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Semidegenerate Congruence-modular Algebras Admitting a Reticulation

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Abstract

The reticulation $L(R)$ of a commutative ring R was introduced by Joyal in 1975, then the theory was developed by Simmons in a remarkable paper published in 1980. $L(R)$ is a bounded distributive algebra whose main property is that the Zariski prime spectrum $Spec(R)$ of R and the Stone prime spectrum $Spec_{Id}(L(R))$ of $L(R)$ are homeomorphic. The construction of the lattice $L(R)$ was generalized by Belluce for each unital ring R and the reticulation was defined by axioms.

In a recent paper we generalized the Belluce construction for algebras in a semidegenerate congruence-modular variety $\mathcal{V}$. For any algebra $A \in \mathcal{V}$ we defined a bounded distributive lattice $L(A)$, but in general the prime spectrum $Spec(A)$ of A is not homeomorphic with the prime spectrum $Spec_{Id}(L(A))$. We introduced the quasi-commutative algebras in the variety $\mathcal{V}$ (as a generalization of Belluce's quasi-commutative rings) and proved that for any algebra $A \in \mathcal{V}$, the spectra $Spec(A)$ and $Spec_{Id}(L(A))$ are homeomorphic.

In this paper we define the reticulation $A \in \mathcal{V}$ by four axioms and prove that any two reticulations of A are isomorphic lattices. By using the uniqueness of reticulation and other results from the mentioned paper, we obtain a characterization theorem for the algebras $A \in \mathcal{V}$ that admit a reticulation: A is quasi-commutative if and only if A admits a reticulation. This result is a universal algebra generalization

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of the following Belluce theorem: a ring R is quasi-commutative if and only if R admits a reticulation.

Another subject treated in this paper is the spectral closure of the prime spectrum $Spec(A)$ of an algebra $A \in \mathcal{V}$, a notion that generalizes the Belluce spectral closure of the prime spectrum of a ring.

Keywords: semidegenerate congruence - modular algebras, axiomatic reticulation, quasi-commutative algebras, spectral algebras, spectral closure

1 Introduction

The reticulation of a commutative ring R is a pair $(L(R), \lambda_R)$ composed of a bounded distributive lattice $L(R)$ and a function $\lambda_R : R \to L(R)$ preserving some operations and constants (see [21, 23, 36]). The most important property of reticulation is that the Zariski prime spectrum $Spec_Z(R)$ of R is homeomorphic with the Stone prime spectrum $Spec_{Id,Z}(L(R))$ of $L(R)$. By using the reticulation we can move some properties from commutative rings to algebras and vice-versa (see [2, 21, 36]). An axiomatic definition of reticulation for arbitrary (unital) rings was proposed by Belluce in [6]. He observed that the reticulation does not exist for any arbitrary ring. In [6] the quasi-commutative and the spectral rings are introduced and it is proven that a ring R admits a reticulation iff R is quasi-commutative iff R is spectral.

The reticulation of a ring inspired a rich literature of reticulation theories for other algebraic structures: F-rings [21], MV-algebras [5], BL-algebras [29, 30], 0-distributive lattices [35], residuated lattices [22, 31, 32, 33], bounded BCK-algebras [10], etc.

These reticulations are used to obtain new results on the algebraic structures. For example, the reticulation of an MV-algebra was the main tool used in [28] for solving the spectrum problem in the MV-algebras theory, i.e the characterization of the topological spaces homeomorphic to the prime spectra of MV-algebras.

The commutator theory, developed by R. Fresee and R. McKenzie in [14] for algebras in congruence-modular varieties, allowed us to endow these algebras with prime spectra having important topological properties (see [1]). By using the ideas of [1], C. Mureşan and the author proposed in [18] a notion of reticulation for the algebras A in a semidegenerate congruence-modular variety $\mathcal{V}$, satisfying the hypothesis (H): the set $K(A)$ of compact congruences of A is closed under commutators. These algebras generalize the Kaplansky neo-commutative rings (cf. [24], p. 73, a ring R

is neo-commutative if the product of two finitely generated ideals of R is finitely generated). Therefore an algebra $A \in \mathcal{V}$ fulfilling (H) will be called a neo-commutative algebra. In [18] it is proven that the prime spectrum $Spec_Z(A)$ of a neo-commutative algebra is homeomorphic with the prime spectrum $Spec_{Id,Z}(L(A))$ of reticulation $L(A)$ of A. Thus $Spec_Z(A)$ is a spectral space in the sense of [12, 19]. The reticulation and its transfer properties were used in [17] for studying the functorial properties of this construction and in [15] for obtaining the characterization theorems for several classes of neo-commutative algebras.

The reticulation theory developed in [18] does not cover the Belluce reticulations for arbitrary rings [6]. Recently, we introduced in [16] the quasi-commutative algebras and the spectral algebras in a semidegenerate congruence-modular variety $\mathcal{V}$. These two classes of algebras coincide. For each algebra $A \in \mathcal{V}$ we built a bounded distributive lattice $L(A)$ and proved that for each quasi-commutative algebra A, the prime spectra $Spec_Z(A)$ and $Spec_{Id,Z}(L(A))$ are homeomorphic.

This paper aims to propose an axiomatic approach of the reticulation for an arbitrary algebra A in a semidegenerate congruence-modular variety $\mathcal{V}$. Firstly we will introduce the notion of pre-reticulation of A by three axioms and develop some elementary matter. Then we define the notion of reticulation of A by adding a fourth axiom that ensures that the prime spectrum of reticulation is homeomorphic with the prime spectrum of A. We prove that all reticulations of A are isomorphic (whenever these reticulations exist), then we obtain a characterization theorem for the algebras $A \in \mathcal{V}$ that admit a reticulation. Another subject treated in this paper is the spectral closure of the prime spectrum $Spec_Z(A)$ of an algebra $A \in \mathcal{V}$, a notion that generalizes the Belluce spectral closure of the prime spectrum of a ring [7].

Now we shall describe the content of this paper. In Section 2 we present some definitions and results on commutators [14], the prime spectra of algebras in a semidegenerate congruence-modular variety $\mathcal{V}$ and some elementary properties of these spectra [1].

Section 3 concerns the axiomatic theory of reticulation for algebras in $\mathcal{V}$. The first piece in this construction is the set $C(A)$ generated by the set $K(A)$ of compact congruences of $A \in \mathcal{V}$, under commutators and finite joins of congruences. The notion of pre-reticulation of A is axiomatically introduced. A pre-reticulation of A is a pair $(L, \lambda : C(A) \to L)$, where L is a bounded distributive lattice and $\lambda : C(A) \to L$ is a function that satisfies some natural axioms. We define two functions $(\cdot)^* : Con(A) \to Id(L)$ and

$(\cdot)_* : Id(L) \to Con(A)$ that connect the congruences of A and the ideals of the lattice L. A reticulation of A is a pre-reticulation of A fulfilling a new axiom that ensures a homeomorphism between the topological spaces $Spec_Z(A)$ and $Spec_{Id,Z}(L(A))$. We introduce the quasi-commutative and the spectral algebras of the variety $\mathcal{V}$ as generalizations of the quasi-commutative and the spectral rings. We prove that any two reticulations of algebra A are isomorphic (whenever they exist) and we characterize the algebras of the variety $\mathcal{V}$ that admit a reticulation: an algebra A admits a reticulation iff A is quasi-commutative iff A is spectral. If we apply this theorem for rings we obtain the Belluce results from [6].

In Section 4 we define the spectral closure $XSpec_Z(A)$ of the prime spectrum $Spec_Z(A)$ of an algebra $A \in \mathcal{V}$. $XSpec_Z(A)$ is a spectral space such that $Spec_Z(A)$ is a dense subspace of $XSpec_Z(A)$. We characterize the elements of $XSpec_Z(A)$ as the locally prime congruences of A. The results of this section can be viewed as a universal algebra generalization of some results obtained by Belluce for the spectral closure of prime spectrum of a ring [7].

2 Preliminaries

Throughout this paper we shall assume that the algebras have a finite signature τ. For any algebra A we shall denote:

- $Con(A)$ is the complete lattice of the congruences of A; Δ_A and ∇_A are the first and the last elements of $Con(A)$;
- $PCon(A)$ is the set of principal congruences of A;
- $K(A)$ is the set of all finitely generated congruences of A (also called compact congruences). We know that $K(A)$ is closed under finite joins of $Con(A)$ and $\Delta_A \in K(A)$.

Recall from [9] that a variety $\mathcal{V}$ of algebras is said to be congruence-modular if for any member of $\mathcal{V}$, $Con(A)$ is a modular lattice.

Let us fix a congruence - modular variety $\mathcal{V}$. According to Definition 3.2 of [14], for each algebra $A \in \mathcal{V}$ we can define a multiplication operation $[\cdot,\cdot]$ on the congruence lattice $Con(A)$, named the commutator operation. This abstract notion extends the commutator operation existing in group theory, as well as the multiplication of ideals in ring theory. The definition of commutator operation for algebras of $\mathcal{V}$ is very technical (see Section 3 of [14]), so we do not recall it here. The proofs of the results in this paper do not use the definition of the commutator, but only its properties mentioned in this section. It is useful to remind the following properties of the commutator operation: $[\cdot,\cdot]$ is commutative, increasing in each argument and distributive with respect to arbitrary joins.

Lemma 1 ([14]) *For any congruence - modular variety $\mathcal{V}$ the following are equivalent:*

(1) $[\nabla_A, \nabla_A] = \nabla_A$, *for all* $A \in \mathcal{V}$;

(2) $[\theta, \nabla_A] = \theta$, *for all* $A \in \mathcal{V}$ *and* $\theta \in Con(A)$.

Recall from [26], that a variety $\mathcal{V}$ is semidegenerate if no nontrivial algebra in $\mathcal{V}$ has one - element subalgebras. According to [26], a variety $\mathcal{V}$ is semidegenerate if and only if for any algebra A in $\mathcal{V}$, the congruence ∇_A is compact.

Proposition 1 ([1]) *If $\mathcal{V}$ is a semidegenerate congruence - modular variety then for each algebra A in $\mathcal{V}$ we have $[\nabla_A, \nabla_A] = \nabla_A$.*

If R is a ring then the lattice $Id(R)$ of its ideals is isomorphic to the lattice $Con(R)$ of congruences of R. The commutator operation in $Id(R)$ is defined by $[I, J] = IJ + JI$, for all ideals I, J of R. It is clear that the class of rings is a semidegenerate congruence-modular variety.

Let us fix a semidegenerate congruence-modular variety $\mathcal{V}$ and $A \in \mathcal{V}$. By Lemma 1 and Proposition 1, for any $\theta \in Con(A)$ we have $[\theta, \nabla_A] = \theta$. Define on the lattice $Con(A)$:

- the residuation operation (implication): $\alpha \rightarrow \beta = \bigvee\{\gamma | [\alpha,\gamma] \subseteq \beta\}$;
- the annihilator operation (polar): $\alpha^{\perp} = \alpha \rightarrow \Delta_A = \bigvee\{\gamma | [\alpha,\gamma] = \Delta_A\}$.

Recall from [1] that the implication $\rightarrow$ fulfills the usual residuation property: for all $\alpha,\beta,\gamma \in Con(A)$, $\alpha \subseteq \beta \rightarrow \gamma$ if and only if $[\alpha,\beta] \subseteq \gamma$. The algebraic structure $(Con(A), \vee, \wedge, [\cdot,\cdot], \rightarrow, \Delta_A, \nabla_A)$ is a commutative and integral complete l - groupoid (see [8, 20, 27, 34]).

Following [14], p.82 or [1], p.582, a congruence $\phi \in Con(A) - \{\nabla_A\}$ is *prime* if for all $\alpha, \beta \in Con(A)$, $[\alpha,\beta] \subseteq \phi$ implies $\alpha \subseteq \phi$ or $\beta \subseteq \phi$. We denote by $Spec(A)$ the set of prime congruences and by $Max(A)$ the set of maximal congruences of $Con(A)$. If $\theta \in Con(A) - \{\nabla_A\}$ then there exists $\phi \in Max(A)$ such that $\theta \subseteq \phi$. By [1], the inclusion $Max(A) \subseteq Spec(A)$ holds. $Spec(A)$ is called the prime spectrum of A and $Max(A)$ is called the maximal spectrum of A. An abstract theory of prime spectra can be found in the very interesting paper [13].

According to [1], p.582, the *radical* $\rho(\theta) = \rho_A(\theta)$ of a congruence $\theta \in A$ is defined by $\rho_A(\theta) = \bigcap\{\phi \in Spec(A) | \theta \subseteq \phi\}$; if $\theta = \rho(\theta)$ then θ is a radical congruence. For the basic properties of radicals, see [1, 18]. In particular, $\rho(\Delta_A) = \bigcap Spec(A)$. The algebra A is *semiprime* if $\rho(\Delta_A) = \Delta_A$.

Let L be a bounded distributive lattice and $Id(L)$ the set of its ideals. Then $Spec_{Id}(L)$ will denote the set of prime ideals in L and $Max_{Id}(L)$ the set of maximal ideals in L. $Spec_{Id}(L)$ (resp. $Max_{Id}(L)$) endowed with Stone topology will be denoted by $Spec_{Id,Z}(L)$ (resp. $Max_{Id,Z}(L)$).

For any ideal I of L we denote $D_{Id}(I) = \{Q \in Spec_{Id}(L) | I \nsubseteq Q\}$ and $V_{Id}(I) = \{Q \in Spec_{Id}(L) | I \subseteq Q\}$. If $x \in L$ then we use the notation $D_{Id}(x) = D_{Id}((x]) = \{Q \in Spec_{Id}(L) | x \notin Q\}$ and $V_{Id}(x) = V_{Id}((x]) = \{Q \in Spec_{Id}(L) | x \in Q\}$, where $(x]$ is the principal ideal of L generated by the set $\{x\}$. Recall from [21] that the family $(D_{Id}(x))_{x\in L}$ is a basis of open sets for the Stone topology on $Spec_{Id}(L)$.

Following [12, 19], a spectral space (or a coherent space in the terminology of [21]) is a topological space X such that the following properties hold:

- (a) X is a compact T_0-space;
- (b) the compact open subsets of X form a basis of the topology of X, closed under finite intersections;
- (c) any irreducible closed subset of X has a generic point.

Let us consider the following property for a topological space X:

- (b') X has a basis of compact open subsets, closed under finite intersections.

Then a topological space X is a spectral space if and only if it satisfies the conditions (a), (b') and (c).

The main examples of spectral spaces arise from commutative rings and bounded distributive lattices: the prime spectrum $Spec_Z(R)$ of a commutative ring R and the prime spectrum $Spec_{Id,Z}(L)$ of a bounded distributive lattice L are spectral spaces (cf. [12, 19, 21]). If L is a bounded distributive lattice then $(D_{Id}(a))_{a\in L}$ is a basis of compact open sets for $Spec_{Id,Z}(L)$.

Let A be an algebra of a semidegenerate congruence-modular variety $\mathcal{V}$. Now we shall recall from [1, 18] some definitions and notations regarding the topology of the prime spectrum $Spec(A)$.

For any $\theta \in Con(A)$ we denote $V_A(\theta) = V(\theta) = \{\phi \in Spec(A) | \theta \subseteq \phi\}$ and $D_A(\theta) = D(\theta) = Spec(A) - V(\theta)$. If $\alpha, \beta \in Con(A)$ then $D(\alpha) \bigcap D(\beta) = D([\alpha, \beta])$ and $V(\alpha) \bigcup V(\beta) = V([\alpha, \beta])$. For any family of congruences $(\theta_i)_{i\in I}$ we have $\bigcup_{i\in I} D(\theta_i) = D(\bigvee_{i\in I} \theta_i)$ and $\bigcap_{i\in I} V(\theta_i) = V(\bigvee_{i\in I} \theta_i)$. Thus $Spec(A)$ becomes a topological space whose open sets are $D(\theta), \theta \in Con(A)$. We remark that this topology is the universal algebra generalization of the Zariski topology (defined on the prime spectra of commutative rings) [3] and the Stone topology (defined on the prime spectra of bounded distributive lattices) [4, 8]. Thus this topology on the prime spectrum $Spec(A)$ of the algebra A will be named Zariski topology and the respective topological space will be denoted by $Spec_Z(A)$. We mention that the family $(D(\alpha))_{\alpha\in K(A)}$ is a basis of open sets for the Zariski topology.

Lemma 2 *Assume that A is an algebra of a semidegenerate congruence-modular variety $\mathcal{V}$. Then the following hold:*

(1) $Spec_Z(A)$ is a T_0-space;

(2) Any irreducible closed subset of $Spec_Z(A)$ has a generic point.

Proof: By Proposition 2.6 of [13], $Spec_Z(A)$ is a sober space, i.e. $Spec_Z(A)$ is a T_0-space and any irreducible closed subset of $Spec_Z(A)$ is the closure of a point set. The compactness of $Spec_Z(A)$ follows from the fact that ∇_A is a compact congruence of A (because $A \in \mathcal{V}$ and $\mathcal{V}$ is a semidegenerate variety). □

In other words, $Spec_Z(A)$ is a compact sober space. In general, $Spec_Z(A)$ is not a spectral space. In virtue of Lemma 2, $Spec_Z(A)$ is a spectral space if and only if it has a basis of compact open sets, closed under finite intersections.

3 Reticulation of a Universal Algebra

Let $\mathcal{V}$ be a semidegenerate congruence - modular variety and A an algebra of $\mathcal{V}$. We shall define the reticulation of A in an axiomatic manner. Our source of inspiration is the Belluce axiomatic definition for the reticulation of an arbitrary (unital) ring [6].

Following [16], let $C(A)$ be the smallest subset of $Con(A)$ with the following properties:

- $K(A) \subseteq C(A)$;
- If $\theta, \chi \in C(A)$ then $\theta \vee \chi \in C(A)$;
- If $\theta, \chi \in C(A)$ then $[\theta, \chi] \in C(A)$.

We remark that the algebraic structure $(C(A), \vee, [\cdot,\cdot], \Delta_A, \nabla_A)$ is similar to a semi-ring, but without the associativity of multiplication $[\cdot,\cdot]$. If A is a ring then $C(A)$ is exactly the commutative semi-ring $Sem(A)$, generated by the principal ideals of A, under the commutator operation and the sum (cf. [6], p. 1856 or [5], p. 1515).

Let us consider a bounded distributive lattice L and a surjective function $\lambda : C(A) \to L$. The pair $(L, \lambda : C(A) \to L)$ is said to be a pre-reticulation of the algebra A if for all $\alpha, \beta \in C(A)$, the following axioms are satisfied:

$(Ax.1)$ $\lambda(\alpha \vee \beta) = \lambda(\alpha) \vee \lambda(\beta)$;

$(Ax.2)$ $\lambda([\alpha, \beta]) = \lambda(\alpha) \wedge \lambda(\beta)$;

$(Ax.3)$ $\lambda(\Delta_A) = 0$; $\lambda(\nabla_A) = 1$.

A pre-reticulation $(L, \lambda : C(A) \to L)$ will be shortly denoted by (L, λ).

The notion of pre-reticulation, defined above, is weaker than that of reticulation (introduced later, by Definition 2). Now we will present some valid results in the abstract framework offered by pre-reticulations.

Lemma 3 *Assume that (L, λ) is a pre-reticulation of A. For all $\alpha, \beta \in C(A)$, $\alpha \subseteq \beta$ implies $\lambda(\alpha) \leq \lambda(\beta)$.*

Proof: If $\alpha \subseteq \beta$ then $\alpha \vee \beta = \beta$, so, by $(Ax.1)$ we have $\lambda(\beta) = \lambda(\alpha) \vee \lambda(\beta)$, hence $\lambda(\alpha) \leq \lambda(\beta)$. □

Proposition 2 *Assume that (L, λ) is a pre-reticulation of A. Let $(\theta_j)_{j \in J}$ be a family of congruences in $C(A)$ such that $\bigvee_{j \in J} \theta_j \in C(A)$. Thus $\lambda(\bigvee_{j \in J} \theta_j) = \bigvee_{j \in J} \lambda(\theta_j)$.*

Proof: Similar to the proof of Proposition 3.3 of [16]. □

If (L, λ) is a pre-reticulation of A, then for all $\theta \in Con(A)$ and $I \in Id(L)$ we shall denote:

$$\theta^* = \{\lambda(\alpha) | \alpha \in C(A), \alpha \subseteq \theta\}; \; I_* = \bigvee\{\alpha \in K(A) | \lambda(\alpha) \in I\}.$$

Lemma 4 *Assume that (L, λ) is a pre-reticulation of A and $\theta \in Con(A)$, $I \in Id(L)$. Then the following hold:*

(1) *θ^* is an ideal of the lattice L and I_* is a congruence of A;*

(2) *If $\theta \in C(A)$ then $\theta^* = (\lambda(\theta)]$, where the second member is the principal lattice ideal generated by $\{\lambda(\theta)\}$ in L.*

Proof:

(1) Let x, y be two elements of θ^*, so there exists $\alpha, \beta \in C(A)$ such that $x = \lambda(\alpha), y = \lambda(\beta)$, $\alpha \subseteq \theta$ and $\beta \subseteq \theta$. Thus $\alpha \vee \beta \subseteq \theta$ and $\alpha \vee \beta \in C(A)$, therefore, by applying $(Ax.1)$ we get $x \vee y = \lambda(\alpha \vee \beta) \in \theta^*$.

In a similar way, by using $(Ax.2)$, one can prove that for all $x, y \in L$, $x \leq y$ and $y \in \theta^*$ imply $x \in \theta^*$. Then θ^* is an ideal of $L(A)$.

That I_* is a congruence of A is obvious.

(2) Similar to the proof of Lemma 3.5 in [16]. □

According to Lemma 4(1) one obtains two order - preserving functions $(\cdot)^* : Con(A) \to Id(L)$ and $(\cdot)_* : Id(L) \to Con(A)$. These two functions will be good vehicles in transferring some properties from congruences of A to ideals of L and vice-versa. This thesis will be illustrated by the following lemmas and propositions.

Lemma 5 *Assume that (L, λ) is a pre-reticulation of A. Then for all $\alpha \in K(A)$ and $I \in Id(L)$, $\alpha \subseteq I_*$ if and only if $\lambda(\alpha) \in I$.*

Proof: Assume that $\alpha \subseteq I_* = \bigvee\{\alpha \in K(A) | \lambda(\alpha) \in I\}$ so there exist an integer $n \geq 1$ and $\beta_1, \cdots, \beta_n \in K(A)$ such that $\alpha \subseteq \beta_1 \vee ... \vee \beta_n$ and $\lambda(\beta_i) \in I$, for all $i = 1, \cdots, n$ (because α is a compact congruence). Thus $\lambda(\alpha) \leq \lambda(\beta_1) \vee ... \vee \lambda(\beta_n) \in I$, so $\lambda(\alpha) \in I$ (because I is an ideal of the lattice L). The converse implication is obvious. □

Lemma 6 *Assume that (L, λ) is a pre-reticulation of A. If $\theta \in Con(A)$ and $I \in Id(L)$ then $\theta \subseteq (\theta^*)_*$ and $I \subseteq (I_*)^*$.*

Proof: According to the definition of the map $(\cdot)_*$, we have the equality $(\theta^*)_* = \bigvee\{\alpha \in K(A) | \lambda(\alpha) \in \theta^*\}$. We remark that $\alpha \in K(A)$ and $\alpha \subseteq \theta$ imply that $\lambda(\alpha) \in \theta^*$, so $\alpha \subseteq (\theta^*)_*$. Then the inclusion $\theta \subseteq (\theta^*)_*$ follows.

In order to prove that $I \subseteq (I_*)^*$, assume that $x \in I$, so $x = \lambda(\varepsilon)$ for some $\varepsilon \in C(A)$. Let α be a compact congruence of A such that $\alpha \subseteq \varepsilon$, hence $\lambda(\alpha) \leq \lambda(\varepsilon)$. Thus $\lambda(\alpha) \in I$, hence, by using Lemma 5, one obtains $\alpha \subseteq I_*$. It follows that $\varepsilon \subseteq I_*$, so $x = \lambda(\varepsilon) \in (I_*)^*$. We conclude that $I \subseteq (I_*)^*$. □

Proposition 3 *Assume that (L, λ) is a pre-reticulation of A, $\alpha \in C(A)$ and $P \in Spec_{Id}(L)$. If $P_* \in Spec(A)$, then $\alpha \subseteq P_*$ if and only if $\lambda(\alpha) \in P$.*

Proof: Assume $P \in Spec_{Id}(L)$. By using the induction on the way in which $C(A)$ is defined, we shall prove that the following sentence is true:

(3.1) $\forall \alpha \in C(A)[\alpha \subseteq P_* \Leftrightarrow \lambda(\alpha) \in P]$.

We shall consider three cases:

(*a*) Assume $\alpha \in K(A)$. The sentence (3.1) holds by Lemma 5.

(*b*) Assume that $\alpha = \alpha_1 \vee \alpha_2$ and the congruences $\alpha_1, \alpha_2 \in C(A)$ fulfill the induction hypothesis: $\alpha_i \subseteq P_*$ iff $\lambda(\alpha_i) \in P$, for $i = 1, 2$. Then the following equivalences hold:

$$\begin{aligned} \alpha \subseteq P_* &\text{ iff } \alpha_1 \subseteq P_* \text{ and } \alpha_2 \subseteq P_* \\ &\text{ iff } \lambda(\alpha_1) \in P \text{ and } \lambda(\alpha_2) \in P \\ &\text{ iff } \lambda(\alpha_1) \vee \lambda(\alpha_2) \in P \\ &\text{ iff } \lambda(\alpha) \in P. \end{aligned}$$

(*c*) Assume that $\alpha = [\alpha_1, \alpha_2]$ and the congruences $\alpha_1, \alpha_2 \in C(A)$ verify the induction hypothesis, that is $\alpha_i \subseteq P_*$ iff $\lambda(\alpha_i) \in P$, for $i = 1, 2$.

By taking into account that $P \in Spec_{Id}(L)$ and $P_* \in Spec(A)$ (by hypothesis) it results that the following equivalences hold:

$$\begin{aligned} \alpha \subseteq P_* &\text{ iff } \alpha_1 \subseteq P_* \text{ or } \alpha_2 \subseteq P_* \\ &\text{ iff } \lambda(\alpha_1) \in P \text{ or } \lambda(\alpha_2) \in P \\ &\text{ iff } \lambda([\alpha_1, \alpha_2]) = \lambda(\alpha_1) \wedge \lambda(\alpha_2) \in P \\ &\text{ iff } \lambda(\alpha) \in P. \end{aligned}$$

□

Example 1 *Following [16], let us consider on $C(A)$ the following equivalence relation $\equiv$: for all $\alpha, \beta \in C(A)$, $\alpha \equiv \beta$ if and only if $\rho(\alpha) = \rho(\beta)$. Let $\hat{\alpha}$ be the equivalence class of $\alpha \in Con(A)$ and the special elements $0 = \hat{\Delta_A}, 1 = \hat{\nabla_A}$. Then $\equiv$ is a congruence on $C(A)$ w.r.t. the join and* *the commutator operations: for all $\alpha, \beta\, \alpha', \beta' \in C(A)$, $\alpha \equiv \alpha'$ and $\beta \equiv \beta'$ implies $\alpha \vee \beta \equiv \alpha' \vee \beta'$ and $[\alpha, \beta] \equiv [\alpha', \beta']$. For all $\alpha, \beta \in C(A)$, define $\hat{\alpha} \vee \hat{\beta} = \alpha \hat{\vee} \beta$ and $\hat{\alpha} \wedge \hat{\beta} = \widehat{[\alpha, \beta]}$. Then the quotient set $L(A) = C(A)/\equiv$ is a bounded distributive lattice. We shall denote by $\lambda_A : C(A) \to L(A)$ the function defined by $\lambda_A(\alpha) = \hat{\alpha}$, for all $\alpha \in C(A)$. By construction, the pair $(L(A), \lambda_A : C(A) \to L(A))$ fulfills the axioms (Ax.1)-(Ax.3), so it is a pre-reticulation of A. We remark that for all $\alpha, \beta \in C(A)$, $\lambda_A(\alpha) = \lambda_A(\beta)$ if and only if $\rho(\alpha) = \rho(\beta)$.*

Lemma 7 ([16]) *Consider an algebra $A \in \mathcal{V}$. For all $\alpha, \beta \in C(A)$, the following hold:*

(1) $\lambda_A(\alpha) = 1$ if and only if $\alpha = \nabla_A$;

(2) If A is semiprime then $\lambda_A(\alpha) = 0$ if and only if $\alpha = \Delta_A$;

(3) $\lambda_A(\alpha) \leq \lambda_A(\beta)$ if and only if for any $\phi \in Spec(A)$, $\beta \subseteq \phi$ implies $\alpha \subseteq \phi$.

Lemma 8 ([16]) *Consider an algebra $A \in \mathcal{V}$, $\theta \in Spec(A)$ and $I \in Id(L(A))$. Then the following hold:*

(1) $\theta \neq \nabla_A$ if and only if θ^* is a proper ideal of $L(A)$;

(2) I_* is a proper ideal of $L(A)$ if and only if $I_* \neq \nabla_A$.

Lemma 9 ([16]) *Consider an algebra $A \in \mathcal{V}$ and $\phi \in Con(A)$. Then the following hold:*

(1) If $\phi \in Spec(A)$ then $(\phi^*)_* = \phi$;

(2) If $\phi \in Spec(A)$ then ϕ^* is a prime ideal of the lattice $L(A)$.

Lemma 10 ([16]) *If $\phi \in Spec(A)$ and $\alpha \in C(A)$ then $\alpha \subseteq \phi$ if and only if $\lambda(\alpha) \in \phi^*$.*

According to Lemma 9(2), one can consider the function $u : Spec(A) \to Spec_{Id}(L(A))$, defined by $u(\phi) = \phi^*$, for any $\phi \in Spec(A)$.

Lemma 11 *The following hold:*

(1) For all $I \in Id(L(A))$ and $\phi \in Spec(A)$, $I_* \subseteq \phi$ if and only if $I \subseteq \phi^*$;

(2) For all $\alpha \in C(A)$ *and* $\phi \in Spec(A)$*,* $\alpha \subseteq \phi$ *if and only if* $(\alpha^*)_* \subseteq \phi$*;*

(3) For any ideal I *of* $L(A)$*,* $u^{-1}(D_{Id}(I)) = D(I_*)$*;*

(4) For any $\alpha \in C(A)$*,* $u^{-1}(D_{Id}(\lambda_A(\alpha))) = D(\alpha)$*.*

Proof:

(1) For all $I \in Id(L(A))$ and $\phi \in Spec(A)$, the following hold:

- $I \subseteq \phi^* \Rightarrow I_* \subseteq (\phi^*)_* = \phi$ (by Lemma 9(1));
- $I_* \subseteq \phi \Rightarrow I \subseteq (I_*)^* \subseteq \phi^*$ (by Lemma 6).

(2) Assume that $\alpha \in C(A)$ and $\phi \in Spec(A)$. If $\alpha \subseteq \phi$ then $(\alpha^*)_* \subseteq (\phi^*)_* = \phi$ (by Lemma 9(1)). Conversely, if $(\alpha^*)_* \subseteq \phi$ then $\alpha \subseteq (\alpha^*)_* \subseteq \phi$ (by Lemma 6).

(3) Let I be an ideal of $L(A)$. By (1), for each $\phi \in Spec(A)$ we have $I_* \subseteq \phi$ if and only if $I \subseteq \phi^*$, therefore

$\phi \in u^{-1}(D_{Id}(I))$ iff $\phi^* \in D_{Id}(I)$ iff $I \not\subseteq \phi^*$ iff $I_* \not\subseteq \phi$ iff $\phi \in D(I_*)$.

It follows that $u^{-1}(D_{Id}(I)) = D(I_*)$.

(4) From *(2)* we infer that $D(\alpha) = D((\alpha^*)_*)$, for any $\alpha \in C(A)$. Then, by using *(3)* and Lemma 4(2), for any $\alpha \in C(A)$ the following equalities hold:

$u^{-1}(D_{Id}(\lambda_A(\alpha))) = u^{-1}(D_{Id}((\lambda_A(\alpha)])) = u^{-1}(D_{Id}(\alpha^*)) = D((\alpha^*)_*) = D(\alpha)$. □

Corollary 1 *u is an injective continuous map.*

Proof: By Lemma 11(3), it follows that u is a continuous map. In order to show that u is injective, assume that $\phi, \psi \in Spec(A)$ and $u(\phi) = u(\psi)$, hence $\phi^* = \psi^*$. According to Lemma 9(1), we get $\phi = \phi^* = \psi^* = \psi$. □

Proposition 4 ([16]) *The following properties are equivalent:*

(1) For any $I \in Id(L(A))$, $I = (I_*)^*$;

(2) For any $P \in Spec_{Id}(L(A))$, P_* is a prime congruence of A.

If the equivalent properties from Proposition 4 hold, then one can define the map $v : Spec_{Id}(L(A)) \rightarrow Spec(A)$ by $v(P) = P_*$, for any $P \in Spec_{Id}(L(A))$.

Lemma 12 ([16]) *If the equivalent properties from Proposition 4 hold, then for any $\theta \in Con(A)$ we have $v^{-1}(D(\theta)) = D_{Id}(\theta^*)$.*

Proposition 5 ([16]) *Assume that the equivalent properties from Proposition 4 hold. Then the functions $u : Spec_Z(A) \rightarrow Spec_{Id,Z}(L(A))$ and $v : Spec_{Id,Z}(L(A)) \rightarrow Spec_Z(A)$ are homeomorphisms, inverse to one another.*

The previous proposition shows that in the presence of the equivalent conditions *(1)* and *(2)* of Proposition 4 the prime spectra of A and $L(A)$ are homeomorphic (recall that this is the principal property of any notion of reticulation).

Let (L, λ) be an arbitrary pre-reticulation of A. Now it is time to introduce a new axiom:

$(Ax.4)$ For any prime ideal P of the lattice L, P_* is a prime congruence of the algebra A and the assignment $P \mapsto P_*$ defines a homeomorphism from $Spec_{Id,Z}(L)$ to $Spec_Z(A)$.

Definition 1 *A pair $(L, \lambda : C(A) \rightarrow L)$ is said to be a reticulation of the algebra A if the axioms $(Ax.1)$-$(Ax.4)$ are satisfied.*

In other words, a reticulation of A is a pre-reticulation of A fulfilling $(Ax.4)$.

We remark that this axiomatic definition of a reticulation of A is a universal algebra generalization of the axiomatic definition of reticulation of rings, introduced by Belluce in [6], p. 1856.

The reticulation $(L, \lambda : C(A) \rightarrow L)$ of A will be shortly denoted by (L, λ).

Assume that $(L(A), \lambda_A)$ is a reticulation of A. We remark that the homeomorphism from $Spec_{Id,Z}(L)$ to $Spec_Z(A)$, given by $(Ax.4)$, is exactly the restriction $v = (\cdot)_*|_{Spec_{Id,Z}(L)} : Spec_{Id,Z}(L) \rightarrow Spec_Z(A)$.

Recall from Example 1 that the pair $(L(A), \lambda_A)$ is a pre-reticulation of A. The following theorem gives some necessary and sufficient conditions for $(L(A), \lambda_A)$ to be a reticulation of A.

Theorem 1 *The following properties are equivalent:*

(1) $(L(A), \lambda_A)$ *is a reticulation of* A*;*

(2) For any $I \in Id(L(A))$*, we have* $I = (I_*)^*$*;*

(3) For any $P \in Spec_{Id}(L(A))$*,* P_* *is a prime congruence of* A*.*

Proof:

(2)⇔(3) By Proposition 4.

(1)⇒(3) By $(Ax.4)$.

(3)⇒(1) By Proposition 5. □

Proposition 6 *Assume that* (L, λ) *is a reticulation of* A*. Then for all* $\alpha, \beta \in C(A)$*,* $\lambda(\alpha) = \lambda(\beta)$ *if and only if* $\rho(\alpha) = \rho(\beta)$*.*

Proof: According to $(Ax.4)$, for any prime ideal P of the lattice L, P_* is a prime congruence of the algebra A and $v : Spec_{Id,Z}(L) \to Spec_Z(A)$ is a bijective map. In fact, u and v are order-isomorphisms between $Spec_{Id,Z}(L)$ and $Spec_Z(A)$. Then, by using Proposition 6, it follous that for all $\alpha, \beta \in C(A)$, the following properties are equivalent:

- $\lambda(\alpha) = \lambda(\beta)$;
- For all $P \in Spec_{Id}(L)$, $\lambda(\alpha) \in P$ iff $\lambda(\beta) \in P$;
- For all $P \in Spec_{Id}(L)$, $\alpha \subseteq P_*$ iff $\beta \subseteq P_*$;
- For all $\phi \in Spec(A)$, $\alpha \subseteq \phi$ iff $\beta \subseteq \phi$;
- $\rho(\alpha) = \rho(\beta)$. □

From the previous proposition, it follows that for any reticulation (L, λ) of A, the following holds:

(3.2) for all $\alpha, \beta \in C(A)$, $\lambda(\alpha) \leq \lambda(\beta)$ if and only if $\rho(\alpha) \subseteq \rho(\beta)$.

Definition 2 *A pre-reticulation* (L, λ) *of* A *is said to be a* $semi\!-\!reticulation$ *of* A *if for all* $\alpha, \beta \in C(A)$*,* $\lambda(\alpha) = \lambda(\beta)$ *if and only if* $\rho(\alpha) = \rho(\beta)$*.*

By Proposition 6, any reticulation of A is a semi-reticulation. We remark that $(L(A), \lambda_A)$ is a semi-reticulation (see Example 1).

Definition 3 *Two pre-reticulations (L_1, λ_1) and (L_2, λ_2) of the algebra A are said to be isomorphic if there exists an isomorphism of bounded distributive lattices $f : L_2 \to L_1$ such that $f \circ \lambda_1 = \lambda_2$. Two reticulations (L_1, λ_1) and (L_2, λ_2) of A are isomorphic if they are isomorphic as pre-reticulations.*

Theorem 2 *Any two semi-reticulations (L_1, λ_1) and (L_2, λ_2) of the algebra A are isomorphic.*

Proof: Assume that (L_1, λ_1) and (L_2, λ_2) are two semi-reticulations of the algebra A. According to Definition 2, for all $\alpha, \beta \in C(A)$, the following equivalences hold:

$$(3.3)\ \ \lambda_1(\alpha) = \lambda_1(\beta) \text{ iff } \rho(\alpha) = \rho(\beta) \text{ iff } \lambda_2(\alpha) = \lambda_2(\beta).$$

In order to define a function $f : L_1 \to L_2$, consider an arbitrary element $x \in L_1$, so there exists $\alpha \in C(A)$ such that $x = \lambda_1(\alpha)$. We set $f(x) = \lambda_2(\alpha)$. By (3.3), the function f is well-defined.

Let x, y be two elements of the lattice L_1, so $x = \lambda_1(\alpha), y = \lambda_2(\beta)$, for some $\alpha, \beta \in C(A)$. By using $(Ax.1)$ we get: $f(x \vee y) = f(\lambda_1(\alpha) \vee \lambda_1(\beta)) = f(\lambda_1(\alpha \vee \beta)) = \lambda_2(\alpha \vee \beta) = \lambda_2(\alpha) \vee \lambda_2(\beta) = f(x) \vee f(y)$. In a similar way we obtain $f(x \wedge y) = f(x) \wedge f(y)$, $f(0) = 0$ and $f(1) = 1$, so f is a morphism of bounded lattices.

By using (3.3) it results that f is an isomorphism in the category of bounded distributive lattices. From the definition of f we get $f \circ \lambda_1 = \lambda_2$, so the pre-reticulations (L_1, λ_1) and (L_2, λ_2) are isomorphic. □

In particular, any two reticulations of A are isomorphic. The following result gives an explicit characterization of the reticulations of A.

Proposition 7 *Let (L, λ) be a reticulation of A. Then $(L(A), \lambda_A)$ is a reticulation of A.*

Proof: Let (L, λ) be a reticulation of A. Then (L, λ) is a semi-reticulation of A, so (L, λ) and $(L(A), \lambda_A)$ are isomorphic (by Theorem 2). Therefore there exists an isomorphism $f : L \to L(A)$ of bounded distributive lattice such that $f \circ \lambda = \lambda_A$. Let P be a prime ideal of the lattice $L(A)$, so $f^{-1}(P)$ is a prime ideal of the lattice L. It is easy to see that for any $\alpha \in K(A)$, the following equivalence holds: $\lambda_A(\alpha) \in P$ if and only if $\lambda(\alpha) \in f^{-1}(P)$. Then the following equality holds:

$$(3.4)\ \bigvee\{\alpha \in K(A) | \lambda_A(\alpha) \in P\} = \bigvee\{\alpha \in K(A) | \lambda(\alpha) \in f^{-1}(P)\}.$$

Since (L, λ) is a reticulation of A and $f^{-1}(P)$ is a prime ideal of L, by $(Ax.4)$ it follows that $\bigvee\{\alpha \in K(A) | \lambda(\alpha) \in f^{-1}(P)\} \in Spec(A)$. In accordance with (3.4), we get $\bigvee\{\alpha \in K(A) | \lambda_A(\alpha) \in P\} \in Spec(A)$, so the pre-reticulation $(L(A), \lambda_A)$ verifies the equivalent conditions of Proposition 4. In virtue of Theorem 1, $(L(A), \lambda_A)$ is a reticulation of A. □

Now we shall recall from [16] the definitions of the quasi-commutative algebras and the spectral algebras. They are universal algebra generalizations of the quasi-commutative rings, respectively the spectral rings, introduced by Belluce in [6].

Definition 4 ([16]) *The algebra A is said to be quasi-commutative if for all $\alpha, \beta \in PCon(A)$ there exists $\gamma \in K(A)$ such that $\gamma \subseteq [\alpha, \beta]$ and $\rho(\gamma) = \rho([\alpha, \beta])$.*

Lemma 13 ([16]) *The following are equivalent:*

(1) A is a quasi-commutative algebra;

(2) For all $\alpha, \beta \in K(A)$ there exists $\gamma \in K(A)$ such that $\gamma \subseteq [\alpha, \beta]$ and $\rho(\gamma) = \rho([\alpha, \beta])$.

Definition 5 ([16]) *The algebra A is said to be a spectral algebra if the following conditions are fulfilled:*

(1) $Spec_Z(A)$ is a spectral space;

(2) For any compact congruence α, $D(\alpha)$ is a compact subset of $Spec_Z(A)$.

Theorem 3 *If the algebra A is quasi-commutative then the pair $(L(A), \lambda_A)$ is a reticulation of A.*

Proof: We know that $(L(A), \lambda_A)$ is a semi-reticulation of A. If A is a quasi-commutative algebra then the equivalent conditions from Proposition 4 are fulfilled (cf. Theorem 3.26 of [16]). By applying Theorem 1 it follows that $(L(A), \lambda_A)$ is a reticulation of A. □

The following theorem shows that the quasi-commutative algebras coincide with the spectral algebras and they are exactly the algebras of $\mathcal{V}$ that admit a reticulation.

Theorem 4 *For any algebra $A \in \mathcal{V}$ the following are equivalent:*

(1) A is a spectral algebra;

(2) A is a quasi-commutative algebra;

(3) For any ideal I of $L(A)$, we have $I = (I_*)^*$;

(4) For any $P \in Spec_{Id}(L(A))$, P_* is a prime congruence of A;

(5) A admits a reticulation;

(6) $(L(A), \lambda_A)$ is a reticulation of A.

Proof:

(1)⇔(2) ⇔(3) See Theorem 3.26 of [16].

(3)⇔(4) ⇔(6) See Theorem 1.

(5)⇔(6) By Proposition 7.

(6)⇔(5) Obviously. □

Remark 1 *According to [5], p.1865, there exists a semiprime ring R which is not quasi-commutative (see also [7], p.1533 and Section 7 of [25]). Therefore there exist semidegenerate congruence-modular varieties $\mathcal{V}$ and semiprime algebras A in $\mathcal{V}$ which are not quasi-commutative. According to the previous theorem, for such algebras A, the semi-reticulation $(L(A), \lambda_A)$ is not a reticulation.*

Corollary 2 *If A is a quasi-commutative algebra then $Spec_Z(A)$ is a spectral space.*

Proof: By Theorem 4, A is a spectral algebra, so the $Spec_Z(A)$ is a spectral space. □

The neo-commutative rings were introduced by Kaplansky in [24]: a ring R is neo-commutative if the product of two finitely generated ideals is a finitely generated ideal. Kaplansky proved that the prime spectrum of a neo-commutative ring is a spectral space.

The notion of neo-commutative ring can be extended to a universal algebra setting: an algebra A in a semidegenerate congruence-modular variety $\mathcal{V}$ is neo-commutative if $K(A)$ is closed under commutator operation. If A is neo-commutative then $C(A) = K(A)$, so A is quasi-commutative (see [16]). Then, for any neo-commutative algebra, the equivalent six conditions from Theorem 4 are fulfilled. In particular, a generalization of the Kaplansky theorem holds: if the algebra A is neo-commutative then its prime spectrum $Spec_Z(A)$ is a spectral space.

Let A be an algebra of $\mathcal{V}$ and $\theta \in Con(A)$. We inductively define $[\theta,\theta]^n$, for any integer $n \geq 0$: $[\theta,\theta]^0 = \theta$, $[\theta,\theta]^1 = [\theta,\theta]$ and $[\theta,\theta]^{n+1} = [[\theta,\theta]^n,[\theta,\theta]^n]$, for all n.

Lemma 14 ([18]) *Assume that A is a neo-commutative algebra. For any congruence θ of A, $\rho(\theta) = \bigvee\{\alpha \in K(A) | [\alpha,\alpha]^n \subseteq \theta$, for all $n\}$.*

By the previous lemma, if A is a neo-commutative algebra, then for all $\theta \in Con(A)$ and $\alpha \in K(A)$ the following equivalence holds: $\alpha \subseteq \rho(\theta)$ if and only if $[\alpha,\alpha]^n \subseteq \theta$, for some integer $n \geq 0$.

Theorem 5 *Let A be a neo-commutative algebra and (L,λ) a pre-reticulation of A. Then the following are equivalent:*

(1) (L,λ) *is a reticulation of A;*

(2) For all $\alpha,\beta \in K(A)$, $\lambda(\alpha) \leq \lambda(\beta)$ if and only if $[\alpha,\alpha]^n \leq \beta$, for some integer $n \geq 0$.

Proof:

(1)⇔(2) Assume that (L,λ) is a reticulation of A. Let α,β be two compact congruences of A. According to (3.2) and Lemma 14, the following equivalences hold: $\lambda(\alpha) \leq \lambda(\beta)$ iff $\rho(\alpha) \subseteq \rho(\beta)$ iff $\alpha \subseteq \rho(\beta)$ iff $[\alpha,\alpha]^n \leq \beta$, for some integer $n \geq 0$.

(2)⇔(1) Let α,β be two compact congruences of A. By using the hypothesis (2) and Lemma 14, the following equivalences hold: $\lambda(\alpha) \leq \lambda(\beta)$ iff $[\alpha,\alpha]^n \leq \beta$, for some integer $n \geq 0$ iff $\alpha \subseteq \rho(\beta)$ iff $\rho(\alpha) \subseteq \rho(\beta)$. Therefore, for all $\alpha,\beta \in K(A)$, $\lambda(\alpha) = \lambda(\beta)$ if and only if $\rho(\alpha) = \rho(\beta)$, i.e. (L,λ) a semi-reticulation of A. According to Theorem 2, the semi-reticulations (L,λ) and $(L(A),\lambda_A)$ are isomorphic. By hypothesis, A is a neo-commutative algebra, so it is quasi-commutative. In virtue of Theorem 3, $(L(A),\lambda_A)$ is a reticulation of A, hence (L,λ) is a reticulation of A (cf. Proposition 7). □

4 The Spectral Closure

Let $\mathcal{V}$ be a semidegenerate congruence - modular variety and A an algebra of $\mathcal{V}$. In Section 3 we defined the map $u : Spec_Z(A) \to Spec_{Id,Z}(L(A))$ by $u(\phi) = \phi^*$, for any $\phi \in Spec_Z(A)$. By Corollary 1, u is an injective continuous map. In general, $Spec_Z(A)$ is not a spectral space, so u is not bijective.

In [5], Belluce studied the spectral closure $XSpec(R)$ of the prime spectrum $Spec(R)$ of an arbitrary (unital) ring R. $XSpec(R)$ is a spectral space that contains $Spec(R)$ as a dense subspace. A quantale version of this spectral closure can be found in [11].

This section contains a generalization of the Belluce spectral closure to a universal algebra framework: we shall enlarge the prime spectrum $Spec_Z(A)$ of the algebra $A \in \mathcal{V}$ to a spectral space $XSpec_Z(A)$ such that $Spec_Z(A)$ is a dense subspace of $XSpec_Z(A)$. The construction and the study of $XSpec_Z(A)$ will use the algebraic and topological transfer properties of the lattice $L(A)$.

We mention that the results obtained in this section are generalizations of some results proven by Belluce for the case of rings [7].

Proposition 8 *Let S be a non-empty subset of $Spec_Z(A)$ such that $\bigcap S = \Delta_A$. Thus $u(S)$ is a dense subset of the space $Spec_{Id,Z}(L(A))$.*

Proof: We observe that $\bigcap S = \Delta_A$ implies $\bigcap Spec(A) = \Delta_A$, hence the algebra A is semiprime. According to the definition of u, we have $u(S) = \{\phi^* | \phi \in S\}$. We have to prove that for any congruence $\alpha \in C(A)$, $D_{Id}(\lambda_A(\alpha)) \neq \emptyset$ implies $D_{Id}(\lambda_A(\alpha)) \cap u(S) \neq \emptyset$.

Assume that $D_{Id}(\lambda_A(\alpha)) \neq \emptyset$, so there exists a prime ideal P of the lattice $L(A)$ such that $\lambda_A(\alpha) \notin P$. Thus $\alpha \neq \Delta_A$, so there exists $\phi \in S$ such that $\alpha \not\subseteq \phi$ (because $\bigcap S = \Delta_A$). By Lemma 10, $\alpha \not\subseteq \phi$ implies $\lambda_A(\alpha) \notin \phi^*$, so $\phi^* \in D_{Id}(\lambda_A(\alpha)) \cap u(S) \neq \emptyset$.

It results that $u(S)$ is a dense subset of the space $Spec_{Id,Z}(L(A))$. $\square$
Let $Min(A)$ be the set of minimal prime congruences of A. $Min(A)$ is called the minimal prime spectrum of the algebra A. If we restrict the topology of $Spec_Z(A)$ to $Min(A)$ then we obtain a topological space, denoted by $Min_Z(A)$. For any $\phi \in Spec(A)$ there exists $\psi \in Min(A)$ such that $\psi \subseteq \phi$ (by Zorn lemma), therefore for any semiprime algebra A we have $\bigcap Min(A) = \Delta_A$.

Corollary 3 *If the algebra A is semiprime then $u(Spec(A))$ and $u(Min(A))$ are dense subsets of $Spec_{Id,Z}(L(A))$.*

Throughout the rest of this section A will be a semiprime algebra.

Definition 6 *[6] Let X, Y be two topological spaces such that Y is a spectral space. A continuous map $f : X \to Y$ is said to be spectral if for any compact subset U of Y, $f^{-1}(U)$ is a compact subset of X.*

Proposition 9 *If $u : Spec_Z(A) \to Spec_{Id,Z}(L(A))$ is a spectral map then u is a homeomorphism.*

Proof: We know from the Stone duality theory [4] that $\mathcal{C} = (D_{Id}(\lambda_A(\alpha)))_{\alpha \in C(A)}$ is the basis of compact open sets of $Spec_{Id,Z}(L(A))$ and $\mathcal{C}$ is closed under finite intersections. Let us consider the family $\mathcal{B} = u^{-1}(D_{Id}(\lambda_A(\alpha)))_{\alpha \in C(A)}$. Since u is a spectral map, $\mathcal{B}$ is a family of compact open subsets of $Spec_Z(A)$. It is clear that $\mathcal{B}$ is closed under finite intersections. In accordance with Lemma 11(4), $\mathcal{B} = (D(\alpha))_{\alpha \in C(A)}$.

An open subset of $Spec_Z(A)$ has the form $D(\theta)$, where θ is an arbitrary congruence of A. We remark that

$$D(\theta) = D(\bigvee\{\alpha \in K(A) | \alpha \subseteq \theta\}) = \bigcup\{D(\alpha) | \alpha \in K(A), \alpha \subseteq \theta\}.$$

It follows that $\mathcal{B}$ is a basis of $Spec_Z(A)$. We proved that $Spec_Z(A)$ has a basis $\mathcal{B}$ of compact open sets, closed under finite intersections and, by Lemma 2, we know that $Spec_Z(A)$ is a compact sober space. Then $Spec_Z(A)$ is a spectral space such that for any $\alpha \in C(A)$, $D(\alpha)$ is a compact open subset of $Spec_Z(A)$, so A is a spectral algebra. By Theorem 4, the equivalent conditions from Proposition 4 are verified, so u is a homeomorphism (cf. Proposition 5). □

Proposition 10 *If $u : Spec_Z(A) \to Spec_{Id,Z}(L(A))$ is a surjective map then u is a homeomorphism.*

Proof: We shall prove that u is a spectral map. We know that the prime spectrum $Spec_{Id,Z}(L(A))$ of the lattice $L(A)$ is a spectral space. Let U be a compact subset of $Spec_{Id,Z}(L(A))$. We have to show that $u^{-1}(U)$ is a compact subset of $Spec_Z(A)$. Assume that $u^{-1}(U) \subseteq \bigcup_{j \in J} D(\alpha_j)$, whenever $(\alpha_j)_{j \in J}$ is a family of compact congruences of A.

Let P be an element of U, hence $P \in Spec_{Id,Z}(L(A))$. Since u is a surjective map, there exists $\phi \in Spec_Z(A)$ such that $P = u(\phi) = \phi^*$, so $\phi \in u^{-1}(U) \subseteq \bigcup_{j \in J} D(\alpha_j)$. Thus $\phi \in D(\alpha_j)$, for some $j \in J$, hence $\alpha_j \not\subseteq \phi$. By Lemma 10, $\alpha_j \not\subseteq \phi$ implies $\lambda_A(\alpha_j) \notin \phi^* = P$, therefore $P \in D_{Id}(\lambda_A(\alpha_j))$. We have proven that $U \subseteq \bigcup_{j \in J} D_{Id}(\lambda_A(\alpha_j))$. But U is a compact subset of $Spec_{Id,Z}(L(A))$, so there exists a finite subset T of J such that $U \subseteq \bigcup_{j\in T} D_{Id}(\lambda_A(\alpha_j))$. Then the following hold:

$$u^{-1}(U) \subseteq u^{-1}(\bigcup_{j\in T} D_{Id}(\lambda_A(\alpha_j))) = \bigcup_{j\in T} u^{-1}(D_{Id}(\lambda_A(\alpha_j))).$$

By applying Lemma 11(4), we obtain $u^{-1}(D_{Id}(\lambda_A(\alpha_j))) = D(\alpha_j)$, for any $j \in T$, therefore $u^{-1}(U) \subseteq \bigcup_{j\in T} D(\alpha_j)$, so $u^{-1}(U)$ is a compact subset of $Spec_Z(A)$. Thus u is a spectral map. By using Proposition 9, it follows that u is a homeomorphism. □

Proposition 11 *Let I be an ideal of the lattice $L(A)$ and P a prime ideal of $L(A)$. If $P_* \subseteq I_*$ then $P \subseteq I$.*

Proof: Assume that $P_* \subseteq I_*$. In order to show that $P \subseteq I$, it suffices to prove that the following sentence is valid:

(4.1) $\forall \alpha \in C(A)[\lambda_A(\alpha) \in P \Rightarrow \lambda_A(\alpha) \in I]$.

We shall prove this fact by induction on how $C(A)$ is defined, so we consider three cases:

(a) Assume that $\alpha \in K(A)$. According to Lemma 5, the following implications hold:

$$\lambda_A(\alpha) \in P \Rightarrow \alpha \subseteq P_* \Rightarrow \alpha \subseteq I_* \Rightarrow \lambda_A(\alpha) \in I.$$

(b) Assume that $\alpha = \alpha_1 \vee \alpha_2$, where the congruences $\alpha_1, \alpha_2 \in C(A)$ verify the sentence (4.1). Thus we have $\lambda_A(\alpha) = \lambda_A(\alpha_1) \vee \lambda_A(\alpha_2)$, therefore the following implications hold: $\lambda_A(\alpha) \in P \Rightarrow \lambda_A(\alpha_1) \in P$ and $\lambda_A(\alpha_2) \in P \Rightarrow \lambda_A(\alpha_1) \in I$ and $\lambda_A(\alpha_2) \in I \Rightarrow \lambda_A(\alpha) \in I$.

(c) Assume that $\alpha = [\alpha_1, \alpha_2]$, where $\alpha_1, \alpha_2 \in C(A)$ verify (4.1), hence $\lambda_A(\alpha) = \lambda_A(\alpha_1) \wedge \lambda_A(\alpha_2)$. Recall that P is a prime ideal of $L(A)$. Thus the following implications hold: $\lambda_A(\alpha) \in P \Rightarrow \lambda_A(\alpha_1) \in P$ or $\lambda_A(\alpha_2) \in P \Rightarrow \lambda_A(\alpha_1) \in I$ or $\lambda_A(\alpha_2) \in I \Rightarrow \lambda_A(\alpha) \in I$. □

Consider the function $w : Spec_{Id}(L(A)) \to Con(A)$, i.e. $w(P) = P_*$, for any $P \in Spec_{Id}(L(A))$.

Corollary 4 *w is an injective function.*

Proof: Let P, Q be two prime ideals of $L(A)$ such that $P_* = Q_*$. By Proposition 11 we get $P = Q$. Therefore the function w is injective. □

By keeping the notation of [7], we set $XSpec(A) = w(Spec_{Id}(L(A))) = \{P_* | P \in Spec_{Id}(L(A))\}$. For each ideal I of $L(A)$ take the set $w(D_{Id}(I)) = \{P_* | P \in Spec_{Id}(L(A)), I \not\subseteq P\}$.

Lemma 15 *The family $\mathcal{T} = (w(D_{Id}(I)))_{I \in Id(L(A))}$ is a topology on $XSpec(A)$.*

Proof: Let $(I_t)_{t\in T}$ be a family of ideals in $L(A)$. Then we obtain the following equality: $\bigcup_{t\in T} w(D_{Id}(I_t))) = w(D_{Id}(\bigvee_{t\in T} I_t))$. Assume now that T is a finite set. Since u is an injective function, we have $\bigcap_{t\in T} w(D_{Id}(I_t))) = w(D_{Id}(\bigwedge_{t\in T} I_t))$. Therefore $\mathcal{T}$ is a topology on $XSpec(A)$. □

The topological space introduced by Lemma 15 will be denoted by $XSpec_Z(A)$. By construction, $w : Spec_{Id,Z}(L(A)) \rightarrow XSpec_Z(A)$ is a homeomorphism. We know that $Spec_{Id,Z}(L(A))$ is a spectral space, so $XSpec_Z(A)$ is also a spectral space. By Lemma 9, for any $\phi \in Spec_Z(A)$ we have $\phi^* \in Spec_{Id,Z}(L(A))$ and $(\phi^*)_* = \phi$, hence $\phi \in XSpec_Z(A)$. It follows that $Spec_Z(A)$ is a subset of $XSpec_Z(A)$.

Lemma 16 *$Spec_Z(A)$ is a subspace of $XSpec_Z(A)$.*

Proof: An open subset of $Spec_Z(A)$ has the form $D(\theta)$, where θ is an arbitrary congruence of A. We shall prove that $D(\theta) = w(D_{Id}(\theta^*)) \cap Spec_Z(A)$. Recall that $w(D_{Id}(\theta^*)) = \{P_* | P \in Spec_{Id}(L(A)), \theta^* \not\subseteq P\}$.

Firstly, we shall prove that $D(\theta) \subseteq w(D_{Id}(\theta^*)) \cap Spec_Z(A)$. Assume that $\phi \in D(\theta)$, so $\phi \in Spec_Z(A)$ and $\theta \not\subseteq \phi$. Assume by absurdum that $\theta^* \subseteq \phi^*$, hence, by using Lemmas 6 and 9, we get $\theta \subseteq (\theta^*)_* \subseteq (\phi^*)_* = \phi$, contradicting $\theta \not\subseteq \phi$. Thus $\theta^* \not\subseteq \phi^*$, hence $\phi^* \in D_{Id}(\theta^*)$. Then we obtain $\phi = (\phi^*)_* \in w(D_{Id}(\theta^*))$. The inclusion $D(\theta) \subseteq w(D_{Id}(\theta^*)) \cap Spec_Z(A)$ is proven.

Conversely, assume that $\phi \in w(D_{Id}(\theta^*)) \cap Spec_Z(A)$, therefore there exists $P \in Spec_{Id}(L(A))$ such that $\phi = P_*$ and $\theta^* \not\subseteq P$. Assume by absurdum that $\theta \subseteq P_*$. Let x be an arbitrary element of θ^*, so $x = \lambda_A(\alpha)$, for some $\alpha \in C(A)$ such that $\alpha \subseteq \theta$. Then $\alpha \subseteq \theta \subseteq P_*$, so $x = \lambda_A(\alpha) \in P$ (cf. Proposition 3). It follows that $\theta^* \subseteq P$, contradicting $\theta^* \not\subseteq P$. Then $\theta \not\subseteq P_*$, resulting that $\phi = P_* \in D(a)$. In conclusion, the inclusion $w(D_{Id}(\theta^*)) \cap Spec_Z(A) \subseteq D(\theta)$ is established. □

Recall that we assumed that the algebra A is semiprime. By applying Lemma 16 and Corollary 3 it follows that $Spec_Z(A)$ is a dense subspace of $XSpec_Z(A)$. Keeping the terminology of [7], the spectral space $XSpec_Z(A)$ will be called the spectral closure of the prime spectrum $Spec_Z(A)$. If A is a ring then we obtain the notion of spectral closure defined in [7].

Definition 7 *A congruence $\phi \neq \nabla_A$ of the algebra A is said to be a locally prime congruence of A if for all congruences $\alpha, \beta \in C(A)$ and $\gamma \in K(A)$,*

$\gamma \subseteq \phi$ and $[\alpha, \beta] \subseteq \rho(\gamma)$ imply that there exists a compact congruence $\gamma' \subseteq \phi$ with $\alpha \subseteq \gamma'$ or $\beta \subseteq \gamma'$.

The notion of locally prime congruence is a generalization of the notion of locally prime ideal in ring theory [7]: an ideal P of a ring R is locally prime if and only if the congruence of R associated with P is a locally prime congruence.

Proposition 12 *Any prime congruence of A is locally prime.*

Proof: Let ϕ be a prime congruence of A. By induction on how the set $C(A)$ is defined we shall prove that the following sentence is true:

$$(4.2)\ \forall \alpha \in C(A)[\alpha \subseteq \phi \Rightarrow \exists \gamma' \in K(A)[(\gamma' \subseteq \phi) \text{ and } (\alpha \subseteq \gamma')]].$$

We shall distinguish three cases:

(a) Assume that $\alpha \in K(A)$. If $\alpha \subseteq \phi$, then, by taking $\gamma' = \alpha$, the sentence (4.2) is obviously satisfied.

(b) Assume that $\alpha = \alpha_1 \vee \alpha_2$ and the congruences $\alpha_1, \alpha_2 \in C(A)$ satisfy the sentence (4.2). If $\alpha \subseteq \phi$ then $\alpha_1 \subseteq \phi$ and $\alpha_2 \subseteq \phi$, so there exist $\gamma_1', \gamma_2' \in K(A)$ such that $\gamma_i' \subseteq \phi$ and $\alpha_i \subseteq \gamma_i'$, for $i = 1, 2$. If we set $\gamma' = \gamma_1' \vee \gamma_2'$ then $\gamma' \in K(A)$, $\gamma' \subseteq \phi$ and $\alpha = \alpha_1 \vee \alpha_2 \subseteq \gamma_1' \vee \gamma_2' = \gamma'$.

(c) Assume that $\alpha = [\alpha_1, \alpha_2]$ and the congruences $\alpha_1, \alpha_2 \in C(A)$ satisfy (4.2). If $\alpha \subseteq \phi$ then $\alpha_1 \subseteq \phi$ or $\alpha_2 \subseteq \phi$ (because ϕ is a prime congruence of A). Assume that $\alpha_1 \subseteq \phi$ so there exists $\gamma_1' \in K(A)$ such that $\gamma_1' \subseteq \phi$ and $\alpha_1 \subseteq \gamma_1'$. If we set $\gamma' = \gamma_1'$ then $\gamma' \in K(A)$, $\gamma' \subseteq \phi$ and $\alpha \subseteq \alpha_1 \subseteq \gamma'$. The case $\alpha_2 \subseteq \phi$ is treated in a similar way.

In order to show that ϕ is locally prime, suppose that the congruences $\alpha, \beta \in C(A)$ and $\gamma \in K(A)$ fulfill $\gamma \subseteq \phi$ and $[\alpha, \beta] \subseteq \rho(\gamma)$. From $[\alpha, \beta] \subseteq \rho(\gamma)$ and $\gamma \subseteq \phi \in Spec(A)$ we get $[\alpha, \beta] \subseteq \phi$, so $\alpha \subseteq \phi$ or $\beta \subseteq \phi$.

Assume now that $\alpha \subseteq \phi$. By applying (4.2), there exists $\gamma' \in K(A)$ such that $\gamma' \subseteq \phi$ and $\alpha \subseteq \gamma'$. The case $\beta \leq \phi$ is treated in a similar way.

□

Proposition 13 *If ϕ is a locally prime congruence of A then the following hold:*

(1) *If* $\phi \in K(A)$ *then* $\phi \in Spec(A)$*;*

(2) *If* A *is quasi-commutative then* $\phi \in Spec(A)$*.*

Proof:

(a) Let α, β be two compact congruences of A such that $[\alpha, \beta] \subseteq \phi$. Since $[\alpha, \beta] \subseteq \phi = \rho(\phi)$ and $\phi \in K(A)$ there exists a compact congruence $\gamma' \subseteq \phi$ such that $\alpha \subseteq \gamma'$ or $\beta \subseteq \gamma'$. Then $\alpha \subseteq \phi$ or $\beta \subseteq \phi$, so $\phi \in Spec(A)$.

(b) Let α, β be two compact congruences of A such that $[\alpha, \beta] \subseteq \phi$. Since A is quasi-commutative and $\alpha, \beta \in K(A)$ there exists $\gamma \in K(A)$ such that $\gamma \subseteq [\alpha, \beta]$ and $\rho(\gamma) = \rho([\alpha, \beta])$ (cf. Lemma 13). Thus $[\alpha, \beta] \subseteq \rho(\gamma)$, $\gamma \in K(A)$ and $\gamma \subseteq \phi$, so, by taking into account that ϕ is a locally prime congruence, it follows that there exists a compact congruence $\gamma' \subseteq \phi$ such that $\alpha \subseteq \gamma'$ or $\beta \subseteq \gamma'$, therefore $\alpha \subseteq \phi$ or $\beta \subseteq \phi$. Conclude that $\phi \in Spec(A)$.

□

The following theorem generalizes Proposition 3.2 of [7].

Theorem 6 *For any* $\phi \in Con(A)$*, the following are equivalent:*

(1) $\phi \in XSpec(A)$*;*

(2) ϕ *is a locally prime congruence.*

Proof:

(1)⇒(2) Assume that $\phi \in XSpec(A)$, so $\phi = P_*$, for some prime ideal P of the lattice $L(A)$. By induction on how the set $C(A)$ is defined we shall prove that the following sentence is true:

$$\forall \alpha \in C(A)[\lambda_A(\alpha) \in P \Rightarrow \exists \gamma' \in K(A)[(\gamma' \subseteq \phi) \text{ and } (\alpha \subseteq \gamma']]. \quad (4.3)$$

We shall distinguish three cases:

(a) Assume that $\alpha \in K(A)$. If $\lambda_A(\alpha) \in P$ then $\alpha \subseteq P_* = \phi$. By taking $\gamma' = \alpha$, the sentence (4.3) is obviously verified.

(b) Assume that $\alpha = \alpha_1 \vee \alpha_2$, whenever the congruences $\alpha_1, \alpha_2 \in C(A)$ satisfy the sentence (4.3). If $\lambda_A(\alpha) \in P$, then $\lambda_A(\alpha_1) \vee \lambda_A(\alpha_2) \in P$, therefore $\lambda_A(\alpha_i) \in P$, for $i = 1, 2$. By applying the induction hypothesis, there exist $\gamma'_1, \gamma'_2 \in K(A)$ such that $\gamma'_i \subseteq \phi$ and $\alpha_i \subseteq \gamma'_i$, for $i = 1, 2$. If we take $\gamma' = \gamma'_1 \vee \gamma'_2$ then $\gamma' \in K(A)$, $\gamma' \subseteq \phi$ and $\alpha = \alpha_1 \vee \alpha_2 \subseteq \gamma'_1 \vee \gamma'_2 = \gamma'$.

(c) Assume that $\alpha = [\alpha_1, \alpha_2]$ and $\alpha_1, \alpha_2 \in C(A)$ satisfy (4.3). If $\lambda_A(\alpha) \in P$, then $\lambda_A(\alpha_1) \wedge \lambda_A(\alpha_2) = \lambda_A(\alpha) \in P$, therefore $\lambda_A(\alpha_1) \in P$ or $\lambda_A(\alpha_2) \in P$. Assume that $\lambda_A(\alpha_1) \in P$. By applying the induction hypothesis, there exists $\gamma'_1 \in K(A)$ such that $\gamma'_1 \subseteq \phi$ and $\alpha_1 \subseteq \gamma'_1$. If we set $\gamma' = \gamma'_1$, then $\gamma' \subseteq \phi$ and $\alpha \subseteq \alpha_1 \subseteq \gamma'$. The case $\lambda_A(\alpha_2) \in P$ is treated similarly.

Then the sentence (4.3) is true. In order to prove that ϕ is a locally prime congruence consider the congruences $\alpha, \beta \in C(A)$ and $\gamma \in K(A)$ such that $\gamma \subseteq \phi$ and $[\alpha, \beta] \subseteq \rho(\gamma)$. By applying Lemma 5, from $\gamma \in K(A)$ and $\gamma \subseteq P_*$ we get $\lambda_A(\gamma) \in P$. We observe that $[\alpha, \beta] \subseteq \rho(\gamma)$ implies $\rho([\alpha, \beta]) \subseteq \rho(\gamma)$, hence $\lambda_A([\alpha, \beta]) \leq \lambda_A(\gamma)$ (cf. (3.2)). Thus $\lambda_A(\alpha) \wedge \lambda_A(\beta) \in P$, so $\lambda_A(\alpha) \in P$ or $\lambda_A(\beta) \in P$.

Assume that $\lambda_A(\alpha) \in P$. By using (4.3) we can find a congruence $\gamma' \in K(A)$ such that $\gamma' \subseteq \phi$ and $\alpha \subseteq \gamma'$, so ϕ is a locally prime congruence. The case $\lambda_A(\beta) \in P$ is treated in a similar way.

(2)⇒(1) Suppose that ϕ is a locally prime congruence. Let Q be the ideal of the lattice $L(A)$ generated by the set $\{\lambda_A(\alpha) | \alpha \in K(A), \alpha \subseteq \phi\}$.

Firstly, we shall prove that Q is a prime ideal of the lattice $L(A)$. Let α, β be two congruences in $C(A)$ such that $\lambda_A(\alpha) \wedge \lambda_A(\beta) \in P$, hence $\lambda_A([\alpha, \beta]) \in Q$. According to the definition of Q, there exist an integer $n \geq 1$ and the congruences $\gamma_1, \cdots, \gamma_n \in K(A)$ such that $\gamma_i \subseteq \phi$, for any $i = 1, \cdots, n$ and $\lambda_A([\alpha, \beta]) \leq \bigvee_{i=1}^n \lambda_A(\gamma_i) = \lambda_A(\bigvee_{i=1}^n \gamma_i)$. Denoting $\gamma = \bigvee_{i=1}^n \gamma_i$, we obtain $\gamma \in K(A)$, $\gamma \subseteq \phi$ and $\lambda_A([\alpha, \beta]) \leq \lambda_A(\gamma)$, therefore $[\alpha, \beta] \subseteq \rho([\alpha, \beta]) \subseteq \rho(\gamma)$ (by the equivalence (3.1)). Since ϕ is a locally prime congruence, there exists a compact congruence $\gamma' \subseteq \phi$ such that $\alpha \subseteq \gamma'$ or $\beta \subseteq \gamma'$. If $\alpha \subseteq \gamma'$ then $\lambda_A(\alpha) \leq \lambda_A(\gamma') \in Q$, so $\lambda_A(\alpha) \in Q$. Similarly, $\beta \subseteq \gamma'$ implies $\lambda_A(\beta) \in Q$, so Q is a prime ideal.

Now we shall prove that $\phi = Q_*$. For any compact congruence α, from $\alpha \subseteq \phi$ we get $\lambda_A(\alpha) \in Q$, so $\alpha \subseteq Q_*$. We proved that $\phi \subseteq Q_*$.

In order to prove the converse inclusion $Q_* \subseteq \phi$, consider a compact congruence β such that $\lambda_A(\beta) \in Q$, so there exist an integer $n \geq 1$ and the congruences $\alpha_1, \cdots, \alpha_n \in K(A)$ such that $\lambda_A(\beta) \leq \bigvee_{i=1}^n \lambda_A(\alpha_i)$ and $\alpha_i \subseteq \phi$, for all $i = 1, \cdots, n$. Denoting $\alpha = \bigvee_{i=1}^n \alpha_i$ we have $\alpha \in K(A)$, $\alpha \subseteq \phi$ and $\lambda_A(\beta) \leq \lambda_A(\alpha)$. In accordance with the equivalence (3.1), we get $\rho(\beta) \leq \rho(\alpha)$.

We observe that $\rho([\beta, \nabla_A]) = \rho(\beta) \leq \rho(\alpha)$ and α, β and ∇_A are compact congruences, hence there exists a compact congruence $\delta \subseteq \phi$ such that $\beta \subseteq \delta$ or $\nabla_A \subseteq \delta$ (because ϕ is a locally prime congruence). But $\nabla_A \subseteq \delta$ implies $\nabla_A \subseteq \delta \subseteq \phi$, contradicting that ϕ is locally prime. It follows that $\beta \subseteq \delta \subseteq \phi$. Then we get $Q_* = \bigvee\{\beta \in K(A) | \lambda_A(\beta) \in Q\} \subseteq \phi$. It follows that $\phi = P_*$ and $Q \in Spec_{Id}(L(A))$, therefore $\phi \in XSpec(A)$. □

We remark that the elements of $XSpec_Z(A)$ are defined by using the lattice $L(A)$ and the map w, while the definition of locally prime congruences does not depend by these notions. Thus the previous theorem offers an intrinsic characterization of the spectral closure $XSpec_Z(A)$.

Corollary 5 *The following are equivalent:*

(1) $Spec(A) = XSpec(A)$*;*

(2) A *is quasi-commutative.*

Proof: *(1)⇒(2)* Let P be a prime ideal of the lattice $L(A)$, hence $P_* \in XSpec(A)$ (by definition). According to the hypothesis (1), it follows that $P_* \in Spec(A)$. Taking into account Theorem 4, A is a quasi-commutative algebra.

(2)⇒(1) Assume that A is quasi-commutative. In order to prove that $XSpec_Z(A) \subseteq Spec_X(A)$, suppose that $\phi \in XSpec_Z(A)$, hence ϕ is a locally prime congruence (by Theorem 6). In virtue of Proposition 13(2), it follows that $\phi \in Spec_Z(A)$, therefore $XSpec_Z(A) \subseteq Spec_Z(A)$.

The converse inclusion $Spec(A) \subseteq XSpec(A)$ was established in Lemma 16, so $Spec_Z(A) = XSpec_Z(A)$. □

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