary constant, and the first integral that corresponds to it is denoted by

$$(w'(\varphi))^2 + \frac{(2\mu + 6K_2)\,w^3(\varphi) + 3\,\lambda\,w^2(\varphi)}{\varrho} = R_1.$$

Returning the ϖ_1 as dependent variable and substituting the solution (3.7), into the first equation of (3.6), concludes

$$\frac{d\varpi_1}{d\varphi} = \frac{\sqrt{\varrho\left(R_1\,\varrho^3 - 2\mu\,\varpi_1^3 - 3\,\lambda\,\varpi_1^2 - 6K_2\,\varpi_1\right)}}{\varrho^2}.$$

One time integration of this ODE, gives the following implicit solution

$$\varphi - \int \frac{\varrho^2 d\varpi_1}{\sqrt{-\varrho\left(-R_1\,\varrho^3 + 2\,\varpi_1^3\mu + 3\,\lambda\,\varpi_1^2 + 6K_2\,\varpi_1\right)}} + R_2 = 0.$$

Assuming $K2 = 0$, and $R_1 = 0$, concludes the following explicit solution

$$w(\varphi) = \varpi_1(\varphi) = -\frac{3\lambda}{2\mu}\left(\tan^2\left(\frac{\sqrt{3\varrho\,\lambda}\,(R_3+\varphi)}{2\varrho^2}\right)+1\right),$$
$$= -\frac{3K_1\nu\left(-\frac{\nu^2}{\varrho}+\varrho-K_1\right)}{\varrho(\varrho+3\nu)}\left(\tan^2\left(\frac{\sqrt{-3K_1\nu\left(-\frac{\nu^2}{\varrho}+\varrho-K_1\right)}\,(R_3+\varphi)}{2\varrho^2}\right)+1\right).$$

Lastly, from (3.2), we obtain

$$v\,(z,t) = -\frac{9\nu^2K_1^2}{2\varrho^2\,(\varrho+3\,\nu)^2}\left(-\frac{\nu^2}{\varrho}+\varrho-K_1\right)^2\left(\tan^2\left(\frac{\nu\,t+\varrho\,z+R_3}{2\varrho^2}\sqrt{-3K_1\nu\left(-\frac{\nu^2}{\varrho}+\varrho-K_1\right)}\right)+1\right)^2$$
$$+\frac{36K_1\nu^2}{\varrho^2\,(\varrho+3\nu)}\left(-\frac{\nu^2}{\varrho}+\varrho-K_1\right)\left(\tan^2\left(\frac{\nu\,t+\varrho\,z+R_3}{2\varrho^2}\sqrt{-3K_1\nu\left(-\frac{\nu^2}{\varrho}+\varrho-K_1\right)}\right)+1\right)+2\,K_1,$$
$$w\,(z,t) = -\frac{3K_1\nu\left(-\frac{\nu^2}{\varrho}+\varrho-K_1\right)}{\varrho(\varrho+3\nu)}\left(\tan^2\left(\frac{\sqrt{-3K_1\nu\left(-\frac{\nu^2}{\varrho}+\varrho-K_1\right)}\,(R_3+\varrho\,z+\nu t)}{2\varrho^2}\right)+1\right).$$

b) Generalized projective Riccati equations method

Using the algorithm for the GPREs method described in Section 3, by performing the balance between the terms $w''(\varphi)$ and, $w^3(\varphi)$ for the formal solution

$$w\,(\varphi) \;\;=\;\; b_0 + \sum_{i=1}^{m}\eta^{i-1}(\varphi)\,(b_{i,0}\kappa(\varphi)+b_{0,i}\eta(\varphi))\,,$$

yields $m = 1$. Therefore, we have

$$w\,(\varphi) = b_0 + b_{0,1}\eta(\varphi) + b_{1,0}\kappa(\varphi). \tag{3.8}$$

Substituting (3.8) into (3.5) and considering the projective Riccati equations (2.9) the left-hand side of (3.5) becomes a polynomial in terms of $\eta(\varphi)$ and $\kappa(\varphi)$.

$$\begin{aligned}
&-\frac{1}{6\varrho}\Big(\varrho\,(3\,(b_{0,1}\eta(\varphi)+b_{1,0}\kappa(\varphi))\,(3\nu\,(b_{0,1}\eta(\varphi)+b_{1,0}\kappa(\varphi))+2K_1)-6K_2)\\
&+3b_0\left(\varrho\left(\varrho\left(3\,(b_{0,1}\eta(\varphi)+b_{1,0}\kappa(\varphi))^2-2\right)+6\nu\,(b_{0,1}\eta(\varphi)+b_{1,0}\kappa(\varphi))+2K_1\right)+2\nu^2\right)\\
&+3\varrho^2\,(b_{0,1}\eta(\varphi)+b_{1,0}\kappa(\varphi))\left((b_{0,1}\eta(\varphi)+b_{1,0}\kappa(\varphi))^2-2\right)\\
&-2\varrho^4\lambda\left(b_{1,0}\kappa(\varphi)\left(-3\zeta\eta(\varphi)+2\Upsilon+2\lambda\kappa(\varphi)^2\right)+b_{0,1}\eta(\varphi)\left(-\zeta\eta(\varphi)+\Upsilon+2\lambda\kappa(\varphi)^2\right)\right)\\
&+9b_0^2\varrho\,(\varrho\,(b_{0,1}\eta(\varphi)+b_{1,0}\kappa(\varphi))+\nu)+6\nu\,(\nu\,(b_{0,1}\eta(\varphi)+b_{1,0}\kappa(\varphi))+K_1)+3b_0^3\varrho^2\Big)=0.
\end{aligned} \tag{3.9}$$

By taking into account the first integral of the projective Riccati method

$$\kappa(\varphi)^2 = -\lambda\left(\frac{\left(\zeta^2-1\right)\eta(\varphi)^2}{\Upsilon}+\Upsilon-2\zeta\eta(\varphi)\right),$$

the algebraic equation (3.9) of the different powers of in $\eta(\varphi)$ and $\kappa(\varphi)$ can be converted into a simplified algebraic equation in terms only of the independent functions $\eta(\varphi)^j$ and $\kappa(\varphi)\eta(\varphi)^\varrho$

$$\begin{aligned}
&\frac{3}{2}b_0\varrho\Upsilon\lambda b_{1,0}^2+\frac{3}{2}\Upsilon\nu\lambda b_{1,0}^2-\frac{b_0\nu^2}{\varrho}-\frac{1}{2}b_0^3\varrho+b_0\varrho-b_0K_1-\frac{3}{2}b_0^2\nu-\frac{K_1\nu}{\varrho}+K_2\\
&+\eta(\varphi)\Big(-\frac{2}{3}\varrho^3\Upsilon\lambda^3b_{0,1}+\frac{1}{3}\varrho^3\Upsilon\lambda b_{0,1}+\frac{3}{2}\varrho\Upsilon\lambda b_{0,1}b_{1,0}^2-\frac{\nu^2b_{0,1}}{\varrho}-3b_0\varrho\zeta\lambda b_{1,0}^2-\frac{3}{2}b_0^2\varrho\, b_{0,1}+\varrho b_{0,1}-K_1b_{0,1}\\
&-3\zeta\nu\lambda b_{1,0}^2-3b_0\nu b_{0,1}\Big)+\eta(\varphi)^2\Big(\frac{3b_0\varrho\zeta^2\lambda b_{1,0}^2}{2\Upsilon}-\frac{3b_0\varrho\lambda b_{1,0}^2}{2\Upsilon}+\frac{4}{3}\varrho^3\zeta\lambda^3b_{0,1}-\frac{1}{3}\varrho^3\zeta\lambda b_{0,1}-3\varrho\zeta\lambda b_{0,1}b_{1,0}^2\\
&-\frac{3}{2}b_0\varrho\, b_{0,1}^2+\frac{3\zeta^2\nu\lambda b_{1,0}^2}{2\Upsilon}-\frac{3\nu\lambda b_{1,0}^2}{2\Upsilon}-\frac{3}{2}\nu b_{0,1}^2\Big)\\
&+\eta(\varphi)^3\left(-\frac{2\varrho^3\zeta^2\lambda^3b_{0,1}}{3\Upsilon}+\frac{2\varrho^3\lambda^3b_{0,1}}{3\Upsilon}+\frac{3\varrho\zeta^2\lambda b_{0,1}b_{1,0}^2}{2\Upsilon}-\frac{3\varrho\lambda b_{0,1}b_{1,0}^2}{2\Upsilon}-\frac{1}{2}\varrho\, b_{0,1}^3\right)\\
&+\kappa(\varphi)\Bigg(-\frac{2}{3}\varrho^3\Upsilon\lambda^3b_{1,0}+\frac{2}{3}\varrho^3\Upsilon\lambda b_{1,0}+\frac{1}{2}\varrho\Upsilon\lambda b_{1,0}^3-\frac{\nu^2b_{1,0}}{\varrho}-\frac{3}{2}b_0^2\varrho\, b_{1,0}+\varrho b_{1,0}-K_1b_{1,0}-3b_0\nu b_{1,0}\\
&+\eta(\varphi)\left(\frac{4}{3}\varrho^3\zeta\lambda^3b_{1,0}-\varrho^3\zeta\lambda b_{1,0}-\varrho\zeta\lambda b_{1,0}^3-3b_0\varrho\, b_{0,1}b_{1,0}-3\nu b_{0,1}b_{1,0}\right)\\
&+\eta(\varphi)^2\left(-\frac{2\varrho^3\zeta^2\lambda^3b_{1,0}}{3\Upsilon}+\frac{2\varrho^3\lambda^3b_{1,0}}{3\Upsilon}+\frac{\varrho\zeta^2\lambda b_{1,0}^3}{2\Upsilon}-\frac{\varrho\lambda b_{1,0}^3}{2\Upsilon}-\frac{3}{2}\varrho\, b_{0,1}^2b_{1,0}\right)\Bigg)=0.
\end{aligned}$$

Setting each coefficient of this polynomial's independent functions $\eta(\varphi)^j$ and $\kappa(\varphi)\eta(\varphi)^\varrho$ to zero, results in a system of algebraic equations.

$$\begin{aligned}
&\frac{3}{2}b_0\varrho\Upsilon\lambda b_{1,0}^2+\frac{3}{2}\Upsilon\nu\lambda b_{1,0}^2-\frac{b_0\nu^2}{\varrho}-\frac{1}{2}b_0^3\varrho+b_0\varrho-b_0K_1-\frac{3}{2}b_0^2\nu-\frac{K_1\nu}{\varrho}+K_2=0,\\
&-\frac{2}{3}\varrho^3\Upsilon\lambda^3b_{0,1}+\frac{1}{3}\varrho^3\Upsilon\lambda b_{0,1}+\frac{3}{2}\varrho\Upsilon\lambda b_{0,1}b_{1,0}^2-\frac{\nu^2b_{0,1}}{\varrho}-3b_0\varrho\zeta\lambda b_{1,0}^2-\frac{3}{2}b_0^2\varrho\, b_{0,1}+\varrho\, b_{0,1}\\
&-K_1b_{0,1}-3\zeta\nu\lambda b_{1,0}^2-3b_0\nu b_{0,1}=0,\\
&\frac{3b_0\varrho\zeta^2\lambda b_{1,0}^2}{2\Upsilon}-\frac{3b_0\varrho\lambda b_{1,0}^2}{2\Upsilon}+\frac{4}{3}\varrho^3\zeta\lambda^3b_{0,1}-\frac{1}{3}\varrho^3\zeta\lambda b_{0,1}-3\varrho\zeta\lambda b_{0,1}b_{1,0}^2-\frac{3}{2}b_0\varrho b_{0,1}^2\frac{3\zeta^2\nu\lambda b_{1,0}^2}{2\Upsilon}\\
&-\frac{3\nu\lambda b_{1,0}^2}{2\Upsilon}-\frac{3}{2}\nu b_{0,1}^2=0,\\
&-\frac{2\varrho^3\zeta^2\lambda^3b_{0,1}}{3\Upsilon}+\frac{2\varrho^3\lambda^3b_{0,1}}{3\Upsilon}+\frac{3\varrho\zeta^2\lambda b_{0,1}b_{1,0}^2}{2\Upsilon}-\frac{3\varrho\lambda b_{0,1}b_{1,0}^2}{2\Upsilon}-\frac{1}{2}\varrho\, b_{0,1}^3=0,\\
&-\frac{2}{3}\varrho^3\Upsilon\lambda^3b_{1,0}+\frac{2}{3}\varrho^3\Upsilon\lambda b_{1,0}+\frac{1}{2}\varrho\Upsilon\lambda b_{1,0}^3-\frac{\nu^2b_{1,0}}{\varrho}-\frac{3}{2}b_0^2\varrho\, b_{1,0}+\varrho\, b_{1,0}-K_1b_{1,0}-3b_0\nu b_{1,0}=0,\\
&\left(\frac{4}{3}\varrho^3\zeta\lambda^3b_{1,0}-\varrho^3\zeta\lambda b_{1,0}-\varrho\zeta\lambda b_{1,0}^3-3b_0\varrho\, b_{0,1}b_{1,0}-3\nu b_{0,1}b_{1,0}\right)=0,\\
&-\frac{2\varrho^3\zeta^2\lambda^3b_{1,0}}{3\Upsilon}+\frac{2\varrho^3\lambda^3b_{1,0}}{3\Upsilon}+\frac{\varrho\zeta^2\lambda b_{1,0}^3}{2\Upsilon}-\frac{\varrho\lambda b_{1,0}^3}{2\Upsilon}-\frac{3}{2}\varrho\, b_{0,1}^2b_{1,0}=0.
\end{aligned}$$

The solution of the above system of equations is given by

$$b_{1,0}=\frac{\sqrt{\varrho^2(\zeta^2-1)(-\lambda^3)}}{\sqrt{3}\sqrt{\lambda-\zeta^2\lambda}},\ \ b_{1,0}=\frac{\sqrt{\varrho^2(\zeta^2-1)(-\lambda^3)}}{\sqrt{3}\sqrt{\lambda-\zeta^2\lambda}},\ \ b_0=\frac{\varrho\zeta\sqrt{\Upsilon}\left(\lambda^2-1\right)}{\sqrt{3}\sqrt{1-\zeta^2}\sqrt{\lambda}}-\frac{\nu}{\varrho},$$
$$\nu=\frac{\sqrt{\varrho}\sqrt{(\zeta^2-1)(\varrho^3\Upsilon+6\varrho\lambda-6K_1\lambda)}}{\sqrt{3}\sqrt{1-\zeta^2}\sqrt{\lambda}},\ \ K_1=\frac{K_2^2}{2\varrho}+\frac{\varrho^3\Upsilon\left(3\zeta^2-2\left(\zeta^2+2\right)\lambda^2+3\lambda^4\right)}{6(\zeta^2-1)\lambda}+\varrho,$$

for the set of parameters $|\zeta|>1,\ \varrho<0,\ K_2>0,\ \lambda=-1,\ \Upsilon>0$ and $|\zeta|<1,\ \varrho<0,\ K_2>0,\ \lambda=1,\ \Upsilon>0$.

For these set of coefficients solutions for the functions $w(\varphi)$ and $v(\varphi)$ are given as

$$w(\varphi)=-\frac{K_2}{\varrho}-\frac{\varrho\left(\Upsilon\lambda^2\kappa(\varphi)+\eta(\varphi)\sqrt{-\Upsilon\lambda(\zeta^2-1)}\right)}{\sqrt{3}\Upsilon\lambda},$$

$$v(\varphi)=-1-\frac{\varrho^2\eta(\varphi)\left(\zeta\Upsilon+\kappa(\varphi)\sqrt{-\Upsilon\lambda(\zeta^2-1)}+-\eta(\varphi)(1-\zeta^2)\right)}{3\Upsilon\lambda}, \tag{3.10}$$

when $|\zeta|>1,\ \varrho<0,\ K_2>0,\ \lambda=-1,\ \Upsilon>0$.

Furthermore, the functions $w(\varphi)$ and $v(\varphi)$ are given as

$$w(\varphi)=\frac{K_2}{\varrho}+\frac{\varrho\left(\sqrt{\Upsilon}\kappa(\varphi)+\eta(\varphi)\sqrt{\lambda(1-\zeta^2}\right)}{\sqrt{3}\sqrt{\Upsilon}\lambda},$$

$$v(\varphi)=-1+\frac{\varrho^2\eta(\varphi)\left(-\zeta\Upsilon+\sqrt{\Upsilon}\lambda\kappa(\varphi)\sqrt{\lambda(1-\zeta^2}+\left(\zeta^2-1\right)\eta(\varphi)\right)}{3\Upsilon\lambda}, \tag{3.11}$$

when $|\zeta|<1,\ \varrho<0,\ K_2>0,\ \lambda=1,\ \Upsilon>0$.

If we consider the solutions (2.14), when $\lambda=-1$,

$$\eta_1(\varphi)=\frac{\Upsilon\mathrm{sech}\left(\sqrt{\Upsilon}\varphi\right)}{\zeta\mathrm{sech}\left(\sqrt{\Upsilon}\varphi\right)+1},\ \ \kappa_1(\varphi)=\frac{\sqrt{\Upsilon}\tanh\left(\sqrt{\Upsilon}\varphi\right)}{\zeta\mathrm{sech}\left(\sqrt{\Upsilon}\varphi\right)+1},$$

solutions (3.10) take the following expressions

$$w_1(\varphi)=\frac{\varrho\left(\frac{\Upsilon^{3/2}\tanh(\varphi\sqrt{\Upsilon})}{\zeta\mathrm{sech}(\varphi\sqrt{\Upsilon})+1}+\frac{\Upsilon\sqrt{(\zeta^2-1)\Upsilon}\mathrm{sech}(\varphi\sqrt{\Upsilon})}{\zeta\mathrm{sech}(\varphi\sqrt{\Upsilon})+1}\right)}{\sqrt{3}\Upsilon}-\frac{K_2}{\varrho},$$

$$v_1(\varphi)=\frac{\varrho^2\mathrm{sech}\left(\varphi\sqrt{\Upsilon}\right)\left(-\frac{\zeta^2\Upsilon\mathrm{sech}(\varphi\sqrt{\Upsilon})}{\zeta\mathrm{sech}(\varphi\sqrt{\Upsilon})+1}+\frac{\sqrt{\zeta^2-1}\Upsilon\tanh(\varphi\sqrt{\Upsilon})}{\zeta\mathrm{sech}(\varphi\sqrt{\Upsilon})+1}+\frac{\Upsilon\mathrm{sech}(\varphi\sqrt{\Upsilon})}{\zeta\mathrm{sech}(\varphi\sqrt{\Upsilon})+1}+\zeta\Upsilon\right)}{3\left(\zeta\mathrm{sech}\left(\varphi\sqrt{\Upsilon}\right)+1\right)}-1. \tag{3.12}$$

By considering the second pair of solutions of the Eq (2.14), when $\lambda = 1$,

$$\eta_2(\varphi) = \frac{\Upsilon \text{csch}\left(\sqrt{\Upsilon}\,\varphi\right)}{\zeta \text{csch}\left(\sqrt{\Upsilon}\,\varphi\right) + 1}, \quad \kappa_2(\varphi) = \frac{\sqrt{\Upsilon} \coth\left(\sqrt{\Upsilon}\,\varphi\right)}{\zeta \text{csch}\left(\sqrt{\Upsilon}\,\varphi\right) + 1},$$

solutions (3.10) take the following expressions

$$w_2(\varphi) = \frac{\varrho\left(\frac{\Upsilon^{3/2}\coth\left(\varphi\sqrt{\Upsilon}\right)}{\zeta \text{csch}\left(\varphi\sqrt{\Upsilon}\right)+1} + \frac{\Upsilon\sqrt{(\zeta^2-1)\Upsilon}\text{csch}\left(\varphi\sqrt{\Upsilon}\right)}{\zeta \text{csch}\left(\varphi\sqrt{\Upsilon}\right)+1}\right)}{\sqrt{3}\Upsilon} - \frac{K_2}{\varrho},$$

$$v_2(\varphi) = \frac{\varrho^2 \text{csch}\left(\varphi\sqrt{\Upsilon}\right)\left(-\frac{\zeta^2\Upsilon\text{csch}\left(\varphi\sqrt{\Upsilon}\right)}{\zeta \text{csch}\left(\varphi\sqrt{\Upsilon}\right)+1} + \frac{\sqrt{\zeta^2-1}\Upsilon\coth\left(\varphi\sqrt{\Upsilon}\right)}{\zeta \text{csch}\left(\varphi\sqrt{\Upsilon}\right)+1} + \frac{\Upsilon\text{csch}\left(\varphi\sqrt{\Upsilon}\right)}{\zeta \text{csch}\left(\varphi\sqrt{\Upsilon}\right)+1} + \zeta\Upsilon\right)}{3\left(\zeta \text{csch}\left(\varphi\sqrt{\Upsilon}\right)+1\right)} - 1. \tag{3.13}$$

Solutions $w_1(z,t)$ and $v_1(z,t)$ are solutions of the classical Boussinesq equations with a well defined behavior of a dark soliton type for $w_1(z,t)$ and one bright soliton solution for $v_1(z,t)$. Figure 1 depicts the shape of the solitary solution $w_1(z,t)$ of the classical Boussinesq equation, with $\Upsilon = 0.995$, $\zeta = 2.098$, $K_2 = 0.45$, $\varrho = 0.37$, $\lambda = -1$, in a 3-D plot. The 2-D plot for different time values is shown in Figure 1a, and the corresponding contour plot is shown in Figure 1c. Figure 2 depicts the shape of the bright soliton solution $v_1(z,t)$ for the classical Boussinesq equation, with $\Upsilon = 0.995$, $\zeta = 2.098$, $K_2 = 0.45$, $\varrho = 0.37$, $\lambda = -1$, in a 3-D plot, Figure 2a, the contour plot is shown in Figure 2c and the 2-D plot for different time values is shown in Figure 2b.

Solutions $w_2(z,t)$ and $v_2(z,t)$ are solutions of the classical Boussinesq equations with a well defined behavior of a singular soliton type for $w_2(z,t)$, where this soliton solution descent to $-\infty$ from one asymptotical state at $\varphi \to -\infty$ and then rises to ∞ and finally descending to another asymptotical state at $\varphi \to \infty$. For the soliton solution $v_2(z,t)$, we observe a different behavior associated with one singular soliton solution showing a symmetrical behavior where this soliton descent to $-\infty$ from one asymptotical state at $\varphi \to -\infty$ and then rises to the same original asymptotical state at $\varphi \to \infty$. Figure 3 illustrates the shape of the solitary solution $w_2(z,t)$ of the classical Boussinesq equation, with $\Upsilon = 0.995$, $\zeta = 2.098$, $K_2 = 0.45$, $\varrho = 0.37$, $\lambda = -1$, in a 3-D plot in Figure 3a, Figure 3b illustrates a 2-D plot for different time values and the contour plot is exhibited in Figure 3c. Figure 4 illustrates the shape of the singular soliton $v_2(z,t)$ of the classical Boussinesq equation, with $\Upsilon = 0.995$, $\zeta = 2.098$, $K_2 = 0.45$, $\varrho = 0.37$, $\lambda = -1$, Figure 4a, shows a 3-D plot, Figure 4b shows a 2-D plot for different time values, and Figure 4c shows a contour plot.

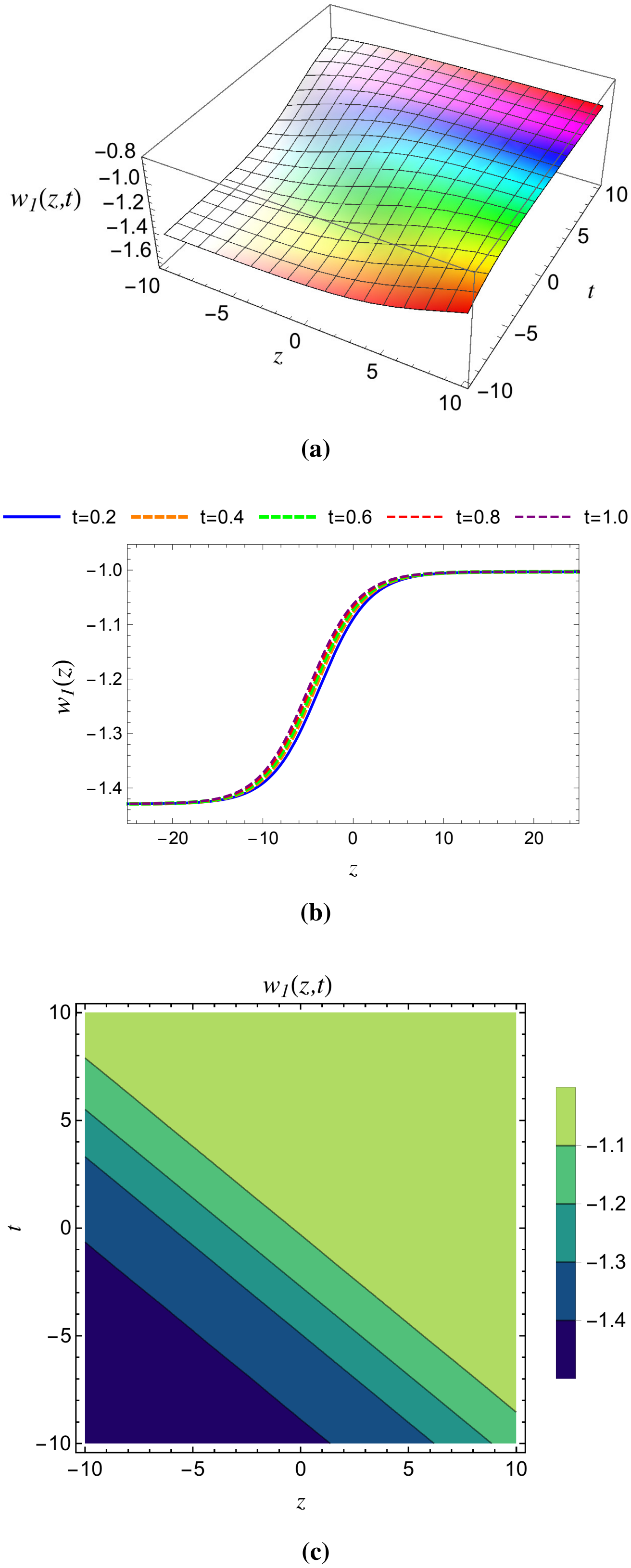
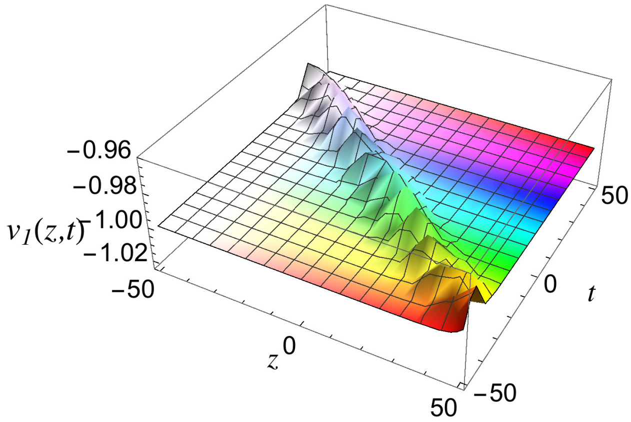
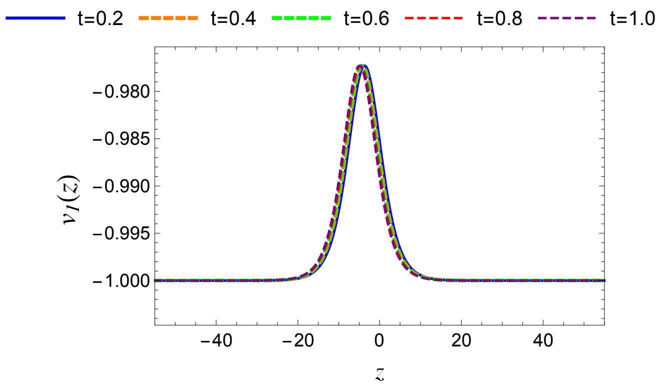


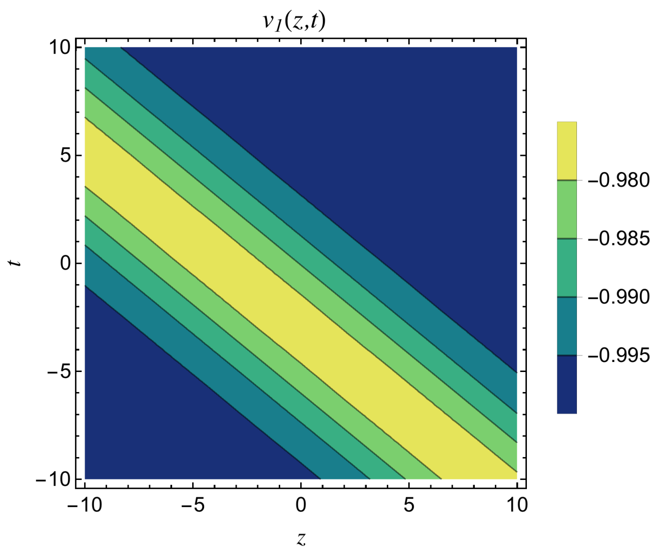
Figure 1. Various representations of $w_1(z,t)$ in (3.12) with $\Upsilon = 0.995$, $\zeta = 2.098$, $K_2 = 0.45$, $\varrho = 0.37$, $\lambda = -1$.



(a)

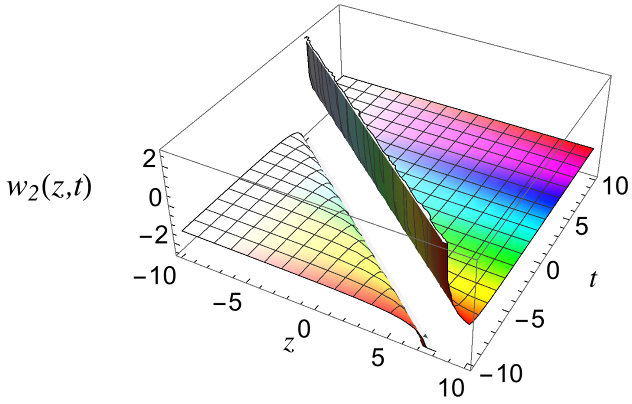


(b)

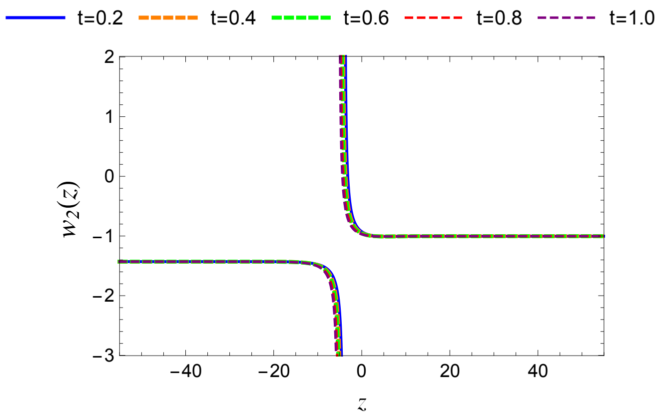


(c)

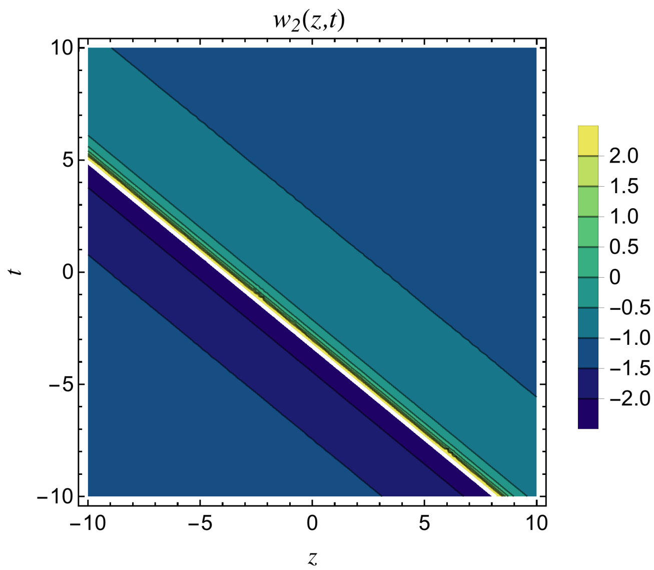
Figure 2. Various representations of $v_1(z,t)$ in (3.12) with $\Upsilon = 0.995$, $\zeta = 2.098$, $K_2 = 0.45$, $\varrho = 0.37$, $\lambda = -1$.



(a)

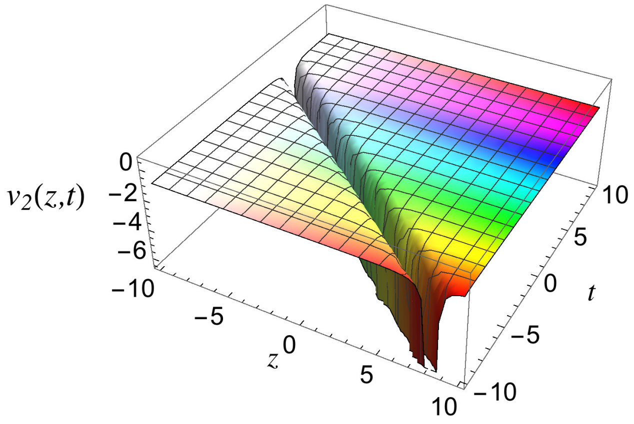


(b)

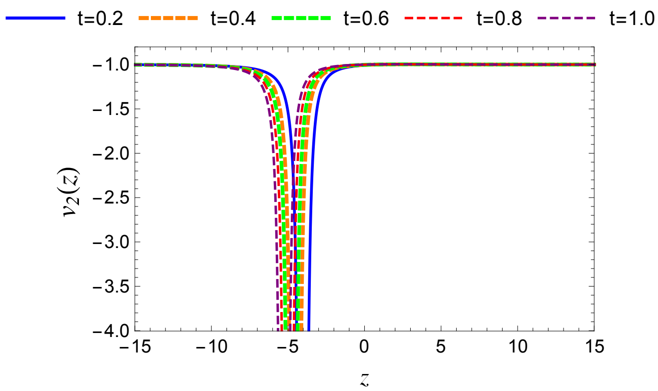


(c)

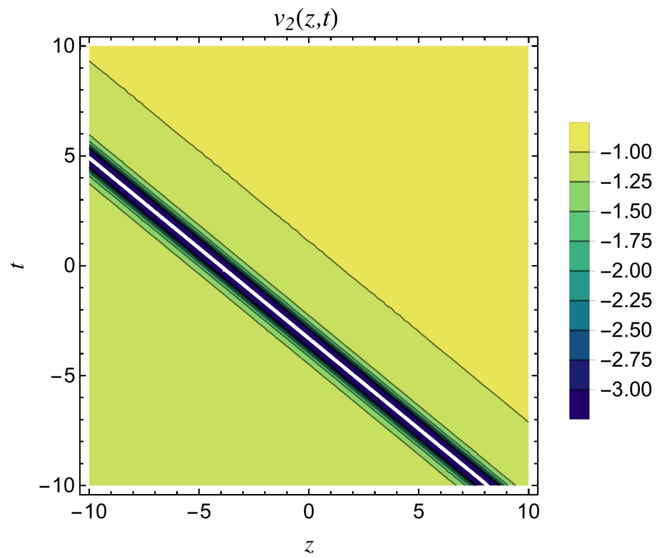
Figure 3. Various representations of $w_2(z,t)$ in (3.12) with $\Upsilon = 0.995$, $\zeta = 2.098$, $K_2 = 0.45$, $\varrho = 0.37$, $\lambda = -1$.



(a)



(b)



(c)

Figure 4. Various representations of $v_2(z,t)$ in (3.12) with $\Upsilon = 0.995$, $\zeta = 2.098$, $K_2 = 0.45$, $\varrho = 0.37$, $\lambda = -1$.

Next if we consider the solutions (2.15), when $\lambda = 1$:

$$\eta_3(\varphi) = \frac{\Upsilon \sec\left(\sqrt{\Upsilon}\varphi\right)}{\zeta \sec\left(\sqrt{\Upsilon}\varphi\right) + 1}, \quad \kappa_3(\varphi) = \frac{\sqrt{\Upsilon}\tan\left(\sqrt{\Upsilon}\varphi\right)}{\zeta \sec\left(\sqrt{\Upsilon}\varphi\right) + 1},$$

solutions (3.11) take the following expressions

$$w_3(\varphi) = \frac{K_2}{\varrho} + \frac{\varrho\left(\frac{\sqrt{1-\zeta^2}\Upsilon \sec(\sqrt{\Upsilon}\varphi)}{\zeta \sec(\sqrt{\Upsilon}\varphi)+1} + \frac{\Upsilon \tan(\sqrt{\Upsilon}\varphi)}{\zeta \sec(\sqrt{\Upsilon}\varphi)+1}\right)}{\sqrt{3}\sqrt{\Upsilon}}$$

$$v_3(\varphi) = -\frac{\varrho^2 \sec\left(\sqrt{\Upsilon}z\right)\left(\zeta\Upsilon - \frac{\zeta^2\Upsilon \sec(\sqrt{\Upsilon}\varphi)}{\zeta \sec(\sqrt{\Upsilon}\varphi)+1} + \frac{\sqrt{1-\zeta^2}\Upsilon \tan(\sqrt{\Upsilon}\varphi)}{\zeta \sec(\sqrt{\Upsilon}\varphi)+1} + \frac{\Upsilon \sec(\sqrt{\Upsilon}\varphi)}{\zeta \sec(\sqrt{\Upsilon}\varphi)+1}\right)}{3\left(\zeta \sec\left(\sqrt{\Upsilon}\varphi\right) + 1\right)} - 1.$$

By considering the second pair of solutions of the Eq (2.15) , when $\lambda = 1$:

$$\eta_4(\varphi) = \frac{\Upsilon \csc\left(\sqrt{\Upsilon}\varphi\right)}{\zeta \csc\left(\sqrt{\Upsilon}\varphi\right) + 1}, \quad \kappa_4(\varphi) = \frac{\sqrt{\Upsilon}\cot\left(\sqrt{\Upsilon}\varphi\right)}{\zeta \csc\left(\sqrt{\Upsilon}\varphi\right) + 1},$$

solutions (3.11) take the following expressions

$$w_4(\varphi) = \frac{K_2}{\varrho} + \frac{\varrho\left(\frac{\sqrt{1-\zeta^2}\Upsilon \csc(\sqrt{\Upsilon}\varphi)}{\zeta \csc(\sqrt{\Upsilon}\varphi)+1} + \frac{\Upsilon \cot(\sqrt{\Upsilon}\varphi)}{\zeta \csc(\sqrt{\Upsilon}\varphi)+1}\right)}{\sqrt{3}\sqrt{\Upsilon}},$$

$$v_4(\varphi) = \frac{\varrho^2 \csc\left(\sqrt{\Upsilon}\varphi\right)\left(\zeta(-\Upsilon) + \frac{(\zeta^2-1)\Upsilon \csc(\sqrt{\Upsilon}\varphi)}{\zeta \csc(\sqrt{\Upsilon}\varphi)+1} + \frac{\sqrt{1-\zeta^2}\Upsilon \cot(\sqrt{\Upsilon}\varphi)}{\zeta \csc(\sqrt{\Upsilon}\varphi)+1}\right)}{3\left(\zeta \csc\left(\sqrt{\Upsilon}\varphi\right) + 1\right)} - 1.$$

Figures 5 and 6 display the singular periodic solutions $w_3(z,t)$ and $v_3(z,t)$, respectively. The singular periodic solution $w_3(z,t)$ exhibits a well-defined $\tan(\varphi)$ pattern and is a solution of the classical Boussinesq equations. On the other hand, the singular periodic solution $v_3(z,t)$ shows a symmetrical behavior of the $\sec(\varphi)$, where the solution descends to $-\infty$ and then rises to its original value. This behavior repeats an infinite number of times.

Figures 7 and 8 show the periodical singular solution of $w_4(z,t)$ and $v_4(z,t)$, respectively. The solution $w_4(z,t)$ is a singular periodical solution of the classical Boussinesq equations with a well defined $\cot(\varphi)$ pattern. For the singular periodic solution $v_4(z,t)$, we observe a typical behavior of the $\csc(\varphi)$ with a symmetrical behavior where this solution descent to $-\infty$ and then rises to its original state with and infinity number of repetitions of this type of singularities.

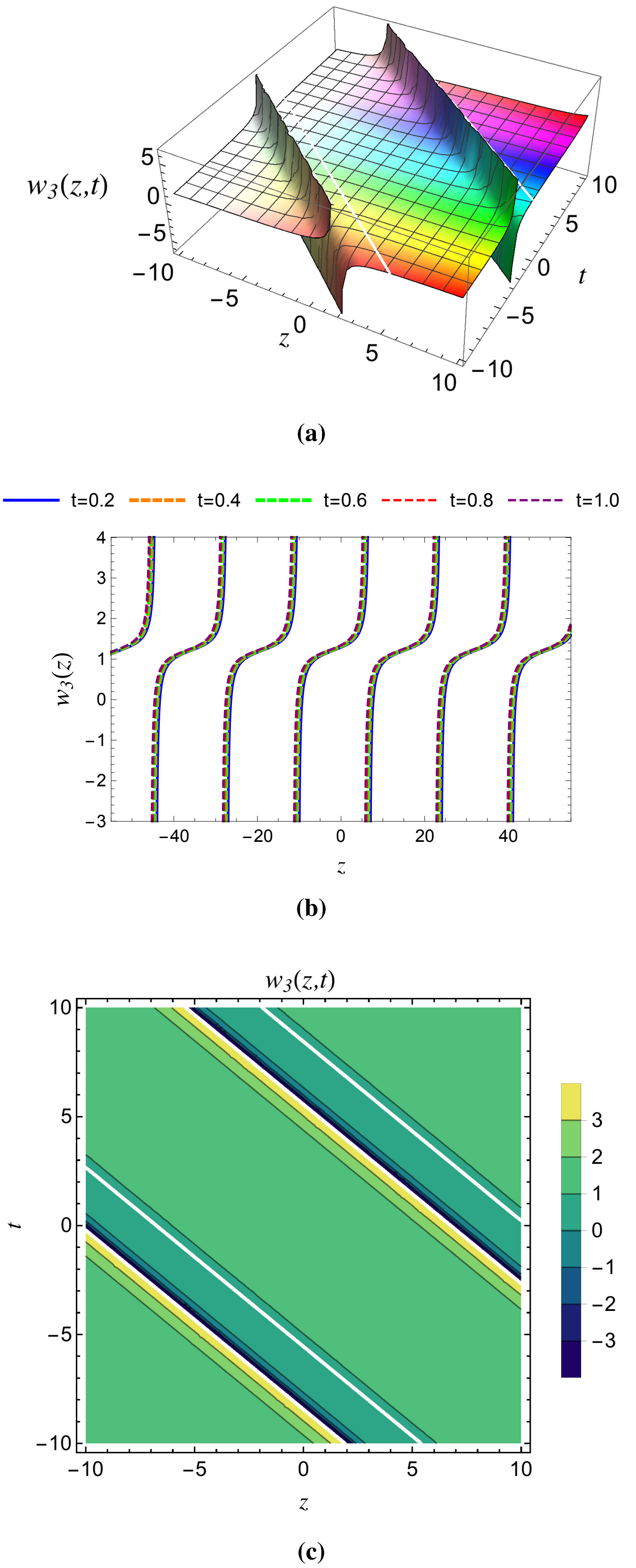
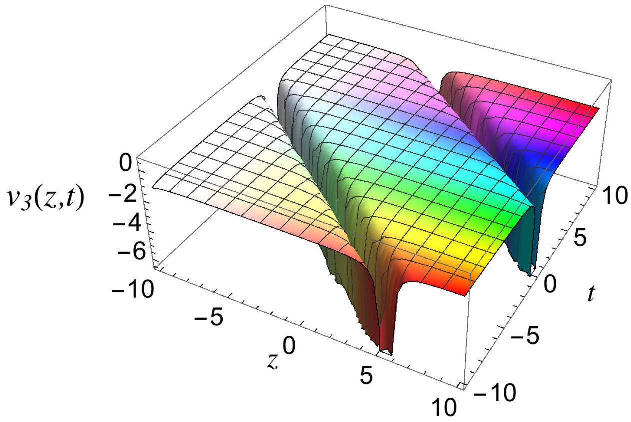
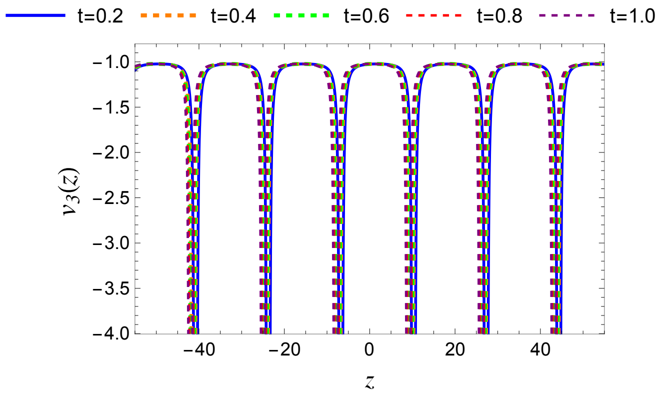


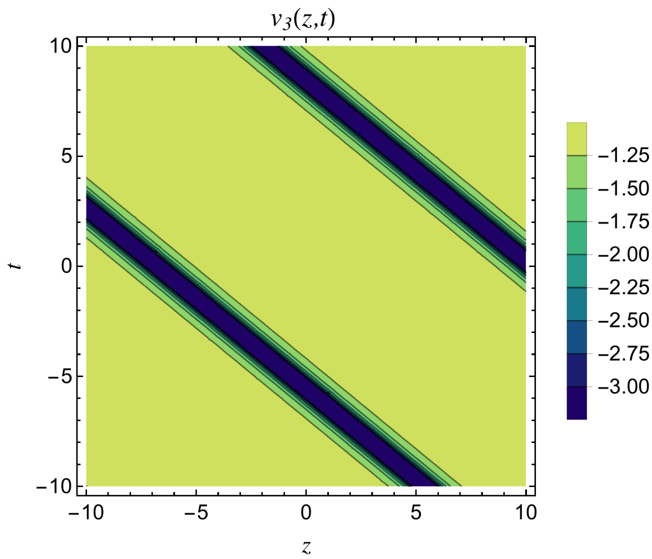
Figure 5. Various representations of $w_3(z,t)$ in (3.12) with $\Upsilon = 0.995$, $\zeta = 0.798$, $K_2 = 0.45$, $\varrho = 0.37$, $\lambda = 1$.



(a)

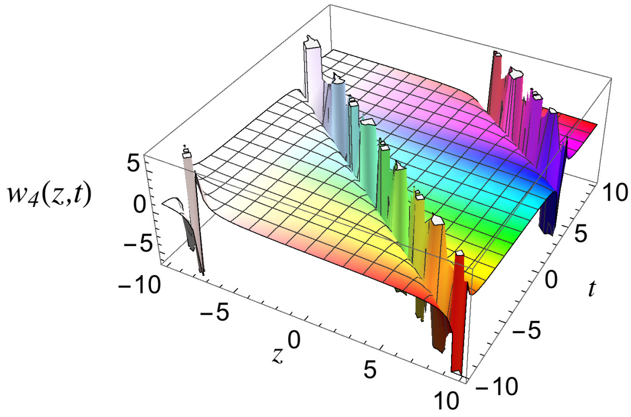


(b)

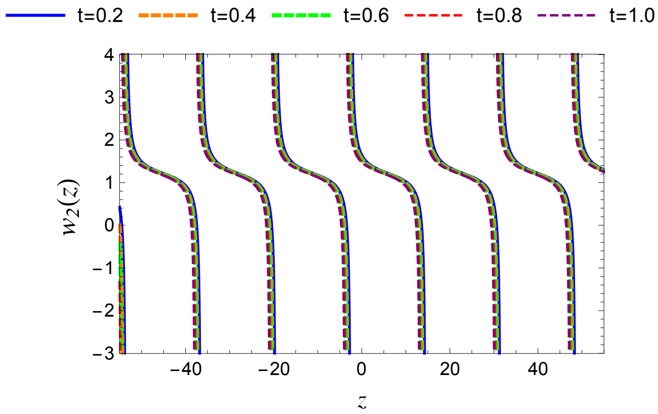


(c)

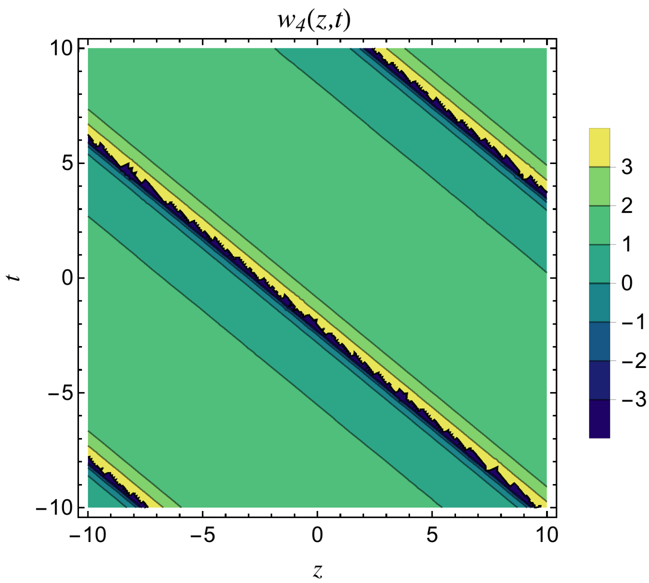
Figure 6. Various representations of $v_3(z,t)$ in (3.12) with $\Upsilon = 0.995$, $\zeta = 0.798$, $K_2 = 0.45$, $\varrho = 0.37$, $\lambda = 1$.



(a)



(b)



(c)

Figure 7. Various representations of $w_4(z,t)$ in (3.12) with $\Upsilon = 0.995$, $\zeta = 0.798$, $K_2 = 0.45$, $\varrho = 0.37$, $\lambda = 1$.

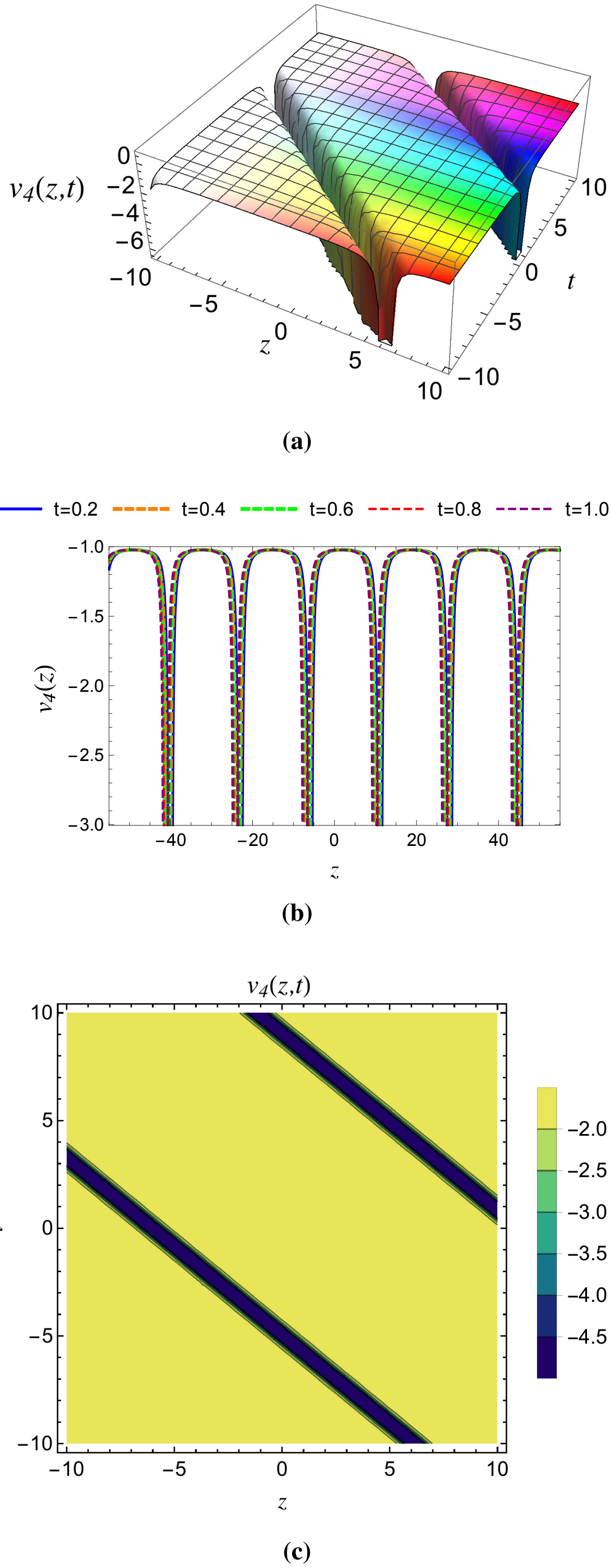


Figure 8. Various representations of $v_4(z,t)$ in (3.12) obtained by the parameters: $\Upsilon = 0.995$, $\zeta = 0.798$, $K_2 = 0.45$, $\varrho = 0.37$, $\lambda = 1$.

3.2. The generalized reaction Duffing model

We have applied the GPREs method to obtain the analytical solutions of the generalized reaction Duffing model [36,37] in the form:

$$\frac{\partial^2 v(z,t)}{\partial t^2} + \alpha \frac{\partial^2 v(z,t)}{\partial z^2} + \beta v(z,t) + \gamma v^3(z,t) = 0, \tag{3.14}$$

where α, β, and γ are constants.

The importance of generalized reaction Duffing model is highlighted by the fact that if certain values are chosen for its coefficients, it produces well known models [38–41].

- For $\alpha = -1, \beta > 0, \gamma > 0$, it produces the Landau-Ginzburg-Higgs equation, which is used to study both low-temperature and high-temperature superconductors.
- For $\alpha = -1, \beta < 0, \gamma < 0$, it is the Klein-Gordon equation, which arises in the applications of relativistic condensed matter physics.
- For $\alpha = -1, \beta = 1, \gamma = -1$, we get the ϕ^4 equation and appears in the applications of quantum mechanics.
- For $\alpha = 1, \beta = 1, \gamma = -1/6$, it results in the sine-Gordon equation, which appears in relativistic field theory and mechanical transmission lines.
- For $\alpha = 0$, it turns into the classical Duffing equation, which is used to detect weak signal of early machinery fault.

If we proceed in a similar way as in the previous example, by considering the wave transformation

$$v(z,t) = v(\varphi), \quad \varphi = \varrho z + \nu t,$$

then Eq (3.14) is reduced to the following ordinary differential equation:

$$\alpha\varrho^2 v''(\varphi) + \nu^2 v''(\varphi) + \beta v(\varphi) + \gamma v(\varphi)^3 = 0. \tag{3.15}$$

a) Nucci's reduction method

By using the change of variables (2.2), the system of (2.3) for the Eq (3.15) can be written as

$$\begin{cases} \varpi_1' = \varpi_2, \\ \varpi_2' = -\dfrac{\beta\varpi_1 + \gamma\varpi_1^3}{\alpha\varrho^2 + \nu^2}. \end{cases} \tag{3.16}$$

If we choice ϖ_1 as a new independent variable, then the system (3.16) converts into

$$\frac{d\varpi_2(\varpi_1)}{d\varpi_1} = -\frac{\beta\varpi_1 + \gamma\varpi_1^3}{(\alpha\varrho^2 + \nu^2)\varpi_2(\varpi_1)}.$$

Solving this equation concludes

$$\varpi_2(\varpi_1) = \sqrt{\frac{-\gamma\,\varpi_1^4 - 2\beta\,\varpi_1^2 + 2(\alpha\,\varrho^2 + \nu^2)R_1}{2(\alpha\,\varrho^2 + \nu^2)}}, \tag{3.17}$$

where R_1 is an arbitrary constant and corresponding first integral can be written by

$$(v'(\varphi))^2 + \frac{\gamma\, v^4(\varphi) + 2\beta\, v^2(\varphi)}{2(\alpha\,\varrho^2 + v^2)} = R_1.$$

Returning the ϖ_1 as dependent variable and substituting the solution (3.17), into the first equation of (3.16), concludes

$$\frac{d\varpi_1}{d\varphi} = \sqrt{\frac{-\gamma\,\varpi_1^4 - 2\beta\,\varpi_1^2 + 2(\alpha\,\varrho^2 + v^2)R_1}{2(\alpha\,\varrho^2 + v^2)}}.$$

One time integration of this ODE, gives the following implicit solution

$$\varphi + \int \sqrt{\frac{2(\alpha\,\varrho^2 + v^2)}{-\gamma\,\varpi_1^4 - 2\beta\,\varpi_1^2 + 2(\alpha\,\varrho^2 + v^2)R_1}} + R_2 = 0. \tag{3.18}$$

Solving this equation with respect to the $\varpi_1(\varphi)$, concludes the following explicit solution

$$\begin{aligned}\varpi_1 &= \sqrt{\frac{2\Xi^3}{\gamma\sqrt{2\,\Xi + \beta^2} + \gamma\beta}}\Bigg[\frac{JCN\,(\Theta_1\varphi, \Theta_2)\,JDN\,(\Theta_1\varphi, \Theta_2)\,JSN\,(\Theta_1 R_2, \Theta_2)}{\left(-\beta\,\sqrt{2\,\Xi + \beta^2} + \Xi + \beta^2\right)(JSN\,(\Theta_1\varphi, \Theta_2))^2\,(JSN\,(\Theta_1 R_2, \Theta_2))^2 + \gamma\,\Xi}\\ &+ \frac{JSN\,(\Theta_1\varphi, \Theta_2)\,JCN\,(\Theta_1 R_2, \Theta_2)\,JDN\,(\Theta_1 R_2, \Theta_2)}{\left(-\beta\,\sqrt{2\,\Xi + \beta^2} + \Xi + \beta^2\right)(JSN\,(\Theta_1\varphi, \Theta_2))^2\,(JSN\,(\Theta_1 R_2, \Theta_2))^2 + \gamma\,\Xi}\Bigg],\end{aligned}$$

where JCN, JDN, JSN, are $JacobiCN$, $JacobiDN$, and $JacobiSN$ functions respectively, and

$$\begin{aligned}\Xi &= R_1\gamma\left(\alpha\,\varrho^2 + v^2\right),\\ \Theta_1 &= \frac{\sqrt{2\,(\alpha\,\varrho^2 + v^2)\left(\sqrt{2\,\gamma\,R_1\,(\alpha\,\varrho^2 + v^2) + \beta^2} + \beta\right)}}{2\,\alpha\,\varrho^2 + 2\,v^2},\\ \Theta_2 &= \sqrt{\frac{\beta\,\sqrt{2\,\gamma\,R_1\,(\alpha\,\varrho^2 + v^2) + \beta^2} - \gamma\,R_1\,(\alpha\,\varrho^2 + v^2) - \beta^2}{\gamma R_1\,(\alpha\,\varrho^2 + v^2)}}.\end{aligned}$$

Lastly, from the transformation (3.2), we obtain the final solution. Moreover, in order to derive a simple solution for the considered equation, we can assume $R_1 = 0$. Then, from (3.18), we get

$$\begin{aligned}\varpi_1 = \frac{4}{\Xi(\varphi)}\Bigg(&-2\,\alpha\,\beta\,\varrho^2 - 2\,\beta\,v^2\\ &-\frac{2\Phi\left(32\,\gamma\,\Phi\,\alpha^2\beta\,\varrho^4 + 64\,\gamma\,\Phi\,\alpha\,\beta\,v^2\varrho^2 + 32\,\gamma\,\Phi\,\beta\,v^4 - \sqrt{-(\Xi(\varphi))^4\,\alpha\,\beta\,\varrho^2 - (\Xi(\varphi))^4\,\beta\,v^2}\right)}{32\,\gamma\,\beta\,\alpha^2\varrho^4 + 64\,\gamma\,\beta\,\alpha\,\varrho^2 v^2 + 32\,\gamma\,\beta\,v^4 - (\Xi(\varphi))^2}\Bigg),\end{aligned}$$

where

$$\Phi = \sqrt{-(\alpha\,\varrho^2 + v^2)\beta}, \quad \Xi(\varphi) = \mathrm{e}^{-\frac{\sqrt{-\left(\alpha\,\varrho^2 + v^2\right)\beta}(\varphi + R_2)}{\alpha\,\varrho^2 + v^2}}.$$

Similarly, we obtain the final solution from the transformation (3.2).

b) Generalized projective Riccati equations method

Based on the issues discussed, we balance $v''(\varphi)$ and $v^3(\varphi)$ in Eq (3.15) for the formal solution

$$v(\varphi) = a_0 + \sum_{i=1}^{m} \eta^{i-1}(\varphi)\left(a_{i,0}\kappa(\varphi) + a_{0,i}\eta(\varphi)\right),$$

yields $m = 1$. Therefore, we have

$$v(\varphi) = a_0 + a_{0,1}\eta(\varphi) + a_{1,0}\kappa(\varphi). \tag{3.19}$$

Substituting (3.19) into (3.15) and considering the projective Riccati equations (2.9) and taking the value $\lambda = -1$, the left-hand side of (3.15) becomes a polynomial in terms of $\eta(\varphi)^j$ and $\kappa(\varphi)\eta(\varphi)^\varrho$. To obtain the values of the coefficients, we set the polynomial's coefficients to zero, resulting in a system of algebraic equations similar to that used for the Boussinesq equation's system of equations.

$$a_{1,0} = \sqrt{2}a_{0,1}\sqrt{\frac{\beta}{-(1-\zeta^2)(\alpha\varrho^2+\nu^2)}},\ \Upsilon = \frac{2\beta}{\alpha\varrho^2+\nu^2},\ \gamma = -\frac{\left(\zeta^2-1\right)\left(\alpha\varrho^2+\nu^2\right)^2}{4\beta a_{0,1}^2},$$

giving the following solution for the generalized reaction Duffing model

$$v(\varphi) = a_{0,1}\left(\frac{\sqrt{2}\beta\kappa(\varphi)}{\sqrt{-\beta(1-\zeta^2)(\alpha\varrho^2+\nu^2)}} + \eta(\varphi)\right), \tag{3.20}$$

with

$$\lambda = -1,\ a_{0,1} > 0,\ \beta > 0,\ \alpha\varrho^2 + \nu^2 > 0,\ 1-\zeta^2 < 0,$$

and

$$a_0 = 0,\ \Upsilon = -\frac{2\beta}{\alpha\varrho^2+\nu^2} < 0,\ \gamma = -\frac{\left(\zeta^2-1\right)\left(\alpha\varrho^2+\nu^2\right)^2}{4\beta a_{0,1}^2},$$

$$a_{1,0} = \sqrt{2}a_{0,1}\sqrt{\frac{\beta}{(\zeta^2-1)(\alpha\varrho^2+\nu^2)}}, \tag{3.21}$$

giving the following solution for the generalized reaction Duffing model

$$v(\varphi) = a_{0,1}\left(\frac{\sqrt{2}\beta\kappa(\varphi)}{\sqrt{\beta(\zeta^2-1)(\alpha\varrho^2+\nu^2)}} + \eta(\varphi)\right), \tag{3.22}$$

with

$$\lambda = 1,\ a_{0,1} < 0,\ \beta < 0,\ \alpha\varrho^2 + \nu^2 < 0, 1-\zeta^2 > 0. \tag{3.23}$$

If we consider the solutions (2.14), when $\lambda = -1$,

$$\eta_1(\varphi) = \frac{\Upsilon \operatorname{sech}\left(\sqrt{\Upsilon}\varphi\right)}{\zeta \operatorname{sech}\left(\sqrt{\Upsilon}\varphi\right)+1},\ \kappa_1(\varphi) = \frac{\sqrt{\Upsilon}\tanh\left(\sqrt{\Upsilon}\varphi\right)}{\zeta \operatorname{sech}\left(\sqrt{\Upsilon}\varphi\right)+1},$$

solution (3.20) takes the following expression

$$v_1(\varphi) = a_{0,1}\left(\frac{\sqrt{2}\beta\sqrt{\Upsilon}\tanh\left(\varphi\sqrt{\Upsilon}\right)}{\left(\zeta\mathrm{sech}\left(\varphi\sqrt{\Upsilon}\right)+1\right)\sqrt{\beta\left(\zeta^2-1\right)\left(\alpha\varrho^2+\nu^2\right)}} + \frac{\Upsilon\mathrm{sech}\left(\varphi\sqrt{\Upsilon}\right)}{\zeta\mathrm{sech}\left(\varphi\sqrt{\Upsilon}\right)+1}\right), \tag{3.24}$$

where Υ is given by the Eq (3.21). By considering the second pair of solutions of the Eq (2.14), when $\lambda = 1$,

$$\eta_2(\varphi) = \frac{\Upsilon\mathrm{csch}\left(\sqrt{\Upsilon}\varphi\right)}{\zeta\mathrm{csch}\left(\sqrt{\Upsilon}\varphi\right)+1}, \quad \kappa_2(\varphi) = \frac{\sqrt{\Upsilon}\coth\left(\sqrt{\Upsilon}\varphi\right)}{\zeta\mathrm{csch}\left(\sqrt{\Upsilon}\varphi\right)+1},$$

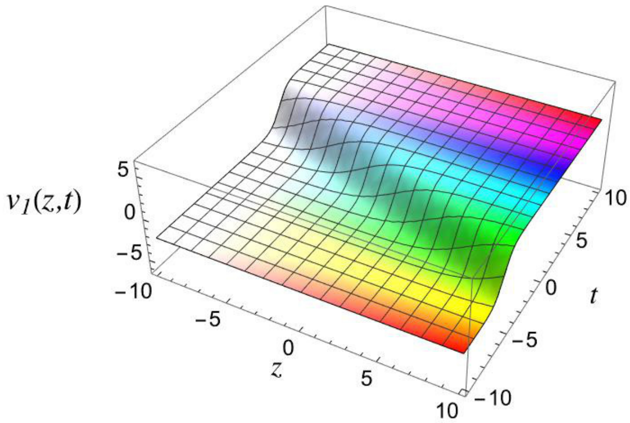
solution (3.20) takes the following expression

$$v_2(\varphi) = a_{0,1}\left(\frac{\sqrt{2}\beta\sqrt{\Upsilon}\coth\left(\varphi\sqrt{\Upsilon}\right)}{\left(\zeta\mathrm{csch}\left(\varphi\sqrt{\Upsilon}\right)+1\right)\sqrt{\beta\left(\zeta^2-1\right)\left(\alpha\varrho^2+\nu^2\right)}} + \frac{\Upsilon\mathrm{csch}\left(\varphi\sqrt{\Upsilon}\right)}{\zeta\mathrm{csch}\left(\varphi\sqrt{\Upsilon}\right)+1}\right), \tag{3.25}$$

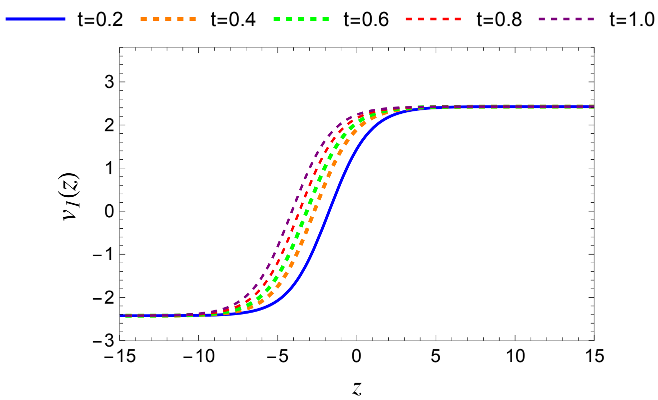
where Υ is given by the Eq (3.21).

The solutions $v_1(z,t)$ and $v_2(z,t)$ obtained from Eqs (3.24) and (3.25) are soliton solutions of the generalized reaction Duffing model equations, exhibiting well-defined behavior. In Figure 9, we observe a dark soliton behavior for $v_1(z,t)$, while in Figure 10, a singular soliton solution for $v_2(z,t)$ is shown. This soliton solution descends to $-\infty$ from one asymptotic state at $\varphi \to -\infty$, then rises to ∞, and finally descends to another asymptotic state at $\varphi \to \infty$. Both types of solutions were obtained using the following parameter values: $\zeta = 2.098$, $\varrho = 0.37$, $a_{0,1} = 1$, $\beta = 2$, $\alpha = 1$, $\nu = 0.87$, $\lambda = -1$ (shown in blue). In Figure 9, we present the properties of $v_1(z,t)$ in a 3-D plot (Figure 9a), a contour plot (Figure 9c), and a 2-D plot for different time values (Figure 9b). Similarly, in Figure 10, we show the properties of $v_2(z,t)$ in a 3-D plot (Figure 2a), a contour plot (Figure 10c), and a 2-D plot for different time values (Figure 10b).

The GPREs method is not able to produce periodic singular solutions with $\lambda = 1$ for the analytical solutions of the generalized reaction Duffing model equations. This is due to the negative value of Υ observed in Eq (3.21), which can be expressed as $\Upsilon = -\frac{2\beta}{\alpha\varrho^2+\nu^2}$. This result stems from the conditions set forth in Eq (3.23). As a consequence, any analytical solutions obtained using the GPREs method and following Eq (3.22) will lack any physical meaning, as the original condition for Υ requires it to be positive in all possible solutions.



(a)



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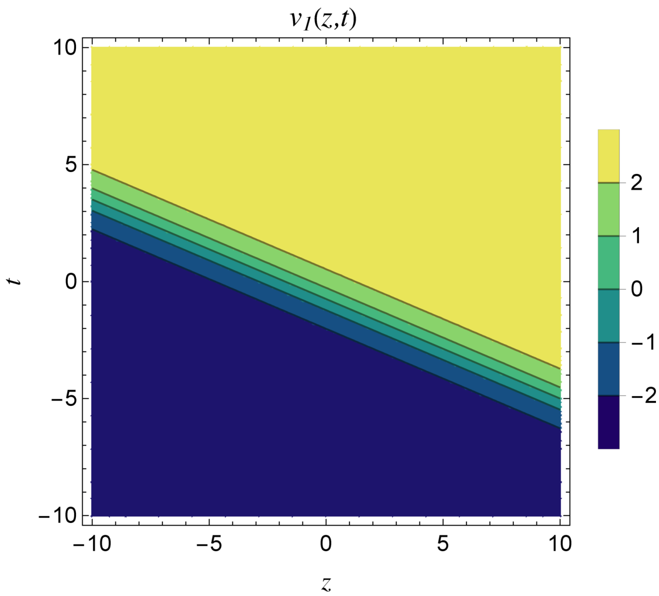


(c)

Figure 9. Various representations of $v_1(z,t)$ in (3.24) obtained by the parameters: $\zeta = 2.098$, $\varrho = 0.37$, $a_{0,1} = 1$, $\beta = 2$, $\alpha = 1$, $\nu = 0.87$, $\lambda = -1$.

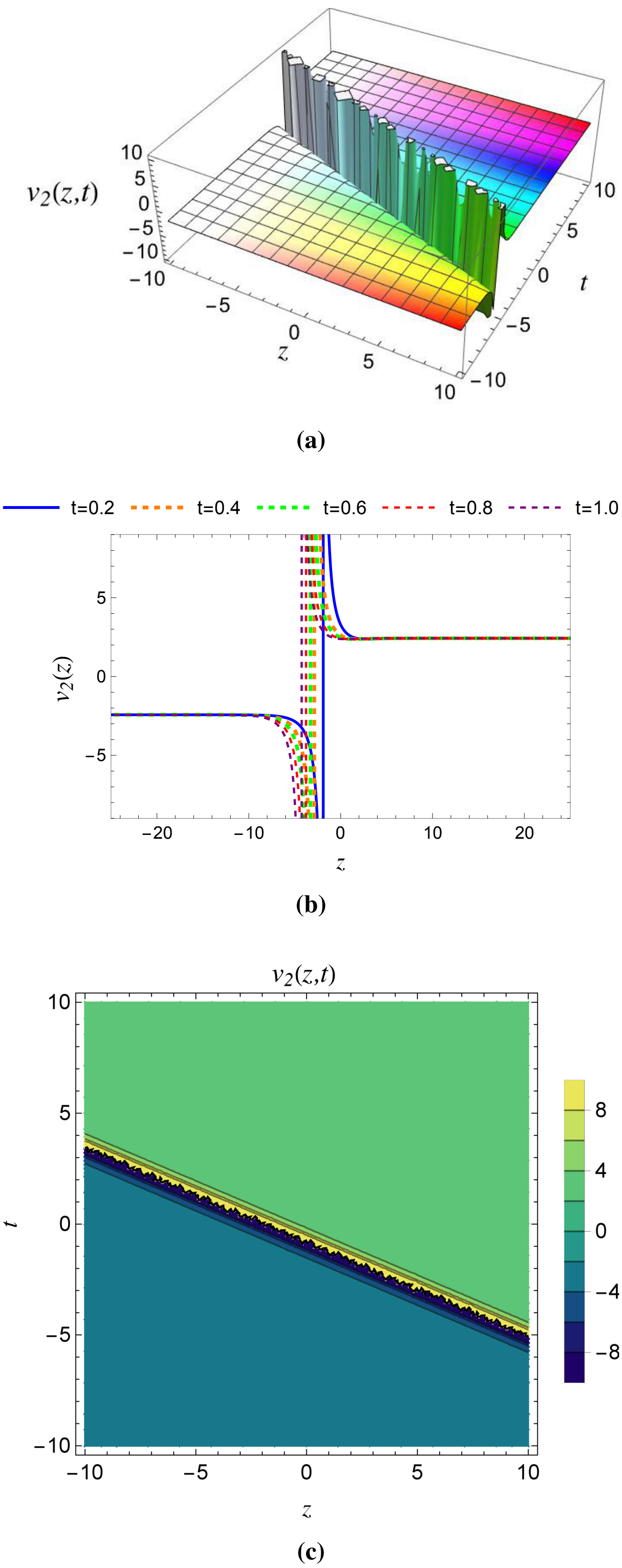


Figure 10. Various representations of $v_2(z,t)$ in (3.25) obtained by the parameters: $\zeta = 2.098$, $\varrho = 0.37$, $a_{0,1} = 1$, $\beta = 2$, $\alpha = 1$, $\nu = 0.87$, $\lambda = -1$.

3.3. The nonlinear Pochhammer-Chree equation

Another important nonlinear system that can be analyzed by the projective Riccati equations method is the Pochhammer-Chree equation [42–44]:

$$\frac{\partial^2\left(-\alpha v(z,t)-\beta v(z,t)^{n+1}-\gamma v(z,t)^{2n+1}\right)}{\partial z^2}-\frac{\partial^4 v(z,t)}{\partial z^2\partial t^2}+\frac{\partial^2 v(z,t)}{\partial t^2}=0, \tag{3.26}$$

where α, β and γ are constants. In the Eq (3.26) $v(z,t)$ describes the nonlinear longitudinal wave propagation into elastic rods and is valid for a cylinder with finite length if the radius is small compared with the wavelength [45,46]. Bifurcations of solitary waves and kink waves for Generalized Pochhammer-Chree Equation are considered in [47]. Travelling wave solutions, which include kink-shaped solitons, bell-shaped solitons, periodic solutions, rational solutions, singular solitons, are obtained in [48] for this equation.

We will apply the GPREs method to obtain the analytical solutions of the nonlinear Pochhammer-Chree equation. We can use the following transformation to get the analytical solutions:

$$v\left(z,t\right)=v\left(\varphi\right),\quad \varphi=\varrho\, z+\nu t. \tag{3.27}$$

This travelling wave change of variable permits us to convert the nonlinear Pochhammer-Chree equation into the following ODE:

$$\begin{aligned}&\frac{\varrho^2\left(\beta(-n-1)v(\varphi)^n\left(nv'(\varphi)^2+v(\varphi)v''(\varphi)\right)\right)}{v(\varphi)}+\frac{\varrho^2\left(-\gamma(2n+1)v(\varphi)^{2n}\left(2nv'(\varphi)^2+v(\varphi)v''(\varphi)\right)\right)}{v(\varphi)}\\&+\frac{\varrho^2\left(-v(\varphi)\left(\nu^2v^{(4)}(\varphi)+\alpha v''(\varphi)\right)\right)}{v(\varphi)}+\nu^2v''(\varphi)=0.\end{aligned} \tag{3.28}$$

By twice integrating the (3.28) with respect to φ and vanishing the integration constants, the following result is obtained:

$$\begin{aligned}&-\varrho^2\left(v(\varphi)\left(v(\varphi)^n\left(\frac{a_0(n+2)\nu^2}{2n^2a_{1,0}^2}-\frac{(n+1)\nu^2v(\varphi)^n}{4n^2a_{1,0}^2}\right)\right)\right)\\&-\varrho^2\left(v(\varphi)\left(+\frac{a_0^2(n+1)\nu^2}{n^2a_{1,0}^2}-\frac{a_0^2(n+2)\nu^2}{n^2a_{1,0}^2}+\frac{\nu^2}{\varrho^2}\right)\right)-\varrho^2\nu^2v''(\varphi)+\nu^2v(\varphi)=0.\end{aligned} \tag{3.29}$$

By balancing with in (3.29), the terms $v''(\varphi)$ and ,$v^{2n+1}(\varphi)$, we can consider the next change for the dependent variable (see for example [28])

$$v(\varphi)=u(\varphi)^{\frac{1}{n}}, \tag{3.30}$$

where $u(\varphi)$ is a new function of φ. Substituting (3.30) into (3.29) we get the new ODE

$$\left(\frac{n^2\nu^2}{\varrho^2}-\alpha n^2\right)u(\varphi)^2-\beta n^2u(\varphi)^3-\gamma n^2u(\varphi)^4+(n-1)\nu^2u'(\varphi)^2-n\nu^2u(z)u''(\varphi)=0. \tag{3.31}$$

a) Nucci's reduction method

By using the change of variables (2.2), the system of (2.3) for the Eq (3.31) can be written as

$$\begin{cases} \varpi_1' = \varpi_2, \\ \varpi_2' = \dfrac{1}{n\nu^2\varpi_1}\left(\left(\dfrac{n^2\nu^2}{\varrho^2} - \alpha n^2\right){\varpi_1}^2 - \beta n^2{\varpi_1}^3 - \gamma n^2{\varpi_1}^4 + (n-1)\nu^2\varpi_2^2\right). \end{cases} \tag{3.32}$$

If we introduce a new independent variable, ϖ_1, the system (3.32), converts into

$$\frac{d\varpi_2(\varpi_1)}{d\varpi_1} = \frac{1}{n\nu^2\varpi_1\varpi_2(\varpi_1)}\left(\left(\frac{n^2\nu^2}{\varrho^2} - \alpha n^2\right){\varpi_1}^2 - \beta n^2{\varpi_1}^3 - \gamma n^2{\varpi_1}^4 + (n-1)\nu^2(\varpi_2(\varpi_1))^2\right).$$

Solving this equation concludes

$$\varpi_2(\varpi_1) = \sqrt{\frac{R_1 n^2\Xi{\varpi_1}^2 + \Phi{\varpi_1}^{\frac{2n+2}{n}} - \gamma\,(n(n+2))^2\varrho^2{\varpi_1}^{\frac{2+4n}{n}} - 2\beta\varrho^2(n(n+1))^2{\varpi_1}^{\frac{2+3n}{n}}}{{\varpi_1}^{\frac{1}{2n}}\Xi}}, \tag{3.33}$$

where

$$\Xi = \nu^2\varrho^2\left(n^2+3n+2\right)^2,\quad \Phi = n^2\left(n^2+3n+2\right)\left(-\alpha\varrho^2+\nu^2\right),$$

and R_1 is an arbitrary constant and corresponding first integral can be written by

$$\frac{u^{\frac{2}{n}}(\varphi)\Psi_1(u'(\varphi))^2 - \Psi_2 u^{\frac{2n+2}{n}}(\varphi) + \gamma\,(n(n+2))^2\varrho^2u^{\frac{2+4n}{n}}(\varphi) + 2\beta\varrho^2(n(n+1))^2(u(\varphi))^{\frac{2+3n}{n}}}{n^2\nu^2(\varrho(n^2+3n+2))^2(u(\varphi))^2} = R_1,$$

where

$$\Psi_1 = \left(n^2+3n+2\right)\nu^2\varrho^2,\quad \Psi_2 = n^2(n+2)(n+1)\left(-\alpha\varrho^2+\nu^2\right).$$

Returning the ϖ_1 as dependent variable and substituting the solution (3.33), into the first equation of (3.32), concludes

$$\frac{d\varpi_1}{d\varphi} = \sqrt{\frac{R_1 n^2\Xi\varpi_1^2 + \Phi{\varpi_1}^{\frac{2n+2}{n}} - \gamma\,(n(n+2))^2\varrho^2{\varpi_1}^{\frac{2+4n}{n}} - 2\beta\varrho^2(n(n+1))^2{\varpi_1}^{\frac{2+3n}{n}}}{\varpi_1^{\frac{1}{2n}}\Xi}}.$$

One time integration of this ODE, by assuming $R_1 = 0$, gives the following implicit solution

$$\varphi + \varrho\nu\,(n+1)(n+2)\int\frac{\varpi_1^{\frac{1}{2n}}}{\Omega}d\varpi_1 + R_2 = 0,$$

where

$$\Omega = \sqrt{n^2(n+2)(n+1)\left(-2\beta\varrho^2(n+1){\varpi_1}^{\frac{4+3n}{n}} + (n+2)\left(-\gamma\varrho^2{\varpi_1}^{\frac{4n+4}{n}} + {\varpi_1}^{\frac{2n+4}{n}}(-\alpha\varrho^2+\nu^2)(n+1)\right)\right)}.$$

Solving this equation with respect to the $\varpi_1(\varphi)$, and different values of n, concludes the following explicit solutions:

- $n = 1$,

$$u(\varphi) = \varpi_1(\varphi) = \frac{12e^{\Lambda_1}\left(-\alpha\,\varrho^2 + \nu^2\right)}{e^{2\Lambda_1} + 4\beta\,\varrho^2 e^{\Lambda_1} + (-18\,\alpha\,\gamma + 4\beta^2)\,\varrho^4 + 18\,\gamma\,\nu^2\varrho^2},$$

where $\Lambda_1 = \frac{-2\,\ln(2)\nu\,\varrho - \ln(3)\nu\,\varrho + \sqrt{\left(-\alpha\,\varrho^2+\nu^2\right)(\varphi+R_2)^2}}{\varrho\,\nu}$.

- $n = 2$,

$$u(\varphi) = \varpi_1(\varphi) = \frac{24e^{\Lambda_1}\left(-\alpha\,\varrho^2 + \nu^2\right)}{6\beta\,\varrho^2 e^{\Lambda_1} + e^{2\Lambda_1} + (-48\,\alpha\,\gamma + 9\beta^2)\,\varrho^4 + 48\,\gamma\,\nu^2\varrho^2},$$

where $\Lambda_1 = \frac{-\ln(2)\nu\,\varrho - \ln(3)\nu\,\varrho + 2\,\sqrt{\left(-\alpha\,\varrho^2+\nu^2\right)(\varphi+R_2)^2}}{\varrho\,\nu}$.

- $n = 3$,

$$u(\varphi) = \varpi_1(\varphi) = \frac{20e^{\Lambda_1}\left(-\alpha\,\varrho^2 + \nu^2\right)}{4\beta\,\varrho^2 e^{\Lambda_1} + e^{\Lambda_2} + (-25\,\alpha\,\gamma + 4\beta^2)\,\varrho^4 + 25\,\gamma\,\nu^2\varrho^2},$$

where $\Lambda_1 = \frac{-2\,\ln(2)\nu\,\varrho - \ln(5)\nu\,\varrho + 3\,\sqrt{\left(-\alpha\,\varrho^2+\nu^2\right)(\varphi+R_2)^2}}{\varrho\,\nu}$.

Similarly, we obtain the final solution from the transformation (3.30) and (3.27).

b) Generalized projective Riccati equations method

Performing the balance in the (3.31), between the terms $u(\varphi)u''(\varphi)$ and $u^4(\varphi)$, we introduce the following guess solution

$$u(\varphi) = a_0 + a_{0,1}\eta(\varphi) + a_{1,0}\kappa(\varphi), \tag{3.34}$$

with a_0, $a_{0,1}$ and $a_{0,1}$ are constants that have to be determined in a similar way as they were obtained in the previous two examples.

Substituting (3.34) into (3.31) and considering the projective Riccati equations (2.9), with the value of $\lambda = -1$ and a procedure similar to the one used for the Boussinesq equation system and the generalized reaction Duffing model, the obtained coefficient values are as follows:

$$\Upsilon = \frac{a_0^2}{a_{1,0}^2},\ a_{0,1} = \frac{\sqrt{\zeta^2 - 1}\,\sqrt{n-2}\,a_{1,0}^2}{\sqrt{a_0^2(n-2)}}, \beta = \frac{a_0(n+2)\nu^2}{2n^2 a_{1,0}^2},\ \gamma = -\frac{(n+1)\nu^2}{4n^2 a_{1,0}^2},\ \alpha = \nu^2\left(\frac{1}{\varrho^2} - \frac{a_0^2}{n^2 a_{1,0}^2}\right), \tag{3.35}$$

which concludes the following solution for the nonlinear Pochhammer-Chree equation

$$u(\varphi) = -\frac{\sqrt{\zeta^2 - 1}\,a_{1,0}^2\eta(\varphi)}{a_0} + a_{1,0}\kappa(\varphi) + a_0, \tag{3.36}$$

with

$$\lambda = -1,\ |\zeta| > 1,\ n > 2.$$

Moreover

$$\Upsilon = -\frac{a_0^2}{a_{1,0}^2},\ a_{0,1} = -\frac{\sqrt{\zeta^2 - 1}\,\sqrt{n-2}\,a_{1,0}^2}{\sqrt{a_0^2(n-2)}}, \beta = \frac{a_0(n+2)\nu^2}{2n^2 a_{1,0}^2},\ \gamma = -\frac{(n+1)\nu^2}{4n^2 a_{1,0}^2},\ \alpha = \nu^2\left(\frac{1}{\varrho^2} - \frac{a_0^2}{n^2 a_{1,0}^2}\right), \tag{3.37}$$

concludes the following solution for the nonlinear Pochhammer-Chree equation

$$u(\varphi) = -\frac{\sqrt{\zeta^2 - 1}a_{1,0}^2\eta(\varphi)}{a_0} + a_{1,0}\kappa(\varphi) + a_0, \tag{3.38}$$

with

$$\lambda = 1,\ |\zeta| < 1,\ \ n > 2. \tag{3.39}$$

If we consider the solution (2.14), when $\lambda = -1$,

$$\eta_1(\varphi) = \frac{\Upsilon \operatorname{sech}\left(\sqrt{\Upsilon}\varphi\right)}{\zeta \operatorname{sech}\left(\sqrt{\Upsilon}\varphi\right) + 1},\ \kappa_1(\varphi) = \frac{\sqrt{\Upsilon}\tanh\left(\sqrt{\Upsilon}\varphi\right)}{\zeta \operatorname{sech}\left(\sqrt{\Upsilon}\varphi\right) + 1},$$

solution (3.36) takes the following expression

$$u_1(\varphi) = a_0 - \frac{a_{1,0}^2\sqrt{\zeta^2 - 1}\Upsilon \operatorname{sech}\left(\varphi\sqrt{\Upsilon}\right)}{a_0\left(\zeta \operatorname{sech}\left(\varphi\sqrt{\Upsilon}\right) + 1\right)} + \frac{a_{1,0}\sqrt{\Upsilon}\tanh\left(\varphi\sqrt{\Upsilon}\right)}{\zeta \operatorname{sech}\left(\varphi\sqrt{\Upsilon}\right) + 1}, \tag{3.40}$$

where Υ is given by the Eq (3.35). By considering the second pair of solutions of the Eq (2.14), when $\lambda = 1$,

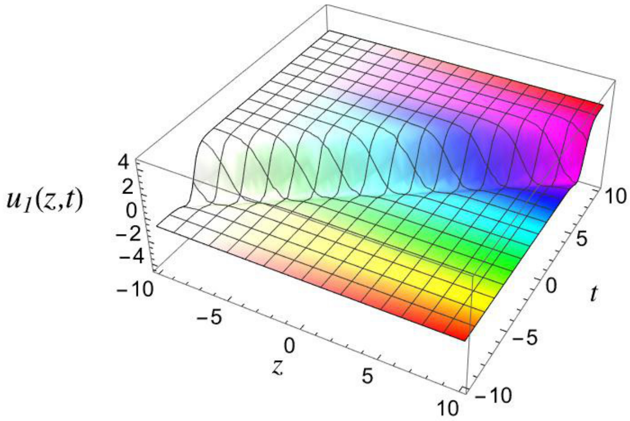
$$\eta_2(\varphi) = \frac{\Upsilon \operatorname{csch}\left(\sqrt{\Upsilon}\varphi\right)}{\zeta \operatorname{csch}\left(\sqrt{\Upsilon}\varphi\right) + 1},\ \kappa_2(\varphi) = \frac{\sqrt{\Upsilon}\coth\left(\sqrt{\Upsilon}\varphi\right)}{\zeta \operatorname{csch}\left(\sqrt{\Upsilon}\varphi\right) + 1}, \tag{3.41}$$

solution (3.20) takes the following expression

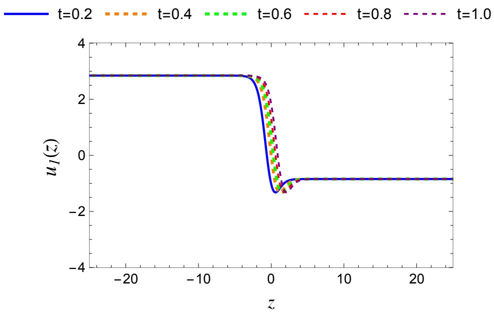
$$u_2(\varphi) = a_0 - \frac{a_{1,0}^2\sqrt{\zeta^2 - 1}\Upsilon \operatorname{csch}\left(\varphi\sqrt{\Upsilon}\right)}{a_0\left(\zeta \operatorname{csch}\left(\varphi\sqrt{\Upsilon}\right) + 1\right)} + \frac{a_{1,0}\sqrt{\Upsilon}\coth\left(\varphi\sqrt{\Upsilon}\right)}{\zeta \operatorname{csch}\left(\varphi\sqrt{\Upsilon}\right) + 1}, \tag{3.42}$$

where Υ is given by the Eq (3.35).

The solutions $u_1(z,t)$ and $u_2(z,t)$ obtained from Eqs (3.40) and (3.42), respectively, are valid solutions of the nonlinear Pochhammer-Chree equation and exhibit distinct behavior. Figure 11 shows a dark soliton-like behavior for $u_1(z,t)$, while Figure 12 demonstrates a singular soliton solution for $u_2(z,t)$. In the latter case, the soliton rises to infinity from one asymptotic state as $\varphi \to -\infty$, then descends to $-\infty$, and finally rises again to another asymptotic state as $\varphi \to \infty$. We used the following parameter values for both types of solutions: $\zeta = 2.098$, $\varrho = 0.37$, $a_{0,1} = 1$, $\beta = 2$, $\alpha = 1$, $\nu = 0.87$, and $\lambda = -1$. Figure 9 illustrates the properties of the solution $v_1(z,t)$ with a 3-D plot (Figure 9a), a 2-D plot for various time values (Figure 9b), and a contour plot (Figure 9c). Similarly, Figure 10 displays the properties of the solution $v_2(z,t)$ with a 3-D plot (Figure 2a), a 2-D plot for various time values (Figure 10b), and a contour plot (Figure 10c).



(a)



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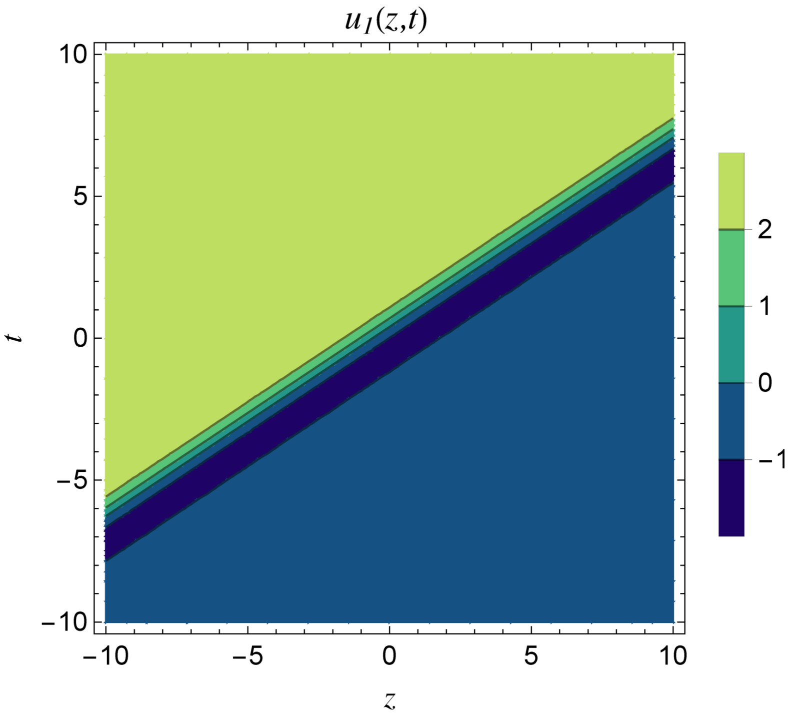
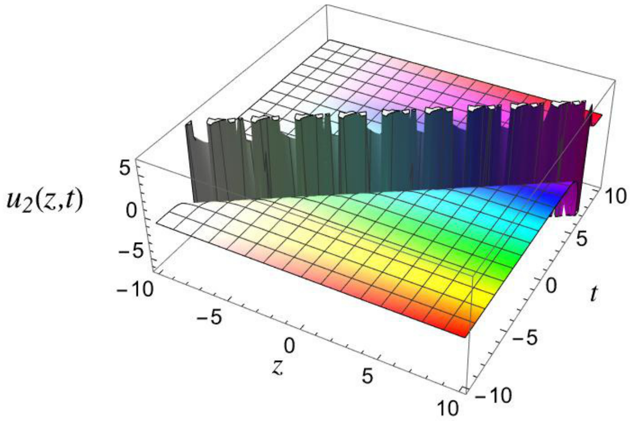
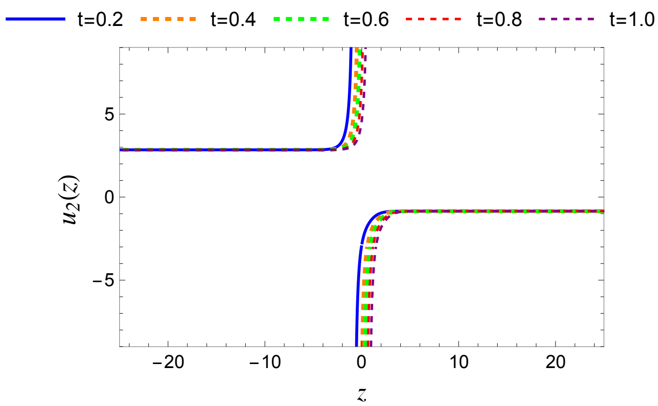


(c)

Figure 11. Various representations of $u_1(z,t)$ in (3.40) obtained by the parameters: ζ = 2.098, $a_{0,1} = 1$, $n = 3, a_0 = 1$, $a_{1,0} = -1$, $\nu = -1.5$, $\varrho = 1$ $\lambda = -1$.



(a)



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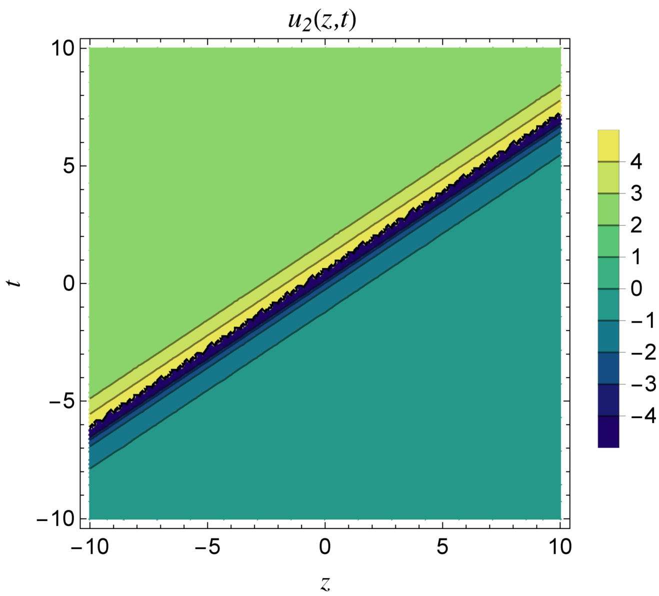

(c)

Figure 12. Various representations of $u_2\,(z,t)$ in (3.40) obtained by the parameters: ζ = 2.098, $a_{0,1}$ = 1, n = 3, a_0 = 1, $a_{1,0}$ = −1, ν = −1.5, ϱ = 1 λ = −1.

The GPREs method is unable to provide us with periodic singular solutions with $\lambda = 1$ for the analytical solutions of the nonlinear Pochhammer-Chree equation. This is explained by analyzing the Eq (3.37), where we observe a negative value of Υ, i.e., $\Upsilon = -\frac{a_0^2}{a_{1,0}^2} < 0$. This result is due to the set of conditions in Eq (3.39). As a result, the analytical solutions (3.38) lack any kind of physical meaning since the original condition for Υ was imposed to be positive in any possible solutions obtained by the GPREs method.

The GPREs method has been applied previously to the nonlinear Pochhammer-Chree equation in [28], but the case $\lambda = 1$ was not analyzed and the condition that all allowed solutions require a positive value for Υ has not been imposed in their analysis.

4. Conclusions and outlook

This study utilized the Nucci's reduction method and the generalized projective Ricatti equations method to obtain novel traveling wave solutions and first integrals for the (1+1)-dimensional classical Boussinesq equations, the generalized reaction Duffing model, and the nonlinear Pochhammer-Chree equation. By directly applying the generalized projective Ricatti equations method to the (1+1)-dimensional classical Boussinesq equations, a new set of analytical solutions was derived. These solutions can be expressed as a combination of hyperbolic or trigonometric functions, which were determined using the solutions of the projective Ricatti equations method described in Eqs (3.10) to (3.11). To ensure the validity of these solutions, certain restrictions must be satisfied for the parameters, such as $|\zeta| > 1$, $\varrho < 0$, $K_2 > 0$, $\lambda = -1$, and $\Upsilon > 0$ for solutions (3.10); and $|\zeta| < 1$, $\varrho < 0$, $K_2 > 0$, $\lambda = 1$, and $\Upsilon > 0$ for solutions (3.11). These newly discovered hyperbolic function solutions are expected to be relevant to physical phenomena and may provide insight into challenging physical aspects of nonlinear systems.

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Conflict of interest

The authors declare no conflicts of interest.

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