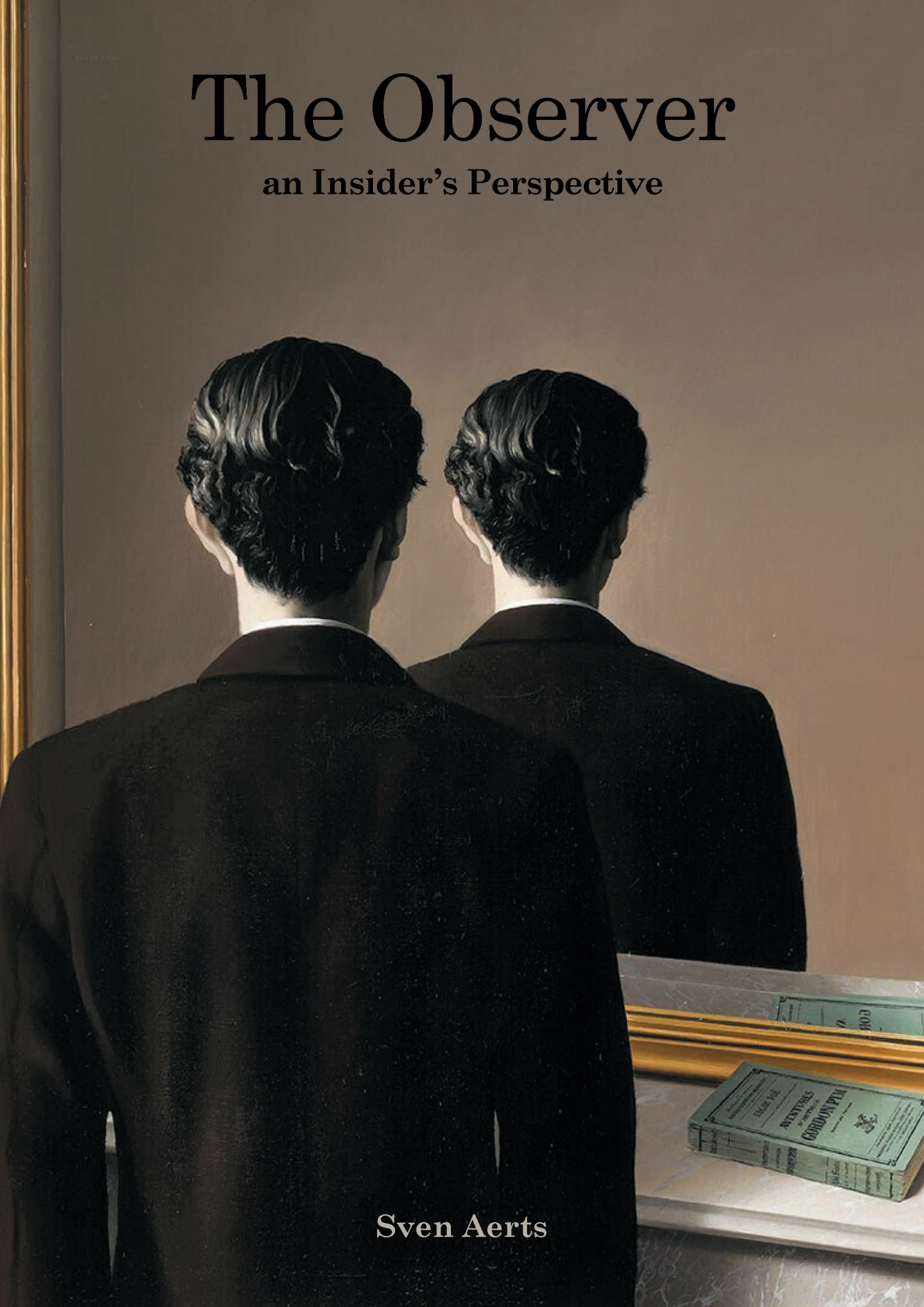
The Observer

an Insider's Perspective



Sven Aerts

The Observer: an Insider's Perspective

An inquiry into the foundations of quantum mechanics and observation

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The Observer: an Insider's Perspective

An inquiry into the foundations of quantum theory and observation

De waarnemer: een blik van binnenuit

Een onderzoek naar de grondslagen van de kwantumtheorie en waarneming

Thesis

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Preface

> "Do you mean ter tell me," he growled at the Dursleys, "that this boy—this boy!—knows nothin' abou'—about ANYTHING?" Harry thought this was going a bit far. He had been to school, after all, and his marks weren't bad. "I know some things," he said. "I can, you know, do math and stuff."
>
> — J.K. Rowling, *Harry Potter and the Sorcerer's Stone*, 1997

I was about seven years old when I bought my first transistor radio. It took what seemed like ages of pocket money saving to collect the necessary amount, but once I had it, I was extremely pleased with its nifty looks and the wonderful way it performed. I recall how I switched it off and on only to realize that songs wouldn't wait for me. Even when I told the little black box to turn off, the song would simply continue in silence. Within a week's time, I completely took it apart and —perhaps needless to say— never got it to function again. That was most unfortunate, but I don't recall having any regrets. I guess that, even if someone would have foretold me the consequences of my bold actions, I would still not have been able to resist the urge to open it up.

Many hobby projects, a Bachelors degree in Electrical Engineering from the University of Ghent and fifteen years later, I had learned that my desire for understanding was much larger than my appreciation of technology. I decided to do my MSc in Mathematical Physics at the University of Antwerp, taking extra courses in nonlinear wave phenomena, general relativity and various branches of quantum theory. On the rare occasion some of the topics got boring, I had my antidote ready: *The Feynman Lectures on Physics.* Reading Feynman would never fail to help me regain my enthusiasm because of his ability to make you understand the subject and at the same time show you why physics was interesting and fun. Nonlinear waves —in spite of their many mathematical intricacies— were relatively easy to grasp. Relativity was a subject to which I devoted more effort, because of the far-reaching and interesting consequences for the universe in which we live. But what really got my mind working overtime, was quantum theory.

Quantum theory is elegant but opaque, it enjoys incredible predictive power, but only in a probabilistic sense. And then there were these fascinating paradoxes. But there was something peculiar about the general attitude many physicists had about the subject. In most subject matters, my teachers rightfully stressed the importance of understanding what is going on in a given physical situation. Physics is not merely equation solving. For instance, one has to set up the

equations before one can even start to do mathematics and this meta-theoretical activity can only be achieved successfully if one has a firm grasp of the subject matter. But when it came to quantum theory, my teachers were claiming one should *not* attempt to understand what is going on. It was a time when merely mentioning the words *interpretation* or *foundations* in connection to quantum physics was sufficient to raise both eyebrows and serious doubts about one's scientific integrity. Even Feynman writes in *The Character of Physical Law* (1967) about the wave-particle duality:

> The difficulty really is psychological and exists in the perpetual torment that results from your saying to yourself, "But how can it be like that?" which is a reflection of [an] uncontrolled but utterly vain desire to see it in terms of something familiar. ... Do not keep saying to yourself, if you can possible avoid it, "But how can it be like that?" because you will get 'down the drain', into a blind alley from which nobody has escaped. Nobody knows how it can be like that.

At least Feynman admits how difficult it is to stop yourself from asking these questions! The two most common reasons I was given as a student to counter questions about foundational issues are echoed in the quote. The first reason is that the attempt to understand was commonly regarded as a "desire to see it in terms of something familiar" and the second "because you will get 'down the drain', into a blind alley from which nobody has escaped". The second reason sounds like the kind of warning parents give to their offspring about drugs. Even if the warning is given out of genuine concern one cannot help but think that if human kind follow this kind of advice, we would not have built airplanes, we would not have a theory of transfinite numbers, and we surely would not have asked ourselves questions about black holes. The first reason, the "desire to see it in terms of something familiar", relates to the idea that understanding is nothing but recognizing parallels with the familiar. It is true that the first models of quanta were semi-classical and that, with the birth of quantum theory in the nineteen-twenties, this search for (semi) classical formulations did not stop, even though they proved unfruitful. I confess that my first naive reformulations of quantum interference phenomena in terms of nonlinear waves were, in fact, little more than an attempt to reconquer some of the lost old intuitions. And for my last disclaimer, I agree with Feynman one should not remain hypnotized over the two-slit experiment because, in the end, nature "just behaves like that". But I beg to differ with the claim that the only way to understand something, is by seeing it in terms of something familiar. One can easily understand the basic laws of probability and this is not because we are *familiar*

with them. Experimental physicists have been *familiar* with quantum behavior for almost a century, and yet the quest for understanding quantum theory has only gained momentum. As a result of the search for a better understanding of quantum theory, we are now in possession of some of the philosophically most meaningful theorems in mathematical physics and I believe it is fair to say that, although many mysteries remain, we understand the theory of quanta and its many fascinating consequences better than ever before. We should never heed ourselves from the urge to understand, but rather from the urge to define what understanding is, because our lack of understanding may denote we are on new grounds, looking for entirely new ways to see the world in which we live and the theories that we require to describe that world. It is my hope that this manuscript is a humble step on the path in that direction.

Sven Aerts,

Amsterdam, September 2021

Chapter 1

Introduction

> All the perceptions both of the senses and the mind bear reference to man and not to the universe, and the human mind resembles those uneven mirrors which impart their own properties to different objects.
>
> —Francis Bacon, *The Great Instauration*, 1620

Galileo Galilei discovered that the swing period of a pendulum is independent of its amplitude (i.e. *isochronous*), by watching a suspended incense lamp swing back and forth in the cathedral of Pisa[1]. This discovery paved the way for the Dutch scientist Christiaan Huygens to formulate his pendulum law and build the first pendulum clock in 1656. Not long thereafter, pendulum driven clocks started to appear all over Europe, to be replaced as the most accurate time keepers by quartz clocks only after 1927. The pendulum is in fact not really isochronous, only approximately so for small amplitudes, but one can hardly blame Galileo for not noticing that the period of a swing increases somewhat with increasing amplitude, because he was timing the phenomenon using his own heart pulse rate as a stopwatch. The deviation from isochronism of the pendulum is well within the variations of a heart pulse rate. This remarkable story shows how Galileo managed to observe the (almost) isochronous character of the pendulum, using a less isochronous 'measurement apparatus'.

[1] According to his pupil and first biographer, Vincenzo Viviani, Galileo's interest in pendulums seems to date back as early as 1582-1583, but the first written record is a letter from Galileo to Guido Ubaldo dal Monte from 1602 [130].

There was of course no need for Galileo to use his pulse. One can easily think of more precise ways to establish the isochronism, such as using a waterclock or simultaneously swinging two pendulums of equal weight and length, but with different starting amplitude and observing whether they are running synchronously after any given number of swings. But right there and then, in the cathedral of Pisa in the middle of Piazza del Duomo, such an experiment —if possible at all— would have provoked reactions from the cathedral visitors. Counting heart beats provided for a discreet and immediately executable alternative. It seems highly improbable Galileo was interested in how many heartbeats a swing takes; in all likelihood he only wanted to verify the perceived periodic regularity of the swing of the incense lamp. This idea seems corroborated by folklore [130]:

> Legend has it that in 1583, a nineteen-year-old student at the University of Pisa attended prayers at the cathedral and, while daydreaming in the pews, noticed one of the altar lamps swaying back and forth. While his companions dutifully recited the Nicene Creed around him, the student became almost hypnotized by the lamp's regular motion. No matter how large the arc, the lamp appeared to take the same amount of time to swing back and forth. As the arc decreased in length, the speed of the lamp decreased as well. To confirm his observations, the student measured the lamp's swing against the only reliable clock he could find: his own pulse.

Galileo's observation can be regarded as a two step process. First he wonders whether the period of the swing is stable. Then he decides to observe whether that is indeed the case. Galileo is testing a hypothesis by means of an experiment; he wants nature to be his judge. This is what makes his act scientific.

This two stage process can be regarded as the basic model we will use to describe observations in the rest of this dissertation: *an observation is an answer to a question posed to a system*. We regard the observer as the system that forms the mediator between system under observation and the answer to the question. Imagine that two weeks after Galileo finds out about the isochronous character of the lamp, he wants to check his pulse. He can then observe whether his heart is beating faster or slower than two weeks ago *using exactly the same experiment as before*, only interpreting the data differently. We see observation is quite an involved concept in that it has to have a bearing on the question one is looking into. In this sense, an observation is an *interpretation* of a phenomenon[2].

[2]In psychology, an 'observation' is supposed to represent an objective acount of

Within the context of quantum mechanics the relation between observation and interpretation was repeatedly stressed by Albert Einstein. In "Der Teil und das Ganze", Heisenberg talks about an encounter with Einstein after a lecture Heisenberg presented in Berlin in 1926. He explained to Einstein that he managed to solve the problems of the harmonic oscillator by looking only at the observable quantities, inspired by a similar argument Einstein himself had used for the definition of simultaneity in special relativity. Much to his surprise, Einstein replied[3]:

> You must appreciate that observation is a very complicated process. The phenomenon under observation produces certain events in our measuring apparatus. As a result, further processes take place in the apparatus, which eventually and by complicated paths produce sense impressions and help us fix the effects in our consciousness— we must be able to tell how nature functions, we must know the natural laws at least in practical terms, before we can claim to have observed anything at all.

Although Einstein was talking about physics, a similar argument can be made with respect to other branches of science. Within the context of the logic of discovery, Hanson [114] coined the term *theory-laden observation*. Though often seen as a term that expresses a limit to the objectivity of observation, Hanson was in fact referring to a more subtle issue, that of the intertwined relation between knowledge and observation, or, how a sound epistemology necessitates an explicit application of concepts to the perceived data. In other words, observation is "concept applied to data", where it remains understood that the "data" themselves are the result of an experiment, constructed in accordance with our understanding of the concepts. It seems we have no alternative but to accept this apparent circular dependence that, on the surface, seems to threaten the objectivity in scientific experimentation.

Indeed, science is known to have been off-track from time to time, in spite of observations that initially seem to support a thesis, such as the claim made by Lawrence Berkeley National Laboratory in California to have discovered element 118 ununoctium[4] in 1999. As Berkeley Lab Director Charles Shank commented on the retraction of the originally

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reality and a subjective interpretation of an observation is called a 'perception', so to the psychologist, the difference between observing and perceiving is somewhat akin to the difference between 'looking' and 'seeing' in the English language. We will not make such a distinction as we consider observation without any form of interpretation void of meaning.

[3] This citation is preceded by the often quoted "It is the theory that decides what can be observed".

[4] Literally "1-1-8". Ununoctium (if it exists at all) has atomic number 118.

stated results "Science is self-correcting, if you get the facts wrong, your experiment is not reproducible." It would be a misconception to think these researchers found puzzling results that upon analysis were attributed to the presence of the long sought super heavy element ununoctium. While Galileo started from the hunch that the swing period looked pretty stable, they were acting on calculations done by the Polish theoretician Robert Smolanczuk, who suggested that by bombarding targets of lead with an intense beam of high-energy krypton ions, one should produce a sequence of decay events that is consistent with theories that have long predicted an "island of stability" for nuclei with approximately 114 protons and 184 neutrons. They hinted at these considerations on the basis of Mendeljev's periodic table and some quantum-mechanical calculations. In itself, there is no reason to claim some sort of scientific deontological breach occurred. In fact, the opposite is true, as a great deal of experiments are performed in clear anticipation of where to look for results. The initially claimed discovery of ununoctium was eventually withdrawn, not because they 'knew what they were looking for', but simply because the experimental results could not be independently verified by other teams. Many elementary particles were claimed to be discovered, both theoretically and experimentally, and many of them didn't stand the test of time. Of those that did survive, a proud fraction, such as the kaon, the positron and the neutrino were predicted on the basis of largely theoretical assumptions. In this respect, theory-laden observation is a powerful tool, especially in assessing layers of reality that escape more direct forms of observation. Much like a detective uses clues to formulate a theory, which in turn helps him to establish the truth of the first obtained clues by matching them with the overall picture that slowly emerges as a consequence of this process, the scientist investigates the validity of hypotheses in order to formulate new hypotheses that encompass a hopefully larger (or more detailed) picture of reality.

This implies one has to be able to anticipate the set of relevant results or hypotheses. Anticipating the set of possible results may seem to increase the danger of 'seeing what is not there', but I believe this is how much of observation actually works. Since the work of Claude Shannon [180] this can be given a precise meaning within the context of information theory, as we now know that a message transmitted over a channel of limited bandwidth but with optimal efficiency, is indistinguishable from random noise if the receiver is unfamiliar with the language of the message[5]. If we want to include situations such as described by

[5] The important word here is "optimal efficiency". If a book in a strange language can be decoded, it is because of much of its information is redundant. For a more physical formulation within the context of electromagnetic radiation we refer to [142], where it is shown that the most information-efficient format for a given

Shannon, we simply have to know what we are looking for. *We bring to the attention of the reader that we will deal only with observations for which we have established a set of relevant and experimentally distinguishable outcomes.* These outcomes may denote numerical values (such as the counting of heart beats), or, more generally, they can be labels of hypotheses (such as "Do pendulums swing isochronously?", or "Did the heart beat three times before the swing was completed?") Hypotheses can be built up from subhypotheses, such as Galileo assuming his pulse is a time measurement device (subhypothesis) to observe whether the pendulum is isochronous (main hypothesis). We will treat both levels of observations mathematically by the same formalization. Hence we will freely talk about the establishment of the isochronicity of pendulums as an example of an observation, but also the outcome an automated measurement apparatus gives, will be considered an observation.

We will not go into how Galileo got the idea the pendulum was swinging with constant periodicity. Getting this idea is a bit like an unconscious observation: such questions dawn upon us and we don't know how we get them. As interesting as this question may be in relation to observation, it is not what we will study here. The main question we address in this dissertation is, *given a question about a system, how a specific outcome is produced by an observer*[6]. This doesn't cover every aspect of observation, albeit we believe it to be an important one, and it is this aspect we have chosen to study. It is also the aspect of observation that, remarkably, quantum mechanics remains silent about, which makes it interesting to study precisely this issue in a more general framework.

To formalize which outcome is obtained in an observation, we will assume *determinism*[7]. This assumption is not made because of some *belief* that the universe evolves deterministically; I am fundamentally agnostic with respect to determinism and will even argue it may be impossible to establish experimentally whether we live in a deterministic universe or not. Determinism is assumed because (contrary to inherently probabilistic formulations) deterministic theories are obliged to tell you *what happens under what conditions*; deterministic theories deliver a narrative of how reality actually proceeds and under what conditions; indeterministic theories are far less committed to telling the

message is indistinguishable from blackbody radiation.

[6] Whether or not an exterior measurement device is used will not be made explicit in our formalization. All that is not system and cooperates to produce an outcome, is compounded under the system that we call "observer".

[7] Altough our observers will not be free to choose their internal state, we will not assume so-called superdeterminism; our observers are at any time free to pose any question they like.

details of such a story. We also consider our approach *realist*, but not in the naive sense that all observable quantities that possibly pertain to a system have pre-determined values. What I do mean to denote with *realism* then, is twofold: (i) that systems *are* in a state and that (ii) this state determines the outcomes of observable quantities which have a predetermined outcome and that this state will determine which observables will not have a predetermined outcome when observed. We also assume *physicalism* as we treat the observer as a physical system, in principle completely described by its state and that all possible mental states supervene on physical states. How do we then reconcile determinism with the assumption that, for a system in a given state, some observables will have predetermined values while others will not? We do so by assuming that two observers, observing the same state can still come to a different conclusion because the observers themselves are not in the exact same state. The same is true for a single observer that makes two subsequent observations: it will, in general, not be in the exact same state.[8] For some observable quantities this will make a difference, for others not.

As an example, recall Galileo pondering the isochronicity of the pendulum. Suppose one swing movement of the incense lamp takes three seconds. When the question is: "Does the swing of the pendulum take longer than one heartbeat?", the answer will be 'yes' to a very large extent independent of how fast Galileo's heart was beating. But the answer to the question "Does the swing of the pendulum take longer than four heartbeats?", will depend on the state of Galileo's heart. If he had to run to reach the cathedral in time to attend mass, the answer could well be 'yes'; if he has been seated on the church bench for some time, the answer is probably 'no'. In both instances the outcome of the observation is in itself correct; the problem is how to interpret the result. If Galileo is already aware of the isochronicity, he will come to the conclusion his heart is beating fast. Galileo would effectively be studying his own state and, given how smart he was, he would probably determine within reasonable bounds how much faster his heart is ticking than it was at the time he discovered the isochronicity[9]. So far we have discussed only the two cases that are most easily interpreted: (a) a stable heartbeat allows to determine the (near) isochronous character of a pendulum, and (b) a stable pendulum allows to determine heart rate variability. Both cases imply repeated measurements and

[8] This is how we can recover the machinery of quantum probability and still maintain determinism and is the essence of the Brussels' approach to quantum probability.

[9] In fact, Galileo's discovery led Sanctorius of Padua, a friend of Galileo and physician by profession, to construct a device, the *pulsilogium* which measured a patient's pulse by varying the length of a pendulum until the periods are equal.

inductive reasoning and can be performed with reasonable scientific validity.

But what about one single observation? Say Galileo is allowed to make only one single comparison between two situations: one in which the lamp was just lit, resulting in a wide swing, and a bit later when the swing is more modest. He counts the heartbeats in the two situations. Now either the number of heartbeats is the same, or the number is not the same. The problem is how to interpret that result. If the two swings are found to count equal heart beats, Galileo can conclude either (i) the pendulum's swing is isochronous, or (ii) his heart rate varied in precisely the same way as the pendulum did. If, however, the timing is not equal, he can conclude that (iii) the pendulum's swing is not isochronous, or maybe that (iv) his heart was beating at a different pace. Perhaps option (ii) seems the least probable option, in fact, there is no option he can logically rule out from his observation. Maybe he missed a heartbeat, maybe his heart missed a beat. But the problem of interpreting the data remains even when all heartbeats are counted properly[10]. The only way to interpret the result correctly is by having knowledge of either the pendulum or of his own heart rate variability at that moment, in which case we're back to cases (a) and (b) described above. But how can Galileo obtain such knowledge? His confidence in correctly judging between options (i), (ii), (iii) and (iv), can only be as big as his confidence in his capability to determine his own heart rate variability. We could simply propose that his state be observed by a secondary observer and communicate the results to the him. But to be certain about the observations the second observer makes, we need to know his state too, leading to an infinite regression of observers. This begs the question what an observer can observe about its own state. To interpret correctly which hypothesis is supported by the observation (hence in our definition of observation, to correctly observe), an observer depends on the ability to verify hypotheses about itself[11]. This in turn implies a fundamental constraint on deterministic[12] formaliza-

[10] We reject the idea that the problem stems from using an inaccurate stopwatch, such as the human heart. When Galileo would have had a more precise quartz stopwatch, the problem of deciding between options i, ii, iii and iv, is indeed far less problematic, but as a matter of principle, the problem has merely been mitigated to the last few digits of the outcome. One can still come up with more sensitive hypotheses that Galileo would not be able to affirm or refute without more detailed knowledge of how and when he starts and stops the stopwatch. We will treat this case extensively in Chapter 3.

[11] A very nice demonstration of this principle can be found in analog meter dials. Such dials come equipped with a mirror behind the scale. The function of the mirror is to eliminate parallax errors by providing a means to the observer to position him directly perpendicular above the scale. In essence, the mirror allows the observer to minimize the effect of his own state on the outcome of the osbervation.

[12] Although an inherently probabilistic formalization of observation can mitigate

tions of observation, because the observation that is required to verify a hypothesis about the observer, is also to be performed by an observer. This apparent epistemic circularity can perhaps be mitigated, but never eliminated and may well be the reason why, in spite of an enormous variety of observational models in statistics (such as estimation theory) and engineering (signal analysis) that deal with observation under specific circumstances, we have not found a general abstract theory of observation in the literature. Rather than circumventing the issue by tackling the problem only in specific instances, we want to take the bull by the horns and expose the problem. To that end we present a diagonal argument that, under plausible conditions, shows even classical observers have properties they cannot observe and hence states that they cannot distinguish between. In absence of epistemic certainty, we resort to probability. We will propose a quite general but conceptually simple probabilistic model of observation that can be illustrated by an urn model. For each question that we can ask about a system, we have an urn filled with observers (possibly in different states) and an urn with copies of that system (possibly in different states). We 'draw' an observer from the first urn and we 'draw' a system from the other urn and let them interact. Their interaction leads deterministically to an outcome. Every possible repeated experiment of a single observable quantity can then be modelled by two probability densities (or measures); one that describes the probability of drawing an observer in a given state, and one that describes the probability of drawing the system in a given state. The probability of a specific outcome is hence the relative fraction of (observer, system) pairs that interact to yield that specific outcome, to the total numbers of (observer, system) pairs. Although conceptually simple we maintain that, under the assumption of determinism, every possible instance of repeated observations can in principle be modelled by this scheme. It is however, in general not possible to operationally obtain the probability distributions for each urn separately. With additional assumptions about these distributions, we show how this formalization of observation can be used to model classical statistical, quantum probabilistic as well as more general situations. We will see even modern Bayesian approaches to model the ambiguities of interpretation in the human visual apparatus fit within this scheme.

Overview

Much of the content of this thesis boils down to two main topics: the construction of a general and formal theory of observation and the investigation of the extent in which the role of observation determines

the problem, it cannot solve it, for there is no room for certainty in probability.

the mathematical structure of quantum theory. In particular, Chapters 1 to 3 deal with the observer from a logical/structural perspective. We begin with a formal theory of properties and states to develop the notion of classical observer. Several fundamental problems related to classical observation are identified and motivate the search for a viable definition of potential properties and the formal introduction of the probabilistic observer. This then is the content of Chapters 4 to 6 which deal with probabilistic observation. We combine notions of potential properties and the probabilistic observer to show that this framework is sufficiently rich to reproduce the probabilistic structure of quantum theory. We work out a model that is more general than classical and quantum mechanical. We end with philosophical ponderings on the nature of observation as essentially a form of inference. To help the reader with the thread of the story, we give an overview of what is treated in each chapter.

Chapter 1 is this introduction and we will not attempt to summarize this summary, lest we fall into infinite regression.

In **Chapter 2**, we introduce the important notions of *system*, *experimental proposition*, *property* and finally *state* following the so-called Geneva approach. This approach, due to Jauch and Piron who built on the seminal work on quantum logic by Birkhoff and von Neumann, had as its main goal the reconstruction of Hilbert space quantum mechanics starting from operational definitions. We introduce barely more from this approach than necessary for our purposes. In particular, we will only introduce the operational part of the approach. We discuss at some length different types of classical properties as we will need these in Chapter 3. We end the chapter with comments and a sketch of the Geneva approach.

Chapter 3 introduces the classical observer. By definition a classical property is either actual or its negation is actual; a classical system is a system that has only classical properties. The classical observer then, is a classical system that can identify whether a given classical property holds for a given system. We will re-introduce the concepts of Chapter 1, but in a form much closer to the Brussels-Geneva approach due to Diederik Aerts and collaborators, which is somewhat more axiomatic than the Geneva approach, although the basic ideas behind the two approaches are similar. This more axiomatic approach is necessary because the observer cannot a priori be assumed to operationally establish its own properties without already having properties. However, the operational procedures of Chapter 1, allow us to formulate precisely what is required of an observer. Hence we introduce a mathematical framework that represents a simple protocol for physical observation and motivate three particular requirements (*consistent*, *free* and *introspective*) for the classical observer. An observer that sat-

isfies all three requirements will be called an *adequate* observer. We prove by a diagonal argument that the adequate observer faces fundamental constraints on the ability to observe itself. For more complex questions, even a consistent and free observer cannot observe all its own actual properties. These results are followed by a brief overview of published results that (although often starting from different angles) lead to similar or related conclusions and some of the implications that are most important from our perspective. In particular, we argue undecidability may offer a possible explanation why quantum mechanics will not give a mechanism for the production of an outcome; it can in some sense be regarded as a formal escape from a far worse problem: inconsistency. We end the chapter pondering the consequences of the result with respect to human observation.

In **Chapter 4,**we explore the notion of a *potential property* and its relation to the *mathematical representation of the state* of a system. As per Chapter 3, no adequate self-observer exists, i.e. it is impossible for an adequate observer to observe all its classical properties. Whereas an actual property is defined in Chapter 2 as a property that is true, regardless of whether it is observed or not, the notion of a potential property is one that *may* be true (or not) and repeated observation may yield different results. This then invites us to look into the notion of a potential property and its relation to the concept of probability. The two most important questions we address in this chapter, is how we can operationally define such a property and how we can mathematically represent it. In a series of steps we come to a definition that, building on a result obtained by Summhammer, leads us to regard potential properties as closed one dimensional subspaces in a finite dimensional Hilbert space. This fascinating result, building solely on classical notions of probability, is strengthened by an argument of Wootters. Whereas we have sought to represent a potential property, Wootters was looking for a sensible measure of the statistical distance between two probability distributions with respect to an experiment with a fixed but finite number of outcomes starting from the definition of statistical distance given by R.A. Fisher in the context of genetic drift studies. Wootters concludes that the Hilbert space angle between the normalized probability distributions uniquely (modulo mere proportionality) fits the desiderata. An extension of this work to cover the case of distinguishing between two quantum states is non trivial, but delivers exactly the same result. The arguments form an intriguing argument that it may well be the impossibility of exact observation which compels us to represent states in a manner conform with orthodox quantum theory. In the beginning of Chapter 5 we will initially regard a state as nothing more than an element of a measurable set, but in subsequent sections and indeed in the rest of the thesis,

we restrict our notion of state to an element of a vector space.

In **Chapter 5**, we introduce the probabilistic observer. As explained at the end of the former section, the inner machinery of our probabilistic observer is still inherently deterministic as outcomes obtained in experiments are without exception regarded as a co-creation between the system under study and the system used to observe. Probability is derived as a consequence of a lack of knowledge about the state of the system, the state of the observer, or both. We introduce a formal mathematical description that allows for the description of an extremely broad class of probabilistic experiments, including many types of biased observation. We propose a criterion that distinguishes a special kind of observing system: the optimal observer. The optimal observer aims to convey in the particular outcome for any given trial of an experiment, as much information about the system observed and as little information about itself as possible. We will only consider optimal observers in the remainder of the work. Whereas classical probability theory is a formalization of repeated measurements on an ensemble of similar but not identically prepared systems, we will treat as most important situation one in which we have lack of knowledge about the precise state of the observer, that is, repeated measurements on identical system states with an ensemble of similar but not identical observer states. We will do so for both epistemic states, which are technically represented as points in the standard simplex, as for ontological states, represented as unit vector states. We show how the Born rule is recovered if the density of observer states is uniformly distributed over the unit sphere. We discuss how this can be regarded as an alternative to orthodox quantum theory by providing a set of axioms that lead to the same experimental consequences as standard quantum theory. The proposal can be falsified, but only if we deviate from the uniform density of observer states.

An example where such a deviation is present forms the content of the entirely technical **Chapter 6.** We first explicitly work out the optimal observer for an experiment with two outcomes and show that, although the decision mechanism is different, the model is very similar to the so-called epsilon model of the Brussels school. Besides reproducing the probabilities of a spin 1/2 measurement, the model allows for a natural parametrization of the density of observer states. We examine our formalization of the probabilistic observer for a non-uniform density of observer states and work this out in the simple case of only two possible outcomes as this already entails considerable mathematical complexity. We calculate parameterized conditional probabilities and show that the two limiting cases of the density of observer states represent a quantum and a Kolmogorovian model for experiments with two outcomes. In between these two extremes there are regions for which no

Hilbert space model, nor a Kolmogorovian model exists, highlighting the scope of probabilistic scenarios this generalized probability model can yield.

In **Chapter 7** we ponder on some more philosophically oriented ramifications. We analyze how both scientific and sensory observation can be regarded as forms of inference, an idea that was strongly advocated by Helmholtz, but that goes back as far as Ptolemy. We propose how the optimal observer may be regarded as a paradigm for observation. Indeed, a proposal to mathematically model human vision from a Bayesian perspective was able to explain a variety of illusions that occur in human visual perception. There too, an observer is an inferential system that seeks to choose an optimal percept from a given set of possible percepts. We discuss the relation to our work and find the proposals to have more in common than they differ. The fact that the same principle governs human perception and quantum mechanical measurement, strengthens the paradigm that observation is a form of inference, one that seeks to maximize information about the system under observation. We end with a discussion on the implications of our proposal for of the interpretation of the quantum state. We argue the state is both an epistemic and an ontological description at the same time: it represents a lack of knowledge situation, but one that is fundamentally irreducible and hence ontic in its very nature.

Chronology

The manuscript that lies before you is by no means a complete story. It is rather an attempt at ordering some of the various lines of research that I have published over close to three decades that can be collected under the leitmotiv "observation". The ordering of the chapters provides a conceptual thread, starting from a very general abstract framework adding more and more mathematical structure as we proceed[13]. As it happens, the chronological order of this research is almost opposite to the ordering presented here to you. To set the record straight I will give you the true story: a brief chronological overview.

It all started back then with the attempt to understand to origin of probabilities in quantum theory. When I came to Brussels in my early twenties, much of the work there was devoted to the study of models, many of them due to Dirk Aerts, in which probability arises as a consequence of a lack of knowledge about the measurement interaction. Although the arisal of probability in these toy models was easy enough to understand, the probabilistic structure was surprisingly not

[13] A good example is the concept of state, which is introduced as an element of a set. This set is first identified as a lattice, later as a measure space and finally as the set of unit rays in complex Hilbert space.

a Kolmogorovian one and these models could reproduce the probability of simple quantum systems quite easily. Besides this surprising mathematical fact, it seemed to me to represent the actual state of affairs in experimental practice. Indeed, experimentalists take great care to produce ensembles of identically prepared states of the quantum system under study; rather obviously no such procedure is possible for the apparata used to measure these carefully prepared systems. The idea that this is indeed the origin of probability in quantum theory has ever since been my working hypothesis and a fruitful one I dare say. My first research was calculating families of conditional probabilities, showing that the limiting cases of these models could be represented in Kolmogorovian and Quantum probabilistic frameworks. Perhaps even more surprising was that the overall probabilistic structure was more general and hence not describable in either one of those frameworks. The essence of that calculation can be found in Chapter 6.

I was very enthused about these results, but what bothered me about the hidden measurement approach, as we would call it in Brussels, is that the hidden measurements were interpreted as representing the state of the measurement apparatus, but these "states" were elements of simplices rather than unit spheres and hence did not have the same mathematical structure as quantum states. I set out to see if this could be remedied. It turned out this was entirely possible and a simple and natural hidden measurement model in an n-dimensional complex vector space emerged. This model is presented in the later sections of Chapter 5. The criterion used to determine which states of the measurement apparatus yield a particular result (for a given system state) turned out to have a beautiful and almost trivial statistical interpretation: the outcome that should be selected is the one that is most likely to tell something about the system state and not about the apparatus state. The technical details of this proposal can be found in the opening sections of Chapter 5. It seemed therefore that the fluctuations in the outcomes of quantum mechanical experiments are far more than a mere inconvenience: from our perspective they are necessary for optimal state estimation.

The idea that the inherent fluctuations in measurement outcomes are at least partially responsible for the mathematical structure of quantum theory was fortified from an entirely different perspective. Wootters showed the Hilbert space angle provides a sensible distance between two probability distributions on the basis of distinguishability in a fixed number of experiments, an idea that goes back to the famous statistician R.A. Fisher. I saw how a reversal of that idea, using an adaptation of an argument by Summhammer, could be used to construct an operational definition of potential properties as a unit vector. As an actual property is a special case of a potential property

and the state of a system is introduced in Chapter 1 as nothing but the 'maximal' actual property of that system, it follows that states are to be described as unit vectors. This can be found in Chapter 4. From where I was standing at that time, the fluctuations in the measurement could provide a clue as to why we represent states as unit vectors and how outcomes are selected in the measurement process such that the resulting probability is in accordance with Born's rule.

The last line of research to be completed was the one that sought to understand the origin of this lack of knowledge about the precise state of the observer. It turned out that classical self-observers face a primitive form of measurement problem. A profound consequence is that a classical observer, living in a classical universe, can never establish with certainty that it is itself a classical system. There will always be a fundamental lack of knowledge about oneself. The most beautiful aspect of this to me, is that it is fundamentally impossible to tell whether this lack of self-knowledge is epistemic or ontological as every bit of knowledge (represented by the epistemic state of affairs) must be encoded in the natural or ontological state of the observer. I have the feeling that, even after all these years of thinking about this, I still do not see all ramifications of this proposal. As I have not entered formal territory here, I dare to go out on a speculative limb to describe two situations which hint that the result may have important consequences. The first situation is one in which we consider a single individual observer, say a famous physicist such as Feynman. Everything he has learned in his life, from the very first words he uttered to the advanced physical theories that won him the Nobel Prize, has come through his senses to become part of his inner state. Every conclusion he draws and every test of every theory that he has constructed are ultimately forms of self-inference. Even if we extrapolate our findings as behavior of the outside world in the end we are self-observers. Many of our extrapolations about the outside world may very well be exactly that: valid inferences about the outside world. But inherent limits to self-observation will spill over into limits regarding the possible descriptions of reality that we can meaningfully discern and I wonder to what extent this is the case in quantum theory. The other speculative picture I like to paint is one in which the observer observes another system, which is perhaps also an observer. As the two learn and communicate about what they have learned, this seems a richer scenario than the first. I believe it indeed is, as errors in inference by one observer can be remarked (and possibly rectified) by communication with another observer. However, according to our definition in Chapter 3, the 'observer' is the complete system used to draw inferences about some system. If two work together to obtain a single outcome, they can be regarded as a single observing system. Every measurement apparatus and every

tool they use becomes by extension a part of that observing system. In our mind we are free to shift the boundary between the observer and the observed ever further until it encompasses the entire observable universe, at which point the universe has become a self-observer. We, as part of that universe, are a manifestation of a universe that tries to understand itself through observation and formalization. There will be questions it can never answer.

By sheer coincidence I recently stumbled upon a paragraph in my high school dissertation [13] on artificial intelligence written 35 years ago that I cannot help but quote: "It may be the case our brains manage to describe themselves without contradiction. They have, however, no means to verify the correctness of their description. This is because the 'observer' that would follow the brain in its description would in turn have to be verified and this verification would in turn have to be verified; etc..."[14]. It seems I have come full circle.

[14] "Het zou kunnen zijn dat onze hersenen zichzelf beschrijven zonder op een contradictie vast te lopen. Zij hebben echter geen mogelijkheid de juistheid van hun beschrijving te toetsen. Dit komt omdat de 'observator' die de hersenen zou volgen in hun beschrijving op zijn beurt weer getoetst moet worden en die toetsing op zijn beurt weer; enz..."

Chapter 2

The description of physical systems

> *It's not denial. I'm just very selective about the reality I accept.*
>
> —Bill Watterson, *Calvin and Hobbes*, 1958 .

To begin with, the most difficult part is to decide where to begin, for this determines the concepts one takes as primitive and the concepts that can be defined in terms of these primitive concepts. Our aim is an abstract formalization of the process of observation, so first of all, we take for granted that there exists a universe and that within this universe there exist observers and phenomena to observe. More often than not, we are not interested in the universe as a whole. We call the part of the universe that is of interest to the observer, the *physical system under study* or simply the *system*[1].

Examples of systems that are currently under scientific study include about anything from electrons to galaxies, and from bacteria to ecosystems. We assume that it makes sense to study the system in relative isolation of the rest of the universe. It may be the case that our system's behavior depends crucially on some other system; as long as this other system is not the whole universe, we can enlarge our system of interest by including it. It may also be the case that the rest of the universe has some overall influence on the system; as long as we

[1] No word is perfect, but alternatives like 'entity' (too alien) or 'phenomenon' (too transient) seem less appropiate.

can capture that overall influence in a small set of global parameters (such as temperature or pressure) we can include these in our description as well. Although it is not at all obvious that every conceivable part of the universe can be studied without the necessity of having a detailed description of the universe as a whole, the effectiveness of science strongly suggests that for a wide variety of phenomena, this can indeed be assumed. Perfect isolation is admittedly an idealization and one that can even be hard to approximate in the laboratory, but it is difficult to imagine how science could proceed if this were not possible. We therefore restrict our attention without further apology to the observation of phenomena that can be isolated in the above sense. Even a single, isolated system may change over the course of time, hence we assume every system *is* at any given time instant in a certain *ontic state*. By saying the system is in an ontic state, we mean that the system exists in reality, as it is in itself, and that 'what it is' may change without altering the fact that it remains the same system. We assume that this ontic state can be represented mathematically and will call such a representation the *ontological state* of the system, or simply *the state* of the system[2] if this representation is formal and supervenes on the ontic state. Ideally, ontic states and ontological states stand in a one-to-one relationship with each other. However, because the representation of a phenomenon is distinct from the represented phenomenon and to accommodate the possibility that even our best knowledge of a system may be incomplete, it is unreasonable to require that every aspect of the ontic state has a corresponding aspect in the ontological state. Hence we insist on the more modest demand that for a state to be called 'ontological', it is required that different ontological states *always* pertain to different ontic states[3]. A mathematical representation of the ontic state of a system is called the ontological state of the system if different representations within the same mathematical representation theory, necessarily correspond to different ontic states. The comparison between these two cannot be made direct, as they belong to different categories: ontological states are mathematical entities and ontic states are natural ones. When we want to consider a theory about a physical system we need to specify which part of the theory corresponds to reality and how this can be tested by an experiment. We shall use the operational link between states and (experimental) propositions that is the operational basis of the so-called

[2] In many cases of practical interest, we have incomplete knowledge of the precise ontological state of the system and we are forced to ascribe an *epistemic state* to the system. We will take the ontological state as the primitive notion and treat epistemic states as mixtures of those.

[3] It is precisely in this respect that epistemic and ontological states differ: two systems described by different epistemic states, can be in the same ontic state.

Geneva-Brussels approach to quantum logic ([5], [14], [24]). We shall not nearly go as far as to impose a quantum logic for systems. In particular, we will not impose axioms that are not operationally motivated (e.g. the covering law or weak modularity), but restrict ourselves to the main operational ingredients of this approach, which we proceed to describe in this chapter.

Experimental propositions and properties

> *Reality is that which, when you stop believing in it, doesn't go away.*
>
> —Philip K. Dick, *How to build a universe*[4], 1978 .

Scientific theory is supposed to tell us something about reality; it must be able to predict something in order to test it. The prediction doesn't need to be revolutionary or original; it may "predict" the voltage drop over a resistor in an electronic circuit. However, an essential requirement is that we can compare the outcome of an experiment with the prediction the theory offers. Only if the two coincide within reasonable limits, one can consider to claim to have described something "real". To give an operational means to this end, we introduce the concept of a *test*. Throughout this chapter we treat the notion of an observer as an unproblematic given: we assume there exist systems that succeed in performing experiments and obtaining the results of the experiment. Only after we have given a precise meaning to the notion of state[5], we direct our attention in the following chapters to the observer as a physical system.

Tests and Properties

For a part of the universe to be identified as a system, we need to be able to recognize it as such. How do we recognize one isolated part of the universe as a system that we call an electron, and another isolated part as say, a proton? To *recognize* implies to be able to *distinguish* successfully among others. To distinguish an electron from a proton, we can expose both to an electromagnetic field and detect which way

[4] *...that doesn't fall apart two days later.*

[5] Unless specified differently, the word "state" is in the remainder of this work used as a shorthand for "ontological state". Epistemic states will be studied in Chapter 5.

they go as they will move in opposite directions. So we have an experimental test at our disposal that will tell us unequivocally whether the system is an electron or a proton. The notion of an experimental test is central to the operational formalism for physical systems of the so-called Geneva School[6], which has its origins in the nineteen seventies under Jauch [126] and Piron [162]. They called a test a *question* or a *definite experimental project.* Related names that one can find in the vast literature on the subject are *experimental propositions* and *yes-no test* or simply *test.* For example, Bogdan Mielnik [150] writes: "*By a question (also: proposition, yes-no experiment) one usually understands any physical arrangement which, when interacting with a micro object, may or may not produce a certain macroscopic effect interpreted as the answer 'yes'.*" Piron's own definition is shorter: "*An experiment is a test iff the result of the experiment can be expressed as an alternative of which the terms are yes and no"* [163]. Notwithstanding the apparent simplicity of this last definition, for Piron this is a complex notion. It involves an apparatus devised to answer that question, an experimentalist to execute the experiment and an instruction manual that shows us how to operate the apparatus and gives a rule that allows the experimentalist to conclude whether the answer was *yes* or *no.* That it is possible at all to distinguish an electron from, say a proton, rests on the fact that it is a *property* of an electron to have an elementary charge and that it is a *property* of the proton that it has the same elementary charge but with opposite sign. So the result of *yes-no* test allows us to establish whether a given property holds or not for system. To make this idea more precise, we first introduce some vocabulary.

True tests and actual properties

For some systems under certain circumstances, we know in advance with certainty that the test will result in a *yes.* In this case we will say the test is "true". We will capture this in the following operational definition[7]:

[6] The Geneva school, later called the Brussels-Geneva approach by Piron, had as its main goal to recover the axioms of quantum logic in an operational way. More on the history of this approach can be found in the last section "Comments and notes" of this chapter.

[7] In compliance with the historic development of the Geneva approach, the majority of the definitions in this chapter will be operational. An operational definition specifies the procedures that define the concept in a sufficiently specific way as to allow for reproduction in an actual experimental setting. After this chapter, most definitions will be regular mathematical definitions. A noteable exception can be found in chapter 4 in which we will formulate a statistical operational definition of a potential property.

Definition 1 (true test) *A test is* true *iff the system is prepared in such a way that if we were to perform the test, the answer would be* yes *with certainty.*

Since *truth* and *falsehood* are traditionally considered as pertaining to a proposition, it may feel awkward to regard a test as being true or false. The reason why tests are called *true* is because in the Geneva School a test is regarded as an "experimental proposition". More precisely, one could say the proposition that corresponds to the question that the test answers is true. We will, however, stick to the usual phrasing of simply calling the test true (or not). A test that is true for S corresponds to a property that holds for S. We will say that such a property is *actual*.

Definition 2 (actual property) *A property is* actual *iff the test that corresponds to it, is true.*

The subtlety of the definition of an actual property lies in the fact that, from the definition of true test, we do not necessarily need to perform the experiment, but that, counter-factually, if we were to perform the experiment, the answer would be *yes*. According to our definition, an actual property is one for which we can predict with certainty the outcome of the test that corresponds to it. This resembles the notion of element of reality given in a seminal paper of 1935 by Einstein, Podolsky and Rosen. In [93], they proposed a sufficient[8] condition for establishing whether a prediction of a value for a physical quantity corresponds to an element of reality:

> "*If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of reality corresponding to that quantity.*"

This is what Piron seems to have had in mind, for he writes in the introduction of [163] "*The stating (sic) point was to take seriously Einstein's criticism of the usual interpretation of quantum mechanics, and thus to describe a physical system in terms of "elements of reality".*" However, he acknowledges that his proposal for an actual property and the above described elements of reality differ markedly in at least one respect: "*without in any way disturbing a system*". We cannot always

[8] In the same paper we find: "It seems to us that this criterion, while far from exhausting all possible ways of recognizing a physical reality, at least provides us with one such way, whenever the conditions set down in it occur. Regarded not as a necessary, but merely as a sufficient, condition for reality, this criterion is in agreement with classical as well as quantum-mechanical ideas of reality."

know whether we test "without in any way disturbing a system". To this end, Piron introduces the following distinction: "*A question β is said to be* ideal *if every proposition, compatible with the proposition b defined by β, which is true beforehand is again true afterwards when the response of the system is "yes" ".* So if b is true before the measurement and the test is of the ideal kind, then the test reveals the element of reality. Piron defends this definition of an ideal test by saying: "*It is a very strong condition which in practice can only approach realization. We have to impose it, however, if we want to describe the measurement process without having to state exactly the mechanism proper to each particular apparatus.*"

Apart from these important caveats, we warn that it would be unnecessarily stringent to demand a scientific theory should deal *only* with elements of reality because many meaningful experiments in virtually every scientific discipline result in a probability, not in certainty. Nevertheless, a *theory* that can predict the probability of the occurrence of an outcome for a given experiment also needs to be able to describe the case in which that probability is one, hence when the outcome corresponds to an element of reality, modulo the fact that it should be an ideal measurement. This allows us to make contact with the experiment in such a way that we can say our state deals with reality, as we required for a proper account of the ontological state of the system. Our notion of state — that, as we later will see, depends only on actual properties— will be realistic, but without demanding that *every* physical observable needs to be an element of reality.

Note also that this definition agrees well with our everyday notion of "having a property". We can meaningfully talk about "an excellent champagne" without uncorking the bottle and tasting it, or about "a strong, safe car" without first crashing it into a wall and checking the state of the driver and car. That this is still meaningful is because we have sufficient knowledge of the system to *expect* to find that these properties manifest themselves *whenever* we decide to test them. That we believe these properties to hold, may be because we have witnessed people survive a crash in the same type of car, or because our memory recollects a delightful occasion in which one uncorked a bottle of champagne with the same label. Such experiences may form one's personal judgement, but it is fair to ask more to meet the criteria of a true test, i.e. to be able to predict with certainty. Still, absolute certainty does not exist and the degree to which our believe is justified, depends crucially on our ability to produce systems in the same state. Ultimately this depends on how far our theoretical knowledge and technological abilities allow us to produce systems in highly similar states, at least with respect to the properties of interest.

Preorder relations

There are questions whose answers imply the answers of other questions. For example, the question "Is it a dog?", if true, implies the question "Is it an animal?" is also true. Using the initial letters of the Greek alphabet $\alpha, \beta, \gamma, ...$ to denote a question, we can capture this in the following definition:

Definition 3 (question preorder) *If it is the case that whenever question α is true, then question β is also true, we will say α* implies *β and write:*

$$\alpha \preccurlyeq \beta. \tag{2.1}$$

Clearly we have $\alpha \preccurlyeq \alpha$ ($\preccurlyeq$ is reflexive) and if $\alpha \preccurlyeq \beta$ and $\beta \preccurlyeq \gamma$ then $\alpha \preccurlyeq \gamma$ ($\preccurlyeq$ is transitive), so that $\preccurlyeq$ is a preorder relation on the set of tests. A preorder can be used to define an equivalence relation denoted $\approx$ in the usual way.

Definition 4 (question equivalence) *Two questions that imply one another will be called* equivalent*:*

$$\alpha \preccurlyeq \beta \text{ and } \beta \preccurlyeq \alpha, \text{ iff } \alpha \approx \beta. \tag{2.2}$$

Apparently if $\alpha \approx \beta$ then α and β test the same property, hence if we call the set of (experimentally meaningful) questions Q, we define the equivalence class of the question α as: $[\alpha] = \{\gamma \in Q : \gamma \approx \alpha\}$. A property is hence more appropriately associated with the equivalence class of tests, than with any particular test within this equivalence class. This is why a property $\mathbf{a}$ (properties are denoted by bold face roman letters $\mathbf{a}, \mathbf{b}, \mathbf{c} \ldots$) in the early literature on the Geneva approach was often identified[9] with the equivalence class of tests: $\mathbf{a} = [\alpha]$. A test β is then valid for the verification of property $\mathbf{a}$ if and only if $\beta \in [\alpha] = \mathbf{a}$. Consequently, the set of properties of a system S can be identified with the quotient class $Q/\approx$. The preorder $\preccurlyeq$ (2.1) on the set of tests induces a preorder $\leq$ between two properties in a straightforward way.

Definition 5 (property preorder)

$$\mathbf{a} \leq \mathbf{b} \text{ iff for every } \alpha \in \mathbf{a}, \beta \in \mathbf{b} : \alpha \preccurlyeq \beta \tag{2.3}$$

If $\mathbf{a} \leq \mathbf{b}$, we will say the actuality of $\mathbf{a}$ *implies* the actuality of $\mathbf{b}$. As is the case for the question preorder, we have that $\leq$ is reflexive ($\mathbf{a} \leq \mathbf{a}$) and transitive ($\mathbf{a} \leq \mathbf{b}$ and $\mathbf{b} \leq \mathbf{c}$ then $\mathbf{a} \leq \mathbf{c}$), which allows us to define an equivalence class of properties.

Definition 6 (property equivalence)

$$\mathbf{a} \leq \mathbf{b} \text{ and } \mathbf{b} \leq \mathbf{a} \text{ iff } \mathbf{a} \approx \mathbf{b}. \tag{2.4}$$

[9] This will do for an operational definition of a property for now; later on we will postulate both the existence of properties and the fact that they correspond to questions.

A neglected problem for the partial order on properties

Historically, it was also imposed that two properties that imply each other are, in fact, one and the same property:

$$\text{if } \mathbf{a} \leq \mathbf{b} \text{ and } \mathbf{b} \leq \mathbf{a}, \text{ then } \mathbf{a} = \mathbf{b}. \tag{2.5}$$

For example, in Jauch ([126], sec. 5-3, p. 74) we find :

> If we set $\mathbf{a} = \mathbf{b}$ we have essentially replaced the proposition by classes of equivalent propositions. In the physical interpretation this means that we have defined a proposition as a class of physical yes-no experiments, all of which measure the same proposition.

Note that Jauch does not seem to make a distinction between "proposition" and "property" here. Piron is clearly more careful when he writes about the implication for questions in [163], p. 20:

> This is an ordering relation if one agrees to identify equivalent questions. By definition, two questions β and γ are equivalent if one has $\beta \prec \gamma$ and $\gamma \prec \beta$.

Nowhere does Piron explicitly write something like (2.5), however, once we *identify* equivalent questions and define a property as an equivalence class of questions, it follows two properties are one and the same property when they pertain to equivalent questions. Similarly, in [5] we find on p. 16: "If $\alpha \approx \beta$ then clearly α and β test the same property." It is only through equating equivalent properties that $\leq$ becomes a partial order and the set of properties of a system forms a *partially ordered set*, also known as a *poset*[10]. However, the *identification* of properties in case the corresponding questions imply each other, is in general not tenable. An example suffices to illustrate this fact. Take as system a flashlight. The properties that correspond to the questions α : "Does it give light?" and β : "Does it radiate heat?", are equivalent: when the switch is 'on' both questions are true and when the switch is 'off' they are both not true; hence these property imply each other for every possible state of the torch. But "giving light" and "giving heat" are from a common sense perspective clearly different properties. Many similar examples can be constructed; one only needs pairs of strongly correlated properties. As a last and perhaps more telling example, consider the the questions α and β that correspond to measuring the spins of

[10] The difference between a preorder and a (weak) partial order lies in the fact that the latter is anti-symmetric, which coincides with the definition of question equivalence.

a compound system consisting of two coupled spin 1/2 particles in the direction z for one subsystem and $-z$ for the other subsystem. If the coupled system is in a singlet state, the outcome of these two measurements will always be the same, hence the two questions are equivalent and correspond to the "same" property[11]. In later papers (see, for example [25]) the approach proceeds more cautiously by first calling two properties *equivalent* if they imply one another. This weaker structure already allows one to talk about the supremum and infimum of a set of properties. Still, the very first *axiom* of state property spaces (cfr. Axiom 1 in Chapter 3, or section 3.1 in [25]), imposes that two properties are the same, if the set of states for which these two properties are actual, are the same. This problem of identifying equivalent, but possibly different, properties seems insufficiently addressed in the literature on quantum logic and the Geneva approach; in fact we have found no explicit reference to it at all. It may be the case that we can distinguish between two properties, even if they are equivalent. The reason that we know the properties of "giving light" and "radiating heat" are not the same, is because we have examples of systems where these properties are not equivalent. For example, boiling water radiates heat but no light. Similarly an LED gives light without substantial heat radiation. Hence we could tentatively propose:

Definition 7 *Properties* $\mathbf{a}$ *and* $\mathbf{b}$ *are* equivalent *for a system* S *iff* $\mathbf{a} \leq \mathbf{b}$ *and* $\mathbf{b} \leq \mathbf{a}$, *in which case we write* $\mathbf{a} \approx \mathbf{b}$.

Definition 8 *Properties* $\mathbf{a}$ *and* $\mathbf{b}$ *are* identical *iff they are equivalent for every system, in which case we write* $\mathbf{a} = \mathbf{b}$.

It is not easy to see the ramifications of such a proposal, in particular in relation to the description of compound systems [198], which has proved to be problematic for quantum logic type approaches to quantum theory. We nevertheless accept (2.5) for now, and will briefly come back to this problem and the consequences it has for our purposes in Chapter 3 on p. 61, when we have developed the necessary concept of state.

Potential properties

The number of all possible questions is staggeringly large. Hence we will agree that the set of experimental propositions about a system

[11] A related problem plagues the concept of "observable". As an example, take inertial and gravitational mass. No known system or experiment can distinguish the two, although the way we measure them is different. It is a matter of both research and debate whether inertial and gravitational mass are identical, or merely have the (exact) same numerical value for our current established spectrum of experiments [134].

are those *that are known to be relevant for that system.* By "relevant" we mean those propositions for which experimental practice has established the applicability and usefulness of the corresponding property with respect to distinguishing different modes of being (states) of a system, or to distinguish between different systems altogether[12]. The partially ordered set of (relevant) properties of a system S will be denoted $\mathcal{L}_S$, for reasons that will become clear soon. If a property can be relevant for a system S, but a test pertaining to the property is not actual, we will say the property is *potential.*

Definition 9 (potential property) *A property* $\mathbf{a} \in \mathcal{L}_S$ *is* potential *for* S *iff it is not actual.*

We see now why we include only relevant properties in $\mathcal{L}_S$; if not "having strawberry flavor" would be a potential property of the sun. As an example, imagine you receive a batch of papers that is just wet enough to make you wonder whether it burns well. You could test the property by taking a piece of paper from the batch and try to set it on fire with a flame. You might find you succeed, perhaps with some difficulty. In doubt, you draw another piece of paper and repeat your experiment. Some papers might be wetter than others, or perhaps the outcome of the test depends on the temperature and time of exposure to the flame that we use to test the property with. Whatever the precise cause may be, if not all of them burn, we will say the property of "burning well when exposed to a burning lighter" is potential rather than actual[13] for this batch of papers. When dried, that property may be actual for the same batch of papers.

[12] To illustrate this, one can think of the classic game *20 questions (*or any of its many variations). In the game, one player -the answerer- thinks of a subject of her own choosing and the other players try to guess what it is. To help them in their guess work, they can ask whatever question they want to, provided that the answer can be given as either "yes" or "no". E.g., they are not allowed to ask "What is its color?", but it is legitimate to ask "Is it green?" It is remarkable that, even with this most simple (binary) set of answers possible, no restriction needs to be set on the things she may think of. It can be an object, a person, a feeling, a concept, whatever. There is an online version of this game in which you are the answerer and the computer attempts to guess what you have in mind. It is remarkable how often the algorithm gets it right in just 20 questions.

[13] Note that a property that is never actual for a given system S (such as the property of burning well for an inflammable piece of paper), is —at this point in the development of the formalism— a potential property according to the definition. We will later introduce more appropriate names for properties that never (or always) hold.

The product test and the meet property

Suppose we have a system S and two properties **a** and **b**. The test that corresponds to property **a** is denoted by α, the test that corresponds to **b** is denoted β. Because α and β correspond in general to entirely different experimental setups, it is often not feasible to perform the two tests simultaneously. Evidently, a system can have more than one actual property at the same time, even if we can neither test these properties simultaneously, nor sequentially. In [5] we find the following example. Assume we want to investigate the properties of a given kind of piece of wood, say pine. Consider the two properties[14]:

$$\mathbf{a} : \textit{Pine burns well}; \text{ and } \mathbf{b} : \textit{Pine floats on water}. \tag{2.6}$$

The test $\alpha \in \mathbf{a}$ could consist of putting the piece of pine into a specified fire for a specified duration of time, taking it out and see whether it burns. The test $\beta \in \mathbf{b}$ could consist of throwing the pine into a pool of water and observe whether it floats. Evidently pine has the property that it burns well, and also the property that it floats on water. Yet it is also evident that one cannot simply perform the two tests that correspond to these properties, neither simultaneously nor sequentially, because wet wood doesn't burn and burned wood is no longer wood[15]. The problem is then, how do we operationally define a test that represents having two properties? The answer can be found by analyzing what we mean when we say that pine has both property **a** and property **b**. We do not mean that it necessarily burns and floats *at the same time*, but that it would float when we would drop it in water, and it would burn when we would expose it to fire, whichever of the two tests we decide to perform. So the solution is as simple as it is subtle. Pick either α or β at random (for example, by throwing a coin), and perform that test. We will say the product test, denoted as $\alpha \cdot \beta$, is true iff the result of the test is *yes* with certainty. Because we picked one of the tests randomly from the product test, *yes* with certainty implies the answer to *both* tests needs to be *yes* with certainty[16]. So $\alpha \cdot \beta$ is

[14] The properties *burning* and *floating* are explicitly dispositional properties. When we say pine burns well, we generally don't intend to say it is burning, but rather that, if we were to set pine on fire, it would burn. Many physical properties, like mass and charge, are not formulated in this way, although they too, imply specific behavior under (often counterfactual) procedures.

[15] Because properties were defined as equivalence classes of questions, there may be tests that are equivalent to burning well and floating on water and that nevertheless can be performed simultaneously. Alternatively, we could break the piece of wood in two pieces and perform one of the tests on each half. Such examples don't render the definition meaningless, for it may be impossible to cut the system in two pieces, or there may be circumstances such that no equivalent but compatible tests exist.

[16] One can rightfully debate whether one can have certainty about the actuality of an untested property. It is nevertheless undisputable we very often attribute a

true iff α is true *and* β is true. This procedure is readily generalized to an arbitrary number of tests α_i, each one of the α_i corresponding to property $\mathbf{a}_i$. The product test, denoted $\Pi_i\alpha_i$, is then operationally defined as follows:

Definition 10 (product test) *We denote by $\Pi_i\alpha_i$ the product of the tests α_i. The test that corresponds to $\Pi_i\alpha_i$ is to choose at random one of the tests α_i and attribute to $\Pi_i\alpha_i$ the result of this test.*

We see that $\Pi_i\alpha_i$ is true only if we are certain that the result of each α_i is true. Product tests induce product properties.

Definition 11 (meet) *The property that corresponds to $\Pi_i\alpha_i$ is denoted as $\wedge_i\mathbf{a}_i$. We call $\wedge_i\mathbf{a}_i$ the* meet *of the properties $\mathbf{a}_i$.*

The meet is always well-defined if the properties $\mathbf{a}_i$ are because we have an operational procedure that can tell us how to observe it. The property $\wedge_i\mathbf{a}_i$ is actual if we are certain that the result of the product test would be *yes,* whichever test α_i we decided to perform. Hence every $\mathbf{a}_i$ must be actual and $\wedge_i\mathbf{a}_i$ implies every $\mathbf{a}_i$. Using the partial order relation for properties, we have $\wedge_{i\in J}\mathbf{a}_i \leq \mathbf{a}_i$ for all $i \in J$, with J an arbitrary index set. We can generalize this directly to the following proposition:

Proposition 12 *Let I and J be two index sets with $I \subset J$ such that for every $i \in J$ we have $\mathbf{a}_i \in \mathcal{L}_S$. Then*

$$\wedge_{i\in J}\mathbf{a}_i \leq \wedge_{i\in I}\mathbf{a}_i. \tag{2.7}$$

Proof. If $\wedge_{i\in J}\mathbf{a}_i$ is actual, then every $\mathbf{a}_i$ with $i \in J$ is actual. Because $I \subset J$, the $\mathbf{a}_i$ with $i \in I$ are actual, which shows that $\wedge_{i\in I}\mathbf{a}_i$ is actual. ■

The set of properties as a complete lattice

We have already mentioned that the set of properties $\mathcal{L}_S$ of a system S together with the (partial) order relation $\leq$, form a partially ordered set or *poset* $(\mathcal{L}_S, \leq)$. If there exists an infimum and a supremum for any subset of this poset and a maximal element exists, the poset is called a *complete lattice*. It is not difficult to show that the meet of an arbitrary set of properties is an infimum with respect to the partial order $\leq$. (Recall that the infimum of a subset V of a preordered set W is the largest element of W that is smaller than all the members of V.) The

property to a system that may not be testable under the experimental circumstances necessary to test another property.

proof is simple. As an immediate corollary to the previous proposition, we already have that $\wedge_i \mathbf{a}_i$ implies (is "smaller" than) every $\mathbf{a}_i$. Suppose we have found $\mathbf{x} \in \mathcal{L}_S$ such that $\forall j : \mathbf{x} \leq \mathbf{a}_j$. As the actuality of $\mathbf{x}$ implies the actuality of each $\mathbf{a}_j$, we have $\mathbf{x} \leq \wedge_i \mathbf{a}_i$. Hence $\wedge_i \mathbf{a}_i$ is the largest element that is smaller than the elements of $\mathbf{a}_i$.

The last requirement to give $\mathcal{L}_S$ the structure of a complete lattice, is to postulate the existence of a (trivial) question τ in Q which is always true[17] and which corresponds to a maximal property denoted by $\mathbf{1} = [\top]$ that is always actual for S. (Technically speaking we should also assume there exists a property $\mathbf{0}$ which is never true. Alternatively, this can be derived after we have introduced the complement in the lattice.) The structure we have then obtained, is a *complete meet-semilattice* $(\mathcal{L}_S, \leq, \wedge)$. To obtain a complete lattice we also need the supremum of arbitrary subsets $\mathcal{L}_S$. This rôle is played by the *join* denoted as $\vee$. The mathematical-canonical way to define the join of a set of properties $\{\mathbf{a}_i : i \in I\}$ from the meet is:

$$\bigvee_{i \in I} \mathbf{a}_i = \bigwedge \{\mathbf{b} \in \mathcal{L}_S | \mathbf{a}_i \leq \mathbf{b}, \forall i \in I\} \tag{2.8}$$

It is unfortunately not always obvious what the join of two properties denotes physically, but at least we know precisely how to construct it if we have the complete meet-semilattice. The structure $(\mathcal{L}_S, \leq, \wedge, \vee, \mathbf{1}, \mathbf{0})$ is then a *complete lattice*, which explains why the set of properties is traditionally given the symbol $\mathcal{L}$. The smallest members of a lattice that are not zero are called *atoms*, as is captured in the following definition.

Definition 13 (atom) *An element* $\mathbf{a} \in \mathcal{L}_S$ *is called an* atom *iff, for every* $\mathbf{x} \in \mathcal{L}_S$*, we have:*

$$\textit{if } \mathbf{0} \leq \mathbf{x} \leq \mathbf{a}, \textit{ then } \mathbf{x} = \mathbf{0} \textit{ or } \mathbf{x} = \mathbf{a}. \tag{2.9}$$

An atom represents the strongest property of a system S in the sense that it is not implied by any other property. In other words: there are no properties that allow for a finer description of S. If we were to add another property $\mathbf{x}$ that implies the atom $\mathbf{a}$, then either $\mathbf{x}$ is not a new property ($\mathbf{x} = \mathbf{a}$), or it represents something that is never true ($\mathbf{x} = \mathbf{0}$). Remark that two systems that have different atoms as actual property cannot be in the same ontological state, as there must exist a weaker property that is actual for one of the systems and not for the other. Remark also that there is no property left that we can pull out of our set of properties to distinguish two systems for which the same atom is actual. Hence it is natural to identify the atoms of the

[17] This question $\top$ can for example be paraphrased as "The system exists".

lattice of properties of a system S as the set of (ontological) states of the system and the actual atomic property as the state of the system.

Definition 14 (state) *The set of states of a system S with property lattice $\mathcal{L}_S$ is the set of atoms of $\mathcal{L}_S$. If an atom is actual for S, we identify the state of S with that atom.*

Of course, at any given instant, only one atom (the state the system is in) can be actual and we see the state can be regarded as a 'maximal' property of a system. Whether each property of the system can be realized by the system S being in a certain state, depends on the completeness of the lattice. Therefore we introduce the following vocabulary.

Definition 15 (atomic and atomistic lattice) *A lattice $\mathcal{L}$ is said to be* atomic *iff there exists at least one atom under each non-zero element, and* atomistic *in case every element of $\mathcal{L}$ is the join of a set of atoms.*

There also exist properties that are primitive properties of the system in the sense that there is no property, except for $\mathbf{1}$ that is implied by it. We call such properties coatoms.

Definition 16 (coatom) *An element $\mathbf{a} \in \mathcal{L}_S$ is called a* coatom *iff, for every $\mathbf{x} \in \mathcal{L}_S$, we have:*

$$\textit{if } \mathbf{a} \leq \mathbf{x} \leq \mathbf{1}, \textit{ then } \mathbf{x} = \mathbf{1} \textit{ or } \mathbf{x} = \mathbf{a}. \tag{2.10}$$

This leads to the following definition:

Definition 17 (coatomic and coatomistic lattice) *A lattice $\mathcal{L}$ is said to be* coatomic *iff every property of $\mathcal{L}$ implies at least one coatom, and* coatomistic *in case every element of $\mathcal{L}$ is the meet of a set of coatoms.*

The inverse question

If α is a test, we define $\sim \alpha$ as the test that corresponds to the *same* experiment as α, but with the roles of *yes* and *no* interchanged. The test $\sim \alpha$ is hence operationally well-defined iff the test α is: if we were to perform the test α and we would find with certainty *no*, then $\sim \alpha$ is *true*. Therefore we always assume Q is closed under the $^\sim$ operation:

$$\alpha \in Q \text{ iff } \sim \alpha \in Q \tag{2.11}$$

For obvious reasons, $\sim \alpha$ will be called the *inverse question* or *inverse test*.

Inverse properties and inverse tests

Even a seemingly innocent operation such as the inversion of the outcome of a test has non-trivial implications for the kind of properties that may be derived from it. Although the set of questions equivalent to a given question defines a property in the way described above, the inverse question does not, in general, define an inverse property. To see this let us take a system S with just three possible states q, r and s and let property $\mathbf{a}$ correspond to question α. Suppose that if S is in the state q, then the outcome of testing question α is always *yes*; if it is in the state r the outcome of testing question α is always *no*, and for S in the state s the outcome of testing question α yields sometimes *yes* and sometimes *no*. Clearly the property $\mathbf{a}$ is actual for S in the state q because testing question α yields *yes* with certainty and hence α is true. Clearly $\mathbf{a}$ is not actual for S in the state r and s. Suppose now we have a second question β that yields always *yes* if S is in the state q and always *no* if S is in the state r. Clearly, if testing question α yields *yes* with certainty, then so does β and vice versa, so that $\alpha \approx \beta$ and both $\alpha \in \mathbf{a}$ and $\beta \in \mathbf{a}$. See the left half of Fig (2-1).

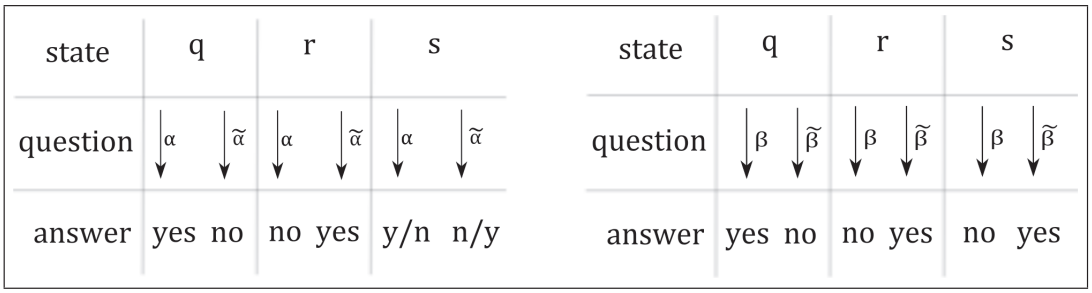

Figure 2-1 We see here for the states q and r that, if α is true, then so is β (and vice versa). So $\alpha \approx \beta$ by construction. However, for the state s we see that $\tilde{\alpha}$ is true does not imply $\tilde{\beta}$ true. This means $\tilde{\alpha}$ and $\tilde{\beta}$ are not equivalent.

However, suppose the outcome of testing question β on S in the state s always gives *no*. Then question $\sim \beta$ yields *yes* with certainty for either S in the state r or in the state s, whereas $\sim \alpha$ only yields *yes* with certainty for S in r. See Fig 2-1. We see that, although $\alpha \approx \beta$ (by construction), this does *not* necessarily imply $\sim \alpha \approx \sim \beta$. For this reason the inverse of a question cannot in general induce a unique inverse property. This problem with the inverse of a question has led to a generalization of the negation that is called the *orthocomplementation*.

Definition 18 *A function* $' : \mathcal{L}_S \to \mathcal{L}_S$, $\mathbf{a} \mapsto \mathbf{a}'-$ *is called the* orthocomplementation *iff for* $\mathbf{a}, \mathbf{b} \in \mathcal{L}_S$ *we have*

$$(\mathbf{a}')' = \mathbf{a}, \tag{2.12}$$

$$\textit{if } \mathbf{a} \leq \mathbf{b}, \textit{ then } \mathbf{b}' \leq \mathbf{a}',$$

$$\mathbf{a} \wedge \mathbf{a}' = 0 \text{ and } \mathbf{a} \vee \mathbf{a}' = I.$$

Of much greater importance to us than the orthocomplementation is the special negation that exists for classical properties.

Classical properties

> *Knowledge is two-fold, and consists not only in an affirmation of what is true, but in the negation of that which is false.*
>
> —Charles Caleb Colton, *Lacon*, 1828

From the example above we see that the inverse of a question does not necessarily define an inverse property because of the existence of states for which the test does not give a predetermined answer. Indeed, if for all states and all tests the answer would be either *yes* with certainty or *no* with certainty, then no such example could have been constructed. It is therefore appropriate to give questions with predetermined answers a special name. We will call them *classical* questions.

Definition 19 (classical questions) *A question $\alpha \in Q_S$ is classical for a system S iff either α is true or $\sim \alpha$ is true.*

As is evident from the definition, the alternatives are exclusive: if $\sim \alpha$ is true, then α is not (and vice versa). This has an important consequence. Suppose $\sim \alpha$ is true. Then for a general question, the result of α could be *yes* or could be *no*; if however α is classical, the result of test α has to be *no* with certainty as it is the inverse of $\sim \alpha$ which is true with certainty. Therefore, classical questions naturally induce classical properties.

Definition 20 (classical properties) *A property* $\mathbf{a}$ *is classical for a system S iff there is a test α in* $\mathbf{a}$ *that is classical. The set of classical properties of a system will be denoted $\mathcal{L}_S^{cl}$.*

We obviously have $\mathcal{L}_S^{\mathrm{cl}} \subset \mathcal{L}_S$. For a classical property, the orthocomplementation yields a *unique* complement, which is the Boolean negation. One can directly verify from the definition that if $\mathbf{a}$ is a classical property, then $\mathbf{a}'$ is also classical. In this case we write $\neg\mathbf{a}$ for $\mathbf{a}'$ as a notational reminder that this complement is unique. If every property in $\mathcal{L}$ has such a negation, we say the lattice is uniquely complemented.

Definition 21 (negation) *A lattice* $\mathcal{L}$ *is* uniquely complemented *iff there exists a function* $\neg : \mathcal{L} \to \mathcal{L}$, $\mathbf{a} \mapsto \mathbf{a}'$, *called a* negation *such that for every* $\mathbf{a} \in \mathcal{L}$ *there exists a unique element in* $\mathcal{L}$ *denoted by* $\neg\mathbf{a}$ *with*

$$\begin{aligned} \mathbf{a} \quad &\vee \quad \neg\mathbf{a} = 1 \\ \mathbf{a} \quad &\wedge \quad \neg\mathbf{a} = 0 \end{aligned} \tag{2.13}$$

Definition 22 (classical system) *A system is called classical iff all of its properties are classical:* $\mathcal{L}_S = \mathcal{L}_S^{cl}$.

We will deal exclusively with classical properties in the next chapter, so it is useful to introduce three quantitatively different kinds of classical properties.

Essential properties

Aristotle saw a clear distinction between the properties of a system that are *accidental* (a chair can be made of wood, but this is not essential to the object being a chair) versus the properties that the system needs to have and without which it would seize to be that system. The latter kind of properties he referred to as *essential* properties. Essential properties can be defined in our approach as classical properties which are always true. For example, an electron always has the same electric charge associated with it. Essential properties are trivial, not in the physical sense *an sich*, but for the system under consideration: to predict the outcome of the test that pertains to an essential property, it is sufficient to know only what kind of system we are dealing with. If we call the system that we study S (say a boson), we already claim implicitly that its essential properties as being S (having integer spin) are always actual. Essential properties are used to define the nature of the system S, but not its state. Attempts at grand unified theories of physics such as string theory or the standard model, seek to explain how different types of physical entities arise in nature. They must deal with reproducing essential properties. But here we are concerned with the far more modest problem of the observation of experimental propositions which we already know to be relevant for that system. As a consequence, we shall not explicitly consider essential properties when we treat the problem of observation in the next chapter.

Classical (accidental) properties

Although essential properties are by definition classical properties, accidental properties may or may not be classical. We define a *classical accidental property* as a classical property that is not essential. Because this is the type of classical property that we need the most, we

will often refer to it as *classical property*. More precisely, the property $\mathbf{a}_j$ with $j \in I$ is *classical accidental* iff $\wedge_{k\in I}\mathbf{a}_k$ is an atom (i.e. a possible state) of S, then there exists an atom (a possible state) of S for which $\neg\mathbf{a}_j$ is actual. A simple example is the state of a computer regarded as the ordered list of zeroes and ones that represent the content of the computer's memory. Every bit can then be regarded as a property with actuality of that property corresponding to the bit being 1. If such a list describes the state of the computer, then inverting the value of any bit corresponds to another state of the computer. Another example of a classical accidental property, is the property of "giving light" for a light bulb. If we are given the full state of the light bulb and the light is on, then there is a state of the light bulb in which the light is off. Note that it may very well be the case that many other properties of the light bulb have changed too. For example, if the presence of the net magnetization of the light bulb is considered as a property, then the actuality of this property will be correlated with the property of giving light, as the current-carrying wire will produce a magnetic field. One last important class of classical properties, are sets of classical properties that —taken together— describe observable quantities with more than two possible outcomes.

Classical observables

An important part of natural science is the prediction of the numerical value of a physical quantity, such as the coordinate of a particle at a given time. In the literature, such quantities are called *observables*[18]. The measurement of an observable can be translated in the language of properties. Let us first give a simple example of a classical test. The state space of a particle in classical physics is the phase space which consists of the particle's position and momentum coordinates. Suppose we have a point particle that is allowed to move on the one dimensional coordinate space in the interval $[x, y]$. We neglect the momentum coordinate and restrict the state space to the interval: $\Sigma_S = [x, y]$. We

[18] In the philosophical literature, the term *observable* usually means something that pertains to a system and that can be directly perceived by our senses. A color is observable, the spin of an electron is not. To physicists, an observable means something that one is able in principle to quantify in an experiment. To a physicist, the orbit of an electron is not observable, but its charge is. In a work that deals with both philosophy and physics, we need to settle on the meaning of the word. We will stick to the physicists' convention for two reasons. Firstly because the term observable is the preferred term in the majority of publications in the foundations of physics. Secondly because one of the claims of this dissertation is that good observations -whether they be made by the senses or by apparata- face similar problems and are subject to the same laws. To us, color and charge are both observable.

partition this restricted state space Σ_S in two sets:

$$\Sigma_S^1 = [x, z[\text{ and } \Sigma_S^2 = [z, y], \tag{2.14}$$

with $z \in]x, y[$. Next we consider the set of questions

$$\alpha_z : \text{Is the particle somewhere in } \Sigma_S^1 \text{ ?} \tag{2.15}$$

The test consists of observing the position of the particle and determine whether the outcome lies in $[x, z[$. If this is the case, we give the answer *yes*; if not, we give the answer *no*. If the observation of the position of the particle is infinitely precise and does not change the particle's position —or more precisely, when an eventual perturbation of the position happens after the recording of the position— then the result was predetermined by the position of the particle before the observation, so the question is a classical one. The corresponding property of "being somewhere in the interval $[x, z[$" is hence also classical. We see that under these conditions (infinite precision and non-disturbance of the observation) a classical property is one that either holds, or doesn't hold, even before we applied our test; the test only serves the purpose to ascertain the status of the property as being actual or not. Let us formalize the notion of this type of property somewhat more. Suppose we have an observable quantity $\mathcal{O}$ with set of possible outcomes $X_\mathcal{O}$. From an operational point of view —and contrary to the idealized example given above— any such set will in practice be a finite, discrete set, as the number we obtain from an experiment will always be finite both in magnitude and in precision[19]. There is always a maximum and a minimum to what can be measured, and there is always at least a practical limit to the maximum resolution beyond which the observer cannot be certain anymore that result is true with certainty. Future generations may be able to improve that resolution; they too will face a lower limit beyond which only statistical considerations can extend the resolution at the cost of introducing uncertainty. We will use the greater part of this dissertation to explore this possibility, but we are here concerned with classical properties that are actual (or their negation is) with certainty, not with some probability. So let us assume the cardinality of the set $X_\mathcal{O}$ is N. The power set $\mathcal{P}(X_\mathcal{O})$ then contains 2^N elements. Consider two different sets X_i, $X_j \in \mathcal{P}(X_\mathcal{O})$ with $1 \leq i, j \leq 2^N$ and associate with each of these sets the questions θ_i and θ_j that ask whether the value x of the observable quantity $\mathcal{O}$ lies

[19] Günther Ludwig [145]: "The infinity of the sets used in all important physical theories does not come from the reality which is described by theories, it comes from nonrealistic idealizations introduced to simplify the theories."

in the set X_i, resp. X_j.

$$\theta_k : x \in X_k? \ (k = i, j). \tag{2.16}$$

The questions θ_i and θ_j correspond to the (classical accidental) properties $\mathbf{o}_i$ and $\mathbf{o}_j$ respectively. If we have $X_i \subset X_j$, then clearly if θ_i is true, θ_j must be true also. Hence the actuality of property $\mathbf{o}_i$ implies the actuality of property $\mathbf{o}_j$. We therefore have by definition (2.3):

$$\mathbf{o}_i \leq \mathbf{o}_j \quad \text{iff} \quad X_i \subset X_j. \tag{2.17}$$

Obviously, $\mathbf{o}_i$ and $\mathbf{o}_j$ can only be both actual if x is in both X_i and X_j :

$$\mathbf{o}_i \wedge \mathbf{o}_j \text{ is actual} \quad \text{iff} \quad x \in X_i \cap X_j. \tag{2.18}$$

We lastly turn to the complement. If $\mathbf{o}_i$ is not actual, then the value x of the observable quantity $\mathcal{O}$ is not in X_i, and because $\mathcal{O}$ is classical, we have $x \in X_i^C = X_{\mathcal{O}} \backslash X_i$.

$$\neg \mathbf{o}_i \text{ is actual} \quad \text{iff} \quad x \in X_i^C. \tag{2.19}$$

We have explained that the join of two properties does not, in general, permit an operational interpretation. However, for classical properties, it is easy to see that, since the join $\mathbf{o}_i \vee \mathbf{o}_j$ is defined (2.8) as the meet of all properties that are implied by $\mathbf{o}_i$ and $\mathbf{o}_j$, we have:

$$\mathbf{o}_i \vee \mathbf{o}_j \text{ is actual} \quad \text{iff} \quad x \in X_i \cup X_j. \tag{2.20}$$

With this correspondence, we see that the algebra of the subsets of $X_{\mathcal{O}}$ generates a Boolean sublattice of $\mathcal{L}_S$ consisting of the set of properties $\mathbf{o}_i$ that x is in the subset X_i. In practice we do not need the entire power set $\mathcal{P}(X_{\mathcal{O}})$. For example, suppose we want to increase the precision of a numerical quantity, such as a fundamental constant, or the energy of a particle. We could simply consider a set of nested intervals, each interval smaller than the previous one, "zooming in" on the numerical value. Another, more common situation occurs when we read a digital meter to measure, for example, a voltage drop. To obtain an operational list of classical properties that allow the determination of the voltage drop in this case, we proceed as follows. We forget about the least significant digits that may fluctuate if we repeat the measurement several times under as similar conditions as we can possibly produce. The smallest quantity that can be measured with near certainty with the remaining digits will be our unit value. All outcomes will be multiples of this unit value. Consider the property $\mathbf{o}_i$ as x representing the i-th multiple of the unit value. The set $D = \{\mathbf{o}_i \in \mathcal{L}_S : 0 \leq i \leq N\}$ is the element of

$\mathcal{P}(X_{\mathcal{O}})$ with the smallest cardinality (and hence the minimal number of classical properties) that allows as good a determination of $\mathcal{O}$ as any other possible choice of properties from $\mathcal{L}_S$. It may be the best possible choice we can make at that present stage in our scientific development. The set D may not be ontologically speaking be true atoms as we have defined them (2.9), as future experimental propositions will, in all likelihood, reveal finer structure; if, however, we take the idea seriously that a property lattice is to be obtained operationally, then these $\mathbf{o}_i$ will serve as our atoms nonetheless. An important difference with (accidental) classical properties, however, lies in the negation. Suppose for simplicity that S has only one observable $\mathcal{O}$ and no other properties. Its set of states has cardinality N, as only one of each of these atomic properties $\mathbf{o}_i$ can be actual. Suppose the only knowledge we have about S is that $\mathbf{o}_i$ is not actual. Then any of the other properties $\mathbf{o}_j$ with $j \neq i$ could be actual. E.g. if you know the position of a particle is not in the origin of your frame of reference, it can be in any other position. In fact, the only property we know with certainty to be actual, $\neg\mathbf{o}_i$, is the join of many atomic properties $\vee_{j\neq i}\mathbf{o}_j$ (the property of being anywhere, except at the origin) and hence not itself an atom. If, on the other hand, a system has only one accidental property, the negation of either one of the two atoms is again an atom. We see however from these examples that it is not difficult to give examples of non-classical properties, to which we turn in the next section.

Non-classical properties

> *On every occasion I thought that, if only I had more information, I could make sense of the whole business; but every time that extra intelligence came into my hands, it again proved insufficient.*
>
> —Iain Pears, *An Instance of the Fingerpost*, 1997

Propositions tested with finite precision

A first example of a class of mildly non-classical properties, can be found when we take into account that measurements are necessarily imprecise. Recall the example above in the section on p.46, but suppose that out of a large number of observations, the vast majority will find the particle to be in some interval

$$\Delta W = [w - \Delta, w + \Delta]. \tag{2.21}$$

If $\Delta W \subseteq \Sigma_S^1$ or $\Delta W \subseteq \Sigma_S^2$, we could still claim we have predetermined answers. However, if $\Delta W \cap \Sigma_S^1$ and $\Delta W \cap \Sigma_S^2$ are both non-empty, the measurement will not have an answer that is predetermined by the state. Hence this type of position measurement is, strictly speaking, not what we have called classical. This would still be generally considered as a classical situation because one can make the measurement error so small, that ΔW fits in either Σ_S^1 or Σ_S^2. In effect this means that for the majority of states in Σ_S, the question α_z is predetermined by the state of S, and the property is classical for almost every state. We could call $\mathbf{a}_z$ *arbitrarily close* to a classical observable. Although Newtonian mechanics is an example of a theory that deals with classical properties and classical observables, a mild form of non-classicality (in the sense that we have defined it here) arises when one is verifying the theory experimentally and measurement errors are introduced.

Measurement induced state transitions

Let us consider a second example. Imagine a person with closed eyes, lying on a couch. The question β we consider is "Is the subject asleep?" In absence of technical equipment, a possible test that corresponds to this question is to simply ask the person if he/she is indeed sleeping. If the subject answers the question, we will give the answer *no*, if the subject remains silent, we give the answer *yes*. If the person is indeed truly in a state of deep sleep, he will not answer the question and the test will reveal that the person is asleep. If the person (assumed here to be honest for the sake of simplicity) is awake, he will answer the

question, which is that he is awake. If however the subject is in a state of mind that is neither deep asleep nor wide awake, then performing the act of asking the question of whether he/she is sleeping may alter his/her state. Clearly, the subject answering "I am awake" does not necessarily reveal that the state before the question was asked was one of being awake (and remaining silent may simply mean the subject is trying very hard to ignore you). So for some states the result is not solely predetermined by the state of the subject, as in the example of Fig. 2-1. As a consequence, the question β is not a classical one and neither is the property $\mathbf{b}$ that corresponds to the question β. One can counter that a more precise measurement is possible using MRI equipment, so that we need not disturb the state of the subject and still know the answer to β. In this case there is some similarity between this example and the first example: if we have access to more precise measurements, we can make the non-classical aspect (almost) disappear. However, there is a big difference between the two if we don't have access to more precise measurements, for only in the first case the point particle would still be in some exact location and its actual state left untouched by our observation. In practice, every possible measurement requires a physical interaction and affects in some way the system under investigation[20]. In some experimental situations this disturbance is minimal, in others it may not be negligible; in some cases it may be fundamental. A quantitative description of this type of situation, as well as the first type, requires a probabilistic treatment of the observer, which is one of the main goals of this manuscript.

Potential properties

An important class of non-classical questions and properties can be found in quantum physics. We will have much more to say about them in Chapter 4. For now, we merely want to point out that the examples show "non-classical", as we have defined it here, does not necessarily imply "quantum".

Comments and Notes

The modern formulation of both quantum theory and (Kolmogorovian) probability theory was developed mainly in the nineteen twenties and thirties. It was recognized already in an early stage that, although

[20] A notable exception is formed by a clock that measures time even if it doesn't interact with anything (but itself). This is of little consequence to us: in physics, time is generally considered a parameter, not a property of a system.

quantum theory deals exclusively with the probability of an event, this probability was not of the same kind as the one that was formalized by Kolmogorov. The Kolmogorovian approach to probability regards probability as a normalized measure on a Boolean lattice of events which represents the classical logical structure of the events. Garret Birkhoff and John von Neumann were the first to investigate the analog structure of the Boolean lattice for quantum theory. In their seminal paper "The Logic of Quantum Mechanics" [52] they point out that in the formal theory of quantum mechanics, an operational proposition of a quantum system is represented by an orthogonal projection operator or, equivalently, the corresponding closed subspace on the Hilbert space $\mathcal{H}$. If we denote the set of all closed subspaces by $\mathcal{P}(\mathcal{H})$, then $\mathcal{P}(\mathcal{H})$ corresponds to the set of *yes/no* questions that one can ask about a quantum system. They noticed that the structure of the space $\mathcal{P}(\mathcal{H})$ is that of a lattice, but not one that satisfies the requirements of a Boolean algebra, and hence one that does not entail a classical logic. In particular, the distributive law between conjunction and disjunction is not valid for propositions about quantum systems. The replacement of this law by a proper generalization, the orthomodular law, led to the conception of quantum logic and a new line of research was born. Whether quantum logic is a genuine kind of logic remains a topic of debate. However, the main paradigm shift in their work was to concentrate on the structure of $\mathcal{P}(\mathcal{H})$, rather than that of $\mathcal{H}$ itself. Because $\mathcal{P}(\mathcal{H})$ corresponds to the set of *yes/no* questions, a direct connection was made between the operational *yes/no* questions and the mathematical structure of quantum mechanics.

In 1963, George Mackey also pursued this line of thought in [146], but started the other way around, namely with the set of all operational propositions $\mathcal{L}$. To obtain a Hilbert space Mackey imposed directly as an axiom that the structure $\mathcal{L}$ is isomorphic with that of $\mathcal{P}(\mathcal{H})$. A year later, Constantin Piron took this line of research one step further [162]. Instead of imposing a single axiom stating equivalence between $\mathcal{L}$ and $\mathcal{P}(\mathcal{H})$, he introduced 5 separate axioms (completeness, atomisticity, orthocomplementation, covering law and weak modularity) and proved his celebrated representation theorem that shows the set of operational propositions $\mathcal{L}$, together with these five axioms, is isomorphic to $\mathcal{P}(\tilde{\mathcal{H}})$, the set of closed subspaces of a generalized Hilbert space $\tilde{\mathcal{H}}$. In later work, Piron [163] aims (and partially succeeds) to introduce the axioms in an operationally motivated way. A sixth axiom called plane transitivity [22] was derived from the mathematical work of M.P. Solèr [184] to get directly the structure of one of the three standard Hilbert spaces (over the field of real, complex or quaternion numbers). The quantum logic approach to quantum theory has attracted considerable attraction over the last few decades and is now a rather mature mathematical sub-

ject with many beautiful results. However, it has also been criticized on several accounts. First, there seems to be no canonical way of obtaining the lattice of properties for a compound system from the lattices of the constituting systems and the notion of separable systems seems only possible when either one of the two lattices is a classical lattice [5], [125], [198]. Second, the approach delivered few, if any, practical results for physics. Another critique is that the approach emphasizes the importance of introducing the concepts operationally, but in later stages imposes axioms (i.e. the covering law, orthomodularity) that are directly inspired mainly by the Hilbert space formulation of quantum theory. Additional points of critique raised by various authors, as well as neatly presented counter arguments to these points, can be found in [181]. As stated before, we have not found earlier reports on the problem of equating properties that imply one another that we identified on p.36, although this is closely related to what philosophers call the intensional character of properties. More recent formulations of the Geneva-Brussels approach [20], [24], [25] introduce the set of states separately and the relation between actuality of operational propositions (what we now call properties) and states is imposed through axioms by means of the Cartan map. We follow this scheme in the next chapter.

Chapter 3

The classical observer

We want to talk about observers, not only about sensors or detectors. We believe it is possible to explain the qualitative difference between the two without taking recourse to concepts like free will, intelligence or consciousness. A detector is sensitive to a particular kind of natural phenomenon and takes input of that kind from the natural world and produces an output that can be used by the observer. An observer, on the other hand, is a system that uses detectors or sensors/senses to construct a model of its environment which allows it to direct its actions. A Geiger-Müller counter produces a very primitive image of the world: it replicates the radioactivity in the surroundings and matches the intensity by a proportional electrical current in a part of its output circuitry. A camera even produces a pretty detailed image of the world. But both systems cannot act on that data because they have no internal representation of what the images mean. For the images to have meaning, the action that follows from the internal representation needs to have a purpose, a motivation. Purpose and motivation are too complex notions for us here, but it seems evident that a necessary requirement to be able to direct any action, is the possibility to *relate* different input states of the senses to each other by means of an internal image of the world. If the lion jumps to catch its prey, it has an estimate where the prey will be a split second later. That the lion is such a successful killer, indicates it "knows" quite well at what precise moment in time its ability to act swiftly overcomes the possible error

in the model[1] of the movement the animal has. The successes of its model were rewarded with the stilling of its hunger and this sensory experience provides the positive feedback necessary to improve its skills. An observer forms relationships between pieces of sensory input that in some way correspond to the relationships in the world. The image of the world that the observer constructs is not the same thing as the real world, but we insist on a structural resonance between the image and the world it is supposed to represent, if only partly. This, then, is to us an observer.

As we have seen in Chapter 2, the minimal structure between two properties is that of an implication. There are plenty of those in our image of the world. If we insist on causality, we as observers, are demanding that for every sensory input, or rather for every complex aggregate of inputs, there was —somewhere out there in the world— a phenomenon that would have produced another complex aggregate of inputs that necessarily predates the first set. There is an implication between the second set of inputs and the first set. An ant that directs its mobile antennae to follow the smell of a pheromone trail is doing just that. It will —depending on the desired action— be led to food or to the colony and classifies as an observer. Again, we don't want to get involved in the question: did the ant deliberately choose to gather food or to go to the colony? It is easy to test this is not haphazard movement: we can pick the ant up and put it down; it will generally try to follow the same course as it did before we picked it up. We may see two ants and as far as we can tell they look the same, but there is clearly some internal working at play as one goes for the food, the other goes for the colony. As a whole, ants seem to survive very well which is evidence that they successfully correlate model with world. We don't see rocks or Geiger-Müller counters do that kind of thing. It is however not a prerequisite for an observer that it is a biological entity: a robot that at the end of the day is using his sensors to locate a socket to reload its battery, qualifies as an observer. Of course there may exist important differences in the way a cat and a robot observe; what I am pointing out is that the formalization we will propose here will not treat the cat and the robot differently. We want to leave this particular problem in this form.

We now have a hierarchy of three levels. At the most basic level we have systems which are just that: systems with all their properties and the possible interactions they may have with other systems. One level higher, we find special systems that are extra sensitive to certain interactions in the sense that these interactions will produce correla-

[1] The word 'model' is used here in a rather loose sense, i.e. any internal function that correlates with the outside world and that allows the animal to steer its actions. A homing pigeon that finds its way home loft is using this type of model.

tions with aggregates of properties of that special system. These we have called detectors or sensors, or senses[2]. On the next level we find systems that *use* the input of sensors to form an image of its surroundings whose internal working structurally resonates with the events in the outside world. It are these systems that we have called observers. Humans and all animals surely belong to this class. The next layer in our hierarchy is a systems that has produced an image of the world that is so complex, that it includes an internal representation of itself. *The logical constraints on the formal description of these systems is the subject of our investigation in this chapter.* Such systems by definition need to observe their own properties and that is, as we shall see, not always feasible. However special it may be, some animals have shown to posses this quality. In the 1970s, Gordon Gallup [103] developed a test known as the mirror self-recognition test also known as MSR or simply the "mark test". He was inspired by Darwin who observed that Gorillas make faces to their own mirror image. But Darwin could not conclude whether the response was one of reacting at what they believed was another animal, or simply playing with their own image. The experiment has been repeated often; although most apes are at first surprised, hostile or afraid, they eventually act as if they are playing with their image. Gallup had the smart idea to repeat the experiment with a simple modification. He put some dye on a spot on their body in a part (such as on their head or back) that is accessible to the animal (for example with their limbs), but not directly visible. The spot would however be clearly visible in the mirror. Some animals clearly show that they understand the image is their own, as they will reach for the place on their body where the mirror shows the spot to be[3]. There is at least one more layer. That is, the class of systems that has an internal representation of the internal image that other observers have about them. It was long thought that this was a distinguishing feature of humans with respect to animals, but again, Gallup [104] claims this

[2]Detectors, contrary to sensors, are usually yes/no outcome apparata and the senses are biological, whereas sensors are usually assumed to be synthetic. However we need a word and perhaps 'sensor' is a word that -if its meaning in the English language is stretched a bit- can cover both detectors and senses.

[3]Both the orangutan and the chimpanzee [103] are known to consistently pass the test. But also bottlenose dolphins [149], orcas and elephants [164] pass it. Suprisingly, the European magpie passes the test [167]. Almost all human childeren pass the test somewhere between their second and third year of life, which virtually coincides with the age that is required to develop long lasting memories. Pigs do not really pass the test, but this does not prove they have no self-awareness: maybe pigs are very unimpressed by spots on their body. Pigs do show the ability to use the information displayed in a mirror. In [62] eight pigs were separately confronted with a situation in which some food was hidden behind a wall such that it could only be seen in a well-placed mirror. Seven of the pigs tested were able to find the food using the mirror. Number eighth looked *behind* the mirror for the food.

is not the case. This is the social observer, about whom we will have nothing to say in this dissertation.

Metaphysical preliminaries

> *We and our measurement instruments are part of nature and so are, in principle, described by an amplitude function satisfying a deterministic equation.*
>
> —Richard Feynman, *Probability in Quantum Mechanics*, 1951

If we are to describe the process of observation in physical terms, we have to be able to attribute a physical state to the observer. Physically speaking this is trivial as an observing system is still a system with properties that can we can study. But philosophically speaking we take no small leap. Somehow, the content of the memory and mind of the experimenter must have a counterpart in the representation of the state of the observing system, as it is necessary for him to have an understanding of the experiment and its purpose in order to be able to decide on many practical issues during the execution of the experiment and in particular in the last stage when the decision has to be taken as to whether the outcome of the test was *yes* or *no*. We do not claim we need a physical theory of the mind, for physical theory need not be the best candidate to describe something as complex as the mind. Neither will we argue with the idea that there may exist a quality in mind that is existentially independent of the physical. But we will maintain that the mind needs the physical in order to have content. More specifically, it is not possible for the observer to have experienced two different states of mind, if it happens to be twice in the exact same physical state. This metaphysical stance is a consequence of what is broadly understood under the term *physicalism*. According to the Stanford Encyclopedia of Philosophy, physicalism is "the thesis that everything is physical, or as contemporary philosophers sometimes put it, that everything supervenes on, or is necessitated by, the physical." Phrased in this language, we assume that the state of mind *supervenes* on the physical state of the observer. Alternatively, one could argue that the whole experimental setup and execution can be made to operate completely autonomous, so that the outcome is produced without any further help from the experimentalist. The state of the observer can then be regarded as the state of this autonomous apparatus. But if one insists on describing the process of observation in its totality,

one must also appreciate the fact that an outcome only means something to someone who understands the question. And one can only claim it makes sense to ascribe a state to the observer with inclusion of its mental state that contains the representation of the world in which the question makes sense, if one admits physicalism to be true in a quite literal sense[4]. In the previous chapter we have explained that a test (which is a particularly simple instance of an observation in that it allows only *yes* or *no* answers) includes the measuring apparatus, an experimenter, an instruction manual and a criterion to decide which outcome was obtained. The assumption of physicalism allows us to group them all together into one rather complex physical system M that is in a state m that —by definition— represents all the actual properties of the observer. A direct mathematical consequence of physicalism is that *the state space of an observing system belongs to the same category as the state space of the observed system.* We can have classical systems observing classical systems and quantum mechanical systems observing quantum mechanical systems, but no mixing is allowed. Another very important metaphysical assumption we make is *determinism.* In particular, we will assume the interaction between the observer and the system is deterministic. By no means we mean this to imply so-called super-determinism in which the observer is not even free in choosing a particular measurement. On the contrary, we leave the question as to *what* kind of experimental proposition the observer observes and *who* makes this choice, completely open. So in this respect, there is room for the physicalist, the dualist that insists on free will and people that adhere to psycho-physical parallelism. What we essentially mean by determinism is that, for any particular question that is chosen for a particular system in a certain state and that is observed by a particular observer in a certain state, the result will be determined. Exact repetition would yield the exact same outcome. We will show this notion of ontological determinism is nevertheless compatible with intrinsic indeterminism from the observer's point of view.

[4]In spite of this decisive sounding statement, many subtle issues arise when considering physicalism and observation, see for example Butterfield in [65]. In a minimalist spirit, the motivation for adopting physicalism can be regarded as the philosophical requirement that grants us the power of formalizing observers.

State property spaces

> *In mathematics our freedom lies in the questions we ask – and how we pursue them – but not in the answers awaiting us.*
>
> —Steven Strogatz, 2012

The advantage of introducing the concepts *state* and *property* in an operational way as we did in Chapter 2, is that we are not merely dealing with some abstract concept of state, but that we have stipulated the rules an experimenter must follow to determine that state. However, if we want to describe the state of the observer, we need to be able to asses all its actual properties. If these are also to be obtained operationally, we are in need of another observation and a complete description of the entire observation process requires the inclusion of this observer too. This would lead to an infinite regression of observations. We will therefore postulate them as is done in more modern versions of the Geneva-Brussels approach, albeit for different reasons. As we have already motivated most of the concepts in the former chapter, this approach can be introduced briefly. So for every physical system we postulate the existence states and properties and of the so-called state-property space[5] that describes the relation of the state to its actual properties. Before we are in a position to do so, we first define what we mean by a state property space.

Definition 23 (state property space) *The triple* $(\Sigma, \mathcal{L}, \kappa)$*, called a* state property space*, consists of the (non-empty) set of states* $\Sigma = \{q, r, s, \ldots\}$*, the (non-empty) set of properties* $\mathcal{L} = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots\}$*, and a function* $\kappa : \mathcal{L} \to \mathcal{P}(\Sigma)$*,* $\mathbf{a} \mapsto \kappa(\mathbf{a})$ *called the Cartan map, such that a* $\in \mathcal{L}$ *is said to be an actual property for the state* q *iff* $q \in \kappa(\mathbf{a})$*.*

Rather than saying "the property a is actual for the system S in the state s", as we did throughout Chapter 2, the state property space formulation allows us to briefly state $s \in \kappa_S(a)$. The Cartan map also allows one to define the map ξ_S that tells us which properties are actual for a given state of a system S:

$$\xi_S \quad : \quad \Sigma_S \to \mathcal{P}(\mathcal{L}_S), s \mapsto \xi_S(s), \tag{3.1}$$

$$\xi_S(q) \quad = \quad \{\mathbf{a} \in \mathcal{L}_S : q \in \kappa_S(\mathbf{a})\}. \tag{3.2}$$

[5] In the literature we find various closely related formulations, with different names such *state property system*, *state property entity* and *state property space*. We have chosen *state property space* because we reserved the word *system* for that part of the universe that we investigate and the use of of the term "state property system" is somewhat unfortunate as it seems to imply some systems would have state properties, and others not.

This map ξ is called the Aristotle map [23]. The partial order relations that we introduced in Chapter 2, can now be rephrased in set-theoretical terms.

Definition 24 (partial order) *Let S be a system with state property space $(\Sigma_S, \mathcal{L}_S, \kappa_S)$. For $\mathbf{a}, \mathbf{b} \in \mathcal{L}_S$ we define*

$$\mathbf{a} \leq \mathbf{b} \text{ iff } \kappa_S(\mathbf{a}) \subset \kappa_S(\mathbf{b}) \tag{3.3}$$

We will say that '$\mathbf{a}$ implies $\mathbf{b}$' if $\mathbf{a} \leq \mathbf{b}$.

As before, $\mathbf{a} \leq \mathbf{b}$ means that whenever $\mathbf{a}$ is actual (for a given state), then $\mathbf{b}$ has to be actual (for that state). As before, we define $\mathbf{a} = \mathbf{b}$ iff $\mathbf{a} \leq \mathbf{b}$ and $\mathbf{b} \leq \mathbf{a}$. We left open the question whether every system S has such a state property system. We will impose this in the form of an axiom.

Axiom 1 (completeness) *To every system S there corresponds a state property space $(\Sigma_S, \mathcal{L}_S, \kappa_S)$ such that for $q, r \in \Sigma_S$ and $\mathbf{a}, \mathbf{b} \in \mathcal{L}_S$, we have:*

$$\begin{aligned} &\text{if } \kappa_S(\mathbf{a}) = \kappa_S(\mathbf{b}), \text{ then } \mathbf{a} = \mathbf{b}, \text{ and} \\ &\text{if } \xi_S(q) = \xi_S(r), \text{ then } q = r. \end{aligned} \tag{3.4}$$

This axiom tells us that every system 'has' a state property space and that this state property space exhausts all there is to know about the system. The name *completeness* refers to two different kinds of completeness: (a) every system has it, (b) it provides a complete description of the system. The first form of completeness is related to the idea of physicalism, the second to the possibility of providing an exhaustive representation of natural systems. The first implication in this axiom tells us that two properties ($\mathbf{a}$ and $\mathbf{b}$) are the same if the set of states for which these properties are actual ($\kappa_S(\mathbf{a})$ and $\kappa_S(\mathbf{b})$) are identical. The second implication tells us that two states of a system are identical iff the set of properties that are actual if the system is in each of these states, are identical. Whereas we agree with the second implication, we have already indicated in Chapter 2, p. 36 the first implication is, in general, not fulfilled. That is, we can easily construct examples of pairs of strongly correlated properties that will be actual for precisely the same set of states, but which do not correspond to the same property, as one can recall from the example of the torch on p. 35. Equating equivalent properties causes a form of degeneracy, in which different properties can have the same label. We have indicated in that same section how one may proceed to lift this degeneracy, but this is not necessary for our purpose here. Indeed, the point of this chapter is to examine how an observer can come to learn about a given

system through observation of the tests that correspond to relevant properties. The ultimate task of an observer is state determination. From the axiom of completeness given above, we have that two properties that are equivalent in the sense of their corresponding questions implying one another, have the same set of states which make them actual. Hence they have exactly the same "resolving power" with respect to any particular state determination problem. Let us reiterate this in a slightly more formal way. Suppose we have observed that property **a** is actual for a system S. The state can then no longer be an arbitrary element from Σ_S, but needs to be an element of $\kappa_S(\mathbf{a}) \subset \Sigma_S$. If property **b** is equivalent to property **a**, then —by equivalence— we have that **b** is actual too. Hence the state needs to be in $\kappa_S(\mathbf{b})$, but —again by equivalence— we have $\kappa_S(\mathbf{a}) = \kappa_S(\mathbf{b})$, indicating we have not improved our knowledge of the state of the system. So instead of changing the content of the axiom, we choose to think of the properties defined here, as *equivalence classes of properties that are equally resolvent with respect to the problem of state determination.*

Perhaps the reader wonders why we went through the effort of defining properties operationally in Chapter 2, only to postulate them now. First, as we already mentioned, this escapes the infinite regression of observations that would be necessary to establish the *existence* of a property. We simply proclaim they exist by the completeness axiom. Secondly, the operational procedures explained in Chapter 2 will serve as a rather precise description of what an observer has to do in order to establish the actuality of properties of a system, and we will use them as a lead to define and motivate the requirements for a system to be an observer. In spite of obvious idealizations we illustrate that state property spaces are a very general framework. We will illustrate this by briefly describing the explicit state property system of two particularly relevant instances.

The state property space of a classical system

The state space of a system described by classical mechanics is phase space. For an n-particle system a point in phase space is determined uniquely by the $3n$ position and $3n$ momentum coordinates, denoted $\vec{q}$ and $\vec{p}$ respectively. Hence we have $\Sigma_S = \mathbb{R}^{6n}$. The observables are represented by real-valued (differentiable) Borel functions from phase space to the (real) value the observable takes when the system is in the state $(\vec{q}, \vec{p})$:

$$f \quad : \quad \mathbb{R}^{6n} \to \mathbb{R} \tag{3.5}$$

$$(\vec{q}, \vec{p}) \quad \longmapsto \quad f(\vec{q}, \vec{p}). \tag{3.6}$$

Experimental propositions correspond to Borel (sub)sets Γ of the real line that represent a set of numbers on the reading scale of some abstract apparatus in the following way. Consider now the set $\mathcal{B}(\mathbb{R}^{6n})$ of all Borel sets of $\mathbb{R}^{6n}$. With every experimental proposition, we identify the unique Borel set $f^{-1}(\Gamma) \in \mathcal{B}(\mathbb{R}^{6n})$:

$$f^{-1}(\Gamma) = \{(\vec{q}, \vec{p}) \in \mathbb{R}^{6n} : f(\vec{q}, \vec{p}) \in \Gamma\}.$$

Clearly $f^{-1}(\Gamma)$ is the set of states for which the reading scale of the apparatus takes a value in Γ. Vice versa, with every $B \in \mathcal{B}(\mathbb{R}^{6n})$ we associate a property that is actual iff $(\vec{q}, \vec{p}) \in B$. For properties we take f as $\mathbf{1}_B$, the indicator function of the set B. If $\mathbf{1}_B$ is explicitly given we can formally (but trivially) rewrite this last equation as

$$B = \{(\vec{q}, \vec{p}) \in \mathbb{R}^{6n} : \mathbf{1}_B(\vec{q}, \vec{p}) = 1\}.$$

Clearly we have $\mathcal{L}_S = \mathcal{B}(\Sigma_S)$ and the Cartan map κ_S is simply the identity mapping Id. Hence the state property system of an n particle system described by classical mechanics is $(\mathbb{R}^{6n}, \mathcal{B}(\mathbb{R}^{6n}), Id)$. States correspond to point sets in $\mathbb{R}^{6n}$ and are atoms of the lattice. Remark that the algebra of the Borel sets naturally induces the implication and the meet as respectively the set-theoretic inclusion and the intersection of the Borel sets that correspond to the properties.

The state property space of a quantum system

For a system described by orthodox quantum mechanics, we take as set of states the unit norm rays in Hilbert space $\mathcal{H}_{\mathbb{C}}$ over the field of complex numbers. A ray $[\psi]$ with representative ψ is the equivalence class of vectors that differ at most an overall phase factor from ψ:

$$\phi \in [\psi] \Leftrightarrow \exists \alpha \in [0, 2\pi[: \phi = e^{\mathrm{i}\alpha}\psi.$$

Hence the state space of a quantum system is the set $\Sigma(\mathcal{H}_{\mathbb{C}}) = \{[\psi] \in \mathcal{H}_{\mathbb{C}} : |\psi| = 1\}$. The set of properties $\mathcal{L}_S$ is given by the set of closed linear subspaces[6], or —equivalently— the set of projectors (onto those subspaces), which we denote by $\mathcal{L}(\mathcal{H}_{\mathbb{C}})$. For any projector $\mathbb{P} \in \mathcal{L}(\mathcal{H}_{\mathbb{C}})$, the Cartan map is then given by:

$$\kappa(\mathbb{P}) = \{[\psi] \in \Sigma(\mathcal{H}_{\mathbb{C}}) \mid \forall \psi \in [\psi] : \mathbb{P}\psi = \psi\}.$$

[6]Linear subspaces are by definition closed under addition of vectors and multiplication by a scalar, so this is not the type of closed we refer to here. Closed is used here in the topological sense, i.e. a subset L of Hilbert space $\mathcal{H}$ is called closed iff for every (Cauchy) sequence of vectors $(\phi_n)_n$ in L, such that $\phi_n \to \phi$ in $\mathcal{H}$, we have that $\phi \in L$. A linear subspace of a vectors space is itself a vector space; a closed linear subspace of a Hilbert space, is itself a Hilbert space.

The state property space of a system that is described by quantum mechanics is then $(\Sigma(\mathcal{H}_{\mathbb{C}}), \mathcal{L}(\mathcal{H}_{\mathbb{C}}), \kappa)$. Note that $\mathbb{P}_1$ implies $\mathbb{P}_2$ iff $\mathbb{P}_1$ is a linear subspace of $\mathbb{P}_2$; the meet of two properties is represented by the intersection of the corresponding closed linear subspaces. The states, the unit norm rays in Hilbert space, correspond to the atoms of the lattice $\mathcal{L}(\mathcal{H}_{\mathbb{C}})$.

The lattice $\mathcal{L}_S^{\mathbf{cl}}$

We have shown in Chapter 2 that $\mathcal{L}_S$ is a complete semi-meetlattice. As explained there, we can use the implication to define the join of a set of properties $\mathbf{a}_i$ with $i \in I$ as

$$\vee_{i\in I}\mathbf{a}_i = \wedge\{\mathbf{b} \in \mathcal{L}_S \mid \mathbf{a}_i \leq \mathbf{b}, \forall i \in I\} \tag{3.7}$$

and $\mathcal{L}_S$ becomes a complete lattice. We will restrict our attention in the rest of this chapter to the observation of systems that have only *classical properties* and will call such systems classical systems.

Definition 25 (classical system) *We will say a system is* classical *if the lattice of properties in its state property system is classical.*

The symbol we will use to indicate this lattice of classical properties is $\mathcal{L}_S^{\mathrm{cl}}$. A classical property $\mathbf{a}$ is either actual, or its negation $\neg\mathbf{a}$ is, so if $\mathbf{a} \in \mathcal{L}_S^{\mathrm{cl}}$, we also have $\neg\mathbf{a} \in \mathcal{L}_S^{\mathrm{cl}}$. One wonders whether such a complete, uniquely complemented lattice is not simply a Boolean lattice, as the only lacking property is distributivity[7]. In 1904, E.V. Huntington [123] stated the following problem: *Is every uniquely complemented lattice distributive?* Many mathematicians in the field (including Garrett Birkhoff) conjectured the answer was affirmative. But the question turned out to be both subtle and difficult, and even today we simply

[7] A Huntington property is any extra assumption that makes a completely negated lattice distributive. The first known Huntington property was discovered by John von Neumann and Garrett Birkhoff who proved that a *modular* lattice with a unique complement is distributive. Birkhoff and Ward proved that if it is atomic and complete, then it is also distributive, making it Boolean. However, history has shown this to be a difficult question. In fact, in a review article "*Two problems that shaped a century of Lattice Theory*" for Notices of the AMS from 2007, George Grätzer calls Huntington's question one of these two problems. Important progress was made in 1945 when R.P. Dilworth gave a highly non-trivial construction to embed *every* lattice into a lattice with unique complement. Now take any non-modular lattice $\mathcal{N}$ and embed it into a lattice $\mathcal{L}$ with unique complements using Dilworth's technique. Then $\mathcal{L}$ cannot be modular since it contains a non-modular sublattice. The present day situation is such that many Huntington properties are known. A Huntington variety is a lattice variety in which every uniquely complemented is distributive and it is known there is a continuum of such varieties [156].

do not know whether there exists a complete lattice with unique complement that is not distributive. However, we want our states to be atoms of the lattice (which, as shown in the examples, is the case for both classical and quantum-mechanical systems), and impose that our lattice is atomistic, in which case it is also atomic[8] and by a theorem of Birkhoff and Ward [53] the set of properties will then form a Boolean lattice. Without repeating the proof in [53], we then have:

Theorem 26 *The structure* $(\mathcal{L}_S^{cl}, \leq, \neg, \wedge, \vee, \mathbf{1}, \mathbf{0})$ *is a Boolean, atomistic lattice.*

Questions and state property systems

In Chapter 2, the primitive notion was the set of experimental propositions Q; properties were regarded as equivalence classes of such experimental propositions. We employ the wording *experimental proposition* here rather than its synonyms *test* or *question* to emphasize that the properties in the lattice are those that we know how to observe experimentally. This means that for each question in Q_S, we are in need of an experiment that can test it. In some cases this may be a single experimental setup, in other cases the experimental proposition can be a complex product of other propositions. Questions are not explicitly represented in the state property space but to define observation we need a way to tell what property the observer is testing. To this end we postulate a correspondence between Q_S and $\mathcal{L}_S^{\mathrm{cl}}$.

Axiom 2 (correspondence) *For every system* S *with a property lattice* $\mathcal{L}_S^{cl}$ *there exists a countable*[9] *set* Q_S *such that to every property in* $\mathcal{L}_S^{cl}$*, there corresponds an experimental proposition in* Q_S*.*

Since Q_S is assumed to be countable, its elements can be numbered by an index that takes values in the natural numbers and we shall indeed often write α_i for an arbitrary question. Still Q_S can be a very big set. Ideally we want to be able to find out what the state s of the system S is and we need a set of questions that is sufficiently rich to manage that.

Definition 27 (distinguishing list) *A set of questions* $D \subset Q_S$ distinguishes *the state* $s \in \Sigma$ *iff no other state in* Σ *will produce the same yes/no answers to the questions in* D *as* s *does.*

[8] Recall from Chapter 1 that a lattice is *atomic* iff there exists at least one atom under each non-zero element, and *atomistic* in case every element of $\mathcal{L}$ is the join of a set of atoms.

[9] A countable set has the same cardinality as some (not necessarily proper) subset of the set of natural numbers.

Definition 28 (complete list) *A set Q_S is a* complete list (of questions) *for S (with state space Σ_S) iff Q_S distinguishes every $s \in \Sigma_S$.*

When the set is complete for S, it can discriminate between any two states of S. By the axiom of completeness of $\mathcal{L}_S^{\text{cl}}$ (see p. 61) and the correspondence axiom, we have that Q_S is *complete*, which means such a complete list always exists by postulate. The questions in a complete list for s at least include those that correspond to the properties in the Aristotle map $\xi(s)$, Eq.(3.1).

The classical observer

We have already explained that strict adherence to physicalism implies the observer is simply a specific type of physical system. As a consequence, within any given physical universe, both the state property space of the system and that of the observer have to belong to the same mathematical category. If this were not the case how would we even start to describe their interaction? We will assume throughout this chapter the system under observation and the observer are both classical systems[10] in the sense of definition 25; this is what we mean when we talk about the "classical observer".

The ultimate goal of observation is the determination of the state of the system under observation. But the state of a system is not carved in stone: it is reconstructed from the registration of the actual properties as evidenced by the observation of the corresponding experimental propositions[11]. Suppose we have a system S in the state s with set of actual properties $\xi(s) = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, ...\}$. For every property $\mathbf{a}_i$ in this list we have, by the axiom of completeness that $s \in \kappa_S(\mathbf{a}_i)$. So each correctly observed α_i shows us whether a property $\mathbf{a}_i$ is actual or not, and brings the observer closer to the determination of the state of S. If this is done for every property in the list, the state is determined. One of the problems the observer faces, is that the mere act of observation of the system requires an interaction and may disturb the actuality of other properties. *This represents a first fundamental problem in the state determination of a single system*: when we succeed in establishing whether a given property is actual, this may (and

[10] In Chapter 5 we will treat a quantum observer observing a quantum system.

[11] Please note that to us observation refers to the whole chain of events that constitute the embodiment of an experimental proposition and that leads to an outcome which is considered the answer to that proposition. We will however often simply refer to "the observation of a property", of even "the observation of a question" in spite of the obvious oversimplification it seems to imply.

often will) affect the actuality of other properties of interest; it may even change the actuality of the property that was tested! This means we need to stipulate what constitutes an observation: do we want to establish the truth value of a property prior or posterior the act of observation? The same problem exists in quantum mechanics where the mere measurement of a property can even destroy the system in which case there seems only one viable option. As we are particularly interested in quantum theory, we will follow this lead and highlight our choice in the following remark[12].

Remark 1 *The aim of observation is to assess the truth value of a property as it was immediately prior to the observation.*

We will come back to this first fundamental problem in the section on compound questions on p. 79, but for now we want to formulate what the best next option in line is. Recall that the state of the system s is given by the meet of all the actual properties of S and is tested by the product question. According to the operational definition of product question, the observer only needs to be able *to verify the actuality of any arbitrarily chosen* α_i. We will call an observer M *free* for S in the state s iff for any property $\mathbf{a}_i$ for which $s \in \kappa_S(\mathbf{a}_i)$, it can observe the actuality of $\mathbf{a}_i$. (A more precise definition will follow soon.) The free observer may not be able to asses the full state of S, but it can correctly observe any particular property at will. This is already a very strong type of observer because, under several benevolent circumstances, such an observer can acquire a lot of information about the system, possibly even its state. If, for example, the free observer is in proud possession of n identical copies of the system and it needs to assess the actuality of any given n properties, it can. If n is greater than the number of questions in a complete list, it can also determine the state. If the free observer is able to "reset" the state of the system to the pre-measurement state, it may observe other properties until it has full knowledge of the state. If the observer is only interested in a subset of the properties and it manages to observe those properties in a way that affects only non-relevant properties, it can assess the actuality of the properties of interest. But if we want to include the possibility that it is not in any of these lucky circumstances, the free observer is the best we can hope for. The definition of the free observer begs the questions what it means to observe correctly the result of the experimental test pertaining to a *single* property. Hence we turn to the most basic of

[12] Other choices can be entirely rational. The introduction of a novel product on the market can be regarded as a question whether there is public interest in this product. In such a case, one would clearly be interested in the truth value of the property *after* measurement.

observations: that of a single observation of the test that pertains to a single property. To formalize the process of observation, it is helpful to fix a minimal protocol. This is what we have in mind: an observer receives an experimental proposition and a system. The goal is to interact with the system under observation according to the rules of the test and indicate correctly its outcome. Let's put it on paper.

Observation as a physical interaction

An observer M with state property space $(\Sigma_M, \mathcal{L}^{\rm cl}_M, \kappa_M)$ in a state $m \in \Sigma_M$ gets a system S with state property space $(\Sigma_S, \mathcal{L}^{\rm cl}_S, \kappa_S)$ in an unknown state $s \in \Sigma_S$ and a question $\alpha \in Q_S$. It has to give as outcome "1" if property **a** that corresponds to question α is actual for S. If not it has to give the outcome "0". We represent this by a mapping o :

$$o : \Sigma_S \times \Sigma_M \times Q_S \to \{1, 0\}, (s, m, \alpha) \mapsto o(s, m, \alpha). \tag{3.8}$$

This mapping o is our main building brick of observation: a truth-valued function on $\Sigma_S \times \Sigma_M \times Q_S$. We will show how this mapping can be implemented in a physical account in a three stage process whose stages we call *preparation*, *interaction* and *indication*. The following exposition is not sufficiently general to encompass every type of observation: it is however sufficiently general to act as a mathematical model of the physical stages involved in setting up the mapping o.

1. The observer first receives a question (or it may itself come up with a question) which is an element of Q_S. It prepares itself (and any necessary equipment that when used becomes part of the observing system) in a state that it deems fit to observe the property that corresponds to the question in the following way. If the question is α, it evolves according to:

$$\begin{aligned} prep \quad &: \quad \Sigma_M \times Q_S \to \Sigma_M, \\ (m, \alpha) \quad &\mapsto \quad prep(m, \alpha) = m_\alpha \end{aligned} \tag{3.9}$$

   The suffix in the notation of m_α serves as a label that the state of M is fit to interact with S in order to observe the experimental proposition related to α. This stage gives the observer the possibility to prepare for the particular measurement that corresponds to α. This preparation cannot depend on the (unknown) state of the observed system, but it may depend on what kind of system S is under consideration[13]. This can be accomplished

[13] E.g., to measure the velocity of an electron, we can measure the curvature of its track when exposed to a magnetic field; a method that is useless for chargeless particles.

by assuming the wording of α always includes reference to the kind of system we test. If it is the case that for some questions α the observer is already in a state m fit to measure α, we can assume without loss of generality that *prep* is the identity; hence if the state is already m_α, then *prep* is the identity in the first argument: $prep(m_\alpha, \alpha) = m_\alpha$. From this we see that

$$\begin{aligned} prep(prep(m,\alpha),\alpha) &= prep(m_\alpha,\alpha) && (3.10)\\ &= m_\alpha = prep(m,\alpha), && (3.11)\end{aligned}$$

which shows that $prep(\cdot, \alpha)$ is idempotent in its first argument:

$$prep(prep(m,\alpha),\alpha) = prep(m,\alpha). \tag{3.12}$$

So preparing the observer once more in the same state, does not change anything. The observer is now ready to interact with the system.

2. In a second stage, the observer M *interacts* with the system and ends up in a different state.

$$\begin{aligned} int &: \Sigma_S \times \Sigma_M \to \Sigma_M && (3.13)\\ (s, m_\alpha) &\mapsto int(s, m_\alpha) = m'_\alpha \end{aligned}$$

We have already acknowledged that the state of S may change as a result of the observation, so we are interested here only in the question whether S had the property **a** *prior* to the measurement, as is customary in most quantum experiments.

3. Finally, the post-measurement state *indicates* whether the outcome is yes or no, depending on the actuality of a preferred classical property of the observer that we call the indicator: $\mathbf{i} \in \mathcal{L}_M^{\mathrm{cl}}$

$$\begin{aligned} ind &: \Sigma_M \to \{0,1\}, m \mapsto ind(m), && (3.14)\\ ind(m'_\alpha) &= 1 \text{ iff } m'_\alpha \in \kappa_M(\mathbf{i})\\ ind(m'_\alpha) &= 0 \text{ iff } m'_\alpha \in \kappa_M(\mathbf{i}) \end{aligned}$$

The outcome mapping (3.15) is the composition of the three mappings (3.9), (3.13) and (3.14):

$$\begin{aligned} o &: \Sigma_S \times \Sigma_M \times Q_S \to \{1,0\} && (3.15)\\ (s,m,\alpha) &\mapsto o(s,m,\alpha) = ind(int(s, prep(m,\alpha))). \end{aligned}$$

We assume the mapping o is onto for every α so that every outcome is the result of at least one s and m. As we assume the interaction is

deterministic, all mappings are (single-valued) functions. From (3.15) and (3.12) we derive a useful identity:

$$\begin{aligned} o(s,m,\alpha) &= ind(int(s, prep(m,\alpha))) && (3.16)\\ &= ind(int(s, prep(prep(m,\alpha),\alpha))) && (3.17)\\ &= ind(int(s, prep(m_\alpha,\alpha))) && (3.18)\\ &= o(s,m_\alpha,\alpha) && (3.19) \end{aligned}$$

The adequate observer

We are now in possession of a mathematical procedure that models the physical protocol, but we still have to formulate the most important ingredient before an interaction deserves to be called an observation, which is that the outcome should correlate with the actuality of the property of the system. This is captured in the next definition.

Definition 29 (Sensitive observer) *Let S be a system, M the observer and α an experimental proposition that corresponds to property $\mathbf{a} \in \mathcal{L}_S^{cl}$. M will be called* **a**-sensitive *iff there exists a mapping $o : \Sigma_S \times \Sigma_M \times Q_S \to \{1,0\}$, given by (3.15) such that for every $m \in \Sigma_M$ and for every $s \in \Sigma_S$*

$$o(s,m,\alpha) = o(s,m_\alpha,\alpha) = \mathbf{1}_{\kappa_S(\mathbf{a})}(s). \tag{3.20}$$

The right-hand side $\mathbf{1}_{\kappa_S(\mathbf{a})}(s)$ in the definition is the indicator function, i.e. $o(s,m,\alpha) = 1$ iff $s \in \kappa_S(\mathbf{a})$ and zero otherwise. The strong requirement in the definition of **a**-sensitiveness lies in the fact that M needs to satisfy the correlation expressed by (3.20) *for every* $s \in \Sigma_S$. We insist this is necessary for the concept of sensitivity to have any meaning at all if we are to infer information about a system in an unknown state. Sensitivity also requires M should be able to observe α for every m, but this is less stringent than may seem to be the case, as the first step in the observation protocol is the function *prep* which is an evolution of m to the state m_α. So in essence we impose that M can always evolve from m to m_α. The concept of sensitivity is the stepping stone that allows us to define what we have previously described as a *free* observer.

Definition 30 (Free observer) *An observer M is* free *with respect to a set of properties $\mathcal{A} \subset \mathcal{L}_S^{cl}$ of a system S iff it is* **a***-sensitive for every* $\mathbf{a} \in \mathcal{A}$. *An observer M will be called* free *(with respect to S) iff M is free for $\mathcal{L}_S^{cl}$.*

It is however not sufficient to assert only correlation (3.20). We need to exclude the possibility that a piece of paper with the word

"*Yes!*" written on it constitutes a good observer of any property that happens to be actual for S at that moment. If, for some reason, the property **a** happens to be no longer actual, then (3.20) already excludes our paper-with-*Yes!* as an observer. But we also want to exclude the possibility that the paper qualifies as a good observer of an *essential property* (see page 45), i.e. in case **a** is always actual. The rationale for such a disqualification is that logically speaking *it makes no sense to assert that we have genuinely observed a classical property to be actual, when we know that we would have received the same outcome if the inverse question had been asked.* So we need to impose $o(s, m, \sim \alpha) \neq o(s, m, \alpha)$. We call such an observer *consistent.*

Definition 31 (Consistent observer) *An observer M is* consistent *for S iff for every pair of questions $\alpha, \sim \alpha \in Q_S$ we have $o(s, m, \sim \alpha) = 1 - o(s, m, \alpha)$.*

The inverse question $\sim \alpha$ was operationally defined in Chapter 2 by the observer following *exactly* the same procedure as for testing α but switching the outcomes of the test. All operational procedures are now to be performed by the observer, who is the physical realization of the mapping o, so this switching of outcomes needs to have some counterpart in terms of observer states.

It is easy to see this role is played by the state $m_{\sim\alpha}$. So we could rephrase the condition of consistency as the requirement that for each state m_α there needs to exist a state $m_{\sim\alpha}$. It is important to notice that, because o is single-valued, m_α and $m_{\sim\alpha}$ can *never* be the same state. Nonetheless m_α and $m_{\sim\alpha}$ are extremely similar states because from an operational perspective, the two states have to perform the same complex series of steps; only in the last step, when they attribute opposite outcomes, they differ. (It is hard to imagine a good observer of classical properties that fails to switch the outcomes when the inverse question is posed.)

The idea is that the consistent observer can give the correct answer to $\sim \alpha$ (provided it gives the correct answer to α) by evolving to the state $m_{\sim\alpha}$ (this could be as simple as knowing or remembering it has to switch the outcomes at the end of the test) and perform the same procedure as it would for test α. Consistent observers also solve the problem we mentioned earlier: our piece of paper with the word "*Yes!*" written on it, doesn't classify as a consistent observer, not even if the property it was supposed to be saying "*Yes!*" about, is actual. Hence if M is **a**-sensitive and consistent implies, besides Eq. (3.20), that:

$$\begin{aligned} s \;&\in\; \kappa_S(\neg\mathbf{a}) \Leftrightarrow o(s, m, \sim \alpha) = 1 \\ s \;&\notin\; \kappa_S(\neg\mathbf{a}) \Leftrightarrow o(s, m, \sim \alpha) = 0 \end{aligned} \tag{3.21}$$

We can formulate this conveniently using the indicator function:

$$o(s, m, \sim \alpha) = \mathbf{1}_{\kappa_S(\neg \mathbf{a})}(s). \tag{3.22}$$

There is one last item we have on our wishlist for the observer. After the first stage in the measurement interaction, the state m is "prepared" in a state m_α fit to observe the experimental proposition α, which we have expressed as $prep(m, \alpha) = m_\alpha$. This means that at this stage, the state of the observer "encodes" the fact that it is fit to observe the experimental proposition α. We require that our observer can extract that information back from its own state by means of a special question φ and call an observer that can do so, *introspective*.

Definition 32 (Introspective observer) *An observer M is* introspective *if there exists a question $\varphi \in Q_M$ such that for every $\alpha \in Q_S$, we have: $o(s, m_\alpha, \varphi) = o(s, m, \alpha)$.*

We see that φ, in some sense, inverts the mapping *prep* given by Eq. (3.9).The observer can physically realize this either by memorizing the question α or by introspectively deducing it from its state. Either way, the existence of φ represents the ability of M to follow through with the procedures directed by the experimental proposition α. In essence, it means the observer either 'knows' what it is doing, or that it can deduce introspectively to what particular question it is sensitive.

The different requirements that we impose on observing system form a ladder of ever higher requirements. The first step on this ladder is formed by the requirement of being *consistent*. It is a *conditio sine qua non* for any type of test, irrespective of the system tested or the system used to test. The second step, requirement of *sensitiveness* with respect to a particular property, is what guarantees that performing the test would yield information about the corresponding property. It is hence also a *conditio sine qua non* for any type of test. *Freeness* requires sensitiveness with respect to more than one property and is a quantitative step up from sensitiveness rather than a qualitative. A laboratory is an example of a system that would satisfy this requirement. An electronic multimeter would be a more modest example. Freeness with respect to S (i.e., to every property of a system) is indeed a strong requirement, but if we want to be able to perform a state determination task for S, we will have to impose it. The highest step on the ladder is the condition of being *introspective*. It really is this requirement that sets apart the laboratories and measuring devices from the observers. Not only needs the system to be able to test any question, the systems needs to be able to answer for itself which question it is testing. It is a rather high level, but very reasonable requirement. As we explained in the introduction of this chapter, the difference between a measuring

device and an observer is that an observer "is able to *relate* different input states of the senses to each other by means of an internal image of the world". To accomplish such feat, it is a necessary precondition that the observing system can tell for which question its state is prepared. Note that, although being introspective is defined with respect to an observer observing an arbitrary system, it is in essence a requirement that needs some form of self-reference. As it is the state of the observer which "encodes" the question for which it is prepared, the observer needs to somehow infer that question from its own state. Ideally, we want an observer to be introspective, free (with respect to S) and consistent. We call such an observer an *adequate observer for* S.

Definition 33 (Adequate observer) *An observer M will be called* adequate *for S iff it is consistent, free (with respect to S) and introspective.*

The adequate observer is able to observe in a consistent way any particular property of S; given the fact that the observer may alter the state of S by the act of observation, this seems to be the best one can hope for without taking recourse to repeated observations.

The classical self-observer

> *I am not yet able, as the Delphic inscription has it, to know myself; so it seems to me ridiculous, when I do not yet know that, to investigate irrelevant things.*
>
> —Socrates as quoted by Plato in *Phaedrus*, 370 BC

Suppose we have a universe with only classical systems and that, in this universe, there is an observer that is adequate with respect to quite complex systems. If such an observer decides to look at itself[14], we will call it a *classical self-observer.* We wonder whether such an observer can be adequate with respect to itself and will grant it every physically possible advantage to help it do so. To begin with, the self-observer may study its own state as if it were just another system. Formally this means the equations of the former section remain valid when we replace S with M, s with m and o becomes a mapping $o : \Sigma_M \times \Sigma_M \times Q_M \to \{1, 0\}$. As the state that M is observing is the

[14] For a more elaborate motivation to observe oneself, we refer to the introduction of this dissertation.

same state as the one it is in, we can unambiguously (but with abuse of notation[15]) abbreviate

$$o(m, m, \alpha) \text{ by } o(m, \alpha). \tag{3.23}$$

Let's collect all we have from the former section and what falls under the definition adequate self-observer.

Definition 34 (Adequate self-observer) *A classical self-observer M with state property space $(\Sigma_M, \mathcal{L}_M^{cl}, \kappa_M)$ is* adequate *iff there exists a mapping $o : \Sigma_M \times Q_M \to \{1, 0\}$ defined by (3.15) such that for every state $m \in \Sigma_M$ and for every question $\alpha \in Q_M$ and corresponding property $\mathbf{a} \in \mathcal{L}_M^{cl}$, M is* free[16]

$$o(m, \alpha) = o(m_\alpha, \alpha) = \mathbf{1}_{\kappa_M(\mathbf{a})}(m), \tag{3.24}$$

and consistent:

$$o(m, \sim \alpha) = 1 - o(m, \alpha). \tag{3.25}$$

Moreover M is introspective*, i.e. $\exists \varphi \in Q_M$ such that for every $\alpha \in Q_M$ we have*

$$o(m_\alpha, \varphi) = o(m, \alpha). \tag{3.26}$$

Unfortunately, there are no adequate classical self-observers as we show in the next theorem.

Theorem 35 *There does not exist a classical self-observer M that is adequate.*

Proof: Assuming M is adequate, it is introspective: $(\exists \varphi \in Q_M)(\forall \alpha \in Q_M) : o(m_\alpha, \varphi) = o(m, \alpha)$. Q_M is closed under negation, hence $\varphi \in Q_M$ implies $\sim \varphi \in Q_M$. Define χ as $\sim \varphi$. M is introspective for every $\alpha \in Q_M$, hence also for χ and we obtain: $o(m_\chi, \varphi) = o(m, \chi)$. Since M is free, it is sensitive with respect to the property that corresponds to χ and we obtain $o(m, \chi) = o(m_\chi, \chi)$. Combining the last two equations, we have $o(m_\chi, \varphi) = o(m_\chi, \chi)$. As χ is defined as $\sim \varphi$, we have that φ is equivalent to $\sim \chi$ and we obtain $o(m_\chi, \sim \chi) = o(m_\chi, \chi)$ showing M is inconsistent.■

As a direct corollary we obtain that no consistent and introspective classical self-observer can be free with respect to all its properties. Every property of an observer trivially partitions its state space into

[15] Unambiguously, as the two distinct mappings can be distinguished by the number of arguments they take. Translation from $o(\cdot, \cdot, \cdot)$ to $o(\cdot, \cdot)$ can be made through equation (3.23).

[16] Note that we before we talked about an observer being "free for S", because it may have been free with respect to a given system, but not with respect to another. We drop this reference now, as we are interested only in one particular system: M.

two sets: those for which it is actual as opposed to those states for which the property is not. Given that there are properties for which the consistent and introspective classical self-observer is not free, implies there are states of the observer that it cannot distinguish between.

A diagonal argument

The proof above is very simple, but due to its formal character perhaps not very transparent. As we put in considerable effort to motivate the ingredients of the argument, it is worthwhile to spell out the proof in detail. This will add to our understanding of the problem and show along the way that the proof is in fact nothing but a diagonal argument. To do so we will make use of a simple coding scheme for states that is conceptually easier, but not unlike the famous Gödel numbering. There is no need for an observer to have access to this coding, we merely use it for our argument. We'll explain the coding of a system S before we do the same for an observer. Let $L_S \subset Q_S$ be a complete and countable set of questions for S, which exists by our correspondence axiom, Axiom 2 on p.65. We order the members of this set in an arbitrary but fixed way as $L_S = \{\alpha_1, \alpha_2, \alpha_3, \ldots\}$. Giving natural numbers as index to the questions is always possible because the cardinality of L_S is at most countable. By definition, the answers to the experimental propositions in L_S determine the state of S. Hence we can write out the states of the system as a list of zeroes and ones depending on whether the question α_i is true or false. Writing 1 for true and 0 for false, we can picture this as:

$$\begin{array}{cccccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \ldots \\ 1 & 0 & 1 & 1 & \ldots \end{array} \tag{3.27}$$

Because the α_i are ordered by means of the indices, the string 1011... (which is simply the concatenation of the entries in the row in (3.27)) can be interpreted as a binary number that encodes the state of s. We will call this number the *code* of s and write $[s] = 1011....$ The entries we have given here are made up, but if the code is to describe s, the i^{th} entry has to correspond to the actuality of property $\mathbf{a}_i$. Writing $[s]_i$ as a symbol for the value of the i^{th} entry of $[s]$,

$$[s]_i = \mathbf{1}_{\kappa_S(\mathbf{a}_i)}(s) \tag{3.28}$$

we can write this table as:

$$\begin{array}{ccccccc} & \alpha_1 & & \alpha_2 & & \alpha_3 & \ldots \\ [s] & 1_{\kappa_S(\mathbf{a}_1)}(s) & & 1_{\kappa_S(\mathbf{a}_2)}(s) & & 1_{\kappa_S(\mathbf{a}_3)}(s) & \ldots \end{array} \tag{3.29}$$

To recover the state of the system from its code $[s]$, we note that for the index set $I = \{i \in \mathbb{N} : [s]_i = 1\}$ we can use the meet and the function

κ (def. 23 on p.60):

$$s = \kappa_S(\bigwedge_{i\in I} \mathbf{a}_i) \tag{3.30}$$

as these $\mathbf{a}_i$ are all actual[17]. Note that an actual property in the list may be redundant in the sense that it is a Boolean function of other actual properties, but this is of no consequence. What is important, is that the list L_S always contains by definition *all* actual properties and hence Eq.(3.30) defines an atom of the lattice $\mathcal{L}_S^{\text{cl}}$: the state s of S. The translation of the above in case the system is an observer is initially straightforward. Let $L_M = \{\alpha_1, \alpha_2, \alpha_3, \ldots\}$ denote a complete list of questions that can define any particular state $m \in \Sigma_M$. We encode the state of the observer as a list of zeroes and ones depending on whether question α_i is true or not:

$$\begin{array}{cccccc} & & \alpha_1 & & \alpha_2 & & \alpha_3 & \ldots \\ [m] & & \mathbf{1}_{\kappa_M(\mathbf{a}_1)}(m) & & \mathbf{1}_{\kappa_M(\mathbf{a}_2)}(m) & & \mathbf{1}_{\kappa_M(\mathbf{a}_3)}(m) & \ldots \end{array} \tag{3.31}$$

We define the special index set $I = \{i \in \mathbb{N} : [m]_i = 1\}$ to generate the state of the observer by:

$$m = \kappa_M(\bigwedge_{i\in I} \mathbf{a}_i) \tag{3.32}$$

As before, we call this the *code* of m and write $[m]$. So far everything looks identical to what we did with a regular system, but in what follows we can see that there is more going on for an observer.

Suppose an observer M is given test α_1. Then, according to our physical account of observation, the first step is to prepare, that is to evolve to a state fit to test α_1, according to Eq.(3.9), on p.68. This state is obtained as $prep(m, \alpha_1)$ and we have denoted this state m_{α_1}. This state m_{α_1} can itself be encoded by writing out the value of every question in L_M to obtain $[m_{\alpha_1}]$. If we do the same for every question in Q_M, we get:

$$\begin{array}{cccccc} & \alpha_1 & & \alpha_2 & & \alpha_3 & \ldots \\ [m_{\alpha_1}] & \mathbf{1}_{\kappa_M(\mathbf{a}_1)}(m_{\alpha_1}) & & \mathbf{1}_{\kappa_M(\mathbf{a}_2)}(m_{\alpha_1}) & & \mathbf{1}_{\kappa_M(\mathbf{a}_3)}(m_{\alpha_1}) & \ldots \\ [m_{\alpha_2}] & \mathbf{1}_{\kappa_M(\mathbf{a}_1)}(m_{\alpha_2}) & & \mathbf{1}_{\kappa_M(\mathbf{a}_2)}(m_{\alpha_2}) & & \mathbf{1}_{\kappa_M(\mathbf{a}_3)}(m_{\alpha_2}) & \ldots \\ [m_{\alpha_3}] & \mathbf{1}_{\kappa_M(\mathbf{a}_1)}(m_{\alpha_3}) & & \mathbf{1}_{\kappa_M(\mathbf{a}_2)}(m_{\alpha_3}) & & \mathbf{1}_{\kappa_M(\mathbf{a}_3)}(m_{\alpha_3}) & \ldots \\ \vdots & \vdots & & \vdots & & \vdots & \ddots \end{array} \tag{3.33}$$

The truth values in the table given by the indicator function completely determine the state m_{α_i} as the collection of all its actual properties. For example, if the third entry for m_{α_2} in the table is 1, then

[17] The Cartan mapping is defined as $\kappa : \mathcal{L} \to \mathcal{P}(\Sigma)$ whereas the image of the mapping in Eq. (3.30) now is a state, i.e. a member of Σ. There is no inconsistency here, as $\Sigma \in \mathcal{P}(\Sigma)$ and the collection of actual properties defines an atom, i.e. a state.

$1_{\kappa_M(\mathbf{a}_3)}(m_{\alpha_2}) = 1$, hence $m_{\alpha_2} \in \kappa_M(\mathbf{a}_3)$. This then indicates property $\mathbf{a}_3$ is actual for the state m_{α_2}. But if the observer is $\mathbf{a}_i$-sensitive, then we have $[m]_i = o(m_{\alpha_i}, \alpha_i)$. If M is free, it is by definition $\mathbf{a}_i$-sensitive for every property in $\mathcal{L}_M^{\mathrm{cl}}$ and we get[18]:

	α_1	α_2	α_3	$\ldots$
$[m_{\alpha_1}]$	$o(m_{\alpha_1}, \alpha_1)$	0	1	$\ldots$
$[m_{\alpha_2}]$	1	$o(m_{\alpha_2}, \alpha_2)$	0	$\ldots$
$[m_{\alpha_3}]$	0	1	$o(m_{\alpha_3}, \alpha_3)$	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Note that $[m_{\alpha_i}]$ encodes for each i the state m_{α_i}, and *not* the initial state of M, which was m. However, remember m_{α_i} was a state specifically prepared to observe α_i for the state m. So if the observer is in the state m_{α_i} and it observes the question α_i, then that outcome *must* coincide with the actuality of that property[19], because the state m_{α_i} was by assumption $\mathbf{a}_i$-sensitive for m. So for each i, the outcome $o(m_{\alpha_i}, \alpha_i)$ must indicate whether property $\mathbf{a}_i$ is actual or not, for *both* m and m_{α_i}. This reasoning can formally be expressed as

$$o(m_{\alpha_i}, \alpha_i) = o(m, \alpha_i) \tag{3.34}$$

but we do not need to impose as a separate condition as it is already guaranteed by the construction of the function o as can be seen in Eqs. (3.16) on p. 70 where we have used (3.23). We obtain:

	α_1	α_2	α_3	$\ldots$
$[m_{\alpha_1}]$	$o(m, \alpha_1)$	0	1	$\ldots$
$[m_{\alpha_2}]$	1	$o(m, \alpha_2)$	0	$\ldots$
$[m_{\alpha_3}]$	0	1	$o(m, \alpha_3)$	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Since $o(m_{\alpha_i}, \alpha_i) = o(m, \alpha_i)$ has to hold for every i, the outcomes on the diagonal of the scheme form the code $[m]$ and hence completely determine the state m. Likewise, the codes $[m_\alpha]$ on the rows determine that state m_α, because —by definition (28)— the answers to the complete list of questions $L_M = \{\alpha_1, \alpha_2, \alpha_3, \ldots\}$ are sufficient to determine *any* state of M.

[18] As the values that are not on the diagonal only tell us something about the state m_{α_i} and not necessarily about m, we have replacede the non-diagonal entries by random number, which we will continue to do clarity.

[19] In other words: the preparation of the state m_{α_i} can alter the actuality of some properties of m, but not of property $\mathbf{a}_i$. The post-measurement state of the observer, what we have called m'_α in (3.13), may well have altered the actuality of $\mathbf{a}_i$. This is entirely possible but of no further consequence here.

We are now in a position to rephrase the proof in somewhat more detail. To do so we set up the table of codes for M in the state m for a complete list of questions $L_M = \{\alpha_1, \alpha_2, \alpha_3, \ldots\}$ (28), with the diagonal elements replaced using (3.24):

$$\begin{array}{cccc} & \alpha_1 & \alpha_2 & \alpha_3 & \ldots \\ [m_{\alpha_1}] & o(m_{\alpha_1},\alpha_1) & \mathbf{1}_{\kappa_M(\mathbf{a}_2)}(m_{\alpha_1}) & \mathbf{1}_{\kappa_M(\mathbf{a}_3)}(m_{\alpha_1}) & \ldots \\ [m_{\alpha_2}] & \mathbf{1}_{\kappa_M(\mathbf{a}_1)}(m_{\alpha_2}) & o(m_{\alpha_2},\alpha_2) & \mathbf{1}_{\kappa_M(\mathbf{a}_3)}(m_{\alpha_2}) & \ldots \\ [m_{\alpha_3}] & \mathbf{1}_{\kappa_M(\mathbf{a}_1)}(m_{\alpha_3}) & \mathbf{1}_{\kappa_M(\mathbf{a}_2)}(m_{\alpha_3}) & o(m_{\alpha_3},\alpha_3) & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

Consider now the question χ defined as $\sim \varphi$, with φ the question defined by $o(m_\alpha, \varphi) = o(m, \alpha)$, $\forall\ \alpha \in Q_M$. We have $\varphi \in Q_M$ because M is introspective by (3.26). Because the set of questions Q_M is closed under negation, we also have $\chi \in Q_M$. By definition of χ we have $o(m_{\alpha_i}, \chi) = o(m_{\alpha_i}, \sim \varphi)$ for every α_i and by (3.25) we have $o(m_{\alpha_i}, \chi) = 1 - o(m_{\alpha_i}, \alpha_i)$. Let us make the question χ explicit as a column in the table with the codes of m_{α_i} for every i given by its definition:

$$\begin{array}{cccccc} & \alpha_1 & \alpha_2 & \ldots & \chi \\ [m_{\alpha_1}] & o(m_{\alpha_1},\alpha_1) & \ldots & \ldots & 1 - o(m_{\alpha_1},\alpha_1) \\ [m_{\alpha_2}] & \ldots & o(m_{\alpha_2},\alpha_2) & \ldots & 1 - o(m_{\alpha_2},\alpha_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{array}$$

We use Eq. (3.34) to substitute the diagonal elements with elements that relate directly to m :

$$\begin{array}{cccccc} & \alpha_1 & \alpha_2 & \ldots & \chi \\ [m_{\alpha_1}] & o(m,\alpha_1) & \ldots & \ldots & 1 - o(m,\alpha_1) \\ [m_{\alpha_2}] & \ldots & o(m,\alpha_2) & \ldots & 1 - o(m,\alpha_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{array}$$

Since L_M is a complete list, there has to be a value j such that $\chi = \alpha_j$. This question α_j corresponds to a property $\mathbf{a}_j$. Because M is assumed free, it is $\mathbf{a}_j$-sensitive (definition (29) on p. 70) and can evolve to the state m_{α_j}. We get the following table:

$$\begin{array}{cccccc} & \alpha_1 & \ldots & \alpha_j & \ldots & \chi \\ [m_{\alpha_1}] & o(m,\alpha_1) & \ldots & 1 - o(m,\alpha_1) & \ldots & 1 - o(m,\alpha_1) \\ \vdots & \vdots & & \vdots & & \vdots \\ [m_{\alpha_j}] & \ldots & \ldots & o(m,\alpha_j) & \ldots & 1 - o(m,\chi) \\ \vdots & \vdots & & \vdots & \ddots & \vdots \end{array} \tag{3.35}$$

By $\mathbf{a}_j$-sensitivity the j^{th}code of $[m_{\alpha_i}]_j$ has to be equal to the outcome: $o(m, \alpha_j)$, but by definition of χ the outcome is $1 - o(m, \chi) = o(m, \sim$

$\chi) = o(m, \sim \alpha_j)$ and M is not consistent (definition (31) on p. 71). This finalizes the proof. What is this property $\mathbf{a}_j$ that corresponds to the question χ? It is formally the negation of the question whose existence was required for an observer to be introspective. We can however give it a more direct, semantical interpretation. The question χ is defined as $o(m_{\alpha_i}, \chi) = 1 - o(m_{\alpha_i}, \alpha_i)$ and hence in natural language denotes the property that distinguishes the state for which it is actual by a "reversal of the outcome to each question". The set of states that makes this property actual is not empty, as this is in fact the defining property of m_χ. When M is in the very state that changes outcomes with respect to the question asked, it cannot possibly be sensitive with respect to the property that defines it, as this particular property is just one more example of a question that can be asked. One may regard the state m_χ as the equivalent of a "liar-state", but the existence of m_χ is an inescapable consequence of the requirements we imposed on the observer. I believe this argument is essentially a "proof-of concept" proof and that more detailed formal descriptions of self-observers will include many more specific and more interesting unobservable properties of self-observers. The proof hints at where we should be on the lookout. In particular, m_χ was a state necessary in the observation protocol and it is plausible that, by the same token, the intermediary states necessary to observe a primitive question are prone to unobservability. We will see that under different premises somewhat different, but in essence similar issues of non-observable properties arise. In experimental science one speaks of "consilience of inductions" when unrelated pieces of evidence support the same hypothesis. In our case different premises and different argumentations can be regarded as a form of "consilience of deductions" that lend greater credibility to the claim no self-observer can observe all its actual properties.

Compound questions for the self-observer

We now turn to compound questions, by which we mean questions that cannot be tested in a single setup. A good example of a complex question, is a question that is a Boolean function of primitive[20] questions. Suppose M is in the state m that is free with respect to a set of properties $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ that correspond to a set of primitive questions $Q_A = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$[21]. We want to observe the compound question α that is an n-ary Boolean function g of the n primitive questions in

[20] Recall that a question is called primitive if it is testable in a single experimental setup.

[21] More precisely, we should write $Q_A = \{\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_n}\}$, but we choose to use the first n natural numbers as indices for notational simplicity; the actual questions involved of course need not be the first n questions in Q_M.

Q_A:

$$\begin{aligned} g &: \{0,1\}^n \to \{0,1\} \\ \alpha &= g(\alpha_1, \alpha_2, \ldots, \alpha_n) \end{aligned}$$

We can enlarge the set of questions to include such questions in a recursive way.

Definition 36 *Let $Q_S^{primitive}$ be a countable[22] set of primitive questions regarding the system S. Then Q_S is recursively defined by:*

1. $\alpha \in Q_S^{primitive}$ *then* $\alpha \in Q_S$
2. $\alpha, \beta \in Q_S$ *then* $\alpha \cdot \beta \in Q_S$
3. $\alpha \in Q_S$ *then* $\sim \alpha \in Q_S$
4. $\alpha \in Q_S$ *then* $(\alpha) \in Q_S$

In this way Boolean expressions of primitive questions such as $\sim (\alpha \cdot \beta)$ are in Q_S by application of rules 2, 4 and 3 respectively. It is well-known the Boolean operators NOT and AND (corresponding in Q_S to $\sim$ and $\cdot$ respectively) are sufficient to generate all Boolean functions, so every possible g will correspond to a specific question $\alpha \in Q_S$. Is it possible and if so under what conditions, can M establish the truth value of α? We will assume here there is no problem in observing the separate tests (as M is assumed free for $\mathcal{A}$), nor in calculating the function g (as we are comfortable to grant the observer the power to perform Boolean functions on data it has already observed). The fundamental problem that cannot be evaded is what the final outcome still has to say about the (pre-measurement) state m.

To illustrate the problem, take the most simple example of a compound question: the product of two questions. (We will assume in this section that there does not exist a question that can be observed in a single setup that is equivalent to the product of the constituent questions.) First of all, as explained in the definition of the meet 11, we require the result of observing $\alpha \cdot \beta$ for the observer in state m to correspond to the actuality of the meet of the corresponding properties $\mathbf{a}$ and $\mathbf{b}$ for the state m :

$$o(m, \alpha \cdot \beta) = \mathbf{1}_{\kappa_M(\mathbf{a} \wedge \mathbf{b})}(m)$$

[22] A countable set has the same cardinality as some (not necessarily proper) subset of the set of natural numbers.

As $\kappa_M(\mathbf{a} \wedge \mathbf{b})$ is the set of states for which both $\mathbf{a}$ and $\mathbf{b}$ are actual, the outcome is 1 iff m is in both $\kappa_M(\mathbf{a})$ and $\kappa_M(\mathbf{b})$. Hence we have

$$\begin{aligned} o(m, \alpha \cdot \beta) &= \mathbf{1}_{\kappa_M(\mathbf{a}) \cap \kappa_M(\mathbf{b})}(m) \\ &= \mathbf{1}_{\kappa_M(\mathbf{a})}(m) \cdot \mathbf{1}_{\kappa_M(\mathbf{b})}(m) \end{aligned}$$

For an observer that is $\mathbf{a}$-sensitive, we have $o(m, \alpha) = \mathbf{1}_{\kappa_M(\mathbf{a})}(m)$. If the same goes for $\mathbf{b}$, we can formally write:

$$o(m, \alpha \cdot \beta) = o(m, \alpha) \cdot o(m, \beta) \tag{3.36}$$

Algebraically speaking, we couldn't have wished for a more transparent formula, but operationally it is anything but straightforward. The problem is not that the formula doesn't hold (it does), but that the observer has to choose which question to observe first and this will change its state. Indeed, observing the answer to any question in $\mathcal{A}$ *must* change the state m, otherwise the observer cannot possibly make any kind of inference after the observation that it could not already have made prior to the observation, making the act of observation superfluous. The epistemic gain of knowledge due to observation must, by physicalism, have a counterpart in the ontological state of the observer. The most economic situation an observer can hope for is one in which only *one single* property of M has changed as a result of observing another property. But if the property that has changed happens to correspond to the next question in line to be observed, the final result will no longer have anything to say about the pre-measurement state m. As a result, no observer can determine its state at any specific instance by following a strategy of observing property after property. Indeed, after the first observation at least one property has changed and the question that corresponds to this property has to be included in the list of questions that determine its state.

Self-observation and the knowledge balance principle

In this section we restrict ourselves to finite observers, that is, observers for which the set of available states to an arbitrary but fixed finite number. For simplicity, we assume the specific function g is simply the product of the arguments. Can we then assume M is in principle able to observe the outcome of properties that are the meet of arbitrary properties in the set $\mathcal{A}$ by performing the corresponding test of each property and forming the product? We explained in Chapter 2 that the meet of properties is operationally well-defined if each of the properties in the meet is well-defined. Let us illustrate why, in general, product

properties belonging to the observing system cannot be observed using the operational procedure of Chapter 2 by picking a simple example.

Assume M is a system completely determined by just 4 classical properties, $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ and $\mathbf{a}_4$, and as such has 2^4 possible states. If the product of interest is $\alpha_1 \cdot \alpha_2$ and the observation of α_1 changes the actuality of $\mathbf{a}_3$, and the observation of α_2 changes the actuality of $\mathbf{a}_4$[23], it has obtained all that is necessary to form the product. But we cannot measure a third property without altering at least one of the previously obtained values, effectively erasing that observation. It is easy to see this generalizes to the following proposition (using $\#(A)$ to denote cardinality of A).

Proposition 37 (Maximal Self Knowledge) *A free, classical self-observer M with $\#(\mathcal{L}_M^{cl}) = 2n$ can observe at most n properties in $\mathcal{L}_M^{cl}$ without changing either the actuality of observed properties, or the result of the observations of those properties.*

The proposition provides an upper bound which can be reached only if the following three conditions are met: (i) the ability of M to change only a single property for each observation it performs, (ii) to leave this altered property unaltered by the subsequent observations, and (iii) to be able to perform Boolean functions on the values of earlier observations that are stored. In essence maximal self knowledge is a trade-off between what is considered unknown and hence candidate for observation on the one hand and the ability to 'store' the result of the observation, i.e. properties now considered as 'known' post observation. Note that the reason why the claim holds is essentially because the observer is self observing. If M were to be observing a system exterior to itself, storing the outcomes of previous observations in its own state would not have direct consequences for its ability to gain more information about the system it is observing. Maximal self knowledge does not a priori rule out that an observer is able to perform the test corresponding to a product question directly in a single measurement setup as long as this does not require obtaining more values for the constituent questions in the product, than bound by maximal self knowledge. The maximum number of properties M can hope to observe is still n and among those n there may or may not be product or other, even more complex, properties.

Maximal self knowledge reminds us of Spekkens' toy model for quantum mechanics where a very similar idea is called the *knowledge*

[23] M cannot change the actuality of $\mathbf{a}_3$ a second time, because then the postmeasurement state would be identical to the state we would have had when α_2 would have been observed directly on m. Using the actuality of $\mathbf{a}_3$ a second time has destroyed any possible gain from the observation of the first property.

balance principle. Spekkens himself formulates the principle in [186] as follows:

> **Knowledge Balance Principle:** "If one has maximal knowledge, then for every system, at every time, the amount of knowledge one possesses about the ontic state of the system at that time must equal the amount of knowledge one lacks."

Effectively the knowledge balance principle says that, if $2n$ questions characterize a system completely, maximally n can be known. One can therefore regard classical self-observers as an example of a system that automatically obeys the knowledge balance principle. Spekkens' model shows many impressive qualitative similarities with quantum theory and it would be interesting to investigate how his toy theory can be reinterpreted in terms of self-observers[24].

On related results

Although to the best of my knowledge, no formal arguments of the kind given above exists in the literature, they are related to a few other results and we will discuss the ones that we feel are most closely related. We will treat Breuer's result because the wording of his result is closest to our result. Next we will discuss similarities and differences between our proof and the results of Gödel, Tarski and other closely related theorems. A group of researchers, among whom we find Karl Popper, Maria Luisa Dalla Chiara, Martin Zwick, Asher Perez, Karl Svozil and others have drawn attention to the similarity between undecidability and the quantum measurement problem and we will argue that from the perspective of our theorem, this is not coincidental.

[24] As discussed by Spekkens, one of the implications of the knowledge balance principle, is that the state of a system that obeys the principle needs to be fixed by an even number of questions of which only half can be known. This in turn means the number of ontic states of an elementary system, defined by two questions, is at least 4. If there are 2n questions, the number of states is 2^{2n}; a rather artifical constraint on the state space. If one allows for the possibility that the observation of a property changes that very same property, this (hard to physically justify) constraint on the allowable size of the state space is no longer necessary when we deal with self-observers. The reason is that if the act of self observation has changed the actuality of the property under observation, the outcome may pertain to the past but not to the present, or vice versa. We cannot know which is the case and we are always left with exactly as many knowns as unknowns.

Breuer's argument

In a series of very interesting papers [61], [60], Thomas Breuer examines the limits of an observer that is contained in the system that it is seeking to observe. Breuer's main *result* is very similar to ours, so it is worthwhile to describe it in some detail[25].

Let the states of S (object) and M (measurement apparatus) refer to the same time instant after the measurement. Let θ be an inference map and with $\sigma_M \in \mathcal{P}(\Sigma_M)$ and $\sigma_S \in \mathcal{P}(\Sigma_S)$ we have:

$$\theta : \mathcal{P}(\Sigma_M) \to \mathcal{P}(\Sigma_S), \quad \sigma_M \longmapsto \theta(\sigma_M) = \sigma_S. \tag{3.37}$$

That is, the map θ allows us to infer that if the apparatus is in one of the states $m \in \sigma_M$, then the system has to be in a state $s \in \sigma_S$. We should not be able to infer states that are not possible, so we have $\theta(\Sigma_M) \subset \Sigma_S$. Obviously, we have

$$\theta(\sigma_M) = \cup_{m \in \sigma_M} \theta(\{m\}). \tag{3.38}$$

A state s_0 is called *exactly measurable* in an experiment with inference map θ if, after the measurement, there exists a set $\sigma_M \in \mathcal{P}(\Sigma_M)$ of apparatus states referring uniquely to the state s or, symbolically: $\theta(\sigma_M) = \{s_0\}$. An experiment with inference map θ is said to be able to *distinguish* the states s_1 and s_2 iff there is one set of final apparatus states referring to s_1 but not to s_2, and another set of final apparatus states referring to s_2 but not to s_1. Note that, if a state is exactly measurable, it can be distinguished from any other possible final state. However, for two states to be distinguishable, it is not necessary that they are exactly measurable. If, however, all possible final states are distinguishable, then they are all exactly measurable.

The central argument of Breuer then goes as follows. We consider the situation where the apparatus M is contained in the system S it is seeking to observe. In other words: the observed system includes the apparatus. This is formulated by the assumption of *proper inclusion*:

$$\exists s, s' \in \Sigma_S : s \neq s' \text{ and } s|_M = s'|_M. \tag{3.39}$$

The notation $s|_M$ denotes the set of states of the apparatus that one obtains if one restricts the state of the observed system to the apparatus subsystem. For finite state systems, this assumption means that there are strictly more states for S than there are for M, and this already makes the exact determination of all the possible states of $S + M$

[25] We stay very close here to Breuer's original presentation. Because it seems harmless and allows for a more continuous reading, we have only changed the symbols for both the states and the set of states of both the system and the observer to match the ones you have become accustomed to over the last few sections.

impossible for the subsystem M. For infinite state spaces, Breuer suggests to tackle the problem on the basis of a consistency condition that he calls the *meshing condition* and that we can formulate as:

$$\forall m \in \Sigma_M : \{s|_M : s \in \theta(\{m\})\} = \{m\}. \tag{3.40}$$

This condition makes sense: for every apparatus state $m \in \Sigma_M$, the restriction of system states $\theta(\{m\})$ to which the inference map refers should be that same apparatus state, lest the reference is contradictory. These few tools suffice to prove an interesting proposition.

Proposition 38 *Not all states of a system can be measured exactly by an internal observer. Symbolically:* $\exists s_0 \in \Sigma_S, \forall \sigma_M \in \mathcal{P}(\Sigma_M) : \theta(\sigma_M) \neq \{s_0\}$.

Proof. (Reductio ad absurdum) Assume $\forall s \in \Sigma_S, \exists \sigma_M \in \mathcal{P}(\Sigma_M) : \theta(\sigma_M) = \{s\}$. From the assumption of proper inclusion, we have $\exists s, s' \in \Sigma_S : s \neq s'$ and $s|_M = s'|_M$. By assumption that the states can be measured exactly, we have: $\exists \sigma_M, \sigma'_M$ such that $\theta(\sigma_M) = \{s\}$ and $\theta(\sigma'_M) = \{s'\}$. Since $\cup_{m\in\sigma_M}\theta(\{m\}) = \theta(\sigma_M) = \{s\}$, there exist a m in σ_M such that $\theta(\{m\}) = \{s\}$. Likewise,there is $m' \in \sigma'_M$ such that $\theta(\{m'\}) = \{s'\}$. But then we have: $\{s\} = \theta(\{m\}) = \theta(\{\theta(\{m\})|_M\}) = \theta(\{s|_M\}) = \theta(\{s'|_M\}) = \theta(\{\theta(\{m'\})|_M\}) = \theta(\{m'\}) = \{s'\}$. But by assumption, we had $s \neq s'$.■

Breuer's reasoning aims to show the existence of distinct states that are indistinguishable for an observer that is contained in the system that it seeks to observe, a result that is extremely close in content to ours. The argument's strong point is its simplicity and the fact that it does not rely on any specific kind of dynamics for the interaction between the system and the apparatus. In particular it does not depend on whether the evolution of the system is deterministic or statistical, although it is not easy to conceive how *statistical* evolution would help an observer to obtain *exact* knowledge. A less attractive feature of the proof is that the assumption of proper inclusion (3.39) is very strong. Although physically certainly reasonable, it already foretells us that there will be distinct states that appear the same when one only considers the state of the apparatus as a proper subsystem. The assumption of proper inclusion can only be made plausible when the observer is studying a system of which it is a part. But what if the observer is looking at it self? The set of states that it has at its disposal is then exactly the same as the set of states that it can be in, and so one of the basic the premises of the proof, the assumption of proper inclusion, is no longer warranted.

Gödel and Tarski.

Breuer writes in [61] that it is safe to say the only thing his result and Gödel's have in common is the aspect of self-reference. Our result uses an explicit diagonal argument and is hence closer to Gödel[26] [109]. It is perhaps still closer to the work of Raymond Smullyan [182], [183] that seeks to simplify and at same time generalize Gödel's results. There is however an important difference. As Gödel's argument is a mathematical precise statement about mathematics itself, it is logically speaking very hard to dispute it: one would have to deny parts that are essential to mathematics itself. We are here dealing with observations, with the inevitable consequence that these observations are about things in the world . We are comparing reality and our formal description of it in one theory, and we do so when we impose that every system has a state property system. One can always escape the theorem by denying observers can be described by a state property system or that one of the premises (consistent, free or introspective) of the theorem is not granted in reality. Our description of the observer is not a physical theory of how systems evolve per sé; it should rather be viewed upon as a logically more primitive theory, containing the necessary basic notions for a realistic description of an observing system. So while the theorem of this kind can never justify itself as firmly as Gödel's can, we do claim it imposes a restriction on the representation of knowledge gained from observation that an observer can have about itself. Although our result relies on the basics of the Geneva-Brussels approach, I believe it is essentially reproducible in any realistic account of self-observation as a physical process. The underlying reason that is essential to the derivation of the result is that the observer is not only the system under study but also the one that —through its observations— makes claims about this system[27]. This particular form of self-reference is related to Tarski's famous result on the undefinability of truth of a sufficiently rich system from within that system [195]. To see this, we only have to realize the requirement of being *introspective* can be semantically reformulated to resemble a *truth* predicate. The question $\varphi \in Q_M$ such that for every $\alpha \in Q_M$ we have $o(m_\alpha, \varphi) = o(m, \alpha)$ can be rephrased as: "Proof you are able to observe for which question your state is primed by giving the correct answer to that very question." The structure resembles that of a truth predicate T and in much the same way the truth of a proposition p has the same truth value as the proposition

[26] Neither Gödel nor Tarski explicitly used the word diagonal argument in their work. It was Carnap [67] who recognized the diagonal argument in both proofs.

[27] This is in essence the same reason why computers cannot (universally) tell when a program they are running will halt, why a system of axioms cannot prove its own completeness and consistency, and why it does not make sense for a book to declare —as part of its contents— its never erroneous origin.

itself ($T(p)$ is true iff p is true), $o(m_\alpha, \varphi)$ will be true if α is.

Self-reference and the quantum measurement problem

> *The observer, when he seems to himself to be observing a stone, is really, if physics is to be believed, observing the effects of the stone upon himself.*
>
> —Bertrand Russell, *An Inquiry Into Meaning and Truth*, 1950

Of the many philosophical issues surrounding the interpretation of quantum theory, none is by many considered quite as important and controversial as the so-called quantum measurement problem. To give an overview of the problem would require a dissertation in itself, and it would be hard to do a better job than one of the classics on the topic by Busch, Lahti and Mittelstaedt [64]. As we believe our result has some bearing on the measurement problem, we will very briefly state the nature of the problem. It has been known since the early days of quantum theory and at least since von Neumann's classic [204] that by the linearity of the dynamic laws, the superposition of two (or more) states of an observed quantum system is transferred to a superposition of the pointer observable of the measurement apparatus. If quantum theory is the whole and complete story, then how do definite pointer positions arise? This is roughly what is called the objectification problem in [64], and much of the book is devoted to describe the work that has been done by various authors to weaken one of the assumptions of standard quantum mechanics in an attempt to solve the problem. Many different approaches pass the revue such the Many-Worlds interpretation [94], the Modal interpretation [86], [199], decoherence based approaches [214], [212], unsharp objectification [63], [179], modified dynamical laws [108], [58], hidden variables [56], [57], and so on. None of these proposals has solved the measurement problem to everybody's satisfaction. Many these attempts share the feature that to a greater or lesser extent, they alter standard (non relativistic) quantum theory, especially with respect to the projection postulate or at least the interpretation or status of this postulate. However, as the authors acknowledge ([64], p. 137), there is a completely different way of looking at the problem:

> ...from a methodological point of view the measuring process does not belong to the domain of quantum mechanics but rather

> serves to constitute the semantics of this theory. It is the requirement of the semantical completeness of quantum mechanics which stipulates that the very (measuring) processes providing operational definitions of the concepts of the theory must be describable in terms of the theory. This semantical completeness ... induces a logical situation similar to that encountered with Gödel's theorem. To avoid inconsistencies within a universally valid quantum mechanics, it is argued, the theory cannot be applied to yield a complete description of a measurement situation. Rather one has to accept that part of the process of measurement remains unanalyzed; in other words, according to this point of view one has to distinguish between two levels of description: the endophysical (ontic) and the exophysical (epistemic) level, analogously to the distinction between object language and metalanguage in logic.

The authors refer to the works of Dalla Chiara [79], Breuer [61] and Perez [158]. The idea of endophysical and exophysical perspectives and their relationship to the quantum measurement problem can also be found in Asher Perez [158]:

> Even if quantum theory is universal, it is not closed. A distinction must be made between endophysical systems—those which are described by the theory—and exophysical ones, which lie outside the domain of the theory (for example, the telescopes and photographic plates used by astronomers for verifying the laws of celestial mechanics). While quantum theory can in principle describe anything, a quantum description cannot include everything. In every physical situation something must remain unanalyzed. This is not a flaw of quantum theory, but a logical necessity in a theory which is self-referential and describes its own means of verification. This situation reminds of Gödel's undecidability theorem: the consistency of a system of axioms cannot be verified because there are mathematical statements that can neither be proved nor disproved by the formal rules of the theory; but they may nonetheless be verified by metamathematical reasoning.

Although it goes nowhere as far as this, we already glimpse a tendency to look in this direction in John von Neumann's classic book of 1932. We quote from the last chapter of [204], p. 352:

> Let us now compare these circumstances with those which actually exist in nature or its observation. First, it is inherently entirely correct that the measurement or the related process of

> the subjective perception is a new entity relative to the physical environment and is not reducible to the latter. Indeed, subjective perception leads us into the intellectual inner life of the individual, which is extra-observational by its very nature (since it must be taken for granted by any conceivable observation or experiment).

Von Neumann then explains at length that the boundary between the system and the observer is to a large extent arbitrary, to conclude that this:

> ...does not change the fact that in each method of description the boundary must be put somewhere, if the method is not to proceed vacuously, i.e., if a comparison with experiment is to be possible. Indeed experience only makes statements of this type: an observer has made a certain (subjective) observation; and never any like this: a physical quantity has a certain value.

Comparing Bohr's position that ultimately all measurements are to be described by classical physics to von Neumann's position, shows von Neumann is heading for an even more radical solution to the measurement problem. According to von Neumann "perception...is extra-observational by its very nature since it must be taken for granted by any conceivable observation or experiment". We leave it to the reader to judge whether von Neumann's remarks have any bearing on the relation between semantical completeness and the measurement problem.

In 1951 Karl Popper [165] attempts to show that Gödel's theorem implies a fundamental form of indeterminism in classical physics. More specifically, Popper gives several informal arguments (using a *predictor*, which is a physical instantiation of Laplace's demon as a supercomputer furnished with initial data) to show no observer can know its own state before this state has already passed. The first two papers that —to the best of my knowledge— are entirely devoted to the point of view that the measurement problem may have more to do with self-reference than with quantum theory *an sich*, are: "Logical self-reference, set theoretical paradoxes and the measurement problem in quantum mechanics" from 1977 by Maria Dalla Chiara [79] and "Quantum measurement and Gödel's proof" of 1978 by Martin Zwick [215]. Dalla Chiara does not dispute the universal applicability of quantum theory per sé, but argues that "...any apparatus which realizes the reduction of the wave function is necessarily only a meta-theoretical object " [79] (p. 340) Together these papers form a strong argument: whereas Dalla Chiara's paper is very general, formal and written in the language of logic (although she does point out some of the more philosophical consequences), Zwick's paper is more physical and philosophical in nature (although there are

some initial attempts at formalization). Zwick is particularly careful to point out it is not after the argument that physics rests on mathematics and since mathematics is incomplete, so is our physical description of reality, which he considers true but trivial. He is more careful with the argument that an incompleteness result could be given *within* some proper axiomatization of quantum theory, although this is not Zwick's goal. Such a result however seems to be what Busch, Lahti and Mittelstaedt had in mind when they end their book with the concluding remark:

> On the other hand, no Gödel-type propositions were formulated in quantum mechanics up to now. In our opinion these ideas deserve to be taken seriously; but they also require further elaboration towards rigorous formalization before their far-reaching implications can be properly estimated.

Although it may be worthwhile to investigate the possibility of Gödel-type propositions within a given axiomatization of quantum theory, we want to stress this is not our point here. As has been pointed out frequently, for all its paradoxes, quantum mechanics is never at odds with experiment and never internally inconsistent. The paradoxes always use some form of counter-factual reasoning or counter factual outcome assignments. An explicit example of this kind was given by Kochen and Specker in [139], in which they identify a set of observables that lead to an inconsistency if value assignments of all observables in this set are assumed to exist[28]. A similar example of counter-factual attribution of values that lead to paradoxes can be found in the Mermin-Peres square [157]. Quantum theory escapes the possibility of an inconsistency because it will not predict what outcome will be obtained. Quantum theory tells us: If *you* happen to have measured a definite outcome, the projection postulate was somehow activated; I will only tell how often on average you will have projected onto a particular subspace. To us, the more interesting question is whether the mathematical structure of quantum theory is (partly) the result of the attempt to produce a physical theory that is universally valid and that avoids inconsistencies. The projection postulate seen from this perspective is a solution to a problem, rather than a problem in itself[29], as it is to much to ask of *any* theory to be complete with respect to deterministic predictions for every possible interaction and

[28] Ernst Specker may have entertained a relation between quantum theory and undecidability, judging from his paper "The logic of non-simultaneously decidable propositions" [185], which makes one suspect it deals with the mathematics of undecidability rather than quantum physics.

[29] Although popular wisdom has it that *in every small problem there is a bigger one struggling to get out.*

be consistent at the same time. This speculative idea seems to find additional support in several papers that indicate how quantum theory allows to "circumvent" some of the classic undecidable problems. The first paper by Karl Svozil assigns a halting amplitude to determine the halting probability of an algorithm [191]. A second result by Diederik Aerts *et al.* [21], describes the evolution of the truth values in the liar paradox by means of a unitary evolution in four dimensional Hilbert space. Lastly, we point to a paper by Paterek, Koer, Prevedel, Klimek, Aspelmeyer, Zeilinger and Brukner where they write [155][154]:

> ..states of elementary quantum systems are capable of encoding mathematical axioms and show that quantum measurements are capable of revealing whether a given proposition is decidable or not within the axiomatic system. Whenever a mathematical proposition is undecidable within the axioms encoded in the state, the measurement associated with the proposition gives random outcomes. Our results support the view that quantum randomness is irreducible and a manifestation of mathematical undecidability.

The paper raised some questions because the title of the paper —*Mathematical undecidability and quantum randomness*— led some people to believe the authors claim that truly undecidable statements can be solved or decided by quantum theory. Two years later a second and revised version of the paper by Paterek *et al.* is somewhat more careful and states

> We demonstrate that quantum systems in the eigenstates of Pauli group operators are capable of encoding mathematical axioms and show that Pauli group quantum measurements are capable of revealing whether or not a given proposition is logically dependent on the axiomatic system.

The notion of undecidability as a stronger form of logical independence is —as the authors point out— related to Chaitin's conception of undecidability in [70], where he writes "Undecidability arises whenever a proposition and a given set of axioms together contain more information than the axioms themselves." This information based notion of undecidability leads Chaitin to suggest that "the incompleteness phenomenon discovered by Gödel is natural and widespread rather than pathological and unusual" [70]. With the caveat pointed out above we subscribe to that point of view. Note however, that the problem we identified with the classical self-observer cannot be rephrased as a problem of mere logical (in)dependence; we cannot add a state or property to the state-property system of the observer such that the self-observer becomes adequate.

Self-reference and human observation

It is the unfortunate fate of many undecidability arguments that their relevance is either exaggerated or marginalized. I hold the relevance of the existing undecidability results in many branches of mathematics to be self-evident. I hope to have explained why I think our result holds philosophical significance with respect to the interpretation of quantum theory. The question we want to address now is how important our argument is with respect to actual observation. Let us first ponder this question in relation to automated or robotic observers. Turing's famous result [80] on the impossibility of determining whether a given program will halt or not on a given input, was the first undecidability result for computers. Anyone who has waited long enough for a computer to finish a job only to pull the plug eventually, knows the problem is more than academic. In 1987, Fred Cohen showed in [73] that no virus detecting software can detect with certainty every virus and not sometimes mistake parts of the operating systems to be a virus. Cohen proves his result by showing it is equivalent to Turing's halting problem; if the latter is undecidable, the former has to be too. A mere three years later Lane Davis and Steven Drake created the first so-called *rootkit*, a type of virus that merges so deeply with fundamental parts of the operating system, that the current expert advice for removal is to reformat the computer's hard drive and completely reinstall all software including the operating system[30]. This is one of the many examples that show undecidable questions are quite often not without practical interest[31]. I believe that simultaneous with the advent of autonomous robots, we will come to appreciate the multitude of interesting questions that involve self-reference in one way or another. These results are in the framework of theoretical informatics; if the conditions are met, the result will hold. We are dealing with a formalization of a function of biological systems, which are vastly more

[30] One wonders whether at some point in the future it would make sense to have a biological equivalent to this theorem, saying that no living organism can have an auto-immune system that eliminates every possible threat and yet is free of autoimmune disorders that mistakingly 'protect' against the organism self.

[31] Another famous example, is the result of Fischer, Lynch and Paterson (commonly known as FLP, [96]) that, surprisingly, a set of processes in an asynchronous distributed system cannot with certainty agree on a binary value if a single processes is unreliable. The proof directly implied the impossibility of a number of related problems, including *consensus*, which is defined as agreement among a number of processes about the numerical value of data. Consensus is necessary to achieve a reliable output in a distributed computer setup and as such the result had an enormous impact in distributed computing, both theoretical and practical.

complex. If however the model of observation we propose is sufficiently general to encompass human observation, we ought to be able to understand the theorem from our everyday experience as observers, at least in some limiting sense. We believe it is not hard to give examples to show this is indeed the case. Suppose we are given a collection of objects and are asked to determine which color each of them has from a list of existing colors. Suppose furthermore that we have a collection of observers that we ask to judge the color of each object. If a great majority of observers judge all the items to be red, we 'objectify' the outcome of the observation and say all members in the collection *are* red, without reference to a particular observer. An observer is then deemed objective (in the language developed here, we would say *red-sensitive*) when it too, observes all these items to be red. When there is no general accord among observers as to what color a particular item has, we cannot objectify the color of that item and we cannot say what it means to be a red-sensitive observer. Let us now turn to the perspective of a single observation made by a single observer, which is closer to the content of the theorem. If the observer attributes the color red to an item, it must have at least one state that attributes the color red to that item. But how is the observer to know whether the object *is* indeed red, or whether it is simply in a state that attributes red to the item. Such states of the observer do exist, for example, after having been exposed to a very bright cyan colored light, or after being punched on the nose so that everything looks red. To know the difference doesn't seem so problematic in this instance. Surely an observer would know if it was exposed to a bright cyan light and even more so when it was punched on the nose. It then realizes it is perhaps not sensitive to that property. But to be able to tell whether a given state of the observer is indeed sensitive to a particular property, requires exactly what we have defined on p.72 as "Introspective". The corollary to the main theorem on p.145 shows that there exist different states that cannot be distinguished by a consistent and introspective observer. Every observer has at least one state (which is sufficient for the argument) that is potentially liable to such an ambiguity, and that is the state that attributes the color red to an object. It follows it is for the observer in principle not possible to tell with certainty whether the observed property is indeed an attribute of the outside world or of itself[32].

In general we do not consider this problem to be undecidable because we believe that in the majority of cases we observe the object to be red or not depending only on whether the object *is* red or not. However, if we are exposed to the object for only a fraction of a second,

[32]In Chapter 5 we will treat this dilemma using likelihood estimators to come to the definition of optimal observer.

or the observation occurred under very dim light conditions, we should (and hopefully would) be more prudent with respect to this belief. In dealing with observations that are made by other observers, especially if the result of the observation seems implausible to us, we take into account that what was claimed to have been observed may well have been the result of the observer simply *believing* it saw what it saw. In the court of law, a lawyer can sometimes succeed to discredit the testimony of a witness by pointing out that some of the circumstances under which the observation happened were unfavorable to observation. Many more examples can be found in psychology, often in relation to what is called projection. According to psychoanalysis projection is particularly present in the neurotic or psychotic person. It is however well-known to be a very common phenomenon. A kind (or rude) person often judges others around her to be kind (or rude), effectively "measuring other people's corn by one's own bushel". The point here is not so much the observation of being surrounded by nice (or rude) people in itself, but rather the impossibility for such a person to really answer the question whether she herself is kind (or rude), or the people encountered that day. Very often the objectification problem for the observer is not whether it is possible to observe an sich, but rather whether it is possible to know for sure that what is observed is indeed an attribute of the outside world or of oneself[33].

We have given several examples to show the problem is indeed present in everyday observation. Yet it cannot be denied that in most practical instances we feel justified in attributing reality to the content of our observations. To explain this apparent tension, we need to appreciate that classical observation consists of an inference that is always based on many more elementary observations. Our eyes sample at least 25 images per second. It is well-known that a very large part of the information that we believe we have seen or heard is in fact not directly observed, but inferred information. Often multiple senses are involved and their signals combined to add plausibility to the observation. All of this suggests that probability may be a tool that can assist us in dealing with this problem. Classical error theory is an example of such a probabilistic tool and we can witness the remains of the undecidability in the minimal resolution of the last digit in the readout of an measurement apparatus. An experimentalist will admit that we do not know whether the value of this last digit is due to the system under

[33] We have illustrated this in the introduction with the example of Galileo using his own heartbeat to measure the timing of the pendulum. As a matter of principle, uncertainty in self-observation can spill over into uncertainty about general observation as every observation is ultimately a form of self-observation; we have to believe our eyes before we can make inferences about the outside world based on what we see.

investigation or to inherent fluctuations in the process of observation. We analyze this situation in detail in the next chapter. We emphasize that undecidability does not amount to impossibility per se. It does call for an additional principle to limit its influence on the content of the observation. We will take a lead from E.T. Jaynes [129]: "From our viewpoint 'undecidability' merely signifies that a problem is one that calls for inference rather than deduction." Jaynes' remark is particularly apt for our situation and we will turn to probabilistic inference in the remaining chapters.

Chapter 4

Potential properties, probability and state space

And shall we leave Gorgias and Tisias undisturbed, who saw that probabilities are more to be esteemed than truths?

—Socrates as quoted by Plato in *Phaedrus*, 370 BC

Uncertainty in measurement

Lord Kelvin once wrote "When you can measure what you are speaking about, and express it in numbers, you know something about it"[1]. That saying was later popularized to: "To measure is to know", but measurement comes with uncertainty and hence our knowledge about what we have measured too. Let us pick a simple example. What do we really mean when we say that a rod has a length of 1 m? We could be *defining* the meter, but more likely we are comparing the size of the rod to a previously defined meter. No two objects are perfectly alike, hence the object in question is at least slightly larger or smaller than the one we compare it with. But what is slightly? This is open to a certain degree arbitrariness, but most people would agree that saying a rod is 1 meter, means at least the rod has to be smaller than 2 m and larger than 0 m. In fact, most would agree it's length should

[1] *Electrical Units of Measurement*, Vol. 1, 1883-05-03.

be between 0,5 m and 1,5 m. This is the idea of *significant digits*: when you express a quantity in numbers, we take it as a convention in scientific notation that —in absence of error margins— the numbers you give are the ones you are confident about. In this sense "having an extension of 1 m" becomes a classical property if we translate it as: "The extension is between 0 m and 2 m." It may be a lot closer to 1 m than this statement expresses; the question is how to turn this into a precise statement. We pick a ruler with finer resolution, say 1 mm, and measure the rod by fixing the zero of the ruler to one end, keeping it close to the rod and read the number on the other end of the ruler that seems to be nearer to the end of the rod than any other number. That is the operational procedure. We do our best. Say we judge that number to be somewhere between 1,013 m and 1,014 m, perhaps slightly closer to 1,013 m. In the judgement of magnitudes like a meter we are more confident than in this case. We may have misaligned the zero of the ruler, we may have introduced parallax errors, the ruler or the rod may have moved after we fixed the zero (but before we read the second number), and so on. To verify our measurement, we repeat it or we let someone else repeat it. We might find the outcome is indeed very close to 1,013 m and we decide that number represents the length of the rod. We can officially proclaim that we have established a classical property of the rod: we claim its length at room temperature is between 1,012 and 1,014 *with certainty*. The uncertainty because of the finite precision of the measurement doesn't really bother us. What matters is that we have a claim that someone else can test. That is what makes observation scientific. Parts that are designed and produced in different parts of the world need to fit: technology needs a high degree of certainty. It is this kind of certainty that promotes a piece of data to stardom; it represents something real.

Suppose then that our second measurement gives us a result that is slightly closer to 1,014 m, or perhaps even 1,016 m. When situations like this occur, and this is very often the case in experimental science, it is common practice to take recourse to error theory. Besides obvious mistakes, there are two important classes of measurement errors: *systematic* errors and *random* errors. A systematic error occurs in our example when our ruler is not very well made: it may consistently give a slightly higher or lower reading. This type of error can be hard to catch, but it has the advantage that, if we find out the ugly truth about our ruler, we can accommodate for this fact and adjust all our experimental data, even without having to repeat our experiments. We will not discuss this type of error. The second type of measurement error, the random error, is more interesting to us. The strategy to fight this error is to repeat the measurement many times under the assumption that each measurement makes an independent random error so that

adding them up cancels them, at least to a certain extent. Besides the independence and randomness, there seem to be two assumptions that are central to this idea. The first assumption is that the true value of the quantity we are measuring, is *a function of the probability of the outcomes* in independent trials. In other words: if the outcome 1,013 m is much more likely than 1,014 m, the result will be close to 1,013 m. The second assumption is that this probability *can be approximated by the relative frequency.* These two assumptions show how this situation can only be treated by a combination of a theoretical model and multiple measurements.

In the language we have developed in the previous chapters, we would say the test that claims the length is 1,013 m does not yield "yes" with certainty and the property is not actual but potential[2]. What we need, then, is an operational definition of a potential property. We start with the assumption that the test yields "yes" with some fixed probability p. It is conceivable that some fluctuating phenomena in nature may fail to converge to a fixed relative frequency; it is however difficult to conceive how such a situation should be treated at all. Luckily a very large and interesting class of phenomena does enjoy the property that the relative frequency converges. Let us naively pursue the analogy with the actual property and tentatively propose:

> (Tentative version 1) If the number of tests α (that corresponds to property **a** of the system S in the state s) that yield "yes" , divided by the total number of performed tests, comes ever closer to a fixed number $p \in]0,1[$ when we increase the number of measurements, then the property **a** will be called potential with probability p for S in the state s.

This tentative version expresses that the relative frequency should converge to p for many repetitions of the test, but how many tests do we need to perform? We encounter a well-known problem in the philosophy of probability. If the sequence of "yes" and "no" answers is truly stochastic, then isn't it possible to choose a starting point of the yes-no sequence of answers and cut it off at a certain point to get pretty

[2]However, a property can only have two outcomes for each trial and length measurements can have many. For a classical property, this is of no consequence, because if one of the lengths (within some interval) is actual, all other lengths (outside that interval) are not. However, for potential properties we have to be more carefull. If the length measurements only give us 1,013 and 1,014, then we proceed by calling one of the two a "yes" outcome and the other a "no" outcome. If more outcomes are possible, we have to divide them into two groups and this is possible in many ways. We will later treat measurements with more outcomes.

close to any p in the unit interval[3]? Clearly for some phenomena a few measurements are sufficient, whereas other phenomena require delicate and numerous measurements. Evidently, the number of measurements in itself doesn't really tell us what we need. What we need is an operational measure of uncertainty. The uncertainty the observer is facing, is not one about the possible outcomes if the mechanism of the outcome production is known in the form of a probability distribution. it wants to establish whether a property is potential or not, and this is only possible, if it has established for itself a procedure to estimate the value of p from the observations. In other words, this is a problem of inverse probability rather than direct probability. Such questions are less apparent in the definition of an actual property because we simply demand that it yields "yes" with certainty. But then again, what is certainty? How many measurements reveal certainty? Coincidentally, the problem has some resemblance to the old problem of how to define the continuity of a function. It is instructive to investigate this parallel. Take for example the following passage written by Cauchy around 1821 in his famous *Cours d'analyse* [69]:

> This granted, the function $f(x)$ will be, between the two limits assigned to the variable x, a continuous function of the variable if, for each value of x intermediate between these two limits, the numerical value of the difference $f(x + a) - f(x)$ decreases indefinitely with that of a.

The definition captures the essential idea, but the reference to "decreases indefinitely" is objectionable. How small is small enough to be sure? The mathematical solution to this problem is both remarkable and simple and has become one of the cornerstones of modern analysis and topology. In 1817, Bolzano writes: *...the difference* $f(x+h)-f(x)$ *can be made smaller than any given quantity, if* h *is taken sufficiently small.* The modern definition, also known as the epsilon-delta definition, is to be found in the writings of Weierstrass[4]. We quote from Kline [138], who writes:

> To remove the vagueness in the phrase "becomes and remains less than any given quantity" which Bolzano and Cauchy

[3]We are not talking about selecting outcomes, which would amount to plain cheating.

[4]The basis for the idea can be traced back at least to Leibniz who writes as early as 1687 that when the differences between two cases can be made smaller than any datum in the given, the differences can be made smaller than any given quantity in the result. It is generally recognized much of the credit should go to Bolzano. As early as 1830 he explained the difference between continuity and uniform continuity: a function is uniform continuous in its domain if one can choose a value of δ that depends only on ε and not on the point in the domain. See [174].

> used in their definitions of continuity and limit of a function, Weierstrass gave the now accepted definition that $f(x)$ is continuous at $x = x_0$ if given any positive number ε, there exists a δ such that for all x in the interval $|x-x_0| < \delta$, $|f(x)-f(x_0)| < \varepsilon$.

The definition is considered a rigorous mathematical definition, but it also has the flavor of an operational definition[5]. To show a function is continuous in x_0, one has to take the prescription of the function $f(x)$ to construct an inequality that relates ε to δ to show that —however small you would like ε to be— I can give you an neighborhood of x_0 that is guaranteed to give a difference in function values $|f(x) - f(x_0)|$ smaller than that ε. Rather than solving the problem how small the difference should be, the definition somewhat provocatively asks us: *how small do you want it to be*? Related to the problem of defining continuity (and in appearance perhaps a bit closer to our problem) is the definition of the limit of a sequence. In Bolzano's *Rein analytischer beweis* of 1817, we find [152]:

> If a sequence of quantities $F_1, F_2, F_3, ..., F_n$ has the property that the difference between its n-th term F_n and every later term F_{n+r}, however far from the former, remains smaller than any given quantity if n has been taken large enough, then there is always a certain constant quantity, and indeed only one, which the terms of the sequence approach, and to which they can come as close as desired if the sequence is continued far enough.

Bolzano does not show the sequence has a limit, he merely states this. In spite of this weakness, the definition of Bolzano contains the essential elements that we find in most modern definition, which all resemble the following:

> A real number x is the limit of the sequence (x_n) if for each $\varepsilon > 0$, there exists a natural number N such that, for every $n \geq N$, we have $|x_n - x| < \varepsilon$.

The similarity with our problem is that, ideally, we would like the relative frequency n/N to converge to the probability p, i.e., we want $p - n/N$ to "decrease indefinitely", or "remain smaller than any given quantity". Suppose we have a sequence of tests of a potential property:

$$0, 1, 1, 0, 1, 0, 0, 0, 1, 0, 1, \ldots$$

[5]It is a great merrit of mathematics that the two can actually coincide. In fact, there are those who believe the only quantities we should be dealing with in mathematics, are those that we know how to construct.

From this sequence, we derive a second sequence which counts the ones and divides by the total number of tests so far:

$$0,\ 1/2,\ 2/3,\ 2/4,\ 3/5,\ 3/6,\ 3/7,\ 3/8,\ 4/9,\ 5/10,\ 6/11, \ldots$$

Let us call this second, derived sequence, the *counting sequence*. It would be nice to be able to write something like:

> (Tentative version 2) A property **a** is potential with probability $p \in]0,1[$ iff the number p is the limit of the counting sequence (c_n), of the sequence of tests α of **a**, i.e. if for each $\varepsilon > 0$, there exists a natural number N such that, for every test after $n \geq N$, we have $|c_n - p| < \varepsilon$.

The improvement with respect to upon Tentative version 1 on p. 99, is that we no longer use the vague phrase "comes ever closer". However, an essential quality of probability is the unpredictability of outcomes and this characteristic renders this proposal rather naive. We do not know if every test after a given test N will keep the difference between c_n and p smaller than a given ε. But the other essential quality of probability is the tendency of relative frequencies to converge. We need a more flexible notion of convergence that takes the fluctuations in the relative frequency into account; a measure of uncertainty and a way of expressing how that the uncertainty becomes smaller with increasing trials.

Probability and the law of large numbers

> *La théorie des probabilités n'est, au fond, que le bon sens réduit au calcul.*
>
> —Laplace, *Théorie Analytique des Probabilités*, 1814

> *Common sense is merely a stupid absence of imagination and mental flexibility.*
>
> —H. P. Lovecraft, *The Unnamable*, 1925

The historical roots of the theory of probability can be traced back to the fifteenth and sixteenth century when a few Italian writers, most notably Paciola (1494), Cardano (1545) and Tartaglia (1556), discussed the problem of how to divide the stake of a gambling game when the

game is ended prematurely. The problem was revived in 1654 when Chevalier de Méré wrote to his friend Blaise Pascal: "In eight throws of a die a player is to attempt to throw a one, but after three unsuccessful trials the game is interrupted. How should he be indemnified?" As a result of the attempt to answer this question, Pascal and Fermat exchanged a series of letters which are now regarded as the starting point for the modern theory of probability. Of course, we cannot know how the game ends -that is why it is a game of chance-, so a fair division of stakes is only possible when the players agree upon the rationale they see behind the patterns of the game. The players have to agree in what fraction of the total number of possible games player 1 would win, and in what fraction of those games player 2 will win. The fact that Fermat and Pascal separately arrived at the exact same conclusions, strengthened their belief in the correctness of the reasoning and results they obtained. This is an experience that many students who encounter probability for the first time share: that the laws are in some sense self-evident and logical to the point of being natural even though the results can be counter intuitive. In this sense the calculus of probabilities —and in particular when we should multiply and when we should add probabilities— is almost trivial. In fact, R.T. Cox has shown how to derive those laws from a few simple and very reasonable assumptions [76] on how one should treat a quantity that expresses a measure of probability. (Bayesians call these: reasoning with measures of belief, Cox itself called these the axioms of probable inference). These are the echoes of Laplace's famous dictum that "Probability is nothing but common sense reduced to calculation". Maybe this was the reason that neither Fermat nor Pascal published their results. However, their correspondence seems to have prompted Christiaan Huygens[6] to publish "*De ratiociniis in ludo aleae*" (On Reasoning in Games of Dice, [124]) in 1657[7] in which Huygens duly pays tribute to both great men but complains they never explained how to obtain their results. He then sets out for itself to solve the problems and explains in detail the gambling principles that are required to obtain the results of Fermat and Pascal and related problems. The first substantial volume in the theory of probability is however Jacob Bernoulli's "*Ars conjectandi*" [51] (or, "Art of Conjecturing") published in 1713, eight years after the death of its author. The early roots of probability sprang forth from

[6]The Dutch translation of this work, "*Van Rekeningh in Spelen van Geluck*" was to appear three years later.

[7]In those days, probability was besides a good subject for endless discussion, also an experimental science. Respectable mathematicians were rolling dice hundreds of times to verify their proposals and calculations, in much the same way as physicists use random numbers generated by computers to check the validity of their statistical models using the Monte Carlo algorithm.

the conviction that, although some natural phenomena (e.g. dice) may escape prediction, their behavior follows a pattern. Cardano stated without proof that the accuracies of empirical statistics tend to improve with the number of trials. This was later formalized and is now known as the law of large numbers. The first law of large numbers for a random variable that can only take two values was given and proved by Jacob Bernoulli. As Bernoulli itself writes in *Ars Conjectandi*, it took it over 20 years to develop a proof and he branded it his "Golden Theorem"[8]. Bernoulli proof was rather complicated when looked upon from a modern point of view. Still, this particular theorem stands as one of the most important landmarks in the history of probability. Let us take a look at the probabilistic process that Bernoulli was studying, when he found the law of large numbers. To introduce the necessary concepts efficiently, we will make use of a more modern language than was available at the time.

Bernoulli trials of potential properties

> *Certainty of things, considered with respect to us, is not the same for all things, but varies diversely and occurs now greater, now lesser.*
>
> —Jacob Bernoulli, *Ars*, 1713

A Bernoulli *trial* is an experiment with a random outcome taken from only two possible outcomes: "Success" and "Failure", "1" and "0", "yes" or "no", or, as Bernoulli itself calls them, *fertile* and *sterile*. A Bernoulli *process* consists of a sequence of independent identically distributed Bernoulli trials. Here is a definition: consider a (finite or infinite) sequence of independent random variables X_1, X_2, X_3,... such that:

1. For each i, the value of X_i is either 0 or 1.
2. For all values of i, the probability that $X_i = 1$ is the same number p. [9]

[8] It is now generally known as "Bernoulli's Theorem".

[9] We eventually want to talk about the relation between limits of measured quantities versus abstract, theoretical constructs. Hence, when we state condition number 2, we talk about probability as a mathematical quantity that we know when and how to mutiply and add. The places where this occurs will be made explicit in the text.

Let us apply this to a sequence of tests of a potential property. If a trial of the property **a** yields an affirmation, we will write `1`, otherwise, we write `0`. A possible Bernoulli process consisting of a sequence of 11 trials of the test α corresponding to the potential property **a**, could look like:

$$\texttt{0, 1, 1, 0, 1, 0, 0, 0, 1, 0, 1.}$$

The number of successes is the number of `1`'s in this sequence. Suppose that a particular process consists of N tests of which precisely k times the result `1` was obtained. To calculate the probability of that happening, we perform a two step process. First we rewrite the sequence but put all the fertile outcomes up front:

$$\texttt{1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0.}$$

What is the probability of this (pretty rare) event occurring? This is simply an application of the product rule for independent random variables: the probability of obtaining twice head in two throws of a coin, is the product of the probability of throwing a head with each throw separately[10]. Repeated application of this rule then tells us the probability of occurrence of this particular sequence, is $p^k(1-p)^{N-k}$. But every rearrangement of this sequence that is produced by switching an arbitrary number of `1`'s and `0`'s in this sequence will yield the same number of both, and hence will also contribute to the occurrence of having k out of N times the result `1`. There are overall $N!$ permutations of the symbols, but none of the $k!$ permutations of the `1`'s, nor the $(n-k)!$ permutation of the `0`'s, will change the sequence, so it is easy to see there are precisely $N!/k!(N-k)!$ such rearrangements. We now apply the sum rule of probability: the probability that either sequence A happens *or* sequence B happens, is the sum of the probability that each of them occur. Hence we have to multiply the *number* of possible arrangements with the probability of each of them occurring. In this way we obtain by sheer combinatorial counting and using the two most basic principles of probability that the probability $P(k)$ of precisely k successes in N trials in (and under the assumptions given above of) a Bernoulli process equals:

$$P(k) = \binom{N}{k} p^k (1-p)^{N-k}. \tag{4.1}$$

This probability distribution is for rather obvious reasons known as the binomial distribution. To obtain the average and variance for the binomial distribution, we need the definition of an expectation value

[10] In symbolic notation we have, for independent X_i : $p(X_i = x_i, X_j = x_j) = p(X_i = x_i).p(X_j = x_j)$.

of a discrete random variable X. Let X be a discrete random variable taking real values $x_1, x_2, ...$ with respective probabilities $p_1, p_2, ...$ (i.e. $p(x_i) = p_i$), then the expectation value E[11] of this random variable is defined as (i runs over the values necessary to include all the x_i):

$$E[X] \equiv \sum_i x_i p_i. \tag{4.2}$$

With this last definition the variance is defined as:

$$V[X] \equiv E[(X - E[X])^2]. \tag{4.3}$$

A well-known and not difficult to prove property of the expectation value, is its linearity: $E[\Sigma_i c_i X_i] = \Sigma_i c_i E[X_i]$. These hold for arbitrary X_i. For the variance, we have $V[cX] = c^2 V[X]$. If the X_i are also independent[12], one can show that the individual variances are additive:

$$V[X] = \sum_i V[X_i]. \tag{4.4}$$

This last property illustrates how useful the definition of variance (4.3) is. That is about all we need. In our case, we have only two outcomes and hence only two probabilities: p and $1 - p$. The random variable X is defined as the number of successes in N trials. We then have for the average number of successes μ in N trials:

$$\begin{aligned} \mu &\equiv E[X] = E[\Sigma_i X_i] \\ &= \Sigma_i E[X_i] = Np. \end{aligned} \tag{4.5}$$

This fact was already known by Cardano, who makes explicit use of it (See [153], p. 152-154). To calculate the variance $V[X]$ for the random variable X which denotes the number of successes in N trials, we evaluate (4.3) using (4.2), where we substitute the probability distribution (4.1):

$$E[(X - E[X])^2] = \sum_{k=0}^{N} (k - Np)^2 \binom{N}{k} p^k (1-p)^{N-k}. \tag{4.6}$$

[11] The idea of expectation goes back to Huygens; that it is denoted by a capital E goes back to the beginning of the twentieth century when it was introduced by W.A. Whitworth. It is no mystery why the symbol took hold, as it is the first letter "Expectation", "Erwartungswert" and "Espérance mathématique"!

[12] Two random variables X and Y are independent if receiving information about one of the two does not change our assessment of the probability distribution of the other. Formally this means the probability of occurrence of one random variable X taking a value x, and the other random variable Y the value y, is simply the product of the probability of X taking value x and Y taking value y.

To avoid doing this summation for all possible values of k in a Bernoulli process, many text books prefer calculating the variance using the moment generating function of the binomial distribution. This is indeed a very concise way of obtaining the result, but has the disadvantage of obscuring what goes into the derivation. We can however perform a clever trick to circumvent the summation entirely, or rather reduce it to a trivial sum. First calculate the variance for a single X_i :

$$\begin{aligned} \sigma_i^2 &= E[(X_i - E[X_i])^2] \\ &= (1-p)^2 p + (0-p)^2(1-p) \\ &= p(1-p)(1-p+p) = p(1-p). \end{aligned} \tag{4.7}$$

We have N contributions of independent random variables to the total variance: $V[X] = \sum_i^N V[X_i]$, and for each of these $V[X_i] = p(1-p)$, (by assumption (2) of the Bernoulli process the p's are equal). Hence we obtain:

$$V[X] = \sum_i^N p(1-p) = Np(1-p). \tag{4.8}$$

In the derivations so far, X was the number of successes in N trials. We now turn to the *fraction of successes* (which will of course be our estimate for p) as a new random variable $\bar{X} = X/N$. It's expectation value is given by

$$E[\bar{X}] = E[X/N] = Np/N = p.$$

This is of course, not very surprising. What we really are after is the *variance* of the *fraction of successes,* which is given by:

$$\sigma^2[\bar{X}] = \sigma^2[X/N] = \frac{1}{N^2} Np(1-p) = \frac{p(1-p)}{N}. \tag{4.9}$$

Note that this function has the characteristics of what one expects of a measure of uncertainty: it becomes very small if either of the following three is satisfied: (a) p is close to 1, (b) p is close to 0 and (c), the number of measurements increases. Maximal uncertainty, on the other hand, occurs when (d) p is (close to) 1/2 and (e) $N = 1$. It is obvious that all these criteria (a)—(e) are desirable properties of a measure of uncertainty. It is also rather obvious σ^2 is algebraically the most simple function which satisfies criteria (a)—(e). But there are more reasons to suggest it is a good measure.

Convergence of Bernoulli processes

We have calculated the expected value of the fraction of success and the expected standard deviation of this fraction. As the latter is proportional to a constant ($\sqrt{p(1-p)}$) and inversely proportional to the square root of the number of measurements, it clearly becomes very small if N is large. In fact the largest value σ can attain for different values of p, is when the nominator is maximal, which occurs when $p = 1/2$, in which case $\sigma = \frac{1}{2\sqrt{N}}$. To understand what this means for our estimates, we turn to a truly classic result: the Chebyshev inequality[13].

Theorem 39 *For any discrete random variable X with mean μ and standard deviation σ, and for any positive number c, we have*

$$P(|X - \mu| > c) < (\frac{\sigma}{c})^2. \tag{4.10}$$

It is instructive to measure the bound on the deviation $|X - \mu|$ in "standard deviation units", i.e. we set $c = k\sigma$. We then obtain $P(|X - \mu| > k\sigma) < 1/k^2$. We then see the inequality for $k \leq 1$ tells us a probability should be a number smaller than 1, which isn't particularly enlightening. However, for larger values of k, Eq.(4.10) does tell us something. In words: the probability that a random variable will differ more than k standard deviations from its mean value is never greater than the square of the reciprocal of k. The true strength of the Chebyshev inequality is not that it provides a tight bound for any particular distribution (it usually does not), but rather that it is extremely general in that we only require of the random variable that its mean and standard deviation *exist* (see however the caveat in the Comments and Notes section, p. 117). In fact, it is the best bound possible considering all possible random variables with given mean and standard deviation. It is not difficult to obtain the (weak) law of large numbers from this theorem. Suppose we have an infinite sequence of identical, independent random variables X_i , each one with expected value $E[X_i] = p$. The focus of our interest is the convergence behavior of the sample average $\bar{X}_N = \sum_i^N X_i/N$. Obviously, we have $E[\bar{X}_N] = p$. As before, we have $V[X_i] = \sigma^2$ for every i. Since the X_i are independent by assumption, we have

$$\begin{aligned} V[\bar{X}_N] &= V[\sum_i X_i/N] = \frac{1}{N^2} V[\sum_i X_i] \\ &= \frac{1}{N^2} \sum_i V[X_i] = \frac{1}{N^2} N\sigma^2 = \frac{\sigma^2}{N}. \end{aligned}$$

[13] See, for example, Kolmogorov's *Foundations of the Theory of Probability*, pp. 42-43.

Substituting these two results in the Chebyshev inequality (4.10), we get:

$$P(|\bar{X}_N - p| \geq \varepsilon) \leq \frac{\sigma^2}{N\varepsilon^2}. \tag{4.11}$$

This is the law of large numbers. To put it into a more familiar form, we note that $P(|\bar{X}_N - p| < \varepsilon) = 1 - P(|\bar{X}_N - p| \geq \varepsilon) = 1 - \sigma^2/N\varepsilon^2$. We see the right hand side goes to 1 as N goes to infinity. This result is hence often summarized as:

$$\lim_{N\to\infty} P(\bar{X}_N \to p) = 1$$

We see that the chance of being off-limit with an estimate for p as the relative frequency, diminishes with increasing N. This is favorable, but the Chebyshev inequality is a limiting theorem for large N. For a finite ensemble, our estimate for the standard deviation depends on our estimate for the probability, so we are still not in a position to claim we have succeeded in our definition of potential property.

Maximum predictive power

In this section we will make use of a rather informal but interesting proposal by Summhammer [187], designed by it for another purpose, which may be of help in our search for a suitable definition of a potential property. Assume that we have performed a dichotomic experiment N times, where the result "1" was obtained n_1 times, and the result "0" was obtained $n_2 = N - n_1$ times. The relative frequency of outcome 1 is then n_1/N, and this also delivers our best estimate for the probability and associated uncertainty interval:

$$p = \frac{n_1}{N}; \qquad \Delta p = \sqrt{\frac{p(1-p)}{N}}. \tag{4.12}$$

From this estimate we want to derive the magnitude of a physical quantity χ by means of an unspecified relation:

$$\chi = \chi(p). \tag{4.13}$$

Its associated uncertainty interval is then given by the propagation of errors formula:

$$\Delta\chi(p) = |\frac{\partial\chi(p)}{\partial p}|\Delta p = |\frac{\partial\chi(p)}{\partial p}|\sqrt{\frac{p(1-p)}{N}} \tag{4.14}$$

We want to concentrate on the observation of those random variables $\chi(p)$ for which additional data always decreases the uncertainty. Indicating that our estimate for p depends on N, we require that

$$\Delta\chi(p_{N+1}) < \Delta\chi(p_N). \tag{4.15}$$

Many functions will satisfy this requirement, and to further constrain the set of possible χ's, Summhammer introduces the notion of *maximum predictive power*.

Definition 40 *(Maximum predictive power) A random variable χ allows for maximum predictive power, if the uncertainty $\Delta\chi$ depends only on the number of trials N, and not on the value of n_1.*

Summhammer calls this maximum predictive power because it allows to know in advance how many times the experiment needs to be repeated to reach a certain prescribed uncertainty level. Remark that N is decided by the experimenter, while n_1 is the number of clicks nature has returned, so maximally predictive random variables are those for which the experimentalist is able to eliminate nature's influence on $\Delta\chi$. So we set, with C some constant:

$$\Delta\chi(p) = \frac{C}{\sqrt{N}}. \tag{4.16}$$

From this we get

$$\sqrt{N}\Delta\chi = |\frac{\partial\chi(p)}{\partial p}|\sqrt{p(1-p)} = C, \tag{4.17}$$

the solution of which is given by:

$$\chi(p) = C\arcsin(2p-1) + D. \tag{4.18}$$

Its inverse is given by

$$p(\chi) = \frac{1}{2}(1+\sin\left((\chi-D)/C\right))) = cos^2(\pi/4-(\chi-D)/2C). \tag{4.19}$$

As remarked in [187], any smooth function $\alpha(\chi)$ of our random variable χ for which equal intervals along χ correspond to equal intervals along α, will also be a solution. To give one particular solution, choose $C = 1$ and D equal to $-\pi/2$, to recover

$$p = \cos^2(\chi/2). \tag{4.20}$$

Let us come back to our initial problem: How many tests do we need to perform for it to be meaningful to say a test yields "yes" with

probability p? The answer we give here is that, for any confidence interval $\Delta\chi$ you require for your estimate of χ, we can tell in advance how many measurements you need, regardless of the state (or p). And this can be done, if we insist that the relation between the random variable and the probability is of the nature of Eq.(4.20). The essential requirement to obtain this result is that the probability is the same fixed number in $[0, 1]$ for each separate trial, which is satisfied for a Bernoulli process. We are now in a position to give our operational definition of potential property, as first published in [42]:

Definition 41 (potential property) *Given a system S and a property $\mathbf{a} \in \mathcal{L}_S$ being tested by a Bernoulli process of tests. If, for any prescribed uncertainty interval Δp, one can say in advance how many measurements are necessary such that test α yields "yes" with probability p within that uncertainty interval Δp, then property $\mathbf{a}$ is called potential with probability p for S.*

What makes our definition of potential property operational is that for any desired uncertainty level the theorist requires for the acceptance or refutation of a conjectured potential property, the experimenter knows in advance how many measurements he needs to come to a conclusion. Certainty still follows only after an infinite number of measurements, but for any allowable uncertainty in the estimate of the degree of potentiality, we know how often we have to question nature for it to be a reliable judge on our trial. Besides the operational character of the concept of a potential property, we get an unexpected clue as to what type of mathematical representation of state space should be used to incorporate potential properties.

Potential properties and state space representation

From the work of Birkhoff and von Neumann [52] we know that the properties of a quantum system correspond precisely to the closed one dimensional subspaces. Because the state of a quantum system is also a property, it is also represented by a closed one dimensional subspace. It follows the representation of a property in Hilbert space quantum mechanics as a closed one-dimensional subspace automatically fulfills our definition of a (potential) property because the probability of the property holding is essentially of the form of Eq.(4.20). To illustrate this, assume we have a system S in a state that we represent by the

unit ket $|\psi\rangle$ in an n-dimensional Hilbert space $\mathcal{H}_n$. Let ϕ be the one dimensional subspace that represents the property to be tested and let $|\phi\rangle$ be a unit ket along the ray ϕ. The probability of finding the property to hold for the system in the state $|\psi\rangle$ is given by the square modulus of the inner product $|\langle\phi|\psi\rangle|^2$ where $\langle\phi|$ is the Hermitian adjoint of $|\phi\rangle$. With α the Hilbert space angle between ψ and ϕ, defined through the inner product $|\langle\phi|\psi\rangle| = \cos(\alpha)|\langle\phi|\phi\rangle||\langle\psi|\psi\rangle|$ and noting both ϕ and ψ are unit norm, we see that indeed $|\langle\phi|\psi\rangle|^2 = \cos^2(\alpha)$[14]. On the other hand it means that for the statistical estimation of observable quantities in quantum theory (at least without additional classical errors), we always have that more data decreases one's uncertainty, as required in Eq.(4.15) and that for any desired uncertainty level, we can say in advance how many measurements are required. Summhammer's proposal implies measurements for unit vector states are in a statistical sense optimal. For our purposes, we reverse that idea: our operational definition of potential properties shows they can be represented by unit vectors in Hilbert space. As the state of a system is a (maximal) property of the system[15], it follows the state space should be the unit sphere. Interestingly, there was a more elaborate argument in the literature which started from quite different considerations, but also mathematically hinges essentially on Bernoulli trials, which hinted at a similar result. We will treat this proposal in the next section.

Optimally distinguishing probability distributions

> *There is definite mathematical connection between the ubiquitous statistical fluctuations in the outcomes of measurements and the geometry of the set of states.*
>
> —William Wootters, *Statistical distance and Hilbert space*, 1981

We start again with an experiment with two possible outcomes: 0 and 1. Depending on the state of the system, the experiment can be characterized by the probability p of the occurrence of outcome 1. Suppose we have prepared four ensembles, E_1, E_2, E_3 and E_4, of systems such

[14] An alternative derivation starts from the trace rule $\Pr_\rho(A = a) = Tr(\rho R)$, where ρ is the state operator and R the projection operator that projects onto the eigenvectors a_k of (the non-degenerate operator) A. Assuming the state vector represents a pure state, $\rho = |\varphi\rangle\langle\varphi|$, we immediately obtain $Tr(\rho R) = |\langle a_k|\varphi\rangle|^2$. Once again, with the definition of α given above, we obtain the result.

[15] We introduced the state as the maximal property of a system on p. 42.

that the first ensemble E_1, is characterized by a value $p_1 = 1$, E_2 with $p_2 = 0.9$, E_3 with $p_3 = 0.55$ and E_4 with $p_4 = 0.45$. By making measurements on members of the ensembles, we are to estimate which of the four values of p pertain to that ensemble. Clearly, it is easier to distinguish (in a finite set of trials for the experiment) E_1 from E_2, than it is to distinguish E_3 from E_4, even though Δp is 0.1 in both cases. This idea can be made qualitative, as Wootters has done. Suppose that we are to distinguish two ensembles, one characterized by p, the other by $p + \Delta p$. For N trials of an experiment with two possible outcomes, the sequence of outcomes constitute again a Bernoulli process with associated standard deviations

$$\begin{aligned} \sigma_1 &= \sqrt{\frac{p(1-p)}{N}}, \\ \sigma_2 &= \sqrt{\frac{(p+\Delta p)(1-p-\Delta p)}{N}}. \end{aligned} \tag{4.21}$$

We will say that two such states are *distinguishable in N trials of the experiment*, iff the difference Δp is greater than the sum of these two standard deviations:

$$\Delta p > \sigma_1 + \sigma_2. \tag{4.22}$$

Define the *statistical distance* between the two states as the reciprocal of the square root of the number N_0 of measurements necessary to make distinguishable, so we get

$$d(p, p + \Delta p) = \frac{1}{\sqrt{N_0}}, \tag{4.23}$$

such that Δp equals the sum of these two standard deviations: $\Delta p = \sigma_1 + \sigma_2$. To the first order in Δp, using Eq.(4.21), this is equal to

$$d(p, p + \Delta p) = \frac{|\Delta p|}{2\sqrt{p(1-p)}}. \tag{4.24}$$

For two arbitrary biases, Δp may be large, and we will denote the respective probabilities as p_1 and p_2. The definition of statistical distance, as given by R.A. Fisher who used this definition to study genetic drift, then is:

$$d(p_1, p_2) = \lim_{N\to\infty} \frac{n}{\sqrt{N}}, \tag{4.25}$$

where n is the maximum number of intermediate probabilities that can still be distinguished in N trials according to Eq.(4.22). Wootters

shows this to be

$$\begin{aligned} d(p_1, p_2) &= \frac{1}{2}\int_{p_1}^{p_2} \frac{dp}{\sqrt{p(1-p)}} \\ &= \arccos(\sqrt{p_1 p_2} + \sqrt{(1-p_1)(1-p_2)}) \end{aligned} \tag{4.26}$$

The result extends to the case for an experiment with k mutually exclusive outcomes $x_1, x_2, ..., x_k$ with respective probabilities $p_1(x_i)$ and $p_2(x_i)$. This is useful because —as Wootters points out— we can then speak about the distance between two differently loaded dice, or the distance between two different preparations of a quantum state. The details of the calculation can be found in ([209], [210]), but we will give a brief exposition to show the gist of the calculation. We start with the observation that the probabilities $p_1(x_i)$ and $p_2(x_i)$ for each outcome x_i can be treated as components of vectors. The probabilities are assumed to be normalized, hence the components of the vectors sum to one and lie in the standard $(k-1)$-simplex in Euclidean k-space. The probabilities can be written as vectors π_1 and π_2 in the following way:

$$\pi_j = (p_j(x_1), p_j(x_2), \ldots, p_j(x_k)), \qquad j = 1, 2. \tag{4.27}$$

We now draw a continuous, but otherwise arbitrary, curve between π_1 and π_2 that lies entirely in the simplex.

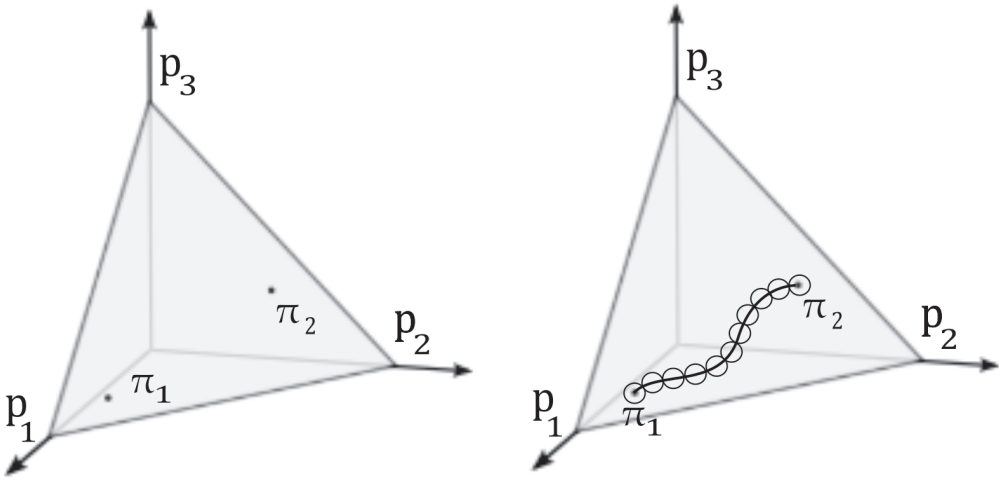

Figure 4-1 The shaded triangle represents the space of allowed probability triples and contains π_1 and π_2. To define the statistical length of a curve we count the number of points along C that are distinguishable in N trials, divide by $\sqrt{N}$ and take the limit for $N \to \infty$.

The statistical length of that curve is *defined* as the maximal number of mutually distinguishable points (in N trials) on that curve, divided by $\sqrt{N}$ in the limit for $N \to \infty$ as shown in Fig. 4-1. The statistical distance is then defined as the statistical length of the shortest (in statistical sense) possible curve. Wootters needs to redefine distinguishable for experiments with k outcomes and he does so by approximating the multinomial distribution with a Gaussian, which is entirely permissible when N is large. As before, two points π_1 and π_2 will be called distinguishable in N trials, if their regions of uncertainty do not overlap. For large N, this is the case if

$$\frac{\sqrt{N}}{2}\sqrt{\sum_{i=1}^{k}\frac{(\Delta p(x_i))^2}{p(x_i)}} > 1. \tag{4.28}$$

Armed with this definition of distinguishability, we can find an explicit expression for the distance by writing the curve that connects the two vectors as a parameter equation $\pi(t) = (p_1(t), p_2(t), \ldots, p_k(t))$, with $\pi(0) = \pi_1$ and $\pi(1) = \pi_2$, where we have abbreviated the probability of outcome number i, as $p_i(t)$. The statistical length is then given by

$$l = \frac{1}{2}\int_0^1 \sqrt{\sum_{i=1}^{k}\frac{1}{p_i(t)}(\frac{dp_i(t)}{dt})^2}dt. \tag{4.29}$$

In his Ph.D. dissertation, Wootters [209] calculates the distance by variational calculus where Lagrange multipliers ensure the variations remain in the simplex. As he points out in his 1981 article, there is an easier road to the same end. If we map (4.27) to the unit sphere using

$$\boldsymbol{\psi}(t) = (\sqrt{p_1(t)}, \sqrt{p_2(t)}, \ldots, \sqrt{p_k(t)}), \tag{4.30}$$

then, abbreviating $\sqrt{p_i(t)}$ as $\psi_i(t)$, Eq. (4.29) reduces to:

$$l = \int_0^1 \sqrt{\sum_{i=1}^{k}(\frac{d\psi_i(t)}{dt})^2}dt$$

which is nothing but the arclength of the curve in the space of the $\boldsymbol{\psi}'s$. The requirement that π_1 and π_2 lie in the simplex translates to the requirement that the $\boldsymbol{\psi}'s$ must lie on the unit sphere:

$$\sum_{i=1}^{k}\psi_i^2(t) = 1.$$

The shortest path between two points on the unit sphere is an arc lying on a great circle. This situation is depicted in Fig. 4-2. If the

sphere has unit radius then the length of this path equals the angle measured in radians. Hence we have:

$$d(\pi_1, \pi_2) = \arccos \sum_{i=1}^{k} \sqrt{\pi_1(x_i)\pi_2(x_i)}. \tag{4.31}$$

Using the bra-ket notation once more, we can rewrite the result of Eq.(4.31) as :

$$\begin{aligned} d(\pi_1, \pi_2) &= \arccos |\langle \boldsymbol{\psi}_1 | \boldsymbol{\psi}_2 \rangle| \\ &= \measuredangle(\boldsymbol{\psi}_1, \boldsymbol{\psi}_2) \end{aligned} \tag{4.32}$$

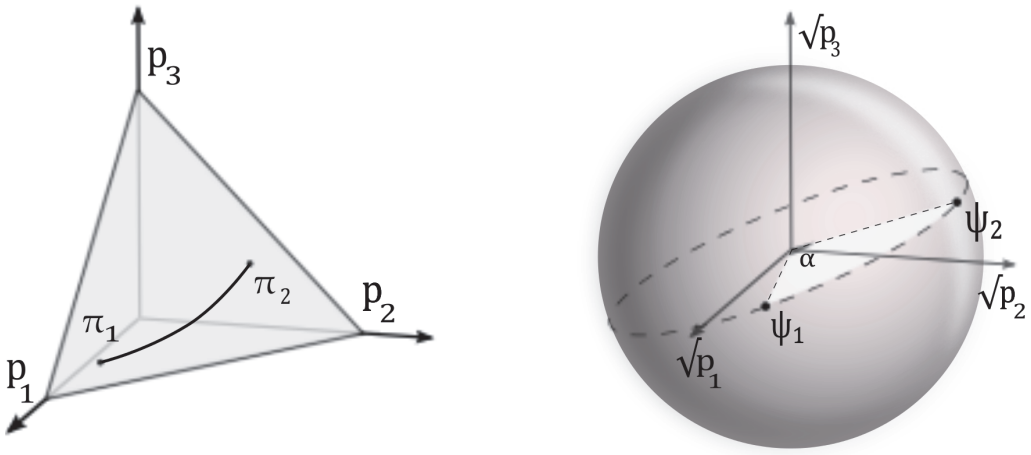

Figure 4-2 By mapping the probability vectors π_1 and π_2 to ψ_1 and ψ_2 on the unit sphere, it becomes evident the statistical distance equals the angle between ψ_1 and ψ_2.

It is well-known that every unitarily invariant Riemannian metric on the unit sphere is proportional to the Hilbert space angle and hence to $d(\boldsymbol{\pi}_1, \boldsymbol{\pi}_2)$, so Eq.(4.30) leads to a representation of the probabilities in such a way that the only metric between the two unit vectors, is proportional to the statistical distance. At no point have we invoked quantum features; yet we obtain a structure that -at least partially- resonates with the formalism of quantum theory[16].

[16] Wootters did extend his argument to the quantum domain. This is necessary and nontrivial because the inferential power of the experimenter in the quantum domain for the distinguishing of two states, depends on whatever observable he chooses to measure. Therefore Wootters defines the statistical distance between two preparations (states) as the largest distance when the preparations are analyzed by the most discriminating apparatus. Again, he recovers Eq.(4.32). Because the Hilbert space angle obtains only for the most discriminating apparatus, this is one more indication that Hilbert space representations in some sense already incorporate maximal statistical performance of observation.

Conclusions and Notes

> *Un physicien éminent me disait un jour à propos de la loi des erreurs: "Tout le monde y croit fermement parce que les mathématiciens s'imaginent que c'est un fait d'observation, et les observateurs que c'est un théorème de mathématiques."*
>
> —Henri Poincaré, *Thermodynamique*, 1908

A third road that leads to the unit sphere as sensible state space is provided by considering what would constitute a proper measure of distance between probability distributions using the Fisher information. Ronald Aylmer Fisher was a famous statistician who observed that, when one has to distinguish between two populations, each one given by a finite probability distribution, it are the square roots of the probability distributions that play a leading role, not the probabilities themselves [98]. We will not repeat the careful analysis presented in Uffink and Hilgevoord [196], as it is hard to improve upon and basically leads to the same end result as Wootters' work; we highly recommend it to the reader. We did not derive any new mathematical result in this chapter. We have merely shown unit norm vectors built from square roots of discrete probabilities have desirable statistical properties that make them good candidates to represent potential properties [42]. Whereas the result of Wootters shows the distance between such states in terms of distinguishability in a fixed set of trials is recovered as the arc length on the unit sphere between those states, our definition of potential properties and the work of Summhammer shows that these representatives automatically satisfy the condition we impose on potential properties. All three results hint that the mathematical structure of the state space of a quantum system reflects the fact that observation is inherently uncertain. If every observer faces a fundamental impossibility to correctly identify its own state, it will unavoidably "coarse grain" some of what it observes. It then makes utter sense to spread the perceived states apart in such a way that equal distance between states denotes equal probability of distinguishability. This leads to representing states on the sphere. If we take the proposal of this chapter seriously, that is, if the requirement of optimal distinguishability promotes a representation of states as unit vectors built from square roots of probabilities, then there should exist other examples of theories of observation outside the quantum domain with similar features. It is well documented ([179], [147] and [99]) and I have published myself on the matter ([37], [28], [42]) that signal analysis, especially when presented in operator form, shows many remarkable similarities with quantum theory, although there are also striking differences [75]. That

both fields employ a similar formalism, in spite of the fact that they deal with different parts of reality, makes that point of view even more plausible, for what unites them on a conceptual level is that both are ultimately concerned with the optimal extraction of information from nature in the face of uncertainties.

Chapter 5

The probabilistic observer

> *My thesis, paradoxically, and a little provocatively, but nonetheless genuinely, is simply this: PROBABILITY DOES NOT EXIST*[1] *.*
>
> —Bruno de Finetti, *The Theory of Probability*, 1970

In this chapter we extend the notion of an observer to deal with an observable quantity rather than a test. The physical nature of the observable is not important and it is not assumed that the outcomes are ordered, only that they are finite in number. As before, we maintain that all physical interactions, including the interaction necessary to produce an outcome, are deterministic. Probability enters our description by including circumstances where there is a lack of knowledge about the state of the system and/or about the state of the observer. After we have described the mathematical framework to cover this general lack of knowledge situation, we turn to repeated measurements on identical system states in which, however, there remains a lack of knowledge about the state of the observer. Important instances of this situation can be found in classical error theory and, we hope to convince the reader, in quantum-mechanical experiments. Following the considerations of the former chapter, we will work out two specific state spaces that seem well suited to deal with this type lack of knowledge: the standard simplex and the (complex) unit sphere. We propose a criterion that an observer needs to satisfy in order to be called *optimal*.

[1] Capitals are copied from the original [81].

We show how such an observer will obtain probabilities that coincide with Born's rule. This shows the principle of optimal observation can be seen as giving an interpretation for Born's probability rule. Because the state of the system is assumed to be identical in each trial of a repeated measurement, our proposal can be regarded as a hidden variable theory, but one that maintains the quantum state represents complete knowledge about the system. The lack of knowledge situation arises in the measurement interaction[2]. We show that every set of probabilities derivable within a finite Hilbert space by the standard machinery of orthodox quantum theory can be reproduced by our model, although we do not claim this is always possible in a local way. We conclude with examining the consequences of the proposal.

General framework

Ignorance gives one a large range of probabilities.

—George Eliot, *Daniel Deronda*, 1876

The deterministic interaction

By an observer we mean in this chapter a physical system M that interacts with a system S to produce an outcome that helps to infer the state of S with respect to a physical quantity. Every experiment has only finitely many outcomes, so the outcome x will be assumed to be a member of a discrete, finite, not necessarily ordered, set of outcomes $X = \{x_1, x_2, \ldots, x_n\}$. In general, this definition of an observer includes the experimental setup, apparata, sensors, as well as the operator and interpreter. It is however quite irrelevant to our purposes whether we consider an apparatus or a detector, an animal or a human being as observer, as long as we agree that it is this system that has produced the outcome. As explained above, we will assume the

[2]This is the reason Diederik Aerts et al. refer to them as *hidden measurements* [7]. Diederik has on several occassions expressed his concern that this choice of vocabulary might suggest the approach assumes there may be "unseen measurements" at work. As John Bell put it: "The usual nomenclature, 'hidden variables' is most unfortunate. Pragmatically minded people can well ask 'why bother about hidden variables that have no effect on anything?' Of course, every time a scintillation occurs on screen, every time an observation yields one thing rather than another, the value of a hidden variable is revealed. Perhaps uncontrolled variable would have been better, for these variables, by hypothesis, for the time being, cannot be manipulated by us." It is in very much the same spirit the name "hidden measurements" was proposed: as an indication there are uncontrollable parameters in the measurement interaction.

observer comes to this outcome through a *deterministic* interaction. Thus if we have perfect knowledge of the initial state of the system and of the observer, the outcome is predetermined. Besides the fact that all fundamental theories of physics (even classical chaotic systems and quantum dynamics[3]) postulate deterministic evolution laws, the requirement of determinism allows to regard probability as a derived concept. So let us assume that the outcome of an observation is the result of a deterministic interaction τ:

$$\tau : \Sigma_S \times \Sigma_M \to X, \qquad \tau(s, m) = x, \tag{5.1}$$

where τ is the *interaction*, Σ_S is the set of states of the observed system, Σ_M the set of states of the observing system and X the set of outcomes that our observable quantity can have. We will deal only with the observation of outcomes that pertain to a *single* observable, which is why we have chosen to have no notational reference to the particular observable. The mapping τ encodes how an observer in a state $m \in \Sigma_M$, observing a system in the state $s \in \Sigma_S$, comes to the outcome $x \in X$. Since our observer is deterministic and every outcome is assumed to be the result of such an interaction, we assume τ is a single-valued surjective mapping. We will later consider the preimage of τ, (sometimes also called the *fibre* of $x \in X$ under τ, or the *level set*, though we will not use this terminology) defined in the standard way:

$$\tau^{-1}(x) = \{(s, m) \in \Sigma_S \times \Sigma_M : \tau(s, m) = x\}. \tag{5.2a}$$

We assume the set of states of S and M are appropriately chosen so that the interaction of every couple (s, m) leads to an outcome in X, hence we have $\tau^{-1}(X) = \Sigma_S \times \Sigma_M$. For the observation to be meaningful, the observer faces the task of selecting an outcome from the set X that tells something about the system under observation. As the result of the interaction is interpreted by the observer as a certain outcome, the outcome has to be encoded somehow in the post-interaction state of the observer. Hence the outcome itself is also an observable quantity of the post-interaction state of the observer. The outcome will then have to share its story among the two participating systems that gave rise to its existence: it will always have something to say about both the observer *and* the system under study. We illustrated in Chapter 1, that the observer cannot have logical certainty with respect to correctness of a single, deterministic self-observation. In Chapter 3 we showed that even in the most simple case of a classical observer, observing only classical properties, there exist properties that cannot be adequately observed. But all observation can be considered as a form of self-observation.

[3] According to standard quantum theory, it is only when we observe a quantum system that the evolution is no longer deterministic.

On the other hand, observation is an absolutely indispensable part of doing science, hence it is only natural that every scientist believes that truthful observation can and does indeed occur. Living in the real world, somewhere between the extremes of the ideal and the impossible, we wonder whether there is an optimal strategy for the observer so that each picked outcome has the largest likelihood of telling something about the system under observation.

Probability in a run of experiments

To increase knowledge (or decrease our uncertainty) about a system, the observer performs a repeated experiment. For each trial in the experiment, the interaction τ will determine the outcome of the experiment. Because τ is single-valued, the outcome will always be the same when the state of both the system under study and the observer are the same. But in a real life situation, no two experiments will be exactly the same. Suppose then that we have a lack of knowledge about the precise state of both the system and apparatus. With $\mathcal{B}(\Sigma_S)$ (and $\mathcal{B}(\Sigma_M)$) as the σ-field of subsets of Σ_S (and Σ_M), our repeated experiment is in its most general form characterized by two probability measures: μ_{Σ_S} as a probability measure from $\mathcal{B}(\Sigma_S) \to [0,1]$ and μ_{Σ_M} as probability measure $\mathcal{B}(\Sigma_M) \to [0,1]$. So we postulate two probability spaces:

$$\mathcal{P}_{\Sigma_S} = (\Sigma_S, \mathcal{B}(\Sigma_S), \mu_{\Sigma_S}),$$
$$\mathcal{P}_{\Sigma_M} = (\Sigma_M, \mathcal{B}(\Sigma_M), \mu_{\Sigma_M}).$$

Since μ_{Σ_S} and μ_{Σ_M} are probability measures, we have:

$$\mu_{\Sigma_S}(\Sigma_S) = \mu_{\Sigma_M}(\Sigma_M) = 1. \tag{5.3}$$

The way the system and the apparatus interact is governed solely by τ: the measures on each set are assumed to be independent. In general, it is impossible to give a direct operational meaning to $(\Sigma_S, \mathcal{B}(\Sigma_S), \mu_{\Sigma_S})$ and $(\Sigma_M, \mathcal{B}(\Sigma_M), \mu_{\Sigma_M})$ separately, as the final probability is a combination of both. To define the probability of the occurrence of an outcome we assume τ is a measurable, independent random variable from $\Sigma_S \times \Sigma_M$ onto X. Let $\mathcal{B}(\Sigma_S \times \Sigma_M)$ be the σ-algebra generated by $\mathcal{B}(\Sigma_S) \times \mathcal{B}(\Sigma_M)$. The measures μ_{Σ_S} and μ_{Σ_M} induce the unique product probability measure $\rho : \mathcal{B}(\Sigma_S \times \Sigma_M) \to [0,1]$, such that for every $\sigma_s \in \mathcal{B}(\Sigma_S)$ and $\sigma_m \in \mathcal{B}(\Sigma_M)$, we define the product measure as

$$\rho(\sigma_s \times \sigma_m) = \mu_{\Sigma_S}(\sigma_s)\mu_{\Sigma_M}(\sigma_m). \tag{5.4}$$

Because of Eq.(5.3), our last equation also gives us $\rho(\Sigma_S \times \Sigma_M) = 1$. We can then construct the probability space

$$\mathcal{P}_{\Sigma_S \times \Sigma_M} = (\Sigma_S \times \Sigma_M, \mathcal{B}(\Sigma_S \times \Sigma_M), \rho). \tag{5.5}$$

In order for an experiment to have a meaningful outcome, experimentalists narrow down the possible states of the systems and they design, construct and fine tune the apparatus used to measure the quantity of interest. As we are able to improve our capability in filtering the set of states of the system we are studying, as well as the set of detector states that we are studying the system with, we increase the precision of the questions that can be asked to nature. The final product of the efforts of the experimentalist are modelled here as the measures μ_{Σ_S} and μ_{Σ_M} and can under ideal circumstances be thought of as the preparations of both the system and the apparatus. However, because preparation is imperfect, we prefer to think of them as mathematical representations of a very general kind of lack of knowledge situation. The probability of obtaining an outcome x in this setting is defined as the probability (given μ_{Σ_S} and μ_{Σ_M}) of picking a couple $s \in \Sigma_S$ and $m \in \Sigma_M$ for which $\tau(s, m) = x$. More precisely, the probability of obtaining an outcome x is the measure ρ of $\tau^{-1}(x) \in \mathcal{B}(\Sigma_S) \times \mathcal{B}(\Sigma_M)$, the subset of the states for system and observer that give rise to the outcome x. This leads to the following definition:

Definition 42 *Given an ensemble of systems S described by the measure μ_{Σ_S} over $\mathcal{B}(\Sigma_S)$ and an ensemble of observers M described by the measure μ_{Σ_M} over $\mathcal{B}(\Sigma_M)$. With ρ the unique product measure 5.4, the probability $\Pr : X \times \mathcal{B}(\Sigma_S \times \Sigma_M) \to [0, 1]$ of the observing the outcome x in the setting described by ρ, and with $\tau^{-1}(x)$ defined by Eq.(5.2a), is defined as*[4]*:*

$$\Pr{}_\rho(x) = \rho(\tau^{-1}(x)). \tag{5.6}$$

Because τ is a surjective function, we have that $\cup_{x \in X} \tau^{-1}(x) = \Sigma_S \times \Sigma_M$, and thus $\sum_{x \in X} \Pr_\rho(x) = 1$. This is our description of the most general[5] lack of knowledge situation: fluctuations in the occurrence of outcomes are a consequence of the inability to prepare identical states for the system, for the apparatus, or both. The quantity $\Pr_\rho(x)$ can model the probability of occurrence of the outcome x in every type experiment, or indeed every combination of different experiments provided they measure the same observable. The problem is that ρ depends on μ_{Σ_M} and we generally want to minimize the observers' influence. If we are able to control or prepare the state of the observer precisely enough, we can do so. But if we have an uncertainty about

[4] Technically speaking, Pr is the image measure or push-forward measure of ρ under τ.

[5] The definition could be made even more general by defining the probability as a map $p : \mathcal{P}(X) \times \mathcal{B}(\Sigma_S \times \Sigma_M) \to [0, 1]$. However, as X is assumed to be a finite, discrete set, we have a trivial counting measure on the set of outcomes. Therefore it is sufficient to consider only the probability for the occurrence of a single outcome.

the precise state of the observer, it is problematic to give an *operational* meaning to both $(\Sigma_S, \mathcal{B}(\Sigma_S), \mu_\Sigma)$ and $(\Sigma_M, \mathcal{B}(\Sigma_M), \mu_{\Sigma_M})$ separately and we have to look for other strategies to obtain a probability that depends only on the system's state.

Repeated measurement on identical system states

Assume the observer is given a large number of identical systems, every one of them in an identical but unknown state. To find out more about the system, the observer will interact with each of the members of this ensemble in turn. For each and every single interaction, the observer picks the outcome that 'has the largest likelihood' of pertaining to the system. By randomizing the probe state and picking the outcomes in this way, the observer aims to restore objectivity, so that the information that is gained pertains solely to the system under observation. To calculate the probability of an outcome if the system is in the given state s within the deterministic setting of the previous section Eq.(5.1) is in principle straightforward. Since there is only a lack of knowledge about the state of the observer and not about the system, we will seize the opportunity to simplify the notation a bit and write μ for μ_{Σ_M}. The experiment our observer will perform is a repeated one, in which the probability measure μ_{Σ_S} on the set of states of the system is reduced to a point measure δ_s on the singleton s, and the probability measure μ on the set of states for the observer, we obtain for Eq.(5.4):

$$\rho = \delta_s \mu_. \tag{5.7}$$

With $\mathcal{B}(\Sigma_M)$ the $\sigma-$algebra of Borel subsets of Σ_M, we have that the probability measure μ acts on the measure space $(\Sigma_M, \mathcal{B}(\Sigma_M))$. For any two disjoint σ_i, σ_j in $\mathcal{B}(\Sigma_M)$, we have

$$\begin{aligned} \mu : \mathcal{B}(\Sigma_M) &\to [0,1] \\ \mu(\sigma_i \cup \sigma_j) &= \mu(\sigma_i) + \mu(\sigma_j) \\ \mu(\Sigma_M) &= 1. \end{aligned} \tag{5.8}$$

Two observer states are equivalent if they both produce outcome x when observing $s : m_1 \approx m_2$ iff $\tau(s, m_1) = \tau(s, m_2)$. Hence, for any given system state s, the mapping τ defines in a natural way a partition of the state space of the observer with each member $eig(x_i, s)$ in the partition belonging to exactly one outcome x_i. This leads to the following important definition:

Definition 43 *The set of states of the observer that give rise to the outcome $x_i \in X$ when they interact with a system under observation in the state s, will be denoted as $eig(x_i, s)$:*

$$eig(x_i, s) = \{m \in \Sigma_M : \tau(s, m) = x_i\}. \tag{5.9}$$

From the single-valuedness of τ in Eq.(5.1), it follows that different outcomes (for the same s) necessarily correspond to different states of the observer, so that for $x_i \neq x_j$:

$$eig(x_i, s) \cap eig(x_j, s) = \varnothing. \tag{5.10}$$

If we assume that every observation leads to an outcome for every state of S, we have:

$$\cup_{i=1}^{n} eig(x_i, s) = \Sigma_M. \tag{5.11}$$

From (5.10) and (5.11) we see that the sets $eig(x_i, s)$ for $i = 1, \ldots, n$ are indeed a partition of Σ_M. To calculate $\Pr_\rho(x_i)$ in this case, we note that

$$\begin{aligned} \tau^{-1}(x_i) &= \{(s, m) \in \{s\} \times \Sigma_M : \tau(s, m) = x_i\} && (5.12) \\ &= \{(s, m) : \{s\} \times eig(x_i, s)\}, && (5.13) \end{aligned}$$

by definition (5.9). The product measure (5.7) then factorizes as

$$\rho(\tau^{-1}(x_i)) = \delta_s(s)\mu(eig(x_i, s)) = \mu(eig(x_i, s)), \tag{5.14}$$

and we obtain from (5.6):

$$\Pr_s(x_i) = \mu(eig(x_i, s)). \tag{5.15}$$

This last formula is fundamental to the remainder of this work. It says that for a repeated experiment on a collection of identically prepared pure system states, the probability $\Pr_s(x_i)$ is the normalized number of observer states that after interaction with the system in the state s, yield outcome x. For our normalized measure μ of Eq.(5.8), this is simply $\mu(eig(x_i, s))$.

Two remarks are in order here. First, the sets $eig(x_i, s)$ are *not* eigenspaces in the algebraic sense of the word[6] but rather a generalization of these. Indeed, if it happens to be the case that for a given s and for (almost) every $m \in \Sigma_M$, we have $\tau(s, m) = x_k$ so that

$$\mu(eig(x_k, s)) = \mu(\Sigma_M) = 1. \tag{5.16}$$

For that particular s, we obviously have $\Pr_s(x_k) = 1$. If the state space is a Hilbert space, the vector s thus defined, will evidently coincide with a regular eigenvector that corresponds to the eigenvalue x_k. Our second remark concerns the outcome x. It is obvious that (5.8) is additive in X:

$$\mu(eig(x_i, s) \cup eig(x_j, s)) = \mu(eig(x_i, s)) + \mu(eig(x_j, s)) \tag{5.17}$$

[6]The sets (5.9) are called in eigensets in accordance with many of the papers from the Brussels group, see for example, [24].

because of (5.10). Therefore we can (and will) restrict our discussion in what follows to the probability of the occurrence of a single outcome. The success of the program to model the probabilities in quantum mechanics as coming from a lack of knowledge about the precise state of the observer stands or falls with the question of defining a natural mapping τ (which determines the outcome and hence $eig(x, s)$) such that the measure μ of the eigenset $eig(x_i, s)$ pertaining to outcome x_i coincides with the probability obtained by the Born rule.

The optimal observer

> *It is easier to make a bad theory of good measurements than to make a good theory of bad measurements.*
>
> —Willis E. Lamb, verbatim[7]

Definition

We can see from Eq.(5.15) that the system state s can be associated with a probability in a fairly trivial way: the probability of a given outcome x when the system is in a pure state s, is the probability the observer attributes x to the outcome of the experiment. In a repeated trial this equals the relative proportion of observer states that attribute outcome x to that state. Even for a repeated measurement on a set of identical pure states, probability can arise from a lack of knowledge concerning the precise state of the observer. Suppose now the observer, considered as a system in its own right, is in a state m. Then in exactly the same way we can associate probabilities with that state too. The operational meaning of this association is given either by a secondary observer observing an ensemble of observers in the state m, or by the observer (mis)identifying its own state m for a state of the system s. We have argued that every outcome will say something about the observer, (that is, about m), and something about the system (that is, about s). The problem is that this information is mixed up in a single outcome. Some outcomes might be more informative about the state of the system, and some more about the state of the apparatus[8]. To

[7] From his talk "*From Newton to Schrödinger and Beyond*" given at the Oviedo Symposium *Fundamental Problems in Quantum Physics* in Spain, 1996.

[8] A typical example arises if it is known that the noise level fluctuates, but the particular noise level for a given outcome is unknown. Eventually, we, as operators of our detection apparatus, will have to decide whether we will retain a given outcome or reject it. Such decisions are a vital part of experimental science. For example, an outcome that is deemed too far off the expected value (so-called

proceed, we regard the state as a parameter that determines the distribution of the outcomes and consider the following binary hypotheses about the investigated system

H_S : s is the state of the system under investigation.

H_M : m is the state of the system under investigation.

If it is possible (with non-vanishing probability) to get an outcome x_i in an experiment under either hypothesis, then the factual occurrence of x_i supports *both* hypotheses simultaneously. What really matters in deciding between H_S and H_M on the basis of the outcome x_i is not the probability of the correctness of each hypothesis itself but rather whether one hypothesis has become *more likely* than the other as a result of getting outcome x_i. All the information in the data that is relevant for deciding between H_S and H_M, is contained in the so-called likelihood ratios or, in the binary case, the *odds:*

$$\Lambda_i = \frac{\mathcal{L}(x_i|H_S)}{\mathcal{L}(x_i|H_M)}. \tag{5.18}$$

The idea goes back to a Lemma[9] of Neyman and Pearson [132] and is an important topic in decision theory (see, for example, Jaynes[129]). The notation $\mathcal{L}(x_i|H_S)$ is common in decision theory and denotes the likelihood of the outcome x_i under the assumption that H_S holds. According to Laplace[10], this can estimated using

$$\mathcal{L}(x_i|H_S) \propto \Pr{}_s(x_i) \tag{5.19}$$

$$\mathcal{L}(x_i|H_M) \propto \Pr{}_m(x_i) \tag{5.20}$$

Under the assumption the constant of proportionality is the same, we obtain

$$\Lambda_i = \frac{\Pr{}_s(x_i)}{\Pr{}_m(x_i)}, \qquad i = 1, \ldots, n. \tag{5.21}$$

In this last two formulas, Pr is determined by Eq.(5.15). Finally, with the function arg defined as

$$\arg\max_{x_i} f(x_i) = \{x_j : f(x_j) > f(x_i), \forall i \neq j\}, \tag{5.22}$$

---

outliers) is rejected and excluded from subsequent analysis. The usual rationale for this exclusion is that an outlier does not contain information about the system we seek to investigate, but rather that it represents a non meaningful peculiarity of that particular measurement.

[9] The Neyman–Pearson lemma states roughly that the likelihood ratio test for deciding between two hypotheses $H_0 : \theta = \theta_0$ and $H_1 : \theta = \theta_1$, is the most *powerful* test. The *power* of a test is the probability of (correctly) rejecting the null-hypothesis (ie., when H_0 doesn't hold).

[10] "If an event can be produced by a number n of different causes, the probabilities of the existence of these causes given the event are to each other as the probabilities of the event given the causes", Laplace in [131], 1774.

we are in position to propose our strategy for the optimal observer.

Definition 44 *We call a system M an* optimal observer *iff, for every state $\mathbf{m} \in \Sigma_M$ an interaction with a system in the state $s \in \Sigma_S$, produces the outcome $x_j \in X$ that maximizes the likelihood ratio*

$$\Lambda_i = \frac{\mathcal{L}(x_i|H_S)}{\mathcal{L}(x_i|H_M)}. \tag{5.23}$$

Equivalently, using (5.19) and (5.22), the optimal observer M observes the outcome:

$$x_j = \arg\max_{x_i} \frac{\Pr_s(x_i)}{\Pr_m(x_i)}. \tag{5.24}$$

Picking the outcome x_j from X that maximizes the corresponding likelihood ratio Λ_j, is optimizing the odds for H_S, given the states s and m and this is what the optimal observer does. This concludes our description of the optimal observer. The quantitative criterion for the optimal observer was obtained using arguments from inverse probability theory, but —as this field is not without its own controversies entirely untouched by our discussion— we are glad to announce we will only need the definition. In fact, much of the remainder of this thesis is devoted to the calculation of $\Pr_s(x_i)$ under various circumstances. To calculate the probability for a repeated experiment when an observer is optimal, we need a state space. Our main interest will be complex Hilbert space, but it is both necessary and instructive to look at epistemic states first.

The optimal observer for epistemic states

Remember that our outcome set X need not be a number field or even a (partially) ordered. It may well be that we are drawing painted balls from an urn and that we decide that $x_1 =$*red* and $x_2 =$*green*. We may for example know that 60% of the balls in the urn are *red* and 40% of the balls are *green*. In this section we want to show how the principles of the foregoing sections work for an epistemic state. Epistemic states are often called *statistical states* and represent *mixtures*. Such states can be represented as weighted sums of ontological states. In our example the epistemic state can be represented as $0,6x_1 + 0,4x_2$. To do things properly, we first define the convex closure of a number of elements $a_1, \ldots, a_n \in A$ as a weighted sum of those elements:

$$[a_1, \ldots, a_n] = \{ \sum_{j=1}^{n} \lambda_j a_j \mid \ \lambda_j \in [0,1] \text{ and } \sum_{j=1}^{n} \lambda_j = 1\}. \tag{5.25}$$

For an arbitrary discrete set C, we denote with $[C]$ the convex closure of the elements in C. For example, the convex closure $[X]$ of the outcome set X is:

$$[X] = [x_1, \ldots, x_n]. \tag{5.26}$$

Geometrically $[X]$ can be thought of as the standard $(n-1)$ simplex Δ_{n-1} in a Euclidean vector space . To see this, we identify with every single outcome x_j one of the base vectors[11] $\mathbf{x}_j$ of $\mathbb{R}^n$. Linear combinations of the $\mathbf{x}_j$ span the whole of $\mathbb{R}^n$ and we call the set:

$$\mathbb{R}^n(X) = \{ \sum_{j=1}^{n} \lambda_j \mathbf{x}_j \mid \ \lambda_j \in \mathbb{R} \text{ and } x_j \in X \}. \tag{5.27}$$

the *free vector space generated by the set* X. Likewise, we call $\Delta_{n-1}(X)$ the standard $(n-1)$ simplex Δ_{n-1} generated by X, and we do so by first forming the convex closure $[X]$ and then identifying each of the x_j with one of the base vectors $\mathbf{x}_j$.

$$\Delta_{n-1}(X) = \{ \sum_{j=1}^{n} \lambda_j \mathbf{x}_j \mid \ \lambda_j \in [0,1] \text{ and } \sum_{j=1}^{n} \lambda_j = 1 \}. \tag{5.28}$$

In what follows we will denote the outcome with x_j and the unit vector that corresponds to it, by $\mathbf{x}_j$. Combining .(5.26) and (5.28), we can write

$$\Delta_{n-1}(X) = [\mathbf{x}_1, \ldots, \mathbf{x}_n]. \tag{5.29}$$

and see from (5.27) that $\Delta_{n-1}(X) \subset \mathbb{R}^n(X)$.

Ideally, an epistemic state is a representation of a very large ensemble with stable relative fractions of systems in different ontological states. Let t_j be the relative fraction of systems in an ontological state that, when properly observed, will yield outcome x_j. Then we can write the epistemic state as

$$\mathbf{s} = t_1 \mathbf{x}_1 + \ldots + t_n \mathbf{x}_n. \tag{5.30}$$

As t_j is a relative fraction, we have for $j = 1, \ldots, n$ that $t_j \in [0,1]$ and

$$\sum_{j=1}^{n} t_j = 1, \tag{5.31}$$

and hence $\mathbf{s} \in \Delta_{n-1}(X)$. We will assume the state space of the observer with respect to this particular observation is the same as that of the system:

$$\Sigma_M = \Delta_{n-1}(X). \tag{5.32}$$

[11] In the remainder of this dissertation we will denote (real or complex) vectors by bold characters. This will cause no confusion with the bold characters we used previously for properties, since we will not derive new theorems about properties.

In this section $\langle , \rangle$ denotes the standard real inner product in a finite dimensional vector space, and with $\langle \mathbf{x}_i, \mathbf{x}_j \rangle = \delta_{ij}$, we have from this last equation

$$t_k = \langle \mathbf{s}, \mathbf{x}_k \rangle. \tag{5.33}$$

With a state space Eq.(5.26), and a rule to extract a probability from a state Eq.(5.33), we are in position to characterize the sets $eig(x_k, \mathbf{s})$. Let $\mathbf{s}$ and $\mathbf{m}$ be states in $\Delta_{n-1}(X)$, written as:

$$\mathbf{s} = \sum_{j=1}^{n} t_j \mathbf{x}_j \text{ and } \mathbf{m} = \sum_{j=1}^{n} r_j \mathbf{x}_j. \tag{5.34}$$

By the definition of optimal observation, we have that the outcome x_k is chosen, if for all $l \neq k$, the corresponding likelihood ratio's satisfy $\Lambda_k > \Lambda_l$. By Eqs. (5.21) and (5.30), x_k is chosen, iff for $l = 1, \ldots, k-1, k+1, \ldots, n$ we have:

$$\frac{\mathcal{L}(x_k|H_S)}{\mathcal{L}(x_k|H_M)} > \frac{\mathcal{L}(x_l|H_S)}{\mathcal{L}(x_l|H_M)}. \tag{5.35}$$

But if t_k is the relative fraction of systems in the ensemble that will yield outcome x_k (if H_S holds) and r_k is the fraction of observer states that will also yield x_k (if H_M holds), then we have

$$\frac{\mathcal{L}(x_k|H_S)}{\mathcal{L}(x_k|H_M)} = \frac{t_k}{r_k}. \tag{5.36}$$

Likewise, we have $\mathcal{L}(x_l|H_S)/\mathcal{L}(x_l|H_M) = t_l/r_l$. This means outcome x_k is chosen, if for all $l \neq k$ we have $t_k/r_k > t_l/r_l$. With $l = 1, \ldots, n$, $j \neq k$ and t_k, r_k, t_l, r_l defined in (5.34), we find for the eigenset (5.9):

$$eig(x_k, \mathbf{s}) = \{\mathbf{m} \in \Delta_{n-1}(X) : \frac{t_k}{r_k} > \frac{t_l}{r_l}\}. \tag{5.37}$$

According to Eq.(5.15), the probability of the outcome x for the repeated experiment on a set of identical system states equals the ratio of observer states that tell the outcome is x, to the total number of observer states. It is natural to take for μ the $(n-1)$-Lebesgue measure in $\Delta_{n-1}(X)$, assumed to be normalized: $\mu(\Delta_{n-1}(X)) = 1$. Because of Eq.(5.15), we have:

$$\Pr_s(x_k) = \mu(eig(x_k, \mathbf{s})). \tag{5.38}$$

However, because of the way we defined the epistemic state, one expects that this probability should also be given directly by the fraction of systems with outcome x_k, that is the k^{th} component of $\mathbf{s}$. So the question is whether the optimal observer Eq.(5.38) can recover that probability, i.e. whether Eq.(5.38) equals Eq.(5.33):

$$\mu(eig(x_k, \mathbf{s})) = \langle \mathbf{s}, \mathbf{x}_k \rangle. \tag{5.39}$$

To see if this is the case, we need to characterize the eigensets for this particular state space. To do so, we first define the open convex closure of a set of vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^n$ as

$$]\mathbf{x}_1, \ldots, \mathbf{x}_n[\quad = \{ \sum_{j=1}^{n} \lambda_j \mathbf{x}_j : \lambda_j \in]0,1[\text{ and } \sum_{j=1}^{n} \lambda_j = 1\}. \tag{5.40}$$

With this definition, we shall characterize $eig(x_k, \mathbf{s})$ for the epistemic state $\mathbf{s}$ as 'almost equal' to the set of open simplices in a simplicial subdivision[12] of $\Delta_{n-1}(X)$ that one obtains when one replaces one vertex of the original simplex $\Delta_{n-1}(X)$ with $\mathbf{s}$, the vector representing the state:

$$C_k^s = \quad]\mathbf{x}_1, \ldots, \mathbf{x}_{k-1}, \mathbf{s}, \mathbf{x}_{k+1}, \ldots, \mathbf{x}_n[. \tag{5.41}$$

The scheme along with the simplicial shape of the eigensets C_k^s are illustrated in Fig. 5-1. The following lemma shows just how $eig(x_k, \mathbf{s})$ 'almost equals' C_k^s.

Lemma 45 *Let C_k^s be defined by Eq.(5.41), $[C_k^s]$ be the convex closure of C_k^s, and $eig(x_k, \mathbf{s})$ by Eq.(5.37), then:*

$$C_k^s \subset \ eig(x_k, \mathbf{s}) \subset [C_k^s]. \tag{5.42}$$

Proof. We start with the first inclusion. Suppose $\mathbf{m}$ is in one of the open $(n-1)-$simplices C_k^s. Then, by definition (5.41) and (5.40), there exist $\lambda_i \in]0,1[$ such that $\lambda_1 + \ldots + \lambda_n = 1$,

$$\mathbf{m} = \sum_{i \neq k}^{n} \lambda_i \mathbf{x}_i + \lambda_k \mathbf{s}. \tag{5.43}$$

On the other hand, we have that $\mathbf{s} \in \Delta_{n-1}(X)$, and hence there exist $t_l \geq 0, \sum_{l=1}^{n} t_l = 1$, such that (5.30) holds:

$$\mathbf{s} = \sum_{l=1}^{n} t_l \mathbf{x}_l. \tag{5.44}$$

Substitution of Eq.(5.43) into Eq.(5.44) yields

$$\mathbf{m} = \lambda_k t_k \mathbf{x}_k + \sum_{i \neq k}^{n} (\lambda_i + \lambda_k t_i) \mathbf{x}_i.$$

Calculating the likelihood ratios Eq.(5.21), we obtain $\Lambda_k = 1/\lambda_k$, and for $i \neq k$ we have:

$$\Lambda_i = \frac{t_i}{\lambda_i + \lambda_k t_i}.$$

[12] A simplicial subdivision is a decomposition of a simplex in smaller simplices.

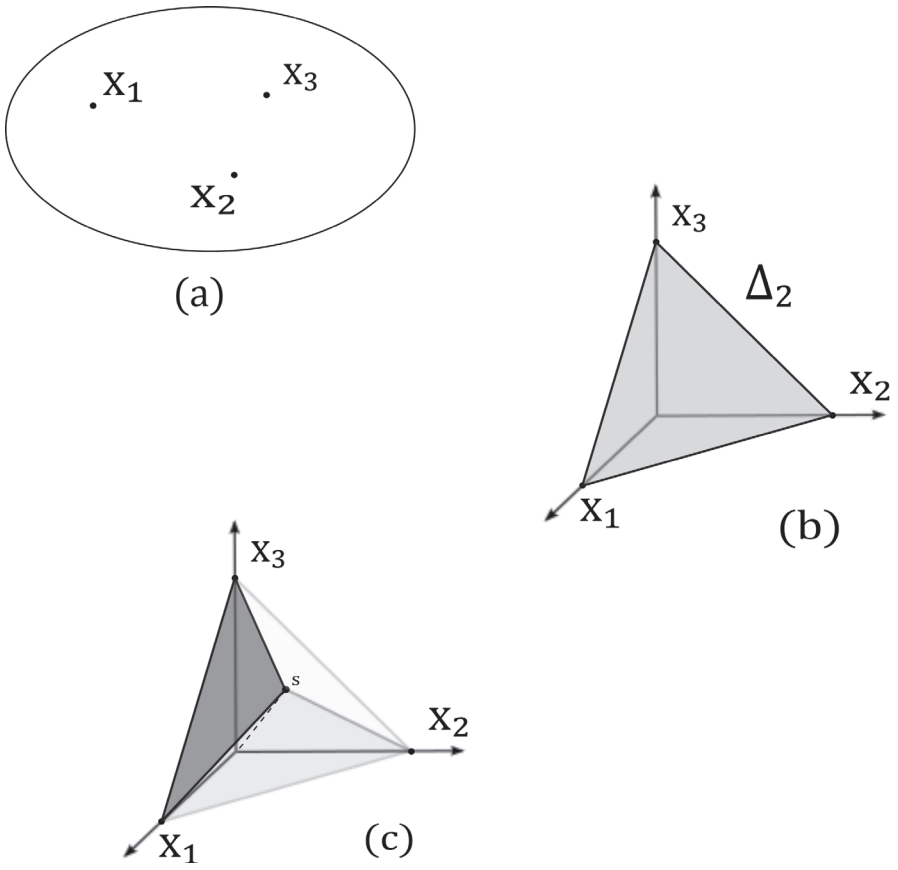

Figure 5-1 Illustration of the scheme in the simplex state space. We start with the discrete outcome set, depicted in figure (a). The (statistical) state space for an outcome set with three outcomes is the standard 2-simplex Δ_2 in the free vector space generated by the outcome set over the field of real numbers as depicted in picture (b). In figure (c) we see the regions of the simplex that show what outcome will be obtained from an optimal measurement. An apparatus state picked from the darkest shaded triangle (which represents $eig(x_2, x(s))$ or C_2^s) will lead to the outcome x_2, in the lightest region to x_1, and the intermediately shaded region leads to the outcome x_3. The probability is the Lebesgue measure over the depicted eigensets. I.e., the probability of obtaining the outcome x_2, is equal to the normalized area of the darkest triangle.

We easily see that $\Lambda_k > \Lambda_i$ because $\lambda_i > 0$, which is satisfied by the definition of C_k^s as an open closure and (5.40). Hence, by Eq.(5.37), every $\mathbf{m} \in C_k^s$ gives an outcome x_k, establishing the result.

For the second inclusion, suppose there exists some $\mathbf{m} \in \Delta_{n-1}$ such that $\mathbf{m} \notin [C_k^s]$. The sets C_k^s in our theorem, as can be seen from the definition of Eq.(5.41), are disjoint open $(n-1)$-simplices. Then we see that, had we defined them by means of the *closed* convex closure, they would maximally have shared

$$[C_j^s] \cap [C_k^s] = \Delta_{n-2}^s(j,k),$$

where $\Delta_{n-2}^s(j,k)$ is the the $(n-2)$ simplex defined as:

$$\Delta_{n-2}^s(j,k) = [\mathbf{s}, \mathbf{x}_1, \ldots, \mathbf{x}_{j-1}, \mathbf{x}_{j+1}, \ldots, \mathbf{x}_{k-1}, \mathbf{x}_{k+1}, \ldots, \mathbf{x}_n]. \quad (5.45)$$

Assume first a is not in the boundary of $[C_k^s]$, i.e. not in one of the lower dimensional sub-simplices $\Delta_{n-2}^s(j,k)$. Then $\mathbf{m} \in C_i^s$ with $i \neq k$. Because of the above demonstrated first inclusion, we have $\mathbf{m} \in eig(x_i, \mathbf{s})$ and hence $\mathbf{m} \notin eig(x_k, \mathbf{s})$. If, on the other hand $\mathbf{m} \in \Delta_{n-2}^s(j,k)$, then the outcome assignment on the basis of the maximum likelihood principle is ambiguous, as there will be two equal maxima, and even more when $\mathbf{m}$ is chosen in a still lower dimensional subsimplex. However, we are free to choose whatever outcome we like as long as it is one of the maxima. Because the maxima coincide, these points lie in the boundary and hence the conclusion remains $eig(x_k, \mathbf{s}) \subset [C_k^s]$. ■

To obtain the probability of Eq.(5.38), we calculate the $\mu-$measure of $[C_k^s]$, which is simply the $(n-1)$-dimensional volume of the simplex $[C_k^s]$.

Lemma 46 *Let μ be the Lebesgue measure with $\mu(\Delta_{n-1}(X)) = 1$, and C_k^s is defined by the convex closure of (5.41), then we have*

$$\mu([C_k^s]) = \langle \mathbf{s}, \mathbf{x}_k \rangle. \quad (5.46)$$

We proof this lemma using determinant calculus, similar to the original derivation in D. Aerts [7]. Let ρ_{n-1} be the (not necessarily normalized) $(n-1)$ -Lebesgue measure in $\Delta_{n-1}(X)$. Then we have:

$$\begin{aligned}
\mu([C_k^s]) &= \frac{\rho_{n-1}([C_k^s])}{\rho_{n-1}(\Delta_{n-1})} \\
&= \frac{\rho_{n-1}([\mathbf{x}_1, \ldots, \mathbf{x}_{k-1}, \mathbf{s}, \mathbf{x}_{k+1}, \ldots, \mathbf{x}_n])}{\rho_{n-1}([\mathbf{x}_1, \ldots, \mathbf{x}_n])} \\
&= \frac{\frac{1}{n!} \det[\mathbf{x}_1, \ldots, \mathbf{x}_{k-1}, \mathbf{s}, \mathbf{x}_{k+1}, \ldots, \mathbf{x}_n]}{\frac{1}{n!} \det[\mathbf{x}_1, \ldots, \mathbf{x}_{k-1}, \mathbf{x}_k, \mathbf{x}_{k+1}, \ldots, \mathbf{x}_n]}
\end{aligned}$$

Substitution of Eq.(5.44) gives

$$
\begin{aligned}
&= \frac{\det[\mathbf{x}_1, \ldots, \mathbf{x}_{k-1}, \sum_{l=1}^{n} t_l \mathbf{x}_l, \mathbf{x}_{k+1}, \ldots, \mathbf{x}_n]}{\det[\mathbf{x}_1, \ldots, \mathbf{x}_{k-1}, \mathbf{x}_k, \mathbf{x}_{k+1}, \ldots, \mathbf{x}_n]} \\
&= \frac{\det[\mathbf{x}_1, \ldots, \mathbf{x}_{k-1}, t_k \mathbf{x}_k, \mathbf{x}_{k+1}, \ldots, \mathbf{x}_n]}{\det[\mathbf{x}_1, \ldots, \mathbf{x}_{k-1}, \mathbf{x}_k, \mathbf{x}_{k+1}, \ldots, \mathbf{x}_n]} \qquad (5.47) \\
&= \frac{t_k \det[\mathbf{x}_1, \ldots, \mathbf{x}_{k-1}, \mathbf{x}_k, \mathbf{x}_{k+1}, \ldots, \mathbf{x}_n]}{\det[\mathbf{x}_1, \ldots, \mathbf{x}_{k-1}, \mathbf{x}_k, \mathbf{x}_{k+1}, \ldots, \mathbf{x}_n]} \qquad (5.48) \\
&= t_k \qquad (5.49)
\end{aligned}
$$

Using Eq.(5.44) once more, we conclude that

$$\mu([C_k^s]) = t_k = \langle \mathbf{s}, \mathbf{x}_k \rangle. \qquad (5.50)$$

Theorem 47 $\mu(eig(x_k, s)) = t_k$.

Proof. By the first lemma (45) we have $C_k^s \subset eig(x_k, \mathbf{s}) \subset [C_k^s]$. Because $A \subset B \implies \mu(A) \leq \mu(B)$, we have

$$\mu(C_k^s) \leq \mu(eig(x_k, \mathbf{s})) \leq \mu([C_k^s]).$$

By the lemma (46) we have $\mu([C_k^s]) = t_k$. To calculate $\mu(C_k^s)$, we note that $\mu(C_k^s) = \mu([C_k^s]) - \mu([C_k^s] \cap C_k^s)$. Because $[C_k^s] \cap C_k^s$ is the collection of faces of C_k^s, a set of finite cardinality whose members have an affine dimension maximally equal to $n-2$, it is $\mu-$negligible, hence we also have $\mu(C_k^s) = t_k$, establishing the result.■

Note that we did not specify what happens on the boundary of $[C_k^s]$. In that case, the principle of optimal observation does not allow us to pick a unique outcome because several of the odds Λ_i (5.21) on p. 127 will have the same magnitude. We will come back to this subtle point in our last chapter, but, as the above theorems show, this is of no consequence with respect to the magnitude of the probability for each outcome. We see that indeed Eq.(5.39) holds and the optimal observer recovers the probability:

$$\Pr{}_s(x_k) = t_k = \mu(eig(x_k, \mathbf{s})). \qquad (5.51)$$

In this way the observer succeeds in obtaining a quantity that, in the limit of an infinite number of measurements, depends only on the state of the system under investigation, and not on its own state. The idea of obtaining the probability in the simplex state space as a uniform measure over the eigensets was first presented in 1986 by Diederik Aerts [7], where it was proposed under the name "hidden measurements" to indicate the origin of the lack of knowledge. We have here extended these ideas in two main ways. First, in publications on hidden measurements, the eigensets are postulated *ad hoc* (i.e. because they yield

the correct probability), whereas we have here derived their simplicial shape from the principle of optimal observation. Second, in D. Aerts [7], the result is presented in a real vector space, with the exception of the two dimensional case where the Bloch sphere representation is used. The generality of the principle of optimal observation and the more abstract approach used here, allows to motivate the shape of the eigensets and extend the results of D. Aerts [7] to systems with a complex state space of arbitrary, albeit finite, dimension. It is important to note another well-developed extension of the hidden measurement formalism that covers complex Hilbert spaces of arbitrary dimension was found by Aerts Diederik and Massimiliano Sassoli de Bianchi [26], [27] a few years later then the results presented here. Their construction is based on a generalization of the Bloch-sphere representation and as such is unrelated to what we do here. Before we turn to complex state spaces, there are two issues of interest that we address first.

On the interpretation of epistemic states

There are two main scenarios in which epistemic states arise naturally. An epistemic state can be the result of mixing ontological states with appropriate weights, or it can represent a statistical tendency of a system to evolve towards one of a set of alternatives. If we can only perform a single experiment on each member of an ensemble described by one epistemic state, we cannot distinguish between these two situations. To illustrate this, suppose we have an urn filled with coins. If we are allowed to inspect each coin only after a single throw of the coin, then we cannot know whether it is a tendency of the coin to show heads with probability 1/2, or whether half of the coins in the urn have both sides printed with heads and half of them have both sides showing tails. If we cannot distinguish between these two situations, our description should be amenable to both scenarios. There is however a big difference between these two situations with respect to the optimal observer. If the epistemic state represents a statistical tendency of a system to evolve towards one of a set of alternatives, the derivation of (5.51) can be repeated step-by-step; we merely interpret the coefficients in (5.34) as *tendencies* towards an alternative, rather than as fractions in an ensemble. The principle of optimal observation can either be considered as a heuristic constraint on observation, or it can be regarded as a condition that must be satisfied by the physical interaction between the observer and the system. However, if we interpret the coefficients as fractions in an ensemble —as we did above— one cannot regard the principle of optimal observation as a *physical* interaction with the epistemic state, because the observer will, on each trial of the experiment, physically interact with a system in a *non-mixed* state $\mathbf{x}_i$

and the probability that the state $\mathbf{x}_i$ was drawn from the ensemble, is t_i. It is however straightforward to show that exactly the same result applies from a perspective where the observer does physically interact with a pure state $\mathbf{x}_i$. Applying the maximum likelihood principle for the optimal observer, we see that, for the trial where the observer physically interacts with the state $\mathbf{x}_i$, the set $eig(x_k, \mathbf{x}_i)$ is either empty, or the whole state space of M. This can readily be seen from Eq.(5.37) applied to $\mathbf{s} = \mathbf{x}_i$:

$$eig(x_k, \mathbf{x}_i) = \{\mathbf{m} \in \Delta_{n-1}(X) : \frac{t_k}{r_k} > \frac{t_l}{r_l}\}. \tag{5.52}$$

Indeed, if $\mathbf{s} = \mathbf{x}_i$ we have $t_i = 1$ and $t_j = 0$ for $i \neq j$. If $k = i$, we have $\frac{1}{r_k} > \frac{0}{r_l}$ for all l and $eig(x_i, \mathbf{x}_i) = \Sigma_M$, so that (5.51) yields $\Pr_s(x_k) = 1$. If, on the other hand, $k \neq i$, we have $\frac{0}{r_k} < \frac{t_l}{r_l}$ and $eig(x_k, \mathbf{x}_i)$ is empty with $\Pr_s(x_k) = 0$ as result. So the optimal observer will always assign outcome x_k when interacting with $\mathbf{x}_k$. If we have an ensemble that consists of a mixture of different states $\mathbf{x}_i$, the probability of obtaining x_k will then be equal to the proportion of states $\mathbf{x}_k$ in the ensemble.

Contextual outcome assignment

It is interesting that, even for the conceptually simple statistical mixtures, the outcome assignment given by the optimal observer is *contextual* in the following sense: given a state for the observer and system that lead to the outcome x_l, then the mere interchanging of the coefficients t_j and t_k (equal to the probability for the outcomes x_j and x_k) can easily result in an outcome different from x_l, even if *neither* x_j, *nor* x_k is equal to x_l! This can readily be verified in Fig. 5-2. However, the probability $\Pr_s(x_k)$ is a function of t_k only as can be seen from (5.51), hence the probability itself is non-contextual because the effect is cancelled by the uniform distribution of the observer states. Conversely, given a state of an observer $\mathbf{m}$ and a system state $\mathbf{s}$ that interact to yield the outcome x_k, it is often possible to change the outcome of the optimal observer to a different outcome by interchanging suitable coefficients of the observer, leaving r_k untouched. This means that changing the observer's preferences over the outcomes x_j and x_l, may let the optimal observer decide x_q is a more optimal outcome than x_k, for j, l, q, k all different!

From Fig. 5-2 we see that, whether a contextual outcome change by interchanging coefficients is possible, depends on how close the state is near the centre of the simplex. The closer to the centre, the closer the coefficients of $\mathbf{s}$ in Eq.(5.30) are to $1/n$, effectively limiting possibilities for producing a contextual outcome change. It seems then that

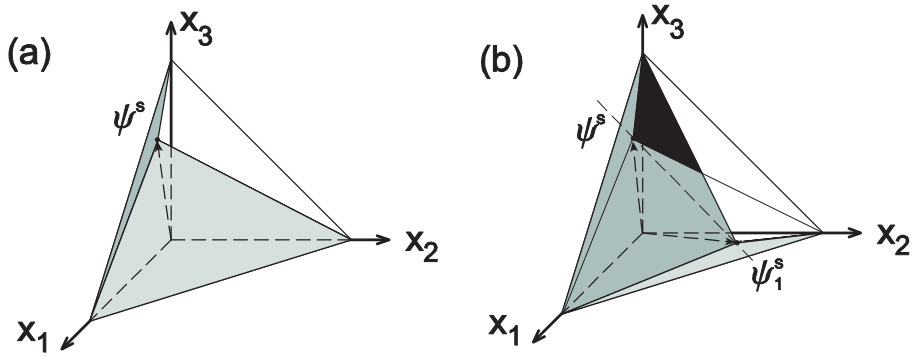

Figure 5-2 (a) Suppose a measurement of the state ψ_s yields outcome x_1. This means the state of the observer was somewhere in the white triangle. (b) If we interchange the second and third component of ψ^s , we obtain ψ^s_1. The probability of obtaining outcome x_1 is the same as in picture (a), because the two triangles have the same area as we did not change the x_1 component in the state ψ^s to obtain ψ^s_1. However, an observer state choosen from the black shaded region would yield outcome x_1 in picture (a), whereas it would yield x_2 in picture (b).

this form of contextuality of the outcome assignment depends on the classical entropy of the state[13].

Optimal observation in Hilbert space

Complex Hilbert spaces are of considerable interest as they arise naturally in many prominent scientific areas including quantum theory [204], signal analysis [147], [99] (both in time-frequency and in wavelet analysis), phasors in electromagnetism and electronic networks[14], and the more recently founded shape theory [135]. We are here particularly interested in states that are used to describe spin-n systems, internal degrees of freedom of a particle, properties of particles, or more generally, any experiment in quantum mechanics that has only a finite number of outcomes. Such states are vectors in a finite dimensional vector space over the complex numbers equipped with an inner product and form a simple example of a Hilbert space[15]. The number of

[13] According to a well-known theorem due to Shannon, the higher the classical entropy of the epistemic state $\mathbf{x}(s)$, the closer each coefficient is to $1/n$.

[14] In these last two fields of research, the complex character of the phasor is seen mostly as convenient and not forced upon us by the physics. Interestingly, the name probability *amplitude*, and indeed the Born interpretation of the wave vector in quantum mechanics as a probability density, were conceived by Born in analogy with electromagnetic waves. In that case, the square of the norm of the amplitude equals the energy in the wave, and conservation of probability is replaced with conservation of energy.

[15] What we actually need is something more than $\mathbb{C}^n$ (which has only a vectorspace structure and needs an inner product) and less than the sequence space

possible outcomes in an experiment equals the dimension of the Hilbert space. The set of states of the observed system that we will consider, is the set of unit vectors in an n-dimensional Hilbert space that we denote as $\mathcal{H}_n$,

$$\Sigma_S = \{\mathbf{s} \in \mathcal{H}_n : ||\mathbf{s}|| = 1\}. \tag{5.53}$$

The norm $||.||$ is defined by the usual sesquilinear Hilbert space product that we will denote $\langle ., . \rangle$. It is well-known Hilbert spaces of the same dimension are isomorphic and this has the interesting consequence that the probabilistic machinery of quantum theory works in exactly the same way for these systems, irrespective of the particular nature of the system and/or the physical quantity involved. In the same way as we did for the epistemic states, one can form the free vector space $\mathbb{C}^n(X)$ over the set of outcomes by identifying each outcome with a base vector in $\mathbb{C}^n$. Every state can then be as a linear combination of these with the constraint that the resulting vector is unit norm. We can, however, make closer contact with quantum theory in the following way. If $\mathcal{L}(\mathcal{H}_n)$ is the set of linear operators that act on the elements of $\mathcal{H}_n$, then an observable $\mathcal{A}$ is represented by a self-adjoint element of $\mathcal{L}(\mathcal{H}_n)$:

$$\mathcal{A} \in \mathcal{L}(\mathcal{H}_n) : \mathcal{A}^\dagger = \mathcal{A}. \tag{5.54}$$

As said, we assume $\mathcal{A}$ has a discrete, finite, non-degenerate spectrum, which implies that eigenvectors belonging to different eigenvalues are mutually orthogonal. Let $F_\mathcal{A}$ be the set of the eigenvectors of $\mathcal{A}$:

$$F_\mathcal{A} = \{\mathbf{x}_k \in \mathcal{H}_n : \mathcal{A}\mathbf{x}_k = c_k \mathbf{x}_k,\ c_k \in \mathbb{R}\}. \tag{5.55}$$

Because $\langle \mathbf{x}_i, \mathbf{x}_j \rangle = \delta_{i,j}$ and $\sum_{j=1}^n \mathbf{x}_j \otimes \mathbf{x}_j = I$,(here $\mathbf{x}_j \otimes \mathbf{x}_j$ is the outer product of $\mathbf{x}$ with itself; it is a projector onto the closed subspace generated by $\mathbf{x}_j$), we have that $F_\mathcal{A}$ is an orthonormal frame. An arbitrary element of $\mathcal{H}_n$ can be written in this frame $F_\mathcal{A}$ as:

$$\mathbf{s} = \sum_{j=1}^n \alpha_j \mathbf{x}_j, \text{ and } \alpha_j \in \mathbb{C}. \tag{5.56}$$

If $\mathbf{s}$ is of unit norm, it satisfies (5.53), then it lies in $\Sigma_S \subset \mathcal{H}_n$, and the $\alpha' s$ obey:

$$\sum_j \alpha_j \alpha_j^* = 1. \tag{5.57}$$

Moreover, one can verify that the observable $\mathcal{A}$ can be written as ($\mathbf{x}_j \otimes \mathbf{x}_j$ is a projector onto the closed subspace generated by $\mathbf{x}_j$) :

$$\mathcal{A} = \sum_{j=1}^n a_j \mathbf{x}_j \otimes \mathbf{x}_j. \tag{5.58}$$

$\ell^2(\ \mathbb{C})$ of square summable series (as we don't need infinite series). Therefore we write $\mathcal{H}_n$ to remind us this is a Hilbert space of finite dimension.

Hence the observable $\mathcal{A}$ is in a one-to-one correspondence with an orthonormal frame $F_{\mathcal{A}}$ of eigenvectors of $\mathcal{A}$ and we will represent the observable by its associated frame. The state of the system S can be expanded in this frame as:

$$\mathbf{s} = \sum_{j=1}^{n} q_j \mathbf{x}_j, \tag{5.59}$$

where the coefficients $q_i \in \mathbb{C}$ and $||\mathbf{s}|| = 1$. The difference with the epistemic states (5.30) is substantial. Whereas in the epistemic case two interpretations of the state were possible, no such possibility exists for states that are represented as unit vectors. The reason lies in the fact that *the points on the sphere cannot be written as a convex combination of other points on the sphere*. Operationally this means you cannot prepare such a state in the laboratory by mixing an appropriate fraction of other states, hinting that the states on the unit sphere represent a deeper ontological rather than epistemic character.

The outcome set X consists of an orthonormal frame of complex vectors $\{\mathbf{x}_j\}$, thought of as the set of (distinct) eigenvectors of the Hermitian operator A that represents the observable $\mathcal{A}$. With this in mind, we model the set of states of the observer as unit vectors in $\mathcal{H}_n$:

$$\Sigma_M = \{\mathbf{m} \in \mathcal{H}_n : ||\mathbf{m}|| = 1\}. \tag{5.60}$$

The state of the observer, to us, always means the subset of the state space that is of relevance to the production of outcomes[16]. This is especially relevant for the interpretation of sentences such as "uniform distribution of initial observer states", which taken too literally, would indicate the observer is probably doing something completely different than observing. The state of an observer with respect to an experiment with outcome set X can be written as ($r_j \in \mathbb{C}$):

$$\mathbf{m} = \sum_{j=1}^{n} r_j \mathbf{x}_j. \tag{5.61}$$

[16]The observing system usually has a very large number of internal degrees of freedom, its state space a Hilbert space of appropriately high dimensionality. According to quantum theory, if we have two systems, one having as state space a Hilbert space $\mathcal{H}_n$ of dimension n and the other one a Hilbert space $\mathcal{H}_m$ of dimension m, then the compound system can be described by a state in $\mathcal{H}_n \otimes \mathcal{H}_m$ (with m possibly much larger then n). However, by the Schmidt bi-orthogonal decomposition theorem, there exist orthonormal bases $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset \mathcal{H}_n$ and $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\} \subset \mathcal{H}_m$ such that the compound state can be written as $\psi = \sum_{j=1}^{n} a_j \mathbf{v}_j \otimes \mathbf{w}_j$. The orthonormal basis in $\mathcal{H}_m$ also consists of only n unit vectors. Hence the correlations between the two systems can –in the appropriate bases– be described as if it where the correlations of two systems, each one living in a Hilbert space of dimension no more than n.

Because the coefficients now assume complex values, they cannot be interpreted as probabilities. This difference also affects the deeper, deterministic level of the description in a profound way. Let us briefly explain why.

For epistemic states, each eigenset is a subsimplex of the state space. This was the essence of lemma (45). Because any two eigensets share at most a lower dimensional face, any two different eigensets (for a fixed system state) can be separated[17] by a single hyperplane. This is essentially the mathematical reason why we can have a linear decision criterion for epistemic states. But in a complex space, a hyperplane does not separate that space in two half-spaces because we do not have a total order relation in the field of complex numbers. To apply the maximum likelihood criterion, one needs to decomplexify the space to restore the order relation. This can be done in a variety of ways[18] and the principle of optimality alone cannot resolve this ambiguity. Let us propose a minimal generalization of the real case Eq.(5.37). For epistemic states we made to following assumption (5.36):

$$\frac{\mathcal{L}(x_k|H_S)}{\mathcal{L}(x_k|H_M)} = \frac{t_k}{r_k}. \tag{5.62}$$

As we can see, the left hand side is estimated as a trivial function of t_k and r_k : the quotient of k^{th} component of the vectors representing $\mathbf{s}$ and $\mathbf{m}$. If we would try the same in the complex case, that quotient would involve a phase factor (i.e. the difference between the phase factors in nominator and denominator). Here we propose that the likelihood ratio is a *monotone* function f of the ratio[19] of the *moduli* of the corresponding coefficients. The nature of f is irrelevant with respect to outcome that will be obtained, and we may as well choose

[17] If C_1 and C_2 are two sets in $\mathbb{R}^n$, then a hyperplane H is said to *separate* C_1 and C_2 iff C_1 is contained in one of the closed halfspaces associated with H and C_2 lies in the opposite closed half-space. Two convex sets in $\mathbb{R}^n$ that share at most an affine set of dimension $n-1$ can be separated by a hyperplane.

[18] On the other hand, this plurality of decomplexifications need not bother us too much. Just as in the case of the epistemic states, the observer can check the statistical validity of his outcome assignment by verifying that the probability (in the sense of a relative frequency) that results from repeated application of his outcome assignment, equals the *anticipated* probability. In the same way, we can postulate or guess a specific form of the probability assignment and justify it *a posteriori*: If the relative frequency of an outcome (as a result of the optimal observers' outcome assignment), converges to a limit that yields (a monotone function of) the very probability assignment he used to obtain those outcomes, the optimal observer knows a posteriori he was optimal. If he wasn't, he can always restart with another guess.

[19] A slightly less general formulation assumes the likelihood is any *monotone* function f of the *modulus* of the corresponding coeficient:$\mathcal{L}(x_k|H_S) = f(|q_k|)$. This does not affect the eigensets.

the identity function:

$$\frac{\mathcal{L}(x_k|H_S)}{\mathcal{L}(x_k|H_M)} = \frac{|q_k|}{|r_k|}. \tag{5.63}$$

With $\mathbf{m}$ defined as in Eqs. (5.59) and (5.61) and $j = 1, \ldots, n; \quad j \neq k$, we then have:

$$eig^{\mathbb{C}}(x_k, \mathbf{s}) = \{\mathbf{m} \in \Sigma_M : \frac{|q_k|}{|r_k|} > \frac{|q_j|}{|r_j|}\}. \tag{5.64}$$

The only difference with Eq.(5.37), is that we take the *modulus* of the coefficients and that the eigenset now contains complex vectors, which is why we have given it the superscript $\mathbb{C}$. To check the consistency of our optimal observer in the complex state space, we evaluate the Lebesgue measure $\nu(eig^{\mathbb{C}}(x_k, \mathbf{s}))$. For this purpose, we regard measure ν in $\mathbb{C}^n$ as the Lebesgue measure μ over $\mathbb{R}^{2n}$. The calculation of the Lebesgue measure of the eigensets by direct integration can be avoided by use of a mapping ω that preserves the measure. Recall that a measurable mapping $\omega : X \to Y$ is called *measure-preserving* between two measure spaces $(X, \mathcal{B}(X), \nu)$ and $(Y, \mathcal{B}(Y), \mu)$ iff for every $A \in \mathcal{B}(Y)$, we have $\nu(\omega^{-1}(A)) = \mu(A)$. In our following lemma we show that the point-wise or Haddamard product $\odot$ of two vectors $\mathbf{v} = (v_1, \ldots, v_n)$ and $\mathbf{w} = (w_1, \ldots, w_n)$, defined as

$$\odot \quad : \quad \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n \tag{5.65}$$

$$\mathbf{v} \odot \mathbf{w} \quad = \quad (v_1 w_1, v_2 w_2, \ldots, v_n w_n), \tag{5.66}$$

can be used to define a mapping ω :

$$\omega \quad : \quad \mathbb{C}^n \to \mathbb{R}^n \tag{5.67}$$

$$\omega(\mathbf{v}) \quad = \quad \mathbf{v} \odot \mathbf{v}^* = (v_1 v_1^*, v_2 v_2^*, \ldots, v_n v_n^*) \tag{5.68}$$

that preserves our measures up to a constant of proportionality. Take notice of the fact that ω maps elements of the (real or complex) unit sphere to the standard simplex.

Lemma 48 *Let $S_n = \{\mathbf{z} \in \mathbb{C}^n : \sum_{i=1}^n z_i z_i^* = 1\}$ be the complex unit n-sphere in $\mathbb{C}^n$, as in (5.60), and let Δ_{n-1} be the standard $(n-1)-$ simplex as in (5.28). Let $\mathbf{z} = (z_1, z_2, \ldots, z_n) \in S_n$. The mapping ω*

$$\omega : S_n \to \Delta_{n-1}$$
$$\omega(\mathbf{z}) = (z_1 z_1^*, z_2 z_2^*, \ldots, z_n z_n^*)$$

is measure-preserving up to a constant of proportionality, i.e. for the measure spaces $(S_n, \mathcal{B}(S_n), \nu)$ and $(\Delta_{n-1}, \mathcal{B}(\Delta_{n-1}), \mu)$ with ν and μ Lebesgue measures and $A \in \mathcal{B}(\Delta_{n-1})$ and $\omega^{-1}(A) \in \mathcal{B}(S_n)$, we have:

$$\nu(\omega^{-1}(A)) = \frac{2\pi^n}{\sqrt{n}}\mu(A).$$

Proof. Let A be an arbitrary open convex set in $\Delta_1 : A = \{(y_1, y_2) : a < y_1 < b,\ y_2 = 1 - y_1\}$ (i.e., an open line segment). Evidently, $\mu(A) = \sqrt{2}(b-a)$. Consider the set B

$$\begin{aligned} B &= \{(z_1, z_2) \in Z_1 \times Z_2 \subset \mathbb{C}^2 \mid Z_1 = \{z_1 : a < |z_1|^2 < b\}, \\ Z_2 &= \{z_2 : z_2 = \sqrt{1-|z_1|^2}\,\mathrm{e}^{\mathrm{i}\theta}, \theta \in [0, 2\pi[\}\}. \end{aligned}$$

Clearly, the members in B are unit norm, hence $B \subset S_n$. Using the definition of ω it is a matter of straightforward verification to see B is the pull-back of A under ω, i.e. $\omega(B) = A$. Its measure factorizes

$$\nu(B) = \nu(Z_1)\nu(Z_2) = \pi(b-a).2\pi = \frac{2\pi^2}{\sqrt{2}}\mu(A).$$

Where we have used $\nu(Z_1) = \pi(b-a)$ because (See Fig.(5-3) the boundary of Z_1 are two concentric cylinders: one of radius $\sqrt{a}$ and one of radius $\sqrt{b}$. Hence the theorem holds for open convex sets in Δ_1. This conclusion can readily be extended to an arbitrary open $(n-1)$-dimensional rectangle set A in Δ_{n-1} $(a_j, y_j, b_j \in \mathbb{R})$:

$$A = \{(y_1, \ldots, y_{n-1}, 1 - \sum_{j=1}^{n-1} y_j) \mid 0 \le a_j < y_j < b_j \le 1, \forall j = 1, \ldots, n-1\}.$$

Its measure factorizes into:

$$\mu(A) = \sqrt{n}\prod_{j=1}^{n-1}(b_j - a_j).$$

Next consider n-tuples of complex numbers:

$$\begin{aligned} B &= \{(z_1, z_2, \ldots, z_n) \in Z_1 \times \ldots \times Z_n\} \\ Z_j &= \{z_j \in \mathbb{C} \mid a_j < |z_j|^2 < b_j,\ j \neq n, \\ z_n &= \sqrt{1 - |z_1|^2 - \ldots - |z_{n-1}|^2}\,\mathrm{e}^{\mathrm{i}\theta_n}, \theta_n \in [0, 2\pi[\}. \end{aligned}$$

Clearly we have $\omega(B) = A$. The measure of B can be factorized as:

$$\begin{aligned} \nu(B) &= \nu(Z_1)\nu(Z_2)\ldots\nu(Z_n) \\ &= 2\pi\prod_{k=1}^{n-1}\pi(b_k - a_k) = \frac{2\pi^n}{\sqrt{n}}\mu(A). \end{aligned}$$

Hence the theorem holds for an arbitrary rectangle set $A \subset \Delta_{n-1}$. As every open set in Δ_{n-1} can be written as a pair-wise disjoint countable union of rectangular sets, it follows $\nu(\omega^{-1}(\cdot)) = 2\pi^n \mu(\cdot)/\sqrt{n}$ for all open sets in Δ_{n-1}. Both ν and μ are finite Borel measures because Δ_{n-1} and S_n are both compact subsets of a vector space of countable dimension. Therefore they must be regular measures[20], which are completely defined by their behavior on open sets. Hence ω is measure preserving for Borel sets.■

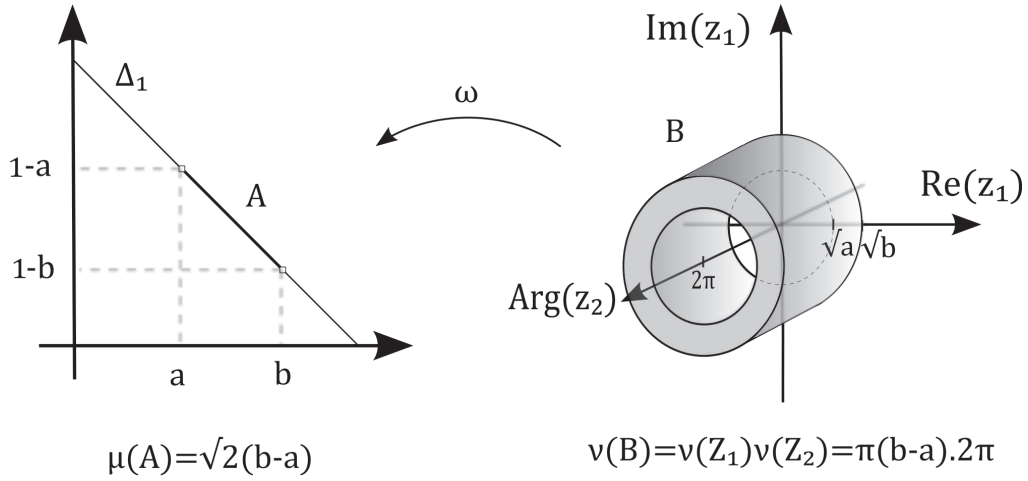

Figure 5-3 On the left we see an illustration of the identity $\mu(A) = \sqrt{2}(b-a)$. On the right side, we see a representation of all the triples $(\mathrm{Im}(z_1), \mathrm{Re}(z_1), \mathrm{Arg}(z_2))$ that are mapped by ω to A. The modulus of z_2 cannot be displayed as this would require an extra dimension, but it does not contribute to the measure as its value is fixed by z_1. We see that $\nu(B) = \nu(Z_1)\nu(Z_2) = \pi(b-a).2\pi = \sqrt{2}\pi^2\mu(A)$.

We have given a graphic representation of the action of ω in Fig. 5-4.

We are almost ready to demonstrate our main result of this section, but first we prove a lemma that shows the complex eigensets are mapped by ω to the real eigensets of the epistemic states, provided one also maps the complex state $\mathbf{s}$ to the simplex by ω. Remark that we don't have to map the $\mathbf{x}_j$ since ω has no effect on the components of base vectors, as can readily be verified by their definition (5.67). This is also geometrically obvious: the base vectors point exactly to the only points where the unit sphere and the simplex meet.

Lemma 49 *Let $eig^{\mathbb{C}}(x_k, \cdot)$ be defined by (5.64) and $eig(x_k, \cdot)$ by (5.37). Then for all unit vectors $\mathbf{s}$ we have*

$$\omega(eig^{\mathbb{C}}(x_k, \mathbf{s})) = eig(x_k, \omega(\mathbf{s})). \tag{5.69}$$

[20] A regular measure is one for which every measurable set can be approximated from above by an open measurable set and from below by a compact measurable set. See, for example, Rudin [173], p. 47.

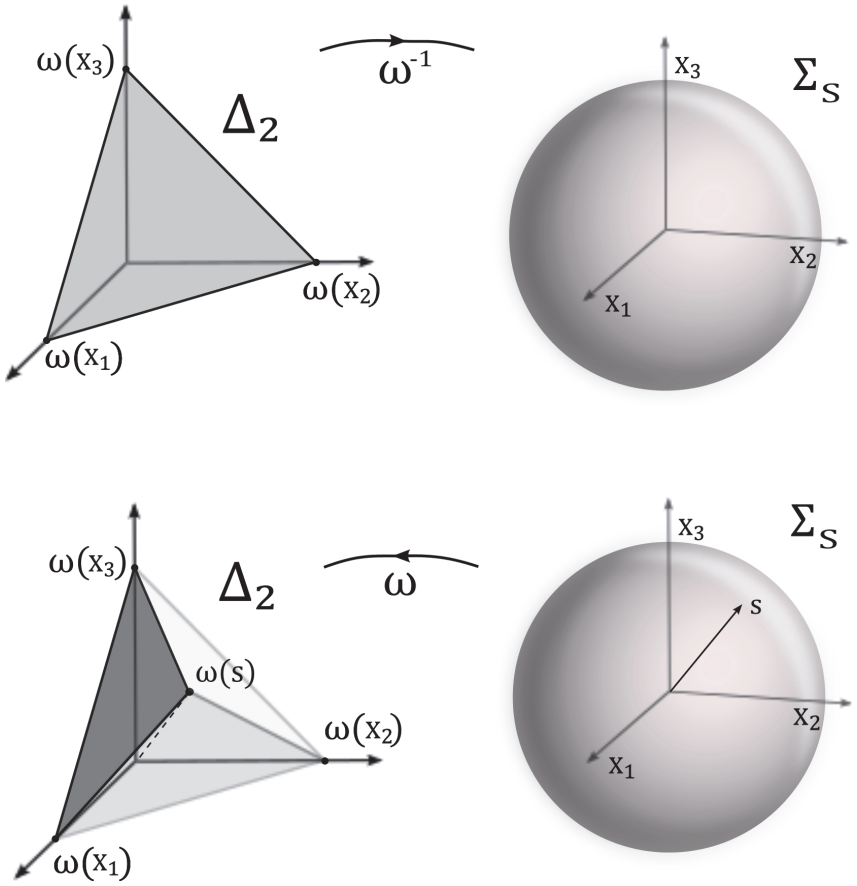

Figure 5-4 The action of the mapping ω sends elements of the unit sphere to the standard simplex (upper figure). The probability for the occurrence of outcome x_k is the measure of the eigenset corresponding to outcome x_k and is calculated in the simplex using the measure preserving mapping ω. The eigensets are depicted in the lower figure for the simplex; it is not possible to show graphically what these sets look like in the complex unit sphere, except for the two dimensional case, as was presented in Fig. (5-3).

Proof. Let $\mathbf{x}_k$ be the base vector that corresponds to outcome x_k $(k = 1, \ldots, n)$ and let $\mathbf{s}$ be given by (5.59)

$$\mathbf{s} = \sum_{j=1}^{n} q_j \mathbf{x}_j. \tag{5.70}$$

Clearly $\omega(\mathbf{s}) = (q_1 q_1^*, q_2 q_2^*, \ldots, q_n q_n^*) \in \Delta_{n-1}(X)$ iff $q_1 q_1^* + \ldots + q_n q_n^* = 1$ and this is true because $||\mathbf{s}|| = 1$. Let $\mathbf{v}$ be an arbitrary vector in $eig^{\mathbb{C}}(x_k, \mathbf{s})$:

$$\mathbf{v} = \sum_{j=1}^{n} r_j \mathbf{x}_j. \tag{5.71}$$

It is clear that $\omega(\mathbf{v})$ is in $eig(x_k, \omega(\mathbf{s}))$ as it satisfies (5.37), since

$$r_1 r_1^* + r_2 r_2^* + \ldots + r_n r_n^* = 1 \tag{5.72}$$

and by squaring both sides of the inequality in (5.64) we obtain:

$$\frac{|q_k|^2}{|r_k|^2} > \frac{|q_j|^2}{|r_j|^2}. \tag{5.73}$$

So every vector in $eig^{\mathbb{C}}(x_k, \mathbf{s})$ is mapped to an element of $eig(x_k, \omega(\mathbf{s}))$ under ω. But clearly the mapping $\omega : eig^{\mathbb{C}}(x_k, \mathbf{s}) \rightarrow eig(x_k, \omega(\mathbf{s}))$ is *onto* as every vector $\mathbf{w} = w_1\mathbf{x}_1 + \ldots + w_n\mathbf{x}_n \in eig(x_k, \omega(\mathbf{s}))$ is the image of $\sqrt{w_1}\mathbf{x}_1 + \ldots + \sqrt{w_n}\mathbf{x}_n$, which is obviously in $eig^{\mathbb{C}}(x_k, \mathbf{s})$ because $(\sqrt{w_1})^2 + \ldots + (\sqrt{w_n})^2 = 1$ and $\sqrt{|q_k|/|w_k|} > \sqrt{|q_j|/|w_j|}$.■

We are now in a position to prove our main result.

Theorem 50 *Let M be an optimal observer, observing a quantity of a system S such that the observation can have one of n possible outcomes. Let both M and S have as states unit norm vectors in an n-dimensional complex Hilbert space. Then the probability for the occurrence of outcome x_k for a repeated observation if the system state is a fixed vector $\mathbf{s}$, observed by a uniform density of observer states, is given by the Born rule:*

$$\Pr{}_s(x_k) = |\langle \mathbf{x}_k, \mathbf{s} \rangle|^2. \tag{5.74}$$

Proof. We first define:

$$C_k^s = \quad]\omega(\mathbf{x}_1), \ldots, \omega(\mathbf{x}_{k-1}), \omega(\mathbf{s}), \omega(\mathbf{x}_{k+1}), \ldots, \omega(\mathbf{x}_n)[. \tag{5.75}$$

With $eig^{\mathbb{C}}(x_k, \mathbf{s})$ defined by Eq.(5.64), it is straightforward to see from the last lemma that:

$$C_k^s \subset \omega(eig^{\mathbb{C}}(x_k, \mathbf{s})) \subset [C_k^s]. \tag{5.76}$$

Let $\tilde{\mu}$ and $\tilde{\nu}$ stand for the normalized versions of the measures μ and ν in the proof of lemma (48): $\tilde{\nu}(S_n) = \tilde{\mu}(\Delta_{n-1}) = 1$, so that their constant of proportionality equals one: $\tilde{\nu}(\omega^{-1}(A)) = \tilde{\mu}(A)$. By definition $\Pr_s(x_k) = \tilde{\nu}(eig^{\mathbb{C}}(x_k, \mathbf{s}))$, and by lemma (48), we have:

$$\tilde{\nu}(eig^{\mathbb{C}}(x_k, \mathbf{s})) = \tilde{\nu}(\omega^{-1}(C_k^{\mathbf{s}})) = \tilde{\mu}(C_k^s). \tag{5.77}$$

The normalized measure $\tilde{\mu}(C_k^s)$ of the real simplex C_k^s was calculated for the real state space. We can repeat the exact same calculation we did to obtain Eq.(5.50) on p. 134, which in this case gives us:

$$\Pr_s(x_k) = \tilde{\mu}(C_k^s) = \langle \omega(\mathbf{x}_k), \omega(\mathbf{s}) \rangle.$$

But $\langle \omega(\mathbf{x}_k), \omega(\mathbf{s}) \rangle = |q_k|^2$ by Eq.(5.59) and the definition of the Haddamard product. Hence, by Eq.(5.70) we get $\Pr_s(x_k) = |q_k|^2 = |\langle x_k, \mathbf{s} \rangle|^2$, which is (5.74).■

We see that indeed the optimal observer recovers the Born rule as a result of maximizing the odds with respect to the outcome that pertains to the system. Although our observer, by Eq.(5.64), used the ratio's

$$\tilde{\Lambda}_k = \frac{|q_k|}{|r_k|}, \tag{5.78}$$

rather than

$$\Lambda_k = \frac{|\langle \mathbf{x}_k, \mathbf{s}\rangle|^2}{|\langle \mathbf{x}_k, \mathbf{m}\rangle|^2} = \frac{|q_k|^2}{|r_k|^2}. \tag{5.79}$$

which we obtain by substitution of the Born rule in Eq.(5.21), it still did maximize the odds of Eq.(5.79). This is obvious if we recognize that Eqs. (5.79) and (5.78) are maximal for the same value of k because one is the square of the other, which is clearly a monotone function. As a consequence, it does not matter whether the optimal observer works with Eq.(5.79) or with Eq.(5.78): repeated application of either strategy on the same pure state will make the relative frequency converge to the Born rule in exactly the same way in both cases. The result should not be understood as a demonstration the Born rule can derived *deus ex machina*. In fact, we have used a cue from quantum theory to get to a suitable form of the likelihood ratio when we imposed Eq. (5.63). That doesn't make the result trivial. Indeed, the optimal observer is forced to give an outcome and we have given an explicit scheme of how and which outcome is produced, something about which quantum theory remains silent. But merely giving a mechanism that produces outcomes is not sufficient: we have to show that the choice of outcomes is such that the probabilities at which one arrives after averaging over the states of the observer are in agreement with empirical data. This is what we have demonstrated and this is how the result should be understood. The fact that the Born rule comes out of the calculation gives us a justification of the ansatz expressed by Eq. (5.63); a wrong ansatz would give rise to a probability that does not match the probabilities used in the likelihood ratio[21].

An interesting link was pointed out to me by Thomas Durt [90]. It turns out the regions of the Bohm-Bub model of 1966 [58], coincide with the eigensets in the complex case, given by Eq.(5.64). Moreover, Bohm and Bub propose a uniform measure of states that they too interpret as apparatus states. They perform the integration directly for the two dimensional case only and mention the integration scheme could be extended to the more dimensional case. Their main result, in spite of using a mathematically very different approach, is also the reproduction

[21] This hints at a difficult but interesting line of research: find the most general mathematical ansatz for $\mathcal{L}(x_k|H_S)$ and $\mathcal{L}(x_k|H_M)$ for complex vectors such that the resulting probabilities after averaging over observer states, are in accordance with the ansatz. We conjecture it is precisely the ansatz of Eq. (5.63).

of the Born rule. For Bohm and Bub, the reproduction of the Born rule is the *a posteriori* justification for the shape the regions take. From our perspective, optimal observation yields an interpretation for the regions employed by Bohm and Bub as the unique (modulo measure null sets) partitioning of the set of states of the observer into subsets such that each state of the observer optimally observes the state of the system.

An alternative to standard quantum theory

Although the word 'quantum' was used on many occasions, we did not focus on describing quantum systems as such before. We will now show how one can make contact with the mathematical structure of standard, non-relativistic quantum theory. Let us summarize the mathematical core of the orthodox theory by listing the following six postulates.

Axiom 1. The properties of a quantum system S are completely defined by specification of its state vector $|\mathbf{s}\rangle$ which is a unit norm element of a complex Hilbert space $\mathcal{H}$ called the state space.

Axiom 2. With every physical observable $\mathcal{X}$ (energy, position, momentum, angular momentum,...) there exists an associated Hermitian operator X, which acts in the state space. The eigenvalues x_k of the operator X are the possible values the physical observable $\mathcal{X}$ can take.

Axiom 3. The probability of finding the outcome x_k when we measure the observable $\mathcal{X}$ on a system S in a state $|\mathbf{s}\rangle$, is given by

$$\Pr_{\mathbf{s}}(\mathcal{X} = x_k) = |\langle \mathbf{x}_k, \mathbf{s}\rangle|^2, \tag{5.80}$$

where $\mathbf{x}_k$ is the eigenvector of the operator X that corresponds to the eigenvalue x_k, or: $X|\mathbf{x}_k\rangle = x_k|\mathbf{x}_k\rangle$.

Axiom 4. The state of the system immediately after the measurement of the physical observable $\mathcal{X}$ is the eigenstate $|\mathbf{x}_k\rangle$ of X that corresponds to the outcome x_k.

Axiom 5. The dynamical evolution of a closed system represented by $|\mathbf{s}(t_0)\rangle$ at t_0 is governed by a unitary time evolution operator U: $|\mathbf{s}(t)\rangle = U(t, t_0)|\mathbf{s}(t_0)\rangle$.

Axiom 6. When two systems interact, the resulting compound state is described by a unitary evolution that acts on a unit vector in the Hilbert space that is the tensor product of the Hilbert spaces of the two separate systems.

What kind of minimal adaptation do we need to regard quantum theory as optimal observation in Hilbert space? We can keep most axioms in the list, replacing $\mathcal{H}$ by $\mathcal{H}_n$ to remind us we have only shown the scheme works in finite dimensional Hilbert spaces. Next we omit Axiom 3 and replace it by two axioms, one describing the production of an outcome, and one describing how to calculate the probability of an outcome.

Axiom 3a. The outcome of the measurement of observable $\mathcal{X}$ for a system S in state $|\mathbf{s}\rangle \in \mathcal{H}_n$ is the outcome x_k that maximizes the likelihood ratio (5.79) among the set of outcomes when S is observed by an observer in the state $|\mathbf{m}\rangle \in \mathcal{H}_n$.

Axiom 3b. The probability for the occurrence of the outcome x_k is given by the normalized uniform measure on the set of states that maximize (5.79) for x_k.

We can easily show how Axioms 3a and 3b lead to Axiom 3. From 3a we conclude that the observer is optimal. The set of states that maximize (5.79) for x_k, are precisely those in $eig^{\mathbb{C}}(x_k, \mathbf{s})$, defined in Eq.(5.64) on p. 141, so from 3b we have that the probability $\Pr_{\mathbf{s}}(\mathcal{X} = x_k)$ is the normalized measure of $eig^{\mathbb{C}}(x_k, \mathbf{s})$. Theorem (50) shows that this leads leads to the Born rule, which is how Axiom 3 is generally referred to. Hence both lists of axioms lead to the same experimental consequences.

Compound systems and singlet correlations

One might wonder whether Axiom 6 is not at odds with Axioms 3a and 3b. Axiom 6 dictates that two systems in interaction are to be described in the Hilbert space that is the tensor product of the two Hilbert spaces of the systems that interact. Our measure theory as introduced on p. 123 used the Cartesian product of the state of the system and the observer. However, if we take determinism seriously, there is no contradiction between Axioms 3a, 3b and Axiom 6. On the condition that the system and the observer are isolated systems

before the measurement interaction takes place, the *initial* state in the tensor product space is uniquely defined by the two elements in the Cartesian product[22]. Once the interaction Hamiltonian kicks in, the evolution of the compound system can no longer be described by a unitary transformations in the separate Hilbert spaces and a Cartesian product is no longer appropriate. But Axioms 3a and 3b do not describe the evolution of the compound system, only the production of the final outcome. And that final outcome is a *deterministic* function of the states of S and M, both under our assumptions, and under the unitary evolution law described in Axiom 5 (but not under Axiom 3). This means that, on the condition that determinism holds, it is possible to produce the outcome either as a *function* of the states of S and M right before the interaction, or as a *function* of the compound state in the compound state space. They will not be the same functions, we only argue that it is possible. Although we consider it a plus our interaction rule defined on p. 121 and the unitary evolution in Axiom 5 are both deterministic (in contradistinction with quantum theory), we cannot find a *unitary* evolution that represents how the interaction leads to a definite outcome. This is a consequence of the measurement problem. But in quantum mechanics outcome states are obtained by a projection as described in Axiom 4; a manifestly non-unitary transformation. We feel we are entitled to our own alternative without doing worse than standard quantum mechanics.

How do we treat compound systems then? Using Axiom 6, we describe two systems that are prepared independently in the pure states $\mathbf{s}$ and $\mathbf{r}$, by forming the initial compound state as $\mathbf{s} \otimes \mathbf{r}$, which is again a pure state. The evolution of that state is governed by Axiom 5. That the compound system is in a pure state indicates there exists a test that one can submit to the compound system and that will deliver a given outcome with certainty[23]. In the language we developed in Chapter 2, we say that in this case the compound system has an actual property, which defines an element of reality. Unfortunately, it is less easy to identify actual properties of the constituent states $\mathbf{s}$ and $\mathbf{r}$ as a result of their interaction. Mathematically this is a consequence of describing the states of the subsystems as a partial trace over the

[22] Or as Peres ([157], p. 115) phrases it: "Our problem is to construct a formalism whereby the state of a composite system is expressed in terms of states of its constituents. The situation is simple if the electron and the proton are widely separated—they may possibly be in different laboratories, where they have been prepared in states u and v, respectively....The state of both particles together can then be represented as a direct product (sometimes called "tensor product") of these two vectors, written as $w = u \otimes v$. ... Direct products can represent the state of two (or more) systems that have been prepared independently.".

[23] See, for exmple, [157] p. 30, where Peres call this the postulate of "Statistical Determinism".

compound state, resulting in subsystems that are no longer described by unit vectors, and hence no longer represent pure states. If this is the whole story, then there do not exist actual properties for the subsystems and we have no elements of reality for the subsystems[24]. A peculiarity of the quantum world is that the connection between subsystems seems not even restricted by space-time separation. The most famous demonstration of this fact is the analysis by Bell of the EPRB experiment [50]. Because we have shown we can reproduce the probabilities given by the Born rule, one wonders whether we can also reproduce the probabilities to violate the Bell inequality with an optimal observer. To investigate this question, we first note that an arbitrary spin-1/2 state $|q\rangle$ can be written as a linear combination of two base states as

$$|q\rangle = \alpha|z^+\rangle + \beta|z^-\rangle. \tag{5.81}$$

Here we denote the base states by the kets $|z^+\rangle$ and $|z^-\rangle$ because they refer to the states "spin up" and "spin down" measured (for example) by a Stern-Gerlach device aligned with a spatial axis z. Being base states, we have:

$$\langle z^+|z^+\rangle = \langle z^-|z^-\rangle = 1 \text{ and } \langle z^-|z^+\rangle = 0. \tag{5.82}$$

Now consider the following vectors, parameterized[25] by the angles θ and ϕ :

$$|u^+\rangle = \mathrm{e}^{-\mathrm{i}\phi/2}\cos(\theta/2)\,|z^+\rangle + \mathrm{e}^{\mathrm{i}\phi/2}\sin(\theta/2)\,|z^-\rangle \tag{5.83}$$

$$|u^-\rangle = -\mathrm{e}^{-\mathrm{i}\phi/2}\sin(\theta/2)\,|z^+\rangle + \mathrm{e}^{\mathrm{i}\phi/2}\cos(\theta/2)\,|z^-\rangle. \tag{5.84}$$

Because of Eq.(5.82), we have

$$\langle u^+|u^+\rangle = \langle u^-|u^-\rangle = 1 \text{ and } \langle u^-|u^+\rangle = 0, \tag{5.85}$$

so $|u^+\rangle$ and $|u^-\rangle$ can also serve as base states[26]. This change of basis can be very convenient. One can, for example, directly determine from (5.81) and (5.83) that the probability of the outcome "spin up" for a system prepared in the pure state $|z^+\rangle$, measured by a Stern-Gerlach device in a direction that makes an angle θ with z, will be $\cos^2(\theta/2)$.

[24] It is however shown in the so-called *modal interpretations* of quantum mechanics (which don't impose Axiom 4) this may still be possible using the bi-orthogonal Schmidt decomposition. For an overview and references, see [87].

[25] This representation of the spin 1/2 state is parameterized by the spherical coördinates θ and ϕ and is closely related to the Bloch sphere representation of a qbit. We will examine the Bloch sphere in more detail in the next Chapter.

[26] Remark that setting $\theta = 0$ makes the two bases identical, whereas setting $\theta = \pi$ identifies $|u^+\rangle$ with $|z^-\rangle$ and $|u^-\rangle$ with $|z^+\rangle$ (both apart from a phase factor $\mathrm{e}^{-\mathrm{i}\phi/2}$). By direct comparison of (5.81) and (5.83) we obtain the relation between them as $\beta/\alpha = \tan(\theta/2)\mathrm{e}^{\mathrm{i}\phi}$.

Next we turn to the singlet state $|\sigma\rangle$. It can be written in a basis aligned with the z axis (the subscripts 1 and 2 indicate the relevant subsystem) as:

$$|\sigma\rangle = \frac{1}{\sqrt{2}}(|z_1^+\rangle \otimes |z_2^-\rangle - |z_1^-\rangle \otimes |z_2^+\rangle). \tag{5.86}$$

This state enjoys rotational invariance due to the fact that singlet states have a vanishing total angular momentum. It can therefore be written in a basis aligned with respect to any other axis, say u, resulting in an expression that has the exact same functional form as (5.86).

$$|\sigma\rangle = \frac{1}{\sqrt{2}}(|u_1^+\rangle \otimes |u_2^-\rangle - |u_1^-\rangle \otimes |u_2^+\rangle). \tag{5.87}$$

If is u tilted at an angle θ with respect to z, we can use Eqs. (5.83) and (5.84) to write $|u_1^+\rangle$ and $|u_1^-\rangle$ in terms of $|z_1^+\rangle$ and $|z_1^-\rangle$ by direct substitution to obtain:

$$\begin{aligned} |\sigma\rangle &= \frac{1}{\sqrt{2}}((e^{-i\frac{\phi}{2}}\cos(\theta/2)\,|z_1^+\rangle + e^{i\frac{\phi}{2}}\sin(\theta/2)\,|z_1^-\rangle) \otimes |u_2^-\rangle \quad (5.88)\\ &\quad -(-e^{-i\frac{\phi}{2}}\sin(\theta/2)\,|z_1^+\rangle + e^{i\frac{\phi}{2}}\cos(\theta/2)\,|z_1^-\rangle) \otimes |u_2^+\rangle) \quad (5.89) \end{aligned}$$

$$\begin{aligned} &= \frac{e^{-i\frac{\phi}{2}}}{\sqrt{2}}(\sin(\theta/2)\,|z_1^+\rangle \otimes |u_2^+\rangle + \cos(\theta/2)\,|z_1^+\rangle \otimes |u_2^-\rangle \quad (5.90)\\ &\quad +e^{i\phi}\sin(\theta/2)\,|z_1^-\rangle \otimes |u_2^-\rangle - e^{i\phi}\cos(\theta/2)\,|z_1^-\rangle \otimes |u_2^+\rangle). \quad (5.91) \end{aligned}$$

This is the singlet state written down in the Hilbert space that has as its 4 base states the vectors $|z_1^+\rangle \otimes |u_2^+\rangle$, $|z_1^+\rangle \otimes |u_2^-\rangle$, $|z_1^-\rangle \otimes |u_2^+\rangle$ and $|z_1^-\rangle \otimes |u_2^-\rangle$. A general state of the observer in this base is given by

$$|m\rangle = a|z_1^+\rangle \otimes |u_2^+\rangle + b|z_1^+\rangle \otimes |u_2^-\rangle + c|z_1^-\rangle \otimes |u_2^+\rangle + d|z_1^-\rangle \otimes |u_2^-\rangle. \tag{5.92}$$

where a, b, c and d are the components of a random, 4 dimensional, complex unit vector. For each picked state of the observer (5.92), the measurement interaction chooses the outcome which maximizes the odds, given the singlet state (5.90). Note that a 'single' outcome in this four dimensional Hilbert space, constitutes two separate outcomes, one for each subsystem. Theorem (50) then tells us that optimal observation will deliver any of the four possible outcomes with a probability that equals the modulus squared of the corresponding component in (5.90). These can be directly inferred from the state (5.90). If we denote by $\Pr(z_1^+ \text{ and } u_2^+)$ the probability we will get the outcome "up" for subsystem 1 measured in the direction given by z, as well as "up"

for subsystem 2 measured in the direction given by u, (and similarly for the other possible events) we have:

$$\Pr(z_1^+ \text{ and } u_2^+) = \Pr(z_1^- \text{ and } u_2^-) = \frac{1}{2}\sin^2(\theta/2) \tag{5.93}$$

$$\Pr(z_1^+ \text{ and } u_2^-) = \Pr(z_1^+ \text{ and } u_2^-) = \frac{1}{2}\cos^2(\theta/2)\,. \tag{5.94}$$

We see the probabilities add up to one and calculate the correlation function $E(z,u)$ as the expectation value for the product of the outcomes:

$$\begin{aligned} E(z,u) &= +\Pr(z_1^+ \text{ and } u_2^+) + \Pr(z_1^- \text{ and } u_2^-) && (5.95)\\ &\quad -\Pr(z_1^+ \text{ and } u_2^-) - \Pr(z_1^- \text{ and } u_2^+) && (5.96)\\ &= \sin^2(\theta/2) - \cos^2(\theta/2) && (5.97)\\ &= -\cos\theta && (5.98) \end{aligned}$$

Because of the rotational invariance of the singlet state, this correlation only depends on the relative angle between the local measurements. This is the quantum mechanical correlation function and Bell's inequality will be violated by the optimal observer in exactly the same way as quantum mechanics predicts. But that as it may, this is not an example of a local hidden variable theory because the maximum likelihood criterion compares *all* components of the states of the singlet and the observer simultaneously to decide which outcome is optimal. Such a feat cannot, in general, be achieved locally. If we were to adapt the procedure to local measurements on each subsystem, each subsystem state has to be compared with a local random unit vector for the observer to apply the likelihood criterion. There are now two possibilities. Either we follow orthodoxy and calculate the states of each subsystem by partial tracing operation of $|\sigma\rangle$ over the subspace of the other particle. It is well-known that for the singlet state the resulting state of each particle expresses complete ignorance and no violation the Bell inequality occurs. Alternatively, we can assume the subsystems were in a pure state, but one that the orthodox theory simply fails to provide. The problem is then that a nonlocal interaction between the subsystems is required to maintain the spherical symmetry of the singlet state after an outcome is obtained in one location. This is provided for in standard quantum mechanics by the projection postulate. Without this constraint, no violation will follow. Bell's theorem can then be rephrased in our interpretation as follows: *It is in general not possible for an observer to perform local optimal measurements on subsystems, such that the resulting outcomes are also optimal for the compound system.*

It is noteworthy that this does not necessarily imply that in our approach the nonlocal correlations of the singlet state are transferred

to a physical nonlocal correlation between the local observers. In a typical experiment that studies quantum entanglement, the number of detected non-entangled states more often than not outnumbers the number detected of entangled states. To eliminate the influence of the non-entangled states on the statistics, it is common practice to use a *coincidence counter* that discards the counts that are not detected at the same moment within some fixed time interval. Although one would not typically describe this removal of non-coincidental counts as "spooky action at a distance", it will nevertheless be obvious that it is only *after* we have collected the locally obtained outcomes, that we can ascertain which ones were detected simultaneously and which were not. Hence this postselection of outcomes cannot be obtained in a local way. It is indeed well-known that the subtraction of events that are considered irrelevant to the demonstration, no longer warrants the exclusion of local hidden variable models to reproduce the quantum mechanical correlations. We have given an explicit example of a local hidden variable model that exploits this loophole to reproduce the quantum mechanical predictions for the well-known Franson two photon interference experiments in [32] and [35]. This demonstrates non-local actions may not always be directly recognizable as such.

Decision invariance and unitarity

The outcome chosen by a optimal observers, is the one that maximizes the corresponding likelihood ratio Λ_i (5.18). Any monotonously increasing function of the likelihood ratio's preserves their relative order and hence their maximum. By Eqs.(5.37) and (5.64), this carries over to the coefficients of the state vectors in both the real and the complex state space. The same is true for multiplication by a phase factor, which is cancelled by taking the moduli in Eq.(5.64). As a result, the state space is a *projective* vector space: if the vectors in the state space are multiplied by $z \in \mathbb{C}$, $0 < |z| < \infty$, this does not change the result of the decision procedure adopted by the optimal observer. This directly implies the scheme works equally well with rays as states rather than unit vectors. But there is a much bigger class of transformations that leave the optimal decision unaltered. For any $\mathbf{s}$, the probability of obtaining x_k is defined as:

$$\Pr{}_s(x_k) = \mu(eig^{\mathbb{C}}(x_k, \mathbf{s})). \tag{5.99}$$

Obviously the vectors in $eig^{\mathbb{C}}(x_k, \mathbf{s})$, as a subset of the complex unit sphere, have finite norm. We can then apply a linear transformation

to the base vectors of the state space:

$$T : \Sigma_S \to \Sigma_S,\ T(\mathbf{x}_j) = \sum_i^n \sigma_{ij}\mathbf{x}_j. \tag{5.100}$$

The eigenset $eig^{\mathbb{C}}(x_k, \mathbf{s})$ will accordingly be transformed by applying T to $\mathbf{x}_k$ and $\mathbf{s}$. By Lebesgue measure theory, the volume of the transformed set is proportional to the volume of the original set, the constant of proportionality being the determinant of the transformation:

$$\mu(T(eig^{\mathbb{C}}(x_k, \mathbf{s}))) = |\det(T)| \cdot \mu(eig^{\mathbb{C}}(x_k, \mathbf{s}))$$

for all $eig^{\mathbb{C}}(x_k, \mathbf{s}) \in \mathcal{B}(\Sigma)$. This is a classic result[27], and we refer the interested reader to Rudin [173], p. 54 for a proof. Note that this would typically be false for a nonlinear transformation. As a result, all transformations with $|\det(T)| = 1$ leave the probabilities invariant, which means we have invariance under *unitary* transformations. Intuitively this is obvious: if the probabilities have their origin in a measure on state space, then unitarily transforming the entire state space does not alter the relative proportions of the eigensets, whence the invariance. Of course, once we have the Born rule, it is trivial to derive that the transition probability is invariant under simultanuous unitary transformation of the states, because the Born rule is the square modulus of an inner product and a unitary transformation is defined as a linear operator that leaves the inner product invariant. However, our invariance principle tells us the same story at a deeper level, for not only the probabilities are invariant under unitary transformation, also each individually obtained outcome will be the same whether or not we transform the state of S and M by the same unitary operator.

Whether the proposed interpretation is scientific or philosophical depends ultimately on its falsifiability. This can be achieved only if we succeed in tailoring the probe states of the apparatus to our needs. If we can produce a non-uniform distribution for the initial states, we would be able to distinguish some pairs of states better, and some pairs of states worse than the usual Born rule allows. This implies such a probe can only be used to our advantage if we posses *some* information about the state prior to the measurement. We will work out the details of such a proposal in the next chapter.

[27] As before, we regard the complex $n-$space as a real $2n-$space, for which the theorem is applicable.

Chapter 6

Between Classical and Quantum

Give a small boy a hammer, and he will find
that everything he encounters needs pounding.

—Abraham Kaplan, *The Conduct of Inquiry*, 1964

By all means, do not use a hammer.

—IBM Manual, 1925

Now that we have shown how the scheme works in the classical simplex as well as quantum mechanical state spaces, we wonder if and how we can treat intermediate cases. To fix our thoughts, let us revisit the simple example we gave on p. 97 in the introduction of Chapter 4: how to determine the length of a rod. In a classical setting we are in principle free within large bounds to choose the number of outcomes and we are allowed to make many observations before we settle on the result of a single observation. For example, we can align the zero of the measuring rod with one end point of the system and read the outcome at the other end point as many times as we want to. If we are not satisfied with the precision that the measuring rod affords we can pick a better one or improve it by adding a nonius (or vernier) system to it. As long as we are able to do this we are still in a classical regime of observation. In the classical regime of observation, the distribution of observer states will be highly non-uniform. Ideally, of all possible

measurements, the only uncertainty we have about the state of the observer that is assumed to be of relevance to the measurement outcome, is an uncertainty of the order of the smallest number the measuring rod can represent. To decrease the uncertainty about the result, even beyond the precision offered by the smallest number the measuring rod can represent, it is common scientific practice to repeat the measurement many times. Assuming identical, independent observations, one can apply standard error theory. In our search for ever more precise measurements or measurements on ever smaller constituents of nature, we eventually reach a region where we cannot repeat measurements without absorbing the system or altering its state dramatically. We may not even be able to choose freely the set of outcomes for a particular measurement as is the case in the quantum regime. It is then no longer possible to obtain directly the "true" value of a physical quantity because the eigenstate of the observing system may not (and in general will not) coincide with the state of the system under investigation. We cannot attempt the same measurement (or one with altered eigenstates) on the same system because the state of the system has been altered or even destroyed. In view of this impossibility, we are led to statistical observation on ensembles. We have shown it is possible to recover an objective probability if the distribution of observer states is uniform. We see that the best possible observation scheme in the classical regime entails a *minimal* uncertainty (i.e. about the interpretation of the last digit only) in the state of the observer, and in the quantum regime a *maximal* uncertainty (any outcome is in principle possible) about the state of the observer. The consequence of such an interpretation is that we will only be able to identify intermediate regions when we allow for a more complete description of the observing system. In essence, we need to describe how to go from this minimal to this maximal uncertainty state. An infinite variety of situations with varying degrees of lack of knowledge about the states of the system and the observer can in principle be modelled by the general formula (5.6) of p. 123. In this chapter we will investigate the probabilistic behavior of arguably the most simple model that can describe a range of qualitatively different experiments which have only two possible outcomes. This model has its origin in Aerts 'sphere model [7] and was elaborated in [12] by introducing a free parameter ϵ, which represents the amount of fluctuations of the measurement state. In particular the value $\epsilon = 1$ —maximal lack of knowledge about the measurement state— produces a model that is isomorphic to a spin-1/2 model. The consequences for the probabilistic structure as well as the axioms of the quantum logic of the system when one varies ϵ, were studied in detail by the Brussels group under Diederik Aerts ([11], [12], [15], [17],

[19], [91], [84][1]). We show here how this model can be reformulated as to fit naturally into the framework we have set out here. Next we will show how to calculate conditional probabilities that allow a continuous transition from the quantum to the classical. The conditional probabilities we obtain here were first derived in [29] and published in [30]. Although the calculation presented here is the same as in [30], the final result is substantially shorter than the one presented there by analytic extension of the final result to $\mathbb{C}$ and taking the real part of the expression.

The Bloch sphere representation

A spin 1/2 particle, the polarization state of a photon, a qubit, or any two level system for that matter, are often described by a vector, usually denoted by a ket $|\psi\rangle$, in $\mathcal{H}_2(\mathbb{C})$, the two-dimensional Hilbert space over the field of complex numbers. Any $|\psi\rangle \in \mathcal{H}_2(\mathbb{C})$ can be written as a linear combination of any two vectors that span $\mathcal{H}_2(\mathbb{C})$. If we denote our base vectors as $|+\rangle$ and $|-\rangle$, we can write any $|\psi\rangle$ with appropriate complex numbers a and b as:

$$|\psi\rangle = a|+\rangle + b|-\rangle. \tag{6.1}$$

As a and b are two complex scalars, we need four real numbers to specify them. Because the probability related to the measurement of a specific observable is the square modulus of the inner product:

$$|a|^2 = |\langle\psi|+\rangle|^2 \text{ and } |b|^2 = |\langle\psi|-\rangle|^2, \tag{6.2}$$

and the sum of the probabilities need to add up to one, the state needs to be unit norm: $|\langle\psi|\psi\rangle| = 1$. This means only three independent numbers specify (6.1). Furthermore, because two states that differ only by an overall phase, i.e. $|\psi'\rangle = e^{i\phi}|\psi\rangle$ cannot by physically distinguished, the physically distinguishable states are formed by the equivalence class of vectors that differ by an overall phase, which leaves only two free parameters in (6.1). Hence without loss of generality, we can write each $|\psi\rangle$ uniquely[2] as:

$$|\psi\rangle = e^{-i\phi/2}\cos\left(\frac{\theta}{2}\right)|+\rangle + e^{i\phi/2}\sin\left(\frac{\theta}{2}\right)|-\rangle, \tag{6.3}$$

[1] Many more references could be given; we cite only a few representative ones and advice the interested reader to consult the references to be found in the publications we cite here.

[2] Except if the vector coincides with one of the base states; the representation is for those two points not unique. Spherical coordinates have the same issue with points at the pole, in which case the longitude is not defined.

where $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$. Interpreting the free parameters θ and ϕ as angular coordinates in a spherical coordinate system $(\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$, we obtain a point on the two-sphere in Euclidean three space. In this way we obtain a one-to-one correspondence between $|\psi\rangle$ and $(\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$, known as the *Bloch sphere* representation[3] of the system. A Stern-Gerlach spin experiment on this spin 1/2 system, of which the state on the Bloch sphere is given by

$$\mathbf{w} = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta), \tag{6.4}$$

denoted by $|\psi_w\rangle$, and the experiment e_u corresponding to the spin experiment in the direction that corresponds to

$$\mathbf{u} = (\cos\gamma\sin\delta, \sin\gamma\sin\delta, \cos\delta) \tag{6.5}$$

on the Bloch sphere, is described respectively by the vector

$$|\psi_w\rangle = (\mathrm{e}^{-\mathrm{i}\phi/2}\cos(\theta/2), \mathrm{e}^{\mathrm{i}\phi/2}\sin(\theta/2)) \tag{6.6}$$

and the self-adjoint operator

$$H_u = \frac{1}{2}\begin{pmatrix} \cos\delta & e^{-i\gamma}\sin\delta \\ e^{i\gamma}\sin\delta & -\cos\delta \end{pmatrix}. \tag{6.7}$$

When we denote by $P_u(+|\psi_w)$ the probability the outcome will be $+$ when we measure H_u if the system is prepared in the state $|\psi_w\rangle$, it is a matter of direct application of the Born rule to show that

$$P_u(+|\psi_w) = \cos^2\left(\frac{\theta}{2}\right) \text{ and } P_u(-|\psi_w) = \sin^2\left(\frac{\theta}{2}\right). \tag{6.8}$$

We have seen in Chapter 5 how to recover the probabilities of a quantum mechanical n-level system. We will now turn to this specific instance of a two level system, using the Bloch sphere representation. This has the advantage of aiding the visualization of the results when we later investigate what happens if we deviate from a uniform distribution of observer states.

[3]This representation is also known in polarimetry where the Bloch sphere is known as the Poincaré sphere and has found many useful applications ranging from 3D movies, radar, photography, stress analysis and so on. As Felix Bloch was just seven years old when Herni Poincaré died in 1912, it is perhaps more appropriate to call it the Poincaré sphere.

Optimal observation in the Bloch sphere

Always test your general reasoning against simple models

—J. S. Bell, *On the impossible Pilot Wave*, 1982

Say we have a pure state $|\psi_w\rangle$ representing a spin 1/2 particle that we want to investigate. We do so by placing a Stern-Gerlach apparatus in a fixed direction that we represent by a vector u. We denote the two of outcomes[4] for this experiment e_u by o_1^u and o_2^u. We could refer to both the state and the apparatus with respect to some arbitrary chosen spherical coordinate system. However, we are free to expand the state in any basis we like and it will be convenient to express the state with respect to the orthogonal vectors that correspond to the eigenstates of H_u. The eigenvectors of H_u are orthogonal in $\mathcal{H}_2$, but by mapping them to the Bloch sphere, we obtain a pair of antipodal points that we label $+u$ and $-u$. Then for state $|\psi_w\rangle$ (6.6) there exist ϕ and θ such that:

$$|\psi_w\rangle = \mathrm{e}^{-\mathrm{i}\phi/2}\cos\left(\frac{\theta}{2}\right)|+u\rangle + \mathrm{e}^{\mathrm{i}\phi/2}\sin\left(\frac{\theta}{2}\right)|-u\rangle, \tag{6.9}$$

and this state is represented on the Bloch sphere by (6.4). We assign a state to the apparatus:

$$|\psi_v\rangle = \mathrm{e}^{-\mathrm{i}\beta/2}\cos\left(\frac{\alpha}{2}\right)|+u\rangle + \mathrm{e}^{\mathrm{i}\beta/2}\sin\left(\frac{\alpha}{2}\right)|-u\rangle \tag{6.10}$$

which has a representation on the Bloch sphere as

$$\mathbf{v} = (\cos\beta\sin\alpha, \sin\beta\sin\alpha, \cos\alpha). \tag{6.11}$$

The situation is depicted in Fig. 6-1.

To pick an outcome for this particular pair $|\psi_w\rangle$ and $|\psi_v\rangle$, the optimal observer compares the likelihood ratios:

$$\frac{|\langle+u|\psi_w\rangle|}{|\langle+u|\psi_v\rangle|} \qquad \text{and} \qquad \frac{|\langle-u|\psi_w\rangle|}{|\langle-u|\psi_v\rangle|}. \tag{6.12}$$

If the first ratio is bigger than the second, the state of the system makes a transition to $+u$, and the outcome is o_1^u; if the second ratio is

[4] In the last section we labelled the outcomes of a spin experiment by + and −. In the following sections we will need to make explicit reference to which measurement an outcome refers, so we need the notation to reflect that.

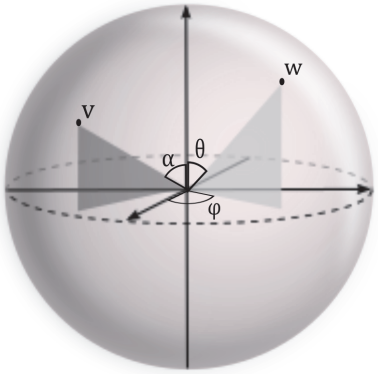

Figure 6-1 Representations of $|\psi_w\rangle$ and $|\psi_v\rangle$ as w and v on the Bloch sphere.

bigger, the system makes a transition to $-u$ and the outcome is o_2^u.[5] Inserting the values of the scalar products using (6.10) and (6.9) using the orthogonality relations $\langle +u|-u\rangle = \langle -u|+u\rangle = 0$ and $\langle +u|+u\rangle = \langle -u|-u\rangle = 1$, the outcome is o_1^u iff

$$\frac{|\cos\left(\frac{\theta}{2}\right)|}{|\cos\left(\frac{\alpha}{2}\right)|} > \frac{|\sin\left(\frac{\theta}{2}\right)|}{|\sin\left(\frac{\alpha}{2}\right)|},$$

and o_2^u otherwise. Multiplying each side by $|\cos\left(\frac{\alpha}{2}\right)\sin\left(\frac{\alpha}{2}\right)|$ and dividing by $|\cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\alpha}{2}\right)|$, gives the outcome o_1^u iff:

$$|\tan\left(\frac{\alpha}{2}\right)| > |\tan\left(\frac{\theta}{2}\right)|. \tag{6.13}$$

Because α and θ both take only values in $[0,\pi]$ on the Bloch sphere, $\alpha/2$ and $\theta/2$ take values in $[0,\pi/2]$. Since the tangent is strictly increasing on $[0,\pi/2]$, (6.13) holds iff $\alpha > \theta$. So the optimal observer in the Bloch sphere has a very simple decision mechanism to pick an outcome:

$$o_1^u \text{ iff } \alpha > \theta \text{ and } o_2^u \text{ iff } \alpha < \theta.$$

According to the main theorem expressed in Eq.(5.74) which we proved in Chapter 5, the criteria (6.13) recover the Born rule probabilities (6.8) when we calculate the probability of each outcome under the

[5]The demand that the system makes a state transition to one of the eigenstates of H_u is not necessary to obtain the probabilities related to single type of experiment. If we don't make any assumption about what happens to the state, then all we can infer from the probabilities refers only to the state of the system before the measurement. It is however the equivalent of the projection postulate in the orthodox treatment of spin in quantum theory. This allows us to makes inferences about the future behavior of the system as revealed by eventual consecutive measurements.

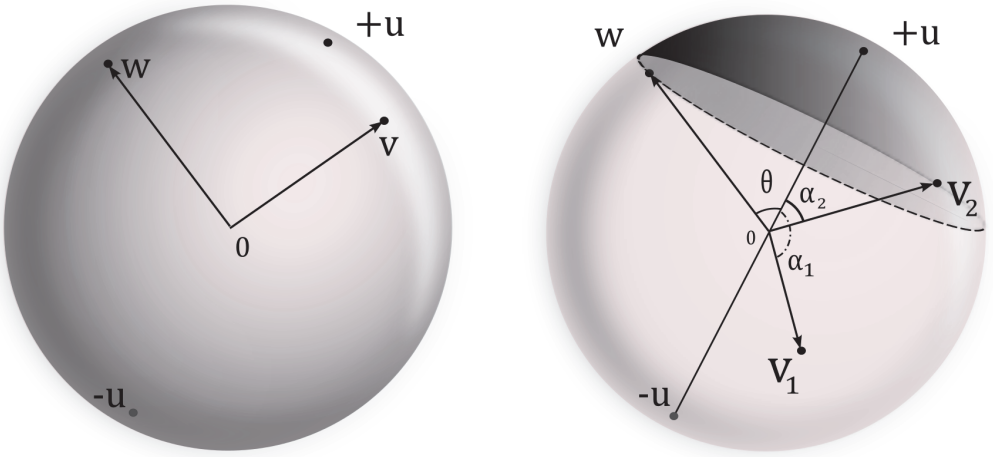

Figure 6-2 On the left we have depicted a possible situation prior to the measurement interaction. On the right a graphical representation of condition (6.13) is shown. The dark spherical cap is the set of Bloch vectors that satisfy $\alpha < \theta$, if the system state is w. A measurement with a Bloch vector v_2 will have $\alpha_2 < \theta$ and will produce outcome $-u$. Likewise, v_1 taken from the lighter spherical cap, has $\alpha_1 > \theta$ and will yield outcome $+u$.

assumption that the system state is the same in each trial and we perform repeated measurement with a uniform distribution of observers states[6]. We saw there that the probability for obtaining the outcome x if the system state is s, is calculated as

$$\Pr{}_s(x) = \mu(eig(x, \mathbf{s}))/\mu(\Sigma_M), \tag{6.14}$$

where $eig(x, s) \subset \Sigma_M$, is the subset of observer states that will give outcome x when presented with the state s for S. In our present situation, $x = o_1^u$, $s = w$ and $\Sigma_M = S$, so that we have:

$$p(o_1^u, w) = \frac{\mu(eig(o_1^u, w))}{\mu(S)}$$

The beauty of the Bloch sphere is that it is embedded in $\mathbb{R}^3$, so we can rely on simple geometry to calculate the probability. Looking at Fig. 6-2, we see that the set of states for the observer that satisfies the criterion of optimality (6.13) and that will give the outcome o_1^u, are the ones that lie in the spherical cap that surrounds $+u$; so $eig(o_1^u, w)$ is

[6] Actually, we have shown that this is true for the complex n-sphere, not for the Bloch sphere. It turns out that the bijection that maps the complex sphere to the Bloch sphere is measure preserving, so the principle works equally well on the Bloch sphere.

the light shaded area in Fig. 6-2.

$$eig(o_1^u, w) = \{v : v \cdot u < w \cdot u\} \tag{6.15}$$

By the assumed uniformity of observer states, the probability of obtaining outcome o_1^u is proportional to the area of this spherical cap. From elementary geometry we know that this area is $2\pi Rh$, with R the radius of the sphere and h the height of the cap. With $R = 1$, $h = 1 + \cos\theta$, and denoting μ for the Lebesgue measure, we have for the real unit sphere $\mu(S) = 4\pi$ and we find:

$$\begin{aligned} p(o_1^u, w) &= \frac{\mu(eig(o_1^u, w))}{\mu(S)} = \frac{2\pi(1+\cos\theta)}{4\pi} = \cos^2\left(\frac{\theta}{2}\right), \\ p(o_2^u, w) &= \frac{\mu(eig(o_2^u, w))}{\mu(S)} = \frac{2\pi(1-\cos\theta)}{4\pi} = \sin^2\left(\frac{\theta}{2}\right). \end{aligned} \tag{6.16}$$

Comparison with (6.8) shows the formulas give the transition probabilities of a Stern-Gerlach experiment on a spin 1/2 quantum particle.

Non uniform density of observer states

Under the assumption that we have minimal knowledge about the state of the probe, the set of observer states is uniform over the Bloch sphere and we recover the spin 1/2 probabilities. The most simple generalization, and we shall further on see this already leads to mathematically quite complex expressions, is to have a uniform density over a proper subset of the Bloch sphere and to set the density zero outside this subset. To make a reasonable choice for such a set, we go back to the measurement of the length of a stick as our guide. It is natural to assume that if the actual length of the stick is very close to a reading of the ruler, we judge it to have just that length. But if the length is close to the middle of two readings of the ruler, the state of the observer will play a role in deciding the outcome. A density that does just that is depicted in Fig. 6-3.

The density of the set of observer states cuts the Bloch sphere into three distinct pieces:

1. a spherical cap with $+u$ as its center, which we call $cap(o_1^u)$;
2. a spherical segment, which we call $sup(o_1^u)$; and
3. a spherical cap with $-u$ as center, which we call $cap(o_2^u)$.

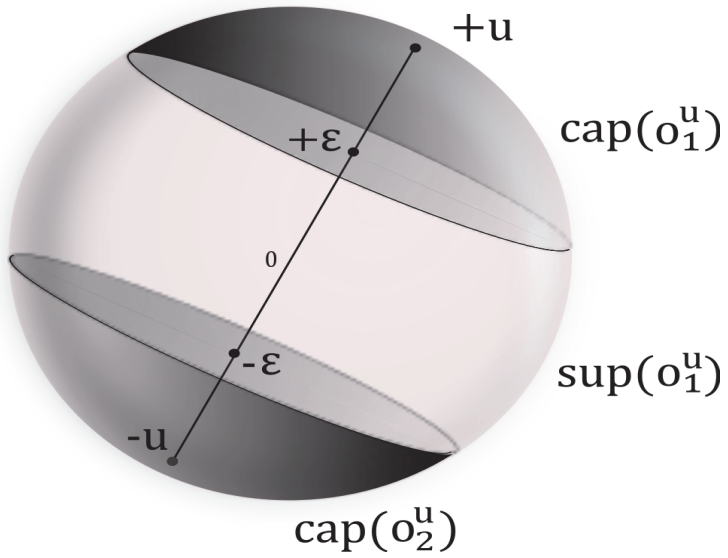

Figure 6-3 The dark shaded spherical caps are the sets in state space where the density of observer states is zero. This means that a system prepared in a state that is in one of these caps, will always yield the outcome that corresponds to the eigenvector of H_u that lies in the cap.

In $cap(o_1^u)$ the density of observer states is zero, so if the state w of the system we want to observe resides within $cap(o_1^u)$, every allowable state of the observer will satisfy $\alpha > \theta$, and hence for such a w the outcome *will always* be o_1^u . On the other side of the sphere, we have for the same reason that any $w \in cap(o_2^u)$ will *always* give as outcome o_2^u. For states w in the spherical segment $sup(o_1^u)$, the outcome depends on the condition (6.13). If we set the width of the segment equal to 2ϵ, its surface area equals $2\pi R 2\epsilon = 4\pi\epsilon$. The width of the sub-segment for which $\alpha > \theta$, equals $\epsilon + \cos(\alpha)$, and hence the surface area of sub-segment is $2\pi(\epsilon + \cos(\alpha))$. The probability of obtaining outcome o_1^u will then be $2\pi(\epsilon + \cos(\alpha))/(4\pi\epsilon) = (\epsilon + \cos(\alpha))/(2\epsilon)$. The probability of obtaining outcome o_2^u will of course be $(\epsilon - \cos(\alpha))/(2\epsilon)$.

Let us summarize. There are three different cases, depending on whether w is in $cap(o_1^u)$, in $sup(o_1^u)$ or in $cap(o_2^u)$.

1. $w \in cap(o_1^u)$. This is the case if $w \cdot u \leq \epsilon$. We have $p_\epsilon(o_1^u, w) = 0$ and $p_\epsilon(o_2^u, w) = 1$.

2. $w \in sup(o_1^u)$. This is the case if $-\epsilon < w \cdot u < \epsilon$. The transition probabilities are then:

$$p_\epsilon(o_1^u, w) = \frac{1}{2\epsilon}(w \cdot u + \epsilon) \text{ and } p_\epsilon(o_2^u, w) = \frac{1}{2\epsilon}(\epsilon - w \cdot u). \quad (6.17)$$

3. $w \in cap(o_2^u)$ This is the case if $\epsilon \leq w \cdot u$. We have $p_\epsilon(o_1^u, w) = 1$ and $p_\epsilon(o_2^u, w) = 0$.

This model replicates the same transition probabilities as Diederik Aerts' epsilon model which is why we have chosen to parametrize the density of observer states in our model with ϵ so as to obtain an exact replica of the results. It was in fact heavily inspired by the former model and the only difference between the two models is the decision mechanism of the outcome production. The epsilon model is so simple, it can be explained in a few lines. It is usually presented as a toy model in the form of a sphere equipped with a special rubber band that connects (in a straight line and through the interior of the sphere) two antipodal points on the sphere. The centered middle portion of length 2ϵ is breakable and will do so with uniform probability over that interval, the rest of the rubber band is unbreakable. When measured, the particle falls orthogonally onto the rubber band, it breaks and the particle gets dragged to one of the endpoints. As outcome we give the point to where the particle was dragged. If we calculate the probabilities, we arrive at (6.17). One of the attractive features of the epsilon model, is that it shows how one can build simple mechanistic models with quite unexpected behavior. A similar, but unfortunately much more complex construction was made for measurements with three outcomes in [18], but it was not obvious at the time how to scale such toy models to higher dimensions. In later publications by Aerts D. and Sassoli di Bianchi M. ([26] and [27]), it was shown how to extend the results using the so-called extended Bloch sphere representation. In contradistinction, the optimal observer works without *any* form of modification in arbitrary finite dimensional Hilbert spaces, both real and complex. The hidden variable of the measurement is no longer an element of an interval, but a vector in a Hilbert space of the same dimension as the system, which explains why it so easily scales with dimension. The two dimensional epsilon model was studied intensively in the nineteen-nineties by the Brussels group under varying degrees of lack of knowledge, parameterizing this variation by a number $\epsilon \in [0, 1]$ such that $\epsilon = 1$ corresponds to the situation of maximal lack of knowledge, giving rise to a quantum structure and $\epsilon = 0$ corresponds to the situation of zero lack of knowledge, generating a classical structure. Other values of ϵ correspond to intermediate situations, giving rise to a structure that is neither quantum nor classical, ([12], [17], [16]). The model was studied probabilistically, with respect to its lattice-theoretic structure and with respect to its algebra of observables ([12], [19], [84], [91]). Interestingly, the notions of "classical" in these different approaches do not always coincide and may occur for different values of epsilon and $w \cdot u$. In particular, there was no value for ϵ which could produce a classical probabilistic limit. To produce a classical probabilistic limit, we must turn to families of conditional probabilities.

The conditional probability for incompatible measurements

The conditional probability is often introduced by means of Bayes' theorem. Let us give a standard definition[7].

Definition 51 *Let A and B be two random variables with possible values $\{a_1, ..., a_n\}$ and $\{b_1, ..., b_m\}$, respectively. The conditional probability $P(A = a_k \,|B = b_l)$ is then defined as:*

$$P(A = a_k|B = b_l) = \frac{P(A = a_k \text{ and } B = b_l)}{P(B = b_l)}. \tag{6.18}$$

The meaning of $P(A = a_k$ and $B = b_l)$ is of interest to us. It denotes the joint probability, which is the probability that A has the value a_k and B has the value b_k. The use of Bayes' theorem only makes sense if the joint probability exists, which is a problem for incompatible observables in quantum theory because the observables cannot be assumed, nor tested, to take their values simultaneously. This problem is sometimes referred to as the non-existence of a joint-probability distribution. ([197], see however [75], [74]) Still it is obvious that the preparation of a system in a state can be regarded as an act of conditioning. To define a more general conditional probability, we once more rely on Piron's notion of product question (see Chapter 2) and condition on "an element of reality". Following [14], we propose the following definition of conditional probability that is operational both in the quantum and in the classical regime and does not rely on the existence of a joint probability.

Definition 52 *We consider a system S and two (possibly incompatible) experiments A and B for S with possible outcome sets X_A and X_B. Let $a \in X_A$ and $b \in X_B$. The conditional probability $P(A \to a \,|B \to b)$ is the probability that a measurement of A will give the result a when we know that if we had chosen to measure the observable B, we would have found the result b with certainty.*

We have used the notation $A \to a$ instead of $A = a$ to stress that A does not need to *have* value a; it may acquire value a when measured. If A and B are compatible, then measuring B does not change the probabilities related to the outcomes X_A. If A and B are incompatible this definition still makes sense, as we are conditioning on something

[7] See Kolmogorov [140] where the Bayes formula is the very first consequence he derives from his five axioms.

that we (counterfactually) know to be true, even if we haven't chosen to measure it. As explained in Chapter 2, we can only give an operational meaning to this conditioning when we are able to prepare states of S for which we know that an eventual measurement of B will yield outcome b. Note that the transition probability of quantum theory from ψ_i to ψ_j (i.e. $|\langle\psi_i|\psi_j\rangle|^2$) is a special case of this conditional probability. If on the other hand, the outcomes of the experiments A and B were pre-existent and hence only revealed by the experiments, the definition amounts to Bayes' rule. In Kolmogorovian probability theory there is no place for a distinction between what is, and what becomes true as a result of observing, so we either end up with Bayes' rule, or with a non existing joint probability. To calculate this probability for the model, we use the general formula Eq.(5.6): $\Pr_\rho(x) = \rho(\tau^{-1}(x))$. The choice of the product measure ρ corresponds to picking a particular of measure for S and one for M. Conditioning then is a special case, in which one can use a measure on $\mathcal{B}(\Sigma_S)$ such that experiment B will give outcome b. In fact one can regard conditioning as a change of measure or generalize this definition to include the situation where we are interested in the probability of the occurrence of an outcome in an arbitrary subset of X_A when we know the outcome for B has to be in a subset of X_B. For more details see [30], [14]. We can apply this operational definition to our model and calculate the conditional probability with a non-uniform distribution of observer states to analyze the resulting probabilistic structures. We shall see how the conditional probability on the ϵ-model evolves continuously from the quantum transition probability (for $\epsilon = 1$), to a classical Kolmogorovian probability satisfying Bayes' formula (for $\epsilon = 0$).

The conditional probability and the ϵ-model.

For a fixed value of ϵ, we are given the two experiments e^ϵ_u and e^ϵ_w, each one with two possible outcomes (o^u_1 and o^u_2 for e^ϵ_u and o^w_1 and o^w_2 for e^ϵ_w), which we think of as spin 1/2 measurements in the directions u and w on the Bloch sphere. The conditional probability we propose to calculate is the probability that the experiment e^ϵ_u gives the outcome o^u_1, if the system S is conditioned in such a way that if we were to perform the experiment e^ϵ_w, the outcome o^w_1 would be obtained with certainty. This conditioning means that the state of the system is such that its

representative vector w is somewhere in the spherical cap $cap(o_1^w)$.[8] We will assume a *uniform distribution* of states within this cap. An example of such a cap can be seen the grey area in Fig. 6-3, only then for w rather than u. The conditional probability (52) that the experiment e_u^ϵ gives the outcome o_1^u (respectively o_2^u), when the system is conditioned for the outcome o_1^w of the experiment e_w^ϵ, is then written as $P(e_u^\epsilon \rightarrow o_1^u | e_w^\epsilon \rightarrow o_1^w)$. In a similar way we should also denote the conditional probability that the experiment e_u^ϵ gives the outcome o_1^u (respectively o_2^u) if the system is conditioned on the outcome o_2^w of the experiment e_w^ϵ by $P(e_u^\epsilon \rightarrow o_2^u | e_w^\epsilon \rightarrow o_1^w)$. However, this is notational a too heavy burden. Anticipating that the probability depends only on the parameter ϵ and the angle between u and w (they are both unit norm and only their scalar product occurs in 6.17), we abbreviate them as:

$$\begin{aligned} P^\epsilon_{\rm cond}(\alpha) &= P(e_u^\epsilon \rightarrow o_1^u | e_w^\epsilon \rightarrow o_1^w), \\ P^\epsilon_{\rm cond}(\pi - \alpha) &= P(e_u^\epsilon \rightarrow o_2^u | e_w^\epsilon \rightarrow o_1^w). \end{aligned} \tag{6.19}$$

This last equation is more than just an abbreviation. Inspection of the meaning of the right hand side will convince you of its correctness. Obviously we have

$$P^\epsilon_{\rm cond}(\alpha) + P^\epsilon_{\rm cond}(\pi - \alpha) = 1, \tag{6.20}$$

which we will need later on. The calculation of the conditional probability (52) becomes a surface integral with the transition probability as the integrand, but now with w as a function of the infinitesimal surface element. The calculation of the integral is complicated by the fact that one has to use a different integrand for different regions. To make this clear in the notation, let us introduce the following abbreviations (*cap* and *sup* are defined on p. 162):

$$\begin{aligned} \Omega_0 &= cap(o_2^u) \cap cap(o_1^w) \\ \Omega_s &= sup(o_1^u) \cap cap(o_1^w) \\ \Omega_1 &= cap(o_1^u) \cap cap(o_1^w) \end{aligned} \tag{6.21}$$

We note that these sets form a partition of $cap(o_1^w) = \Omega_0 \cup \Omega_s \cup \Omega_1$. Let us look at these sets and what they mean.

1. One can easily see in Fig.6-4 that Ω_0 is not empty iff $\epsilon < \sin(\alpha/2)$. If the state of the system is in the region $\Omega_0 = cap(o_1^w) \cap cap(o_2^u)$,

[8] Although we make no notational reference to it, the sets *cap* and sup obviously depend on the parameter ϵ. In [30] this set is called *eig*, but this would introduce unnecessary confusion with the former chapter were another definition of *eig* was used.

it will *never* give outcome o_1^u. This means that for this region the integrand is 0.

2. Likewise we find that Ω_1 is not empty iff $\epsilon > \cos(\alpha/2)$. As can be seen from Figs.6-4 and 6-5, the region Ω_1, where $cap(o_1^w)$ and $cap(o_1^u)$ intersect, contributes in a deterministic way to the probability in the sense that if the state of the system is in Ω_1 it will *always* give rise to the outcome o_1^u. This means that for this first region the integrand becomes 1.

3. As can be seen from Fig. 6-4, the set Ω_s will only be empty if $\epsilon = 0$. If the state of the system is in Ω_s it will produce outcome o_1^u or o_2^u depending on condition (6.13). In the region Ω_s the integrand is hence given by the transition probabilities (6.17).

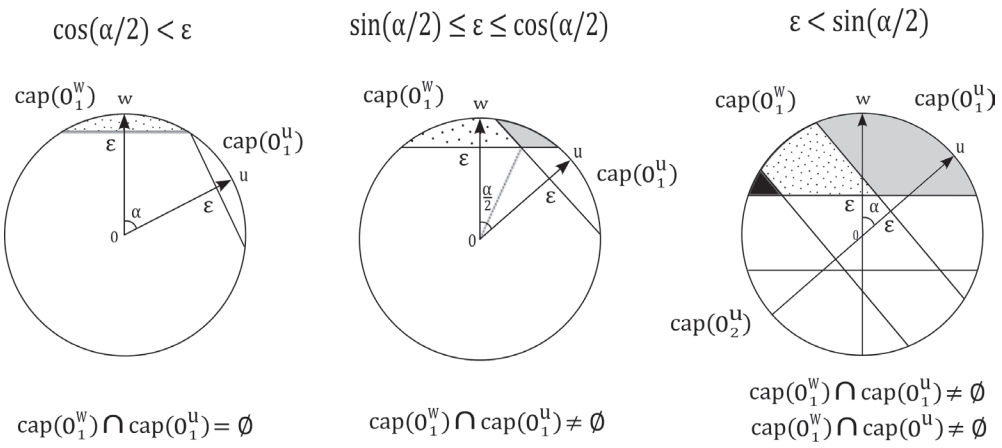

Figure 6-4 Depending on the relation between ϵ en α, there are three distinct situations to consider. The dotted areas represent Ω_s, the grey areas Ω_1 and the black area Ω_0. In the first situation, the whole spherical cap $cap(o_1^w)$ is equal to Ω_s. In the second situation $cap(o_1^w)$ it is partitioned in Ω_s and Ω_1. In the third situation $cap(o_1^w)$ consists of Ω_0, Ω_s and Ω_1.

For any $A \subset S$, we denote by $\mu(A) = \iint_A dA$ the Lebesgue measure or surface area. The conditional probability is then given by:

$$P^{\epsilon}_{\text{cond}}(\alpha) = \frac{1}{\mu(cap(o_1^w))}\left[\iint_{\Omega_0} 0\, d\Omega_0 + \iint_{\Omega_s} \frac{\mathbf{u}\cdot\mathbf{w}+\epsilon}{2\epsilon} d\Omega_s + \iint_{\Omega_1} 1\, d\Omega_1\right] \quad (6.22)$$

We remind the reader we use bold typesetting to indicate vectors (such as $\mathbf{u}$); it is crucial to make a notational difference between vectors

and scalars in the integration. Putting this together we find that the quantity we need to calculate, is:

$$P^{\epsilon}_{\text{cond}}(\alpha) = \frac{1}{4\pi(1-\epsilon)}(2\mu(\Omega_1) + \mu(\Omega_s) + \epsilon^{-1}\iint_{\Omega_s} \mathbf{u}\cdot\mathbf{w}\, d\Omega_s). \qquad (6.23)$$

Observing that $\mathbf{w}$ is perpendicular to the surface of the sphere and has unit norm we regard it as a surface normal:

$$\mathbf{w}\, d\Omega_s = d\mathbf{\Omega_s}. \qquad (6.24)$$

By use of Gauss' theorem, we may re-write the integral to contain only surfaces that are related to the sphere, which eliminates the difficult problem of integrating the scalar product. To apply Gauss' theorem we need to close Ω_s by arbitrary surfaces, preferably surfaces with an area that is easy to calculate.

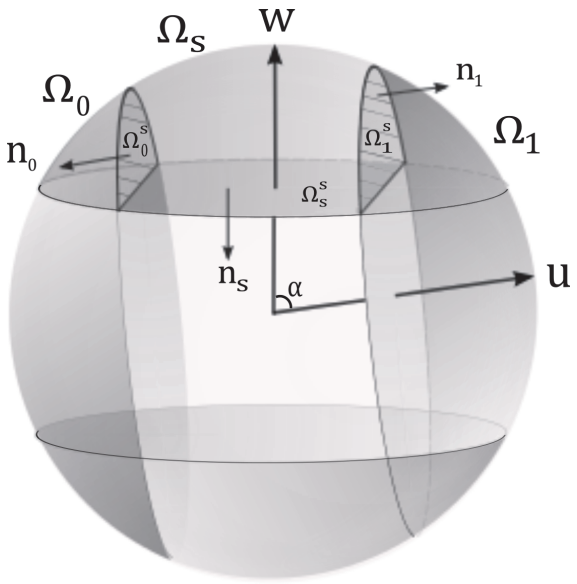

Figure 6-5 The spherical cap $cap(o_1^w)$ is divided into three sets Ω_0, Ω_s and Ω_1 depending on which integrandum we have to use. In the interior of the closure of $cap(o_1^w)$ we find three circlesegments: Ω_0^s, Ω_s^s and Ω_1^s that, together with Ω_s, they enclose a volume to which we can apply Gauss' theorem. For this purpose, we have drawn the normals to Ω_0^s, Ω_s^s and Ω_1^s. The picture is drawn for $\epsilon = 1/2$ and α close to $\pi/2$, for which these sets are non-empty. For other values, Ω_0 and Ω_0^s or even Ω_1 and Ω_1^s may be empty.

If we take three segments of disks that close the superposition zone, as shown in Fig. 6-5, and call these surfaces Ω_0^s, Ω_s^s and Ω_1^s, we obtain a closed surface as follows:

$$\Omega_{\text{closed}} = \Omega_s \cup \Omega_0^s \cup \Omega_1^s \cup \Omega_s^s. \qquad (6.25)$$

We see that Ω_{closed} is indeed compact with piecewise smooth boundaries, as is required for Gauss' theorem. If we denote by V the set of points that is enclosed by Ω_{closed}, and remark that $\mathbf{u}$ is continuously differentiable in V, we may apply Gauss' theorem:

$$\int\int_{\Omega_{\text{closed}}} \mathbf{u} \cdot d\,\mathbf{\Omega}_{\text{closed}} = \iiint_V \boldsymbol{\nabla} \cdot \mathbf{u} dV \tag{6.26}$$

As $\mathbf{u}$ is a constant vector, $\boldsymbol{\nabla}\cdot\ \mathbf{u} = 0$. Therefore we have:

$$\iint_{\Omega_s} \mathbf{u} \cdot d\mathbf{\Omega_s} + \iint_{\Omega_o^s} \mathbf{u} \cdot d\mathbf{\Omega}_0^s + \iint_{\Omega_1^s} \mathbf{u} \cdot d\mathbf{\Omega}_1^s + \iint_{\Omega_s^s} \mathbf{u} \cdot d\mathbf{\Omega}_s^s = 0. \tag{6.27}$$

With the $\mathbf{n}_0$, $\mathbf{n}_s$ and $\mathbf{n}_1$ the normals to Ω_0^s, Ω_1^s and Ω_s^s, respectively, we can write $d\mathbf{\Omega}_0^s = \mathbf{n}_0 d\Omega_0^s$, $d\mathbf{\Omega}_1^s = \mathbf{n}_1 d\Omega_1^s$ and $d\mathbf{\Omega}_s^s = \mathbf{n}_s d\Omega_s^s$. Hence we have

$$\iint_{\Omega_s} \mathbf{u} \cdot d\mathbf{\Omega_s} = -\iint_{\Omega_o^s} \mathbf{u} \cdot \mathbf{n}_0 d\Omega_0^s - \iint_{\Omega_1^s} \mathbf{u} \cdot \mathbf{n}_1 d\Omega_1^s - \iint_{\Omega_s^s} \mathbf{u} \cdot \mathbf{n}_s d\Omega_s^s$$

If we call α the angle between the two vectors $\mathbf{u}$ and $\mathbf{w}$, it is easy to see that $\mathbf{u} \cdot \mathbf{n}_0 = -1$, $\mathbf{u} \cdot \mathbf{n}_1 = 1$ and $\mathbf{u} \cdot \mathbf{n}_s = -\cos\alpha$.We obtain:

$$\iint_{\Omega_s} \mathbf{u} \cdot d\mathbf{\Omega_s} = \mu(\Omega_0^s) - \mu(\Omega_1^s) - \cos(\alpha)\mu(\Omega_s^s). \tag{6.28}$$

The probability can now be expressed in terms of areas that relate to surfaces on and in the sphere. We rewrite (6.23) as:

$$P_{\text{cond}}^{\epsilon}(\alpha) = \frac{1}{4\pi(1-\epsilon)}(2\mu(\Omega_1)+\mu(\Omega_s)+\epsilon^{-1}(\mu(\Omega_0^s)-\mu(\Omega_1^s)-\cos(\alpha)\mu(\Omega_s^s))) \tag{6.29}$$

Using

$$\mu(\Omega_1) + \mu(\Omega_s) + \mu(\Omega_0) = 2\pi(1-\epsilon), \tag{6.30}$$

we can write (6.29) more symmetrically as:

$$P_{\text{cond}}^{\epsilon}(\alpha) = \frac{1}{2}+\frac{1}{4\pi(1-\epsilon)}(\mu(\Omega_1)-\mu(\Omega_0)+\epsilon^{-1}(\mu(\Omega_1^s)-\mu(\Omega_0^s)+\cos(\alpha)\mu(\Omega_s^s)) \tag{6.31}$$

Thus we have managed to transform the complex integrations (6.22) to the calculation of the areas of the different Ω which can be accomplished by direct integration and the use of spherical trigonometry[9].

[9]The calculation is rather long but straightforward. For details we refer to [29].

The result can conveniently be written using the Heaviside function[10] $H(x)$ as:

$$\begin{aligned} P^{\epsilon}_{\rm cond}(\alpha) &= p^{\epsilon}_1(\alpha)\cdot H(\epsilon-\cos(\frac{\alpha}{2})) \\ &\quad +H(\epsilon-\sin(\frac{\alpha}{2}))\cdot p^{\epsilon}_2(\alpha)\cdot H(\cos(\frac{\alpha}{2})-\epsilon) \\ &\quad +p^{\epsilon}_3(\alpha)\cdot H(\sin(\frac{\alpha}{2})-\epsilon), \end{aligned} \tag{6.32}$$

where

$$p^{\epsilon}_1(\alpha)=\frac{1}{2}+\frac{\cos(\alpha)(1+\epsilon)}{4\epsilon} \tag{6.33}$$

$$p^{\epsilon}_2(\alpha)=\frac{1}{2}+p^{\epsilon}_1(\alpha)+\frac{\omega^{\epsilon}(\alpha)+\cos^2(\alpha/2)\sigma^{\epsilon}(\alpha)}{2\pi(1-\epsilon)} \tag{6.34}$$

$$p^{\epsilon}_3(\alpha)=p^{\epsilon}_1(\alpha)+\frac{\omega^{\epsilon}(\alpha)-\omega^{\epsilon}(\pi-\alpha)+\sin^2(\frac{\alpha}{2})\cdot\sigma^{\epsilon}(\pi-\alpha)+\cos^2(\frac{\alpha}{2})\cdot\sigma^{\epsilon}(\alpha)}{2\pi(1-\epsilon)}, \tag{6.35}$$

with

$$\omega^{\epsilon}(\alpha)=2\epsilon\arccos\sqrt{\frac{1-(\frac{\epsilon}{\cos(\alpha/2)})^2}{1-\epsilon^2}}-2\arcsin\frac{\sin(\alpha/2)}{\sqrt{1-\epsilon^2}}, \tag{6.36}$$

and

$$\sigma^{\epsilon}(\alpha)=\tan(\frac{\alpha}{2})\sqrt{1-(\frac{\epsilon}{\cos(\alpha/2)})^2}-(\frac{1-\epsilon^2}{\epsilon})\arccos(\frac{\epsilon\cdot\tan(\alpha/2)}{\sqrt{1-\epsilon^2}}). \tag{6.37}$$

For the calculations we have restricted α to $[0,\pi/2]$, which greatly facilitates the manipulation of the inverse trigonometric functions in the calculations. It is however easy to see from Eq.(6.20) that for $\alpha\in[\pi/2,\pi]$, the probability is given by $1-P^{\epsilon}_{\rm cond}(\pi-\alpha)$.

[10] The Heaviside function can be introduced in many ways. For our purposes, the most simple definition will do: $H(x)=0$ iff $x<0$ and $H(x)=1$ iff $x\geq 0$.

A substantial simplification which we haven't reported on before, is to extend the domain of $P^{\epsilon}_{\rm cond}(\alpha)$ by analytic continuation in the complex plane. The intersections regions of the spherical caps will become complex and are filtered out taking the real part of $p^{\epsilon}_3(\alpha)$; this contains $p^{\epsilon}_2(\alpha)$ and $p^{\epsilon}_1(\alpha)$ as special cases. With the following definitions $\epsilon' = \sqrt{1-\epsilon^2}$ and $\delta^2_\epsilon(\alpha) = 1 - \epsilon^2/\cos^2(\alpha/2)$, we can rewrite $\omega^\epsilon(\alpha)$ and $\sigma^\epsilon(\alpha)$:

$$\omega^\epsilon(\alpha) = 4\epsilon \arccos\frac{\delta_\epsilon(\alpha)}{\epsilon'} - 4\arcsin\frac{\sin(\alpha/2)}{\epsilon'} \tag{6.38}$$

$$\sigma^\epsilon(\alpha) = \tan(\alpha/2)\delta_\epsilon(\alpha) - \epsilon'^2\epsilon^{-1}\arccos\left((\epsilon/\epsilon')\tan(\alpha/2)\right) \tag{6.39}$$

We further define

$$\begin{aligned}
\Delta\omega^\epsilon(\alpha) &= \omega^\epsilon(\alpha) - \omega^\epsilon(\pi-\alpha),\\
\Delta\sigma^\epsilon(\alpha) &= \sigma^\epsilon(\alpha) - \sigma^\epsilon(\pi-\alpha),\\
\Sigma\sigma^\epsilon(\alpha) &= \sigma^\epsilon(\alpha) + \sigma^\epsilon(\pi-\alpha).
\end{aligned}$$

With Re the real part of a complex number, a simple calculation shows we can write the conditional probability in the following form which holds for $\alpha \in [0,\pi]$:

$$P^{\epsilon}_{\rm cond}(\alpha) = \frac{1}{2} + {\rm Re}\,\left(\frac{\Delta\omega^\epsilon(\alpha) + \Delta\sigma^\epsilon(\alpha) + (\Sigma\sigma^\epsilon(\alpha) + \pi\epsilon'^2\epsilon^{-1})\cos(\alpha)}{4\pi(1-\epsilon)}\right) \tag{6.40}$$

We note that there is no necessity to use complex numbers here as (6.32) shows; it merely allows to discard the three Heaviside functions to yield a much more concise way to write $P^{\epsilon}_{\rm cond}(\alpha)$. We have presented a graph of this function in Fig. 6-6.

To interpret the graph we first consider the two extreme cases: $\epsilon = 0$, and $\epsilon = 1$. It is evident that (6.40) is continuous (and differentiable) for $(\epsilon,\alpha) \in \,]0,1[\times]0,\pi[$ and continuously extendable to $[0,1]\times[0,\pi]$. Therefore the limits are well-defined and can be calculated analytically from (6.32). We will follow a much less rigorous, but shorter and more instructive path, by starting from (6.31) and see what happens to the areas involved.

The Bayesian limit

In this case, we have to take the limit $\epsilon \to 0$. Let us first look what $\lim_{\epsilon\to 0} P^{\epsilon}_{\rm cond}(\alpha)$ means in terms of the model. If $\epsilon \to 0$, we see in Fig. 6-5 that the middle zone Ω_s disappears. We then obtain a situation such as depicted in Fig. 6-7.

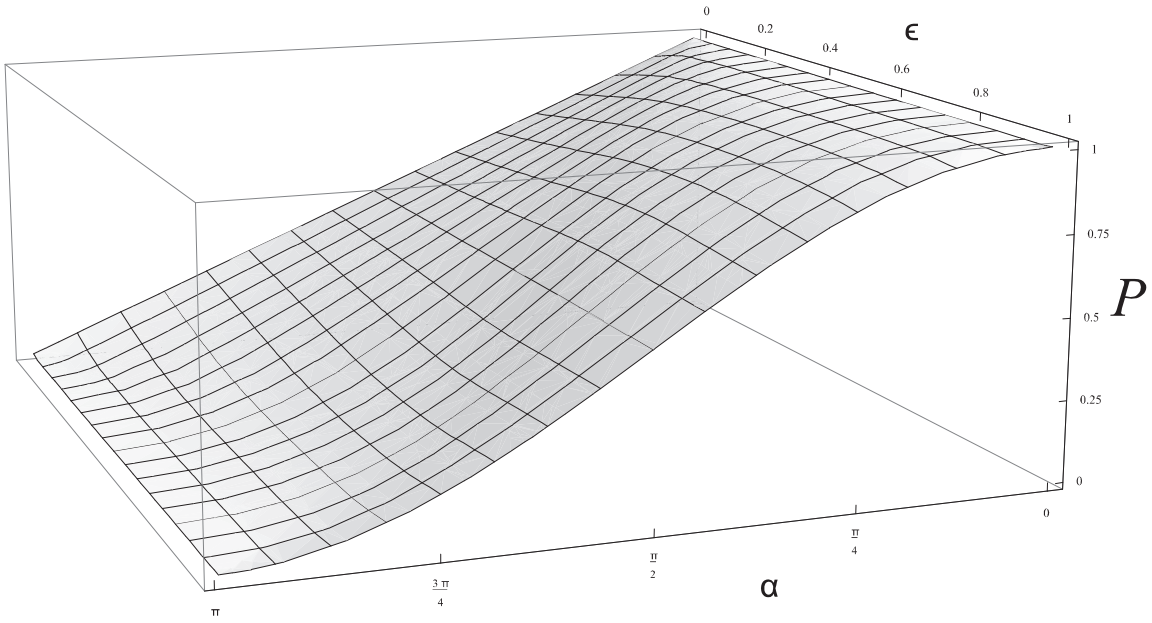

Figure 6-6 A plot of the conditional probability. For $\epsilon = 1$ we recognize the $\cos^2(\alpha/2)$ shape, while for $\epsilon = 0$ we have a straight line representing Bayes rule.

This has two consequences: (i) the fact that the state is conditioned on o_1^w, means that the system state has to be in the hemisphere that contains w; (ii) the only states of the observer that are allowed are those that lie on the equator when o_1^u and o_2^u are the poles. So a measurement simply decides if the system was in the upper hemisphere (with respect to the direction $+u$) or in the lower hemisphere. It is evident that the probability of getting the result o_1^u is then proportional to the overlap of the hemispheres $cap(o_1^u)$ and $cap(o_1^w)$. We should be able to calculate this by direct application of Bayes. Let's see whether we can get the same result starting from (6.31). We have $\lim_{\epsilon\to 0}\mu(\Omega_s^s) = \lim_{\epsilon\to 0}\mu(\Omega_s) = 0$ and $\lim_{\epsilon\to 0}\mu(\Omega_0^s) = \lim_{\epsilon\to 0}\mu(\Omega_1^s)$. We also have $\mu(\Omega_1) + \mu(\Omega_s) + \mu(\Omega_0) = 2\pi(1-\epsilon)$. Substitution of these into (6.31) gives:

$$\begin{aligned}
\lim_{\epsilon\to 0} P^{\epsilon}_{\rm cond}(\alpha) &= \frac{1}{2} + \lim_{\epsilon\to 0}\frac{1}{4\pi(1-\epsilon)}[\mu(\Omega_1) - \mu(\Omega_0)] \\
&= \frac{1}{2} + \lim_{\epsilon\to 0}\frac{1}{4\pi(1-\epsilon)}[\mu(\Omega_1) + \mu(\Omega_1) - 2\pi(1-\epsilon)] \\
&= \lim_{\epsilon\to 0}\frac{1}{4\pi(1-\epsilon)}[2\mu(\Omega_1)] \\
&= \frac{2\mu(\Omega_1)}{2\mu(cap(o_1^w))} \\
&= \frac{\mu(cap(o_1^u)\cap cap(o_1^w))}{\mu(cap(o_1^w))}.
\end{aligned}$$

This last formula is recognizable as Bayes' formula (6.18) in definition

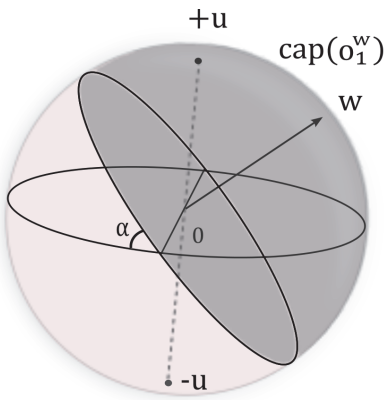

Figure 6-7 In the limit $\epsilon \to 0$, the conditioning on o_1^w, means the state of the system is in the dark hemisphere. If we next apply a measurement along u, then the outcome will be o_1^u if the state was in the upper hemisphere. We then see that the conditional probability is $(\pi - \alpha)/\pi$.

(51) on p. 165. In Fig. 6-7, we see that the angle α between u and w is also the angle between the equator planes (as each one stands perpendicular to the vectors u and w). We then obtain by application of Bayes (6.18):

$$\lim_{\epsilon \to 0} P^{\epsilon}_{\text{cond}}(\alpha) = 1 - \frac{\alpha}{\pi}, \tag{6.41}$$

which is linear function of the angle.

The quantum limit

For this case we take the limit $\epsilon \to 1$. We then have $\lim_{\epsilon \to 1} \mu(\Omega_1) = \lim_{\epsilon \to 1} \mu(\Omega_0) = \lim_{\epsilon \to 1} \mu(\Omega_0^s) = \lim_{\epsilon \to 1} \mu(\Omega_1^s) = 0$. Substitution into (6.31) gives:

$$\begin{aligned} \lim_{\epsilon \to 1} P^{\epsilon}_{\text{cond}}(\alpha) &= \frac{1}{2} + \lim_{\epsilon \to 1} \frac{1}{4\pi(1-\epsilon)} [\cos(\alpha) \mu(\Omega_s^s)] \\ &= \frac{1}{2} + \lim_{\epsilon \to 1} \frac{1}{4\pi(1-\epsilon)} [\cos(\alpha) 2\pi(1-\epsilon)] \\ &= \cos^2(\alpha/2) \end{aligned}$$

A more rigorous approach here is quite simple: we only have to take into account the contribution of p_1 in Eq.(6.32) because the Heaviside functions that are multiplied with p_2 and p_3 are zero and hence we can directly calculate that:

$$P^{\epsilon}_{\text{cond}}(\alpha) = \frac{1}{2} + \lim_{\epsilon \to 1} \frac{\cos(\alpha)(1+\epsilon)}{4\epsilon} = \cos^2(\alpha/2) \tag{6.42}$$

which is the well known quantum transition probability. Now that we have identified the two extreme cases, we can interpret the graph in

Fig. 6-6 and see that the conditional probability $P^{\epsilon}_{\text{cond}}(\alpha)$ evolves continuously from the quantum transition probability between the states u and w to a linear function of the angle between the two vectors u and w which satisfies Bayes'-axiom.

The intermediate case

It is interesting to see that for a substantial interval I of values ϵ close to 1, the graph of $P^{\epsilon}_{\text{cond}}(\alpha)$ becomes stronger curved than the quantum limiting case. This can clearly be seen in Fig. 6-6. Without presenting a proof, we have

$$|\frac{d^2 P^{\epsilon\in I}_{\text{cond}}(\alpha)}{d\alpha^2}| \geqslant ||\frac{d^2 \cos^2(\alpha/2)}{d\alpha^2}|| = \frac{1}{2}|\cos(\alpha)|$$

with equality holding only for $\alpha = \frac{\pi}{2}$ in which case the curvature of both graphs is zero. It was shown in [30] that for this region the set of conditional probabilities cannot be reproduced in a Hilbert space framework, nor in a Kolmorogovian probability space, demonstrating the vast array of probabilities this model can reproduce.

Chapter 7

Consequences of optimal observation

> *For it was my master who taught me not only how very little I knew but also that any wisdom to which I might ever aspire could consist only in realizing more fully the infinity of my ignorance*

—Karl Popper in *Unended Quest: An Intellectual Autobiography*, 1974

The last few chapters have mainly been used to derive mathematical results. In the remainder, we briefly sketch further avenues worthy of investigation that have more of an interpretative character.

Undecidability for the optimal observer

We have argued in Chapter 3 a rudimentary kind of measurement problem occurs when we allow a universal theory to describe its own means of verification. Quantum theory, from this perspective, escapes the unsolvable problem of how to attribute consistent values to all observables, by not giving any physical mechanism for the production of an outcome. However, if this analysis is correct, then there have to exist situations where our model is not able to produce definite outcomes

either. It is indeed the case that there is a form of undecidability in the scheme (and one that has increased the complexity of our proof building). The problem can be identified most easily using Eq.(5.24):

$$x_j = \arg\max_{x_i} \frac{\Pr{}_s(x_i)}{\Pr{}_m(x_i)}, \tag{7.1}$$

which in turn gives rise to (5.37), but (5.64) suffers exactly the same problem. We see that, if there exist i and j such that $\Pr{}_s(x_i)/\Pr{}_m(x_i) = \Pr{}_s(x_j)/\Pr{}_m(x_j)$, then there is no unique maximum and hence no recipe for deciding whether outcome x_i or x_j should be produced. This lack of a criterion to choose an outcome occurs if m lies in the boundary of the eigensets, as was shown and discussed when we arrived at (5.45). Luckily the Lebesgue measure of the boundary is $\mu-$negligible; apart from pathological distributions where the support of the measure is a subset of the boundary, this is of no consequence for the derived probability. This is the way we have chosen to treat this issue; by showing how a regular measure can be defined on the open eigenset and this is sufficient to determine the probability. One can imagine treating the problem by arbitrarily assigning outcomes for states of M that lie on the boundary of the eigenset, but there seems to be no rationale for choosing an outcome. The worst situation occurs if all likelihood ratio's are equal; the state of the system and that of the observer coincide and any particular outcome assignment flouts the inherent symmetry of this situation.

The optimal observer as a paradigm for observation

The proposed principle of observation is based on a maximum likelihood criterion applied to a binary decision problem. Even so, we do not use it in its usual decision-theoretic form. In decision theory we seek to establish which of the hypotheses enjoys the strongest support in evidence of the data. In our case, there is no data to feed the likelihood with, because we produce the data by means of the odds. The reason for this reversal is that we are not seeking how to interpret the data from observation, but we are looking for how to produce data. So the way we employ the principle is rather like an inverse decision problem, as if anticipating that the result will be judged afterwards by a decision procedure performed by one with absolute knowledge of the system and observer states prior to the measurement. This makes observation effectively a form of inference. The roots of this idea go

at least as far back as Ptolemy. Although Euclid wrote the apparent size of an object is determined by the visual angle subtended at the eye, Ptolemy argues that two objects with the same subtended visual angle may be seen to be of different size when there is a perception or knowledge of the distance at which those objects are from the observer. That an object generally appears to have the same size, irrespective of variations in the visual angle (called *size constancy* by twentieth century researchers), has a parallel in color vision. *Color constancy* occurs when we perceive something to have the same color under quite wide variations in the intensity and color of the ambient lighting. We find a very early account of this color constancy by Ibn al-Haytham Alhazen around 1021 A.D. Alhazen noticed that the light that reflects from an object depends on the color of the object. He writes ([46], II.3.48):

> ...from perceiving the variations of lights falling upon visible objects, and from perceiving that objects are sometimes luminous and sometimes not, the faculty of judgement perceives that the colors in these objects are not the same as the lights that supervene upon them. Then, as this notion is repeated, it is established in the soul, as a universal, that colors in colored objects are not the same as their lights.

That the observer perceives the color of the object irrespective of the color of the lighting, is —according to Alhazen— because the "faculty of judgement" learns the characteristics of illumination through experience. That perception is an automated form of judgement is taken even more seriously in the influential work of the physicist and mathematician Hermann von Helmholtz [118], who proposed the notion of *unconscious inference*[1]. Helmholtz analyzes situations where facts and perception disagree, such as for example at sunset when the sun seems to go down (while it is the earth that moves), or when an actor can persuade our senses he is much older than we consciously know it to be. The problem of color constancy is perhaps an even better example that these perceptual impressions "can never once be elevated to the plane of conscious judgments" and therefore "lack the purifying and scrutinizing work of conscious thinking" ([118], p. 27). However unconscious the process may be, according to Helmholtz, it has the characteristics of an inferential process [119]:

> It is clearly possible, using the sensible images of memory instead of words, to produce the same kind of combination which, when expressed in words, would be called a proposition or judgment.

[1] "Unbewusster Schluss" [118].

Helmholtz's ideas have been criticized precisely because, to some, inference is *per sé* a conscious activity, but, as Daniel Gilbert has pointed out [116], "Helmholtz presaged many current thinkers not only by postulating the existence of such (unconscious inferential) operations, but also by describing their general features" . It was only in the last decade that attempts were made to translate Helmholtz's ideas into a mathematical framework, not in the least because computer scientists who wanted to model the human vision system were faced with the apparent complexity that underlies human perception. The Bayesian framework provides the tools necessary to understand and explain a wide variety of sometimes baffling visual illusions that occur in human perception [105]. The names that are given in this literature to the observer are *ideal* and *bayes-optimal*. Like us, they treat observation as inference, seeking to choose an optimal percept from the set of possible percepts, judged on the basis of how likely a real-world situation is, given a prior probability and a sensory state. There are however some differences in the application of the principle with respect to our proposal. In the literature on visual perception, the prior distributions are often derived from real world statistics. Of course, this begs the question how these prior distributions were obtained in the first place. There are two basic possibilities to obtain a prior: (i) either a prior distribution is based on some theoretical assumption, or (ii) it is established by looking at the relative frequency of actual recordings.

Option (i) is the one we pursued here, where we assumed a uniform distribution of observer states[2]. In the second case, which is the one adopted in the literature on perception, one has the advantage of being able to explain a wide variety of visual effects in human perception and how the priors can be adapted through the use of Bayesian updating, but we cannot explain observation itself. The relative frequency needed to obtain the prior is rooted in the observation of data, which requires another prior and so on ad infinitum. One can break away from this loop by reconsideration of what a state is. In the literature on perception, states are considered only as (real) statistical mixtures, severely limiting both the applicability and the philosophical scope of the paradigm. The state, as we have defined it here, can be a complex vector, not obtainable as a mixture in principle, and yet gives rise to probabilities if we attempt to observe the system's state as good as possible. So the state is simultaneously a description of the 'mode of being' (the pure state that physically interacts), and a 'catalogue of information' (the probabilities the optimal observer obtains). The possibility that the same principle governs human perception and quantum-mechanical

[2]The absence of a more informative prior distribution effectively reduces the criterion of Bayes-optimality to a Neyman-Pearson maximum-likelihood criterion, making the two approaches even more alike.

observation strengthens the idea that observation can be considered a form of optimal inference. Measurement apparata and human perception can be rooted in the same principle: the attempt to relate the outcome to the object under investigation as unambiguously as possible by choosing the outcome that has the largest odds in Eq.(5.21).

Each separate trial in a series of observations is treated in our proposal by selecting at random a new state for the observer. This is reminiscent of the idea that observation needs variation of the senses to discover the invariants that pertain to real things in the world, as corroborated by psychological accounts of perception, in particular those of James Gibson, who devoted much of his scientific career to the question how we perceive things. Before Gibson, it was assumed that perception is essentially a passive process, a reception of stimuli in neural receptors. Perception, was supposed to stem form a stimulus that "*...in general has migrated from the external world to the retina*" [59]. The act of observation, according to this view, was an integration of sensory data into a representation of the thing perceived. Philosophically, this theory was called the *sense-datum* theory, which states that the perception of things is born out of a logical assembly of sense data. As always, the experiments to test a theory, come from that theory itself. Hence an attempt was made in the late 19th century to retain the subject in an as passive condition is possible and to add the sensory stimuli one by one to verify how they add up to the thing perceived. Remarkably, the result of these experiments was that, holding a subject in a rigid frame, completely passive, not only resulted in a different perception, but eventually led to the subject ceasing to be a perceptor all together[3]. Something important was missing in this kind of description. The work of Gibson changed all of this radically. No longer insisting on a static frame and instead supposing there was a dynamic element at work, Gibson noted that, while many changes occur as the subject moves in his environment, there are certain higher order variables of stimulus energy that retain their proportions. It is these invariants that pertain to permanent properties of the environment. The main conjecture following from this assumption that, perhaps, the registration of invariants of the stimuli as the subject changes his orientation of his sense organs, is what is essential to perception. The new theory that emerged from these observations, is described by Gibson in [106]:

> The active observer gets invariant perceptions despite varying sensations. He perceives a constant object by vision despite

[3]In absence of sufficiently strong stimuli, the visual apparatus produces hallucinations, a phenomenon known in psychology as *visual release syndrome* or the *Charles Bonnet syndrome*. There is an auditory variant, known as the Sacks syndrome [175].

> changing sensations of light; he perceives a constant object by feel despite changing sensations of pressure; he perceives the same source of sound despite changing sensations of loudness in his ears. The hypothesis is that constant perception depends on the ability of the individual to detect the invariant... The movement of the eyes, the mouth, and the hands...seem to keep changing...the input of sensation, just so as to isolate over time the invariants of the level of input at the level of the perceptual system.

If we take this view seriously, then it seems we perceive the world by actively seeking the invariants in the relations between the changing sensations that result from changes in our state as observer. A simple experiment serves to illustrate this. If you move your head, the images cast on the retina will change. Yet the world is perceived as being stationary, not moving. If, on the other hand, the world is moving (i.e., we look out of the window of a train), then a very similar change in images will occur. Yet we will perceive the world as moving and accept our head as fixed. This means that we derive that information from more than just the images on the retina. Gibson explains this paradox as "Motion of the retina is a misconception, motion *in* the retinal image, change of pattern, is not displacement with reference to the retina." It is a well-known fact that the eyeball vibrates 50 times per second. This so-called visual tremor allows the fovea, the most sensitive part of the retina, to *scan* the image for invariants, such as constant proportions. It seems variations in the state of the observer are called for to account for observation, not only in human observation, but also in devices constructed to observe quantum phenomena[4].

Are state vectors epistemic or ontic?

The search for a Bayesian or epistemic interpretation of quantum states has been subject of an increasing number of interesting publications

[4]Interestingly, a recent publication [1] investigates the construction of a sensor capable of tracking charged particles at the LHC, using construction inspired by the human visual apparatus: " We present the results of an R&D study for a specialized processor capable of precisely reconstructing events with hundreds of charged-particle tracks <...> suitable for processing LHC events. For this purpose we design and test a massively parallel pattern-recognition algorithm, inspired to the current understanding of the mechanisms adopted by the primary visual cortex of mammals in the early stages of visual-information processing. "

([71], [82], [112], [144], [177], [186], [208]). An important motivation for seeking such an interpretation is that it allows for a subjective interpretation of the state vector by regarding it as a mathematical representation of the knowledge an agent has about a system. The collapse of the wave function is certainly much less surprising if the wave function represents some agents' knowledge rather than some physical entity or process; Bayesian updating is a testimony to that. But the epistemic interpretation has been met with well-founded critique as well. Some argue they regard the relative frequencies obtained in actual experiments as something that relates more directly to the physical system irrespective of what any agent happens to believe. There are many other valid points of criticism and we refer to [168] and [115] for an analysis of the issues that challenge epistemic interpretations. The principle of optimal observation allows for an epistemic interpretation of quantum probability too, and we have shown there are interesting connections between, for example, [186] and the constraints we derived for classical self-observers. But our proposal is also different from the above mentioned research. To understand observation as a primitive concept, one cannot assume to be in possession of *a priori* knowledge about the world. This is translated here as the uniform distribution of initial observer states and optimal observation of an ensemble of identical states will then result in an unbiased probability, i.e. a quantity that pertains only to the state of the system under investigation. In this sense, the state vector (and hence also the probabilities derived from it) *can* be assigned in an objective way to a system. The ontological character of the state vector also follows from assumption Eq.(5.1); that the state is a complete and realistic description of the system, and it is the state of the system and the observer that physically and deterministically interact to produce the measurement outcome. Systems *are* in a state that determines how it will interact. The state vector represents complete information about a system, not merely as a collection of objective attributes, but as a representation of the possible deterministic interactions with any other system, in particular observing systems. But the source of probability in observation, the randomness in the state of the observer immediately prior to an observation, may very well be fundamental. We have shown that for every single measurement outcome, there is a trade-off between the information that an observer can choose to extract about itself and about the system it is observing[5].

If this is indeed the underlying reason for probabilistic structure of quantum mechanics, then the nature of the probabilities in quantum

[5]It is argued in [194] on thermodynamical grounds that any gain in information about a system is accompanied by an equal increase of entropy in the state of the observing system.

mechanics is both ontic and epistemic. From an omniscient perspective, probability is epistemic and arises only because there is a lack of knowledge situation; it can be represented as a measure over deterministic interactions between the observer and the observed. But to the one who observes within the universe, being part of what it aims to describe, there is always a lack of knowledge that is fundamentally irreducible. Need we add that we ourselves are part of the universe we seek to describe? And how relevant is it that all our theories live essentially within the boundaries of mind; the mind of the observer who is the one that formulates and verifies them? We leave it to the reader to ponder these questions

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Theorem: There does not exist a classical self-observer M with state property space $(\Sigma^{\mathrm{M}}, \mathrm{L}^{\mathrm{M}}_{\mathrm{cl}}, \kappa^{\mathrm{M}})$ and a truth-valued mapping $\mathrm{o} : \Sigma^{\mathrm{M}} \times \mathcal{Q}^{\mathrm{M}} \to \{1, 0\}$ such that for every state $m \in \Sigma^{\mathrm{M}}$ and for every question $\alpha \in \mathcal{Q}^{\mathrm{M}}$ and corresponding property $a \in \mathrm{L}^{\mathrm{M}}_{\mathrm{cl}}$, M is `free` `consistent`

$$\mathrm{o}(m, \alpha) = \mathrm{o}(m^{\alpha}, \alpha) = \mathbf{1}^{\kappa^{\mathrm{M}}(\mathbf{a})}(m),$$

and `introspective`, i.e. $\exists \phi \in \mathcal{Q}^{\mathrm{M}}$ such that for every $\alpha \in \mathcal{Q}^{\mathrm{M}}$ we have

$$\mathrm{o}(m, \backsim \alpha) = 1 - \mathrm{o}(m, \alpha)$$

$$\mathrm{o}(m^{\alpha}, \phi) = \mathrm{o}(m, \alpha).$$

Proof: Assume M is `introspective`:

$$(\exists \phi \in \mathcal{Q}^{\mathrm{M}})(\forall \alpha \in \mathcal{Q}^{\mathrm{M}}) : \mathrm{o}(m^{\alpha}, \phi) = \mathrm{o}(m, \alpha).$$

Because $\mathcal{Q}^{\mathrm{M}}$ is closed under negation

$$\phi \in \mathcal{Q}^{\mathrm{M}} \text{ implies } \backsim \phi \in \mathcal{Q}^{\mathrm{M}}.$$

Define χ as $\backsim \phi$. M is `introspective` for every $\alpha \in \mathcal{Q}^{\mathrm{M}}$, hence also for χ:

$$\mathrm{o}(m^{\chi}, \phi) = \mathrm{o}(m, \chi).$$

M, being `free`, is also `sensitive` to the property that corresponds to χ:

$$\mathrm{o}(m, \chi) = \mathrm{o}(m^{\chi}, \chi).$$

From these last two equations we see

$$\mathrm{o}(m^{\chi}, \phi) = \mathrm{o}(m^{\chi}, \chi).$$

As χ is defined as $\backsim \phi$, we have that ϕ is equivalent to $\backsim \chi$ and we obtain

$$\mathrm{o}(m^{\chi}, \backsim \chi) = \mathrm{o}(m^{\chi}, \chi),$$

showing M is `inconsistent`. ■