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Research article

Rough fractional integral and its multilinear commutators on p-adic generalized Morrey spaces

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Abstract: In this paper, we establish the boundedness of rough p-adic fractional integral operators on p-adic generalized Morrey spaces, as well as the boundedness of multilinear commutators generated by rough p-adic fractional integral operator and p-adic generalized Campanato functions. Moreover, the boundedness in classical Morrey is given as corollaries.

Keywords: rough p-adic fractional integral; multilinear commutator; p-adic generalized Morrey spaces; p-adic generalized Campanato function
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1. Introduction

In the past few decades, p-adic analysis has gathered a lot of attention by its applications in many aspects of mathematical physics, such as quantum mechanics, the probability theory and the dynamical systems [1]. Significantly, the geometry of the field of p-adic numbers is surprisingly unlike the geometry of the real numbers field $\mathbb{R}$, in particular the Archimedean axiom is not true in the field of p-adic numbers [2]. Therefore, p-adic analysis has also gained impeccable attraction in harmonic analysis [3–7].

In p-adic harmonic analysis, fractional calculus is a key area because of its heap of applications in engineering science and technology, see for instance [8, 9]. Also, fractional integral operators (Riesz potentials) are significant in the mathematical analysis as they construct and formulate inequalities which have several applications in scientific areas that can be found in the existing literature [10, 11]. The boundedness criteria of fractional integral operators on different functional spaces is a key area not only in harmonic analysis but also in partial differential equations, differentiation theory and potential theory [12, 13]. In this connection, the fractional integral operator in p-adic analysis is defined by

$$T_\beta^p f(\mathbf{x}) = \frac{1-p^{-\beta}}{1-p^{\beta-n}} \int_{\mathbb{Q}_p^n} \frac{f(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|_p^{n-\beta}} d\mathbf{y}, \quad 0 < \beta < n.$$

Here, $\mathbb{Q}_p^n$ consists of all points $\mathbf{x} = (x_1, \cdots, x_n)$ for $n \in \mathbb{N}$, where $x_j \in \mathbb{Q}_p (j = 1, \cdots, n)$ and $\mathbb{Q}_p$ is the field of p-adic numbers.

When $n = 1$, Haran [3, 4] not only obtained the explicit formula of the fractional integral operator T_β^p on $\mathbb{Q}_p$ but also developed the analytical potential theory on $\mathbb{Q}_p^n$. Taibleson [2] gave the fundamental analytic properties of T_β^p on local fields, including $\mathbb{Q}_p^n$, as well as the classical Hardy-Littlewood-Sobolev theorem (also see [5]). Moreover, Volosivets [6, 7] showed that T_β^p is bounded on radial Morrey spaces. In 2015, Wu and Fu [14] established Hardy-Littlewood-Sobolev inequalities on p-adic central Morrey spaces and λ-central BMO estimates for commutators of T_β^p. In 2018, Mo et al. [15] showed the boundedness of T_β^p on p-adic generalized Morrey spaces, as well as the boundedness of multilinear commutators generated by T_β^p and generalized Campanato functions. In 2022, both Shi et al. [16] and Sarfraz et al. [17] studied the boundedness of T_β^p and its commutators on Morrey-Herz spaces. At the same time, Sarfraz and Jarad [18] considered the roughness of the operator T_β^p, they introduced rough fractional integral operator $T_{\beta,\Omega}^p$. In the form, $T_{\beta,\Omega}^p$ has the following integral expression

$$T_{\beta,\Omega}^p f(\mathbf{x}) = \int_{\mathbb{Q}_p^n} \frac{\Omega(|\mathbf{y}|_p \mathbf{y}) f(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|_p^{n-\beta}} d\mathbf{y}, \tag{1.1}$$

for suitable measurable mappings $f : \mathbb{Q}_p^n \to \mathbb{R}$ and $\Omega : S_0(\mathbf{0}) \to \mathbb{R}$. When $f \in L^q(\mathbb{Q}_p^n)$, $1 \le q < \infty$, by the same way in the book [2], Sarfraz and Jarad [18] showed the boundedness of $T_{\beta,\Omega}^p$ on Lebesgue spaces (see Lemma 2.2 in Section 2). Furthermore, they obtained the boundedness of $T_{\beta,\Omega}^p$ on p-adic central Morrey spaces, as well as the λ-central BMO estimates for commutator $T_{\beta,\Omega}^{p,b}$ defined by

$$T_{\beta,\Omega}^{p,b} f(\mathbf{x}) = \int_{\mathbb{Q}_p^n} \frac{(b(\mathbf{x}) - b(\mathbf{y}))\Omega(|\mathbf{y}|_p \mathbf{y}) f(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|_p^{n-\beta}} d\mathbf{y}.$$

In [19], Sarfraz and Aslam showed the boundedness of $T_{\beta,\Omega}^p$ and $T_{\beta,\Omega}^{p,b}$ on p-adic Herz spaces.

We observe above works, the boundedness of $T_{\beta,\Omega}^p$ on generalized Morrey spaces are remains open. Therefore, in this paper, we are going to devote to the boundedness of $T_{\beta,\Omega}^p$ on p-adic generalized Morrey spaces. Moreover, let $\mathbf{b} = (b_1, b_2, \cdots, b_m)$ with $b_i \in L_{\text{loc}}(\mathbb{Q}_p^n)$ for $1 \le i \le m$, $m \in \mathbb{N}$, we will consider multilinear commutator defined by

$$T_{\beta,\Omega}^{p,\mathbf{b}} f(\mathbf{x}) = \int_{\mathbb{Q}_p^n} \frac{\prod_{i=1}^m (b_i(\mathbf{x}) - b_i(\mathbf{y}))\, \Omega(|\mathbf{y}|_p \mathbf{y}) f(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|_p^{n-\beta}} d\mathbf{y},$$

and investigate the boundedness of $T_{\beta,\Omega}^{p,\mathbf{b}}$ on p-adic generalized Morrey spaces with symbols in Campanato spaces. It should be emphasized that our results are new and cover some existing results of T_β^p and $T_{\beta,\Omega}^p$.

Our paper is organized as follows. In Section 2, we present some notations and preliminaries. In Section 3, we present our main results. In Section 4, we will give the proof of main results. Throughout this paper, $q' = q/(q-1)$ for $1 < q < \infty$ and $q' = \infty$ when $q = 1$, the letter C will be used to denote various constants,.

2. Preliminaries

We begin this section with recalling some preliminaries of p-adic analysis pertaining to our work. For a prime number p, let $\mathbb{Q}_p$ be the field of p-adic numbers defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to non-Archimedean p-adic normal $|\cdot|_p$. This normal $|\cdot|_p$ is defined as follows: if $x = 0$, $|0|_p = 0$; if $x \neq 0$ is an arbitrary rational number with the unique representation $x = p^{\gamma}m/n$, where m, n are not divisible by p, $\gamma = \gamma(x) \in \mathbb{Z}$, then $|x|_p = p^{-\gamma}$. It's not hard to see that the norm satisfies the following properties:

(i)$|x|_p \geq 0$, $\forall x \in \mathbb{Q}_p$ and $|x|_p = 0 \Leftrightarrow x = 0$;

(ii) $|xy|_p = |x|_p|y|_p$, $\forall x, y \in \mathbb{Q}_p$;

(iii) $|x + y|_p \leq \max(|x|_p, |y|_p)$, $\forall x, y \in \mathbb{Q}_p$ and when $|x|_p \neq |y|_p$, we have $|x + y|_p = \max(|x|_p, |y|_p)$.

It is also well known that any non-zero p-adic number $x \in \mathbb{Q}_p$ can be uniquely represented in the canonical series

$$x = p^{\gamma}(x_0 + x_1 p + x_2 p^2 + \cdots), \tag{2.1}$$

where $\gamma = \gamma(x) \in \mathbb{Z}$, $x_k \in \{0, 1, \cdots, p-1\}$, $x_0 \neq 0$, $k = 0, 1, \cdots$. The series (2.1) converges in the p-adic norm because $|x_k p^k|_p = p^{-k}$.

The p-adic norm of $\mathbb{Q}_p^n = \mathbb{Q} \times \mathbb{Q} \times \cdots \times \mathbb{Q}$ is defined by

$$|\mathbf{x}|_p = \max_{1\leq j\leq n} |x_j|_p, \quad \mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{Q}_p^n. \tag{2.2}$$

Denote by

$$B_{\gamma}(\mathbf{a}) = \{\mathbf{x} \in \mathbb{Q}_p^n : |\mathbf{x} - \mathbf{a}|_p \leq p^{\gamma}\},$$

the ball of radius p^{γ} with center at $\mathbf{a} \in \mathbb{Q}_p^n$ and write $\mathbb{B} = \left\{B_{\gamma}(\mathbf{a}) : \mathbf{a} \in \mathbb{Q}_p^n, \gamma \in \mathbb{Z}\right\}$. If let

$$S_{\gamma}(\mathbf{a}) = B_{\gamma}(\mathbf{a}) \setminus B_{\gamma-1}(\mathbf{a}) = \{\mathbf{x} \in \mathbb{Q}_p^n : |\mathbf{x} - \mathbf{a}|_p = p^{\gamma}\},$$

the sphere of radius p^{γ} with center at $\mathbf{a} \in \mathbb{Q}_p^n$, it is easy to see that

$$B_{\gamma}(\mathbf{a}) = \bigcup_{k\leq\gamma} S_k(\mathbf{a}).$$

Since the space $\mathbb{Q}_p^n$ is a locally compact commutative group under addition, there exists the Haar measure dx on the additive group of $\mathbb{Q}_p^n$ normalized by $\int_{B_0} dx = |B_0| = 1$, where $B_0 := B_0(\mathbf{0})$ and $|E|$ denotes the Haar measure of a measurable set $E \subset \mathbb{Q}_p^n$. Then by a simple calculation, the Haar measures of any balls and spheres can be obtained. Especially, we frequently use

$$\left|B_{\gamma}(\mathbf{a})\right| = p^{n\gamma}, \quad \left|S_{\gamma}(\mathbf{a})\right| = p^{n\gamma}(1 - p^{-n}), \quad \forall \mathbf{a} \in \mathbb{Q}_p^n.$$

For a more complete introduction to the p-adic analysis, we refer the readers to [2] and the references therein.

Now, let us give the definitions of generalized Morrey spaces and generalized Campanato spaces on the p-adic number field as follows.

Definition 2.1. *[15] Let* $1 \le q < \infty$*,* $B_\gamma(\mathbf{a})$ *be a ball in* $\mathbb{Q}_p^n$ *and* $\omega(\mathbf{x})$ *be a non-negative measurable function in* $\mathbb{Q}_p^n$*. A function* $f \in L^q_{\mathrm{loc}}(\mathbb{Q}_p^n)$ *is said to belong to the generalized Morrey space* $L^{q,\omega}(\mathbb{Q}_p^n)$*, if*

$$\|f\|_{L^{q,\omega}(\mathbb{Q}_p^n)} = \sup_{\gamma\in\mathbb{Z},\mathbf{a}\in\mathbb{Q}_p^n} \frac{1}{\omega(B_\gamma(\mathbf{a}))}\left(\frac{1}{\left|B_\gamma(\mathbf{a})\right|_H}\int_B |f(\mathbf{y})|^q d\mathbf{y}\right)^{1/q} < \infty,$$

where $\omega(B_\gamma(\mathbf{a})) = \int_{B_\gamma(\mathbf{a})}\omega(\mathbf{x})d\mathbf{x}$.

Notice that if let $\omega(B_\gamma(\mathbf{a})) = |B_\gamma(\mathbf{a})|^\lambda$, then $L^{q,\omega}(\mathbb{Q}_p^n)$ is the classical Morrey spaces $M_p^\lambda(\mathbb{Q}_p^n)$. Moreover, if let $\lambda \in \mathbb{R}$ and $B_\gamma(\mathbf{a}) = B_\gamma(\mathbf{0})$, then $L^{q,\omega}(\mathbb{Q}_p^n)$ is the central Morrey spaces $\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$ (see [14, 18]) defined by

$$\|f\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)} = \sup_{\gamma\in\mathbb{Z}}\left(\frac{1}{\left|B_\gamma(\mathbf{0})\right|_H^{1+\lambda q}}\int_{B_\gamma(\mathbf{0})}|f(\mathbf{y})|^q d\mathbf{y}\right)^{1/q} < \infty.$$

Definition 2.2. *[15] Let* $1 \le q < \infty$*,* $B_\gamma(\mathbf{a})$ *be a ball in* $\mathbb{Q}_p^n$ *and* $\omega(\mathbf{x})$ *be a non-negative measurable function in* $\mathbb{Q}_p^n$*. A function* $f \in L^q_{\mathrm{loc}}(\mathbb{Q}_p^n)$ *is said to belong to the generalized Campanato space* $\mathcal{L}^{q,\omega}(\mathbb{Q}_p^n)$*, if*

$$\|f\|_{\mathcal{L}^{q,\omega}(\mathbb{Q}_p^n)} = \sup_{\gamma\in\mathbb{Z},\mathbf{a}\in\mathbb{Q}_p^n} \frac{1}{\omega(B_\gamma(\mathbf{a}))}\left(\frac{1}{\left|B_\gamma(\mathbf{a})\right|_H}\int_B \left|f(\mathbf{y}) - f_{B_\gamma(\mathbf{a})}\right|^q d\mathbf{y}\right)^{1/q} < \infty,$$

where $f_{B_\gamma(\mathbf{a})} = \frac{1}{|B_\gamma(\mathbf{a})|_H}\int_{B_\gamma(\mathbf{a})} f(\mathbf{x})d\mathbf{x}$.

We invoke the following result.

Lemma 2.1. *[15, 20] Let* $1 \le q < \infty$ *and* ω *be a non-negative measurable function. Suppose that* $b \in \mathcal{L}^{q,\omega}(\mathbb{Q}_p^n)$*, then*

$$\left|b_{B_k(\mathbf{a})} - b_{B_j(\mathbf{a})}\right| \le |k-j|\,\|b\|_{\mathcal{L}^{q,\omega}(\mathbb{Q}_p^n)} \max\left\{\omega\left(B_k(\mathbf{a})\right), \omega\left(B_j(\mathbf{a})\right)\right\}$$

for any $j, k \in \mathbb{Z}$ *and any fixed* $\mathbf{a} \in \mathbb{Q}_p^n$*. Thus, for* $j > k$*, we have*

$$\left(\int_{B_j(\mathbf{a})}\left|b(y) - b_{B_k(\mathbf{a})}\right|^q\right)^{1/q} \le (j+1-k)\left|B_j(\mathbf{a})\right|_H^{1/q}\omega\left(B_j(\mathbf{a})\right)\|b\|_{\mathcal{L}^{q,\omega}(\mathbb{Q}_p^n)}.$$

In addition, for $\lambda < 1/n$, if let $B_\gamma(\mathbf{a}) = |B_\gamma(\mathbf{a})|^\lambda$ in Definition 2.2, then $\mathcal{L}^{q,\omega}(\mathbb{Q}_p^n) = \mathrm{BMO}^{q,\lambda}(\mathbb{Q}_p^n)$. Moreover, let $B_\gamma(\mathbf{a}) = B_\gamma(\mathbf{0})$, then $\mathcal{L}^{q,\omega}(\mathbb{Q}_p^n)$ is the λ-central BMO space $\mathrm{CBMO}^{q,\lambda}(\mathbb{Q}_p^n)$ (see [14, 18]) defined by

$$\|f\|_{\mathrm{CBMO}^{q,\lambda}(\mathbb{Q}_p^n)} = \sup_{\gamma\in\mathbb{Z}}\left(\frac{1}{\left|B_\gamma(\mathbf{0})\right|_H^{1+\lambda q}}\int_{B_\gamma(\mathbf{0})}\left|f(\mathbf{y}) - f_{B_\gamma(\mathbf{0})}\right|^q d\mathbf{y}\right)^{1/q} < \infty.$$

Furthermore, when $\lambda = 0$, the particular case of $\mathrm{CBMO}^{q,\lambda}(\mathbb{Q}_p^n)$ is $\mathrm{CBMO}^{q}(\mathbb{Q}_p^n)$ defined in [21].

Now, we present two desired lemmas which will be used in the proof of our main results.

Lemma 2.2. *[18] Let* $0<\beta<n$, $1\le q<r<\infty$, $1/r=1/q-\beta/n$, $\Omega\in L^{q'}(S_0(\mathbf{0}))$ *and* $f\in L^q(\mathbb{Q}_p^n)$.

(i) If $q>1$, *then*

$$\|T^p_{\beta,\Omega}f\|_{L^r(\mathbb{Q}_p^n)}\le C\|f\|_{L^q(\mathbb{Q}_p^n)}.$$

(ii) If $q=1$, *for any* $\sigma>0$, *then*

$$\left|\left\{\mathbf{x}\in\mathbb{Q}_p^n : |T^p_{\beta,\Omega}f(\mathbf{x})|>\sigma\right\}\right|_H\le C\left(\frac{\|f\|_{L^1(\mathbb{Q}_p^n)}}{\sigma}\right)^r.$$

Lemma 2.3. *Let* $0<\beta<n$, $1\le q<r<\infty$, $1/r=1/q-\beta/n$, $\Omega\in L^{q'}(S_0(\mathbf{0}))$, $f\in L^{q,\nu}(\mathbb{Q}_p^n)$ *and* ν *is a non-negative measurable functionin* $\mathbb{Q}_p^n$. *For any* $B_\gamma(\mathbf{a})\in\mathbb{B}$, *then*

$$\int_{B^c_\gamma(\mathbf{a})}\frac{\left|\Omega(|\mathbf{y}|_p\mathbf{y})\right||f(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|_p^{n-\beta}}d\mathbf{y}\le C\|f\|_{L^{q,\nu}(\mathbb{Q}_p^n)}\sum_{k=\gamma+1}^{\infty}p^{k\beta}\nu(B_k(\mathbf{a})),\quad \mathbf{x}\in B_\gamma(\mathbf{a}).$$

Proof. For any $\mathbf{x}\in B_\gamma(\mathbf{a})$, we have $|\mathbf{x}-\mathbf{a}|_p\le p^\gamma$. For any $\mathbf{y}$ satisfying $p^{k-1}<|\mathbf{y}-\mathbf{a}|_p\le p^k$ for some $k\ge\gamma+1$, the property (ii) of $|\cdot|_p$ shows that $p^{k-1}<|\mathbf{y}-\mathbf{a}|_p\le\max(|\mathbf{x}-\mathbf{y}|_p,|\mathbf{x}-\mathbf{a}|_p)$, the inequality $|\mathbf{x}-\mathbf{a}|_p\le p^\gamma\le p^{k-1}$ guarantees that $|\mathbf{x}-\mathbf{y}|_p>p^{k-1}$. Consequently, by Hölder's inequality, we have

$$\begin{aligned}\int_{B^c_\gamma(\mathbf{a})}\frac{\left|\Omega(|\mathbf{y}|_p\mathbf{y})\right||f(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|_p^{n-\beta}}d\mathbf{y}&=\int_{|\mathbf{y}-\mathbf{a}|_p>p^\gamma}\frac{\left|\Omega(|\mathbf{y}|_p\mathbf{y})\right||f(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|_p^{n-\beta}}d\mathbf{y}\\
&\le\sum_{k=\gamma+1}^{\infty}\int_{p^k\ge|\mathbf{y}-\mathbf{a}|_p>p^{k-1}}\frac{\left|\Omega(|\mathbf{y}|_p\mathbf{y})\right||f(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|_p^{n-\beta}}d\mathbf{y}\\
&\le C\sum_{k=\gamma+1}^{\infty}p^{-k(n-\beta)}\int_{S_k(\mathbf{a})}\left|\Omega(|\mathbf{y}|_p\mathbf{y})f(\mathbf{y})\right|d\mathbf{y}\\
&\le C\sum_{k=\gamma+1}^{\infty}p^{-k(n-\beta)}\left(\int_{S_k(\mathbf{a})}\left|\Omega(|\mathbf{y}|_p\mathbf{y})\right|^{q'}d\mathbf{y}\right)^{1/q'}\left(\int_{S_k(\mathbf{a})}|f(\mathbf{y})|^q\,d\mathbf{y}\right)^{1/q}\\
&\le C\sum_{k=\gamma+1}^{\infty}p^{-k(n-\beta)}\left(\int_{S_k(\mathbf{a})}\left|\Omega(|\mathbf{y}|_p\mathbf{y})\right|^{q'}d\mathbf{y}\right)^{1/q'}\left(\int_{B_k(\mathbf{a})}|f(\mathbf{y})|^q\,d\mathbf{y}\right)^{1/q}.\end{aligned}$$

Let nonzero $\mathbf{y}\in\mathbb{Q}_p^n$ has a form $\mathbf{y}=(y_1,\cdots,y_n)$, applying (2.1), we proceed as

$$y_i=p^{\gamma_i}(\alpha_{0,i}+\alpha_{1,i}p+\alpha_{2,i}p^2+\cdots),\quad i=1,\cdots,n.$$

Then there exists $i_0\in\{1,\cdots,n\}$ such that $|y_{i_0}|_p=p^{-\gamma_{i_0}}\ge p^{-\gamma_i}=|y_i|_p$, whenever $y_i\ne0$. Using (2.2), we obtain $|\mathbf{y}|_p=p^{-\gamma_{i_0}}$. It follows that

$$\left||\mathbf{y}|_p\mathbf{y}\right|_p=\left|p^{-\gamma_{i_0}}\mathbf{y}\right|_p=\max_{1\le i\le n,y_i\ne0}p^{\gamma_{i_0}-\gamma_i}=p^{\gamma_{i_0}-\gamma_{i_0}}=1.$$

Thus, for every nonzero $\mathbf{y}\in\mathbb{Q}_p^n$, the vector $|\mathbf{y}|_p\mathbf{y}$ belongs to sphere $S_0(\mathbf{0})=\left\{\mathbf{y}\in\mathbb{Q}_p^n:|\mathbf{y}|_p=1\right\}$. Notice that $\Omega\in L^{q'}(S_0(\mathbf{0}))$, then

$$\int_{S_k}\left|\Omega(|\mathbf{y}|_p\mathbf{y})\right|^{q'}d\mathbf{y}=\int_{|\mathbf{x}|_p=1}|\Omega(\mathbf{z})|^{q'}p^{kn}d\mathbf{z}\le Cp^{kn}.$$

Hence, we obtain

$$\begin{aligned}\int_{B_\gamma^c(\mathbf{a})}\frac{\left|\Omega(|\mathbf{y}|_p\mathbf{y})\right||f(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|_p^{n-\beta}}d\mathbf{y}&\le C\sum_{k=\gamma+1}^{\infty}p^{-k(n-\beta)+kn/q'}\left(\int_{B_k(\mathbf{a})}|f(\mathbf{y})|^q\,d\mathbf{y}\right)^{1/q}\\&\le C\sum_{k=\gamma+1}^{\infty}p^{-k(n-\beta)+kn/q'}\nu(B_k(\mathbf{a}))|B_k(\mathbf{a})|_H^{1/q}\|f\|_{L^{q,\nu}(\mathbb{Q}_p^n)}\\&\le C\|f\|_{L^{q,\nu}(\mathbb{Q}_p^n)}\sum_{k=\gamma+1}^{\infty}p^{k\beta}\nu(B_k(\mathbf{a})).\end{aligned}$$

Lemma 2.3 is proved.

3. Main results

Before giving the main results in this paper, according to the idea of the article [22], we will first state how to define the action of $T^p_{\beta,\Omega}$ on generalized Morrey spaces.

Definition 3.1. *Let $0<\beta<n$, $1\le q<\infty$, $\Omega\in L^{q'}(S_0(\mathbf{0}))$, $T^p_{\beta,\Omega}$ be a fractional integral operator defined by (1.1), ν be a non-negative measurable function such that*

$$\sup_{\gamma\in\mathbb{Z},\mathbf{a}\in\mathbb{Q}_p^n}\sum_{k=\gamma+1}^{\infty}p^{k\beta}\nu(B_k(\mathbf{a}))<\infty. \tag{3.1}$$

For any $f\in L^{q,\nu}(\mathbb{Q}_p^n)$ and any fixed $B_\gamma(\mathbf{a})\in\mathbb{B}$, define

$$T^p_{\beta,\Omega}f(\mathbf{x})=T^p_{\beta,\Omega}(f\chi_{B_\gamma(\mathbf{a})})(\mathbf{x})+T^p_{\beta,\Omega}(f\chi_{B^c_\gamma(\mathbf{a})})(\mathbf{x}),\quad \mathbf{x}\in B_\gamma(\mathbf{a}).$$

Remark 3.1. For any $B_\gamma(\mathbf{a})\in\mathbb{B}$ and $f\in L^{q,\nu}(\mathbb{Q}_p^n)$, write $f=f\chi_{B_\gamma(\mathbf{a})}+f\chi_{B^c_\gamma(\mathbf{a})}$. We can see that the definition of $L^{q,\nu}(\mathbb{Q}_p^n)$ assures that $f\chi_{B_\gamma(\mathbf{a})}\in L^q(\mathbb{Q}_p^n)$, so Lemma 2.3 guarantees that $T^p_{\beta,\Omega}(f\chi_{B_\gamma(\mathbf{a})})$ is well defined. Besides that, if ν satisfies (3.1), Lemma 2.3 implies

$$\int_{B_\gamma^c(\mathbf{a})}\frac{\left|\Omega(|\mathbf{y}|_p\mathbf{y})\right||f(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|_p^{n-\beta}}d\mathbf{y}<\infty$$

for $\mathbf{x}\in B_\gamma(\mathbf{a})$. That is, $T^p_{\beta,\Omega}(f\chi_{B^c_\gamma(\mathbf{a})})$ is well defined when ν satisfies (3.1). Consequently, the linearity of $T^p_{\beta,\Omega}$ on $L^q(\mathbb{Q}_p^n)$ yields

$$T^p_{\beta,\Omega}(f\chi_{B_\gamma(\mathbf{a})})(\mathbf{x})+T^p_{\beta,\Omega}(f\chi_{B^c_\gamma(\mathbf{a})})(\mathbf{x})=T^p_{\beta,\Omega}f(\mathbf{x})$$

for $f\in L^{q,\nu}(\mathbb{Q}_p^n)$ and $\mathbf{x}\in B_\gamma(\mathbf{a})\in\mathbb{B}$.

Now we give the boundedness result of $T^p_{\beta,\Omega}$ on generalized Morrey spaces in the following.

Theorem 3.1. *Let $0<\beta<n$, $1\le q<r<\infty$, $1/r=1/q-\beta/n$, $\Omega\in L^{q'}(S_0(\mathbf{0}))$ and $f\in L^{q,\nu}(\mathbb{Q}_p^n)$. Suppose that ω and ν are non-negative measurable functions such that*

$$\sup_{\gamma\in\mathbb{Z},\mathbf{a}\in\mathbb{Q}_p^n}\sum_{k=\gamma}^{\infty}\frac{\nu(B_k(\mathbf{a}))}{\omega(B_\gamma(\mathbf{a}))}p^{k\beta}<\infty. \tag{3.2}$$

(i) If $q > 1$, then

$$\|T^p_{\beta,\Omega}f\|_{L^{r,\omega}(\mathbb{Q}_p^n)} \leq C\|f\|_{L^{q,\nu}(\mathbb{Q}_p^n)}.$$

(ii) If $q = 1$, for any $\sigma > 0$, $\gamma \in \mathbb{Z}$ and $\mathbf{a} \in \mathbb{Q}_p^n$, then

$$\frac{\left|\left\{\mathbf{x} \in B_\gamma(\mathbf{a}) : |T^p_{\beta,\Omega}f(\mathbf{x})| > \sigma\right\}\right|_H}{\omega(B_\gamma(\mathbf{a}))^r \left|B_\gamma(\mathbf{a})\right|_H} \leq C\left(\frac{\|f\|_{L^{1,\nu}(\mathbb{Q}_p^n)}}{\sigma}\right)^r.$$

Remark 3.2. Notice that ν satisfy (3.1) if ω and ν satisfy (3.2). That is, (3.2) assures that $T^p_{\beta,\Omega}$ is well defined on $L^{q,\nu}(\mathbb{Q}_p^n)$.

Significantly, our results not only extend Theorem 1 in [18] to generalized Morrey spaces, but also extend Theorem 3.1 in [15] to rough p-adic fractional integral operator. At the same time, for $\lambda < -\beta/n$ and $\mu = \lambda + \beta/n$, if we take $\omega(B_\gamma(\mathbf{a})) = |B_\gamma(\mathbf{a})|_H^\mu$, $\nu(B_\gamma(\mathbf{a})) = |B_\gamma(\mathbf{a})|_H^\lambda$ for any fixed $B_\gamma(\mathbf{a})$, it is easy to check that ω and ν satisfy (3.2). Hence, by Theorem 3.1, we can obtain the following corollary.

Corollary 3.1. *Let $0 < \beta < n$, $1 \leq q < r < \infty$, $1/r = 1/q-\beta/n$, $\lambda+\beta/n < 0$, $\mu = \lambda+\beta/n$, $\Omega \in L^{q'}(S_0(\mathbf{0}))$ and $f \in M_q^\lambda(\mathbb{Q}_p^n)$.*

(i)If $q > 1$, then

$$\|T^p_{\beta,\Omega}f\|_{M_r^\mu(\mathbb{Q}_p^n)} \leq C\|f\|_{M_q^\lambda(\mathbb{Q}_p^n)}.$$

(ii)If $q = 1$, for any $\sigma > 0$, $\gamma \in \mathbb{Z}$ and $\mathbf{a} \in \mathbb{Q}_p^n$, then

$$\frac{\left|\left\{\mathbf{x} \in B_\gamma(\mathbf{a}) : |T^p_{\beta,\Omega}f(\mathbf{x})| > \sigma\right\}\right|_H}{\left|B_\gamma(\mathbf{a})\right|_H^{1+r\mu}} \leq C\left(\frac{\|f\|_{M_1^\lambda(\mathbb{Q}_p^n)}}{\sigma}\right)^r.$$

Our second main result is the boundedness of multilinear commutators generated by rough p-adic fractional integral operator and p-adic generalized Campanato functions.

Theorem 3.2. *Let $m \in \mathbb{N}$, $0 < \beta < n$, $1 < q, q_1, \cdots, q_m < \infty$, $r > n/(n-\beta)$, $1/r = 1/q_1 + \cdots + 1/q_m + 1/q - \beta/n$ and $\Omega \in L^{q'}(S_0(\mathbf{0}))$. Suppose that ω, ν and ν_i $(i = 1, 2, \cdots, m)$ are non-negative measurable functions, and satisfy*

$$\sup_{\gamma\in\mathbb{Z},\mathbf{a}\in\mathbb{Q}_p^n} \frac{\nu(B_\gamma(\mathbf{a}))\prod_{i=1}^m \nu_i(B_\gamma(\mathbf{a}))}{\omega(B_\gamma(\mathbf{a}))} p^{\gamma\beta} < \infty \tag{3.3}$$

and

$$\sup_{\gamma\in\mathbb{Z},\mathbf{a}\in\mathbb{Q}_p^n} \sum_{k=\gamma+1}^{\infty} \frac{\nu(B_k(\mathbf{a}))\prod_{i=1}^m \nu_i(B_k(\mathbf{a}))}{\omega(B_\gamma(\mathbf{a}))}(k-\gamma+1)^m p^{\gamma\beta} < \infty, \tag{3.4}$$

for any $\gamma \in \mathbb{Z}$ and $\mathbf{a} \in \mathbb{Q}_p^n$. If $b_i \in \mathcal{L}^{q_i,\nu_i}(\mathbb{Q}_p^n)$, $f \in L^{q,\nu}(\mathbb{Q}_p^n)$, then

$$\|T^{p,\mathbf{b}}_{\beta,\Omega}f\|_{L^{r,\omega}(\mathbb{Q}_p^n)} \leq C\prod_{i=1}^m \|b_i\|_{\mathcal{L}^{q_i,\nu_i}(\mathbb{Q}_p^n)}\|f\|_{L^{q,\nu}(\mathbb{Q}_p^n)}.$$

For $0 \leq \lambda_1, \lambda_2, \cdots, \lambda_m < 1/n$, $\lambda+\Sigma\lambda_i+\beta/n < 0$ and $\mu = \lambda+\Sigma\lambda_i+\beta/n$, if we take $\omega(B_\gamma(\mathbf{a})) = |B_\gamma(\mathbf{a})|_H^\mu$, $\nu(B_\gamma(\mathbf{a})) = |B_\gamma(\mathbf{a})|_H^\lambda$ and $\nu_i(B_\gamma(\mathbf{a})) = |B_\gamma(\mathbf{a})|_H^{\lambda_i}$ for any $B_\gamma(\mathbf{a})$, it is not difficult to check that ω, ν and ν_i satisfy (3.3) and (3.4). Hence, Theorem 3.2 implies the following corollary.

Corollary 3.2. *Let $m \in \mathbb{N}$, $0 < \beta < n$, $1 < q, q_1, \cdots, q_m < \infty$, $r > n/(n-\beta)$, $1/r = 1/q_1 + \cdots + 1/q_m + 1/q - \beta/n$ and $\Omega \in L^{q'}(S_0(\mathbf{0}))$. Suppose that $0 \le \lambda_1, \lambda_2, \cdots, \lambda_m < 1/n$, $\lambda + \Sigma\lambda_i + \beta/n < 0$, $\mu = \lambda + \Sigma\lambda_i + \beta/n$. If $b_i \in \mathrm{BMO}^{q_i,\lambda_i}(\mathbb{Q}_p^n)$, $f \in M_q^{\lambda}(\mathbb{Q}_p^n)$, then*

$$\|T_{\beta,\Omega}^{p,\mathbf{b}} f\|_{M_r^{\mu}(\mathbb{Q}_p^n)} \le C \prod_{i=1}^{m} \|b_i\|_{\mathrm{BMO}^{q_i,\lambda_i}(\mathbb{Q}_p^n)} \|f\|_{M_q^{\lambda}(\mathbb{Q}_p^n)}.$$

If we let $b_i \in \mathrm{CBMO}^{q_i,\lambda_i}(\mathbb{Q}_p^n)$, by Corollary 3.2, we will obtain the follwing boundedness of $T_{\beta,\Omega}^{p,\mathbf{b}}$ on central Morrey spaces.

Corollary 3.3. *Let $m \in \mathbb{N}$, $0 < \beta < n$, $1 < q, q_1, \cdots, q_m < \infty$, $r > n/(n-\beta)$, $1/r = 1/q_1 + \cdots + 1/q_m + 1/q - \beta/n$ and $\Omega \in L^{q'}(S_0(\mathbf{0}))$. Suppose that $0 \le \lambda_1, \lambda_2, \cdots, \lambda_m < 1/n$, $\lambda + \Sigma\lambda_i + \beta/n < 0$, $\mu = \lambda + \Sigma\lambda_i + \beta/n$. If $b_i \in \mathrm{CBMO}^{q_i,\lambda_i}(\mathbb{Q}_p^n)$, $f \in \dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$, then*

$$\|T_{\beta,\Omega}^{p,\mathbf{b}} f\|_{\dot{B}^{r,\mu}(\mathbb{Q}_p^n)} \le C \prod_{i=1}^{m} \|b_i\|_{\mathrm{CBMO}^{q_i,\lambda_i}(\mathbb{Q}_p^n)} \|f\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)}.$$

Here we point out that Corollary 3.3 extends Theorem 2 in [18] to the multilinear case.

4. Proof of theorems

The proof of Theorem 3.1. As non-negative measurable functions ω and ν satisfy (3.2), ν fulfills (3.1), so Definition 3.1 assures that $T_{\beta,\Omega}^{p}$ is well defined on $L^{q,\nu}(\mathbb{Q}_p^n)$. For $f \in L^{q,\nu}(\mathbb{Q}_p^n)$ and any fixed $B_\gamma(\mathbf{a}) \in \mathbb{B}$, it follows that

$$T_{\beta,\Omega}^{p} f(\mathbf{x}) = T_{\beta,\Omega}^{p}(f\chi_{B_\gamma(\mathbf{a})})(\mathbf{x}) + T_{\beta,\Omega}^{p}(f\chi_{B_\gamma^c(\mathbf{a})})(\mathbf{x}), \quad \mathbf{x} \in B_\gamma(\mathbf{a}).$$

Consequently, we only need to estimate $T_{\beta,\Omega}^{p}(f\chi_{B_\gamma(\mathbf{a})})(\mathbf{x})$ and $T_{\beta,\Omega}^{p}(f\chi_{B_\gamma^c(\mathbf{a})})(\mathbf{x})$ respectively.

(i) If $q > 1$, for fixed $B_\gamma(\mathbf{a})$, we have

$$\begin{aligned}
\frac{1}{\omega(B_\gamma(\mathbf{a}))}\left(\frac{1}{|B_\gamma(\mathbf{a})|_H}\int_{B_\gamma(\mathbf{a})} |T_{\beta,\Omega}^{p} f(\mathbf{x})|^r d\mathbf{x}\right)^{1/r} &\le \frac{1}{\omega(B_\gamma(\mathbf{a}))}\left(\frac{1}{|B_\gamma(\mathbf{a})|_H}\int_{B_\gamma(\mathbf{a})} |T_{\beta,\Omega}^{p}(f\chi_{B_\gamma(\mathbf{a})})(\mathbf{x})|^r d\mathbf{x}\right)^{1/r} \\
&\quad + \frac{1}{\omega(B_\gamma(\mathbf{a}))}\left(\frac{1}{|B_\gamma(\mathbf{a})|_H}\int_{B_\gamma(\mathbf{a})} |T_{\beta,\Omega}^{p}(f_{B_\gamma^c(\mathbf{a})})(\mathbf{x})|^r d\mathbf{x}\right)^{1/r} \\
&=: I + II.
\end{aligned}$$

For I, by the L^r-boundedness of $T_{\beta,\Omega}^{p}$ (see Lemma 2.2) and (3.2), it follows that

$$\begin{aligned}
I &= \frac{1}{\omega(B_\gamma(\mathbf{a}))}\left(\frac{1}{|B_\gamma(\mathbf{a})|_H}\int_{B_\gamma(\mathbf{a})} |T_{\beta,\Omega}^{p}(f\chi_{B_\gamma(\mathbf{a})})(\mathbf{x})|^r d\mathbf{x}\right)^{1/r} \\
&\le \frac{1}{\omega(B_\gamma(\mathbf{a}))}\frac{1}{|B_\gamma(\mathbf{a})|_H^{1/r}}\left(\int_{B_\gamma(\mathbf{a})} |f(\mathbf{x})|^q d\mathbf{x}\right)^{1/q}
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{\nu(B_\gamma(\mathbf{a}))}{\omega(B_\gamma(\mathbf{a}))} \frac{\left|B_\gamma(\mathbf{a})\right|_H^{1/q}}{\left|B_\gamma(\mathbf{a})\right|_H^{1/r}} \frac{1}{\nu(B_\gamma(\mathbf{a}))} \left(\frac{1}{\left|B_\gamma(\mathbf{a})\right|_H} \int_{B_\gamma(\mathbf{a})} |f(\mathbf{x})|^q d\mathbf{x} \right)^{1/q} \\
&\le \frac{\nu(B_\gamma(\mathbf{a}))}{\omega(B_\gamma(\mathbf{a}))} p^{\gamma\beta} \|f\|_{L^{q,\nu}(\mathbb{Q}_p^n)} \\
&\le \|f\|_{L^{q,\nu}(\mathbb{Q}_p^n)}.
\end{aligned}
$$

For II, Lemma 2.3 yields that

$$
\begin{aligned}
II &= \frac{1}{\omega(B_\gamma(\mathbf{a}))} \left(\frac{1}{\left|B_\gamma(\mathbf{a})\right|_H} \int_{B_\gamma(\mathbf{a})} |T^p_{\beta,\Omega}(f_{B^c_\gamma(\mathbf{a})})(\mathbf{x})|^r d\mathbf{x} \right)^{1/r} \\
&\le \|f\|_{L^{q,\nu}(\mathbb{Q}_p^n)} \sum_{k=\gamma+1}^{\infty} p^{k\beta} \frac{\nu(B_k(\mathbf{a}))}{\omega(B_\gamma(\mathbf{a}))} \\
&\le \|f\|_{L^{q,\nu}(\mathbb{Q}_p^n)}.
\end{aligned}
$$

Consequently, combining the estimation of I and II, we have

$$
\frac{1}{\omega(B_\gamma(\mathbf{a}))} \left(\frac{1}{\left|B_\gamma(\mathbf{a})\right|_H} \int_{B_\gamma(\mathbf{a})} |T^p_{\beta,\Omega} f(\mathbf{x})|^r d\mathbf{x} \right)^{1/r} \le \|f\|_{L^{q,\nu}(\mathbb{Q}_p^n)},
$$

which implies the desired inequality $\|T^p_{\beta,\Omega} f\|_{L^{r,\omega}(\mathbb{Q}_p^n)} \le C\|f\|_{L^{q,\nu}(\mathbb{Q}_p^n)}$.

(ii) If $q = 1$, for fixed $B_\gamma(\mathbf{a})$ and $T^p_{\beta,\Omega}(f\chi_{B_\gamma(\mathbf{a})})(\mathbf{x})$, by the weak L^r-boundedness of $T^p_{\beta,\Omega}$ (see Lemma 2.2) and (3.2), we get

$$
\begin{aligned}
&\left|\left\{\mathbf{x} \in B_\gamma(\mathbf{a}) : |T^p_{\beta,\Omega}(f\chi_{B_\gamma(\mathbf{a})})(\mathbf{x})| > \sigma/2\right\}\right|_H \\
&\le C \left(\frac{2\|f\chi_{B_\gamma(\mathbf{a})}\|_{L^1(\mathbb{Q}_p^n)}}{\sigma} \right)^r \\
&= \frac{C}{\sigma^r} \omega(B_\gamma(\mathbf{a}))^r \left|B_\gamma(\mathbf{a})\right|_H \left(\frac{\nu(B_\gamma(\mathbf{a}))}{\omega(B_\gamma(\mathbf{a}))} \frac{\left|B_\gamma(\mathbf{a})\right|_H}{\left|B_\gamma(\mathbf{a})\right|_H^{1/r}} \right)^r \left(\frac{1}{\nu(B_\gamma(\mathbf{a}))} \frac{1}{\left|B_\gamma(\mathbf{a})\right|_H} \int_{B_\gamma(\mathbf{a})} |f(\mathbf{x})| d\mathbf{x} \right)^r \\
&\le C\omega(B_\gamma(\mathbf{a}))^r \left|B_\gamma(\mathbf{a})\right|_H \left(\frac{\nu(B_\gamma(\mathbf{a}))}{\omega(B_\gamma(\mathbf{a}))} p^{\gamma\beta} \right)^r \left(\frac{\|f\|_{L^{1,\nu}(\mathbb{Q}_p^n)}}{\sigma} \right)^r \\
&\le C\omega(B_\gamma(\mathbf{a}))^r \left|B_\gamma(\mathbf{a})\right|_H \left(\frac{\|f\|_{L^{1,\nu}(\mathbb{Q}_p^n)}}{\sigma} \right)^r.
\end{aligned}
$$

For $T^p_{\beta,\Omega}(f\chi_{B^c_\gamma(\mathbf{a})})(\mathbf{x})$, by Chebychev's inequality, Lemma 2.3 and (3.2), we have

$$
\begin{aligned}
&\left|\left\{\mathbf{x} \in B_\gamma(\mathbf{a}) : |T^p_{\beta,\Omega}(f\chi_{B^c_\gamma(\mathbf{a})})(\mathbf{x})| > \sigma/2\right\}\right|_H \\
&\le \frac{C}{\sigma^r} \int_{B_\gamma(\mathbf{a})} |T^p_{\beta,\Omega}(f\chi_{B^c_\gamma(\mathbf{a})})(\mathbf{x})|^r d\mathbf{x}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{C}{\sigma^r}\int_{B_\gamma(\mathbf{a})}\left|\int_{B^c_\gamma(\mathbf{a})}\frac{\left|\Omega(|\mathbf{y}|_p\mathbf{y})\right||f(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|_p^{n-\beta}}d\mathbf{y}\right|^r d\mathbf{x}\\
&\leq \frac{C}{\sigma^r}\left|B_\gamma(\mathbf{a})\right|_H \|f\|^r_{L^{1,\nu}(\mathbb{Q}_p^n)}\left(\sum_{k=\gamma+1}^{\infty}p^{k\beta}\nu(B_k(\mathbf{a}))\right)^r\\
&\leq \frac{C}{\sigma^r}\omega(B_\gamma(\mathbf{a}))^r\left|B_\gamma(\mathbf{a})\right|_H \|f\|^r_{L^{1,\nu}(\mathbb{Q}_p^n)}\left(\sum_{k=\gamma+1}^{\infty}p^{k\beta}\frac{\nu(B_k(\mathbf{a}))}{\omega(B_\gamma(\mathbf{a}))}\right)^r\\
&\leq C\omega(B_\gamma(\mathbf{a}))^r\left|B_\gamma(\mathbf{a})\right|_H\left(\frac{\|f\|_{L^{1,\nu}(\mathbb{Q}_p^n)}}{\sigma}\right)^r.
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
&\left|\left\{\mathbf{x}\in B_\gamma(\mathbf{a}) : |T^p_{\beta,\Omega}f(\mathbf{x})|>\sigma\right\}\right|_H\\
&\leq \left|\left\{\mathbf{x}\in B_\gamma(\mathbf{a}) : |T^p_{\beta,\Omega}(f\chi_{B_\gamma(\mathbf{a})})(\mathbf{x})|>\sigma/2\right\}\right|_H+\left|\left\{\mathbf{x}\in B_\gamma(\mathbf{a}) : |T^p_{\beta,\Omega}(f\chi_{B^c_\gamma(\mathbf{a})})(\mathbf{x})|>\sigma/2\right\}\right|_H\\
&\leq C\omega(B_\gamma(\mathbf{a}))^r\left|B_\gamma(\mathbf{a})\right|_H\left(\frac{\|f\|_{L^{1,\nu}(\mathbb{Q}_p^n)}}{\sigma}\right)^r.
\end{aligned}
$$

Ultimately,

$$
\frac{\left|\left\{\mathbf{x}\in B_\gamma(\mathbf{a}) : |T^p_{\beta,\Omega}f(\mathbf{x})|>\sigma\right\}\right|_H}{\omega(B_\gamma(\mathbf{a}))^r\left|B_\gamma(\mathbf{a})\right|_H}\leq C\left(\frac{\|f\|_{L^{1,\nu}(\mathbb{Q}_p^n)}}{\sigma}\right)^r,
$$

for any $\sigma>0$, $\gamma\in\mathbb{Z}$ and $\mathbf{a}\in\mathbb{Q}_p^n$. This completes the proof of Theorem 3.1.

The proof of Theorem 3.2. In order to simplify the proving process, for a positive integer m and $0\leq i\leq m$, we denote by C_i^m the family of all finite subsets $\theta=\{\theta_1,\theta_2,\cdots,\theta_i\}$ of $\{1,2,\cdots,m\}$ of i different elements, let $\theta^c=\{1,2,\cdots,m\}\backslash\theta$ for any $\theta\in C_i^m$. For $\mathbf{b}=(b_1,b_2,\cdots,b_m)$ and $\theta=\{\theta_1,\theta_2,\cdots,\theta_i\}\in C_i^m$, set $\mathbf{b}_\theta=(b_{\theta_1},b_{\theta_2},\cdots,b_{\theta_i})$ and the product $b_\theta=b_{\theta_1}b_{\theta_2}\cdots b_{\theta_i}$. With this notation, we write

$$
(b(\mathbf{x})-b_{B_\gamma})_\theta=(b_{\theta_1}(\mathbf{x})-b_{\theta_1B_\gamma})\cdots(b_{\theta_i}(\mathbf{x})-b_{\theta_iB_\gamma}),
$$

$$
(b_{B_k}-b_{B_\gamma})_\theta=(b_{\theta_1B_k}-b_{\theta_1B_\gamma})\cdots(b_{\theta_iB_k}-b_{\theta_iB_\gamma}),
$$

and

$$
\|\mathbf{b}_\theta\|_{\mathcal{L}_{\bar q,\bar\nu}}=\|b_{\theta_1}\|_{\mathcal{L}_{\bar q_1,\bar\nu_1}}\cdots\|b_{\theta_i}\|_{\mathcal{L}_{\bar q_i,\bar\nu_i}},
$$

where $1/\bar q=1/\bar q_1+\cdots+1/\bar q_i$ and $\bar\nu(B_\gamma(\mathbf{a}))=\bar\nu_1(B_\gamma(\mathbf{a}))\cdots\bar\nu_i(B_\gamma(\mathbf{a}))$ for any ball $B_\gamma(\mathbf{a})$. Especially, when $i=m$, $\theta=\{1,2,\cdots,m\}$ and $\theta^c=\emptyset$, we have $\mathbf{b}_\theta=\mathbf{b}$, hence

$$
(b(\mathbf{x})-b_{B_\gamma})_\theta=(b_1(\mathbf{x})-b_{1B_\gamma})\cdots(b_m(\mathbf{x})-b_{mB_\gamma}),\quad (b(\mathbf{x})-b_{B_\gamma})_{\theta^c}=1.
$$

When $i=0$, $\theta=\emptyset$ and $\theta^c=\{1,2,\cdots,m\}$, we have$\mathbf{b}_{\theta^c}=\mathbf{b}$, hence

$$
(b(\mathbf{x})-b_{B_\gamma})_\theta=1,\quad (b(\mathbf{x})-b_{B_\gamma})_{\theta^c}=(b_1(\mathbf{x})-b_{1B_\gamma})\cdots(b_m(\mathbf{x})-b_{mB_\gamma}).
$$

We write $b_i(\mathbf{x})-b_i(\mathbf{y})=(b_i(\mathbf{x})-b_{iB_\gamma})+(b_i(\mathbf{y})-b_{iB_\gamma})$ for $i=1,2,\cdots,m$, then

$$
T^{p,\mathbf{b}}_{\beta,\Omega}f(\mathbf{x})=\int_{\mathbb{Q}_p^n}\frac{\prod_{i=1}^m(b_i(\mathbf{x})-b_i(\mathbf{y}))\,\Omega(|\mathbf{y}|_p\mathbf{y})f(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|_p^{n-\beta}}d\mathbf{y}
$$

$$
\begin{aligned}
&= \int_{\mathbb{Q}_p^n} \frac{\Omega(|\mathbf{y}|_p\mathbf{y})f(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|_p^{n-\beta}} \prod_{i=1}^{m} \Big[(b_i(\mathbf{x}) - b_{iB_\gamma}) + (b_i(\mathbf{y}) - b_{iB_\gamma})\Big]\, d\mathbf{y} \\
&= \prod_{i=1}^{m}(b_i(\mathbf{x}) - b_{iB_\gamma}) T^p_{\beta,\Omega} f(\mathbf{x}) + (-1)^m T^p_{\beta,\Omega}\left(\prod_{i=1}^{m}(b_i - b_{iB_\gamma})f\right)(\mathbf{x}) \\
&\quad + \sum_{i=1}^{m-1}\sum_{\theta\in C_i^m}(-1)^{m-i}(b(\mathbf{x}) - b_{B_\gamma})_\theta \int_{\mathbb{Q}_p^n} \frac{\Omega(|\mathbf{y}|_p\mathbf{y})f(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|_p^{n-\beta}} \Big(b(\mathbf{y}) - b_{B_\gamma}\Big)_{\theta^c}\, d\mathbf{y} \\
&= \prod_{i=1}^{m}(b_i(\mathbf{x}) - b_{iB_\gamma}) T^p_{\beta,\Omega} f(\mathbf{x}) + (-1)^m T^p_{\beta,\Omega}\left(\prod_{i=1}^{m}(b_i - b_{iB_\gamma})f\right)(\mathbf{x}) \\
&\quad + \sum_{i=1}^{m-1}\sum_{\theta\in C_i^m}(-1)^{m-i}(b(\mathbf{x}) - b_{B_\gamma})_\theta T^p_{\beta,\Omega}\Big((b - b_{B_\gamma})_{\theta^c} f\Big)(\mathbf{x}) \\
&= \sum_{i=0}^{m}\sum_{\theta\in C_i^m}(-1)^{m-i}(b(\mathbf{x}) - b_{B_\gamma})_\theta T^p_{\beta,\Omega}\Big((b - b_{B_\gamma})_{\theta^c} f\Big)(\mathbf{x}).
\end{aligned}
$$

For $f \in L^{q,\nu}(\mathbb{Q}_p^n)$, $q > 1$, let $f = f\chi_{B_\gamma(\mathbf{a})} + f\chi_{B^c_\gamma(\mathbf{a})} =: f_1 + f_2$, then we get

$$
\begin{aligned}
&\frac{1}{\omega(B_\gamma(\mathbf{a}))}\left(\frac{1}{\big|B_\gamma(\mathbf{a})\big|_H}\int_{B_\gamma(\mathbf{a})}\left|T^{p,\mathbf{b}}_{\beta,\Omega} f(\mathbf{x})\right|^r d\mathbf{x}\right)^{1/r} \\
&\le \frac{1}{\omega(B_\gamma(\mathbf{a}))\big|B_\gamma(\mathbf{a})\big|_H^{1/r}}\left(\int_{B_\gamma(\mathbf{a})}\left|\sum_{i=0}^{m}\sum_{\theta\in C_i^m}(b(\mathbf{x}) - b_{B_\gamma})_\theta T^p_{\beta,\Omega}\Big((b - b_{B_\gamma})_{\theta^c} f_1\Big)(\mathbf{x})\right|^r d\mathbf{x}\right)^{1/r} \\
&\quad + \frac{1}{\omega(B_\gamma(\mathbf{a}))\big|B_\gamma(\mathbf{a})\big|_H^{1/r}}\left(\int_{B_\gamma(\mathbf{a})}\left|\sum_{i=0}^{m}\sum_{\theta\in C_i^m}(b(\mathbf{x}) - b_{B_\gamma})_\theta T^p_{\beta,\Omega}\Big((b - b_{B_\gamma})_{\theta^c} f_2\Big)(\mathbf{x})\right|^r d\mathbf{x}\right)^{1/r} \\
&=: J_1 + J_2.
\end{aligned}
$$

To facilitate estimates J_1 and J_2, set

$$
1/s = \sum_{\theta_i\in\theta} 1/q_i, \quad 1/h = \sum_{\theta_i\in\theta^c} 1/q_i, \quad 1/g = 1/h + 1/q, \quad 1/t = 1/g - \beta/n
$$

and

$$
\nu'(B_\gamma(\mathbf{a})) = \prod_{\theta_i\in\theta}\nu_i(B_\gamma(\mathbf{a})), \quad \nu''(B_\gamma(\mathbf{a})) = \prod_{\theta_i\in\theta^c}\nu_i(B_\gamma(\mathbf{a}))
$$

then $1/r = 1/s + 1/t$ and $g > 1$.

First, using Minkowski's inequality, Hölder's inequality and L^t-boundedness of $T^p_{\beta,\Omega}$, we obtain

$$
\begin{aligned}
&\left(\int_{B_\gamma(\mathbf{a})}\left|\sum_{i=0}^{m}\sum_{\theta\in C_i^m}(b(\mathbf{x}) - b_{B_\gamma})_\theta T^p_{\beta,\Omega}\Big((b - b_{B_\gamma})_{\theta^c} f_1\Big)(\mathbf{x})\right|^r d\mathbf{x}\right)^{1/r} \\
&\le C\sum_{i=0}^{m}\sum_{\theta\in C_i^m}\left(\int_{B_\gamma(\mathbf{a})}\left|(b(\mathbf{x}) - b_{B_\gamma})_\theta T^p_{\beta,\Omega}\Big((b - b_{B_\gamma})_{\theta^c} f_1\Big)(\mathbf{x})\right|^r d\mathbf{x}\right)^{1/r}
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\sum_{i=0}^{m}\sum_{\theta\in C_i^m}\left(\int_{B_\gamma(\mathbf{a})}\left|(b(\mathbf{x})-b_{B_\gamma})_\theta\right|^s d\mathbf{x}\right)^{1/s}\left(\int_{B_\gamma(\mathbf{a})}\left|T^p_{\beta,\Omega}\left((b-b_{B_\gamma})_{\theta^c}f_1\right)(\mathbf{x})\right|^t d\mathbf{x}\right)^{1/t}\\
&\leq C\sum_{i=0}^{m}\sum_{\theta\in C_i^m}\left(\int_{B_\gamma(\mathbf{a})}\left|(b(\mathbf{x})-b_{B_\gamma})_\theta\right|^s d\mathbf{x}\right)^{1/s}\left(\int_{B_\gamma(\mathbf{a})}\left|(b(\mathbf{x})-b_{B_\gamma})_{\theta^c}f_1(\mathbf{x})\right|^g d\mathbf{x}\right)^{1/g}\\
&\leq C\sum_{i=0}^{m}\sum_{\theta\in C_i^m}\left(\int_{B_\gamma(\mathbf{a})}\left|(b(\mathbf{x})-b_{B_\gamma})_\theta\right|^s d\mathbf{x}\right)^{1/s}\left(\int_{B_\gamma(\mathbf{a})}\left|(b(\mathbf{x})-b_{B_\gamma})_{\theta^c}\right|^h d\mathbf{x}\right)^{1/h}\left(\int_{B_\gamma(\mathbf{a})}|f(\mathbf{x})|^q d\mathbf{x}\right)^{1/q}\\
&\leq C\nu(B_\gamma(\mathbf{a}))\left|B_\gamma(\mathbf{a})\right|_H^{1/q}\|f\|_{L^{q,\nu}(\mathbb{Q}_p^n)}\sum_{i=0}^{m}\sum_{\theta\in C_i^m}\nu'(B_\gamma(\mathbf{a}))\left|B_\gamma(\mathbf{a})\right|_H^{1/s}\|\mathbf{b}_\theta\|_{\mathcal{L}^{s,\nu'}}\nu''(B_\gamma(\mathbf{a}))\left|B_\gamma(\mathbf{a})\right|_H^{1/h}\|\mathbf{b}_{\theta^c}\|_{\mathcal{L}^{h,\nu''}}\\
&\leq \nu(B_\gamma(\mathbf{a}))\left|B_\gamma(\mathbf{a})\right|_H^{1/q}\|f\|_{L^{q,\nu}(\mathbb{Q}_p^n)}\prod_{i=1}^{m}\|b_i\|_{\mathcal{L}^{q_i,\nu_i}(\mathbb{Q}_p^n)}\nu_i(B_\gamma(\mathbf{a}))\left|B_\gamma(\mathbf{a})\right|_H^{1/q_i}.
\end{aligned}
$$

Then, it is not difficult for us to get

$$
\begin{aligned}
J_1&\leq C\frac{\nu(B_\gamma(\mathbf{a}))\left|B_\gamma(\mathbf{a})\right|_H^{1/q}\|f\|_{L^{q,\nu}(\mathbb{Q}_p^n)}}{\omega(B_\gamma(\mathbf{a}))\left|B_\gamma(\mathbf{a})\right|_H^{1/r}}\prod_{i=1}^{m}\|b_i\|_{\mathcal{L}^{q_i,\nu_i}(\mathbb{Q}_p^n)}\nu_i(B_\gamma(\mathbf{a}))\left|B_\gamma(\mathbf{a})\right|_H^{1/q_i}\\
&\leq C\frac{p^{\gamma\beta}\nu(B_\gamma(\mathbf{a}))\|f\|_{L^{q,\nu}(\mathbb{Q}_p^n)}}{\omega(B_\gamma(\mathbf{a}))}\prod_{i=1}^{m}\|b_i\|_{\mathcal{L}^{q_i,\nu_i}(\mathbb{Q}_p^n)}\nu_i(B_\gamma(\mathbf{a}))\\
&\leq C\prod_{i=1}^{m}\|b_i\|_{\mathcal{L}^{q_i,\nu_i}(\mathbb{Q}_p^n)}\|f\|_{L^{q,\nu}(\mathbb{Q}_p^n)}\frac{\nu(B_\gamma(\mathbf{a}))\prod_{i=1}^m\nu_i(B_\gamma(\mathbf{a}))}{\omega(B_\gamma(\mathbf{a}))}p^{\gamma\beta}\\
&\leq C\prod_{i=1}^{m}\|b_i\|_{\mathcal{L}^{q_i,\nu_i}(\mathbb{Q}_p^n)}\|f\|_{L^{q,\nu}(\mathbb{Q}_p^n)}.
\end{aligned}
$$

Second, we will turn to the estimation of J_2. Given $\mathbf{x}\in B_\gamma(\mathbf{a})$, by Hölder's inequality, Minkowski's inequality and Lemma 2.1, we have

$$
\begin{aligned}
&\left|T^p_{\beta,\Omega}\left((b-b_{B_\gamma})_{\theta^c}f_2\right)(\mathbf{x})\right|\\
&=\int_{B^c_\gamma(\mathbf{a})}\frac{\left|\Omega(|\mathbf{y}|_p\mathbf{y})\right||f(\mathbf{y})|\left|(b(\mathbf{y})-b_{B_\gamma})_{\theta^c}\right|}{|\mathbf{x}-\mathbf{y}|_p^{n-\beta}}d\mathbf{y}\\
&=\sum_{k=\gamma+1}^{\infty}p^{-k(n-\beta)}\int_{S_k(\mathbf{a})}\left|\Omega(|\mathbf{y}|_p\mathbf{y})\right||f(\mathbf{y})|\left|(b(\mathbf{y})-b_{B_\gamma})_{\theta^c}\right|d\mathbf{y}\\
&\leq\sum_{k=\gamma+1}^{\infty}p^{-k(n-\beta)}\left(\int_{S_k(\mathbf{a})}\left|\Omega(|\mathbf{y}|_p\mathbf{y})\right|^{q'}d\mathbf{y}\right)^{1/q'}\left(\int_{S_k(\mathbf{a})}|f(\mathbf{y})|^q d\mathbf{y}\right)^{1/q}\\
&\quad\times\left(\int_{S_k(\mathbf{a})}\left|(b(\mathbf{y})-b_{B_\gamma})_{\theta^c}\right|^h d\mathbf{y}\right)^{1/h}|S_k(\mathbf{a})|_H^{-1/h}\\
&\leq\sum_{k=\gamma+1}^{\infty}p^{-k(n-\beta)-kn/h+kn/q'}\nu(B_k(\mathbf{a}))|B_k(\mathbf{a})|_H^{1/q}\|f\|_{L^{q,\nu}(\mathbb{Q}_p^n)}
\end{aligned}
$$

$$\times\left[\left(\int_{B_k(\mathbf{a})}\left|(b(\mathbf{y})-b_{B_k})_{\theta^c}\right|^h d\mathbf{y}\right)^{1/h}+\left(\int_{B_k(\mathbf{a})}\left|(b_{B_k}-b_{B_\gamma})_{\theta^c}\right|^h d\mathbf{y}\right)^{1/h}\right]$$
$$\leq \|f\|_{L^{q,\nu}(\mathbb{Q}_p^n)}\sum_{k=\gamma+1}^{\infty}p^{k\beta-kn/h}\nu(B_k(\mathbf{a}))\|\mathbf{b}_{\theta^c}\|_{\mathcal{L}^{h,\nu''}}|B_k(\mathbf{a})|_H^{1/h}\nu''(B_k(\mathbf{a}))(k-\gamma+1)^{m-i}$$
$$\leq \|f\|_{L^{q,\nu}(\mathbb{Q}_p^n)}\sum_{k=\gamma+1}^{\infty}p^{k\beta}\nu(B_k(\mathbf{a}))\|\mathbf{b}_{\theta^c}\|_{\mathcal{L}^{h,\nu''}}\nu''(B_k(\mathbf{a}))(k-\gamma+1)^{m}.$$

Then, it follows that

$$J_2=\frac{1}{\omega(B_\gamma(\mathbf{a}))\left|B_\gamma(\mathbf{a})\right|_H^{1/r}}\left(\int_{B_\gamma(\mathbf{a})}\left|\sum_{i=0}^{m}\sum_{\theta\in C_i^m}(b(\mathbf{x})-b_{B_\gamma})_\theta T^p_{\beta,\Omega}\left((b-b_{B_\gamma})_{\theta^c}f_2\right)(\mathbf{x})\right|^r d\mathbf{x}\right)^{1/r}$$
$$\leq \frac{C}{\omega(B_\gamma(\mathbf{a}))\left|B_\gamma(\mathbf{a})\right|_H^{1/r}}\sum_{i=0}^{m}\sum_{\theta\in C_i^m}\left(\int_{B_\gamma(\mathbf{a})}\left|(b(\mathbf{x})-b_{B_\gamma})_\theta\right|^s d\mathbf{x}\right)^{1/s}\left(\int_{B_\gamma(\mathbf{a})}\left|T^p_{\beta,\Omega}\left((b-b_{B_\gamma})_{\theta^c}f_2\right)(\mathbf{x})\right|^t d\mathbf{x}\right)^{1/t}$$
$$\leq \frac{C}{\omega(B_\gamma(\mathbf{a}))\left|B_\gamma(\mathbf{a})\right|_H^{1/r}}\sum_{i=0}^{m}\sum_{\theta\in C_i^m}\nu'(B_\gamma(\mathbf{a}))\left|B_\gamma(\mathbf{a})\right|_H^{1/s}\|\mathbf{b}_\theta\|_{\mathcal{L}^{s,\nu'}}\|f\|_{L^{q,\nu}(\mathbb{Q}_p^n)}\left|B_\gamma(\mathbf{a})\right|_H^{1/t}$$
$$\times\sum_{k=\gamma+1}^{\infty}p^{k\beta}\nu(B_k(\mathbf{a}))\|\mathbf{b}_{\theta^c}\|_{\mathcal{L}^{h,\nu''}}\nu''(B_k(\mathbf{a}))(k-\gamma+1)^m$$
$$\leq \frac{C\|f\|_{L^{q,\nu}(\mathbb{Q}_p^n)}}{\omega(B_\gamma(\mathbf{a}))}\sum_{k=\gamma+1}^{\infty}\nu(B_k(\mathbf{a}))(k-\gamma+1)^m p^{k\beta}\sum_{i=0}^{m}\sum_{\theta\in C_i^m}\nu'(B_\gamma(\mathbf{a}))\|\mathbf{b}_\theta\|_{\mathcal{L}^{s,\nu'}}\nu''(B_k(\mathbf{a}))\|\mathbf{b}_{\theta^c}\|_{\mathcal{L}^{h,\nu''}}$$
$$\leq \frac{C\|f\|_{L^{q,\nu}(\mathbb{Q}_p^n)}}{\omega(B_\gamma(\mathbf{a}))}\sum_{k=\gamma+1}^{\infty}\nu(B_k(\mathbf{a}))(k-\gamma+1)^m p^{k\beta}\prod_{i=1}^{m}\|b_i\|_{\mathcal{L}^{q_i,\nu_i}(\mathbb{Q}_p^n)}\nu_i(B_k(\mathbf{a}))$$
$$\leq C\prod_{i=1}^{m}\|b_i\|_{\mathcal{L}^{q_i,\nu_i}(\mathbb{Q}_p^n)}\|f\|_{L^{q,\nu}(\mathbb{Q}_p^n)}\sum_{k=\gamma+1}^{\infty}\frac{\nu(B_k(\mathbf{a}))\prod_{i=1}^{m}\nu_i(B_k(\mathbf{a}))}{\omega(B_\gamma(\mathbf{a}))}(k-\gamma+1)^m p^{k\beta}$$
$$\leq C\prod_{i=1}^{m}\|b_i\|_{\mathcal{L}^{q_i,\nu_i}(\mathbb{Q}_p^n)}\|f\|_{L^{q,\nu}(\mathbb{Q}_p^n)}.$$

At last, combining the estimation of J_1 and J_2, we obtain

$$\frac{1}{\omega(B_\gamma(\mathbf{a}))}\left(\frac{1}{\left|B_\gamma(\mathbf{a})\right|_H}\int_{B_\gamma(\mathbf{a})}\left|T^{p,\mathbf{b}}_{\beta,\Omega}f(\mathbf{x})\right|^r d\mathbf{x}\right)^{1/r}\leq C\prod_{i=1}^{m}\|b_i\|_{\mathcal{L}^{q_i,\nu_i}(\mathbb{Q}_p^n)}\|f\|_{L^{q,\nu}(\mathbb{Q}_p^n)}$$

for any $B_\gamma(\mathbf{a})\subset\mathbb{Q}_p^n$ and $f\in L^{q,\nu}(\mathbb{Q}_p^n)(q>1)$, and the proof of Theorem 3.2 is finished.

5. Conclusions

In this article, the boundedness of the rough p-adic fractional integral operator on p-adic generalized Morrey spaces is studied. In addition, the boundedness for multilinear commutators generated by rough p-adic fractional integral operator and p-adic generalized Campanato functions is also obtained. Moreover, the boundedness in classical Morrey is given as corollaries.

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Conflict of interest

The authors declare that they have no conflict of interest.

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