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Teaching Bayesian Model Comparision with the Three-Sided Coin

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Determining the Geometry of a Three-sided Fair Coin: Exploring the Probability of a Coin Landing on its Edge

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Abstract

A three-sided fair coin is defined to be a cylinder which, when flipped like a coin, has an equal probability of landing on heads, tails or the edge. We present an analysis of this problem from several perspectives.

1 Introduction

A three-sided fair coin is defined to be a cylinder which, when flipped like a coin, has an equal probability of landing on heads, tails or the edge. A regular coin, like a penny, has almost zero chance of landing on the edge. On the opposite extreme, a long narrow cylinder like an unsharpened new pencil, has almost zero chance of landing on "heads" or "tails". Somewhere in between these extremes should exist a cylinder with equal probability of landing on its edge or an end. What is the size of this cylinder or, more precisely, how does the radius, R, of the circular cross section compare with the height, h, of the edge (Figure 1)? Although easily stated, this problem has many possible solutions, with varying degrees of complexity. In addition, there are a number of intuitive hypotheses that turn out to be incorrect. As a result, this problem is ideal to introduce students to probability and mechanics.

Some definitions:

- $\eta \equiv h/R$
- $l = \sqrt{R^2 + (h/2)^2}$

We will concern ourselves with the more general question of

What is the probability, p_{edge}, for the coin to land on the edge, as a function of the radius, R, and the height, h?

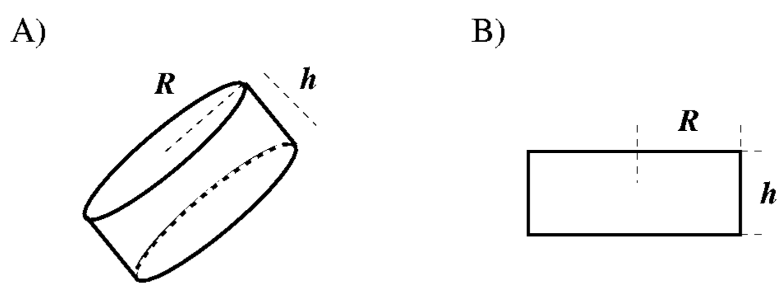

Figure 1: Three-sided coin. (A) A cylinder with radius, R, of the circular cross section and height, h, of the edge. (B) The cross section of the cylinder.

The more restricted question of the fair three-sided coin will be *What values of h and R yield* $p_{\text{edge}}(h, R) = 1/3$*?*. The result should probably be scale invariant, and thus should be expressible in terms of the ratio, $\eta \equiv h/R$, although due to the contraints of flipping, this invariance may not always hold.

2 Some of the Previous Methods for Solving the Problem

The following methods have been reported in a number of sources(**?**; Murray and Teare, 1993), with a varying degree of thoroughness and experimental corroboration.

2.1 Equal Areas

In analogy with dice, one presumes that the probability is proportional to the surface area of the edge.

$$\begin{aligned} p_{\text{edge}}(h, R) &= \frac{2\pi R h}{2\pi R^2 + 2\pi R h} \\ p_{\text{edge}}(\eta) &= \frac{\eta}{1+\eta} \end{aligned}$$

For the fair coin we obtain

$$\begin{aligned} \frac{\eta}{1+\eta} &= \frac{1}{3} \\ \eta &= \frac{1}{2} \end{aligned}$$

2.2 Equal Length

Again, in analogy with dice, one presumes that the probability is proportional to the cross sectional length. A square cross section would then have equal probability.

$$\begin{aligned} p_{\text{edge}}(h, R) &= \frac{h}{2(2R) + h} \\ p_{\text{edge}}(\eta) &= \frac{\eta}{4+\eta} \end{aligned}$$

For the fair coin we obtain

$$\begin{aligned} p_{\text{edge}}(\eta) &= \frac{1}{3} = \frac{\eta}{4+\eta} \\ \eta &= 2 \end{aligned}$$

2.3 Equal Solid Angle

Edward Pegg Jr in his Master's thesis(Edward Pegg, 1997) introduces a Geometrical Model, where the probability of certain faces of dice being produced (coins are refered to as "cylindrical dice") is proportional to the solid angle subtended by the face. For the edge probabilities we obtain

$$\begin{aligned} \beta &\equiv \operatorname{atan}\left(\frac{h}{2R}\right) = \operatorname{atan}(\eta/2) \\ \text{edge area subtended} &= \int_0^{2\pi} d\phi \int_{\pi/2-\beta}^{\pi/2+\beta} \sin\theta d\theta d\phi \end{aligned}$$

$$
\begin{aligned}
&= 4\pi \sin\beta \\
p_{\rm edge}(\eta) &= \sin\beta \\
&= \frac{\eta}{\sqrt{\eta^2+4}}
\end{aligned}
$$

2.4 Activation Energy Motivation

In (Levin, 1983) a similar problem is addressed for regular cubical dice loaded with a brass cylinder to offset the center of mass. They also consider a similarly altered cylinder as an example. Due to the symmetries of our problem, their result takes on a particularly simple form. If we label the states "heads", "edge", and "tails" with 1,2, and 3, the probability for each state depends on the gravitational energy of the state, U_i, and the activation energy, AE_i, or the energy needed to leave one state for another one. Levin empirically suggests a form of the following

$$
\begin{aligned}
p_i &\propto e^{-\frac{U_i}{AE_i}} \\
p_1 = p_3 &\propto e^{-\frac{h/2}{l-h/2}} \\
p_2 &\propto e^{-\frac{R}{l-R}}
\end{aligned}
$$

so

$$
p_{\rm edge}(h,R) = \frac{e^{-\frac{R}{l-R}}}{2e^{-\frac{h/2}{l-h/2}} + e^{-\frac{R}{l-R}}}
$$

2.5 Center-of-Mass Argument

For a solid object with one point of contact with the floor, the location of the center of mass with respect to the vertical line through the contact point determines which way the object will tip. If the center of mass is to the left of the vertical line then the object will tip to the left. Likewise, if the center of mass is to the right, then the object will tip to the right. If we assume there is no bouncing, then the coin lands on a contact point at a particular angle, θ. Comparing the landing angle to the angle at which the center of mass is directly above the contact point (Figure 2), α, will determine the direction the coin will tip over and come to rest:

- if $0^\circ < \theta < \alpha$ then the coin will land on heads
- if $\alpha < \theta < 90^\circ$ then the coin will land on the edge

where $\alpha = \operatorname{atan}\left(\frac{R}{h/2}\right)$. For $h/R \sim 0$ then $\alpha \sim 90^\circ$. For larger h/R, then α gets smaller. Symmetry of the problem allows us to then write that the probability for landing on edge is

$$
p_{\rm edge}(h,R) = 1 - \frac{\alpha}{90^\circ} \equiv p_e
$$

where the definition of p_e as the edge probability from the center-of-mass argument will be used later.

For a fair coin we obtain

$$
\begin{aligned}
p_{\rm edge}(h,R) &= \frac{1}{3} = 1 - \frac{\alpha}{90^\circ} \\
\alpha &= 60^\circ \\
h/R &= \frac{2}{\sqrt{3}} \approx 1.155
\end{aligned}
$$

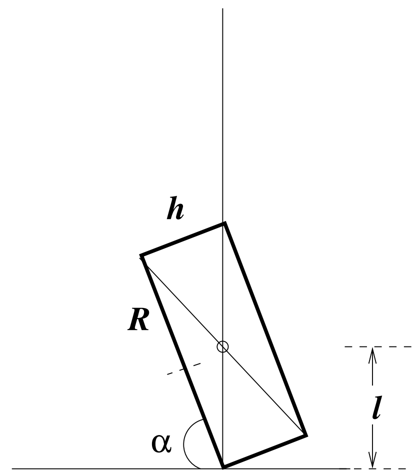

Figure 2: Coin with the center of mass directly above the impact point. The angle $\alpha = \text{atan}(2R/h)$ is the angle at which this particular configuration is achieved. It is a geometric property of the coin itself.

2.6 Murray and Teare, 1993

In Murray and Teare, 1993, a dynamical model of a coin is simulated. They use a straightforward application of Newton's laws, on a model cylinder allowed to move in one dimension (vertically), rotate and bounce on the surface. In dimensionless units the position of the center of mass, $Z(t)$, the vertical speed, $V(t)$, and the orientation angule, $\theta(t)$ are given by

$$\begin{aligned} Z(t) &= Z_o + V_o t + \frac{1}{2}t^2 \\ V(t) &= V_o - t \\ \theta(t) &= \theta_o + \omega t \end{aligned}$$

The coin is numerically simulated until it strikes the ground. At which point the vertical and angular speeds are modified by the bounce. Defining V' and V'' as the vertical speeds just before and after the bounds, and ω' and ω'' as the angular speeds just before and after the bounce, they present

$$\begin{aligned} V'' &= V' - (1+\gamma)k^2 \frac{V' + x\omega'}{(k^2 + x^2)} \\ \omega'' &= \omega' - (1+\gamma)x \frac{V' + x\omega'}{(k^2 + x^2)} \end{aligned}$$

where x is the position of the contact point and γ is coefficient of restitution parameter. For $\gamma = 1$ there is no energy loss, and $\gamma = 0$ there is no bounce (all energy is lost on the first bounce). One can show that the velocity of the corner of the coin that strikes the floor, before and after the impact, are related by $U'' = -\gamma U'$.

2.7 Summary of Methods

Figure 3 displays the methods described thus far. Table 1 shows the predictions for the probability of real coins landing on the edge(**?**).

3 Potential Energy of a Coin: Incorporating Bounce

The more detailed simulation of the coin seems to have the following two properties:

1. much lower probability of edge landing for thin coins

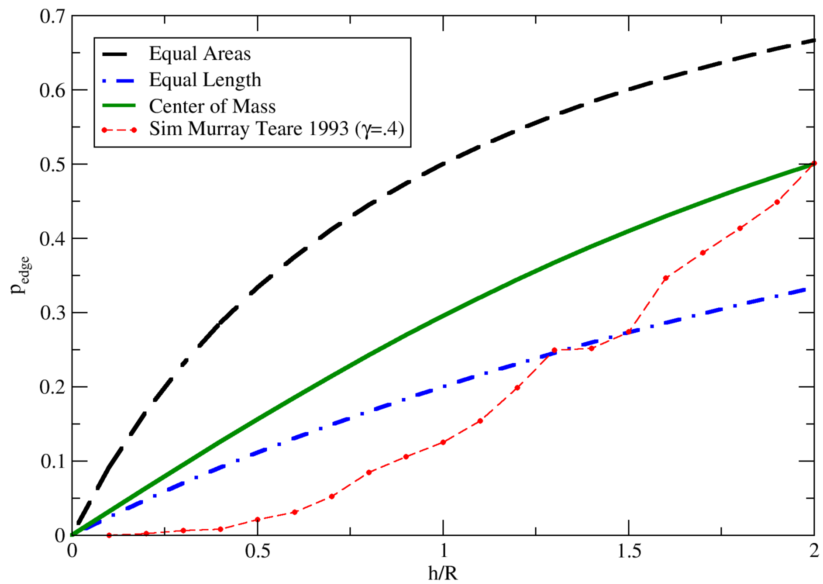

Figure 3: Theories for the Probability of an Edge-falling Coin. Plotted are the edge probabilities as a function of the h/R ratio of the coin. The Murray-Teare results are from simulation, with a $\gamma = 0.4$.

A)

Coin	**Radius (mm)**	**Height (mm)**
Quarter	12.1	1.75
Dime	8.95	1.35
Nickel	10.6	1.95
Penny	9.5	1.5

B)

Number of Edge Landings in 10,000 Flips

Coin	**Equal Area**	**Equal Length**	**Center of Mass**	**Murray and Teare, 1993***
Quarter	1264	349	460	3
Dime	1311	363	479	5
Nickel	1554	440	584	17
Penny	1364	380	502	7

* extrapolated from data. The original Murray and Teare, 1993 paper quoted 1/6000 tosses for a nickel.

Table 1: (A) Size Data for US Coins. (B) Number of Edge Landings in 10,000 Flips Predicted by Different Methods

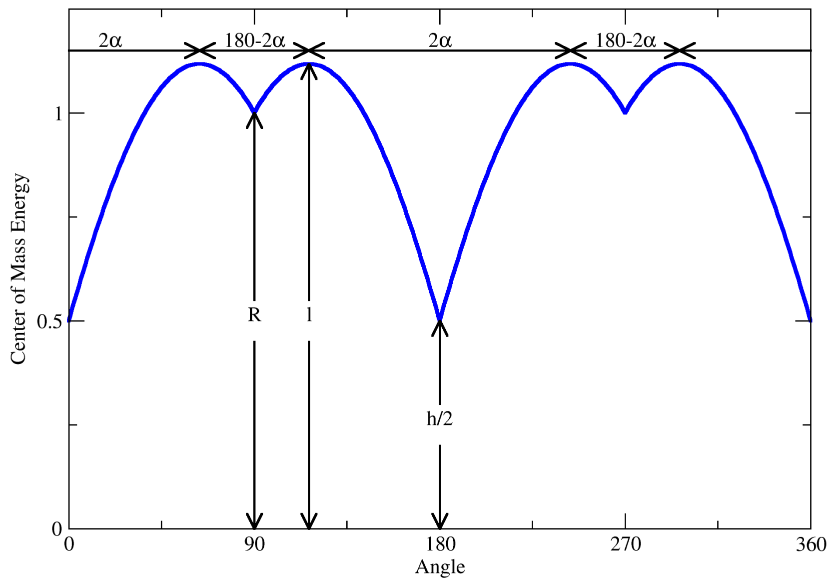

Figure 4: Energy (or Height) of the Center of Mass of a Coin versus Orientation Angle.

2. higher relative probability than other models for thick coins

Point 1 seems to be more closely related to ones intuition: flipping a quarter has much less than 1/10 chance of landing on edge (Equal Areas), and even less than 1/30 chance (Equal Length and Center of Mass). Is there a simpler way to account for this? The physics, given the assumptions, for the Center of Mass argument is correct. The error must lie in the assumptions. There is one obvious assumption: no bouncing is included. What is the effect of bouncing?

In order to address this problem in a simple way, we introduce an energy function for the coin. This is the height of the center of mass, as one rolls the coin along the surface, starting in a "heads" position. This height, or gravitational energy (in dimensionless units) is shown in Figure 4. It has local minima at the positions of "heads", "tails" or the "edge", and maxima when the coin is in the configuration shown earlier in Figure 2.

We can incorporate bouncing in the following way. Let $E_{\rm CM}(\theta)$ be the center of mass energy as a function of the orientation of the coin as it strikes the floor, E be the incoming energy of the coin, and γ be the fractional energy loss on each bounce, i.e. on each bounce $E \to \gamma E$. Note that this is slightly different than the γ used by Murray and Teare, 1993. In their case the *velocities* behaved like $U \to \gamma U$ at a bounce. Since the kinetic energy of the coin goes as the square of the velocity, our γ should be compared to the square of the γ used by Murray and Teare, 1993.

As the coin bounces, it loses energy. As long as it has enough energy to escape the energy wells for the three cases, it will keep randomizing its angle. Eventually it will not have enough energy to escape one of the energy wells. In general (for $h/R < 2$) the energy well for "edge" is smaller than that for "heads" or "tails". There will come a point when the energy will be large enough to escape the "edge" well, but not a "heads" well. After that point, if the coin has a heads-inducive fall, then it will stay on heads forever. Thus, to land on edge, the coin must has a series of edge-inducive falls in a row until the energy falls to the point where it cannot escape the "edge" energy well.

We have the following relationships

$$\alpha = \text{atan}\left(\frac{R}{h/2}\right)$$

$$\text{(probability for an edge-inducive fall)} \qquad p_e = 1 - \frac{\alpha}{90}$$

$$\text{(coin energy after } n \text{ bounces)} \qquad E_n = E_o\gamma^n$$

$$\text{(number of bounces, given initial and final energy)} \qquad n = \log(E_n/E_o)/\log\gamma + 1$$

$$\text{(half-length of the diagonal of the coin)} \qquad l = \sqrt{R^2 + (h/2)^2}$$

$$\text{(depth of the ``heads'' energy well)} \qquad E_h = l - h/2$$

$$\text{(depth of the ``edge'' energy well)} \qquad E_e = l - R$$

If we start with energy just able to escape the largest well (E_h), and bounce until we cannot escape the smallest well (E_e), then we will bounce n times where n is

$$n \quad = \quad \log(E_h/E_r)/\log\gamma + 1$$

The probability of bouncing n times, each time with an edge-inducing fall, is simply

$$p_{\text{edge}} = p_e^n$$

Figure 5 compares this simple bounce model with the Murray and Teare, 1993 results. The similarity is quite striking, given the simplicity of the bounce model.

3.1 Correlated Bouncing

In this model, we ignore the energy of the bounce, and simply posit that sequential edge-conducive bounces are correlated, and sequential heads-conducive bounces are correlated. This model is comparable to the urn problem discussed in (**?**). Using his notation, we assume that an edge-conducive bounce increases the probability of another edge-conducive bounce by an amount $\epsilon > 0$, while a heads-conducive bounce *decreases* the probability for an edge-conducive bounce by an amount $\delta > 0$. Thus we have:

$$\begin{aligned}
P(E_k|E_{k-1}I) &= p_e + \epsilon \\
P(E_k|H_{k-1}I) &= p_e - \delta \\
P(H_k|E_{k-1}I) &= 1 - p_e - \epsilon \\
P(H_k|H_{k-1}I) &= 1 - p_e + \delta
\end{aligned}$$

If we write the probabilities for the kth trial as a vector,

$$V_k \equiv \begin{pmatrix} P(E_k|I) \\ P(H_k|I) \end{pmatrix}$$

Then this Markov process can be described by the matrix equation

$$V_k = \begin{pmatrix} (p_e + \epsilon) & (p_e - \delta) \\ (1 - p_e - \epsilon) & (1 - p_e + \delta) \end{pmatrix}^{k-1} V_1$$

The general solution becomes(**?**),

$$P(E_k|I) = \frac{(p_e - \delta) - (\epsilon + \delta)^{k-1}(p_e\epsilon - (1 - p_e)\delta)}{1 - \epsilon - \delta}$$

The center-of-mass argument becomes the limit of $\epsilon \to 1 - p_e$ and $\delta \to p_e$: once the first bounce occurs, the state remains constant forever.

In this model, there doesn't seem to be any obvious criterion to determine the number of bounces. This parameter, n, should be treated as a nuisance parameter, between zero and a handful of bounces in realistic situations. Otherwise, we could the calculated version above.

4 Statistical Model of Bouncing

- simulation
  - choose an angle, θ
  - if $E > E_{\rm CM}(\theta)$ then choose another angle
  - if $E < E_{\rm CM}(\theta)$ then stuck in that well
- recursive formulation
  - given a uniform distribution over angle
  - given one particular well, with minimum m
  - probability of obtaining a depth of $l - E_{\rm CM}(\theta)$ *less than* an energy ϵ is approximately

$$F(\epsilon) = \begin{cases} 0 & \epsilon < 0 \\ \sqrt{\frac{\epsilon}{m}} & 0 < \epsilon < m \\ 1 & \epsilon > m \end{cases}$$

    this is also the probability of escape, given energy ϵ, and a particular well
  - define F_e and F_h as the wells for edge and heads respectively. $m_e = l - R$ and $m_h = l - h/2$
  - the probability for finally landing on edge on the ith bounce, given a sequence of bounce energies as before $E_n = \gamma^n E_o$, is

$$\begin{aligned} p_{\rm edge}(i) &= p_e(1 - F_e(E_i)) + p_e F_e(E_i) p_{\rm edge}(i+1) + p_h F_h(E_i) p_{\rm edge}(i+1) \\ &= p_e(1 - F_e(E_i)) + p_{\rm edge}(i+1)(p_e F_e(E_i) + p_h F_h(E_i) p_{\rm edge}(i+1)) \end{aligned}$$

  - Implemented something like

```
if (i>i_max) {
  p=pe;
} else {
  p=pe*(1-F(E,me))+p_edge(gamma*E,h,R,gamma,i+1)*(pe*F(E,me)+pht*F(E,mh));
}
```

5 Data

5.1 PVC Solid Stock, Flipped by Hand

5.2 Murray Teare 1993, Brass Nuts

5.3 Murray 2005, Automatic Flipping

6 Model Comparison

$$\begin{aligned} P(M_i|D,I) &= \frac{P(D|M_i,I)P(M_i|I)}{P(D|I)} \\ \frac{P(M_i|D,I)}{P(M_j|D,I)} &= \frac{P(D|M_i,I)P(M_i|I)}{P(D|M_j,I)P(M_j|I)} \end{aligned}$$

if all of the models are *a priori* equally likely, and we look at the logarithm, we have

$$\log P(M_i|D,I) - \log P(M_j|D,I) = \log P(D|M_i,I) - \log P(D|M_j,I)$$

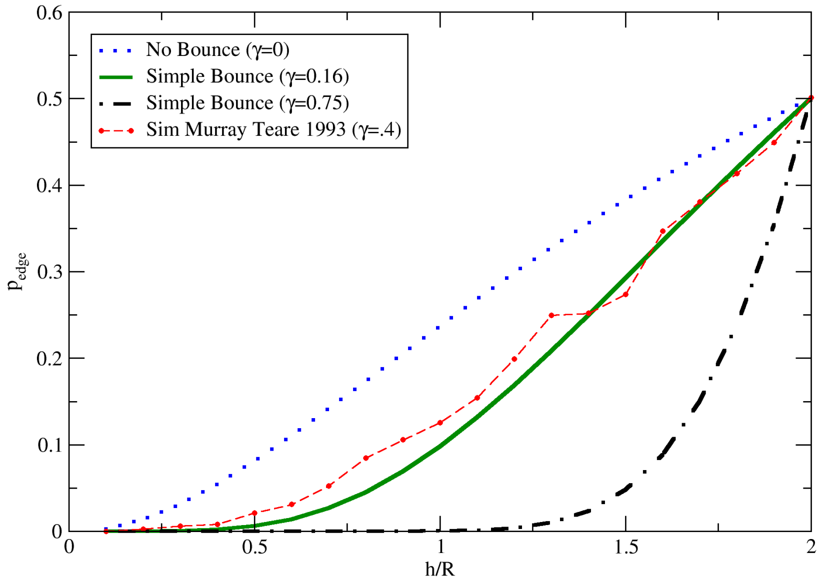

Figure 5: Theories for the Probability of an Edge-falling Coin: Simple Bounce model compared to Murray and Teare, 1993 simulations. Note that the simple bounce model value of γ should be compared to the Murray and Teare, 1993 value of γ^2.

For models with extra parameters,= we need to marginalize over those parameters, such as

$$P(M_i|D,I) \quad = \quad \int d\xi P(D|\xi, M_i, I)P(\xi|I)$$

this has the benefit of automatically penalizing more complex models unless they can be justified with a particularly better fit (i.e. a quantitative Occam's razor).

7 The Problem with Flipping

- rolling

8 Conclusions

9 Bibliography

daveweb: `http://users.frii.com/davejen/coin3.htm`

References

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Levin, E. (1983). Experiments with loaded dice. *American Journal of Physics*, 51(2):149–152.

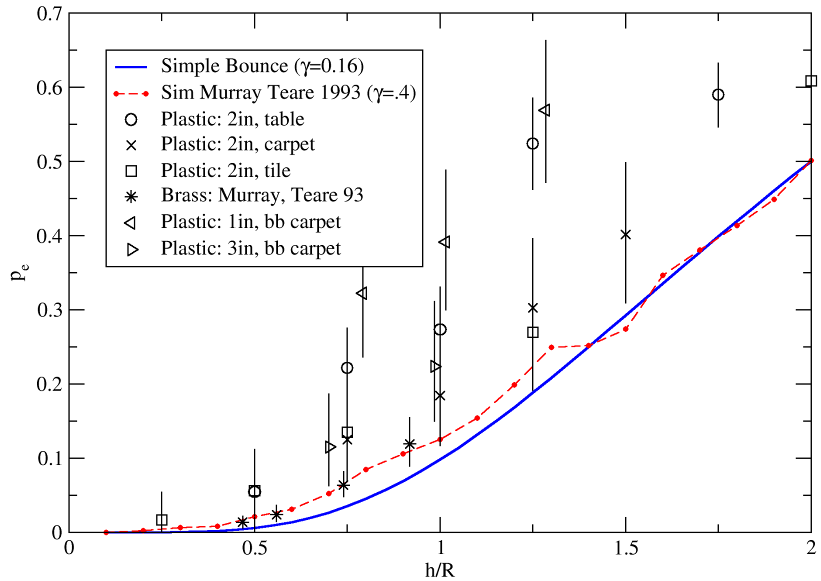

Figure 6: All of the Data For Coin Flipping

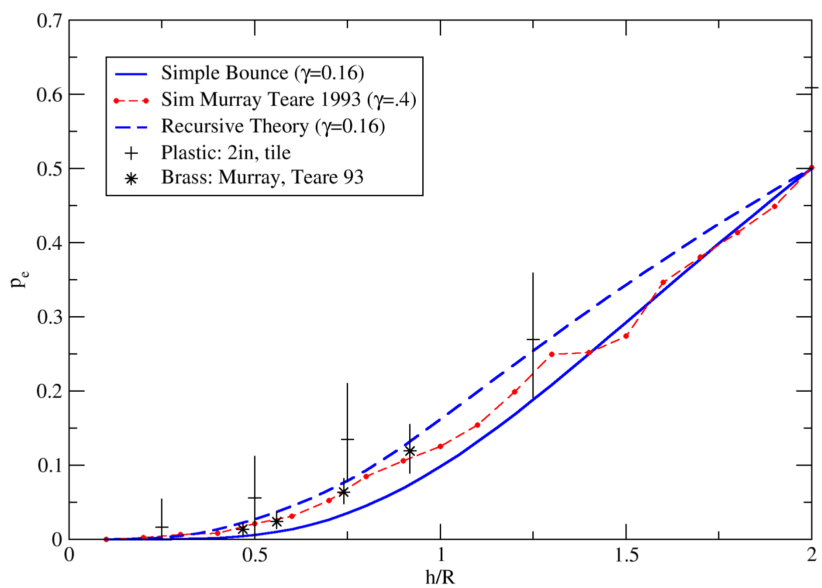

Figure 7: Subset of the Data.

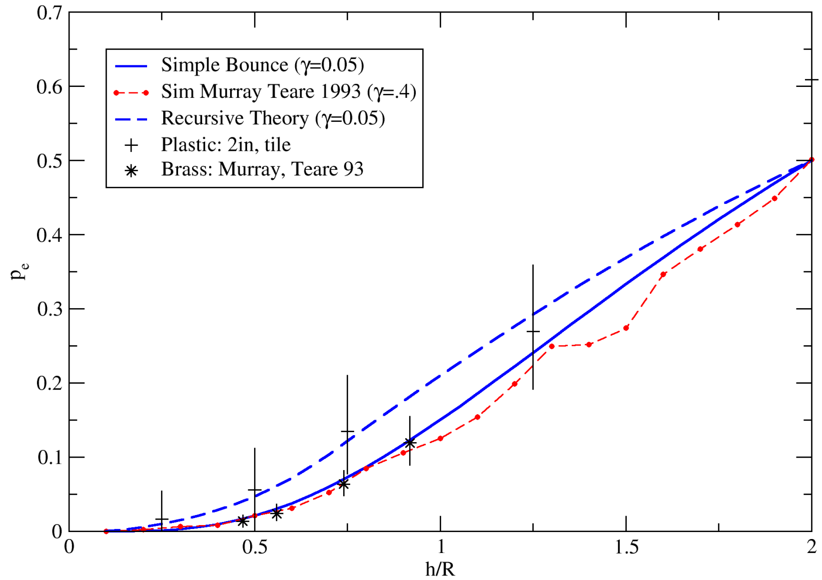

Figure 8: Subset of the Data. Different γ.

Murray, D. B. and Teare, S. W. (1993). Probability of a tossed coin landing on edge. *Physical Review E*, 48(4):2547–2552.