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A Study on Centralizing Monoids with Majority Operation Witnesses[1]

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Abstract

A centralizing monoid M is a set of unary operations which commute with some set F of operations. Here, F is called a witness of M. On a 3-element set, a centralizing monoid is maximal if and only if it has a constant operation or a majority minimal operation as its witness.

In this paper, we take one such majority operation, which corresponds to a maximal centralizing monoid, on a 3-element set and obtain its generalization, called m_{b}, on a k-element set for any $k \geq 3$. We explicitly describe the centralizing monoid $\mathsf{M}(m_{\mathrm{b}})$ with m_{b} as its witness and then prove that it is not maximal if $k > 3$, contrary to the case for $k = 3$.

Keywords: clone; centralizer; centralizing monoid; majority operation; minimal operation

1 Introduction

1.1 Overview

Let A be a finite set with $|A| > 2$, and $\mathcal{O}_A$ be the set of operations on A. A *majority operation* $m \in \mathcal{O}_A$ is a ternary operation, i.e., $m : A^3 \rightarrow A$, which

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takes the majority value among the elements in the argument, i.e., $m(x,x,y) = m(x,y,x) = m(y,x,x) = x$ holds for all $x, y \in A$.

A centralizer $C\,(\subseteq \mathcal{O}_A)$ is the set of operations which commute with all members of some set $F \subseteq \mathcal{O}_A$, and a centralizing monoid M is the unary part of a centralizer C. We call F a witness of the centralizing monoid M. (For the precise definition of some terms, refer to the next subsection.)

A remarkable fact on a 3-element set A is that a centralizing monoid is maximal if and only if it has a constant operation or a majority minimal operation as its witness [3].

Up to conjugacy, there are three majority operations on a 3-element set which are minimal and serve as witnesses of maximal centralizing monoids. In this article, they are called m_1, m_2 and m_3.

The aim of our study is to know how these properties can be generalized from a 3-element set to a k-element set with $3 \le k < \omega$. For m_1 and m_3, generalizations were presented in [4], which are summarized in Section 3.

The main part of this paper is Section 4 where we generalize the remaining majority operation m_2 on a 3-element set to a majority operation on a k-element set for any $3 \le k < \omega$. A generalization is successfully achieved, but it fails to inherit the property of maximality from a 3-element case.

1.2 Basic Terminology

Let $k > 1$ be a fixed integer and E_k be the initial segment of $\mathbb{N}$ with k elements, i.e., $E_k = \{0, 1, \ldots, k-1\}$. Denote by $\mathcal{O}_k^{(n)}$, $n > 0$, the set of n-ary operations on E_k, that is, functions from E_k^n into E_k, and by $\mathcal{O}_k$ the set of all operations on E_k, i.e., $\mathcal{O}_k = \bigcup_{n=1}^{\infty} \mathcal{O}_k^{(n)}$.

The n-ary i-th *projection* e_i^n on E_k, for $1 \le i \le n$, is an operation in $\mathcal{O}_k^{(n)}$ which is defined by $e_i^n(x_1, \ldots, x_n) = x_i$ for $x_1, \ldots, x_n \in E_k$. Denote by $\mathcal{J}_k$ the set of projections on E_k.

Let $\mathrm{CONST}_k\,(\subseteq \mathcal{O}_k^{(1)})$ (or simply CONST) be the set of unary constant operations on E_k and $\mathcal{S}_k$ be the symmetric group on E_k.

For f, $g \in \mathcal{O}_A^{(n)}$, g is *conjugate* to f if there exists a permutation σ on A for which $g(x_1, \ldots, x_n) = \sigma^{-1}(f(\sigma(x_1), \ldots, \sigma(x_n)))$ holds for all $x_1, \ldots, x_n \in A$. In other words, g is conjugate to f if g is obtained from f by renaming the elements of the base set E_k.

A subset C of $\mathcal{O}_k$ is a *clone* on E_k if C contains all the projections, i.e., $\mathcal{J}_k \subseteq C$, and is closed under (functional) composition. The set of clones on E_k forms a lattice with respect to set inclusion and is denoted by $\mathcal{L}_k$. For

$F \subseteq \mathcal{O}_k$, $\langle F \rangle$ denotes the smallest clone containing F. We say F *generates* a clone C if $C = \langle F \rangle$. When $F = \{f\}$, we often write $\langle f \rangle$ instead of $\langle F \rangle$.

An atom of $\mathcal{L}_k$ is called a *minimal clone*. In other words, C $(\in \mathcal{L}_k)$ is a minimal clone if $\mathcal{J}_k \subset C' \subseteq C$ implies $C = C'$ for any C' in $\mathcal{L}_k$. Clearly, a minimal clone is generated by a singleton set. An operation $f\,(\in \mathcal{O}_k)$ is called a *minimal operation* if f generates a minimal clone C and its arity is minimum among the arities of all operations which generate C.

For n-ary operation $f \in \mathcal{O}_k^{(n)}$ and m-ary operation $g \in \mathcal{O}_k^{(m)}$ for any m, $n \geq 1$, we say that f *commutes* with g, or f and g *commute*, if

$$\begin{aligned} & g(f(x_{11}, x_{12}, \ldots, x_{1n}), \ldots, f(x_{m1}, \ldots, x_{mn})) \\ = \;& f(g(x_{11}, x_{21}, \ldots, x_{m1}), \ldots, g(x_{1n}, \ldots, x_{mn})) \end{aligned}$$

holds for all $x_{ij} \in E_k$ where $1 \leq i \leq m$ and $1 \leq j \leq n$. We write $f \perp g$ when f commutes with g.

In particular, for $m = 1$, $f \perp g$ means that

$$f(g(x_1), \ldots, g(x_n)) \;=\; g(f(x_1, \ldots, x_n))$$

holds for all $x_1, \ldots, x_n \in E_k$.

For any subset F of $\mathcal{O}_k$, let F^* be the set of operations which commute with all members of F, i.e.,

$$F^* \;=\; \{\, g \in \mathcal{O}_k \mid \; g \perp f \;\text{ for all } f \in F \,\}.$$

A subset C of $\mathcal{O}_k$ is a *centralizer* if $C = F^*$ for some $F \subseteq \mathcal{O}_k$. We also say that F^* is the centralizer of F. It is easy to see that F^* is a clone for any subset F of $\mathcal{O}_k$. When $F = \{f\}$, we write f^* for F^*.

As extreme examples, $\mathcal{O}_k$ is a centralizer since $\mathcal{J}_k^* = \mathcal{O}_k$ and $\mathcal{J}_k$ is a centralizer since $(\mathcal{O}_k)^* = \mathcal{J}_k$.

A subset M of $\mathcal{O}_k^{(1)}$ is a *centralizing monoid* on E_k if

$$M \;=\; F^* \,\cap\, \mathcal{O}_k^{(1)}$$

for some $F \subseteq \mathcal{O}_k$. Thus a centralizing monoid is the unary part of a centralizer. Since the centralizer F^* is a clone, the set M defined above is a monoid.

In the above definition, a subset F of $\mathcal{O}_k$ is called a *witness* of M. The centralizing monoid M with F as its witness is denoted by $\mathsf{M}(F)$, i.e.,

$\mathsf{M}(F) = F^* \cap \mathcal{O}_k^{(1)}$. When F is a singleton, i.e., $F = \{f\}$, we write $\mathsf{M}(f)$ for $\mathsf{M}(F)$.

The "top" elements of the set of centralizing monoids are of special interest. A centralizing monoid M is *maximal* if $\mathcal{O}_k^{(1)}$ is the only centralizing monoid properly containing M. Maximal centralizing monoids have a strong connection to minimal clones.

Proposition 1 ([3]) *For every maximal centralizing monoid M there exists a minimal operation f such that $M = \mathsf{M}(f)$.*

On the 3-element set the result is more striking, as is shown below.

2 Results on E_3

There are 84 minimal clones on E_3 [1]. Among them, three are generated by constant operations and seven by majority operations. A striking fact is the following.

Proposition 2 ([3]) *For a centralizing monoid M on E_3, the following are equivalent.*

(1) M is a maximal centralizing monoid.

(2) $M = \mathsf{M}(f)$ for some $f \in \mathcal{O}_k$ which is a constant operation or a majority minimal operation.

Thus, there are 10 maximal centralizing monoids on E_3. Three of them have unary constant operations as their witnesses and seven have majority minimal operations as their witnesses.

Below are majority minimal operations on E_3. Only the values on (x, y, z) for mutually distinct $x, y, z \in E_3$ need to be specified.

- $m_1(x, y, z) = 0 \quad \text{if} \quad |\{x, y, z\}| = 3$
- $m_2(x, y, z) = \begin{cases} 0 & \text{if} \quad (x, y, z) \in \sigma \\ 1 & \text{if} \quad (x, y, z) \in \tau \end{cases}$
- $m_3(x, y, z) = x \quad \text{if} \quad |\{x, y, z\}| = 3$

Here σ and τ are the sets defined by $\sigma = \{(0, 1, 2), (1, 2, 0), (2, 0, 1)\}$ and $\tau = \{(0, 2, 1), (1, 0, 2), (2, 1, 0)\}$.

Among seven majority minimal operations on E_3, two are conjugate to m_1 and the other two are conjugate to m_2.

As m_2 is the main target of this article, we explicitly present the centralizing monoid $\mathsf{M}(m_2)$ having m_2 as its witness:

$$\begin{aligned}\mathsf{M}(m_2) &= \text{CONST} \cup \{s \in \mathcal{O}_3^{(1)} \mid s(0) = s(1) \neq s(2)\} \\ &\quad \cup \{s \in \mathcal{S}_3 \mid s(2) = 2\}.\end{aligned}$$

Naturally, we are lead to investigate how much the above results on E_3 can be generalized to E_k for $3 \leq k < \omega$.

3 Generalization of Majority Operations

Hereafter we assume $3 \leq k < \omega$, unless otherwise stated.

Regarding the generalization, we have already obtained the results for m_1 and m_3 (submitted; [4]), which we shall present here without proof.

Let W ($\subset E_k^3$) be the set of triples on E_k whose components are mutually distinct, i.e.,

$$W = \{(a, b, c) \in E_k^3 \mid |\{a, b, c\}| = 3\}.$$

By definition, a majority operation m is completely determined by the values of m on W.

For a unary operation $s \in \mathcal{O}_k^{(1)}$, $\ker(s)$ is defined by

$$\ker(s) = \{(x, y) \in E_k^2 \mid s(x) = s(y)\}.$$

Clearly, $\ker(s)$ is an equivalence relation on E_k. An equivalence class containing $x \in E_k$ will be denoted by $[x]_{\ker(s)}$.

3.1 Constant-like Majority

A generalization of m_1 is trivial. Let $m_c \in \mathcal{O}_k^{(3)}$, $k \geq 3$, be a majority operation on E_k which takes the constant value 0 on W:

$$m_c(x, y, z) = 0 \qquad \text{for all } (x, y, z) \in W.$$

It is known that m_c is a minimal operation for any $k \geq 3$ [5]. The centralizing monoid $\mathsf{M}(m_c)$ is characterized as follows.

Lemma 1 **([4])** *$\mathsf{M}(m_c)$ is exactly the set of unary operations $s \in \mathcal{O}_k^{(1)}$ satisfying one of the following:*

(1) $|\mathrm{Im}(s)| = 1$ *(i.e.,* $s \in \mathrm{CONST}$*)*

(2) $|\mathrm{Im}(s)| = 2$ *and* $|[0]_{\ker(s)}| = k - 1$

(3) $|\mathrm{Im}(s)| \geq 3$, $s(0) = 0$ *and* $|[x]_{\ker(s)}| = 1$ *for any* $x \notin [0]_{\ker(s)}$

After a bit of elaborate discussion, we get:

Proposition 3 ([4]) $\mathsf{M}(m_{\mathrm{c}})$ *is a maximal centralizing monoid.*

3.2 Projection-like Majority

A generalization of m_3 on E_3 to E_k, $k \geq 3$, is also straightforward. Let $m_{\mathrm{p}} \in \mathcal{O}_k^{(3)}$, $k \geq 3$, be a majority operation which behaves like a projection e_i^3 $(1 \leq i \leq 3)$ on W. Here, let us assume $i = 1$:

$$m_{\mathrm{p}}(x, y, z) \;=\; x \qquad \text{for all } (x, y, z) \in W.$$

The centralizing monoid $\mathsf{M}(m_{\mathrm{p}})$ is easily obtained (e.g., [4]).

Lemma 2 $\mathsf{M}(m_{\mathrm{p}}) = S_k \cup \mathrm{CONST}$.

Regarding the maximality, $\mathsf{M}(m_{\mathrm{p}})$ is maximal in most cases, but not always, as shown below.

Proposition 4 ([4])

(1) For $k = 3$ *or* $k \geq 5$, $\mathsf{M}(m_{\mathrm{p}})$ *is a maximal centralizing monoid.*

(2) For $k = 4$, $\mathsf{M}(m_{\mathrm{p}})$ *is* not *a maximal centralizing monoid.*

For the case of $k = 4$, we shall see further what is happening there.

Let $M_2\,(\subset \mathcal{O}_4^{(1)})$ be the monoid which consists of unary operations s satisfying one of the following: (1) $|E_4/\ker s| = 4$ (i.e., permutation), (2) $|E_4/\ker s| = 1$ (i.e., constant) and (3) $|E_4/\ker s| = 2$ with two blocks of size 2. The following fact is well-known.

Lemma 3 M_2 *is a centralizing monoid.*

Proof: Let $g \in \mathcal{O}_4^{(2)}$ be the binary operation on E_4 defined by the following Cayley table.

$x \backslash y$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Notice that g is commutative and associative. Denote $g(x, y)$ by $x \oplus y$. (Note: $(E_4; \oplus)$ is an elementary 2-group.) Let $f \in \mathcal{O}_4^{(3)}$ be defined by $f(x, y, z) = x \oplus y \oplus z$. Then, M_2 is proved to be a centralizing monoid with f as its witness, i.e., $M_2 = \mathsf{M}(f)$. The proof is carried out by elementary calculations which verify $s \perp f$ for all $s \in M_2$ while $s \not\perp f$ for all $s \in \mathcal{O}_4^{(1)} \setminus M_2$. The details are omitted. (Clearly, $s \in \mathcal{O}_4^{(1)} \setminus M_2$ if and only if $|E_4/\ker s| = 3$ or $|E_4/\ker s| = 2$ with a block of size 1 and a block of size 3.) □

Apparently, $\mathsf{M}(m_\mathrm{p}) \subset M_2$ is a proper inclusion (Lemma 2) and, therefore, $\mathsf{M}(m_\mathrm{p})$ is not a maximal centralizing monoid.

4 Balanced Majority

Now we shall make an attempt to generalize the majority operation m_2 on E_3 to a majority operation on E_k, $k \geq 3$. A generalization of m_2 is less obvious than that of m_1 and m_3.

4.1 Definition of m_b

Let σ be the operation on W, i.e.,

$$\sigma : W \longrightarrow W$$

defined by $\sigma(x, y, z) = (y, z, x)$ for (x, y, z) in W. By convention, we write $\sigma(x, y, z)$ in place of $\sigma((x, y, z))$. Clearly, the order of σ is 3.

Let a binary relation $\sim$ on W be defined as follows:

For any $(a, b, c), (a', b', c') \in W$, $(a, b, c) \sim (a', b', c')$ if and only if $(a', b', c') = \sigma^i(a, b, c)$ for some $0 \leq i \leq 2$. It is clear that the relation $\sim$ is an equivalence relation on W and the equivalence class of (a, b, c), denoted by $[(a, b, c)]_\sim$, consists of 3 triples in W.

E_k is considered as the initial segment of $\mathbb{N}$. Let the order $\leq$ on $\mathbb{N}$ be naturally introduced into E_k, i.e., $0 < 1 < \cdots < k-1$.

A triple $(a, b, c) \in W$ will be called *even* if $a < b < c$ and *odd* if $a < c < b$. Let W_even and W_odd be subsets of W defined by

$$W_\mathrm{even} = \bigcup \{\, [(a, b, c)]_\sim \mid (a, b, c) : \text{even} \,\}$$

and

$$W_{\rm odd} = \bigcup \{ [(a,b,c)]_\sim \mid (a,b,c) : \text{odd} \}.$$

We shall extend the meaning of even and odd and call any element $(a,b,c) \in W$ *even* if it is in $W_{\rm even}$ and *odd* if it is in $W_{\rm odd}$.

$W_{\rm even}$ and $W_{\rm odd}$ can be characterized in the following way. For $\boldsymbol{a} = (a_1, a_2, a_3) \in W$ the number of *reversed pairs* in $\boldsymbol{a}$, denoted by $r(\boldsymbol{a})$, is given by

$$r(\boldsymbol{a}) = |\{(i,j) \mid i,j \in \{1,2,3\},\ i<j,\ a_i > a_j\}|.$$

The following property is easy to see, from which the terms even and odd stem.

Lemma 4 *For any $\boldsymbol{a} \in W$,*

(1) $\boldsymbol{a} \in W_{\rm even} \iff r(\boldsymbol{a})$ is even.

(2) $\boldsymbol{a} \in W_{\rm odd} \iff r(\boldsymbol{a})$ is odd.

Proof: For $\boldsymbol{a} = (a,b,c) \in W$, if $a<b<c$ then $\boldsymbol{a} \in W_{\rm even}$ by definition and, also, $r(\boldsymbol{a})$ is even as $r(\boldsymbol{a}) = 0$. Similarly, if $a<c<b$ then $\boldsymbol{a} \in W_{\rm odd}$ and, also, $r(\boldsymbol{a})$ is odd as $r(\boldsymbol{a}) = 1$. Then the proof follows from an observation that the process of applying σ to $\boldsymbol{x} \in W$ does not alter the parity, i.e., $r(\sigma(\boldsymbol{x})) = r(\boldsymbol{x})$. □

The following is essential in defining the majority operation $m_{\rm b}$, a generalization of m_2.

Lemma 5 *$\{W_{\rm even}, W_{\rm odd}\}$ is a partition of W, i.e.,*

(1) $W = W_{\rm even} \cup W_{\rm odd}$, and

(2) $W_{\rm even} \cap W_{\rm odd} = \emptyset$.

Proof: Obvious from Lemma 4. □

Definition 1 *The majority operation $m_{\rm b}$ on E_k is defined as follows:*

$$m_{\rm b}(x,y,z) = \begin{cases} 0 & \text{if } (x,y,z) \in W_{\rm even} \\ 1 & \text{if } (x,y,z) \in W_{\rm odd} \end{cases}$$

The subscript 'b' stands for *b*alanced. As is easily seen, $m_{\rm b}$ is a generalization of the majority operation m_2 on E_3.

4.2 Centralizing Monoid of m_{b}

We shall determine the centralizing monoid $\mathsf{M}(m_{\mathrm{b}})$ which has m_{b} as its witness.

For an equivalence relation θ on E_k, we call an equivalence class U of θ a *cluster block* (or, simply a *cluster*) of θ if it contains two or more elements, i.e., $|U| \geq 2$. For a unary operation $s \in \mathcal{O}_k^{(1)}$, we abuse the term and say that s has a cluster if there exists a cluster of $\ker(s)$.

Lemma 6 *Let $s \in \mathcal{O}_k^{(1)}$. If $s \perp m_{\mathrm{b}}$ then s satisfies one of the following:*

(1) s has a cluster U of size k (i.e., $U = E_k$),

(2) s has a cluster U of size $k-1$ which contains 0 and 1, i.e., $\{0,1\} \subseteq U$,

(3) s has no cluster.

Note that (1) is equivalent to saying that s is a constant operation, i.e., $s \in \mathrm{CONST}$, while (3) is equivalent to saying that s is a permutation, i.e., $s \in \mathcal{S}_k$.

Proof: The proof follows from three Claims.

Claim 1. If U is a cluster of s, then $\{0,1\} \subseteq U$.

Proof of Claim 1. Let $\alpha \in E_k$ be the value of s on U, i.e., $s(U) = \{\alpha\}$.

Suppose $0 \notin U$. Let $a, b \in U$ be elements such that $a < b$. Since $(0,a,b) \in W$ and $0 < a < b$, we have $m_{\mathrm{b}}(0,a,b) = 0$. Hence, by $0 \notin U$, $s(m_{\mathrm{b}}(0,a,b)) = s(0) \neq \alpha$, whereas $m_{\mathrm{b}}(s(0),s(a),s(b)) = m_{\mathrm{b}}(s(0),\alpha,\alpha) = \alpha$, implying $s \not\perp m_{\mathrm{b}}$. Therefore, $0 \in U$.

Next, suppose $1 \notin U$. Take $a \in U \setminus \{0\}$. Since $(0,1,a) \in W$ and $0 < 1 < a$, we have $m_{\mathrm{b}}(1,0,a) = 1$. The rest is analogous to the above. By $1 \notin U$, $s(m_{\mathrm{b}}(1,0,a)) \neq \alpha$, whereas $m_{\mathrm{b}}(s(1),s(0),s(a)) = m_{\mathrm{b}}(s(1),\alpha,\alpha) = \alpha$ and we get $s \not\perp m_{\mathrm{b}}$. Therefore, $1 \in U$. ◇

As an immediate consequence of Claim 1, we obtain:

Claim 2. For any $s \in \mathcal{O}_k^{(1)}$, there exists *at most* one cluster of s. ◇

Next, we discuss the size of a cluster.

Claim 3. A cluster U of s satisfies $|E_k \setminus U| < 2$.

Proof of Claim 3. Again, we let $\alpha \in E_k$ satisfy $s(U) = \{\alpha\}$. In particular, $s(0) = s(1) = \alpha$ by Claim 1. Assume to the contrary that there exist two elements c, d ($c \neq d$) in $E_k \setminus U$. Let $s(c) = \gamma$ and $s(d) = \delta$ for $\gamma, \delta \in E_k$. Since U is a unique cluster and $c, d \notin U$, we have $\gamma \neq \delta$.

Since both $(0, c, d)$ and $(0, d, c)$ are in W and the values of m_{b} on W are in $\{0, 1\}$, we have

$$s(m_{\mathrm{b}}(0, c, d)) = s(m_{\mathrm{b}}(0, d, c)) \, (= \alpha).$$

On the other hand, we have

$$\begin{aligned} m_{\mathrm{b}}(s(0), s(c), s(d)) &= m_{\mathrm{b}}(\alpha, \gamma, \delta) \\ &\neq m_{\mathrm{b}}(\alpha, \delta, \gamma) \\ &= m_{\mathrm{b}}(s(0), s(d), s(c)). \end{aligned}$$

It follows that s and m_{b} do not commute, i.e., $s \not\perp m_{\mathrm{b}}$, against the assumption. ◇

Evidently, Claims 1, 2, 3 together prove the lemma. □

We shall take a closer look at the cases (2) and (3) in Lemma 6. First, the case (2).

For any $U \in \mathcal{P}(E_k)$ and $c \in E_k \setminus U$ such that $\{0, 1\} \subseteq U$ and $E_k = U \cup \{c\}$ and any $\alpha, \beta \in E_k$ such that $\alpha \neq \beta$, we define a unary operation $s_{\triangleright}(c; \alpha, \beta) \in \mathcal{O}_k^{(1)}$ by

$$s_{\triangleright}(c; \alpha, \beta)(x) = \begin{cases} \alpha & \text{if } x \in U\,, \\ \beta & \text{if } x = c\,. \end{cases}$$

We often write $s_{\triangleright}$ in place of $s_{\triangleright}(c; \alpha, \beta)$ when c, α, β are understood.

Remark 1 *Any $s \in \mathcal{O}_k^{(1)}$ satisfying (2) in Lemma 6 can be expressed as $s_{\triangleright}(c; \alpha, \beta)$ for some $c \in E_k \setminus \{0, 1\}$ and $\alpha, \beta \in E_k$ with $\alpha \neq \beta$.*

Lemma 7 *For any $U \in \mathcal{P}(E_k)$, $c \in E_k \setminus U$ and $\alpha, \beta \in E_k$ as above, $s_{\triangleright}(c; \alpha, \beta)$ and m_{b} commute, i.e., $s_{\triangleright}(c; \alpha, \beta) \perp m_{\mathrm{b}}$.*

Proof: We are to show that $s_{\triangleright}(m_{\mathrm{b}}(x, y, z)) = m_{\mathrm{b}}(s_{\triangleright}(x), s_{\triangleright}(y), s_{\triangleright}(z))$ holds for all (x, y, z) in W, which easily follows from (i) and (ii) below. (i) Because of $m_{\mathrm{b}}(x, y, z) \in \{0, 1\}$ by definition and $\{0, 1\} \subseteq U$ by assumption, it is immediate to see that $s_{\triangleright}(m_{\mathrm{b}}(x, y, z)) = \alpha$. (ii) Among x, y, z for which $(x, y, z) \in W$, at least two of them are in U and, hence, at least two of $s_{\triangleright}(x), s_{\triangleright}(y), s_{\triangleright}(z)$ are α, which implies $m_{\mathrm{b}}(s_{\triangleright}(x), s_{\triangleright}(y), s_{\triangleright}(z)) = \alpha$. □

Denote by $\Sigma_{\triangleright}$ the set of $s_{\triangleright}(c;\alpha,\beta)\,(\in \mathcal{O}_k^{(1)})$ over all $c \in E_k \setminus \{0,1\}$ and $\alpha,\beta \in E_k$ with $\alpha \neq \beta$.

Next, we move to the case (3) in Lemma 6. Let $\widehat{s} \in \mathcal{O}_k^{(1)}$ be the permutation defined by

$$\widehat{s}(x) = \begin{cases} 1 & \text{if } x = 0\,, \\ 0 & \text{if } x = 1\,, \\ k+1-x & \text{if } 2 \leq x < k\,. \end{cases}$$

Lemma 8 *It holds that* $\widehat{s} \perp m_{\mathrm{b}}$.

Proof: The assertion is easy to check for $k = 3$. Assume $k > 3$. Then $\widehat{s}$ is (strictly) decreasing on $E_k \setminus \{0,1\}$, that is, $a < b$ implies $\widehat{s}(a) > \widehat{s}(b)$ for any $a, b \in E_k \setminus \{0,1\}$.

We are to show $\widehat{s}(m_{\mathrm{b}}(x,y,z)) = m_{\mathrm{b}}(\widehat{s}(x), \widehat{s}(y), \widehat{s}(z))$ for all $(x,y,z) \in W$.

First, take the case of $\{x,y,z\} \cap \{0,1\} = \emptyset$. In this case, $(\widehat{s}(x), \widehat{s}(y), \widehat{s}(z))$ $= (k+1-x,\, k+1-y, k+1-z)$ and it is readily checked that the parity (i.e., even or odd) of (x,y,z) and $(\widehat{s}(x), \widehat{s}(y), \widehat{s}(z))$ are different. On the other hand, the parity of $\widehat{s}$ on $\{0,1\}$ is also switched (i.e., $\widehat{s}(0) = 1$ and $\widehat{s}(1) = 0$). Hence, we obtain $\widehat{s}(m_{\mathrm{b}}(x,y,z)) = m_{\mathrm{b}}(\widehat{s}(x), \widehat{s}(y), \widehat{s}(z))$.

The remaining case of $\{x,y,z\} \cap \{0,1\} \neq \emptyset$ is easy and left to the reader.

□

Lemma 9 *Let* $s \in \mathcal{O}_k^{(1)}$ *be a permutation. Then,* $s \perp m_{\mathrm{b}}$ *if and only if* s *is* $\varepsilon\,(= \mathrm{id})$ *or* $\widehat{s}$.

Proof: Sufficiency follows from Lemma 8.

Necessity: Assume $s \perp m_{\mathrm{b}}$. Since $m_{\mathrm{b}}(W) = \{0,1\}$ by the definition of m_{b}, it follows from $s \perp m_{\mathrm{b}}$ that $s(\{0,1\}) \subseteq \{0,1\}$. Since s is a permutation, this implies that either $s(x) = x$ or $s(x) = 1 - x$ holds on $\{0,1\}$. Note that, if $k = 3$, then $s(2) = 2$ and, hence, $s = \varepsilon$ or $\widehat{s}$. Hereafter, we assume $k > 3$.

Case 1. $s(x) = x$ on $\{0,1\}$: Let a, b be any elements in $E_k \setminus \{0,1\}$ such that $a < b$. Since s is a permutation, we have $s(a) \neq s(b)$ and $s(a), s(b) \in E_k \setminus \{0,1\}$. Clearly, we have $s(m_{\mathrm{b}}(0,a,b)) = s(0) = 0$ on the one hand and $m_{\mathrm{b}}(s(0), s(a), s(b)) = m_{\mathrm{b}}(0, s(a), s(b))$ on the other hand. Hence, the assumption $s \perp m_{\mathrm{b}}$ implies $m_{\mathrm{b}}(0, s(a), s(b)) = 0$, which means $s(a) < s(b)$. Thus, it is verified that the restriction of s to $E_k \setminus \{0,1\}$ is a permutation and strictly *increasing*. Therefore, together with the assumption that $s(x) = x$ on $\{0,1\}$, s must be the identity ε on E_k.

Case 2. $s(x) = 1 - x$ on $\{0, 1\}$: A similar argument as above works. Let a, b be in $E_k \setminus \{0, 1\}$ such that $a < b$. Since s is a permutation, we have $s(a) \neq s(b)$ and $s(a), s(b) \in E_k \setminus \{0, 1\}$. Now, we have $s(m_{\rm b}(0, a, b)) = s(0) = 1$ and $m_{\rm b}(s(0), s(a), s(b)) = m_{\rm b}(1, s(a), s(b))$. Hence, $s \perp m_{\rm b}$ implies $m_{\rm b}(1, s(a), s(b)) = 1$, which means $s(a) > s(b)$. Thus, the restriction of s to $E_k \setminus \{0, 1\}$ is a permutation and strictly *decreasing*. In other words, we have a sequence $s(2) > s(3) > \cdots > s(k-1)$. This property, combined with $s(0) = 1$ and $s(1) = 0$, can be satisfied only by $\widehat{s}$. The proof is complete. □

Proposition 5 *The centralizing monoid* $\mathsf{M}(m_{\rm b})$ *which has* $m_{\rm b}$ *as its witness is:*

$$\mathsf{M}(m_{\rm b}) \;\; = \;\; \mathrm{CONST} \cup \Sigma_{\triangleright} \cup \{\varepsilon, \widehat{s}\}$$

Proof: Since all constant operations commute with $m_{\rm b}$, the proof follows from Lemma 6, Lemma 7 (together with Remark 1) and Lemma 9. □

4.3 Is It Maximal?

Is $\mathsf{M}(m_{\rm b})$ a maximal centralizing monoid? Or, does there exist an operation $f \in \mathcal{O}_k^{(n)}$, $n \geq 1$, which satisfies $\mathsf{M}(m_{\rm b}) \subset \mathsf{M}(f) \subset \mathcal{O}_k^{(1)}$?

To begin with, we ask if there exists a monoid (not necessarily a centralizing monoid) M which sits strictly between $\mathsf{M}(m_{\rm b})$ and $\mathcal{O}_k^{(1)}$, i.e., $\mathsf{M}(m_{\rm b}) \subset M \subset \mathcal{O}_k^{(1)}$. The answer is yes. In fact, there are many.

We take one such example, which happens to be minimal among those M's concerned.

Define two permutations $\widehat{s}_1, \widehat{s}_2 \in \mathcal{O}_k^{(1)}$ by:

(i) $\widehat{s}_1 = (01)$, i.e., $\widehat{s}_1(0) = 1$, $\widehat{s}_1(1) = 0$ and $\widehat{s}_1(x) = x$ otherwise,

(ii) $\widehat{s}_2(x) = \begin{cases} x & \text{if } x = 0, 1\,, \\ k + 1 - x & \text{if } 2 \leq x < k\,. \end{cases}$

Note that $\widehat{s} = \widehat{s}_1 \circ \widehat{s}_2 = \widehat{s}_2 \circ \widehat{s}_1$ where $\circ$ denotes the composition of unary operations. In fact, $\{\varepsilon, \widehat{s}, \widehat{s}_1, \widehat{s}_2\}$ is a subgroup of $\mathcal{S}_k$.

Let $\widetilde{M}$ be a subset of $\mathcal{O}_k^{(1)}$ defined by

$$\widetilde{M} \;\; = \;\; \mathrm{CONST} \cup \Sigma_{\triangleright} \cup \{\varepsilon, \widehat{s}, \widehat{s}_1, \widehat{s}_2\}.$$

Clearly, $\widetilde{M} = \mathsf{M}(m_{\rm b}) \cup \{\widehat{s}_1, \widehat{s}_2\}$. For $k > 3$, we have $\mathsf{M}(m_{\rm b}) \subset \widetilde{M}$ since $\widehat{s}_1, \widehat{s}_2 \notin \mathsf{M}(m_{\rm b})$. It should be noted, however, that in case of $k = 3$ we have $\widehat{s}_1 = \widehat{s}$ and $\widehat{s}_2 = \varepsilon$ and, accordingly, $\widetilde{M} = \mathsf{M}(m_{\rm b})$.

The next lemma is straightforward. (The second inclusion is proper because many permutations, for example, are not in $\widetilde{M}$.)

Lemma 10 *Let $k > 3$. $\widetilde{M}$ is a monoid and $\mathsf{M}(m_{\rm b}) \subset \widetilde{M} \subset \mathcal{O}_k^{(1)}$.*

An operation $f \in \mathcal{O}_k^{(n)}$, $n \geq 3$, is a *semiprojection* if there exists $1 \leq i \leq n$ such that $f(a_1, \ldots, a_n) = a_i$ whenever $|\{a_1, \ldots, a_n\}| < n$.

Let $\widetilde{p} \in \mathcal{O}_k^{(3)}$ be the semiprojection defined by $\widetilde{p}(x, y, z) = x$ for $(x, y, z) \in E_k^3 \setminus W$ and

$$\widetilde{p}(x, y, z) = \begin{cases} 1 & \text{if } x = 0\,, \\ 0 & \text{if } x = 1\,, \\ x & \text{if } x \notin \{0, 1\}\,. \end{cases}$$

for $(x, y, z) \in W$.

Lemma 11 *The semiprojection $\widetilde{p}$ commutes with all members of $\widetilde{M}$, i.e., $\widetilde{p} \perp \widetilde{M}$. Hence, $\widetilde{M} \subseteq \mathsf{M}(\widetilde{p})$.*

Proof: First, $\widetilde{p} \perp$ CONST is trivial. To show $\widetilde{p} \perp \Sigma_{\triangleright}$, it is sufficient to observe that $s(\{0, 1\}) = \alpha$ for any $c \in E_k \setminus \{0, 1\}$ and $s = s_{\triangleright}(c; \alpha, \beta)$. Next, since $\widehat{s}_1$ (resp., $\widehat{s}_2$) is a permutation, $(x, y, z) \in W$ implies $(\widehat{s}_1(x), \widehat{s}_1(y), \widehat{s}_1(z)) \in W$ (resp., $(\widehat{s}_2(x), \widehat{s}_2(y), \widehat{s}_2(z)) \in W$). Then it is easy to see that $\widetilde{p} \perp \{\widehat{s}_1, \widehat{s}_2\}$. Finally, $\widetilde{p} \perp \widehat{s}$ is clear as $\widehat{s} = \widehat{s}_1 \circ \widehat{s}_2$. □

Corollary 1 *Let $k > 3$. $\mathsf{M}(m_{\rm b})$ is not a maximal centralizing monoid.*

The proof follows from Lemmas 10 and 11. (Note that $\mathsf{M}(\widetilde{p}) \subset \mathcal{O}_k^{(1)}$ is assured by the existence of, for example, $t \in \mathcal{O}_k^{(1)}$ which is defined by $t(0) = 1$ and $t(x) = 0$ if $x > 0$.)

Remark 2 *One might argue that $m_{\rm b}$ is not a proper generalization of m_2 and a proper one would yield a maximal centralizing monoid. We do not deny such possibility. However, we assert the properness of $m_{\rm b}$ since it is defined in such a natural way.*

A remaining question on $m_{\rm b}$ is whether it is a minimal operation. At the current stage we do not know the answer to it.

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