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Integrating advanced discrete choice models in mixed integer linear optimization

Meritxell Pacheco Paneque[a,e,*], Michel Bierlaire[a], Bernard Gendron[b,c], Shadi Sharif Azadeh[d]

[a] *Transport and Mobility Laboratory (TRANSP-OR), School of Architecture, Civil and Environmental Engineering (ENAC), École Polytechnique Fédérale de Lausanne (EPFL), Lausanne, Switzerland*
[b] *Département d'Informatique et de Recherche Opérationnelle, Université de Montréal, Montréal, Canada*
[c] *Centre interuniversitaire de recherche sur les réseaux d'entreprise, la logistique et le transport (CIRRELT), Montréal, Canada*
[d] *Operations Research & Logistics group, Econometric department, School of Economics Erasmus University, Rotterdam, the Netherlands*
[e] *Decision Support & Operations Research Group (DS&OR), Department of Informatics, Université de Fribourg, Fribourg, Switzerland*

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ABSTRACT

In many transportation systems, a mismatch between the associated design and planning decisions and the demand is typically encountered. A tailored system is not only appealing to operators, which could have a better knowledge of their operational costs, but also to users, since they would benefit from an increase in the level of service and satisfaction. Hence, it is important to explicitly allow for the interactions between the two in the model governing the decisions of the system. Discrete choice models (DCM) provide a disaggregate demand representation that is able to capture the impact on the behavior of these decisions by taking into account the heterogeneity of tastes and preferences of the users, as well as subjective aspects related to attitudes or perceptions. Despite their advantages, the demand expressions derived from DCM are non-linear and non-convex in the explanatory variables, which restricts their integration in optimization problems. In this paper, we overcome the probabilistic nature of DCM by relying on simulation in order to specify the demand directly in terms of the utility functions (instead of the choice probabilities). This allows us to define a mixed-integer linear formulation that characterizes the preference structure and the behavioral assumption of DCM, which can then be embedded in a mixed-integer linear programming (MILP) model. We provide an overview of the extent of the framework with an illustrative MILP model that is designed to solve a profit maximization problem of a parking services operator. The obtained results show the potential of the proposed methodology to adjust supply-related decisions to the users.

1. Introduction

A mismatch between supply and demand is often observed in transportation systems, which yields an imbalance between the amount of supplies of a product or service with the corresponding willingness or need in the market. For instance, ineffective and poorly executed overbooking strategies in the context of airlines can lead to financial penalties and

* Corresponding author.
E-mail addresses: meritxell.pacheco@epfl.ch (M. Pacheco Paneque), michel.bierlaire@epfl.ch (M. Bierlaire), bernard.gendron@cirrelt.ca (B. Gendron), sharifazadeh@ese.eur.nl (S. Sharif Azadeh).

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might even damage customer goodwill (Ma et al., 2019). In vehicle sharing systems, the demand and the strategic location decisions of the operator are also closely related to each other, resulting in relocation strategies throughout the day and even in demand loss when there are not enough vehicles at a station (Huang et al., 2018). Such a mismatch raises a concern about its impact not only in transportation, but also in application areas like supply chain management, health care management and manufacturing, to name a few.

It is therefore of great importance to explicitly take into account the interactions between the users (demand) and the design and planning decisions to be made (supply). Hence, we need to move from the simplistic, and sometimes unrealistic, assumptions that are usually associated with the demand to more appropriate representations that consider the individuals as the ultimate decision makers, whose heterogeneous behavior has a direct impact on the features of the system to be adjusted.

Discrete choice models (DCM) based on the random utility principle are the state-of-the-art of demand modeling at the disaggregate level. Among various advantages, these models enable to capture the heterogeneity of tastes and preferences in high detail, and combined with the more and more available individual-based datasets, they allow to predict a wide range of behaviors in multiple contexts. Unfortunately, their integration in the optimization problems governing the supply-related decisions that require a disaggregate demand representation is a challenging task. This often makes DCM not to be considered or, when considered, they might not include the supply-related decisions, which prevents from capturing the interplay between supply and demand.

The main reason for this lack of integration is the different perspective on each side. On the one hand, optimization problems typically focus on the tractability of the mathematical formulations, which requires linearity or convexity of the involved functions, and the availability of solution techniques. This is why mixed-integer linear programming (MILP) formulations represent a significant share of the models reported in the literature. On the other hand, DCM focus on behavioral realism, which results in complex mathematical formulations, certainly not linear or even convex, that are difficult to embed in MILP models.

The objective of this paper is to define a methodology that allows to incorporate a disaggregate demand representation that interacts with the supply-related decisions to be made. To this end, we propose a general framework that enables the integration of DCM based on the random utility principle in MILP. The only condition that we impose on the DCM is that the decision variables of the MILP model that have an impact on the behavior of individuals, and therefore are also present in the DCM (such as the price of a service or the frequency of a transportation mode), appear linearly in the utility function.

The key idea is to express the demand in terms of the utility function (instead of the probability expressions of DCM), and to rely on simulation to overcome the stochastic nature of the associated random component. Notwithstanding the potentially large size of the formulation, the trade-off between model accuracy and tractability can be explicitly controlled by the modeler thanks to simulation. Furthermore, additional strategies that involve the discretization of continuous variables and the decomposable structure of the framework can be implemented to handle large instances, and individuals with homogeneous behavior can be grouped in order to reduce the size of the mathematical model.

The contribution of this research is twofold. First, we develop a mixed-integer linear formulation of DCM that can be embedded in MILP. For the sake of illustration, we define the problem of an operator that wants to maximize its profit, but other MILP models with different objectives and restrictions could be specified. Second, we show that the framework can be directly applied with existing DCM from the literature without modifications, i.e., only the original variables that appear in the DCM are considered. We also illustrate the flexibility of the framework by adapting the resulting formulation to test different contexts (e.g., price differentiation) and assumptions (e.g., grouping of individuals with homogeneous behavior).

The remainder of the paper is organized as follows. Section 2 reviews several works from the literature where DCM are incorporated within optimization problems in the context of transportation. Section 3 describes the general framework and characterizes the mathematical model. Section 4 depicts a concrete application that illustrates the use of the formulation, and the case study used as a proof-of-concept is detailed in Section 5. Finally, the conclusions and future research directions are discussed in Section 6.

2. Literature review

Facility location, airline revenue management (RM), road tolling and railway timetabling are some general transportation applications where the demand plays a key role in the supply-related decisions of the associated optimization problems. In this section, we provide an overview of numerous optimization problems that rely on DCM to represent the demand and illustrate the challenges being faced.

In the context of facility location, the complexity of the formulations modeling the decisions on spatial resource allocation (Laporte et al., 2015) has historically limited the research to deterministic problems, where all input parameters (including the demand) are considered as known. Since in reality these parameters may broadly fluctuate, researchers typically assume that the (aggregate) demand follows a probability distribution or changes its patterns under different hypothetical scenarios. In several transportation networks, travelers may modify their travel arrangements (e.g., departure time, route) depending on the level of service of the network or the price associated with the available alternatives, which in turn has a direct impact on the features being modeled. This is the case of airline RM systems, as demand forecasting influences the booking limits, which determine the profits. Despite its relevance, the demand is assumed to be isolated from its market environment (van Ryzin, 2005), and the lack of information about customers' preferences and the difficulty of the resulting mathematical formulations make disaggregate forecasting extremely challenging and infrequently used in practice (Talluri and Van Ryzin, 2006).

These application areas represent a small collection of examples illustrating the necessity of a better demand representation that allows for the heterogeneous behavior of users by modeling the demand (either deterministic or stochastic) in a disaggregate way, so that the individuals constitute the fundamental unit of demand. Since the popularization of the logit model (McFadden, 1974), DCM have become the most advanced and operational behavioral models. They are able to predict the choice behavior of individuals in detail, taking into account not only attributes of the goods, such as price or quality, but also socioeconomic characteristics of the individuals, like age or income. Consequently, they allow to capture the heterogeneity of the behavioral patterns in the population, which generate the demand. These models also enable to include other aspects of the demand, such as complex substitution patterns, and to investigate other related phenomena, such as demand elasticity. We review below several works on the integration of DCM in optimization problems, with a special emphasis on MILP, in the above-mentioned subjects.

In facility location, the problem of a firm that wants to enter a competitive market by locating p new facilities to maximize its market share has been well studied. In Benati (1999), the problem is reformulated as a p-median problem and solved by Lagrangian relaxation and branch-and-bound, and in Benati and Hansen (2002), it is reformulated as an MILP model, showing that only moderate size problems can be solved up to optimality. The behavior of customers is predicted by means of a logit model in both examples. Outer approximation (OA), an algorithm used to solve mixed-integer non-linear programming (MINLP) models based on decomposition principles (Duran and Grossmann, 1986), is considered in Mai and Lodi (2017) and Ljubić and Moreno (2018) to tackle the same type of problem. In both cases, the proposed OA methods dominate the corresponding MILP approaches. Notice that in these examples the decision variables associated with the facility location problem are not included in the utility functions. Other examples can be found in school location with free school choice. Müller et al. (2009) describe a two-step procedure to minimize the location and transportation costs over a given time horizon with respect to students choosing the school with the highest utility for a mixture of logit models. Haase and Müller (2013) propose an MILP model to maximize the standardized expected utility of all students, whose values are simulated, and show that real problems can be solved optimally (or close to optimality) within a few minutes with state-of-the-art solvers.

Customer-behavior-oriented models of demand constitute a promising methodology for RM, and especially DCM (Shen and Su, 2007; van Ryzin, 2005). In the context of airline booking, these models were first introduced by Andersson (1998). A logit model is assumed to compute the probability of a passenger that was rejected at one flight-class combination to request a seat at another flight-class, called the recapture rate (or buy-up rate). An example is considered to show that the revenue increases when implementing recapture and buy-up. Talluri and Van Ryzin (2004) provide a general analysis of the impact of choice behavior in RM by explicitly modeling consumer choice behavior with a general choice model where the probabilities of purchasing each fare product depend on the set of available fare products, and show that significant improvements in the revenue can be achieved with respect to the traditional methods. More recently, Korfmann (2018) develops a single-leg choice-based RM model with flexible demand substitution patterns between fare classes, where the demand is represented by the individual utility values (using Monte Carlo simulation), and the objective is the optimal allocation of bookings to the offered fare classes.

The toll setting problem was introduced by Labbé et al. (1998) and consists of a bilevel model where the authority (upper level) wants to maximize its revenues from a taxation scheme on a transportation network at the same time that the users (lower level) minimize their generalized travel costs (i.e., the tolls and the travel costs) while allowing for those tax levels. As discussed by the authors, the implemented deterministic representation of user behavior is too simplistic, as it assumes no dispersion of traffic along the routes and the value of time is uniform throughout the population. The stochastic version of the problem that assigns users to paths according to a DCM has received less attention in the literature. In Gilbert et al. (2014a), a logit route choice model is used to account for users' awareness of the network conditions. The optimization problem is non-linear and non-convex, and may have several local optima. An exhaustive numerical study of this problem is carried out in Gilbert et al. (2015), where it is concluded that the problem can be solved for a near-optimal solution by a combination of mixed-integer approximations and local ascent methods. In Gilbert et al. (2014b), the use of a mixture of logit models (price sensitivity distributed across users) makes this approach numerically challenging as no closed-form solution is available for the assignment of users to paths. This is why approximation schemes that provide starting points from which a local search converges to a near-optimal solution are implemented.

The inclusion of DCM in the toll setting problem is of special interest when it involves a welfare measure. For instance, in Wu et al. (2012), a pricing or credit scheme that simultaneously maximizes the equity and the social welfare in a general multimodal transportation network is described. Travelers' choices of modes and routes are represented by a nested logit model, and the resulting formulation is solved with an iterative derivative-free algorithm due to the presence of a numerical integration. Similarly, de Palma et al. (2018) propose a methodology to compute and compare (in terms of social welfare) optimal tolling systems in dollars and tokens or permits in the presence of static congestion when both the demand (governed by a mixture of logit models) and the capacity are stochastic.

More examples can be found in the context of railway timetabling. Cordone and Redaelli (2011) consider the problem of designing a regular timetable (i.e., the trains arrive and depart at constant intervals) in order to maximize the demand captured by the train when the travelers' choice between bus, car and train is modeled with a logit model. The resulting MINLP formulation is solved with a branch-and-bound algorithm. In Robenek et al. (2018), a logit model is also considered in the presence of an opt-out option representing a competing operator. The train timetabling problem is integrated with a ticket pricing problem and it is solved using a simulated annealing heuristic on a real case study. The results show that the generated revenues can be increased by up to 15% when accounting for passengers' behavior.

In the discussed instances, the probabilistic representation of the choice is either deterministic (i.e., all the explanatory variables of the DCM included in the utility function are known and therefore its value can be preprocessed), or the supply-related decisions (decision variables of the optimization problem) are part of the set of explanatory variables of the DCM and appear in the utility function. The second approach is clearly more challenging, as it leads to non-linear and non-convex formulations. Nevertheless, the reviewed works show that the supply-related decisions can be better adjusted when the interplay between them and the demand is explicitly formulated. In order to come up with tractable and more efficient solutions, various authors have made different assumptions on the DCM in order to simplify the problem, which might be unrealistic. However, more advanced DCM (e.g., based on mixtures of logit models) have shown to better forecast the behavior of individuals. As such models might not have closed-form, they are difficult to integrate in optimization problems.

In conclusion, the review of the literature illustrates that the momentum for including DCM into optimization problems while capturing the interactions between the demand and the supply-related decisions is building up. In this paper, we propose a mathematical formulation that is designed to integrate DCM into MILP. The formulation is linear, in order to ensure the tractability of the optimization model, and remains flexible, in the sense that it can be used with practically any DCM and any optimization problem formulated as an MILP model.

3. General framework

The goal of this section is to develop a general methodology to incorporate a disaggregate demand representation in supply-oriented optimization problems that allows to capture the interplay between the behavior of individuals and the decisions to be made. We assume that the demand is characterized by a DCM, whose parameters are estimated outside of the optimization problem. Notice that the DCM can be borrowed directly from the literature, so that it has been properly tested and validated. The supply-related decisions to be made by the operator are assumed to be governed by an MILP formulation.

We define three types of variables within the framework: the exogenous variables explaining the choice and not involved in the MILP formulation $x^d \in \mathbb{R}^D$, the exogenous optimization variables not involved in the DCM $x^s \in \mathbb{R}^S$, and the endogenous variables $x^e \in \mathbb{R}^E$, which are involved in both models, i.e., they are operator's decisions that are also part of the DCM. The exogenous variables appear in one of the two models, but not in both. Notice that the exogenous variables x^d are explanatory variables of the DCM, and therefore their values are directly obtained from available data, whereas the variables x^s are decision variables of the MILP model. The endogenous variables are present in both, and characterize the interactions. Hence, the exogenous optimization variables and the endogenous variables are decision variables of the MILP model, and for its definition they are assumed to be bounded:

$$\ell^s \leq x^s \leq m^s, \tag{1}$$

$$\ell^e \leq x^e \leq m^e, \tag{2}$$

where $\ell^s \in \mathbb{R}^S$ and $\ell^e \in \mathbb{R}^E$ are the vectors of lower bounds on x^s and x^e, respectively, and $m^s \in \mathbb{R}^S$ and $m^e \in \mathbb{R}^E$ are the vectors of upper bounds.

A typical example of an endogenous variable is the price of a service. The operator decides on a price in order to maximize its revenue, and the user reacts to the price to decide if they buy the service or not. If the price is too high, few users will access the service, and a low revenue will be generated. If it is too low, many users will use the service, but the generated revenue will also be low. This example is treated extensively in Section 5. Other examples of endogenous variables are the schedule of an event (e.g., departure of a train) and the capacity of a facility (e.g., number of coaches in a train).

3.1. The discrete choice model

The set of all potential alternatives is called the choice set and is denoted by $\mathcal{C}$. The alternatives in $\mathcal{C}$ are indexed by i. For each alternative i, we denote by $c_i \geq 1$ its capacity, that is, the maximum number of individuals who can choose it. We allow for a population of N individuals, indexed by $n \geq 1$. Generally, it is impossible to have access to the full population, and a sample must be used. A synthetic population, which is constructed by combining different data sources, is also convenient here (Farooq et al., 2013). The following description, based on the full population, can be easily adapted to a representative sample.

The choice set of two different individuals may not be the same. The choice set of individual n is denoted by $\mathcal{C}_n \subseteq \mathcal{C}$, and it contains the alternatives considered by them. Notice that in some cases, even if considered by the individual, some alternatives may not be offered to them for certain reasons (see Section 3.2). For instance, from a profit maximization point of view, a service that is not profitable will not be proposed. These decisions are modeled with the binary variables y_{in}, which are 1 if alternative i is considered by and offered to individual n, and 0 otherwise. These variables are endogenous, i.e., they belong to the vector x^e. Hence, the set of offered alternatives is flexible in the sense that it is possible not to propose some alternatives to some specific individuals, or in a more practical manner, groups of individuals. This feature allows the operator to investigate different marketing solutions and business models.

The preference structure of individuals is represented with a utility function, which associates a score with each alternative $i \in \mathcal{C}_n$. This utility is defined as

$$U_{in}(x^d_{in}, x^e_{in}; \varepsilon_{in}) = V_{in}(x^d_{in}, x^e_{in}) + \varepsilon_{in}, \tag{3}$$

where $V_{in} : \mathbb{R}^{D+E} \to \mathbb{R}$ is the systematic component of the utility function, that includes everything that can be modeled by the analyst, and ε_{in} is the random component, that captures everything that has not been included explicitly in the model and is independent of the exogenous demand variables x^d_{in} and endogenous variables x^e_{in} associated with alternative i and individual n. As ε_{in} are random variables, $U_{in}(x^d_{in}, x^e_{in}; \varepsilon_{in})$ are also random variables.

The behavioral assumption is that individual n chooses alternative i if the corresponding utility is the largest within the choice set $\mathcal{C}_n$ (Manski, 1977). We assume that each individual chooses one and only one alternative. The probability that individual n chooses alternative $i \in \mathcal{C}_n$ is

$$P_n(i|x^d_{in}, x^e_{in}) = \Pr(U_{in}(x^d_{in}, x^e_{in}; \varepsilon_{in}) \geq U_{jn}(x^d_{jn}, x^e_{jn}; \varepsilon_{jn}), \forall j \in \mathcal{C}_n). \tag{4}$$

Throughout the paper, we assume that V_{in} is linear in the endogenous variables x^e_{in}. This is not required as such for the derivation of the DCM, but important in our context for its integration in an MILP formulation. For this reason, the deterministic term in (3) is written as

$$V_{in}(x^d_{in}, x^e_{in}) = \sum_k \beta_{ink} x^e_{ink} + g_{in}(x^d_{in}), \tag{5}$$

where x^e_{ink} is the kth endogenous variable associated with alternative i and individual n and β_{ink} are the associated coefficients. The functions $g_{in} : \mathbb{R}^D \to \mathbb{R}$ do not need to be linear since the variables x^d_{in} are not involved in the optimization model, i.e., $g_{in}(x^d_{in})$ is a quantity that can be preprocessed. The parameters β_{ink} of the DCM are estimated beforehand (outside of the optimization scheme), i.e., they are not variables of the optimization model. Nevertheless, the described framework can be adapted to address the parameter estimation problem. We refer the reader to Fernández-Antolín et al. (2017) for further details.

Operational DCM are obtained by assuming a distribution for the random term ε_{in}. For example, the logit model is obtained by assuming that ε_{in} are independent and identically distributed (across both i and n), with an extreme value distribution. It can be shown that, for the logit model, (4) is written as

$$P_n(i|x^d_{in}, x^e_{in}) = \frac{y_{in} e^{V_{in}(x^d_{in}, x^e_{in})}}{\sum_{j \in \mathcal{C}_n} y_{jn} e^{V_{jn}(x^d_{jn}, x^e_{jn})}}. \tag{6}$$

Advanced DCM, which aim at relaxing the unrealistic assumptions associated with the logit model and have shown a better prediction power, can also be accommodated within this framework. In the case study described in Section 5, a mixture of logit models, also known as mixed logit model, is considered. Mixtures of logit models are highly flexible models that can approximate any random utility model (Train, 2003). They can be derived under a variety of behavioral specifications whose choice probabilities take a specific functional form, and each derivation provides a particular interpretation. For some derivations of the mixed logit model (and specifically the one in Section 5), the deterministic part of the utility specification of the standard logit model is generalized by allowing one or some of the coefficients β_{ink} in (5) to be randomly distributed across the population, which captures heterogeneity among individuals. The vector of coefficients β_{nk} associated with individual n is therefore a random vector with probability density function $f(\beta_k|\theta)$, where the vector θ contains the parameters of the distribution of β_{nk}, such as their mean and variance. The probability that individual n chooses alternative i is given by the standard logit formula conditional on β_{nk}. As β_{nk} is distributed, the (unconditional) choice probability (4) is the integral of the logit formula over the density of β_{nk}:

$$P_n(i|x^d_{in}, x^e_{in}) = \int \frac{y_{in} e^{V_{in}(x^d_{in}, x^e_{in}; \beta_{nk})}}{\sum_{j \in \mathcal{C}_n} y_{jn} e^{V_{jn}(x^d_{jn}, x^e_{jn}; \beta_{nk})}} f(\beta_k|\theta) d\beta_k. \tag{7}$$

Latent factors, such as personal attitudes and perceptions, also allow for a more realistic representation of the behavior inherent in the choice process. They have been integrated in DCM through two main approaches: models with latent variables, which explicitly model the unobserved psychological characteristics of the individual, and latent class models, which assume that the population can be probabilistically segmented into discrete groups that have different choice behaviors. A special case of the latter corresponds to DCM with latent choice sets (Ben-Akiva and Boccara, 1995), which model individual choice behavior as a two-stage process consisting of choice set generation first followed by a choice from the resulting given choice set. This enables to incorporate an explicit probabilistic representation of the availability of the alternatives, instead of assuming them as given, as is the case in standard DCM.

The expected demand D_i for each alternative $i \in \mathcal{C}$ is given by

$$D_i = \sum_{n=1}^{N} P_n(i|x^d_{in}, x^e_{in}). \tag{8}$$

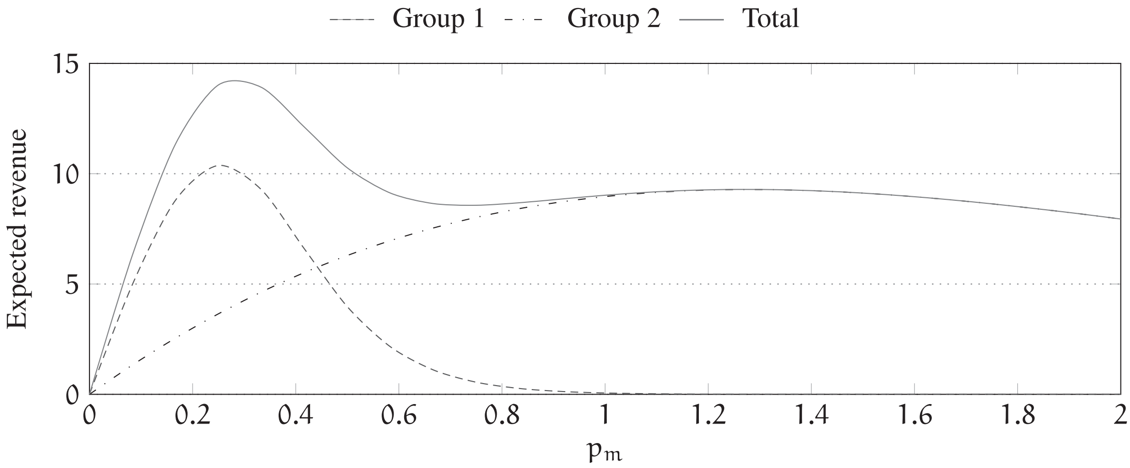

Fig. 1. Expected revenue from group 1, group 2 and total as a function of the price for alternative m (p_m)

Notice that as (6) and (7) are non-linear functions in the variables x^e_{in}, so is the associated expression of the demand (8). It can be used to derive other quantities, such as the expected gain (or revenue) obtained from alternative i: $G_i = p_i D_i$, where p_i denotes the price associated with alternative i (see Section 4). The price is typically an endogenous variable, which makes the formulation of the expected gain even more complex.

We illustrate the non-linearity and non-convexity of the previous expressions by means of an example for a simple logit model. Consider a choice set composed of two alternatives: the same service offered by a certain operator (m) and its competitor (c). A population of $N = 100$ individuals consists of two groups with different behavior: group 1 (2/3 of the population) and group 2 (1/3 of the population). Both services are considered by and offered to all individuals. The systematic term of the utility function for alternative m is defined as $V_{mg} = \beta_g p_m + s_{mg}$, where g denotes the group, β_g the price sensitivity of an individual in group g, p_m the price to access service m and s_{mg} the term associated with service m and group g capturing the socioeconomic characteristics of interest. We assume that group 1 is highly sensitive to price ($\beta_1 = -10$) and has an intrinsic preference towards alternative m ($s_{m1} = 3$), whereas group 2 has a lower price sensitivity ($\beta_2 = -1$) and does not have a preference for alternative m ($s_{m2} = 0$). Furthermore, we assume that no information is available about the competitor, and we therefore define $V_{cg} = 0$ for $g = 1, 2$.

Fig. 1 shows the expected gain for alternative m obtained from the two groups (separately), and the sum of the two, as a function of price. The revenue function is unimodal within each group, while the total gain curve is bimodal: the first local optima is reached when both groups are attracted to m because of the low price, and the second one is related to group 2 only, as group 1 has decided to leave the market due to the high price.

In general, when real models involving heterogeneity in the population are considered, the associated objective functions are multi-modal, as illustrated in Fig. 1 for a simple logit model with only two groups in the population. In this work, we define a framework that avoids the use of the probability formulas by specifying the DCM in terms of the utility functions, which enables its integration in an MILP formulation.

Note that the binary variables y_{in} are endogenous variables, as they belong to both the DCM (see probability expressions (6) and (7)) and the MILP model. However, they do not appear in $V_{in}(x^d_{in}, x^e_{in})$. The presence of endogenous variables in the utility function makes the formulation more complex, since $V_{in}(x^d_{in}, x^e_{in})$ cannot be preprocessed. We propose a general approach that allows for both types of endogenous variables, as described in Section 3.2.

3.2. Simulation-based linearization

Commonly, when a probabilistic model like the one introduced in Section 3.1 is integrated with an MILP formulation, either simulation is used (to approximate the probability expressions, especially when they do not have a closed-form) or linearization techniques are applied to tackle non-linearity. We consider the first approach here in order to address the stochasticity of the random component in (3). This allows to approximate the expected demand in terms of the utility function with a mixed-integer linear formulation based on the sample average approximation (SAA) principle in the space of the utilities.

For each ε_{in} in (3), we generate R draws $\xi_{in1},...,\xi_{inR}$ from its distribution (e.g., Gumbel, normal) outside of the optimization procedure (known as exterior approach in stochastic programming), where ξ_{inr} denotes the rth draw. Each draw can be seen as an independent behavioral scenario. We notice that variance reduction techniques (e.g., linear control random variables method, importance sampling) could be used to enhance convergence of the SAA estimators.

Once the draws have been generated, for each draw r, we obtain the utility associated by individual n with alternative $i \in \mathcal{C}_n$, which is denoted by $U_{inr}(x^d_{in}, x^e_{in})$, or simply U_{inr}. For specification (5), we have

$$U_{inr} = V_{in} + \xi_{inr} = \sum_k \beta_{ink} x^e_{ink} + g_{in}(x^d_{in}) + \xi_{inr}. \tag{9}$$

As the variables x^e_{ink} are bounded (see (2)), and the values $g_{in}(x^d_{in})$ are given, we can derive lower and upper bounds on U_{inr}, denoted respectively by ℓ_{inr} and m_{inr}:

$$\ell_{inr} \leq U_{inr} \leq m_{inr}, \quad \forall i \in \mathcal{C}_n, n, r. \tag{10}$$

Notice that the idea of relying on scenarios to represent uncertainty is typically exploited in stochastic programming, and particularly in the two-stage stochastic programming model. In a standard two-stage model, first-stage variables are decided upon before the actual realization of the random parameters such that the expected value of an objective function (which is the optimal value of the second-stage optimization problem) is optimized. To solve the two-stage problem, the vector of random parameters is often assumed to have a discrete distribution with a finite number of possible outcomes (scenarios) with respective probability masses, which yields a deterministic equivalent formulation of the problem. The expectation function of the formulation is then approximated with the SAA method, which enables to solve the problem using deterministic algorithms.

In this case, by generating R draws (scenarios) of the random component $\varepsilon_{in}, \forall i \in \mathcal{C}_n, n,$ we associate a deterministic utility function with each scenario (see (9)). These utilities are considered to determine a choice for each individual and scenario, which allows to approximate the individual choice probabilities, and consequently the expected demand, with the SAA method. A detailed description is provided hereunder.

Availability of alternatives An alternative may be unavailable for three reasons. First, the operator decides that the alternative is not made available to individual n. This decision is modeled with the binary variables y_{in} introduced in Section 3.1, which are equal to 1 if alternative $i \in \mathcal{C}$ is offered to individual n, and 0 otherwise. Second, an alternative might not be considered by the individual, i.e., $i \notin \mathcal{C}_n$. This decision (coming from data) can be explicitly included in the MILP model by adding the following constraint:

$$y_{in} = 0, \quad \forall i \notin \mathcal{C}_n, n. \tag{11}$$

Third, the alternative may be unavailable because its capacity has been reached. This type of unavailability is more complex to model, as it is not a direct decision as such, but the result of the decisions of multiple individuals. Note that in this framework this can vary from one scenario to the next. Indeed, an alternative might be more attractive in one scenario, generating more demand than its capacity, and less attractive in another one.

We model the availability of alternative i to individual n in scenario r with the binary variables y_{inr}. The variables y_{in} and y_{inr} are related as follows:

$$y_{inr} \leq y_{in}, \quad \forall i \in \mathcal{C}, n, r, \tag{12}$$

which implies that alternative i is not available at the scenario level ($y_{inr} = 0$) if the alternative is not offered to or considered by individual n (i.e., $y_{in} = 0$). If the alternative is offered and considered (i.e., $y_{in} = 1$), and there is still room for individual n, then $y_{inr} = 1$, but if its capacity has been reached, then $y_{inr} = 0$.

Discounted utility The behavioral assumption states that the individual selects the alternative associated with the largest utility in each scenario. To avoid that an unavailable alternative is related to the highest utility, we introduce the concept of discounted utility, which is the utility itself when the alternative is available, and a low value otherwise. The discounted utility associated with alternative i by individual n in scenario r is defined as

$$z_{inr} = \begin{cases} U_{inr} & \text{if } y_{inr} = 1, \\ \ell_{nr} & \text{if } y_{inr} = 0, \end{cases} \forall i \in \mathcal{C}, n, r, \tag{13}$$

where $\ell_{nr} = \min_{j \in \mathcal{C}_n} \ell_{jnr}$ is the smallest lower bound across all alternatives.

The linear formulation of (13) is given by

$$\begin{aligned} \ell_{nr} &\leq z_{inr}, \forall i \in \mathcal{C}, n, r, \\ z_{inr} &\leq \ell_{nr} + M_{inr} y_{inr}, \forall i \in \mathcal{C}, n, r, \\ U_{inr} - M_{inr}(1 - y_{inr}) &\leq z_{inr}, \forall i \in \mathcal{C}, n, r, \\ z_{inr} &\leq U_{inr}, \forall i \in \mathcal{C}, n, r, \end{aligned} \tag{14}$$

where

$$M_{inr} = m_{inr} - \ell_{nr}. \tag{15}$$

Choice The choice of individual n in scenario r is characterized by the binary variables w_{inr}, which are equal to 1 if alternative $i \in \mathcal{C}$ is chosen by individual n in scenario r, and 0 otherwise. As each individual is choosing exactly one alternative, we impose

$$\sum_{i \in \mathcal{C}} w_{inr} = 1, \quad \forall n, r. \tag{16}$$

Moreover, since an alternative that is not available cannot be selected, we add the following constraint:

$$w_{inr} \leq y_{inr}, \quad \forall i \in \mathcal{C}, n, r. \tag{17}$$

In terms of the discounted utility, the chosen alternative corresponds to the one with the highest discounted utility. We introduce the continuous variable U_{nr} to represent it, which is defined as

$$U_{nr} = \max_{i \in \mathcal{C}} z_{inr}, \quad \forall n, r. \tag{18}$$

The linear formulation of (18) is given by

$$z_{inr} \leq U_{nr}, \quad \forall i \in \mathcal{C}, n, r, \tag{19}$$

$$U_{nr} \leq z_{inr} + M_{nr}(1 - w_{inr}), \quad \forall i \in \mathcal{C}, n, r, \tag{20}$$

where

$$M_{nr} = m_{nr} - \ell_{nr} \tag{21}$$

is the difference between the largest upper bound and the smallest lower bound, where the largest upper bound is defined as $m_{nr} = \max_{j \in \mathcal{C}_n} m_{jnr}$. Notice that we cannot exclusively rely on constraint (17) and ignore the discounted utility variables because constraints (19) and (20) written in terms of the utility variables U_{inr} (instead of z_{inr}) lead to infeasibility if the highest utility U_{nr} for individual n and scenario r is associated with an alternative that is not available.

Note that it might happen that the highest utility is achieved by two alternatives. In this case, constraint (16) guarantees that only one of them is chosen, and the actual choice is governed by the specific optimization problem, which is not behaviorally realistic. However, thanks to simulation, such an issue is happening sufficiently rarely to be ignored. Also note that the last constraints in (14) and constraints (19) are equivalent. We have decided to keep both in the model as it has proven to be computationally more efficient (by performing some preliminary tests that ignore one of the two constraints at a time). Furthermore, the characterization of the so-called big M constraints (constraints (14) and (19) and (20)) is tailored to our problem. Indeed, the values of the constants defined in (15) and (21) are the tightest possible values for the associated constraints.

Expected demand The complexity of the probability distributions of the random variables involved in the DCM and their correlation structure are irrelevant in this context as long as it is possible to draw from these distributions (which is performed outside of the optimization scheme). Given an independent and identically distributed sample $\xi_{in1}, \ldots, \xi_{inR}$ of the random variable ε_{in}, the choice variables w_{inr} allow to count the number of times that the behavioral assumption associated with $P_n(i|x^d_{in}, x^e_{in})$ (see (4)) in terms of z_{inr} is met, which enables to compute the sample average

$$\frac{1}{R}\sum_{r=1}^{R} w_{inr}. \tag{22}$$

Hence, as a consequence of the law of large numbers, the relative frequency calculated in (22) provides an estimation of $P_n(i|x^d_{in}, x^e_{in})$, and the total expected demand of alternative $i \in \mathcal{C}$ is approximated by aggregating such estimates across individuals:

$$D_i \approx \frac{1}{R}\sum_{r=1}^{R}\sum_{n=1}^{N} w_{inr}. \tag{23}$$

Notice that the law of large numbers is valid if the expectation of the probability distribution associated with the error component ε_{in} is finite, which is the case for the operational DCM used in practice.

Capacity allocation If the demand for alternative i is larger than its capacity, it is necessary to decide which individuals have access to the alternative. We have decided to model it exogenously, using an externally defined priority list of individuals, similar to Binder et al. (2017). This list determines the way in which the individuals are processed (which does not necessarily coincide with the order in which individuals arrive to the system), and as soon as capacity is reached, the remaining (unprocessed) individuals will not have access to the alternative. An individual has access to an alternative if all individuals before them in the list for which the alternative is offered have also access to it. Thus, the numbering of individuals reflects the priority list. Note that the construction of this priority list can consider various aspects of the relationship between the operator and the individuals, such as fidelity programs, VIP users, etc. It can also be randomly generated, as is assumed in the experiments performed in Section 5.

The capacity restrictions are expressed by the following constraints:

$$\sum_{m=1}^{n-1} w_{imr} \leq (c_i - 1)y_{inr} + (n-1)(1 - y_{inr}), \quad \forall i \in \mathcal{C}, n > c_i, r, \tag{24}$$

$$c_i(y_{in} - y_{inr}) \leq \sum_{m=1}^{n-1} w_{imr}, \quad \forall i \in \mathcal{C}, n > 1, r. \tag{25}$$

Constraint (24) forbids the access of individuals to a certain alternative when its capacity has been reached, whereas constraint (25) ensures the availability of the alternative when the capacity has not been exceeded. They can be verified by considering two cases when $y_{in} = 1$ (i.e., alternative i is offered to and considered by individual n):

1. $\sum_{m=1}^{n-1} w_{imr} < c_i$, and
2. $\sum_{m=1}^{n-1} w_{imr} \geq c_i$.

In the first case, constraint (24) is always satisfied (for both $y_{inr} = 0$ and $y_{inr} = 1$), and constraint (25) forces y_{inr} to be equal to 1 in order to be verified, which means that the number of individuals up to and including n who have chosen alternative i does not exceed c_i, so there is still room for individual n. In the second case, constraint (25) is always satisfied, and constraint (24) implies $y_{inr} = 0$, which means that the capacity has been reached due to the choices of the individuals up to and including $n - 1$, and even if the alternative is proposed to individual n by the operator, there is no room left for them.

Note that if $y_{in} = 0$ (i.e., alternative i is not offered to or not considered by individual n), the variables y_{inr} and w_{inr} are equal to 0 (due to constraints (12) and (17), respectively), and therefore constraints (24) and (25) are always satisfied.

This formulation can be easily extended to include capacity as a decision variable. In order to avoid the non-linearity that would appear in constraints (24) and (25) (due to the product of the capacity and the availability variables), we propose a predefined list of Q feasible values for each decision variable c_i: $c_{i1}, \ldots, c_{iQ}$. Then, alternative i is duplicated Q times, each copy being associated with the same utility function, but with a different capacity level. We define the binary variables y_{iq}, which take value 1 if alternative i is offered with capacity c_{iq}, and 0 otherwise. We also define the binary variables y_{iqn}, which represent the extension of the variables y_{in}, and are therefore equal to 1 if alternative i with capacity c_{iq} is offered to and considered by individual n, and 0 otherwise. It is sufficient to include the following constraints in the formulation:

$$\sum_{q=1}^{Q} y_{iq} \leq 1, \qquad \forall i \in \mathcal{C}, \tag{26}$$

$$y_{iqn} \leq y_{iq}, \qquad \forall i \in \mathcal{C}, q, n. \tag{27}$$

Constraint (26) guarantees that at most one of the duplicates is actually available. Note that it is still possible for the operator to decide not to offer alternative i at all. In that case, the sum of the left hand side of (26) is equal to zero. Constraint (27) ensures that the variables y_{iqn} are set to 0 if alternative i is not offered with capacity c_{iq}.

The remaining variables specified throughout this section need to be extended to account for the capacity levels. We can simplify the notation by redefining $\mathcal{C}$ as the set of the duplicates of the original alternatives. The dimension of $\mathcal{C}$ would then be JQ, where J is the number of original alternatives.

3.3. Mixed-integer linear formulation

The methodology introduced in Section 3.2 yields a mixed-integer linear formulation that describes the preference structure and behavioral assumption associated with DCM based on the random utility principle. This formulation constitutes a set of constraints that can be embedded into any MILP model consisting of a linear objective function that relates the decision variables (exogenous optimization variables and endogenous variables, as defined at the beginning of Section 3) and the expected demand to an aggregate performance of the system, and a set of linear constraints that identifies the feasible configurations of the decision variables. This provides the optimization problem with a disaggregate demand representation able to capture the supply-demand interplay, and can be employed to model numerous applications, such as the profit maximization problem described in Section 4.

Given a sequence of optimal solutions of the optimization problem defined by the mixed-integer linear formulation of DCM with respect to R, it is possible to prove its convergence (when R tends to infinity) to an optimal solution of the optimization problem that relies on the probability-based demand representation of DCM (as defined in (8)). Appendix B includes additional details on this convergence property and the corresponding proof.

The number of constraints comprised in the MILP formulation is of the order of $JQNR$. In real applications, where the number of individuals can be large, this comes with a high computational price. To reduce the size of the model, individuals can be grouped into classes of homogeneous behavior (see Section 5.2), even though this technique requires additional assumptions to handle the access of groups to the alternatives in order to fulfill the capacity restrictions. Moreover, the proposed formulation allows to explicitly model the trade-off between the number of draws and the accuracy of the approximation (see Section 5.1).

As pointed out in Section 3.2, the complexity associated with the DCM is only affected by the number of draws, and not by the complexity of the probability distributions of the random variables involved in the DCM and their correlation structure. This is a strength of the framework, that it is relevant for any existing complex DCM, and for other models to be developed in the future.

Finally, we want to highlight the fact that the tremendous development of advanced mathematical formulations and efficient algorithms allow to solve MILP models of gigantic sizes. Furthermore, the structure of the proposed formulation is particularly well suited for decomposition methods, since most constraints are independent across individuals and scenarios.

Such a structure can be exploited in order to define iterative procedures that efficiently approximate the optimal solution by relying on the discretization of the continuous supply-related decision variables, as illustrated in Section 5.4 for the profit maximization problem, or, more generally, to characterize decomposition methods that do not involve any discretization. Investigating decomposition techniques, however, is out of the scope of the paper, and left for future research.

4. Profit maximization problem

We consider a profit maximization problem to illustrate how the framework described in Section 3 can be used. This application is particularly interesting because the calculation of the expected revenue involves a non-linearity that needs to be addressed, and can be found in many different contexts (e.g., airline RM, road tolling).

The operator aims at finding the best strategy in terms of pricing and capacity allocation to maximize its profit by selling services, each of them at a certain price and with a certain capacity, both to be decided. Regarding the cost of each service, we assume that it is composed of a fixed cost associated with operating the service and a variable cost associated with each unit of the service sold.

The market is composed of N individuals, which are assumed to be heterogeneous and price elastic, in the sense that each user may have a different behavior and sensitivity towards price. The operator is considering a set $\mathcal{C}$ composed of J services, each of them representing a service potentially offered by the operator and its associated capacity level.

In a profit maximization context, we need to model competition. If we do not account for competitive services, users are captive, and the problem becomes unbounded. Competitive services can be explicitly modeled in the choice set, or grouped into an opt-out option that captures users leaving the market, either because they choose a competitor's service or because they do not choose anything at all. To keep the illustrative example simple, we consider the second approach. The main assumption is that the decisions of the competitors are given, and not adjusted as a consequence of the decisions of the operator. The opt-out option is denoted by $i = 0$, and it is always available to all users, i.e., $0 \in \mathcal{C}_n, \forall n \ (y_{0n} = 1)$.

We consider the price as the only endogenous variable (x^e) in the utility function (9). We define $p_{in} \in \mathbb{R}$ as the price that user n must pay to access service $i \in \mathcal{C}_n \setminus \{0\}$, where the index i refers to the duplicates of the original services as introduced in Section 3.2. Note that the index n allows the operator to propose different prices to different users or, more realistically, to different groups of users (e.g., students, seniors, families). In that case, the model includes as many p_i variables as the number of groups. For the sake of illustration, exogenous optimization variables are not considered in this example.

The expected gain obtained from service $i \in \mathcal{C} \setminus \{0\}$ can be derived directly from the demand expression (8) and the price specification:

$$G_i = \frac{1}{R} \sum_{n=1}^{N} \sum_{r=1}^{R} p_{in} w_{inr}. \tag{28}$$

As the price is an endogenous variable, (28) is non-linear. The product of a binary and a continuous variable can be linearized if an upper bound for the latter is known, which can be set by the operator. Assume that the price p_{in} is bounded between a lower bound $a_{in} \in \mathbb{R}$ and an upper bound $b_{in} \in \mathbb{R}$. We define the variables $\eta_{inr} = p_{in} w_{inr}$ with linearizing constraints (29):

$$\begin{aligned}
a_{in} w_{inr} &\leq \eta_{inr}, && \forall i \in \mathcal{C} \setminus \{0\}, n, r,\\
\eta_{inr} &\leq b_{in} w_{inr}, && \forall i \in \mathcal{C} \setminus \{0\}, n, r,\\
p_{in} - (1 - w_{inr}) b_{in} &\leq \eta_{inr}, && \forall i \in \mathcal{C} \setminus \{0\}, n, r,\\
\eta_{inr} &\leq p_{in} - (1 - w_{inr}) a_{in}, && \forall i \in \mathcal{C} \setminus \{0\}, n, r.
\end{aligned} \tag{29}$$

The expected gain G_i is then obtained by replacing the product $p_{in} w_{inr}$ by the variables η_{inr} in (28).

Motivated by the definition of the variables η_{inr}, we can derive the following valid inequality:

$$d_{0nr} w_{0nr} + \sum_{j \in \mathcal{C}_n \setminus \{0\}} (d_{jnr} w_{jnr} + \beta_{jn} \eta_{jnr}) \geq z_{inr}, \qquad \forall i \in \mathcal{C}_n, n, r, \tag{30}$$

where $d_{jnr} = g_{jn}(x^d_{jn}) + \xi_{jnr}, \forall j \in \mathcal{C}$. It corresponds to the linearized version of the constraint

$$\sum_{j \in \mathcal{C}_n} U_{jnr} w_{jnr} \geq z_{inr}, \qquad \forall i \in \mathcal{C}_n, n, r. \tag{31}$$

Indeed, constraint (30) is equivalent to constraints (19) and (20), which set the choice variable of the service with the highest discounted utility (for user n and scenario r) equal to 1. We have compared the computational times for different instances with and without (30), and we observe that it helps to obtain a better LP relaxation, which allows to solve the MILP formulation more efficiently.

It is easy to prove the equivalence with constraints (19) and (20). Assume that z_{i^*nr} is the largest discounted utility associated with user n and scenario r, i.e., $z_{i^*nr} \geq z_{inr}, \ \forall i \in \mathcal{C}_n, i \neq i^*$. Given that only one of the variables w_{inr} associated with user n and scenario r can be equal to 1 due to constraint (16), if $w_{j^*nr} = 1$ for $j^* \neq i^*$, then constraint (30) can be written as $U_{j^*nr} \geq z_{jnr}$ for any $j \in \mathcal{C}_n$, and in particular for $j = i^*$. Since $y_{j^*nr} = 1$ (using (17)), the valid inequality (30) can be written as $z_{j^*nr} \geq z_{i^*nr}$ for service i^*, which implies, together with the initial assumption $z_{i^*nr} \geq z_{j^*nr}$, that $z_{j^*nr} = z_{i^*nr}$. This is equivalent to (19) and (20), since $U_{nr} = \max_{i \in C} z_{inr} = z_{j^*nr}$.

Finally, we assume that the operating cost of service $i \in \mathcal{C} \setminus \{0\}$ is calculated as

$$C_i = (f_i + v_i c_i) y_i, \tag{32}$$

where f_i is the fixed cost and v_i is the cost per sold unit of service i with capacity level c_i. The expected profit is computed by subtracting the total operating costs from the generated revenues. These quantities are obtained by considering the gains and costs from all the services proposed by the operator except the opt-out option (i.e., the services proposed by the competitors are excluded). The resulting objective function is the following:

$$\max \sum_{i \in \mathcal{C} \setminus \{0\}} (G_i - C_i). \tag{33}$$

The constraints of the MILP model are itemized next:

- Utility: (9),
- Availability: (11), (12),
- Discounted utility: (14),
- Choice: (16), (17), (19), (20),
- Capacity allocation: (24)–(27),
- Pricing: (29), and
- Valid inequality: (30).

Table 12 in Appendix A summarizes the main notations used in the model for the reader's convenience, organized by sets, parameters, variables and aggregated quantities. For the sake of simplicity, we do not make the distinction between the terms alternatives and services, as is done in Sections 3 and 4, respectively, and we refer to them simply as alternatives. From now on, we will use both terms interchangeably.

5. Case study

The objective of this case study is twofold. First, we deal with the integration of a DCM that is taken as such from the literature in the MILP formulation characterized in Section 4. Second, we apply the resulting formulation in a realistic context, which allows to assess the generalization capability of the framework by performing different experiments.

Discrete choice model The challenge of the first goal consists in embedding a non-trivial DCM that has been externally developed in the profit maximization problem. We rely on the case study of a parking services operator, which is motivated by a published disaggregate demand model for parking choice (Ibeas et al., 2014), whose data was kindly provided by the authors and used to perform the experiments discussed next.

The parking choice model aims at addressing the economic viability of an underground parking in the area of study. In order to adapt the case study to the application described in Section 4, we assume a park and ride situation, where the parking facilities have public transportation connections and the users leave their vehicles during the day in order to commute to their final destination with public transportation. In this way, we circumvent the inherent dynamic behavior of general parking facilities, where the spots are occupied and liberated indistinctly, and we define a setting where as soon as the parking spot is taken, it will not be available for the users subsequently arriving (within the time horizon being evaluated).

We note that this characterization assumes fully informed drivers, i.e., drivers know if there is a remaining parking spot in each parking facility, and where this spot is located, which is not always the case. A more comprehensive analysis of a problem handling parking facilities should include additional features, such as the fact that finding the last free parking space is a considerable time-consuming effort, while other parking spaces can be freed. In any case, the focus here is not on parking management, but on showing that a DCM from the literature can be directly accommodated in an MILP formulation thanks to the proposed framework.

The choice set consists of three services: paid on-street parking (PSP), paid parking in an underground car park (PUP) and free on-street parking (FSP). Since the latter does not provide any revenue to the operator, it represents the opt-out option, and therefore has unlimited capacity. In view of the nature of this application, we assume that the parking facilities are either opened to everyone or not offered at all. Furthermore, we assume that $\mathcal{C}_n = \mathcal{C}, \forall n$, i.e., all users consider all facilities when deciding where to park, and therefore the availability variables y_{in} are reduced to y_i, i.e., $y_{in} = y_i, \forall n$. We also assume that the price is the same for everyone, i.e., the price variables p_{in} are simplified to p_i.

The utility specification of the DCM is given by Ibeas et al. (2014). They define a mixture of logit models (see (7) for the associated choice probability expression) to describe the behavior of potential car park users, with two of the parameters assumed to be normally distributed and correlated (see Table 13 in Appendix C for the specification table of the DCM).

MILP model The second goal is addressed with several experiments for the proof-of-concept of the introduced methodology. For the sake of illustration, and to avoid solving huge optimization problems, we define a sample of $N = 50$ users, which are randomly selected among the 197 users available in the provided dataset. Despite the reduction in size, this sample still represents a realistic example to test the formulation on. We also provide some insights into computational complexity

Table 1
Computational results of the revenue maximization problem (uncapacitated case)

R	Expected optimal revenue				Computational time (min)		
	Minimum	Average	Maximum	Standard deviation	Minimum	Average	Maximum
2	25.794	26.790	28.284	1.030	0.002	0.003	0.003
5	26.574	27.258	27.906	0.565	0.007	0.009	0.013
10	26.547	27.262	27.642	0.425	0.042	0.050	0.060
25	26.653	26.947	27.087	0.170	0.159	0.307	0.464
50	26.749	26.867	27.071	0.126	1.364	3.267	4.844
100	26.787	26.872	27.059	0.111	15.195	24.173	32.159
250	26.792	26.889	26.967	0.077	55.721	105.445	198.026

when solving the problem for larger instances that rely on the full dataset. In any case, the priority list is defined as the order of the users in the considered sample, which can be interpreted as a random arrival.

As mentioned in Section 4, we assume that the price is the only endogenous variable. It appears linearly in the utility function (as required), as well as the exogenous demand variables, even though linearity on these is not necessary because they are not decision variables of the MILP model. The values of p_{PSP} and p_{PUP} will be determined by the model, whereas the exogenous demand variables will be replaced by their corresponding values in the data. As a stated preference (SP) survey was conducted, we consider one of the choice situations presented to the respondents in order to characterize the attributes of the alternatives.

In the first two experiments (Sections 5.1 and 5.2), we assume a fixed capacity that is large enough in accordance with the size of the sample, but restrictive enough to force some users to opt-out because there is not enough room for everyone. Moreover, the fixed and variable costs of the paid alternatives are set to 0, which turns the objective function into the expected revenue obtained from the offered services, and implies $y_i = 1, \forall i \in \mathcal{C} \setminus \{0\}$ (maximization problem). More precisely, Section 5.1 provides an assessment of the computational time and obtained solutions, and Section 5.2 deals with the segmentation of individuals by differentiating the price among market segments and by grouping users with similar behavior. The third experiment (Section 5.3) deals with the profit maximization problem as such and analyzes the computational expense when capacities are decision variables. Finally, the last experiment (Section 5.4) tackles large instances of the problem by exploring an iterative solution algorithm that relies on the discretization of the price variables. All the experiments have been implemented in C++ using ILOG Concert Technology to access CPLEX 12.8, and all the instances were performed using 12 threads in a 3.33 GHz Intel Xeon X5680 server running a 64-bit Ubuntu 16.04.2.

5.1. Price calibration

In this section, we determine the price of PSP and PUP so that the revenue of the operator is maximized, both with and without capacity restrictions on such services. Based on the values of the price variable in the data, we assume $p_{\mathrm{PSP}} \in [0.5, 0.65]$ and $p_{\mathrm{PSP}} \in [0.7, 0.85]$. We analyze the performance of the framework with respect to the number of simulation draws by running 5 replications for each value of R, where each replication corresponds to an independent generation of R draws of the random terms $\varepsilon_{in}, \forall i \in \mathcal{C}, n$.

Uncapacitated case We assume first that both PSP and PUP have unlimited capacity, i.e., constraints (24) and (25) are ignored. The goal is to compare the obtained results with the capacitated case in order to analyze the expected increase in computational time (due to the increase in complexity) and the differences with respect to the optimal prices and expected demand.

Table 1 presents aggregate statistics of the expected optimal revenue and computational time. We observe that as R increases, the standard deviation decreases. Regarding the computational time, we observe the exponential growing with respect to R, as expected, which becomes particularly noticeable from $R = 100$ (average computational time of 24 min) to $R = 250$ (average computational time of 105 min). Figs. 2 and 3 provide the boxplots for the optimal prices and expected demand, respectively. In all cases, the interquartile range (difference between the upper and lower quartiles) decreases with respect to R.

Capacitated case Table 2 shows the increase in computational time in reference to the uncapacitated case, which suggests that the implementation of the priority list and the tracking of the occupancy for each alternative hugely complicate the solution approach. The jump in average computational time is in this case from 5 h ($R = 100$) to 21 h ($R = 250$). As in the uncapacitated case, we can observe in Figs. 4 and 5 a decreasing variability of the optimal prices and expected demand as R increases.

Both PSP and PUP are more expensive in the capacitated case. Since the demand of PSP was already higher than its current capacity in the uncapacitated case, its price can be increased so that the operator obtains a higher revenue from the users accessing the facility. The price of PUP is also higher, but the demand is similar to the one obtained in the uncapacitated case, which might also be influenced by the capacity restriction on PSP, since normally the opt-out option is the least attractive alternative. However, FSP is experiencing an increase in its demand because it is capturing the users that cannot be allocated due to capacity limitations or that are not willing to pay the current price of the paid alternatives.

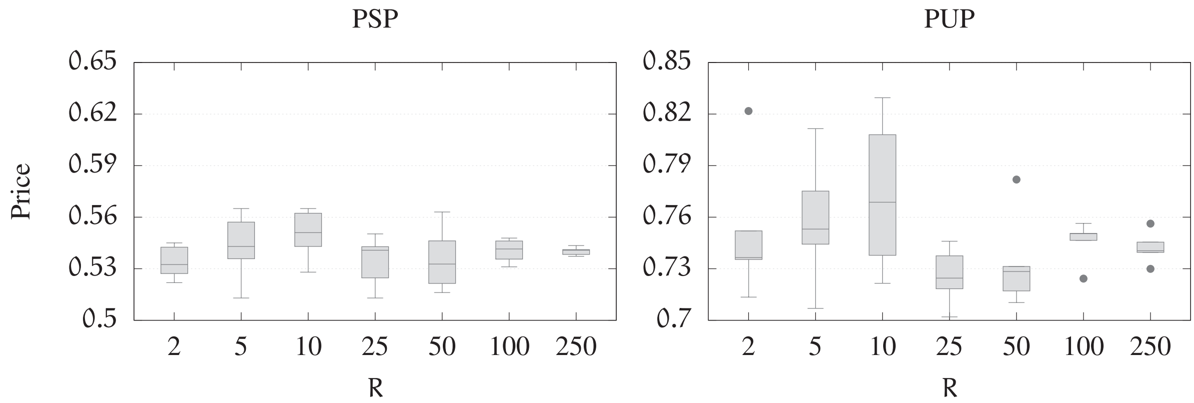

Fig. 2. Boxplot of the optimal prices for the revenue maximization problem (uncapacitated case)

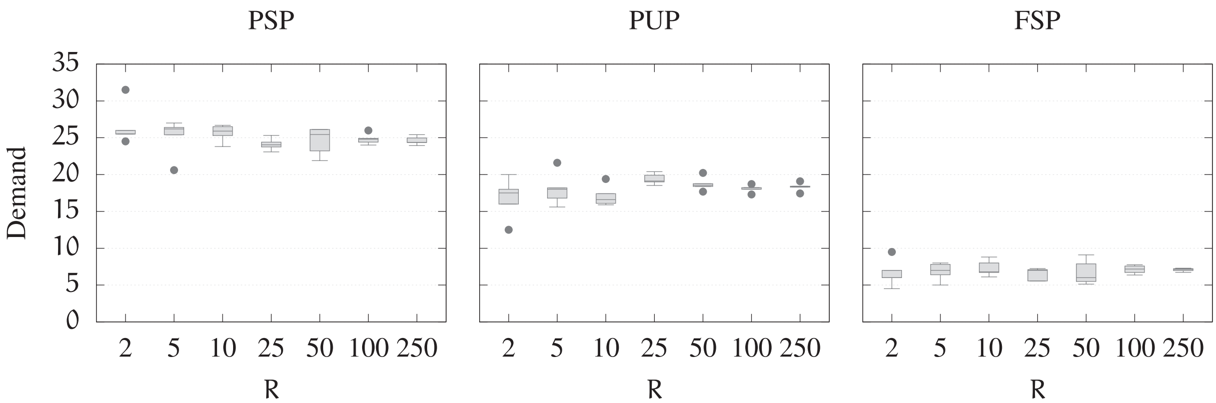

Fig. 3. Boxplot of the expected demand for the revenue maximization problem (uncapacitated case)

Table 2
Computational results of the revenue maximization problem (capacitated case)

R	Expected optimal revenue				Computational time (min)		
	Minimum	Average	Maximum	Standard deviation	Minimum	Average	Maximum
2	24.972	26.183	27.699	1.011	0.006	0.010	0.022
5	25.706	26.246	26.779	0.504	0.093	0.115	0.142
10	25.616	26.404	26.810	0.486	1.157	1.656	2.155
25	25.738	26.029	26.297	0.205	7.714	13.160	22.793
50	25.676	26.000	26.280	0.244	38.119	59.885	69.313
100	25.835	26.015	26.131	0.109	204.940	300.442	450.428
250	25.933	25.977	26.067	0.053	657.679	1261.398	2020.310

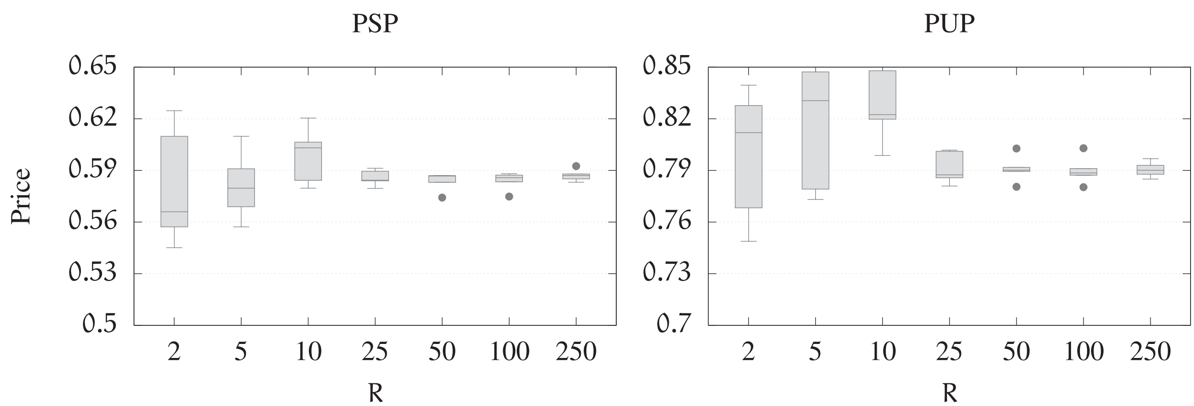

Fig. 4. Boxplot of the optimal prices for the revenue maximization problem (capacitated case)

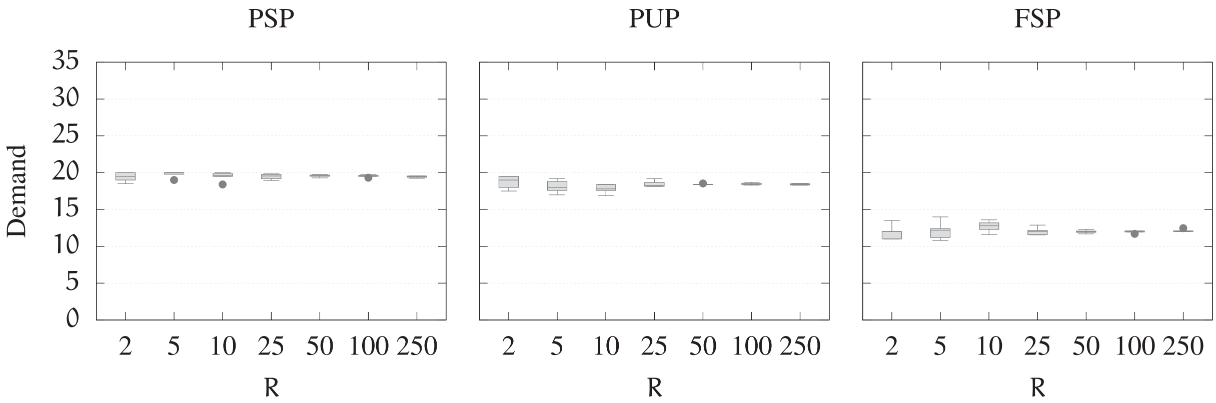

Fig. 5. Boxplot of the expected demand for the revenue maximization problem (capacitated case)

Table 3
Relative differences of the expected optimal revenues obtained by solving the revenue maximization problem with respect to the expected optimal revenues obtained by evaluating the optimal prices with $R^* = 10^6$

R	Relative difference (%) uncapacitated case			Relative difference (%) capacitated case		
	Minimum	Average	Maximum	Minimum	Average	Maximum
2	0.803	2.972	5.214	0.060	3.258	8.139
5	0.048	2.048	4.254	3.377	3.856	4.252
10	1.132	2.217	3.461	0.937	2.952	4.550
25	0.376	0.570	0.898	0.131	0.707	1.424
50	0.077	0.356	0.732	0.167	0.727	1.320
100	0.057	0.302	0.688	0.192	0.443	0.849
250	0.151	0.249	0.360	0.020	0.171	0.503

Evaluation of optimal solutions We now evaluate the obtained solutions by solving the revenue maximization problem for a very large number of draws, namely $R^* = 10^6$, with the price variables fixed to the values obtained with a lower number of draws. Table 3 includes the relative differences to assess the quality of such solutions with the expected optimal revenues obtained with evaluation used as reference values. The relative differences decrease as R increases, which indicate that the expected gain becomes more robust with larger values of R. Indeed, when $R \geq 25$, the average relative difference is in both cases lower than 1%.

Required number of draws The number of draws to be considered in order to obtain accurate results is closely related to the variance of the random components of the DCM. That is, if the DCM is highly deterministic (low variance of the random components), the choices will mostly be driven by the systematic component of the utility function, and a low number of draws will be required. We illustrate the level of stochasticity in this experiment by allowing for an error rate defined as the ratio of the choices estimated by the model that do not match the deterministic choice (i.e., the choice obtained in the absence of the random component) to the total number of estimated choices. Notice that we set the distributed parameters of the mixture of logit models equal to the mean of the associated probability distributions in order to generate the deterministic choices. The error rate fluctuates between 50% and 52% for the different tested values of R in the capacitated case, which indicates that the level of stochasticity of this DCM is considerable.

Nevertheless, as seen in the performed analyses, the obtained quantities are stable enough for $R \geq 25$, in the sense that a low variability is experienced. For the sake of illustration, we consider $R = 50$ for the experiments in Section 5.2, as it provides a good compromise between computational time and accuracy of the results. Since the problem containing the capacity as a decision variable is computationally more expensive, we allow for $R = 25$ in Section 5.3. Furthermore, in Sections 5.2 and 5.4 we evaluate multiple values of R in order to test various instances.

5.2. Population segmentation

The disaggregate representation of the demand provides a large deal of flexibility when it comes to integrate population segmentation strategies within the framework. In this section, we test a price differentiation scheme for two market segments based on the residency in the area of interest, and a grouping approach that gathers individuals with similar behavior. In the first experiment, the users are modeled individually, whereas in the second experiment the consideration of user groups reduces the size of the formulation as a single user represents multiple users with similar characteristics. In both cases, capacity restrictions are assumed and 5 replications for each value of R are run. Finally, we analyze the impact of a random priority list on the obtained results when the users are individually represented and gathered into groups.

Table 4
Average expected optimal revenue and range of results (5 replications) for different discount percentages in situation 1

%	Revenue	Prices (NR)		Demand (PSP)		Demand (PUP)		Demand (FSP)	
		PSP	PUP	R	NR	R	NR	R	NR
20	28.94	[0.63,0.66]	[0.86,0.89]	[10.1,11.0]	[8.7,9.3]	[7.0,7.9]	[10.7,11.4]	[3.5,4.1]	[7.7,8.2]
25	29.74	[0.67,0.70]	[0.89,0.94]	[10.4,11.1]	[7.8,8.7]	[7.5,8.1]	[10.0,10.9]	[3.1,3.7]	[8.7,9.7]
30	30.57	[0.69,0.71]	[0.91,0.95]	[10.8,11.5]	[7.7,8.4]	[7.9,8.7]	[9.8,10.4]	[2.2,2.8]	[9.5,9.9]
40	32.12	[0.76,0.77]	[0.99,1.05]	[11.3,12.8]	[6.1,6.8]	[7.8,9.0]	[8.7,9.4]	[1.3,1.7]	[12.1,13.0]
50	33.91	[0.84,0.95]	[1.14,1.24]	[10.9,12.6]	[2.6,5.6]	[8.6,9.5]	[6.7,8.1]	[0.7,1.7]	[15.0,17.4]

Table 5
Average expected optimal revenue and range of results (5 replications) for different discount percentages in situation 2

%	Revenue	Prices (NR)		Demand (PSP)		Demand (PUP)		Demand (FSP)	
		PSP	PUP	R	NR	R	NR	R	NR
20	26.26	[0.63,0.66]	[0.86,0.89]	[10.1,11.0]	[8.7,9.3]	[7.0,7.9]	[10.7,11.4]	[3.5,4.1]	[7.7,8.2]
25	26.13	[0.67,0.69]	[0.90,0.93]	[10.4,11.4]	[7.8,8.7]	[7.1,8.1]	[10.3,10.7]	[3.3,3.6]	[8.7,9.7]
30	25.93	[0.69,0.71]	[0.91,0.94]	[10.8,11.5]	[7.7,8.6]	[7.9,8.7]	[10.2,10.5]	[2.2,2.8]	[9.2,9.9]
40	25.08	[0.71,0.76]	[0.96,0.99]	[11.3,12.2]	[6.1,8.2]	[8.7,9.6]	[9.2,9.8]	[1.0,1.4]	[10.5,12.4]
50	23.77	[0.71,0.77]	[1.01,1.05]	[11.4,12.3]	[6.7,8.4]	[9.4,10.3]	[7.9,9.0]	[0.2,0.6]	[11.1,12.6]

Price differentiation Imagine that the municipality provides reduced fees to residents (R) who want to access one of the paid alternatives. This is actually done in many cities, where residents get reduced prices for common parking services or even exclusive areas, where only them have the right to park. In this case, we assume a discount factor that is applied to the prices offered to non-residents (NR).

Regarding the operator's revenue, two situations are considered: (1) the difference between the actual price of the service and the contribution of the resident is paid by the municipality in the form of a subsidy and, therefore, contributes to the revenue of the operator, and (2) the operator is obliged by the municipality to offer reduced fees to residents, without any other compensation than the right to operate the parking. In both situations, the reduced prices have an impact on the utility functions of the residents, and consequently on their choice. In situation 1, the revenue is not impacted whereas in situation 2 the reduced fares cause a decrease in the total gain.

Since residents only pay a part of the price that non-residents pay, the former users might be attracted to higher prices, so we expect them to increase. We modify the price bounds of PSP and PUP as follows: $p_{PSP} \in [0.6, 1.2]$ and $p_{PUP} \in [0.8, 1.4]$. The average expected optimal revenue as well as the interval defined by the lowest and highest values obtained for the optimal prices and expected demand for 5 replications are included in Tables 4 and 5 for situations 1 and 2, respectively. In both situations, the higher the discount, the higher the prices being offered, as expected. However, this increase is more moderate in situation 2 because it leads to a decrease in the expected revenue.

In terms of the expected demand, we observe that the higher the discount, the lower the expected non-resident demand of PSP and PUP, and the higher the number of non-residents deciding to opt-out, as they are not willing to pay the offered fees and choose FSP instead. Certainly, the expected resident demand experiences the opposite, since it decreases for FSP and increases for PSP and PUP as the discount increases. Note that low discount percentages have limited impact on the optimal prices and expected demand, which indicates that larger discounts need to be put in place to alter the choices of the users.

Grouping of users In this experiment, we define $\mathcal{G}$ groups of individuals with homogeneous behavior. We keep the same notation and denote the groups by n and their size (number of individuals) by θ_n. We assume that $\theta_n \leq c_i, \forall i \in \mathcal{C} \setminus \{0\}, n,$ i.e., the size of the groups does not exceed the capacity of the alternatives.

We rely on the socioeconomic variables that are present in the DCM to define groups with homogeneous behavior. More precisely, we allow for all possible combinations of the values of the binary variables Resident (residency), $\text{Origin}_{\text{INT\_FSP}}$ (origin of the trip), LowInc (level of income) and $\text{AgeVeh}_{\leq 3}$ (age of the vehicle). A description of these variables is provided in Appendix C. Table 6 includes the user groups derived from such combinations and the sizes within the considered sample ($N = 50$). Notice that 4 of the groups are not represented in the sample, so $\mathcal{G} = 12$. Thus, as $\mathcal{G} < N$, the number of variables and constraints of the resulting MILP formulation is reduced.

The values of the remaining variables present in the DCM (the attributes of the alternatives) are the same across individuals because they correspond to one of the choice situations of the SP experiment performed in Ibeas et al. (2014). This might not be the case in other contexts (e.g., revealed preference data). If so, each variable needs to be set to a single value representing the individuals in the group (e.g., the average).

Concerning the formulation, we need to make some assumptions on the access of groups to the alternatives with respect to capacity restrictions. In order to remain as consistent as possible with the individual-based model, we assume that groups cannot be split, i.e., a group does not have access to an alternative if there is not enough capacity to accommodate all its individuals. Furthermore, the priority list needs to be respected, which is determined at random, but we allow group $n + 1$

Table 6
Definition of user groups

n	θ_n	Resident	Origin$_{INT_FSP}$	LowInc	AgeVeh$_{\leq 3}$	n	θ_n	Resident	Origin$_{INT_FSP}$	LowInc	AgeVeh$_{\leq 3}$
1	2	1	1	1	1	9	0	0	1	1	1
2	10	1	1	1	0	10	0	0	1	1	0
3	1	1	1	0	1	11	1	0	1	0	1
4	2	1	1	0	0	12	0	0	1	0	0
5	1	1	0	1	1	13	7	0	0	1	1
6	5	1	0	1	0	14	11	0	0	1	0
7	1	1	0	0	1	15	5	0	0	0	1
8	0	1	0	0	0	16	4	0	0	0	0

to have access to an alternative provided that it fits even if group n could not access because its size exceeded the remaining capacity.

Hence, we only need to replace constraints (24) and (25) with their adaptation at the group level given by constraints (34) and (35). Constraint (34) becomes active when there is no room for group n, i.e., $\sum_{m=1}^{n-1} \theta_m w_{imr} > c_i - \theta_n$, as it forces y_{inr} to be equal to 0. Similarly, when there is room for group n, i.e., $\sum_{m=1}^{n-1} \theta_m w_{imr} \leq c_i - \theta_n$, constraint (35) implies $y_{inr} = 1$. Notice that there is always room for group $n = 1$ (first group in the priority list) thanks to the assumption $\theta_n \leq c_i, \forall i \in \mathcal{C} \setminus \{0\}, n$.

$$\sum_{m=1}^{n-1} \theta_m w_{imr} \leq (c_i - \theta_n) y_{inr} + \left(\sum_{m=1}^{n-1} \theta_m \right) (1 - y_{inr}), \qquad \forall i \in \mathcal{C}, n > 1, r, \tag{34}$$

$$(c_i - \theta_n + 1)(y_{in} - y_{inr}) \leq \sum_{m=1}^{n-1} \theta_m w_{imr}, \qquad \forall i \in \mathcal{C}, n > 1, r. \tag{35}$$

Table 7 presents the aggregate statistics of the expected optimal revenue and computational time with the user groups defined in Table 6. The computational times are much lower than the ones obtained at the individual level (see Table 2), which also suggests an exponential growing with respect to the number of individuals. For example, the average computational time for $R = 250$ is drastically reduced from 21 h to less than 3 min. We observe that the expected optimal revenues fluctuate within lower values than in the individual case, and the associated standard deviation, though it decreases as R increases, is larger.

The decrease on the obtained values for the revenue is closely related to the group access condition. As shown in Figs. 8 and 9 included in Appendix D, the expected demand for both PSP and PUP is lower, and even though the associated prices are higher, it is not enough to reach the gains obtained at the individual level. The interquartile range of the prices of both services tends to be wider. The increase in prices is again due to the fact that the number of individuals that want to access the paid alternatives is larger than the available capacity, although in this case it is not reached because of the group access condition.

Impact of the arrival of users The priority list introduced in Section 3.2 states the order in which the users are considered to access the alternatives. In this case study, it is defined at random. We expect that a random ordering will have a higher impact on the expected optimal revenue at the group level because of the aggregation of individuals into groups and the fact that the resulting groups are heterogeneous in size. We analyze this effect by evaluating multiple realizations of the priority list, both at the individual level and at the group level.

For the sake of illustration, we consider situation 2 (the operator is forced by law to offer a discount on the fees for residents) and a 30% discount rate. The idea is to test whether the distribution of users (groups) in the priority list affects the expected optimal revenue, as residents benefit from a discount on the price paid by non-residents. We construct 100

Table 7
Computational results for the revenue maximization problem with the user groups defined in Table 6 (capacitated case)

R	Expected optimal revenue				Computational time (min)		
	Minimum	Average	Maximum	Standard deviation	Minimum	Average	Maximum
2	22.950	26.577	29.275	2.501	0.000	0.001	0.001
5	23.314	24.822	25.809	0.995	0.002	0.003	0.003
10	24.139	24.524	25.060	0.355	0.004	0.006	0.008
25	23.446	23.884	24.596	0.477	0.023	0.032	0.051
50	23.299	23.670	24.447	0.473	0.093	0.111	0.143
100	23.328	23.608	24.225	0.366	0.357	0.413	0.479
250	23.493	23.662	23.912	0.158	1.846	2.182	2.828

Fig. 6. Histogram of the expected optimal revenue in situation 2 for a discount rate of 30% (individual level)

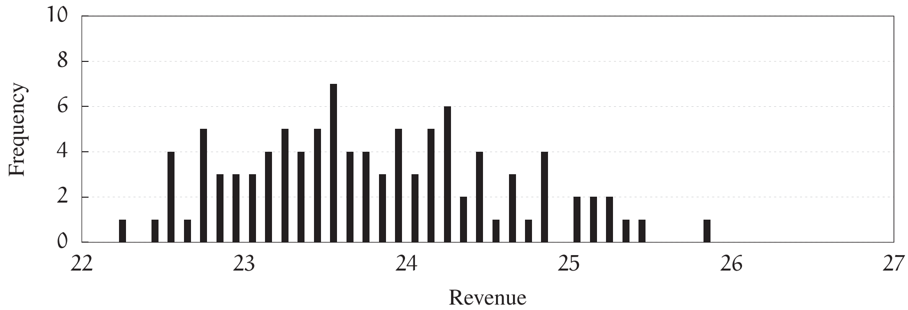

Fig. 7. Histogram of the expected optimal revenue in situation 2 for a discount rate of 30% (group level)

realizations of the priority list by shuffling the users (groups), so that they might arrive in a different order. We run these instances with $R = 50$, $p_{\text{PSP}} \in [0.6, 1.2]$ and $p_{\text{PUP}} \in [0.8, 1.4]$.

Figs. 6 and 7 present the histograms of the expected optimal revenue when the users are individually and jointly modeled, respectively. As already anticipated, the expected optimal revenue at the group level is more dispersed, with values that fluctuate between 22 and 26 (standard deviation equal to 0.79), whereas it ranges from 25 to 27 (standard deviation equal to 0.30) at the individual level. Similar to the previous experiment, lower values for the expected optimal revenue are obtained (see Table 7).

We perceive a higher concentration of values around an expected optimal revenue of about 26 at the individual level and about 23.5 at the group level. The former is in line with the average value 25.928 previously obtained (see Table 5). For the latter, and given that a 30% discount in situation 2 does not have a huge impact on the expected optimal revenue at the individual level, we observe that is also in line with the average value 23.670 previously obtained (see Table 7). This is consistent with the findings of Binder et al. (2017), who show that aggregate indicators are stable across realizations of a random priority list.

It is worth noticing that the average computational time of the individual approach (approximately 2 h) is much larger than its grouping counterpart (approximately 1 min). Hence, one way to smooth the dispersion manifested at the group level might consist in increasing the number of draws while keeping an advantageous computational time. Nevertheless, we note that additional assumptions (e.g., split of groups) might need to be implemented to address the fact that lower values for the expected optimal revenue are achieved.

5.3. Profit maximization through capacity allocation

In this section, we test different capacity levels for both services. As described in Section 3.2, we replicate the services as many times as capacity levels we want to evaluate. We consider 5, 10, 15 and 20 parking spots ($Q = 4$) for both services, which makes four copies of PSP and four of PUP, each of them with the same utility function but a different capacity level. Together with FSP, this experiment contains 9 different “alternatives.”

Note that constraint (26) does not force the opening of both paid services, since it might be more convenient from a profit maximization point of view to allocate all operator's resources to only one of the facilities. If we want to make sure that both PSP and PUP are offered, we can replace this constraint by:

$$\sum_{q=1}^{Q} y_{iq} = 1, \quad \forall i \in \mathcal{C} \setminus \{0\}. \tag{36}$$

Table 8
Results for the profit maximization problem with capacities as decision variables (in brackets the number of replications for which the associated capacity level was obtained)

	Avg. comp. time (h)	Optimal capacity		Expected demand			Optimal prices		Avg. optimal profit
		PSP	PUP	PSP	PUP	FSP	PSP	PUP	
(29)	9.72	20 (5)	NA	[18.2,19.6]	NA	[30.4,31.9]	[0.78,0.85]	NA	6.94
(36)	11.4	20 (4)	10 (5)	[14.8,20.0]	[8.76,9.4]	[20.9,26.2]	[0.65,0.73]	[1.00,1.09]	5.99

As mentioned in Section 4, the cost associated with operating a parking facility is composed of a fixed cost and a variable cost (in this case, a cost per parking spot). We assume that both types of cost are the same among capacity levels. More precisely, $f_{PSP} = 1.5$, $v_{PSP} = 0.35$, $f_{PUP} = 3$ and $v_{PUP} = 0.5$. We set the price bounds to $p_{PSP} \in [0.6, 1.2]$ and $p_{PUP} \in [0.8, 1.4]$.

Table 8 includes the results for both approaches (considering constraints (26) and (36), respectively) and 5 replications each. We see that it is beneficial to close PUP and only open PSP with the highest level of capacity. Indeed, when we impose that both facilities must be opened, PUP is offered with a low capacity level (10 in all tested replications), and a lower expected optimal profit is achieved. The average solution time gives us an idea of the increase in complexity with respect to the revenue maximization problem with fixed capacity. For $R = 25$, it goes from approximately 1 h to almost 10 h with constraint (26) and to more than 11 h with constraint (36).

5.4. Increasing the problem size

As illustrated in Section 5.1, the revenue maximization problem with fixed capacity is computationally complex, even for a population of $N = 50$ users and a number of simulation draws that does not exceed $R = 250$. The experiments performed in Sections 5.1 and 5.2 evidence the exponential growing with respect to the number of scenarios as well as the number of individuals. Indeed, additional tests for $N = 197$ (the size of the sample under consideration), $R = 50$ and an adjusted capacity $c_{PSP} = c_{PUP} = 80$ took on average 5.25 days (see Table 9).

One way to cope with such complexity involves the discretization of the price variables and the decomposable structure of the framework. More precisely, we characterize a procedure that iterates over a set of predetermined price levels and for each price configuration solves the revenue maximization problem with fixed prices, which is notably less challenging. The revenue maximization problem with fixed prices can be solved by iterating over the scenarios and the individuals in the order given by the priority list. For each scenario, the occupancy of the paid alternatives is initialized to 0 and is gradually updated with the choices performed by the individuals. As soon as the occupancy is equal to the capacity, the alternative is no longer available for the upcoming individuals. Notice that the behavioral assumption related to the highest utility can be directly verified as the prices are the only endogenous variables and their values are fixed to the price levels being evaluated. The price configuration providing the highest revenue is then selected as the optimal solution. The pseudocode of this iterative approach is presented in Algorithm 1 .

Table 9 shows the comparison between the two approaches (continuous and discrete prices) for $N = 197$, $R = 50$ and 5 replications. Notice that $p_{PSP} \in [0.5, 0.65]$ and $p_{PSP} \in [0.7, 0.85]$ in the continuous case and $p_{PSP} \in \{0.5, 0.51, \ldots, 0.65\}$ and $p_{PUP} \in \{0.6, 0.61, \ldots, 0.75\}$ in the discrete case, i.e., 16 price levels have been considered for both PSP and PUP. The difference in computational time is tremendous, and the obtained optimal prices and expected optimal revenue are in line with their continuous counterparts. The expected demand presents a larger variability, especially for the opt-out option. As the computational time is more advantageous for discrete prices, it would be possible to refine the discretization from the current spacing between levels to 10^{-3}, which generates 151 price levels for each alternative. The average computational time in this case is 10.6 s, and the average expected optimal revenue is 104.63, which is closer to the value obtained for continuous prices. Hence, when a discretization of the endogenous variables can be implemented, this strategy provides a very efficient technique to handle problems of large size, which can in turn be combined with additional strategies such as the grouping of individuals introduced in Section 5.2.

Table 10 presents the results for $p_{PSP} \in \{0.5, 0.51, \ldots, 0.65\}$ and $p_{PUP} \in \{0.6, 0.61, \ldots, 0.75\}$ and various values of R. The average computational time is extremely manageable, even for very large numbers of draws. Furthermore, as R increases, we hardly observe any change in the obtained results among replications. Indeed, the same optimal prices are obtained in all replications, and the range for the values obtained for the expected demand becomes more and more narrow. Concerning

Table 9
Results for the revenue maximization problem with fixed capacities for continuous and discrete prices for $N = 197$ and $R = 50$ (in brackets the number of replications for which the associated price level was obtained)

	Avg. comp. time	Expected demand			Optimal prices		Avg. optimal revenue
		PSP	PUP	FSP	PSP	PUP	
Continuous	5.25 days	[79.2,79.6]	[75.9,77.0]	[40.8,41.5]	[0.57,0.58]	[0.77,0.78]	104.71
Discrete	0.11 s	[77.4,78.9]	[75.3,76.6]	[41.7,43.3]	0.58 (4), 0.59 (1)	0.77 (3), 0.78 (2)	104.53

Algorithm 1: Revenue maximization problem with discrete prices and fixed capacity

Input: Price levels: $\{p_i^\ell\}_\ell, \forall i \in \mathcal{C} \setminus \{0\}$;
Output: Optimal solution for the revenue maximization problem with fixed capacity and optimal objective function G;
Initialize the optimal objective function $G = 0$;
for *all combinations of* $\bar{p}_i \in \{p_i^\ell\}_\ell, \forall i \in \mathcal{C} \setminus \{0\}$ **do**
  Initialize the current optimal objective function $G^c = 0$;
  for $r = 1 \ldots R$ **do**
    Initialize occupancy level $o_{ir} = 0, \forall i \in \mathcal{C} \setminus \{0\}$ and availability variables $y_{inr} = 1$, $\forall i \in \mathcal{C}_n, n$;
    for $n = 1 \ldots N$ **do**
      for $i \in \mathcal{C}_n$ **do**
        if $o_{ir} < c_i$ **then**
          Calculate $U_{inr} = \beta_{in}\bar{p}_i + g_{in}(x_{in}^d) + \xi_{inr}$ ($\beta_{in} = 0$ for $i = 0, \forall n$);
        else
          Set the alternative unavailable: $y_{inr} = 0$;
          Set the utility to the corresponding lower bound: $U_{inr} = \ell_{nr}$;
      Determine $U_{nr} = \max_{\{i \in \mathcal{C}_n | y_{inr} = 1\}} U_{inr}$ and $j = \arg\max U_{nr}$;
      Set $w_{jnr} = 1$ and $w_{inr} = 0, \forall i \in \mathcal{C}_n \setminus \{j\}$;
      Update the current objective function $G^c = G^c + \sum_{i \in \mathcal{C} \setminus \{0\}} \frac{1}{R} w_{inr}\bar{p}_i$;
      Update the occupancy level $o_{jr} = o_{jr} + 1$;
  if $G^c > G$ **then**
    $G = G^c$ and update the optimal solution accordingly;

Table 10
Results for the revenue maximization problem with fixed capacities and discrete prices for $N = 197$ (in brackets the number of replications for which the associated price level was obtained)

R	Avg. comp. time (s)	Expected demand			Optimal prices		Avg. optimal revenue
		PSP	PUP	FSP	PSP	PUP	
250	0.72	[78.6,78.8]	[75.9,76.6]	[41.7,42.4]	0.58 (5)	0.77 (5)	104.38
500	1.60	[78.5,78.8]	[76.2,76.4]	[42.0,42.2]	0.58 (5)	0.77 (5)	104.38
1000	4.32	[78.6,78.8]	[76.2,76.6]	[41.9,42.1]	0.58 (5)	0.77 (5)	104.42
2500	11.69	[78.6,78.7]	[76.2,76.4]	[42.0,42.1]	0.58 (5)	0.77 (5)	104.40
5000	24.32	[78.6,78.7]	[76.3,76.3]	[42.0,42.0]	0.58 (5)	0.77 (5)	104.40

the average optimal revenue, it remains stable for the different values of R. The obtained values are below the average optimal revenue for continuous prices and $R = 50$. As discussed before, the discretization can be refined, but notice that the average optimal revenue in the continuous case might as well decrease for larger values of R.

Finally, we can perform the same comparison for the profit maximization problem. To this end, we define a procedure that iterates over the capacity levels and for each configuration runs Algorithm 1 with the expected profit instead of the expected revenue as objective function. We also consider $N = 50$, $R = 25$, $p_{\text{PSP}} \in \{0.6, 0.61, \ldots, 1.2\}$ and $p_{\text{PUP}} \in \{0.8, 0.81, \ldots, 1.4\}$ as price levels, and $c_{\text{PSP}} \in \{0, 5, 10, 15, 20\}$ and $c_{\text{PUP}} \in \{0, 5, 10, 15, 20\}$ as capacity levels (if a 0 capacity level is selected, it means that the alternative is not offered). As expected, the computational time is remarkably more advantageous, and the expected optimal profit is notably close to the one obtained for continuous prices. Regarding the capacity, PSP is also offered with 20 spots and PUP is not offered at all. Notice that thanks to the proposed approach the number of draws can be increased in order to enhance the accuracy of the results (see Table 11).

Table 11
Results for the profit maximization problem with capacities as decision variables for continuous and discrete prices for $N = 50$ and $R = 25$ (in brackets the number of replications for which the associated capacity level and price level was obtained)

	Avg. comp. time	Optimal capacity		Expected demand			Optimal prices		Avg. optimal profit
		PSP	PUP	PSP	PUP	FSP	PSP	PUP	
Continuous	9.72 h	20 (5)	NA	[18.2,19.6]	NA	[30.4,31.9]	[0.78,0.85]	NA	6.94
Discrete	4.3 s	20 (5)	NA	[18.9,19.5]	NA	[30.5,31.2]	0.78 (1), 0.8 (1), 0.81 (3)	NA	6.91

6. Conclusions and future work

We have proposed a mixed-integer linear formulation of a DCM based on the random utility principle that is designed to be included in an MILP model in order to provide it with a disaggregate representation of the demand that captures its interactions with the supply-related decisions to be made by the operator. It is general in the sense that it is not limited to simple DCM and it can be embedded in any MILP formulation. The stochasticity of the model is captured by drawing from the distribution of the involved random variables. This enables to avoid the explicit formulation of the choice probabilities and to work directly with the utility functions, using the first principles of utility maximization and SAA.

Concerning the supply side, an illustrative MILP model of the profit maximization problem is characterized. The formulation accounts for the preferences of individuals when deciding on the prices via the utility functions. Since the individuals are not captive, i.e., they can leave the market by choosing the opt-out option, there exists a trade-off between the price and their choices.

The results of the case study show that this methodology allows to configure the features of a system (e.g., the price) based on the heterogeneous behavior of individuals. Additional features such as complex behavioral patterns and alternative supply configurations can also be investigated within the proposed framework. Nevertheless, the disaggregate representation of the preferences of individuals and the linear nature of the formulation result in a high-dimensional problem, and therefore solving it is computationally expensive. This is an issue that needs to be addressed because in practice populations are large and a high number of draws is desirable to be as close as possible to the true values. Notice that the latter depends on the accuracy of the DCM. For the parking choice model, we can rely on a relatively low number of draws to obtain stable estimates of the quantities of interest despite its substantial level of stochasticity.

The flexibility of the framework enables the development of different strategies to deal with the complexity of the resulting mathematical models. The gathering of individuals with similar behavior, which results in a reduction in size of the formulation, has been explored. Notwithstanding the clear advantages in computational time, the price to pay is the fact that the realism of the grouping assumption decreases as the size of the group increases. In a different way, we could add valid inequalities to the MILP model, but they are very likely to depend on the case study. Decomposition techniques are convenient in this case, and represent an alternative to valid inequalities because they can be applied in a general way, although both could also be combined. The model, by design, is built on two dimensions that can be addressed separately. Indeed, each simulation draw constitutes an independent scenario, and each individual has an associated optimization problem, which consists of choosing the alternative among the available ones maximizing their utility. A procedure that iterates over these two dimensions and involves the discretization of the endogenous variables has proven to provide a good approximation of the optimal solution and to be very effective from a computational point of view. Alternatively, a Lagrangian decomposition scheme motivated by scenario decomposition in stochastic programing can be implemented in order to induce separability at the scenario level (without discretization of the endogenous variables).

In summary, in this paper we develop a methodology that integrates (advanced) DCM in MILP in hopes of preventing a mismatch between the demand and the supply-related decisions to be adjusted in a transportation system. The framework can be used in multiple applications, such as the ones reviewed in Section 2. In addition to the development of dedicated algorithms designed to solve large instances of the model, we plan to explore the realm of modeling possibilities of this methodology.

CRediT authorship contribution statement

Meritxell Pacheco Paneque: Conceptualization, Methodology, Software, Data curation, Writing - original draft, Writing - review & editing. **Michel Bierlaire:** Conceptualization, Methodology, Software, Writing - original draft, Writing - review & editing, Funding acquisition. **Bernard Gendron:** Conceptualization, Methodology, Writing - original draft, Writing - review & editing, Funding acquisition. **Shadi Sharif Azadeh:** Conceptualization, Methodology, Software, Writing - original draft, Writing - review & editing, Funding acquisition.

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Appendix A. Main notations

Table 12
Main notations used in the profit maximization problem (Section 4), organized by sets, parameters, variables and aggregated quantities

Name	Description	Section
$\mathcal{C}$	Set of all potential alternatives (indexed by i, $i = 0$ denotes the opt-out option)	3.1
N	Number of users in the population (indexed by $n \geq 1$)	3.1
$\mathcal{C}_n$	Set of the alternatives considered by user n	3.1
R	Number of draws from the distribution of ε_{in} (indexed by r)	3.2
Q	Number of capacity levels (indexed by q)	3.2
J	Number of alternatives in $\mathcal{C}$	3.2
ξ_{inr}	Draw from the distribution of ε_{in}	3.2
ℓ_{inr}	Lower bound on U_{inr}	3.2
m_{inr}	Upper bound on U_{inr}	3.2
ℓ_{nr}	Smallest lower bound across alternatives	3.2
m_{nr}	Largest upper bound across alternatives	3.2
M_{inr}	$m_{inr} - \ell_{nr}$	3.2
M_{nr}	$m_{nr} - \ell_{nr}$	3.2
c_i	Capacity associated with alternative $i \in \mathcal{C} \setminus \{0\}$	3.2
a_{in}	Lower bound on p_{in}	4
b_{in}	Upper bound on p_{in}	4
f_i	Fixed cost associated with alternative $i \in \mathcal{C} \setminus \{0\}$	4
v_i	Cost per sold unit of alternative $i \in \mathcal{C} \setminus \{0\}$	4
U_{in}	Utility associated with alternative i by user n	3.1
V_{in}	Deterministic part of the utility function U_{in}	3.1
ε_{in}	Error term of the utility function U_{in}	3.1
U_{inr}	Utility associated with alternative i by user n in draw r	3.2
y_{in}	Availability at operator level of alternative i to user n	3.1, 3.2
y_{inr}	Availability at draw level of alternative i to user n in draw r	3.2
z_{inr}	Discounted utility associated with alternative i of user n in draw r	3.2
w_{inr}	Choice variable associated with alternative i by user n in draw r	3.2
U_{nr}	Highest value of z_{inr}	3.2
p_{in}	Price that user n has to pay to access alternative $i \in \mathcal{C}_n \setminus \{0\}$	4
η_{inr}	Continuous variable capturing $p_{in} w_{inr}$	4
D_i	Expected demand of alternative i	3.1, 3.2
G_i	Expected gain obtained from alternative $i \in \mathcal{C} \setminus \{0\}$	4
C_i	Total cost associated with alternative $i \in \mathcal{C} \setminus \{0\}$	4

Appendix B. Convergence of the general framework

This appendix outlines the proof of the convergence property described in Section 3.3. The key idea is that the sequence of optimal solutions of the approximated optimization problems (where the demand is being approximated with the mixed-integer linear formulation proposed in Section 3.2) converges to a feasible solution of the original problem (where the demand is obtained via the choice probabilities).

B1. Definition of the problems

We consider a general optimization problem $\mathcal{P}$ defined as:

$$\min_{x_s, x_e, D} f(x_s, x_e, D) \tag{37}$$

$$\text{subject to } D = h(x_e), \tag{38}$$

$$(x_s, x_e) \in X, \tag{39}$$

$$(x_s, x_e, D) \in Y, \tag{40}$$

where the demand model $D \in \mathbb{R}^{\Delta}$ in constraint (38) is characterized with a continuous function:

$$\begin{array}{llll} h: \mathbb{R}^E & \longrightarrow & \mathbb{R}^{\Delta} \\ x_e & \longmapsto & h(x_e) = D. \end{array} \tag{41}$$

Notice that h may also depend on other variables that are exogenous to the optimization problem (x_d) and, therefore, not mentioned here. The objective function f relates the decision variables to an aggregate performance of the system:

$$\begin{array}{lll} f: \mathbb{R}^S \times \mathbb{R}^E \times \mathbb{R}^{\Delta} & \longrightarrow & \mathbb{R}, \\ (x_s, x_e, D) & \longmapsto & f(x_s, x_e, D), \end{array} \tag{42}$$

and is assumed to be continuous. The constraints of the optimization problem are expressed via sets. We assume that $X \subseteq \mathbb{R}^S \times \mathbb{R}^E$ is a closed set that represents the constraints that do not involve the demand model (integrality constraints can be characterized by this set), and $Y \subseteq \mathbb{R}^S \times \mathbb{R}^E \times \mathbb{R}^\Delta$ is a closed and bounded set (and consequently compact) that represents the constraints that involve the demand model and the bounds on the decision variables. We also assume that the problem is feasible, and that at least one global optimal solution exists, denoted by $(x_s^*,\ x_e^*,\ D^*)$.

In our particular case, we assume that the demand is represented with a discrete choice model (DCM). Hence, the function $h = (h^1, \ldots, h^J)$, where J is the number of alternatives in the choice set $\mathcal{C}$, is defined as

$$\begin{array}{lll} h^i : \mathbb{R}^E & \longrightarrow & \mathbb{R} \\ x_e & \longmapsto & D^i = \sum_{n=1}^N P_n(i|x_{in}^d, x_{in}^e), \end{array} \tag{43}$$

where $P_n(i|x_{in}^d, x_{in}^e)$ is the choice probability as defined in (4). Furthermore, we assume a linear objective function f and the sets X and Y consisting of linear constraints. This specification is motivated by the need to solve to global optimality, which requires a convex formulation, or better, linear.

We define the approximated optimization problem $\mathcal{P}_R$ as:

$$\min_{x_s, x_e, D} f(x_s, x_e, D) \tag{44}$$

$$\text{subject to } D = h_R(x_e), \tag{45}$$

$$(x_s, x_e) \in X, \tag{46}$$

$$(x_s, x_e, D) \in Y, \tag{47}$$

where the demand model is approximated with a continuous function

$$\begin{array}{lll} h_R : \mathbb{R}^E & \longrightarrow & \mathbb{R}^\Delta \\ x_e & \longmapsto & h_R(x_e) = D \end{array} \tag{48}$$

such that, for each $x_e \in \mathbb{R}^E$, we have

$$\lim_{R \to \infty} h_R(x_e) = h(x_e). \tag{49}$$

We assume that there exists $\bar{R}$ such that $\mathcal{P}_R$ is feasible $\forall R \geq \bar{R}$, and that at least an optimal solution exists, denoted by $(x_s^R,\ x_e^R,\ D^R)$. In the following, when we refer to R we implicitly assume $R \geq \bar{R}$.

The function h_R that approximates the demand model in our case is defined as

$$\begin{array}{lll} h_R^i : \mathbb{R}^E & \longrightarrow & \mathbb{R} \\ x_e & \longmapsto & D_R^i = \frac{1}{R} \sum_{n=1}^N \sum_{r=1}^R w_{inr}, \end{array} \tag{50}$$

where w_{inr} are the binary variables that determine the choice (see Section 3.2). As discussed in Section 3.2, the choice probabilities calculated with the simulation-based linearization of the DCM converge when $R \to \infty$ to the choice probabilities of the DCM (as a consequence of the law of large numbers). Therefore, h_R defined by (50) satisfies condition (49).

B2. Asymptotic feasibility

Lemma 1. *Denote by $(x_s^R,\ x_e^R,\ D^R)$ the optimal solution of problem $\mathcal{P}_R$. As the sequence $\{(x_s^R, x_e^R, D^R)\}_R$ is bounded because $(x_s^R, x_e^R, D^R) \in Y, \forall R$, it contains convergent subsequences. Consider (x_s^+, x_e^+, D^+) any of the accumulation points. Then, (x_s^+, x_e^+, D^+) is feasible for $\mathcal{P}$.*

Proof. As X and Y are closed, $(x_s^+, x_e^+) \in X$ and $(x_s^+, x_e^+, D^+) \in Y$. So we just need to show that $D^+ = h(x_e^+)$. We have

$$\begin{aligned} |h_R(x_e^R) - h(x_e^+)| &= |h_R(x_e^R) - h_R(x_e^+) + h_R(x_e^+) - h(x_e^+)| \\ &\leq |h_R(x_e^R) - h_R(x_e^+)| + |h_R(x_e^+) - h(x_e^+)|. \end{aligned}$$

The term $|h_R(x_e^R) - h_R(x_e^+)|$ converges to 0 by continuity of h_R, and the term $|h_R(x_e^+) - h(x_e^+)|$ converges to 0 by (49). Therefore, we have

$$D^+ = \lim_{R \to \infty} D^R = \lim_{R \to \infty} h_R(x_e^R) = h(x_e^+), \tag{51}$$

where the second equality comes from the feasibility of D^R for problem $\mathcal{P}_R$. □

B3. Asymptotic optimality

The asymptotic optimality of the accumulation point (x_s^+, x_e^+, D^+) follows from the fact that we have a sequence $\{(x_s^R, x_e^R, D^R)\}_R$ of optimal solutions of the approximated problems $\mathcal{P}_R$ that converges to a feasible point of the original problem $\mathcal{P}$, as shown in Lemma 1.

Appendix C. Discrete choice model specification for the case study

The discrete choice model (DCM) considered in Section 5 is the mixture of logit models estimated in Ibeas et al. (2014) for parking choice behavior. The random coefficients are the ones associated with the access time to the parking place once the user arrives to the parking area (AT) and the parking fee (FEE), and are denoted by β_{AT} and β_{FEE}, respectively. As mentioned by the authors, the latter is related to an hour of use of the parking place, regardless of the time that the spot was needed. The units are not specified. Both parameters are assumed to be normally distributed and correlated, with $cov(\text{AT}, \text{FEE}) = -12.8$. Notice that the FEE variables correspond to the price variables p_i when the DCM is embedded in the profit maximization problem.

Table 13
Specification table of the mixture of logit models in Ibeas et al. (2014)

		FSP	PSP	PUP
ASC_{PSP}	32	0	1	0
ASC_{PUP}	34	0	0	1
β_{AT}	$\sim N(-0.788, 1.06)$	AT_{FSP}	AT_{PSP}	AT_{PUP}
β_{TD}	-0.612	TD_{FSP}	TD_{PSP}	TD_{PUP}
$\beta_{\text{Origin}_{\text{INT\_FSP}}}$	-5.76	$\text{Origin}_{\text{INT\_FSP}}$	0	0
β_{FEE}	$\sim N(-32.3, 14.2)$	0	FEE_{PSP}	FEE_{PUP}
$\beta_{\text{FEE}_{\text{PSP(LowInc)}}}$	-11	0	$\text{FEE}_{\text{PSP}}\text{LowInc}$	0
$\beta_{\text{FEE}_{\text{PSP(Resident)}}}$	-11.4	0	$\text{FEE}_{\text{PSP}}\text{Resident}$	0
$\beta_{\text{FEE}_{\text{PUP(LowInc)}}}$	-13.7	0	0	$\text{FEE}_{\text{PUP}}\text{LowInc}$
$\beta_{\text{FEE}_{\text{PUP(Resident)}}}$	-10.7	0	0	$\text{FEE}_{\text{PUP}}\text{Resident}$
$\beta_{\text{AgeVeh}\leq 3}$	4.04	0	0	$\text{AgeVeh} \leq 3$

The other variables appearing in the utility specification are the following: access time to the destination from the parking spot (TD), an indicator variable that is 1 if the origin of the trip is internal to the town ($\text{Origin}_{\text{INT\_FSP}}$), an indicator variable that is 1 if the income of the user is below 1200€/month (LowInc), an indicator variable that is 1 if the user is resident (Resident), and an indicator variable that is 1 if the age of the vehicle is lower than 3 years ($\text{AgeVeh}_{\leq 3}$). Two interactions to address the variations in taste among users are considered: FEE with having a low income and FEE with being resident. Table 13 provides the specification table of the described DCM.

Appendix D. Exhaustive results for the grouping of users in Section 5.2

The boxplots for the optimal prices and expected demand for the user groups described in Section 5.2 are included in Figs. 8 and 9, respectively.

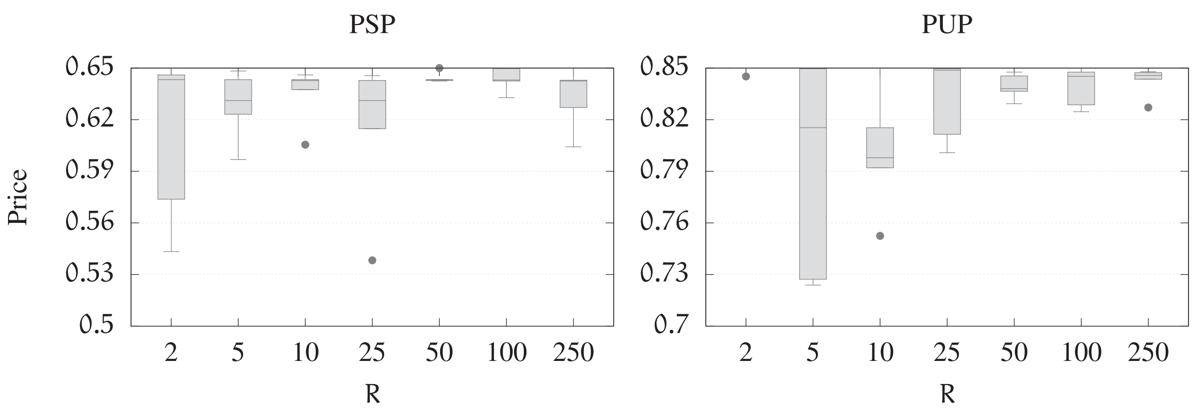

Fig. 8. Optimal prices for the user groups described in Section 5.2 (capacitated case)

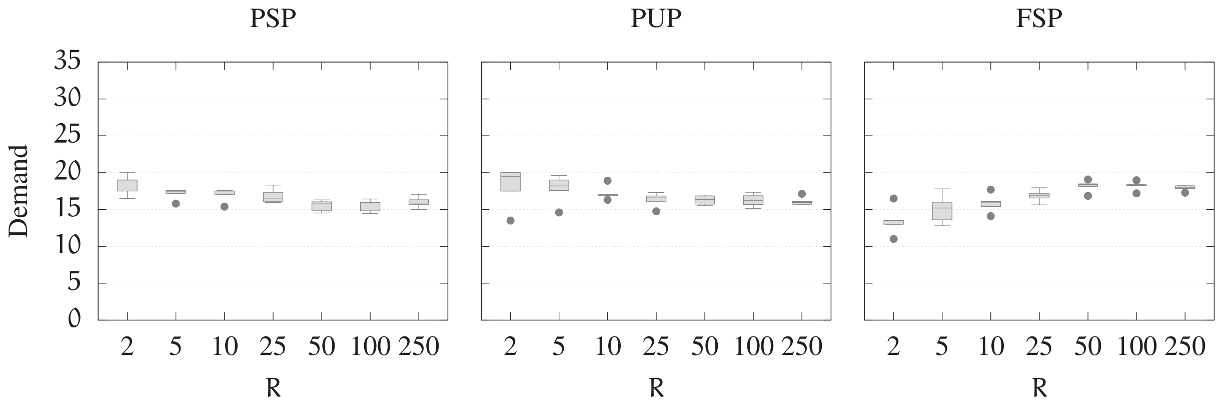

Fig. 9. Expected demand for the user groups described in Section 5.2 (capacitated case)

References

Andersson, S.-E., 1998. Passenger choice analysis for seat capacity control: a pilot project in scandinavian airlines. Int. Trans. Oper. Res. 5 (6), 471–486. doi:10.1111/j.1475-3995.1998.tb00130.x.

Ben-Akiva, M., Boccara, B., 1995. Discrete choice models with latent choice sets. Int. J. Res. Mark. 12 (1), 9–24. doi:10.1016/0167-8116(95)00002-J. Consideration sets.

Benati, S., 1999. The maximum capture problem with heterogeneous customers. Comput. Oper. Res. 26 (14), 1351–1367. doi:10.1016/S0305-0548(99)00040-4.

Benati, S., Hansen, P., 2002. The maximum capture problem with random utilities: problem formulation and algorithms. Eur. J. Oper. Res. 143 (3), 518–530. doi:10.1016/S0377-2217(01)00340-X.

Binder, S., Maknoon, Y., Bierlaire, M., 2017. Exogenous priority rules for the capacitated passenger assignment problem. Transp. Res. Part B 105, 19–42. doi:10.1016/j.trb.2017.08.022.

de Palma, A., Proost, S., Seshadri, R., Ben-Akiva, M., 2018. Congestion tolling - dollars versus tokens: acomparative analysis. Transp. Res. Part B 108, 261–280. doi:10.1016/j.trb.2017.12.005.

Duran, M.A., Grossmann, I.E., 1986. An outer-approximation algorithm for a class of mixed-integer nonlinear programs. Math. Program. 36 (3), 307–339. doi:10.1007/BF02592064.

Farooq, B., Bierlaire, M., Hurtubia, R., Flötteröd, G., 2013. Simulation based population synthesis. Transp. Res. Part B 58 (C), 243–263. doi:10.1016/j.trb.2013.09.012.

Fernández-Antolín, A., Lurkin, V., de Lapparent, M., Bierlaire, M., 2017. Discrete-continuous maximum likelihood for the estimation of nested logit models. In: Proceedings of the 16th Swiss Transport Research Conference. Ascona, Switzerland.

Gilbert, F., Marcotte, P., Savard, G., 2014a. Mixed-logit network pricing. Comput. Optim. Appl. 57 (1), 105–127. doi:10.1007/s10589-013-9585-0.

Gilbert, F., Marcotte, P., Savard, G., 2014b. Logit network pricing. Comput. Oper. Res. 41, 291–298. doi:10.1016/j.cor.2013.05.010.

Gilbert, F., Marcotte, P., Savard, G., 2015. A numerical study of the logit network pricing problem. Transp. Sci. 49 (3), 706–719. doi:10.1287/trsc.2014.0560.

Haase, K., Müller, S., 2013. Management of school locations allowing for free school choice. Omega 41 (5), 847–855. doi:10.1016/j.omega.2012.10.008.

Huang, K., de Almeida Correia, G.H., An, K., 2018. Solving the station-based one-way carsharing network planning problem with relocations and non-linear demand. Transp. Res. Part C 90, 1–17. doi:10.1016/j.trc.2018.02.020.

Ibeas, A., dell'Olio, L., Bordagaray, M., Ortúzar, J., 2014. Modelling parking choices considering user heterogeneity. Transp. Res. Part A 70 (C), 41–49. doi:10.1016/j.tra.2014.10.001.

Korfmann, F., 2018. Essays on Advanced Discrete Choice Applications. Universität Hamburg Ph.D. thesis.

Labbé, M., Marcotte, P., Savard, G., 1998. A bilevel model of taxation and its application to optimal highway pricing. Manag. Sci. 44 (12-NaN-1), 1608–1622. doi:10.1287/mnsc.44.12.1608.

Laporte, G., Nickel, S., Saldanha da Gama, F., 2015. Location Science. Springer.

Ljubić, I., Moreno, E., 2018. Outer approximation and submodular cuts for maximum capture facility location problems with random utilities. Eur. J. Oper. Res. 266 (1), 46–56. doi:10.1016/j.ejor.2017.09.023.

Ma, J., Tse, Y.K., Wang, X., Zhang, M., 2019. Examining customer perception and behaviour through social media research - an empirical study of the united airlines overbooking crisis. Transp. Res. Part E 127, 192–205. doi:10.1016/j.tre.2019.05.004.

Mai, T.T.C., Lodi, A., 2017. Solving large-scale competitive facility location under random utility maximization models. Technical Report. CERC, Polytechnique Montréal, Canada.

Manski, C.F., 1977. The structure of random utility models. Theory Decis. 8 (3), 229. doi:10.1007/BF00133443.

McFadden, D., 1974. Conditional logit analysis of qualitative choice behavior. In: Frontiers in Econometrics. Academic Press, New York, pp. 105–142.

Cordone, R., Redaelli, F., 2011. Optimizing the demand captured by a railway system with a regular timetable. Transp. Res. Part B 45 (2), 430–446. doi:10.1016/j.trb.2010.09.001.

Müller, S., Haase, K., Kless, S., 2009. A multiperiod school location planning approach with free school choice. Environ. Plan. A 41 (12), 2929–2945. doi:10.1068/a40285.

van Ryzin, G.J., 2005. Future of revenue management: models of demand. J. Rev. Pricing Manag. 4 (2), 204–210. doi:10.1057/palgrave.rpm.5170140.

Robenek, T., Azadeh, S.S., Maknoon, Y., de Lapparent, M., Bierlaire, M., 2018. Train timetable design under elastic passenger demand. Transp. Res. Part B 111, 19–38. doi:10.1016/j.trb.2018.03.002.

Shen, Z.-J.M., Su, X., 2007. Customer behavior modeling in revenue management and auctions: a review and new research opportunities. Prod. Oper. Manag. 16 (6), 713–728. doi:10.1111/j.1937-5956.2007.tb00291.x.

Talluri, K.T., Van Ryzin, G.J., 2004. Revenue management under a general discrete choice model of consumer behavior. Manag. Sci. 50 (1), 15–33. doi:10.1287/mnsc.1030.0147.

Talluri, K.T., Van Ryzin, G.J., 2006. The Theory and Practice of Revenue Management. International Series in Operations Research & Management Science. Springer US.

Train, K.E., 2003. Mixed Logit. Cambridge University Press, pp. 138–154.

Wu, D., Yin, Y., Lawphongpanich, S., Yang, H., 2012. Design of more equitable congestion pricing and tradable credit schemes for multimodal transportation networks. Transp. Res. Part B 46 (9), 1273–1287. doi:10.1016/j.trb.2012.05.004.