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Research article

A novel generalized symmetric spectral Galerkin numerical approach for solving fractional differential equations with singular kernel

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Abstract: Polynomial based numerical techniques usually provide the best choice for approximating the solution of fractional differential equations (FDEs). The choice of the basis at which the solution is expanded might affect the results significantly. However, there is no general approach to determine which basis will perform better with a particular problem. The aim of this paper is to develop a novel generalized symmetric orthogonal basis which has not been discussed in the context of numerical analysis before to establish a general numerical treatment for the FDEs with a singular kernel. The operational matrix with four free parameters was derived for the left-sided Caputo fractional operator in order to transform the FDEs into the corresponding algebraic system with the aid of spectral Galerkin method. Several families of the existing polynomials can be obtained as a special case from the new basis beside other new families generated according to the value of the free parameters. Consequently, the operational matrix in terms of these families was derived as a special case from the generalized one up to a coefficient diagonal matrix. Furthermore, different properties relevant to the new generalized basis were derived and the error associated with function approximation by the new basis was performed based on the generalized Taylor's formula.

Keywords: symmetric orthogonal polynomials; operational matrix; spectral Galerkin method; error analysis; fractional differential equations
Mathematics Subject Classification: 35G05, 65N35

1. Introduction

Fractional modeling has demonstrated its effectiveness in describing various time-evolving dynamical systems [1–3]. Over the years, researchers have developed numerous numerical techniques to approximate the solutions of these fractional models [4–8]. Among these methods, the most commonly used techniques involve orthogonal functions, in which the solution is expanded as a weighted infinite sum of basis functions [9]. By truncating the polynomial series, the fractional model under consideration can be transformed into a system of algebraic equations. This transformation is achieved by constructing a matrix representation of the differential operator, known as an operational matrix. Spectral methods can be employed to address the residual associated with the fractional model, aiding in the derivation of the algebraic system that governs the unknowns in the finite weighted sum of the solution.

However, most of the developed numerical schemes depend on the non generalized orthogonal polynomials such as Legendre [10], Laguerre [11], Hermite [12], and different kinds of Chebyshev polynomials [13–15]. For example, Secer and Altun [16] utilized Legendre wavelets operational matrix technique to solve a system of fractional differential equations (FDEs). Baishya and Veeresha [17] construed a numerical scheme based on Lagurre polynomials operational matrix together with spectral collocation method for solving FDE with Mittag-Leffler kernel. Tural-Polat and Dincel [14] used the third-kind Chebyshev polynomials along with spectral collocation method to approximate the solution of multi-term variable order FDEs. In [18], Abd-Elhameed and Youssri used the modified tau method with the shifted Chebyshev polynomials of the fifth-kind to examine the solution of FDEs. Sadri and Aminikhah developed a numerical scheme based on the shifted Chebyshev polynomials of the sixth-kind operational matrix for studying a class of delay fractional order partial differential equations (FPDEs) [19]. Due to the mathematical difficulties, relatively few studies involve the generalized families of orthogonal polynomials, such as Jacobi and Ultraspherical polynomials (also known as Gegenbauer polynomials), compared with the researches involving the traditional polynomial [20–25]. This deficiency may affect the development of accurate numerical schemes by limiting the number of basis available for certain applications.

Recently, Masjed-Jamei [26] introduced a more general family of orthogonal polynomials with four free parameters known as the basic class of symmetric orthogonal polynomials (BCSPs) due to its symmetry property. This family not only includes a vast number of traditional orthogonal polynomials even the generalized one it also can be used to generate new classes of polynomials. The development of the BCSPs started with studying a generalization of the regular Sturm-Liouville problem known as:

$$\frac{d}{dt}\left(p(t)\frac{dy_i}{dt}(t)\right)+(\lambda_i q(t)-w(t))\,y_i(t)=0, \quad p(t)>0,\ q(t)>0, \tag{1.1}$$

which, defined on the open interval (a, b) and satisfies certain boundary conditions. Whereas for any two eigenfunctions $y_i(t), y_j(t)$, the following orthogonality relation holds

$$\int_{(a,b)} w(t)\,y_i(t)\,y_j(t)\,dt=\left(\int_{(a,b)} w(t)\,y_i^2(t)\,dt\right)\delta_{i,j}, \tag{1.2}$$

where $\delta_{i,j}$ is the Kronecker function. Many non generalized orthogonal polynomials that have been mentioned earlier are eigenfunction of Eq (1.1). Considering the symmetric functions $\Theta_i(t)$ satisfying

$\Theta_i(-t) = (-1)^i \Theta_i(t)$, the regular Sturm-Liouville equation can be generalized to the form:

$$\mathcal{A}(t)\Theta_i''(t) + \mathcal{B}(t)\Theta_i' + \left(\lambda_i C(t) + \mathcal{D}(t) + \frac{1-(-1)^i}{2}\mathcal{E}(t)\right)\Theta_i(t) = 0, \tag{1.3}$$

where $\mathcal{A}(t), C(t), \mathcal{D}(t)$ and $\mathcal{E}(t)$ are even arbitrary functions, and $\mathcal{B}(t)$ is an odd function. The functions $\Theta_i(t)$ obey the orthogonality relation with respect to the weight function $W(t)$ over the symmetric closed interval $[-\theta, \theta]$:

$$\int_{[-\theta,\theta]} W(t)\Theta_i(t)\Theta_j(t)\,dt = \left(\int_{[-\theta,\theta]} W(t)\Theta_i^2 dt\right)\delta_{i,j}, \tag{1.4}$$

where the function $W(t)$ is defined as:

$$W(t) = C(t)\cdot\exp\left(\int \frac{\mathcal{B}(t) - \mathcal{A}'(t)}{\mathcal{A}(t)}dt\right). \tag{1.5}$$

As a special case of (1.3), the generalized BCSPs with four free parameters, will denoted as $\mathcal{S}_i^{(r,s,p,q)}(t)$ through this paper, can be constructed, whereas the arbitrary function in (1.3) takes the form [26]:

$$\begin{aligned}&\mathcal{A}(t) = t^2\left(pt^2 + q\right),\ \mathcal{B}(t) = t\left(rt^2 + s\right),\\ &C(t) = t^2,\ \mathcal{D}(t) = 0, \mathcal{E}(t) = -s;\end{aligned}$$

where r, s, p and q are free parameters.

However, since the time BCPs were introduced, it had never been used in any numerical techniques. The main aim of this study is to establish a novel shifted subclass of this family applied to a spectral Galerkin scheme for examining the numerical solution of the multi-term FDE of the form:

$$\sum_{k=0}^{N} \gamma_k(t)\frac{d^{\alpha_k}u(t)}{dt^{\alpha_k}} = f(t, u(t)), \quad t \in [0, T],\ T \in \mathbb{R},\ N \in \mathbb{N}_0, \tag{1.6}$$

where the fractional differentiation of order, $\alpha_k \in \mathbb{R}^+$, is defined in Caputo sense, $u(t)$ satisfies the boundary conditions $u(0) = u_0, u(T) = u_1$ and $\gamma_N(t) \neq 0$. Moreover, the proposed Caputo operational matrix introduced in the numerical scheme is a novel generalized matrix from which we will inferred different operational matrices with respect to other families of orthogonal polynomials.

The rest of this study is organized as follows: The next section provides the relevant definitions of fractional calculus used in this paper. In Section 3, we begin the development of the shifted family of polynomials after illustrating the main results of the BCSPs. After that, Section 4 provides the generalized operational matrix, also, the convergence analysis will be introduced in Section 5. Finally, in Section 6 the numerical scheme will be discussed, supported with different numerical examples.

2. Preliminaries

In this section, some basic definition of fractional calculus is listed which will be used further in this paper.

Definition 2.1. Let $g(t) : [a, b] \to \mathbb{R}$, $a < b$, be Lebesgue square integrable function, and let α be the order of integration. The Riemann-Liouville fractional integral operator is defined as [27]:

$$\mathcal{I}_c^{\alpha} g(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_c^t \frac{g(\tau)}{(t-\tau)^{1-\alpha}} d\tau, & \alpha > 0, \\ g(t), & \alpha = 0. \end{cases}$$

Definition 2.2. The left Caputo's fractional derivative with singular kernel of order $\alpha > 0, \alpha \notin \mathbb{N}_0$ for the function $g(t) : [a, b] \to \mathbb{R}$ is defined as:

$$^{C}\mathcal{D}_{c+}^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \int_c^t \frac{g^{(n)}(\tau)}{(t-\tau)^{1+\alpha-n}} d\tau, \tag{2.1}$$

for $n = -\lceil -\alpha \rceil$, [27].

Remark 2.1. The left Caputo's fractional derivative of the power function t^n, for $n \in \mathbb{N}_0$, has the following two cases:

$$^{C}\mathcal{D}_{c+}^{\alpha} t^n = \begin{cases} 0, & for\; n < \lceil \alpha \rceil, \\ \dfrac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} t^{n-\alpha}, & for\; n \geq \lceil \alpha \rceil. \end{cases} \tag{2.2}$$

Similar to normal derivatives, the left sided Riemann-Liouville and Caputo fractional derivatives obey the linearity relation:

$$^{RL}\mathcal{D}_{a+}^{\alpha} \sum_{s=0}^{m} b_s g_s(t) = \sum_{s=0}^{m} b_s\, {}^{RL}\mathcal{D}_{a+}^{\alpha} g_s(t), \tag{2.3}$$

and

$$^{C}\mathcal{D}_{a+}^{\alpha} \sum_{s=0}^{m} b_s g_s(t) = \sum_{s=0}^{m} b_s\, {}^{C}\mathcal{D}_{a+}^{\alpha} g_s(t), \tag{2.4}$$

where $\{b_s\}_{s=0}^{m}$ are constants.

The following generalized Taylor's formula was introduced by Odibat and Momani in [28], which have the following statement, (see also [29, 30]).

Definition 2.3. Suppose that $g(t)$ is differentiable, and ${}^{C}\mathcal{D}_{0+}^{k\alpha} g(t) \in C(0, 1]$ for $k = 0, 1, \cdots, m$, where $0 < \alpha \leq 1$, the generalized Taylor's formula takes the form:

$$g(t) = \sum_{i=0}^{m-1} \frac{t^{i\alpha}}{\Gamma(i\alpha+1)}\, {}^{C}\mathcal{D}_{0+}^{\alpha} g(0^+) + \frac{t^{m\alpha}}{\Gamma(m\alpha+1)}\, {}^{C}\mathcal{D}_{0+}^{\alpha} g(\xi), \tag{2.5}$$

with $0 \leq \xi \leq t$ for all $t \in (0, 1]$.

3. Basic symmetric polynomials class

This section is concerned with recalling the relations relevant to the basic class of symmetric polynomials (BCSPs) along with developing the main concepts of the novel shifted subclass.

3.1. BCSPs properties

As mentioned earlier, the BSCPs are the eigenfunctions of the symmetric generalization of the Sturm-Liouville problem in (1.3) of the form:

$$t^2\left(pt^2+q\right)\mathcal{S}_i''^{(r,s,p,q)}(t)+t\left(rt^2+s\right)\mathcal{S}_i'^{(r,s,p,q)}(t) - \left[i\left(r+(i-1)\,p\right)t^2+\left(1-(-1)^i\right)\frac{s}{2}\right]\mathcal{S}_i^{(r,s,p,q)}(t)=0. \quad (3.1)$$

The BCSPs can be defined over the symmetric interval $[-\rho,\rho]$, which fulfill certain conditions that will be mentioned, through the following analytic form.

Definition 3.1. Suppose that $t\in[-\rho,\rho]$, and $i\in\mathbb{N}_0$. For the parameters $r,s,p,q\in\mathbb{R}$ such that s and q nor r and p can vanish together, the BCSPs of degree i, can be defined as (see [26]):

$$\mathcal{S}_i^{(r,s,p,q)}(t)=\sum_{k=0}^{\lfloor\frac{i}{2}\rfloor}\binom{\lfloor\frac{i}{2}\rfloor}{k}\left(\prod_{\mu=0}^{\lfloor\frac{i}{2}\rfloor-(k+1)}\xi_{\mu,i}^{r,s,p,q}\right)t^{i-2k}, \quad (3.2)$$

where $\xi_{\mu,i}^{r,s,p,q}$ is given by:

$$\xi_{\mu,i}^{r,s,p,q}=\frac{\left(2\mu+(-1)^{i+1}+2\lfloor\frac{i}{2}\rfloor\right)p+r}{\left(2\mu+(-1)^{i+1}+2\right)q+s}. \quad (3.3)$$

In the purpose of defining the monic type of this family, the leading term is multiplied by the inverse of the coefficient $\xi_{\mu,i}^{r,s,p,q}$ as follows [26].

Definition 3.2. The monic type of BCSPs (MBCSPs) is defined through the following representation:

$$\bar{\mathcal{S}}_i^{(r,s,p,q)}(t)=\left(\prod_{\nu=0}^{\lfloor\frac{i}{2}\rfloor-1}\frac{1}{\xi_{\nu,i}^{r,s,p,q}}\right)\mathcal{S}_i^{(r,s,p,q)}(t). \quad (3.4)$$

Figure 1 indicates the graph of $\bar{\mathcal{S}}_5^{(r,s,p,q)}(t)$ for different values of s, with a fixed r,q and p.

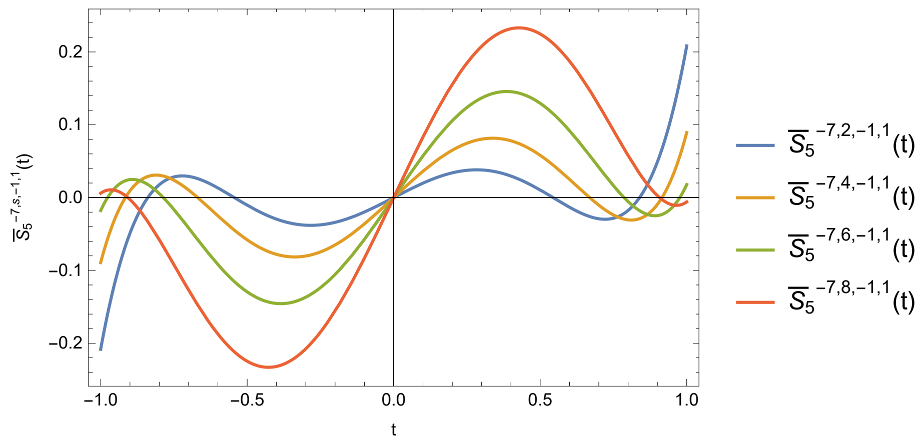


Figure 1. The graph of MBCSP with parameter $r=-7$, $p=-1=-q$ and $s\in\{2,4,6,8\}$.

By the means of the following three-term recurrence relation, the monic type of BSCPs can be generated as:

$$\bar{\mathcal{S}}_{i+1}^{(r,s,p,q)}(t) = t\bar{\mathcal{S}}_{i}^{(r,s,p,q)}(t) + \varrho_i^{r,s,p,q}\bar{\mathcal{S}}_{i-1}^{(r,s,p,q)}(t), \tag{3.5}$$

with the leading members $\bar{\mathcal{S}}_{0}^{(r,s,p,q)}(t) = 1, \bar{\mathcal{S}}_{0}^{(r,s,p,q)}(t) = t$, and $\varrho_i^{r,s,p,q}$ is defied by:

$$\varrho_i^{r,s,p,q} = \frac{(2pq)\,i^2 + 2\left((r-2p)\,q - (-1)^i\,ps\right)i + (r-2p)\,s\left(1-(-1)^i\right)}{2\,(2pi + r - p)\,(2pi + r - 3p)}. \tag{3.6}$$

According to Favard's theorem [31], the above three-term recurrence relation implies the next orthogonality property [26]:

$$\begin{aligned}\langle\bar{\mathcal{S}}_{i}^{(r,s,p,q)}(t), \bar{\mathcal{S}}_{j}^{(r,s,p,q)}(t)\rangle &= \int_{-\rho}^{\rho}\Xi^{(r,s,p,q)}(t)\,\bar{\mathcal{S}}_{i}^{(r,s,p,q)}(t)\,\bar{\mathcal{S}}_{j}^{(r,s,p,q)}(t)\,dt\\ &= \left((-1)^i\prod_{\nu=0}^{i}\varrho_\nu^{r,s,p,q}\int_{-\rho}^{\rho}\Xi^{(r,s,p,q)}(t)\,dt\right)\delta_{i,j},\end{aligned} \tag{3.7}$$

where, $\varrho_\nu^{r,s,p,q}$ as in (3.6), $\delta_{i,j}$ is the Kronecker function, and the weight function $\Xi^{(r,s,p,q)}(t)$ is defined as:

$$\Xi^{(r,s,p,q)}(t) = \exp\left(\int\frac{s + t^2\,(r-2p)}{t\,(q+pt^2)}\right), \tag{3.8}$$

although, the expression $\left(pt^2 + q\right)\Xi^{(r,s,p,q)}(t)$ should vanishes at the boundaries of the orthogonality interval $[-\rho, \rho]$.

3.2. The shifted class of MBCSPs

In this part of the present article, we introduce the shifted class of the MBCSPs defined over the interval $\Omega = [0, \rho]$ by means of a certain transformation. To the best of our knowledge, this class of polynomials are being discussed here for the first time. The shifted class will be denoted by $\mathcal{G}_i^{(r,s,p,q)}(t)$, for $i \in \mathbb{N}_0$.

Definition 3.3. The shifted class of the MBCSPs of order $i \in \mathbb{N}_0$, defined over the interval Ω, is related to the MBCSPs via the following transformation:

$$\mathcal{G}_{i,\rho}^{(r,s,p,q)}(t) = \bar{\mathcal{S}}_{i}^{(r,s,p,q)}(2t - \rho), \tag{3.9}$$

where, $\rho \in \mathbb{R}$.

The set $\left\{\mathcal{G}_{i,\rho}^{(r,s,p,q)}(t)\right\}_{i=0}^{\infty}$ forms a complete set of orthogonal functions with respect to the weight function $\mathcal{W}_{\rho}^{(r,s,p,q)}(t)$ defined as:

$$\mathcal{W}_{\rho}^{(r,s,p,q)}(t) = \exp\left(\int\frac{s + t^2\,(r-2p)}{t\,(q+pt^2)}dt\right)\Big|_{2t-\rho}, \tag{3.10}$$

which obeys the inner product of the form:

$$\langle \mathcal{G}_{i,\rho}^{(r,s,p,q)}(t), \mathcal{G}_{j,\rho}^{(r,s,p,q)}(t)\rangle = \int_{\Omega} \mathcal{W}_{\rho}^{(r,s,p,q)}(t)\, \mathcal{G}_{i,\rho}^{(r,s,p,q)}(t)\, \mathcal{G}_{j,\rho}^{(r,s,p,q)}(t)\, dt$$
$$= \left((-1)^i \prod_{\nu=0}^{i} \varrho_{\nu}^{r,s,p,q} \int_{\Omega} \mathcal{W}_{\rho}^{(r,s,p,q)}(t)\, dt\right) \delta_{i,j}. \quad (3.11)$$

In order to establish the analytic form of the polynomials $\mathcal{G}_{i,\rho}^{(r,s,p,q)}(t)$, we introduce the series representation in the next lemma.

Lemma 3.1. Suppose that $t \in \Omega$ and $i \in \mathbb{N}_0$, the polynomials $\mathcal{G}_{i,\rho}^{(r,s,p,q)}(t)$ can be defined through the following series representation:

$$\mathcal{G}_{i,\rho}^{(r,s,p,q)}(t) = \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} \sum_{l=0}^{i-2k} h_{k,l,i}^{r,s,p,q} t^l, \quad (3.12)$$

where

$$h_{k,l,i}^{r,s,p,q} = \binom{\lfloor \frac{i}{2} \rfloor}{k} \binom{i-2k}{l} \prod_{\nu=0}^{\lfloor \frac{i}{2} \rfloor - 1} \frac{1}{\xi_{\nu,i}^{r,s,p,q}} \prod_{\mu=0}^{\lfloor \frac{i}{2} \rfloor - (k+1)} \xi_{\mu,i}^{r,s,p,q} \frac{(-1)^{i-l}}{2^{-l} \rho^{2k+l-i}}. \quad (3.13)$$

Proof. According to the power series of $\mathcal{S}_i^{(r,s,p,q)}(t)$ in (3.2) and the connection between the monic class (3.4). By substituting the transfomation in (3.3), we arrive at the relation:

$$\mathcal{G}_{i,\rho}^{(r,s,p,q)}(t) = \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} \binom{\lfloor \frac{i}{2} \rfloor}{k} \prod_{\nu=0}^{\lfloor \frac{i}{2} \rfloor - 1} \frac{1}{\xi_{\nu,i}^{r,s,p,q}} \prod_{\mu=0}^{\lfloor \frac{i}{2} \rfloor - (k+1)} \xi_{\mu,i}^{r,s,p,q} (2t-\rho)^{i-2k},$$

by applying the binomial theorem, we get the desired result. □

Considering the column vector of the shifted polynomials $\mathcal{G}_{i,\rho}^{(r,s,p,q)}(t)$ defined by:

$$\mathfrak{G}_{\rho,n}^{(r,s,p,q)}(t) = \left[\mathcal{G}_{0,\rho}^{(r,s,p,q)}(t), \mathcal{G}_{1,\rho}^{(r,s,p,q)}(t), \cdots, \mathcal{G}_{n,\rho}^{(r,s,p,q)}(t)\right]_{1\times(n+1)}^T, \quad (3.14)$$

this vector has a matrix representation that will be derived in the next lemma.

Lemma 3.2. Let $\mathfrak{G}_{\rho,n}^{(r,s,p,q)}(t)$ be the shifted MBCSPs vector, where $n \in \mathbb{N}_0$. For $t \in \Omega$, the vector $\mathfrak{G}_{\rho,n}^{(r,s,p,q)}(t)$ can be written as:

$$\mathfrak{G}_{\rho,n}^{(r,s,p,q)}(t) = Q_{\rho,n}^{r,s,p,q} \mathfrak{T}_n(t), \quad (3.15)$$

where, $\mathfrak{T}_n(t) = \left[t^0, t^1, \cdots, t^n\right]^T$, and $Q_{\rho,n}^{r,s,p,q}$ is the $(n+1 \times n+1)$ lower triangle matrix and

$$Q_{\rho,n}^{r,s,p,q^T} = \begin{pmatrix} q_{\rho,00}^{r,s,p,q} & q_{\rho,10}^{r,s,p,q} & q_{\rho,20}^{r,s,p,q} & \cdots & q_{\rho,n0}^{r,s,p,q} \\ 0 & q_{\rho,11}^{r,s,p,q} & q_{\rho,21}^{r,s,p,q} & \cdots & q_{\rho,n1}^{r,s,p,q} \\ 0 & 0 & q_{\rho,22}^{r,s,p,q} & \cdots & q_{\rho,n2}^{r,s,p,q} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & q_{\rho,nn}^{r,s,p,q} \end{pmatrix}, \quad (3.16)$$

and, the entries $q_{\rho,ij}^{r,s,p,q}$ are given by:

$$q_{\rho,ij}^{r,s,p,q} = \sum_{k=0}^{\lfloor \frac{j}{2} \rfloor - \varrho_i} h_{k,i,j}^{r,s,p,q}, \tag{3.17}$$

where, the ϱ_i is defined as:

$$\varrho_i = \begin{cases} \lceil \frac{i}{2} \rceil & i\ even, \\ \lceil \frac{i}{2} \rceil - 1 & i\ odd, \end{cases} \tag{3.18}$$

and, the coefficients $h_{k,i,j}^{r,s,p,q}$ as in (3.13).

Proof. By careful examination for the coefficients of the power form given in Lemma 3.1, the lemma can be proven. □

For instance, if $r = -5, s = 8, q = 1 = -p, n = 5$, and $\rho = 1$, the vector $\mathfrak{G}_{1,5}^{(-5,8,-1,1)}(t)$ has the form:

$$\mathfrak{G}_{1,5}^{(-5,8,-1,1)}(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & -4 & 4 & 0 & 0 & 0 \\ -\frac{1}{8} & \frac{17}{4} & -12 & 8 & 0 & 0 \\ \frac{3}{80} & -\frac{12}{5} & \frac{92}{5} & -32 & 16 & 0 \\ -\frac{1}{40} & \frac{41}{20} & -22 & 68 & -80 & 32 \end{pmatrix} \begin{pmatrix} t^0 \\ t^1 \\ t^2 \\ t^3 \\ t^4 \\ t^5 \end{pmatrix}.$$

The essence advantage of working with the shifted MBCSPs that it includes as a special cases most of the traditional shifted orthogonal polynomials. The following propositions gives the relation between $\mathcal{G}_{i,\rho}^{(r,s,p,q)}(t)$ and some of other families.

Proposition 3.1. The shifted Legendre polynomials over the interval $[0, 1]$ is defined in terms of the shifted MBCSPs as:

$$P_i^*(t) = \frac{(2i)!}{(i!)^2\, 2^i} \mathcal{G}_{i,1}^{(-2,0,-1,1)}(t). \tag{3.19}$$

Proof. For the even case, taking $i = 2n$ in equation (3.4) and setting the parameters r, s, p, q, ρ to be $-2, 0, -1, 1, 1$ respectively, we get

$$\mathcal{G}_{2n,1}^{(-2,0,-1,1)}(t) = \left(\prod_{\nu=0}^{n-1} \frac{2\nu + 1}{-(2\nu + 2n - 1) - 2} \right) \mathcal{S}_{2n}^{(-2,0,-1,1)}(2t - 1).$$

Thus, the analytic form of the MBCSPs reads:

$$\mathcal{G}_{2n,1}^{(-2,0,-1,1)}(t) = \sum_{k=0}^{n} \frac{(-1)^k\, n!\Gamma\left(\frac{1}{2} + n\right)\left(\frac{3}{2} + n\right)_{-1-k+n}}{k!\,(n-k)!\Gamma\left(\frac{1}{2} + n - k\right)\left(\frac{3}{2} + n\right)_{n-1}} (2t - 1)^{2n-2k},$$

where $(s)_k$ is the Pochhammer symbol (for $s \in \mathbb{C}$) see [32]. Using the identities

$$(s)_m = \frac{\Gamma(s + m)}{\Gamma(s)}, \text{ and } \Gamma\left(\frac{1}{2} + m\right) = \frac{(2m)!\sqrt{\pi}}{2^{2m} m!}, \tag{3.20}$$

we get

$$\mathcal{G}_{2n,1}^{(-2,0,-1,1)}(t)=\sum_{k=0}^{n}\frac{(-1)^k(2n!)^2(4n-2k)!}{k!(4n)!(2n-k)!(2n-2k)!}(2t-1)^{2n-2k},$$

multiplying both sides by $\frac{(4n)!}{(2n!)^2 2^{2n}}$, we have

$$\frac{(4n)!}{(2n!)^2\,2^{2n}}\mathcal{G}_{2n,1}^{(-2,0,-1,1)}(t)=\sum_{k=0}^{n}\frac{(-1)^k(4n-2k)!}{2^{2n}k!(2n-k)!(2n-2k)!}(2t-1)^{2n-2k}, \tag{3.21}$$

where the right hand side is the analytic form of the shifted Legendre polynomials $P_{2n}^{*}(t)$. Similarly, the odd case can be obtained and the proof is complete. □

Proposition 3.2. The shifted Chebyshev polynomials of the first and second-kind over the interval $[0,1]$ are defined in terms of the shifted MBCSPs as:

$$T_i^{*}(t)=2^{i-1}\mathcal{G}_{i,1}^{(-1,0,-1,1)}(t), \tag{3.22}$$

$$U_i^{*}(t)=2^{i}\mathcal{G}_{i,1}^{(-3,0,-1,1)}(t). \tag{3.23}$$

Proposition 3.3. The shifted generalized Ultraspherical polynomials over the interval $[0,1]$ is defined in terms of the shifted MBCSPs as:

$$C_i^{\lambda^*}(t)=\frac{2^i(\lambda)_i}{i!}\mathcal{G}_{i,1}^{(-2\lambda-1,0,-1,1)}(t). \tag{3.24}$$

Also, another significant monic types of orthogonal polynomials can be obtained directly from the shifted MBCSPs.

Proposition 3.4. The shifted monic Chebyshev polynomials of the fifth and sixth-kind over the interval $[0,1]$ are defined in terms of the shifted MBCSPs as:

$$\bar{X}_i^{*}(t)=\mathcal{G}_{i,1}^{(-3,2,-1,1)}(t), \tag{3.25}$$

$$\bar{Y}_i^{*}(t)=\mathcal{G}_{i,1}^{(-5,2,-1,1)}(t). \tag{3.26}$$

Remark 3.1. For obtaining the matrix representation of all other shifted orthogonal polynomials related to the shifted MBCSPs in terms of $Q_{\rho,n}^{r,s,p,q}$ the coefficients correlating these polynomials with $\mathcal{G}_{i,\rho}^{(r,s,p,q)}(t)$ must be taken into account. As an example, by using of Proposition 3.2, the coefficient matrix of the shifted Chebyshev polynomials of the second-kind, for $n=4$, can be written as:

$$diag\left(2^j,4\right)Q_{1,4}^{-3,0,-1,1}=\begin{pmatrix}2^0&0&0&0&0\\0&2^1&0&0&0\\0&0&2^2&0&0\\0&0&0&2^3&0\\0&0&0&0&2^4\end{pmatrix}\begin{pmatrix}1&0&0&0&0\\-1&2&0&0&0\\\frac{3}{4}&-4&4&0&0\\-\frac{1}{2}&5&-12&8&0\\\frac{5}{16}&-5&21&-32&16\end{pmatrix}, \tag{3.27}$$

where $diag(\cdot,n)$ denotes the $(n+1)\times(n+1)$ diagonal matrix.

According to the value of r, s, p, and q. The four parameter weight function in (3.10) reduces to the existing weight function of the related polynomials. For example, in the first-kind Chebyshev polynomials case, we have

$$\begin{aligned}\mathcal{W}_1^{(-1,0,-1,1)}(t) &= \exp\left(\int \frac{t}{1-t^2}dt\right)\Big|_{2t-1}\\ &= exp\left(-\frac{1}{2}\ln\left(1-(2t-1)^2\right)\right)\\ &= \left(1-(2t-1)^2\right)^{\frac{-1}{2}},\end{aligned}$$

which is the weight function of the shifted Chebychev polynomials of the first-kind.

Lemma 3.3. The shifted monic BCSPs has the following analytic form:

$$\mathcal{G}_{i,\rho}^{(r,s,p,q)}(t) = \sum_{k=0}^{i} q_{\rho,ik}^{r,s,p,q} t^k, \tag{3.28}$$

where $q_{\rho,ij}^{r,s,p,q}$, is the entry in (3.17).

Proof. Having the matrix representation in Lemma 3.2, the present lemma can be proven. □

Let $u(t) \in L^2(\Omega)$, the function $u(t)$ can be expanded as a linear combination terms of the orthogonal basis $\left\{\mathcal{G}_{i,\rho}^{(r,s,p,q)}(t)\right\}_{i=0}^{\infty}$ as follows:

$$u(t) = \sum_{i=0}^{\infty} c_i \mathcal{G}_{i,\rho}^{(r,s,p,q)}(t), \tag{3.29}$$

where c_i are the coefficients of the series (3.29). By truncating the series at $i = n \in \mathbb{N}_0$, the first $n+1$ terms are taken as an appropriate approximation of $u(t)$ as:

$$u(t) \simeq u_n(t) = \sum_{i=0}^{n} c_i \mathcal{G}_{i,\rho}^{(r,s,p,q)}(t) = \Lambda^T \mathfrak{G}_{\rho,n}^{(r,s,p,q)}(t) \tag{3.30}$$

where $\Lambda^T = [c_0, c_1, \cdots, c_n]^T$ is the coefficient vector having the entries:

$$c_i = \frac{1}{\sigma_{i,\rho}^{r,s,p,q}} \int_\Omega \mathcal{W}_\rho^{(r,s,p,q)}(t)\, u(t)\, \mathcal{G}_{i,\rho}^{(r,s,p,q)}(t)\, dt, \tag{3.31}$$

and the coefficients $\sigma_i^{r,s,p,q}$ are defined as:

$$\sigma_{i,\rho}^{r,s,p,q} = (-1)^i \prod_{\nu=0}^{i} \varrho_\nu^{r,s,p,q} \int_\Omega \mathcal{W}_\rho^{(r,s,p,q)}(t)\, dt. \tag{3.32}$$

Through the upcoming section we introduce the shifted MBCSPs operational matrix of the left-sided Caputo's fractional operator in the purpose of investigating the numerical solution of the FDEs in (1.6) with the help of spectral Galerkin technique.

4. Shifted MBCSPs operational matrix

The shifted monic operational matrix of the left-sided Caputo fractional operator will be introduced in this section. Consequently, the operational matrix of this operator with respect to other orthogonal polynomials can be constructed as a special case of the generalized one.

Theorem 4.1. *Suppose that* ${}^{C}\mathcal{D}_{c+}^{\alpha}$ *is the Caputo's fractional differentiation of order* $\alpha \in \mathbb{R}$. *The derivative of the shifted MBSCPs vector* ${}^{C}\mathcal{D}_{c+}^{\alpha}\mathfrak{G}_{\rho,n}^{(r,s,p,q)}(t)$ *is given by:*

$$ {}^{C}\mathcal{D}_{c+}^{\alpha}\mathfrak{G}_{\rho,n}^{(r,s,p,q)}(t) = \mathfrak{O}_{\rho,n,\alpha}^{(r,s,p,q)}\mathfrak{G}_{\rho,n}^{(r,s,p,q)}(t), \tag{4.1} $$

where $\mathfrak{O}_{\rho,n,\alpha}^{(r,s,p,q)}$ *is the shifted monic BCSPs operational matrix of the Caputo fractional derivative of order* α, *defined as:*

$$ \mathfrak{O}_{\rho,n,\alpha}^{(r,s,p,q)} = Q_{\rho,n}^{r,s,p,q}\mathcal{H}_{n,\alpha}(t)\left(Q_{\rho,n}^{r,s,p,q}\right)^{-1}, $$

and $\mathcal{H}_{n,\alpha}(t)$ *is the diagonal matrix of the form:*

$$ \mathcal{H}_{n,\alpha}(t) = \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{\Gamma(\lceil\alpha\rceil+1)}{\Gamma(\lceil\alpha\rceil-\alpha+1)}t^{-\alpha} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}t^{-\alpha} \end{pmatrix}. \tag{4.2} $$

Proof. According to the matrix equation in *Lemma* 3.2, the derivative of $\mathfrak{G}_{\rho,n}^{(r,s,p,q)}(t)$ can be written as:

$$ {}^{C}\mathcal{D}_{c+}^{\alpha}\mathfrak{G}_{\rho,n}^{(r,s,p,q)}(t) = Q_{\rho,n}^{r,s,p,q}\,{}^{C}\mathcal{D}_{c+}^{\alpha}\mathfrak{T}_{n}(t), \tag{4.3} $$

whereas the derivative of $\mathfrak{T}_{n}(t)$, by applying (2.2), is given by:

$$ \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{\Gamma(\lceil\alpha\rceil+1)}{\Gamma(\lceil\alpha\rceil-\alpha+1)}t^{\lceil\alpha\rceil-\alpha} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}t^{n-\alpha} \end{pmatrix}, \tag{4.4} $$

which reads $\mathcal{H}_{n,\alpha}(t)\,\mathfrak{T}_{n}(t)$. Since the vector $\mathfrak{T}_{n}(t)$ can be written terms of $\mathfrak{G}_{\rho,n}^{(r,s,p,q)}(t)$ as:

$$ \mathfrak{T}_{n}(t) = \left(Q_{\rho,n}^{r,s,p,q}\right)^{-1}\mathfrak{G}_{\rho,n}^{(r,s,p,q)}(t). $$

Substituting the last equation in (4.3), we get

$$ {}^{C}\mathcal{D}_{c+}^{\alpha}\mathfrak{G}_{\rho,n}^{(r,s,p,q)}(t) = Q_{\rho,n}^{r,s,p,q}\mathcal{H}_{n,\alpha}(t)\left(Q_{\rho,n}^{r,s,p,q}\right)^{-1}\mathfrak{G}_{\rho,n}^{(r,s,p,q)}(t). $$

The previous equation completes the proof. □

The operational matrix with respect to vast families of orthogonal polynomials related to the shifted monic BCSPs can be expressed in terms of the generalized operational matrix given in Theorem 4.1. The following corollaries provide the methodology taken to derive such operational matrices.

Corollary 4.1. Let $\mathcal{P}_n^*(t) = \left[P_0^*(t), P_1^*(t), \cdots, P_n^*(t)\right]$ be the shifted Legendre polynomials vector defined over the interval $[0,1]$. The shifted Legendre operational matrix of the Caputo fractional operator of order α is given by:

$$ {}^{C}\mathcal{D}_{c+}^{\alpha}\mathcal{P}_n^*(t) = \left(\mathfrak{D}_{p,n} Q_{1,n}^{-2,0,-1,1}\right)\mathcal{H}_{n,\alpha}(t)\left(\mathfrak{D}_{p,n} Q_{1,n}^{-2,0,-1,1}\right)^{-1}\mathcal{P}_n^*(t), \tag{4.5}$$

where $\mathfrak{D}_{p,n}$ is the $(n+1)\times(n+1)$ diagonal matrix given by:

$$\begin{pmatrix} \frac{(2\cdot 0)!}{(0!)^2 2^0} & 0 & \cdots & 0 \\ 0 & \frac{(2\cdot 1)!}{(1!)^2 2^1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{(2\cdot n)!}{(n!)^2 2^n} \end{pmatrix}.$$

Proof. Since the shifted Legendre polynomials is related to $\mathcal{G}_{i,\rho}^{(r,s,p,q)}(t)$ as in (3.19), by applying Remark 3.1, the matrix representation of $\mathcal{P}_n^*(t)$ would be:

$$\mathcal{P}_n^*(t) = diag\left(\frac{(2i)!}{(i!)\,2^i}, n\right) Q_{1,n}^{-2,0,-1,1}\mathfrak{T}_n(t). \tag{4.6}$$

Acting on the vector $\mathfrak{T}_n(t)$ as in (4.4), we get the desired result. □

Corollary 4.2. Let $\mathcal{U}_n^{\lambda^*}(t) = \left[C_0^{\lambda^*}(t), C_1^{\lambda^*}(t), \cdots, C_n^{\lambda^*}(t)\right]$ be the shifted Ultraspherical polynomials vector defined over the interval $[0,1]$. The shifted Ultraspherical operational matrix of the Caputo fractional operator of order α is given by:

$$ {}^{C}\mathcal{D}_{c+}^{\alpha}\mathcal{U}_n^{\lambda^*}(t) = \left(\mathfrak{D}_{u,n} Q_{1,n}^{-2\lambda-1,0,-1,1}\right)\mathcal{H}_{n,\alpha}(t)\left(\mathfrak{D}_{u,n} Q_{1,n}^{-2\lambda-1,0,-1,1}\right)^{-1}\mathcal{U}_n^{\lambda^*}(t), \tag{4.7}$$

where $\mathfrak{D}_{u,n}$ is the $(n+1)\times(n+1)$ diagonal matrix given by:

$$\begin{pmatrix} \frac{2^0(\lambda)_0}{(0!)} & 0 & \cdots & 0 \\ 0 & \frac{2^1(\lambda)_1}{(1!)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{2^n(\lambda)_n}{(n!)} \end{pmatrix}.$$

Corollary 4.3. Let $\mathcal{T}_n^*(t) = \left[T_0^*(t), T_1^*(t), \cdots, T_n^*(t)\right]$ be the first-kind shifted Chebyshev polynomials vector defined over the interval $[0,1]$. The shifted Chebyshev of the first-kind operational matrix of the Caputo fractional operator of order α is given by:

$$ {}^{C}\mathcal{D}_{c+}^{\alpha}\mathcal{T}_n^*(t) = \left(\mathfrak{D}_{t,n} Q_{1,n}^{-1,0,-1,1}\right)\mathcal{H}_{n,\alpha}(t)\left(\mathfrak{D}_{t,n} Q_{1,n}^{-1,0,-1,1}\right)^{-1}\mathcal{T}_n^*(t), \tag{4.8}$$

where $\mathfrak{D}_{t,n}$ is the $(n+1)\times(n+1)$ diagonal matrix given by:

$$\begin{pmatrix} 2^{0-1} & 0 & \cdots & 0 \\ 0 & 2^{1-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 2^{n-1} \end{pmatrix}.$$

Corollary 4.4. Let $\bar{\mathcal{X}}_n^*(t) = \left[\bar{X}_0^*(t), \bar{X}_1^*(t), \cdots, \bar{X}_n^*(t)\right]$ be the shifted moinc Chebyshev polynomials vector of the fifth-kind defined over the interval [0, 1]. The shifted moinc Chebyshev of the fifth-kind operational matrix of the Caputo fractional operator of order α is given by:

$$ {}^{C}\mathcal{D}_{c+}^{\alpha}\bar{\mathcal{X}}_n^*(t) = Q_{1,n}^{-3,2,-1,1}\mathcal{H}_{n,\alpha}(t)\left(Q_{1,n}^{-3,2,-1,1}\right)^{-1}\bar{\mathcal{X}}_n^*(t). \tag{4.9}$$

Corollary 4.5. Let $\bar{\mathcal{Y}}_n^*(t) = \left[\bar{Y}_0^*(t), \bar{Y}_1^*(t), \cdots, \bar{Y}_n^*(t)\right]$ be the shifted moinc Chebyshev polynomials vector of the sixth-kind defined over the interval [0, 1]. The shifted moinc Chebyshev of the sixth-kind operational matrix of the Caputo fractional operator of order α is given by:

$$ {}^{C}\mathcal{D}_{c+}^{\alpha}\bar{\mathcal{Y}}_n^*(t) = Q_{1,n}^{-5,2,-1,1}\mathcal{H}_{n,\alpha}(t)\left(Q_{1,n}^{-5,2,-1,1}\right)^{-1}\bar{\mathcal{Y}}_n^*(t). \tag{4.10}$$

5. Discussion of error estimate

Here, the error estimate will be examined by the help of the generalized Taylor's formula.

Theorem 5.1. *Assume that* ${}^{C}\mathcal{D}_{0+}^{i\alpha}g(t) \in C(\Omega)$ *for* $i = 0, 1, \cdots, n$, *and let* $u(t) \in C^n[0,\rho]$. *Consider* $\mathfrak{I}_n = Span\left\{\mathcal{G}_{0,\rho}^{(r,s,p,q)}(t), \mathcal{G}_{1,\rho}^{(r,s,p,q)}(t), \cdots, \mathcal{G}_{n,\rho}^{(r,s,p,q)}(t)\right\}$. *If* $u_n(t) = \Lambda^T\mathfrak{G}_{\rho,n}^{(r,s,p,q)}(t)$ *is the best approximation of the function* $u(t)$ *out of* $\mathfrak{I}_n$, *then the error bound is estimated by:*

$$\|u(t) - u_n(t)\|_{L^2_{\mathcal{W}}(\Omega)} \leq \frac{M}{\Gamma(n\alpha+1)}\sqrt{I_{r,s,p,q}^{n,\rho,\alpha}}, \tag{5.1}$$

where $I_{r,s,p,q}^{n,\rho,\alpha}$ *is defined as:*

$$I_{r,s,p,q}^{n,\rho,\alpha} = \int_\Omega t^{2n\alpha}\mathcal{W}_\rho^{(r,s,p,q)}(t)\,dt, \tag{5.2}$$

which converges for all $t \in \Omega$, *and* $M = \sup_{t\in\Omega}\left|{}^{C}\mathcal{D}_{0+}^{n\alpha}u(t)\right|$.

Proof. By expanding $u(t) \in L^2(\Omega)$ in terms of the generalized Taylor's formula as given in Definition 2.3

$$u(t) = \sum_{i=0}^{n-1}\frac{t^{i\alpha}}{\Gamma(i\alpha+1)}\,{}^{C}\mathcal{D}_{0+}^{\alpha}u(0^+) + \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}\,{}^{C}\mathcal{D}_{0+}^{\alpha}u(\xi),$$

for $\xi \in [0, t]$. Consider the function $\varphi(t) \in \mathfrak{I}_n$ defined as:

$$\varphi(t) = \sum_{i=0}^{n-1}\frac{t^{i\alpha}}{\Gamma(i\alpha+1)}\,{}^{C}\mathcal{D}_{0+}^{\alpha}u(0^+),$$

then

$$|u(t) - \varphi(t)| \leq \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}\left|{}^{C}\mathcal{D}_{0+}^{\alpha}u(\xi)\right| = \frac{Mt^{n\alpha}}{\Gamma(n\alpha+1)}.$$

Having that $u_n(t)$ is the best approximation of $u(t)$ in $\mathfrak{I}_n$, then

$$\begin{aligned}\|u(t) - u_n(t)\|^2_{L^2_{\mathcal{W}}(\Omega)} \leq \|u(t) - \varphi(t)\|^2_{L^2_{\mathcal{W}}(\Omega)} &= \int_\Omega |u(t) - \varphi(t)|^2\,\mathcal{W}_\rho^{(r,s,p,q)}(t)\,dt \\ &\leq \frac{M^2}{\Gamma(n\alpha+1)^2}\int_\Omega t^{2n\alpha}\mathcal{W}_\rho^{(r,s,p,q)}(t)\,dt,\end{aligned}$$

which proves the theorem. □

The next theorem gives the residual error associated with the FDEs considered in Eq (1.6).

Theorem 5.2. *Suppose that $f : [0,T] \times \Omega \to \mathbb{R}$ satisfies a Lipschitz condition in u with Lipschitz constant L, that is*

$$\|f(t,u) - f(t,\tilde{u})\|_{L^2_{\mathcal{W}}(\Omega)} \leq L \|u(t) - \tilde{u}(t)\|_{L^2_{\mathcal{W}}(\Omega)}. \tag{5.3}$$

Then, the residual error is bounded by:

$$\|R(t)\|_{L^2_{\mathcal{W}}(\Omega)} \leq \left(\sum_{k=0}^{N} \vartheta_k \xi_k + L \right) \|u_n(t) - u(t)\|_{L^2_{\mathcal{W}}(\Omega)}, \tag{5.4}$$

where $\|u_n(t) - u(t)\|_{L^2_{\mathcal{W}}(\Omega)}$ is the truncation error in Theorem 5.1.

Proof. Consider the approximate spectral solution $u_n(t)$ of (1.6). Since the residual error is defined as:

$$R(t) = \sum_{k=0}^{N} \gamma_k(t)\, {}^{C}\mathcal{D}_{0+}^{\alpha_k} u_n(t) - f(t, u_n(t)),$$

then

$$\begin{aligned}
\|R(t)\|_{L^2_{\mathcal{W}}(\Omega)} &= \left\| \sum_{k=0}^{N} \gamma_k(t)\, {}^{C}\mathcal{D}_{0+}^{\alpha_k} u_n(t) - f(t,u_n) \right\|_{L^2_{\mathcal{W}}(\Omega)} \\
&= \Bigg\| \sum_{k=0}^{N} \gamma_k(t)\, {}^{C}\mathcal{D}_{0+}^{\alpha_k} u_n(t) - f(t,u_n) \\
&\qquad\qquad - \left(\sum_{k=0}^{N} \gamma_k(t)\, {}^{C}\mathcal{D}_{0+}^{\alpha_k} u(t) - f(t,u) \right) \Bigg\|_{L^2_{\mathcal{W}}(\Omega)} \\
&= \left\| \sum_{k=0}^{N} \gamma_k(t)\, {}^{C}\mathcal{D}_{0+}^{\alpha_k} (u_n(t) - u(t)) - (f(t,u_n) - f(t,u)) \right\|_{L^2_{\mathcal{W}}(\Omega)} \\
&\leq \left\| \sum_{k=0}^{N} \gamma_k(t)\, {}^{C}\mathcal{D}_{0+}^{\alpha_k} (u_n(t) - u(t)) \right\|_{L^2_{\mathcal{W}}(\Omega)} + \|f(t,u_n) - f(t,u)\|_{L^2_{\mathcal{W}}(\Omega)} \\
&\leq \sum_{k=0}^{N} \vartheta_k \left\| {}^{C}\mathcal{D}_{0+}^{\alpha_k} (u_n(t) - u(t)) \right\|_{L^2_{\mathcal{W}}(\Omega)} + L \|u_n(t) - u(t)\|_{L^2_{\mathcal{W}}(\Omega)},
\end{aligned}$$

where

$$\vartheta_k = \sup_{t \in \Omega} \|\gamma_k(t)\|_{L^2_{\mathcal{W}}(\Omega)}. \tag{5.5}$$

Now, from the boundedness of ${}^{C}\mathcal{D}_{0+}^{\alpha_k}$ (see [33]), for each k in $\{0, 1, \cdots, N\}$, we have

$$\left\| {}^{C}\mathcal{D}_{0+}^{\alpha_k} (u_n(t) - u(t)) \right\|_{L^2_{\mathcal{W}}(\Omega)} \leq \xi_k \|u_n(t) - u(t)\|_{L^2_{\mathcal{W}}(\Omega)}. \tag{5.6}$$

Hence,

$$\|R(t)\|_{L^2_{\mathcal{W}}(\Omega)} \leq \left(\sum_{k=0}^{N} \vartheta_k \xi_k + L \right) \|u_n(t) - u(t)\|_{L^2_{\mathcal{W}}(\Omega)},$$

which tends to zero as shown in Theorem 5.1 when n is sufficiently large. □

6. Numerical scheme

In this section, we introduce the method of solution of the multi-term FDE in (1.6). Unlike other traditional numerical schemes, the proposed scheme based on the shifted MBCSPs can be used to study the numerical solution with respect to numerous orthogonal polynomials without performing further calculations.

6.1. Method of solution

Consider the multi-term FDE in (1.6), by approximating the function $u(t) \in L^2_{\mathcal{W}}(\Omega)$ as in (3.30), the FDE reads

$$\sum_{k=0}^{N} \gamma_k(t)\, {}^{C}\mathcal{D}_{0+}^{\alpha_k} \Lambda^T \mathfrak{G}_{\rho,n}^{(r,s,p,q)}(t) = f\left(t, \Lambda^T \mathfrak{G}_{\rho,n}^{(r,s,p,q)}(t)\right),$$

now, applying *Theorem* 4.1 implies

$$\sum_{k=0}^{N} \gamma_k(t)\, \Lambda^T \mathfrak{D}_{\rho,n,\alpha_k}^{(r,s,p,q)} \mathfrak{G}_{\rho,n}^{(r,s,p,q)}(t) = f\left(t, \Lambda^T \mathfrak{G}_{\rho,n}^{(r,s,p,q)}(t)\right),$$

hence, the residual of the problem would be

$$R(t) = \sum_{k=0}^{N} \gamma_k(t)\, \Lambda^T \mathfrak{D}_{\rho,n,\alpha_k}^{(r,s,p,q)} \mathfrak{G}_{\rho,n}^{(r,s,p,q)}(t) - f\left(t, \Lambda^T \mathfrak{G}_{\rho,n}^{(r,s,p,q)}(t)\right). \tag{6.1}$$

The application of spectral Galerkin method yields

$$\langle \frac{\mathcal{G}_{i,\rho}^{(r,s,p,q)}(t)}{\mathcal{W}_{\rho}^{(r,s,p,q)}(t)}, R(t)\rangle = \int_{\Omega} \mathcal{G}_{i,\rho}^{(r,s,p,q)}(t)\, R(t)\, dt = 0, \tag{6.2}$$

for $i = 0, 1, \cdots, n-2$. the above equation generates $n-1$ dimensional system of equation in the unknowns c_i. Making use of the boundary conditions, we get the following additional equation

$$\Lambda^T \mathfrak{G}_{\rho,n}^{(r,s,p,q)}(0) = u_0, \quad \textit{and} \quad \Lambda^T \mathfrak{G}_{\rho,n}^{(r,s,p,q)}(T) = u_1. \tag{6.3}$$

By solving the obtained system using a suitable solver, the approximate solution can be expressed in terms of the shifted MBCSPs.

6.2. Numerical experiments

In the purpose of illustrating the accuracy and applicability of the proposed generalized approach, a numerical treatment for a different FDEs supported with comparison with other techniques will be present in this section.

The error estimate between the exact solution $u(t)$ and the numerical solution $\tilde{u}(t)$, will be measured by calculating the absolute value $e_i = |u(t_i) - \tilde{u}(t_i)|$ for $i = 1, 2, \cdots, n$.

Example 6.1. Consider the following Bagley-Torvik initial-value problem (IVP) [18, 34]:

$$ {}^{C}\mathcal{D}_{0+}^{2}u(t) + {}^{C}\mathcal{D}_{0+}^{\alpha}u(t) + u(t) = f(t), \quad t \in [0, \rho = 1], \tag{6.4} $$

subject to $u(0) = 1$ which has the exact solution $u(t) = 1 + t$. For $\alpha = \frac{1}{2}$, and $n = 2$, writing $u(t)$ as in (3.30)

$$ u_2(t) = c_0 \mathcal{G}_0^{(r,s,p,q)}(t) + c_1 \mathcal{G}_1^{(r,s,p,q)}(t) + c_2 \mathcal{G}_2^{(r,s,p,q)}(t), \tag{6.5} $$

by applying (6.1), the residual of given equation reads

$$ R(t) = \Lambda^{T}\left(\mathfrak{D}_{1,2,2}^{(r,s,p,q)} \mathfrak{G}_{1,2}^{(r,s,p,q)}(t) + \mathfrak{D}_{1,2,\frac{1}{2}}^{(r,s,p,q)} \mathfrak{G}_{1,2}^{(r,s,p,q)}(t) + \mathfrak{G}_{1,2}^{(r,s,p,q)}(t)\right) - f(t), \tag{6.6} $$

where $f(t) = t + \frac{2\sqrt{t}}{\sqrt{\pi}} + 1$. Choosing $r = -5$, $s = 4$, $p = -q = -1$, the proposed scheme with the weights $\mathcal{G}_k^{(-5,4,-1,1)}(t)$, $k = 0, 1, 2$, provide the coefficients

$$ u_2(t) = \left(\frac{3}{2}\right) \mathcal{G}_0^{(-5,4,-1,1)}(t) + \left(\frac{1}{2}\right) \mathcal{G}_1^{(-5,4,-1,1)}(t) + (0)\, \mathcal{G}_2^{(-5,4,-1,1)}(t) = 1 + t, $$

which is the exact solution. For small values of $\alpha = 0.1, 0.05$, and 0.01 with $n = 2$ the following results are obtained in Table 1.

Table 1. The numerical results of Example 6.1 for different values of α at $n = 2$.

t	Exact solution	Absolute error ($\alpha = 0.1$)	Absolute error ($\alpha = 0.05$)	Absolute error ($\alpha = 0.01$)
0.0	1.00000	2.66454×10^{-15}	9.10383×10^{-15}	1.11022×10^{-15}
0.2	1.20000	3.10862×10^{-15}	9.99201×10^{-15}	1.11022×10^{-15}
0.4	1.40000	3.33067×10^{-15}	1.04361×10^{-14}	1.33227×10^{-15}
0.6	1.60000	3.33067×10^{-15}	1.04361×10^{-14}	1.11022×10^{-15}
0.8	1.80000	3.10862×10^{-15}	9.99201×10^{-15}	1.11022×10^{-15}
1.0	2.00000	3.10862×10^{-15}	9.10383×10^{-15}	8.88178×10^{-16}

Example 6.2. Consider the linear fractional IVP of the form [35]:

$$ {}^{C}\mathcal{D}_{0+}^{2}u(t) - 2\,{}^{C}\mathcal{D}_{0+}^{1}u(t) + {}^{C}\mathcal{D}_{0+}^{\frac{1}{2}}u(t) = f(t, u(t)), \quad t \in [0, \rho = 1], \tag{6.7} $$

where $f(t, u(t))$ is defined as:

$$ f(t, u(t)) = t^7 - 14t^6 + 42t^5 - t^2 + \frac{8\left(256t^5 - 143\right)t^{\frac{3}{2}}}{429\sqrt{\pi}} + 4\left(t - \frac{1}{2}\right) - u(t), $$

having the initial conditions $u(0) = u'(0) = 0$ with the exact solution $u(t) = t^2\left(t^5 - 1\right)$. Similarly as in Example 6.1, the residual of the problem can be derived. Assuming that

$$ u(t) \simeq u_7(t) = \sum_{i=0}^{7} c_i \mathcal{G}_i^{(-4,5,-1,1)}(t), $$

by constructing the algebraic system with the aid of Galerkin technique, the approximate solution $u_7(t)$ is found to be

$$u_7(t) = t^2\left(t^5 - 1\right),$$

which is the same as the exact solution. Table 2 shows the comparison between the maximal absolute error of the present method and other polynomial based techniques such as Tau-Shifted Chebyshev (TC), Legendre-Gauss quadrature (LGQ), triangular functions (TF), and block pulse functions (BPF).

Table 2. The maximal absolute error of Example 6.2.

Present method ($n = 8$)	TSC ($n = 8$)	LGQ ($n = 8$)	TF ($h = 0.01$)	BPF ($h = 0.01$)
0.00000	3.07566×10^{-15}	1.22696×10^{-15}	3.96626×10^{-4}	2.61473×10^{-2}

Example 6.3. Consider the Bagley-Torvik boundary value problem (BVP) [34]:

$$ {}^{C}\mathcal{D}^{\alpha}_{c+}u(t) = f(t, u(t)), \quad t \in [0, \rho = 1], \tag{6.8}$$

for $\alpha = \frac{3}{2}$, the function $f(t, u(t))$ takes the form:

$$f(t, u(t)) = t^5 - t^4 + \frac{128t^{\frac{7}{2}}}{7\sqrt{\pi}} - \frac{64t^{\frac{5}{2}}}{5\sqrt{\pi}} - u(t),$$

governed by the boundary conditions $u(0) = u(1) = 0$, with the exact solution $u(t) = t^4(t-1)$. By applying the proposed method for $r = -5, s = 4$ and $p = -q = -1$, the exact solution can be found with $n = 5$, i.e.,

$$u_5(t) = \sum_{i=0}^{5} c_i \mathcal{G}_i^{(-5,4,-1,1)}(t) = t^4(t-1).$$

The graph of the exact and numerical solutions is given in Figure 2(a), in addition, the comparison between the absolute error of the present method and the Gegenbauer wavelet method [34] is given graphically in Figure 2(b). Therefore, the numerical results is compared with the Gegenbauer wavelet method [34] in Table 3. Also, Table 4 provides the absolute error for small values of α.

One of the main advantages of the proposed scheme appears from the ability of providing the approximate solution in terms of different families of polynomials. So, Figure 3 compares the absolute error between the shifted Legendre (SL), first-kind shifted Chebyshev (FSC), second-kind shifted Chebyshev (SSC) and the shown configuration of the MBCSPs for Example 6.3 with $\alpha = \frac{1}{4}$. Table 5 provides the numerical results obtained with different r, s, p and q configurations representing other families of polynomials for $n = 5$.

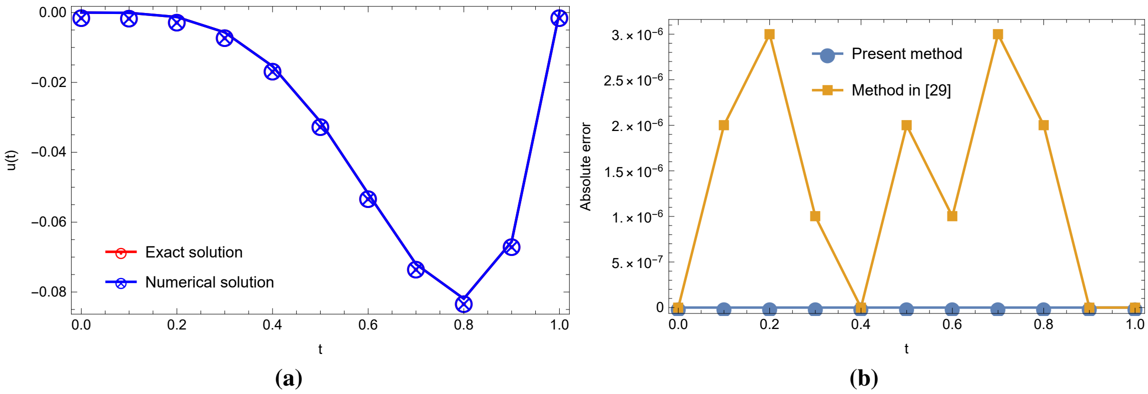


Figure 2. (a) The graph of exact and numerical solutions, (b) comparison between the absolute error of the present method and the method in [34].

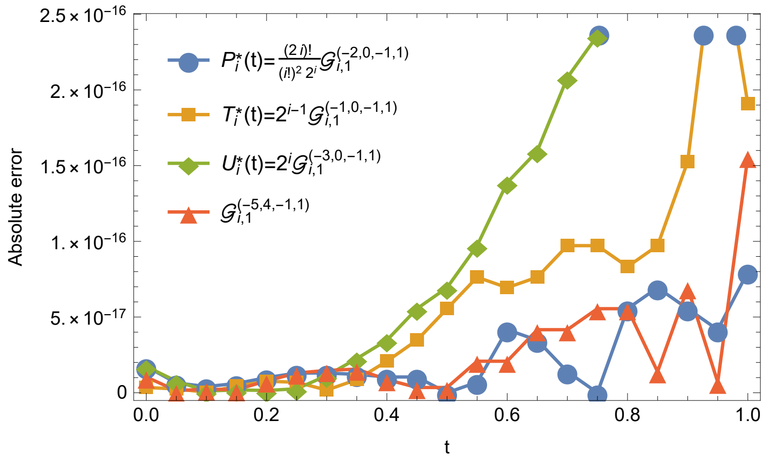


Figure 3. Comparison of the absolute error between, SL, FSC, SSC, and MBCSP.

Table 3. The numerical results of Example 6.3 for $n = 5$.

t	Exact solution	Present method	Absolute error ($n = 5$)	Absolute error in [34]
0.0	0.00000	0.00000	0.00000	0.00000×10^{-6}
0.1	-0.00009	-0.00009	0.00000	2.00000×10^{-6}
0.2	-0.00128	-0.00128	0.00000	3.00000×10^{-6}
0.3	-0.00567	-0.00567	0.00000	1.00000×10^{-6}
0.4	-0.01536	-0.01536	0.00000	0.00000×10^{-6}
0.5	-0.03125	-0.03125	0.00000	2.00000×10^{-6}
0.6	-0.05184	-0.05184	0.00000	1.00000×10^{-6}
0.7	-0.07203	-0.07203	0.00000	3.00000×10^{-6}
0.8	-0.08192	-0.08192	0.00000	2.00000×10^{-6}
0.9	-0.06561	-0.06561	0.00000	0.00000×10^{-6}
1.0	0.00000	0.00000	0.00000	0.00000×10^{-6}

Table 4. The numerical Absolute error of Example 6.3 for different values of α at $n = 2$.

t	Exact solution	Absolute error ($\alpha = 0.25$)	Absolute error ($\alpha = 0.1$)	Absolute error ($\alpha = 0.05$)
0.0	0.00000	1.04083×10^{-17}	3.46945×10^{-18}	1.73472×10^{-18}
0.2	-0.00218	7.80626×10^{-18}	1.30104×10^{-18}	2.58040×10^{-17}
0.4	-0.01536	8.67362×10^{-18}	2.77556×10^{-17}	3.12250×10^{-17}
0.6	-0.05184	2.08167×10^{-17}	8.32667×10^{-17}	9.71445×10^{-17}
0.8	-0.08192	5.55112×10^{-17}	6.93889×10^{-17}	8.32667×10^{-17}
1.0	0.00000	1.56125×10^{-16}	2.25514×10^{-16}	2.23779×10^{-16}

Table 5. The numerical results of Example 6.3 with $\alpha = 0.25$ and $n = 5$ for different families of polynomials.

t	Exact solution	SL solution	FSC solution	SSC solution
0.0	0.00000	1.73472^{-17}	3.46945^{-18}	1.73472^{-17}
0.2	-0.00128	9.97466^{-18}	7.58942^{-18}	1.51788^{-18}
0.4	-0.01536	1.04083^{-17}	2.08167^{-17}	3.46945^{-17}
0.6	-0.05184	4.16334^{-17}	6.93889^{-17}	1.38778^{-16}
0.8	-0.08192	5.55112^{-17}	8.32667^{-17}	2.63678^{-16}
1.0	0.00000	7.97973^{-17}	1.90820^{-16}	5.44703^{-16}

Example 6.4. Consider the linear boundary value problem [36]:

$$(10 + t)^2 \, {}^{C}\mathcal{D}_{0+}^{\frac{5}{2}} u(t) + \frac{5}{2}(10 + t)\, {}^{C}\mathcal{D}_{0+}^{\frac{3}{2}} u(t) + \frac{1}{2}\, {}^{C}\mathcal{D}_{0+}^{\frac{1}{2}} u(t) = \frac{t^{\frac{3}{2}}}{100\sqrt{\pi}}, \quad t \in [0, \rho = 1], \tag{6.9}$$

governed by the boundary conditions $u(0) = \ln(10), u(1) = \ln(11)$, with the exact solution $u(t) = \ln(10 + t)$. By applying the proposed method for $r = -7$, $s = 4$ and $p = -q = -1$. Figure 4(a), obtained The graph of exact and numerical solutions at $n = 12$, whereas, Figure 4(b) gives the absolute error for different values of n, which reflects the convergence of the proposed method by increasing the number of basis. Also, Table 6 gives The numerical results of the present example at $n = 8, 10$ and 12.

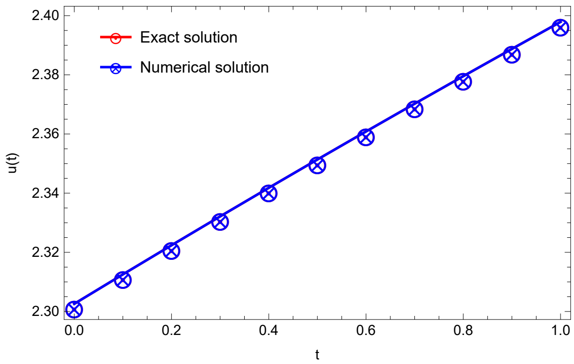
In case that we want to provide the approximate solution in terms of different families of polynomials. Figure 5 shows the absolute error of the approximate solution in terms of the SL, SFC, SSC, and the monic shifted sixth-kind Chebyshev polynomials (MSSC) obtained as a special case of $\mathcal{G}_{i,\rho}^{(r,s,p,q)}(t)$. It is clear that the Chebyshev family has similar performance in this problem. Finally, Table 7 gives the absolute error of different polynomials (SL solution, SFC solution, SSC solution, and MSSC solution) at $n = 8$.

Table 6. The numerical results of Example 6.4 for $n = 8, 10$ and 12.

t	Exact solution	Present method	Absolute error ($n = 8$)	Absolute error ($n = 10$)	Absolute error ($n = 12$)
0.0	2.30259	2.30259	8.88178×10^{-16}	8.88178×10^{-16}	4.44089×10^{-16}
0.1	2.31254	2.31254	2.22045×10^{-15}	4.44089×10^{-16}	0.00000×10^{-16}
0.2	2.32239	2.32239	1.33227×10^{-15}	4.44089×10^{-16}	0.00000×10^{-16}
0.3	2.33214	2.33214	3.10862×10^{-15}	4.44089×10^{-16}	0.00000×10^{-16}
0.4	2.34181	2.34181	2.66454×10^{-15}	4.44089×10^{-16}	0.00000×10^{-16}
0.5	2.35138	2.35138	2.66454×10^{-15}	4.44089×10^{-16}	0.00000×10^{-16}
0.6	2.36085	2.36085	5.32907×10^{-15}	4.44089×10^{-16}	0.00000×10^{-16}
0.7	2.37024	2.37024	8.88178×10^{-16}	4.44089×10^{-16}	0.00000×10^{-16}
0.8	2.37955	2.37955	3.10862×10^{-15}	8.88178×10^{-16}	4.44089×10^{-16}
0.9	2.38876	2.38876	4.44089×10^{-16}	4.44089×10^{-16}	0.00000×10^{-16}
1.0	2.39789	2.39789	0.00000×10^{-16}	1.33227×10^{-15}	0.00000×10^{-16}

Table 7. The absolute error in Example 6.4 of different polynomials for $n = 8$.

t	SL Absolute Error	SFC Absolute Error	SSC Absolute Error	MSSC Absolute Error
0.0	0.00000^{-16}	8.88178^{-16}	4.44089^{-16}	4.44089^{-16}
0.2	4.44089^{-16}	1.33227^{-16}	0.00000^{-16}	0.00000^{-16}
0.4	3.55271^{-15}	2.66454^{-15}	3.99680^{-15}	3.99680^{-15}
0.6	4.44089^{-15}	5.32907^{-15}	3.99680^{-15}	4.44089^{-15}
0.8	3.99680^{-15}	3.10862^{-15}	3.99680^{-15}	3.99680^{-15}
1.0	4.44089^{-16}	8.88178^{-16}	4.44089^{-16}	0.00000^{-16}



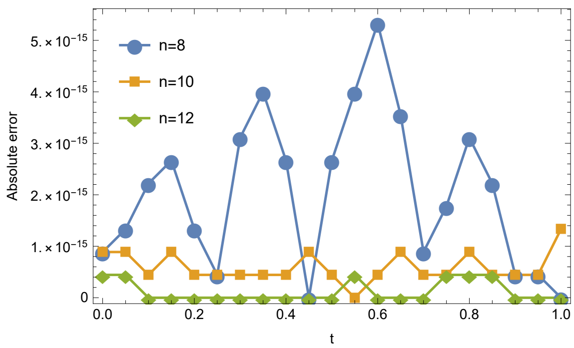


Figure 4. (a) The graph of exact and numerical solutions, (b) the absolute error of the present method for $n = 8, 10$ and 12.

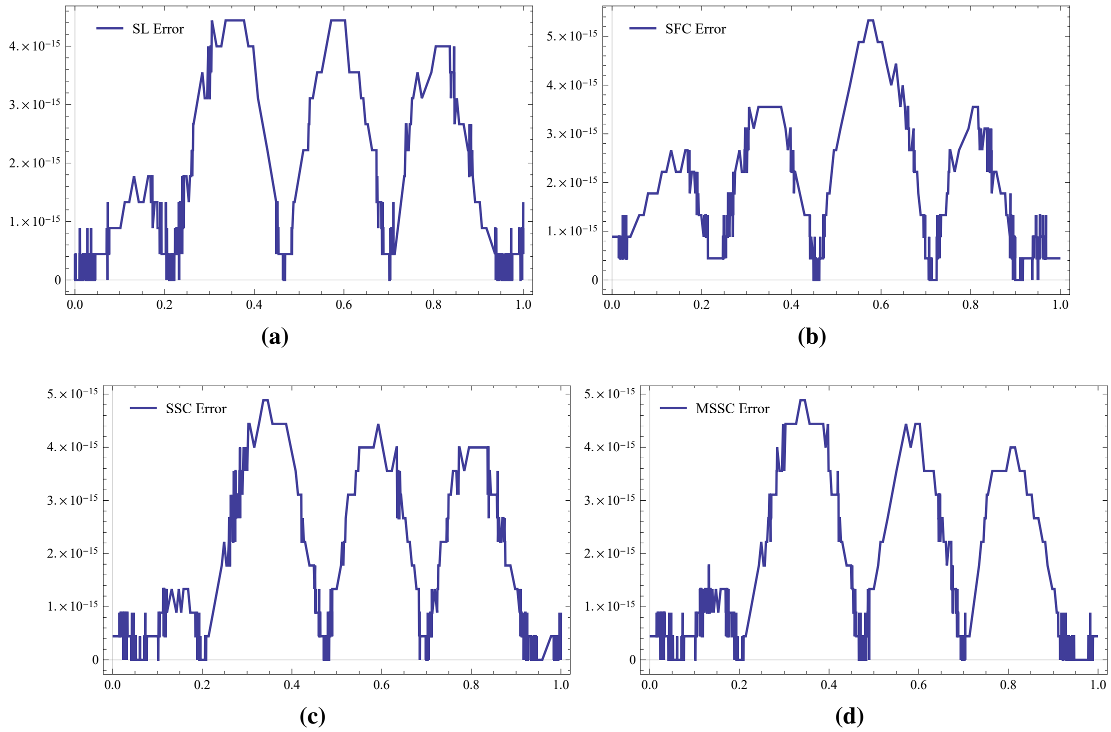


Figure 5. Graph of the absolute error in Example 6.4 of different families of polynomials for $n = 8$, where, (a) SL solution, (b) SFC solution, (c) SSC solution, (d) MSSC solution.

7. Conclusions

Through the discussion in this paper, the authors have introduced a novel generalized shifted symmetric orthogonal basis which was used later for obtaining a general operational matrix for the left-sided Caputo fractional derivative. This class of polynomials presumably used for the first time as trial functions with spectral methods in a numerical scheme concerning the solution of fractional differential equations (FDEs). The obtained operational matrix with four parameters can be used for expressing the operational matrix with respect to other families without needing additional calculations. For the sake of demonstration, the proposed basis were used for solving multiple FDEs showing its accuracy and applicability along with comparing it with other different techniques. The advantage of these orthogonal polynomials is that it include numerous choices for the basis of solution with different domains of definitions as well as generating a new orthogonal basis. This makes the task of choosing the most appropriate basis for the problem under study much easier by comparing the error for different four parameters configurations.

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Conflict of interest

The authors declare that they have no competing interests.

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