SIA Matrices and Non-Negative Stationary Subdivision

Dissertation zur Erlangung des Doktorgrades der Naturwissenschaften (Dr. rer. nat)

Fakultät Naturwissenschaften
Universität Hohenheim

Institut für Angewandte Mathematik
und Statistik (110)

vorgelegt von
Xianjun Li

aus Liaocheng, Shandong, VR China

2012

Dekan:	Prof. Dr. H. Breer
1. berichtende Person:	Prof. Dr. K. Jetter
2. berichtende Person:	Prof. Dr. U. Jensen
Eingereicht am:	2. März 2012
Mündliche Prüfung am:	21. Juni 2012

Die vorliegende Arbeit wurde am 6. Juni 2012 von der Fakultät Naturwissenschaften der Universität Hohenheim als "Dissertation zur Erlangung des Doktorgrades der Naturwissenschaften" angenommen.

Contents

1 Introduction **1**

2 Basics of subdivision schemes **5**
2.1 Convergence concept . . . 5
2.2 The form of the limiting surface of uniformly convergent subdivision schemes . . . 6
2.3 The finite support of the mask . . . 7

3 Non-negative subdivision **10**
3.1 The univariate case . . . 10
3.1.1 Notation . . . 10
3.1.2 The corresponding non-homogeneous Markov processes . . . 12
3.1.3 The connection of uniform convergence to left convergent matrix products . . . 13
3.2 Bivariate non-negative subdivision . . . 14
3.2.1 Notation . . . 14
3.2.2 The corresponding non-homogeneous Markov processes . . . 15
3.2.3 The connection of uniform convergence to left convergent matrix products . . . 17

4 SIA Matrices **19**
4.1 The SIA property of families of non-negative matrices . . . 19
4.2 The directed graph of row stochastic matrices . . . 20
4.3 The directed graph of SIA matrices . . . 25
4.4 The ergodic coefficient and the scrambling property . . . 26
4.5 Equivalent conditions to the SIA property . . . 28
4.6 The Anthonisse – Tijms result . . . 30

5 Uniform convergence **33**
5.1 The univariate case . . . 33
5.1.1 Pointwise definition of the basic limit function . . . 34
5.1.2 Hölder continuity of the basic limit function . . . 37
5.1.3 Characterization of convergence . . . 39
5.1.4 The SIA property of the master matrix **A** . . . 41
5.1.5 The GCD-condition . . . 42
5.1.6 Melkman's univariate string condition . . . 43

5.2 The bivariate case 47
5.2.1 The SIA property in bivariate subdivision 47
5.2.2 Pointwise definition of the basic limit function 51
5.2.3 Hölder continuity of the basic limit function 53
5.2.4 Uniform convergence 55
5.3 Sufficient conditions for uniform convergence 56
5.3.1 Masks with scrambling matrices $\mathbf{A}_\epsilon$ 56
5.3.2 The bivariate string condition 57
5.4 Uniform convergence is a support property of the mask 59
5.5 Convex combinations of non-negative masks 60

6 Tensor product subdivision schemes 61
6.1 Preliminaries 61
6.2 Application 63

7 Extensions 65
7.1 Zonotopes and box spline subdivision 65
7.2 Joint spectral radius 67

8 Appendix 68

Bibliography 71

Zusammenfassung 74

Summary 77

Chapter 1

Introduction

Subdivision is a process of recursively refining discrete data using a set of subdivision rules (called subdivision scheme) which generates a continuous or even smooth limit. It has numerous applications and is used, e.g., in signal denoising, in image compression, in the design of curves and surfaces, and for the approximation of arbitrary functions. It has been around in the theory of spline functions before multiresolution analysis and discrete wavelet transforms were recognized as powerful concepts and fast numerical alternatives in space-frequency analysis of signals, and it is at the core of these methods.

The main mathematical issues to be studied in the theory of subdivision schemes are uniform convergence, the type of regularity or the smoothness of the basic limit function, the stability of the subdivision process, and the properties of the associated functional equation. In the stationary case which we consider here, the subdivision scheme is defined by a real-valued masks $a = (a(\alpha))_{\alpha\in\mathbb{Z}^s}$ of finite support. For a given starting sequence $c = c^{(0)} = (c^{(0)}(\alpha))_{\alpha\in\mathbb{Z}^s}$, the algorithm proceeds iteratively, where at step k the k-th iterated sequence results from convolving an upsampled version of the $(k-1)$st sequence with the mask according to

$$c^{(k)}(\alpha) = \sum_{\beta\in\mathbb{Z}^s} a(\alpha - 2\beta)\, c^{(k-1)}(\beta)\ , \quad \alpha \in \mathbb{Z}^s\ .$$

The theory of such schemes is rather well elaborated. We refer to the early surveys [4, 11], and will give further reference in later chapters. A necessary condition for convergence is the following property of the submasks that

$$\sum_{\alpha\in\mathbb{Z}^s} a(2\alpha + \epsilon) = 1 \quad \text{for} \quad \epsilon \in \{0,1\}^s\ .$$

This property is sometimes called the sum rule.

In this dissertation we study the property of uniform convergence of non-negative, stationary subdivision, for the univariate case $s = 1$, and for the bivariate case $s = 2$ as an example of multivariate subdivision, where non-negativity refers to the property of the mask being non-negative, *i.e.*,

$$a(\alpha) \geq 0, \qquad \alpha \in \mathbb{Z}^s\ .$$

Such subdivision schemes have a long tradition, since the first examples of B-spline subdivision, and their multivariate counterpart of box spline subdivision are of this type. Their convergence properties have been studied for a long time. For instance in the univariate case $s = 1$, if we assume w.l.o.g. (we may shift the mask with no impact on the property of a subdivision scheme being convergent) that

$$I = \text{supp } a := \{\alpha \in \mathbb{Z} : a(\alpha) \neq 0\} \subseteq \{0, 1, \dots, N\} \quad \text{and} \quad a(0)a(N) \neq 0$$

for some integer N, then the univariate sum rule

$$\sum_{\alpha \in \mathbb{Z}} a(2\alpha) = \sum_{\alpha \in \mathbb{Z}} a(2\alpha + 1) = 1$$

implies $N \geq 2$ and $0 < a(0), a(N) < 1$, since otherwise the limit function cannot be continuous. Consider now the two $(N \times N)$-matrices

$$\mathbf{A}_0 = (a(-\alpha + 2\beta))_{\alpha,\beta=0}^{N-1} \quad \text{and} \quad \mathbf{A}_1 = (a(-\alpha + 2\beta + 1))_{\alpha,\beta=0}^{N-1},$$

respectively. Then, if the sum rule is fulfilled, the convergence of non-negative subdivision only depends on the support of the mask, and not on the actual values of the mask coefficients. More precisely, the following result has been proven in [21, 26]:

The subdivision scheme with the non-negative mask a – and the stated restriction on the support and the values of the mask coefficients – converges if and only if

(1) the sum rule is fulfilled, and

(2) for any bits $\epsilon_i \in \{0, 1\}$, $i = 1, 2, \dots, k$, with $k = 2^{N^2}$, the matrix $\mathbf{A}_{\epsilon_k} \cdots \mathbf{A}_{\epsilon_1}$ has a strictly positive column.

There are various partial results, which simplify the second condition. For example, Micchelli and Prautzsch [26] prove that if (2) is replaced by $I = \{0, 1, \dots, N\}$ and $N \geq 2$, then the convergence follows. Gonsor [15] shows that $I = \{0, 1, \dots, N\}$ and $N \geq 2$ can be weakened by $\{0, 1, N - 1, N\} \subseteq I$. Melkman [27] proves that if instead of (2) the support of the mask has the property $I \supseteq \{0, p, q, p + q\}$ with the greatest common divisor $\text{GCD}\{p, q\} = 1$, or if I contains two successive integers and $0 < a(0), a(N) < 1$ then the convergence follows. Wang gives a further modification [34]: The subdivision scheme with the non-negative mask a converges if instead of (2) there exist $\{r, p, q\} \subseteq I$ such that $\text{GCD}\{p - r, q - r\} = 1$ and $q - r$ is even.

Recently, uniform convergence of non-negative univariate subdivision has been finally characterized by Xin-Long Zhou in [36], by verifying the long-standing GCD conjecture. He shows that the subdivision scheme with the non-negative mask a which satisfies $a(0), a(N) \neq 0$, converges if and only if

(a) the sum rule is fulfilled and $0 < a(0), a(N) < 1$,

(b) the greatest common divisor of $\{\alpha \mid a(\alpha) \neq 0\}$ is 1.

The theory of (stationary) subdivision was very much influenced by the seminal paper of Micchelli and Prautzsch [26]. The paper refers to [31], and uses properties of non-negative matrices, although it does not really exploit those in full detail. When studying [6] and [21], who use properties of the joint spectral radius of a family of matrices which appears in stationary subdivision, we got interested in the connection to finite non-homogeneous Markov processes which was mentioned there. In univariate subdivision the considered family of matrices is given by $\mathcal{A} = \{\mathbf{A}_0, \mathbf{A}_1\}$ from above, which is a pair of row stochastic matrices in case the sum rule holds. The non-homogeneous Markov process leads here to the problem of convergence of infinite products of type

$$\lim_{k\to\infty} \mathbf{A}_{\epsilon_k}\mathbf{A}_{\epsilon_{k-1}}\cdots\mathbf{A}_{\epsilon_1}$$

to a rank one row stochastic matrix, for any sequence $(\epsilon_\kappa)_{\kappa=0}^{\infty}$ of bits. In its application to non-negative subdivision, these bits refer to the dyadic expansion of $x = \sum_{\kappa=1}^{\infty} \epsilon_\kappa/2^\kappa$, for given $x \in [0,1]$, and in the convergent case – subject to a compatibility condition involving the left eigenvectors for the dominant eigenvalue $\lambda = 1$ of the matrix pair – the infinite product convergences to a matrix with equal rows of type

$$(\ \phi(x),\ \phi(x+1),\ \cdots,\ \phi(x+N-1)\ )\ ,$$

with ϕ the basic limit function of the subdivision process. This was the starting point for our research in this topic, leading to the results presented here.

From Numerical Linear Algebra we know that the powers of a row stochastic matrix $\mathbf{A}$ converge to a rank one matrix, if and only if $\lambda = 1$ is a strictly dominant eigenvalue, and the limit is a row stochastic matrix with the common row vector being a left eigenvector for this dominant eigenvalue. In the theory of finite Markov processes – where $\mathbf{A}$ describes the transition probabilities between states as the process develops – the equivalent notion of an SIA matrix (refering to the properties of being stochastic, indecomposable, and aperiodic) was introduced by Wolfowitz [35]. It proved to be useful for the study of infinite products of a finite family of row stochastic matrices, a non-homogeneous Markov process, by requiring that each finite section of the product has this SIA property. Concerning the theory of finite Markov chains, we refer to the discussion in [35] and to the books [9], [23] and [31]. Also, the interpretation of the SIA property in terms of the directed graph of a non-negative matrix has proved to be useful for studying this rank one convergence; this connection of SIA matrices with directed graphs was used in the paper by Ren and Beard [30], among others.

This dissertation is concerned with SIA matrices and non-negative (stationary) subdivision, and is organized as follows: In Chapter 2, we introduce basic notation for subdivision schemes and give the concept of uniform convergence.

In Chapter 3, we discuss non-negative subdivision in the univariate and in the bivariate case. The emphasis is put here on elaborating the connection of non-negative subdivision to non-homogenous Markov processes, refering to the subdivision family of matrices $\mathcal{A}$. Among other results we state here, for further use, Lemma 3.1 and Lemma 3.2 which relate the coefficients of the iterated masks to matrix products, as well as the limit cases, respectively, relating the values of the fundamental limit function to the entries in an infinite product of matrices.

Chapter 4 and Chapter 5 are the core of this dissertation. In Chapter 4, we review the spectral properties and the graph properties of row-stochastic matrices and, in particular, of SIA matrices. We point to the notion of scrambling power, see Hajnal [16], and of the related coefficient of ergodicity. We also consider the directed graph of such matrices, and we improve upon a condition given by Ren and Beard in [30]. Finite families of SIA matrices, the properties of their indicator matrices or, equivalently, the connectivity of their directed graphs is also studied. This chapter is an important contribution to non-negative subdivision, since the convergence result of Anthonisse and Tijms [2] which we reprove in Section 4.6 characterizes rank one convergence of infinite products of row stochastic matrices, without refering to the joint spectral radius but using the (equivalent) coefficient of ergodicity instead. Properties equivalent to SIA are listed in Lemma 4.7 and in the subsequent Lemma 4.8; they connect the SIA property to conditions, as they appear in the existing literature dealing with convergence of non-negative subdivision.

The fifth chapter of the dissertation contains the full proof of the uniform convergence for non-negative univariate and bivariate subdivision schemes, respectively. It uses the pointwise definition of the limit function at dyadic points using the Anthonisse - Tijms pointwise convergence and the proper extension of the Micchelli - Prautzsch compatibility condition taking care of the ambiguity of representation of dyadic points. As a consequence of the Anthonisse - Tijms convergence result, we find the Hölder exponent for the Hölder continuity of the basic limit function, with the Hölder exponent expressed in terms of the coefficient of ergodicity. Our convergence theorems, in Theorem 5.1 and Theorem 5.8, include the existing characterizations of uniform convergence for non-negative univariate and bivariate subdivision from the literature.

Chapter 5 also refers to some attempts where we have tried to extend some conditions from univariate subdivision, which are sufficient for convergence, to the bivariate case. So far, we were not able to find the bivariate version of Zhou's GCD characteration for convergence, but we successfully could extend the univariate string condition from Melkman's paper [27] to the bivariate case; the rectangular string condition dealt with in Section 5.3.2 is such an extension result. The chapter concludes with a hint to the support property of convergence in non-negative subdivision, and with an application of this support property to convex combination of subdivision masks.

The last two short chapters deal with tensor product subdivision, and with box spline subdivision. Section 7.2 relates the restricted joint spectral radius to the coefficient of ergodicity. The dissertation ends with an Appendix where we give some definitions and state some basic lemmas and theorems about matrix and graph theory without proofs.

Chapter 2

Basics of subdivision schemes

Subdivision methods in computer graphics constitute a large class of recursive schemes for computing curves and surfaces. Refinable functions are encountered in computer aided geometric design where subdivision schemes are used to construct smooth curves and surfaces.

2.1 Convergence concept

Let s be a fixed natural number and $\mathbb{Z}^s$ the integer s-dimensional lattice. For us a subdivision scheme is determined by any fixed sequence

$$a = (a(\alpha))_{\alpha \in \mathbb{Z}^s}, \tag{2.1}$$

where $s = 1$ in the curve case, and $s = 2$ in the case of surfaces defined by control nets with the topology of a regular grid. Unless otherwise stated we assume that

$$\text{supp } a := \{\alpha \in \mathbb{Z}^s : a(\alpha) \neq 0\} \tag{2.2}$$

is a finite subset of $\mathbb{Z}^s$. We define a *subdivision operator* on the space $\ell^\infty(\mathbb{Z}^s)$ of bounded sequences:

$$S_a : \ell^\infty(\mathbb{Z}^s) \to \ell^\infty(\mathbb{Z}^s), c \longmapsto S_a c, \tag{2.3}$$

where

$$(S_a c)(\alpha) = \sum_{\beta \in \mathbb{Z}^s} a(\alpha - 2\beta) c(\beta), \qquad \alpha \in \mathbb{Z}^s. \tag{2.4}$$

In this setting, a is called the *mask* of the subdivision operator.

We are interested in conditions on the mask to guarantee uniform convergence of the subdivision scheme derived from iterating the subdivision operator according to the now standard

Definition 2.1. *(Uniform convergence) The (stationary) subdivision scheme*

$$c^{(k+1)} := S_a c^{(k)}, \qquad k = 0, 1, \ldots \tag{2.5}$$

is said to be (uniformly) convergent, if for any starting sequence $c = c^{(0)} \in \ell^\infty(\mathbb{Z}^s)$, *there is a continuous function* f_c *satisfying*

$$\lim_{k\to\infty} \sup_{\alpha\in\mathbb{Z}^s} \left| f_c\left(\frac{\alpha}{2^k}\right) - c^{(k)}(\alpha) \right| = 0, \tag{2.6}$$

and if $f_c \neq 0$ *for at least some initial data* $c^{(0)}$.

Remark 2.1. We describe the definition of the stationary subdivision scheme above. There are some differences between stationary subdivision scheme and non-stationary subdivision scheme. For non-stationary subdivision schemes, we know that they consist of recursive refinements of an initial sparse sequence with the use of masks that may vary from one scale to the next finer one but are the same everywhere on the same scale.

2.2 The form of the limiting surface of uniformly convergent subdivision schemes

A simple necessary condition for the subdivision scheme to be uniformly convergent is the following:

Proposition 2.1. *(see [4], Proposition 2.1) Suppose that the subdivision scheme is uniformly convergent. Then the mask satisfies*

$$\sum_{\beta\in\mathbb{Z}^s} a(\alpha - 2\beta) = 1, \qquad \alpha \in \mathbb{Z}^s. \tag{2.7}$$

We note that Proposition 2.1 is equivalent to the following special case

Proposition 2.2. *(see [11], Proposition 2.2) Suppose the subdivision scheme is uniformly convergent, then*

$$\sum_{\beta\in\mathbb{Z}^s} a(\gamma - 2\beta) = 1, \qquad \gamma \in E_s, \tag{2.8}$$

where $E_s = \{0,1\}^s$ *denotes the set of representers of* $\mathbb{Z}^s/2\mathbb{Z}^s$.

In order to check the convergence of subdivision scheme, it is sufficient to show that, for the initial data $c^{(0)} = \delta$, the Kronecker delta, convergence holds with a continuous limit function $f_\delta = \phi$, with ϕ not identically zero. We now provide a theorem identifying the limit of any uniformly convergent subdivision scheme:

Theorem 2.1. *(see [4], Theorem 2.1 and [11], Theorem 2.5) Suppose that the subdivision scheme* (2.4) *is uniformly convergent for all* $c \in \ell^\infty(\mathbb{Z}^s)$. *Then its mask* $a = (a(\alpha))_{\alpha\in\mathbb{Z}^s}$ *determines a unique compactly supported continuous function* ϕ *with the following properties:*

$$\phi(x) = \sum_{\alpha\in\mathbb{Z}^s} a(\alpha)\phi(2x - \alpha), \qquad x \in \mathbb{R}^s \tag{2.9}$$

and

$$\sum_{\alpha\in\mathbb{Z}^s}\phi(x-\alpha)=1. \tag{2.10}$$

Moreover,

$$f_c(x)=\sum_{\alpha\in\mathbb{Z}^s}c(\alpha)\phi(x-\alpha), \qquad x\in\mathbb{R}^s, \quad c\in\ell^\infty(\mathbb{Z}^s). \tag{2.11}$$

Remark 2.2. (see [4], pp.14) We refer to (2.9) as the *two-scale equation* associated with the mask $a=(a(\alpha))_{\alpha\in\mathbb{Z}^s}$. The function ϕ is called the S_a*-refinable function* or the *basic limit function.* Given Proposition 2.1, (2.10) is a special case of (2.11) and (2.11) identifies algebraically the limit (2.6) of the subdivision scheme (2.5). Note also that f_c is uniformly continuous on $\mathbb{R}^s$.

2.3 The finite support of the mask

If we start the subdivision scheme with the delta sequence, the iterates are the so-called *iterated masks*,

$$a^{(k)}=(S_a)^k\delta, \qquad k=1,2,\ldots,$$

with $a^{(0)}=\delta, a^{(1)}=a$, and

$$a^{(k)}(\alpha)=\sum_{\beta\in\mathbb{Z}^s}a(\alpha-2\beta)a^{(k-1)}(\beta), \qquad k>1. \tag{2.12}$$

Lemma 2.1. *For the iterated masks, eq.* (2.12) *is equivalent to*

$$a^{(k)}(\alpha)=\sum_{\substack{\beta_0,\beta_1,\ldots,\beta_{k-1}\in\mathbb{Z}^s\\ \alpha=\beta_0+2\beta_1+\cdots+2^{k-1}\beta_{k-1}}}a(\beta_0)a(\beta_1)\cdots a(\beta_{k-1}) \tag{2.13}$$

Proof. It proceeds by induction on k. For k=2, eq.(2.12) is equivalent to

$$\begin{aligned}a^{(2)}(\alpha)&=\sum_{\beta_1\in\mathbb{Z}^s}a(\alpha-2\beta_1)a(\beta_1)\\ &=\sum_{\substack{\beta_0,\beta_1\in\mathbb{Z}^s\\ \alpha=\beta_0+2\beta_1}}a(\beta_0)a(\beta_1)\end{aligned}$$

so the assumption is satified. Suppose $k>2$ and the lemma has been verified for

$k-1$. Then

$$\begin{aligned} a^{(k)}(\alpha) &= \sum_{\beta\in\mathbb{Z}^s} a(\alpha-2\beta)a^{(k-1)}(\beta) \\ &= \sum_{\beta\in\mathbb{Z}^s}\left(\sum_{\substack{\beta_1,\ldots,\beta_{k-1}\in\mathbb{Z}^s \\ \beta=\beta_1+2\beta_2+\cdots+2^{k-2}\beta_{k-1}}} a(\beta_1)a(\beta_2)\cdots a(\beta_{k-1})\right)a(\alpha-2\beta) \\ &= \sum_{\substack{\beta_0,\beta_1,\ldots,\beta_{k-1}\in\mathbb{Z}^s \\ \alpha=\beta_0+2\beta_1+\cdots+2^{k-1}\beta_{k-1}}} a(\beta_0)a(\beta_1)\cdots a(\beta_{k-1}), \\ & \qquad (let\ \alpha-2\beta=\beta_0) \end{aligned}$$

thereby completing the induction. □

By induction with respect to k, the support of these iterated masks is found:

Lemma 2.2. *(see [4], pp.19) For $I := supp\ a$, the supports $I^{(k)} := supp\ a^{(k)}$, for $k > 1$, satisfy*

$$I^{(k)} \subseteq I + 2\cdot I^{(k-1)} \subseteq I + 2\cdot I + 2^2\cdot I + \cdots + 2^{k-1}\cdot I. \tag{2.14}$$

In case of a non-negative mask a, i.e., if $a(\alpha) \geq 0$ for all $\alpha \in \mathbb{Z}^s$, we have

$$I^{(k)} = I + 2\cdot I^{(k-1)} = I + 2\cdot I + 2^2\cdot I + \cdots + 2^{k-1}\cdot I. \tag{2.15}$$

Proof. It proceeds by induction on k. When $k = 2$, let $\alpha \in I^{(2)}$, *i.e.*, $a^{(2)}(\alpha) \neq 0$. By eq. (2.12) there is a $\gamma \in I$ such that

$$a(\gamma)a(\alpha-2\gamma) \neq 0.$$

Thus, $\alpha - 2\gamma \in I$ and $\alpha \in I + 2\cdot I$. Hence $I^{(2)} \subseteq I + 2\cdot I$.

Suppose $k > 2$ and the lemma has been verified for $k-1$. Let $\alpha \in I^{(k)}$, *i.e.*, $a^{(k)}(\alpha) \neq 0$. By eq. (2.12), there is a $\gamma \in I^{(k-1)}$ such that

$$a^{(k-1)}(\gamma)a(\alpha-2\gamma) \neq 0.$$

Therefore, $\alpha - 2\gamma \in I$, whence $\alpha \in I + 2\gamma \subseteq I + 2\cdot I^{(k-1)}$. Correspondingly, we have

$$\begin{aligned} I^{(k)} &\subseteq I + 2\cdot I^{(k-1)} \\ &\subseteq I + 2\cdot(I + 2\cdot I + \cdots + 2^{k-2}\cdot I) \\ &= I + 2\cdot I + 2^2\cdot I + \cdots + 2^{k-1}\cdot I. \end{aligned}$$

In case of a non-negative mask a, for any $\alpha \in I + 2\cdot I + 2^2\cdot I + \cdots + 2^{k-1}\cdot I$, we have $\gamma_0, \gamma_1, \ldots, \gamma_{k-1} \in I$ such that

$$\alpha = \sum_{j=0}^{k-1} 2^j\gamma_j.$$

As $a(\gamma_j) \neq 0$ we conclude from (2.13)

$$a^{(k)}(\alpha) \geq a(\gamma_0) \cdots a(\gamma_{k-1}) > 0,$$

in other words, $\alpha \in I^{(k)}$ and

$$I^{(k)} \supseteq I + 2 \cdot I + 2^2 \cdot I + \cdots + 2^{k-1} \cdot I.$$

Therefore,

$$I^{(k)} = I + 2 \cdot I^{(k-1)} = I + 2 \cdot I + 2^2 \cdot I + \cdots + 2^{k-1} \cdot I$$

when the mask a is non-negative. □

Chapter 3

Non-negative subdivision

In this chapter we will present the connection of non-negative subdivision scheme to non-homogeneous Markov processes in the univariate and in the bivariate case. We will also introduce the connection of uniform convergence to left convergent matrix products.

3.1 The univariate case

3.1.1 Notation

We assume throughout that the mask (2.1) is non-negative,

$$a(\alpha) \geq 0, \qquad \alpha \in \mathbb{Z},$$

as well as

$$\operatorname{supp} a := \{\alpha \in \mathbb{Z} : a(\alpha) \neq 0\} \subset \{0, 1, \ldots, N\}, \quad \text{and} \quad a(0)a(N) \neq 0, \tag{3.1}$$

for some integer $N \geq 1$.

We also assume the necessary condition of Proposition 2.1 or Proposition 2.2 as the following property of the two (even- and odd-indexed) submasks:

$$\sum_{\alpha \in \mathbb{Z}} a(2\alpha) = \sum_{\alpha \in \mathbb{Z}} a(2\alpha + 1) = 1. \tag{3.2}$$

We will frequently refer to the $(N+1) \times (N+1)$-matrix

$$\begin{aligned} \mathbf{A} &= (a(-\alpha + 2\beta))_{\alpha,\beta=0}^{N} \\ &= \begin{pmatrix} a(0) & a(2) & a(4) & \cdots & & 0 \\ 0 & a(1) & a(3) & a(5) & \cdots & 0 \\ 0 & a(0) & a(2) & a(4) & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & & \cdots & a(N-3) & a(N-1) & 0 \\ 0 & & \cdots & a(N-4) & a(N-2) & a(N) \end{pmatrix}, \end{aligned} \tag{3.3}$$

(where the mask appears on the diagonal), and to the two $N \times N$ principle submatrices

$$\mathbf{A}_0 = (a(-\alpha + 2\beta))_{\alpha,\beta=0}^{N-1} \tag{3.4}$$

and

$$\mathbf{A}_1 = (a(-\alpha + 2\beta))_{\alpha,\beta=1}^{N} = (a(-\alpha + 2\beta + 1))_{\alpha,\beta=0}^{N-1}. \tag{3.5}$$

Property (3.2) tells that these matrices are all non-negative and row stochastic.

If the subdivision scheme is uniformly convergent, then the initial data $c = a^{(0)} = \delta = (\delta(\alpha))_{\alpha\in\mathbb{Z}}$, with $\delta(\alpha)$ denoting the Kronecker delta, *i.e.*,

$$\delta(\alpha) := \begin{cases} 1, & \alpha = 0; \\ 0, & \text{otherwise}, \end{cases}$$

leads to the iterated masks

$$a^{(k)} = (S_a)^k \delta, \qquad k = 1, 2, \ldots,$$

where $a^{(1)} = a$. In this case the limit function $\phi = f_\delta$ (the basic limit function) satisfies the two-scale equation

$$\phi(x) = \sum_{\alpha\in\mathbb{Z}} a(\alpha)\phi(2x - \alpha), \qquad x \in \mathbb{R}, \tag{3.6}$$

and it turns out that supp $\phi \subset [0, N]$. From this, for any integer β, we find

$$\phi(x + \beta) = \sum_{\alpha\in\mathbb{Z}} a(\alpha)\phi(2x + 2\beta - \alpha) = \sum_{\alpha\in\mathbb{Z}} a(-\alpha + 2\beta)\phi(2x + \alpha).$$

We consider these equations for $x + \beta \in [0, N]$, *i.e.*, for $x \in [0, 1]$ and for $\beta \in \{0, 1, \ldots, N - 1\}$. For $x \in [0, 1]$ we shall employ its dyadic expansion

$$x = \sum_{i=1}^{\infty} \epsilon_i \frac{1}{2^i} \qquad \text{with} \quad \epsilon_i \in \{0, 1\}, \quad i = 1, 2, \ldots. \tag{3.7}$$

Remark 3.1. For dyadic numbers

$$x = \frac{\alpha}{2^k}, \quad 0 \le \alpha < 2^k,$$

we have two equivalent expansions, one ending with a constant sequence of bits $\epsilon_i = 0$, and another one ending with a constant sequence of bits $\epsilon_i = 1$. In order to overcome this ambiguity, we agree to the following restriction: *If $x \in [0, 1)$ is a dyadic number, i.e., $x = \frac{\beta}{2^{i_0}}$ for some integer $\beta, 0 \le \beta < 2^{i_0}$, then $\epsilon_i = 0$ for $i > i_0$.*

With this restriction in mind, we find-using the fact ϕ vanishes outside the open interval $(0, N)$:

in case $x \in [0, 1/2)$, *i.e.*, $\epsilon_1 = 0$:

$$\phi(x + \beta) = \sum_{\alpha=0}^{N-1} a(-\alpha + 2\beta)\phi(y + \alpha),$$

where $y = 2x$, since here $2x \in [0, 1)$, and

in case $x \in [1/2, 1)$, *i.e.*, $\epsilon_1 = 1$:

$$\begin{aligned}\phi(x+\beta) &= \sum_{\alpha=-1}^{N-2} a(-\alpha+2\beta)\phi(2x+\alpha) \\ &= \sum_{\alpha=0}^{N-1} a(-\alpha+2\beta+1)\phi(y+\alpha)\end{aligned}$$

where $y = 2x - 1$, since here $2x \in [1, 2)$.

In both cases, the row vector $\Phi(x) = (\phi(x), \phi(x+1), \cdots, \phi(x+N-1))$ is obtained from the row vector $\Phi(2x)$ and $\Phi(2x-1)$, respectively, through right multiplication with $\mathbf{A}_0$ and $\mathbf{A}_1$, respectively. We refer to [26], where this description was used for the first time.

3.1.2 The corresponding non-homogeneous Markov processes

The non-homogeneous Markov chain which we are talking about in univariate subdivision, deals with the pair of row stochastic $(N \times N)$-matrices

$$\mathcal{A} := \{\mathbf{A}_0, \mathbf{A}_1\} \tag{3.8}$$

from eq. (3.4) and (3.5), and with the convergence of matrix products $\mathbf{A}_{\delta_1} \cdots \mathbf{A}_{\delta_k}$ (right convergence) and $\mathbf{A}_{\epsilon_k} \cdots \mathbf{A}_{\epsilon_1}$ (left convergence) as $k \to \infty$, where $\delta_i, \epsilon_i \in \{0, 1\}$ are chosen at random. The connection of these products with the iterated masks from (2.12) with $s = 1$ is described by

Lemma 3.1. *For any integer $k > 0$ and $\delta_i \in \{0, 1\}, i = 1, \ldots, k$, we have*

$$\mathbf{A}_{\delta_1}\mathbf{A}_{\delta_2} \cdots \mathbf{A}_{\delta_k}(\alpha, \beta) = a^{(k)}(-\alpha + \lambda + 2^k\beta), \qquad 0 \le \alpha, \beta \le N-1,$$

where $\lambda = \delta_1 + 2\delta_2 + \cdots + 2^{k-1}\delta_k$.

Proof. The proof proceeds by induction on k. For $k = 1$, putting $\delta_1 = 0$ or 1, the statement is just the definition of $\mathbf{A}_0$ and $\mathbf{A}_1$ given in eq. (3.4) and (3.5).

Suppose $k > 1$, and assume that the lemma has been verified for products of length less than k. By the definition of the iterated masks in (2.12), here let $s = 1$, we have

$$a^{(k)}(-\alpha + \lambda + 2^k\beta) = \sum_{j\in\mathbb{Z}} a(-\alpha + \lambda + 2^k\beta - 2j)a^{(k-1)}(j) \tag{3.9}$$

for any $\alpha, \beta, \lambda \in \mathbb{Z}$. We check for the possibilities where

$$-\alpha + \lambda + 2^k\beta - 2j \in \text{supp } a \subset \{0, 1, \ldots, N\},$$

for given α, β satisfying $0 \le \alpha, \beta \le N-1$ and $\lambda = \delta_1 + 2\delta_2 + \cdots + 2^{k-1}\delta_k = \delta_1 + 2\lambda'$.

Let $\gamma := \delta_2 + 2\delta_3 + \cdots + 2^{k-2}\delta_k + 2^{k-1}\beta - j$. Then the assumption

$$-\alpha + \lambda + 2^k\beta - 2j = -\alpha + \delta_1 + 2\gamma \in \{0, 1, \ldots, N\}$$

implies, since $\alpha \in \{0, 1, \ldots, N-1\}$,

$$\begin{aligned}\delta_1 + 2\gamma &\in \alpha + \{0, 1, \ldots, N\} \subset \{0, 1, \ldots, 2N-1\} \\ &= 2 \cdot \{0, 1, \ldots, N-1\} \cup (\{1\} + 2 \cdot \{0, 1, \ldots, N-1\}),\end{aligned}$$

hence $\gamma \in \{0, 1, \ldots, N-1\}$, since $\delta_1 \in \{0, 1\}$. Thus, eq. (3.9) reduces to

$$a^{(k)}(-\alpha + \lambda + 2^k\beta) = \sum_{0 \le \gamma \le N-1} a(-\alpha + \delta_1 + 2\gamma) a^{(k-1)}(-\gamma + \lambda' + 2^{k-1}\beta), \quad (3.10)$$

with $\lambda' = \delta_2 + 2\delta_3 + \cdots + 2^{k-2}\delta_k$ as above.

Now, by the induction hypothesis, for $0 \le \alpha, \beta, \gamma \le N-1$, we have

$$\mathbf{A}_{\delta_2} \cdots \mathbf{A}_{\delta_k}(\gamma, \beta) = a^{(k-1)}(-\gamma + \lambda' + 2^{k-1}\beta)$$

and

$$\mathbf{A}_{\delta_1}(\alpha, \gamma) = a(-\gamma + \delta_1 + 2\gamma).$$

This in connection with (3.10) gives

$$\begin{aligned}\mathbf{A}_{\delta_1}\mathbf{A}_{\delta_2} \cdots \mathbf{A}_{\delta_k}(\alpha, \beta) &= \sum_{0 \le \gamma \le N-1} \mathbf{A}_{\delta_1}(\alpha, \gamma)\mathbf{A}_{\delta_2} \cdots \mathbf{A}_{\delta_k}(\gamma, \beta) \\ &= a^{(k)}(-\alpha + \lambda + 2^k\beta),\end{aligned}$$

where $0 \le \alpha, \beta \le N-1$, thereby completing the induction. □

3.1.3 The connection of uniform convergence to left convergent matrix products

The connection with the basic limit function now follows from the convergence assumption

$$\lim_{k\to\infty} \max_{\alpha,\beta=0,\ldots,N-1} \left| \phi\left(\frac{-\alpha + \lambda + 2^k\beta}{2^k}\right) - a^k(-\alpha + \lambda + 2^k\beta) \right| = 0, \quad (3.11)$$

where $\lambda = \delta_1 + 2\delta_2 + \cdots + 2^{k-1}\delta_k$ and $\delta_i \in \{0, 1\}, i = 1, \ldots, k$. Putting here

$$\epsilon_i = \delta_{k-i+1}, \qquad i = 1, \ldots, k,$$

we see that

$$\frac{1}{2^k}\lambda = \sum_{i=1}^{k} \frac{\epsilon_i}{2^i}.$$

Take $x \in [0, N]$, which we may represent as

$$x = \beta + \sum_{i=1}^{\infty} \epsilon_i \frac{1}{2^i}$$

with $\beta = 0, 1, \ldots, N-1$. If we put

$$\frac{-\alpha + \lambda + 2^k \beta}{2^k} = x_k = -\frac{\alpha}{2^k} + \beta + \sum_{i=1}^{k} \epsilon_i \frac{1}{2^i}$$

then $x_k \to x$ as $k \to \infty$, and by Lemma 3.1, $\phi(x_k)$ will be close to

$$a^{(k)}(-\alpha + \sum_{i=1}^{k} 2^{k-i} \epsilon_i + 2^k \beta) = \mathbf{A}_{\epsilon_k} \mathbf{A}_{\epsilon_{k-1}} \cdots \mathbf{A}_{\epsilon_1}(\alpha, \beta). \tag{3.12}$$

Note that the indices of the bits ϵ_i from the binary expansion of $x - \lfloor x \rfloor$ appear in this matrix product in a decreasing manner, while in the above Lemma 3.1 the indexing of the δ_i's goes the reverse way. Also, since the representation for dyadic numbers is ambiguous (it may end with a sequences of zeros, or with a sequence of ones) we have two different sequences of bits in eq. (3.12), if x is dyadic.

3.2 Bivariate non-negative subdivision

3.2.1 Notation

We again assume that the mask is non-negative,

$$a(\alpha) \geq 0, \qquad \alpha \in \mathbb{Z}^2,$$

and compactly supported, *i.e.*,

$$\text{supp } a := \{\alpha \in \mathbb{Z}^2 : a(\alpha) \neq 0\}$$

is a finite subset of $\mathbb{Z}^2$. By shifting the support, we can arrange for

$$\text{supp } a \subseteq R_{N_1,N_2} := \{0, 1, \ldots, N_1\} \times \{0, 1, \ldots, N_2\} \tag{3.13}$$

for some positive integers N_1 and N_2; this shift will not affect the essential property of convergence of the scheme which arises from iterating the subdivision operator.

In order to check the uniform convergence of bivariate subdivision scheme, it is sufficient to show that, for the initial data $c^{(0)} = \delta$, convergence holds with a continuous limit function $f_\delta = \phi$, with ϕ not identically zero. Here, $\delta = (\delta(\alpha))_{\alpha \in \mathbb{Z}^2}$ is the bivariate Kronecker delta, with

$$\delta(\alpha) := \begin{cases} 1, & \alpha = (0,0); \\ 0, & \text{otherwise.} \end{cases}$$

In this case, the limit function satisfies the two-scale equation,

$$\phi(x) = \sum_{\alpha \in \mathbb{Z}^2} a(\alpha) \phi(2x - \alpha), \qquad x \in \mathbb{R}^2, \tag{3.14}$$

and the support of ϕ is a subset of the closed rectangle $[0, N_1] \times [0, N_2]$.

3.2.2 The corresponding non-homogeneous Markov processes

In order to describe the values of the iterated mask in terms of a specific non-homogeneous Markov process, we look now at the matrix

$$\mathbf{A}(\alpha,\beta)=a(-\alpha+2\beta),\qquad \alpha=(\alpha_1,\alpha_2),\beta=(\beta_1,\beta_2)\in R_{N_1,N_2}. \tag{3.15}$$

Here, α stands for the row index, while β accounts for the column index. We assume that the $(N_1+1)(N_2+1)$ points from the discrete rectangle R_{N_1,N_2} have been put into some order (*e.g.*, using lexicographic order), which we assume to prevail also in subsequent formulas where the components of row vectors, or of column vectors, are indexed by pairs $\alpha\in R_{N_1,N_2}$.

With respect to the matrix (3.15), we consider the family

$$\mathcal{A}:=\{\mathbf{A}_\epsilon:\epsilon\in E\},\qquad E:=\{0,1\}^2=\{(0,0),(1,0),(0,1),(1,1)\}, \tag{3.16}$$

of $(N\times N)$-matrices, with $N:=N_1\cdot N_2$, where

$$\mathbf{A}_\epsilon(\alpha,\beta)=a(-\alpha+\epsilon+2\beta),\qquad \alpha,\beta\in R_{N_1-1,N_2-1},\quad \epsilon\in E. \tag{3.17}$$

The choice of ϵ refers to the cosets of $\mathbb{Z}^2/2\mathbb{Z}^2$. Since

$$a(-\alpha+\epsilon+2\beta)=a(-\alpha'+2\beta')$$

with

$$\alpha'=\alpha+\epsilon,\quad \beta'=\beta+\epsilon,$$

we see that the matrices $\mathbf{A}_\epsilon$ are submatrices of $\mathbf{A}$. Following a useful notation from finite automata, we will call the family $\mathcal{A}$ our alphabet, where we build words from, *i.e.*, finite products

$$\mathbf{A}_{\delta_1}\mathbf{A}_{\delta_2}\cdots\mathbf{A}_{\delta_k},\qquad \delta_i\in E,\quad i=1,\dots,k.$$

The number k of factors in such a product is the length of this word. The following well-known fact connecting the iterated mask $a^{(k)}$ with such words of length k, is the bivariate occasion of Lemma 3.1:

Lemma 3.2. *For any integer $k>0$ and any sequence $\delta_i\in E, i=1,\dots,k$, we have*

$$\mathbf{A}_{\delta_1}\mathbf{A}_{\delta_2}\cdots\mathbf{A}_{\delta_k}(\alpha,\beta)=a^{(k)}(-\alpha+\lambda+2^k\beta),\qquad \alpha,\beta\in R_{N_1-1,N_2-1},$$

where $\lambda=\delta_1+2\delta_2+\cdots+2^{k-1}\delta_k$.

Proof. For completeness, we present a proof for this important observation. It again proceeds by induction on k. For $k=1$, putting $\delta_1=\epsilon$, the statement is just the definition given in (3.17).

Suppose $k>1$, and assume that the lemma has been verified for words of length less than k. By the definition of the iterated masks in (2.12), here let $s=2$, we have

$$a^{(k)}(-\alpha+\lambda+2^k\beta)=\sum_{j\in\mathbb{Z}^2}a(-\alpha+\lambda+2^k\beta-2j)a^{(k-1)}(j) \tag{3.18}$$

for any $\alpha, \beta, \lambda \in \mathbb{Z}^2$. We check for the possibilities where

$$-\alpha + \lambda + 2^k\beta - 2j \in \operatorname{supp} a \subset R_{N_1,N_2},$$

for given $\alpha, \beta \in R_{N_1-1,N_2-1}$ and $\lambda = \delta_1 + 2\delta_2 + \cdots + 2^{k-1}\delta_k = \delta_1 + 2\lambda'$.

Let $\gamma := \delta_2 + 2\delta_3 + \cdots + 2^{k-2}\delta_k + 2^{k-1}\beta - j$. Then the assumption

$$-\alpha + \lambda + 2^k\beta - 2j = -\alpha + \delta_1 + 2\gamma \in R_{N_1,N_2}$$

implies, since $\alpha \in R_{N_1-1,N_2-1}$

$$\delta_1 + 2\gamma \in \alpha + R_{N_1,N_2} \subset R_{2N_1-1,2N_2-1} = \bigcup_{\epsilon \in E}(\epsilon + 2 \cdot R_{N_1-1,N_2-1}),$$

whence $\gamma \in R_{N_1-1,N_2-1}$, since $\delta_1 \in E$. Thus, eq. (3.18) reduces to

$$a^{(k)}(-\alpha + \lambda + 2^k\beta) = \sum_{\gamma \in R_{N_1-1,N_2-1}} a(-\alpha + \delta_1 + 2\gamma) a^{(k-1)}(-\gamma + \lambda' + 2^{k-1}\beta), \quad (3.19)$$

with $\lambda' = \delta_2 + 2\delta_3 + \cdots + 2^{k-2}\delta_k$ as above.

Now, by the induction hypothesis, for $\alpha, \beta, \gamma \in R_{N_1-1,N_2-1}$, we have

$$\mathbf{A}_{\delta_2} \cdots \mathbf{A}_{\delta_k}(\gamma, \beta) = a^{(k-1)}(-\gamma + \lambda' + 2^{k-1}\beta)$$

and

$$\mathbf{A}_{\delta_1}(\alpha, \gamma) = a(-\gamma + \delta_1 + 2\gamma).$$

Thus, the expression in (3.19) is the stated matrix product. This completes the proof. □

The connection of (stationary) bivariate non-negative subdivision with a non-negative mask to non-homogeneous Markov processes is now apparent. The convergence of the iterated mask is related to the convergence of the matrix product in Lemma 3.2 as the length of the word goes to infinity. Here, the factors (the alphabet of our words) are chosen from the family $\mathcal{A}$ which is a family of row stochastic matrices, as we shall verify now: if the necessary conditions of convergence of bivariate non-negative subdivision are supposed to hold, by Proposition 2.1 and Proposition 2.2, we have

$$\sum_{\beta \in \mathbb{Z}^2} a(\epsilon + 2\beta) = 1, \qquad \epsilon \in E. \quad (3.20)$$

Eq. (3.20) is equivalent to

$$\sum_{\beta \in \mathbb{Z}^2} a(-\alpha + \epsilon + 2\beta) = 1$$

for $\alpha \in R_{N_1-1,N_2-1}$ and $\epsilon \in E$. Now if $\alpha \in R_{N_1-1,N_2-1}$ and $\epsilon \in E$ are fixed, we see that $-\alpha + \epsilon + 2\beta \in \operatorname{supp} a \subseteq R_{N_1,N_2}$ implies that

$$\epsilon + 2\beta \in \alpha + R_{N_1,N_2} \subseteq R_{2N_1-1,2N_2-1},$$

which implies that $\beta \in R_{N_1-1,N_2-1}$. Then

$$\sum_{\beta \in \mathbb{Z}^2} a(-\alpha+\epsilon+2\beta) = \sum_{\beta \in R_{N_1-1,N_2-1}} a(-\alpha+\epsilon+2\beta)$$

showing that eq. (3.20) is indeed equivalent to the property of the matrices $\mathbf{A}_\epsilon \in \mathcal{A}, \epsilon \in E$, being row stochastic.

3.2.3 The connection of uniform convergence to left convergent matrix products

If we specialize the condition (2.6) of convergence to the fundamental limit function, here $s = 2$, Lemma 3.2 yields

$$\lim_{k\to\infty} \max_{\alpha,\beta \in R_{N_1-1,N_2-1}} \left| \phi\left(\frac{-\alpha+\lambda+2^k\beta}{2^k}\right) - \mathbf{A}_{\delta_1}\mathbf{A}_{\delta_2}\cdots\mathbf{A}_{\delta_k}(\alpha,\beta)\right| = 0,$$

where $\lambda = \delta_1 + 2\delta_2 + \cdots + 2^{k-1}\delta_k$. Putting here

$$\epsilon_i = \delta_{k-i+1}, \qquad i = 1, \ldots, k,$$

we see that

$$\frac{1}{2^k}\lambda = \sum_{i=1}^{k} \frac{\epsilon_i}{2^i},$$

and

$$\lim_{k\to\infty} \left| \phi\left(-\frac{\alpha}{2^k} + \sum_{i=1}^{k} \frac{\epsilon_i}{2^i} + \beta\right) - \mathbf{A}_{\epsilon_k}\mathbf{A}_{\epsilon_{k-1}}\cdots\mathbf{A}_{\epsilon_1}(\alpha,\beta)\right| = 0, \tag{3.21}$$

for $\alpha, \beta \in R_{N_1-1,N_2-1}$. This relates the entries of the matrix product to the values of the fundamental limit function as follows.

Let $x = (x_1, x_2) \in [0,1]^2$, and consider the dyadic expansion of x,

$$x = \sum_{i=1}^{\infty} \frac{\epsilon_i}{2^i}, \qquad \epsilon_i = (\epsilon_{i,1}, \epsilon_{i,2}) \in E, \quad i = 1, 2, \ldots \tag{3.22}$$

This expansion is unique except for the cases where x_1 or x_2 is dyadic, *i.e.*, is of type $k/2^\kappa$ with $k \in \mathbb{N}, 0 \le k \le 2^\kappa$. In that case, we have two different dyadic expansions for this component of x, one ending with a sequence of zeros, and the other one ending with a sequence of ones. This ambiguity has to be observed carefully, see Section 5.2.2. Now, for given x as in (3.22), condition (3.21) for the continuous function ϕ can be rewritten as

$$\lim_{k\to\infty} \mathbf{A}_{\epsilon_k}\mathbf{A}_{\epsilon_{k-1}}\cdots\mathbf{A}_{\epsilon_1} = \mathbf{e}\ \Phi(x) \tag{3.23}$$

for the N-column vector $\mathbf{e} = (1, \cdots, 1)^T$, and the N-row vector

$$\Phi(x) = (\phi(x+\beta))_{\beta \in R_{N_1-1,N_2-1}}, \qquad x \in [0,1]^2. \tag{3.24}$$

Eq. (3.23) will be referred to as left convergence of the matrix product to the rank one matrix with all rows equal to the row vector (3.24)

Following the seminal paper of Micchelli and Prautzsch [26], the row vector $\Phi(x)$ can be also determined as the limit of a vector iteration coming from the two-scale equation (3.14). Put x as in eq. (3.22), whence $x = \frac{\epsilon_1}{2} + y$ with $y \in [0, 1/2]^2$. Then, for $\beta \in R_{N_1-1,N_2-1}$,

$$\begin{aligned}\phi(x+\beta) &= \sum_{\alpha\in\mathbb{Z}^2} a(\alpha)\phi(2(x+\beta)-\alpha) \\ &= \sum_{\alpha\in R_{N_1-1,N_2-1}} a(-\alpha+\epsilon_1+2\beta)\phi(2y+\alpha)\end{aligned}$$

This shows that

$$\Phi(x) = \Phi(2y)\mathbf{A}_{\epsilon_1} = \Phi(2x-\epsilon_1)\mathbf{A}_{\epsilon_1}$$

for

$$x = \sum_{i=1}^{\infty} \frac{\epsilon_i}{2^i} \in [0,1]^2 \quad \text{and} \quad y = \sum_{i=1}^{\infty} \frac{\epsilon_i}{2^i} \in [0,1/2]^2.$$

Iterating this process relates $\Phi(x)$ again to the left convergence of the matrix products in (3.23).

Chapter 4

SIA Matrices

In this chapter, we will use indices for elements of matrices, or components of vectors, in order to make formulas easier to read. In the subsequent chapters where these indices are sometimes given by composed formulas, we will turn back to write these as function arguments. We will deal with $(N \times N)$-matrices, for some given positive integer N, and we let

$$\mathbf{e} := (1, \cdots, 1)^T \in \mathbb{R}^N.$$

4.1 The SIA property of families of non-negative matrices

Recall that an $(N \times N)$-matrix $\mathbf{P} = (p_{i,j})_{i,j=1}^N$ is called row stochastic, if $p_{i,j} \geq 0$ for all $i, j = 1, \ldots, N$, and $\sum_{j=1}^N p_{i,j} = 1$ for all $i = 1, \ldots, N$. It is clear from the Gerschgorin theorem, that all eigenvalues λ of such a matrix are inside the closed unit circle, and $\lambda = 1$, together with the N-column vector $\mathbf{e}$ is a (right) eigenpair of $\mathbf{P}$.

A peculiar property of stochastic matrices is the fact that each extremal eigenvalue λ (with the property $|\lambda| = 1$) has an eigenspace of full dimension equal to the algebraic multiplicity of λ. So each Jordan block refering to an extremal eigenvalue has dimension one. Using this fact, it is not hard to show that the following convergence holds (see the ergodic theorem, Hauptsatz 3.5, in [13]): *For a row stochastic matrix* $\mathbf{P}$ *the limit*

$$\mathbf{Q} = \lim_{k \to \infty} \frac{1}{k} \sum_{\kappa=0}^{k} \mathbf{P}^\kappa \tag{4.1}$$

always exists, and it is a (row stochastic) projection matrix satisfying $\mathbf{Q}^2 = \mathbf{Q} = \mathbf{QP} = \mathbf{PQ}$. *Its rank is given by the multiplicity of the eigenvalue* $\lambda = 1$.

The question whether the powers $\mathbf{P}^k$ converge as $k \to \infty$ thus reduces to the case that $\lambda = 1$ is the only extremal eigenvalue. In this case

$$\mathbf{Q} = \lim_{k\to\infty} \mathbf{P}^k$$

for the limit in (4.1). If in addition, the eigenvalue $\lambda = 1$ is simple then the powers converge to a rank one row stochastic matrix

$$\mathbf{Q} = \lim_{k\to\infty} \mathbf{P}^k = \mathbf{e}\mathbf{p}^T = \begin{pmatrix} p_1 & p_2 & \cdots & p_N \\ \vdots & \vdots & \ddots & \vdots \\ p_1 & p_2 & \cdots & p_N \end{pmatrix} \tag{4.2}$$

for a row vector $\mathbf{p}^T = (p_1, \cdots, p_N)$ which is the unique left eigenvector of $\mathbf{P}$ for the eigenvalue $\lambda = 1$ satisfying $p_i \geq 0$ and $p_1 + \cdots + p_N = 1$. Vectors of this type will be called probability vectors from now on.

It is the latter property of rank one convergence which we will focus on. In the notation of numerical analysis this is the assumption on $\mathbf{P}$ for the eigenvalue $\lambda = 1$ being dominant, *i.e.*, it is simple and all other eigenvalues have absolute value less than one. Following the early paper of Wolfowitz [35], we call such row stochastic matrices $\mathbf{P}$ to have property SIA (stochastic, indecomposable, aperiodic). Concerning this notation which comes from the application of such matrices in the theory of finite Markov chains, we refer to the discussion in [35] and to the books [9], [23] and [31].

We point to the different meaning of 'indecomposable' (here we require the presence of only one single ergodic class) in various papers which apparently led to the misinterpretation of being equivalent to 'irreducible' (referring to the property of the directed graph of a matrix to be strongly connected). In particular, we warn the reader that in [13] the notion 'unzerlegbar' (='indecomposable') refers to the graph property.

We give one criterion for checking whether $\mathbf{P}$ is SIA, see Satz 2.14 in [13] (the property 'sehr gut' referred to there is just our SIA property):

Lemma 4.1. *A row stochastic matrix* $\mathbf{P}$ *is SIA if there is a power* $\mathbf{P}^k$ *which has at least one column with all entries positive.*

For a generalization of this, see Lemma 4.7 below. It implies that the converse statement of the lemma holds as well.

4.2 The directed graph of row stochastic matrices

It is useful to formulate the positive column property as a property of the directed graph $G(\mathbf{P}) = \{V, E\}$ associated with the (non-negative) $(N \times N)$-matrix $\mathbf{P}$. This graph is described by the set of its N nodes or vertices $V = \{1, \ldots, N\}$ and the set of its (directed) edges as the set of ordered pairs

$$E = \{(i, j) : i, j = 1, \ldots, N \text{ with } p_{j,i} > 0\}.$$

Note that we use the notation as in the theory of finite Markov processes since in this case the graph describes the way how the information flows. For an edge (i, j), i and j may be called the parent and the child node, respectively, and in place of $(i, j) \in E$ we may also write $i \to j$, meaning that i connects to j. A (directed) path in the graph, of length k, is a given set of edges $(i_0, i_1), (i_1, i_2), \ldots, (i_{k-1}, i_k)$ connecting vertex i_0 with vertex i_k as

$$i_0 \to i_1 \to \cdots \to i_k.$$

Here, we allow a repetition of edges, and also loops which are edges of type (i, i). Since edges in the graph $G(\mathbf{P}^k)$ correspond to paths of length k in the graph $G(\mathbf{P})$, the following is clear:

Lemma 4.2. *For a row stochastic matrix* $\mathbf{P}$ *and* $k \in \mathbb{N}$ *the following are equivalent:*

(a) $\mathbf{P}^k$ *has a strictly positive column* i_0*, say.*

(b) The graph $G(\mathbf{P})$ *has the property that vertex* i_0 *connects to all vertices* $j = 1, \ldots, N$ *via a path of length* k*.*

Property (b) tells that the graph $G(\mathbf{P})$ is connected (there is a vertex which connects to each other vertex by a path), but not necessarily strongly connected (meaning that i connects to j by a path, for any pair (i, j) of vertices). For a connected graph, we can reduce the connectivity by keeping all vertices but omitting certain edges until we arrive at a directed tree (which is a graph where each vertex has a unique parent vertex, except just one vertex i_0 - the so-called root of the tree). Such a tree connecting all vertices is called a *spanning tree* of the graph $G(\mathbf{P})$.

Remark 4.1. In some applications of graph theory to non-negative matrices, edges (i, j) are related to elements $p_{i,j} > 0$ rather than $p_{j,i} > 0$. In that case, the graph refers to properties of $G(\mathbf{P}^T)$ in our notation. We note that the transformation $\mathbf{P} \to \mathbf{P}^T$ does not affect the spectrum, but row-stochastic matrices are transformed into column-stochastic ones, and the graph $G(\mathbf{P}^T)$ is obtained from $G(\mathbf{P})$ by reversing the direction of edges.

Instead of the graph $G(\mathbf{P})$ one can also consider the graph of $\mathbf{Q} = \mathbf{P} - \mathbf{I}$. This transformation

$$\mathbf{P} \mapsto \mathbf{Q} = \mathbf{P} - \mathbf{I} \tag{4.3}$$

transforms the spectrum of $\mathbf{P}$ into a subset of the circle in the complex plane with center at -1 and with radius 1. The SIA property is then the property of $\mathbf{Q}$ that all eigenvalues of $\mathbf{Q}$ are strictly inside this circle except the eigenvalue $\lambda = 0$ which, in addition, must be simple.

The graph $G(\mathbf{Q})$ of $\mathbf{Q}$ has the same vertices as $G(\mathbf{P})$, and also the same edges except for loops. Since loops are not essential for the graph to contain a spanning tree, $G(\mathbf{P})$ and $G(\mathbf{Q})$ share the same spanning trees.

Let $\mathcal{N}_N$ denote the set of $(N \times N)$-matrices $\mathbf{Q}$ with non-negative coefficients outside the diagonal such that all row sums are zero, whence

$$q_{i,i} = -\sum_{i\neq j=1}^{N} q_{i,j}, \qquad i = 1, \dots, N.$$

The transformation $\mathbf{P} \mapsto \mathbf{Q} = \mathbf{P} - \mathbf{I}$ transforms row stochastic matrices into the set $\mathcal{N}_N$ (but not conversely), and the following property of the elementary symmetric functions of the eigenvalues of such matrices may be applied.

Lemma 4.3. *For the characteristic polynomial*

$$\chi_{\mathbf{Q}}(\lambda) = \det(\lambda \mathbf{I}_N - \mathbf{Q}) = \lambda^N + p_{N-1}\lambda^{N-1} + \cdots + p_1\lambda + p_0 \tag{4.4}$$

of a matrix $\mathbf{Q} \in \mathcal{N}_N$ the following holds true:

(a) $p_0 = 0$ *and* $p_1 \geq 0$.

(b) If we modify $\mathbf{Q}$ *by increasing one off-diagonal entry and decreasing the corresponding diagonal entry accordingly,*

$$q_{\ell,m} \mapsto q_{\ell,m} + \Delta \quad \textit{and} \quad q_{\ell,\ell} \mapsto q_{\ell,\ell} - \Delta$$

for some $1 \leq \ell, m \leq N$ with $\ell \neq m$ and some $\Delta \geq 0$, then the coefficient p_1 is non-decreasing.

(c) $p_1 > 0$ *if and only if the directed graph $G(\mathbf{Q})$ has a spanning tree.*

Proof. (a) We note that $\mathbf{Q}$ has zero row sum, and non-positive diagonal coefficients. Therefore, $\mathbf{Q}$ has at least one zero eigenvalue and from Gerschgorin theorem (see Theorem 1.11 in [33]), we know that all the other non-zero eigenvalues $\mathbf{Q}$ are in the open left-half plane. Hence, $p_0 = 0$. The characteristic polynomial of $\mathbf{Q}$ can also be denoted as

$$\det(\lambda \mathbf{I}_N - \mathbf{Q}) = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_N),$$

where $\lambda_1, \lambda_2, \dots, \lambda_N$ are N eigenvalues of matrix $\mathbf{Q}$. Since,

$$p_1 = \frac{d(\det(\lambda \mathbf{I}_N - \mathbf{Q}))}{d\lambda}\Big|_{\lambda=0}, \tag{4.5}$$

$p_1 \geq 0$ follows from the fact that the non-zero eigenvalues of $\mathbf{Q}$ have negative real part, and non-real eigenvalues come in complex-conjugate pairs.

(b) Without loss of genarality, we assume that $\ell = 1$ and $m > 1$. Let $\mathbf{Q}' = (q'_{i,j})_{i,j=1}^N$, where

$$q'_{1,m} = q_{1,m} + \Delta, \quad q'_{1,1} = q_{1,1} - \Delta,$$

and otherwise, $q'_{i,j} = q_{i,j}$. Let

$$\mathbf{Q}_\lambda = \lambda \mathbf{I}_N - \mathbf{Q} \quad \text{and} \quad \mathbf{Q}'_\lambda = \lambda \mathbf{I}_N - \mathbf{Q}'.$$

Denote

$$\det \mathbf{Q}_\lambda = \det(\lambda \mathbf{I}_N - \mathbf{Q}) \quad \text{and} \quad \det \mathbf{Q}'_\lambda = \det(\lambda \mathbf{I}_N - \mathbf{Q}').$$

Given any matrix $\mathbf{M}$, denote $\mathbf{M}([i,j])$ as the submatrix of $\mathbf{M}$ formed by deleting the i-th row and j-th column.

Assume that $\mathbf{Q}_\lambda = (q_{i,j}^{(\lambda)})_{i,j=1}^N$, and $\mathbf{Q}'_\lambda = (q_{i,j}^{\prime(\lambda)})_{i,j=1}^N$. Accordingly, it can be seen that

$$q_{1,1}^{\prime(\lambda)} = \lambda - q'_{1,1} = \lambda - q_{1,1} + \Delta = q_{1,1}^{(\lambda)} + \Delta$$

and

$$q_{1,m}^{\prime(\lambda)} = -q'_{1,m} = -q_{1,m} - \Delta = q_{1,m}^{(\lambda)} - \Delta;$$

and otherwise,

$$q_{i,j}^{\prime(\lambda)} = q_{i,j}^{(\lambda)}.$$

Also note that

$$\det \mathbf{Q}'_\lambda([1,j]) = \det \mathbf{Q}_\lambda([1,j]), \qquad j = 1, \dots, N.$$

Then, we know that

$$\begin{aligned}\det \mathbf{Q}'_\lambda &= \sum_{j=1}^N (-1)^{1+j} q_{1,j}^{\prime(\lambda)} \det \mathbf{Q}'_\lambda([1,j]) \\ &= \sum_{j=1}^N (-1)^{1+j} q_{1,j}^{(\lambda)} \det \mathbf{Q}'_\lambda([1,j]) \\ &\quad + \Delta \det \mathbf{Q}'_\lambda([1,1]) - (-1)^{1+m} \Delta \det \mathbf{Q}'_\lambda([1,m]) \\ &= \det \mathbf{Q}_\lambda + \Delta(\det \mathbf{Q}'_\lambda([1,1]) + (-1)^m \det \mathbf{Q}'_\lambda([1,m])).\end{aligned}$$

Consider a matrix $\mathbf{E} = (e_{i,j})_{i,j=1}^{N-1}$, given by adding the $(N-1)$-vector $(q_{2,1}, q_{3,1}, \dots, q_{N,1})$ to the $(m-1)$-th column of matrix $\mathbf{Q}'([1,1])$. Matrix $\mathbf{E}$ can be denoted as

$$\mathbf{E} = \begin{pmatrix} q_{2,2} & q_{2,3} & \cdots & q_{2,m} + q_{2,1} & \cdots & q_{2,N} \\ q_{3,2} & q_{3,3} & \cdots & q_{3,m} + q_{3,1} & \cdots & q_{3,N} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ q_{N,2} & q_{N,3} & \cdots & q_{N,m} + q_{N,1} & \cdots & q_{N,N} \end{pmatrix}. \tag{4.6}$$

Thus,

$$e_{i,(m-1)} = q_{(i+1),m} + q_{(i+1),1}, \qquad i = 1, \dots, N-1.$$

Using the linear properties of determinants, it can be verified that

$$\det(\lambda \mathbf{I}_{N-1} - \mathbf{E}) = \det \mathbf{Q}'_\lambda([1,1]) + (-1)^m \det \mathbf{Q}'_\lambda([1,m]).$$

Obviously, matrix $\mathbf{E}$ has zero row sum and non-positive diagonal coefficients. From the Gerschgorin theorem again, we know that $\mathbf{E}$ has at least one zero eigenvalue and all the other non-zero eigenvalues are on the left-half plane. Let

$$\det(\lambda \mathbf{I}_{N-1} - \mathbf{E}) = \lambda^{N-1} + p'_{N-2}\lambda^{N-2} + \cdots + p'_1\lambda + p'_0.$$

From (a), we get $p'_1 \geq 0$.

Noting that

$$\begin{aligned}\det \mathbf{Q}'_\lambda &= \det \mathbf{Q}_\lambda + \Delta \det(\lambda \mathbf{I}_{N-1} - \mathbf{E}) \\ &= \lambda^N + (p_{N-1} + \Delta)\lambda^{N-1} + \cdots + (p_1 + \Delta p'_1)\lambda + (p_0 + \Delta p'_0).\end{aligned}$$

This verifies that the coefficient p_1 is non-decreasing.

(c) From (a) and eq. (4.5), we see that $p_1 > 0$ if and only if $\lambda = 0$ is an algebraically simple eigenvalue of $\mathbf{Q}$.

First we will show that if the directed graph associated with $\mathbf{Q}$ is itself a spanning tree, then $\lambda = 0$ is an algebraically simple eigenvalue of $\mathbf{Q}$. We renumber the vertices consecutively by depth in the spanning tree, with the root numbered as vertex 1. In other words, children of 1 are numbered 2 to ℓ_1, children of 2 to ℓ_1 are labeled $\ell_1 + 1$ to ℓ_2 and so on. Hence, by choosing an appropriate permutation matrix $\mathbf{P}_0$ we get

$$\mathbf{P}_0^T \mathbf{Q} \mathbf{P}_0 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ q_{2,1} & q_{2,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ q_{N,1} & q_{N,2} & \cdots & q_{N,N} \end{pmatrix},$$

where $q_{i,i} < 0, i = 2, \ldots, N$. Hence, in this case, $\lambda = 0$ is an algebraically simple eigenvalue of $\mathbf{Q}$. Therefore, $p_1 > 0$.

Finally, if the directed graph associated with $\mathbf{Q}$ has a spanning tree, from (b), since the coefficient p_1 is non-decreasing, and we also have $p_1 > 0$.

Conversely, if the directed graph $G(\mathbf{Q})$ does not have a spanning tree, then there exist at least two separate subgraphs or at least two vertices in the directed graph $G(\mathbf{Q})$ who do not have any one parent vertex. For the first case, there exists an appropriate permutation matrix $\mathbf{P}_1$ such that

$$\mathbf{P}_1^T \mathbf{Q} \mathbf{P}_1 = \begin{pmatrix} Q_{1,1} & 0 & \cdots & 0 \\ 0 & Q_{2,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Q_{s,s} \end{pmatrix},$$

where $Q_{i,i}, i = 1, 2, \ldots, s$ are $s \geq 2$ diagonal block submatrices. Therefore, $\mathbf{Q}$ has at least two zero eigenvalues. For the second case, $\mathbf{Q}$ has at least two zero rows, *i.e.,* there exists some permutation matrix $\mathbf{P}_2$ that

$$\mathbf{P}_2^T \mathbf{Q} \mathbf{P}_2 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ q_{3,1} & q_{3,2} & \cdots & q_{3,N} \\ \vdots & \vdots & \ddots & \vdots \\ q_{N,1} & q_{N,2} & \cdots & q_{N,N} \end{pmatrix},$$

which implies that $\mathbf{Q}$ has at least two zero eigenvalues. This is a contradiction. Therefore, the directed graph $G(\mathbf{Q})$ has a spanning tree. □

Property (c) of Lemma 4.3 is Lemma 3.3 in [30], and statement (b) -although not stated this way - can be read off the proof there. We also refer to Theorem 8.3.

4.3 The directed graph of SIA matrices

Our considerations in the last section yield the following partial characterization of SIA matrices in term of properties of its graph $G(\mathbf{P})$.

Proposition 4.1. *For a row stochastic matrix* $\mathbf{P}$ *the following hold:*

(a) If $\mathbf{P}$ *is SIA, then* $G(\mathbf{P})$ *contains a spanning tree.*

(b) Conversely, if $G(\mathbf{P})$ *contains a spanning tree, and if the root* i_0 *of this spanning tree has a loop* (i_0, i_0)*, then* $\mathbf{P}$ *is SIA.*

Proof. The first statement is a combination of the statements of Lemma 4.1 and Lemma 4.2. In order to prove the second statement, we know from Lemma 4.3, that under the given assumption, $\lambda = 1$ is an algebraically simple eigenvalue of $\mathbf{P}$, and we have to verify that each other eigenvalue satisfies $|\lambda| < 1$.

We may assume that the root of the given spanning tree is given by vertex i_0, and we look at the maximal strongly connected component of the graph containing vertex i_0, with vertices

$$V_1 = \{i_0, i_1, \dots, i_{N_1-1}\} \subset V = \{1, \dots, N\}.$$

V_1 is not empty, since $i_0 \to i_0$ by assumption, and without loss of generality - by renumbering the vertices appropriately - we may assume that $V_1 = \{1, 2, \dots, N_1\}$. We write $\mathbf{P}$ in block form

$$\mathbf{P} = \left(\begin{array}{c|c} \mathbf{P}_{1,1} & \mathbf{P}_{1,2} \\ \hline \mathbf{P}_{2,1} & \mathbf{P}_{2,2} \end{array}\right)$$

with quadratic diagonal blocks of size $N_1 \geq 1$ and $N - N_1 \geq 0$, respectively.

Consider first the spectral properties of the block $\mathbf{P}_{1,1}$ which is irreducible, due to $G(\mathbf{P}_{1,1})$ being strongly connected, see Theorem 1.17 in [33]. Since the vertex $i_0 = 1$ has a loop, the diagonal entry $p_{1,1}$ of the matrix is positive. This implies that the matrix $\mathbf{P}_{1,1}$ is primitive; see Exercise 1 in Section 2.2 of [33]. Thus $\mathbf{P}_{1,1}$ is SIA if it is row stochastic. But this follows from the fact that, in case $N_1 < N$, $\mathbf{P}_{1,2}$ must be a zero block. Namely, if $p_{j,i} > 0$ for some $1 \leq j \leq N_1$ and $N_1 < i \leq N$, then

$$i \to j \to \cdots \to k \to \cdots \to 1 \to \cdots \to i$$

for $k = 1, \dots, N_1$, since $G(\mathbf{P}_{1,1})$ is strongly connected and $G(\mathbf{P})$ contains a spanning tree with root $i_0 = 1$. This contradicts the maximality of the strongly connected component.

From this we conclude that, in case $N_1 < N$, $\lambda = 1$ cannot be an eigenvalue of the lower diagonal block $\mathbf{P}_{2,2}$. But the spectral radius $\lambda = \rho(\mathbf{P}_{2,2})$ must be an eigenvalue of that block, due to the Perron-Frobenius theorem for non-negative matrices, see Theorem 2.20 in [33]. Since $\mathbf{P}_{2,2}$ is a subblock of a row stochastic matrix, all eigenvalues satisfy $|\lambda| \leq 1$, and therefore $\rho(\mathbf{P}_{2,2}) < 1$.

Finally, in case $N_1 < N$, $\rho(\mathbf{P}_{2,2}) < 1$ and $\mathbf{P}_{1,2}$ is a zero block. Since the spectrum of $\mathbf{P}$ is the union of the spectra of the diagonal blocks, we have that $\lambda = 1$ is an algebraically simple eigenvalue of $\mathbf{P}$ and each other eigenvalue of $\mathbf{P}$ satisfies $|\lambda| < 1$. Therefore, we conclude that $\mathbf{P}$ is SIA. This completes the proof. □

Remark 4.2. For the reader's convenience, in order to understand Proposition 4.1 sufficiently and completely, we give some definitions of matrix and graph theory and introduce some useful lemmas and theorems as an appendix in Chapter 8, since we need these lemmas and theorems in the process of the proof of Proposition 4.1.

4.4 The ergodic coefficient and the scrambling property

For a row stochastic matrix $\mathbf{P} = (p_{i,j})_{i,j=1}^N$ we put

$$\gamma(\mathbf{P}) := \min_{i_1,i_2} \sum_j \min(p_{i_1,j}, p_{i_2,j}),$$

and we define the ergodic coefficent as

$$\tau(\mathbf{P}) := \frac{1}{2} \max_{1\le i_1<i_2\le N} \sum_{j=1}^N |p_{i_1,j} - p_{i_2,j}| = 1 - \gamma(\mathbf{P}). \tag{4.7}$$

The identity $\gamma(\mathbf{P})+\tau(\mathbf{P}) = 1$ will become clear in a moment. The notions are used in the theory of Markov chains. The coefficient appears in an important inequality, see the following lemma, and is also more or less implicitly used in convergence proofs for subdivision. As discussed in [35]- who refers to Hajnal [16]- this coefficient satisfies $0 \le \tau(\mathbf{P}) \le 1$, and is less than 1 if and only if $\mathbf{P}$ has the property that for each pair of row indices i_1, i_2 there is a column index $j = j(i_1, i_2)$ such that

$$p_{i_1,j} > 0 \quad \text{and} \quad p_{i_2,j} > 0.$$

Such row stochastic matrices are called scrambling, a notation introduced by Hajnal in [16]. In his paper, $\gamma(\mathbf{P}) = \{\mathbf{P}\}$ is called the scrambling power of $\mathbf{P}$, and in his Lemma 3 he proves an inequality similar to the following one which we take from Theorem 3.1 in [31].

Lemma 4.4. *For a row stochastic matrix $\mathbf{P} = (p_{i,j})_{i,j=1}^N$ and any vector $\mathbf{x} = (x_i)_{i=1}^N$, the vector $\mathbf{z} = \mathbf{P}\mathbf{x} = (z_i)_{i=1}^N$ satisfies the estimate*

$$\max_i z_i - \min_i z_i \le \tau(\mathbf{P}) \left(\max_i x_i - \min_i x_i \right), \tag{4.8}$$

with the ergodic coefficient $\tau(\mathbf{P})$ as in (4.7).

Proof. Since the estimate is used in the proof of Theorem 4.1, and is essential for our entire analysis, we provide the proof along the lines of [2]. For real numbers

a, b, we put $a = a^+ - a^-$, with $a^+, a^- \geq 0$, and we will use the fact that $(a-b)^+ = a - \min(a,b)$. For $1 \leq i_1, i_2 \leq N$ we have

$$\begin{aligned} z_{i_1} - z_{i_2} &= \sum_{j=1}^{N} (p_{i_1,j} - p_{i_2,j})^+ x_j - \sum_{j=1}^{N} (p_{i_1,j} - p_{i_2,j})^- x_j \\ &\leq \sum_{j=1}^{N} (p_{i_1,j} - p_{i_2,j})^+ \left(\max_i x_i - \min_i x_i \right). \end{aligned}$$

Here, we have used the identity

$$0 = \sum_{j=1}^{N} (p_{i_1,j} - p_{i_2,j}) = \sum_{j=1}^{N} (p_{i_1,j} - p_{i_2,j})^+ - \sum_{j=1}^{N} (p_{i_1,j} - p_{i_2,j})^-,$$

showing in addition that

$$\sum_{j=1}^{N} (p_{i_1,j} - p_{i_2,j})^+ = \frac{1}{2} \sum_{j=1}^{N} |(p_{i_1,j} - p_{i_2,j})|.$$

Now,

$$\begin{aligned} \tau(\mathbf{P}) &= \max_{i_1,i_2} \sum_{j=1}^{N} (p_{i_1,j} - p_{i_2,j})^+ \\ &= \max_{i_1,i_2} \sum_{j=1}^{N} (p_{i_1,j} - \min(p_{i_1,j}, p_{i_2,j})) \\ &= 1 - \min_{i_1,i_2} \sum_{j=1}^{N} \min(p_{i_1,j}, p_{i_2,j}) = 1 - \gamma(\mathbf{P}), \end{aligned}$$

showing that

$$z_{i_1} - z_{i_2} \leq \tau(\mathbf{P}) \left(\max_i x_i - \min_i x_i \right)$$

for arbitrary i_1, i_2. The result follows by chosing these indices appropriately. □

The lemma tells that for scrambling matrices, $\tau(\mathbf{P})$ serves as a contraction coefficient in the 'gain inequality' (4.8). As a subclass of the family of row stochastic matrices, the scrambling matrices have interesting properties. *E.g.*, the property is dominant if we take products of row stochastic matrices:

Lemma 4.5. *(see [16], Lemma 2) If* $\mathbf{P}_1$ *and* $\mathbf{P}_2$ *are row stochastic and either of them is a scrambling matrix, so is* $\mathbf{P} = \mathbf{P}_1\mathbf{P}_2$.

A similar dominant property is the positive column property of row stochastic matrices refered to in Lemma 4.1. This class was also mentioned in Hajnal's paper, and was discussed in more detail by Anthonisse and Tijms in [2]. As a special case of Lemma 4.7 below, which we will prove in full detail, we mention the following:

Proposition 4.2. *For a row stochastic matrix* $\mathbf{P}$, *the scrambling property (or the positive column property) implies the property SIA. Conversely, if* $\mathbf{P}$ *has property SIA, then there is a power* $\mathbf{P}^k$ *which is scrambling (or has the positive column property).*

For an interpretation of this result in terms of properties of the directed graph $G(\mathbf{P})$, see Lemma 4.2. For another characterization of scrambling matrices in terms of a restricted ℓ_1 matrix norm see Lemma 2.1 of [21]; note that their result must be applied to the column-stochastic matrix $\mathbf{P}^T$ in our notation.

In [21] and in [6], convergence of products of column stochastic matrices is studied using the notion of joint spectral radius. Refering to Lemma 2.1 in [21] we can refer the scrambling property of a row stochastic matrix $\mathbf{P} = (p_{i,j})_{i,j=1}^N$ to the norm of the operator

$$L_{\mathbf{P}} : \mathbb{R}^N \to \mathbb{R}^N, \ \mathbf{x} \mapsto \mathbf{x}\,\mathbf{P}$$

operating on row vectors, when restricted to the subspace

$$V = \left\{ \mathbf{x} = (x_1, \cdots, x_N) : \sum_{j=1}^N x_j = 0 \right\}.$$

It is clear that V is an invariant subspace of the operator $L_{\mathbf{P}}$, due to the property that the matrix $\mathbf{P}$ is row stochastic.

Let $\|\cdot\|$ denote the 1-norm of row vectors, $\|\mathbf{x}\| = \sum_{j=1}^N |x_j|$. Then the norm of the restricted operator $L_{\mathbf{P}}|_V$ is given by

$$\|\mathbf{P}\|_V = \max_{\mathbf{x}\in V, \|\mathbf{x}\|=1} \|\mathbf{x}\mathbf{P}\| = \tau_{\|\cdot\|}(\mathbf{P}),$$

this norm is called the $\|\cdot\|$-*coefficient of ergodicity* of $\mathbf{P}$ in [19]. Lemma 2.1 in [21] states that

Lemma 4.6. *A row stochastic* $(N\times N)$*-matrix* $\mathbf{P}$ *is scrambling if and only if* $\|\mathbf{P}\|_V < 1$.

4.5 Equivalent conditions to the SIA property

For products of stochastic matrices which are words

$$\mathbf{P} = \mathbf{P}_k\mathbf{P}_{k-1}\cdots\mathbf{P}_1, \qquad \mathbf{P}_\kappa \in \mathcal{P}, \quad \kappa = 1, \ldots, k,$$

from an alphabet $\mathcal{P}$, the SIA property for words can be characterized as follows:

Lemma 4.7. *For a finite family* $\mathcal{P}$ *of row stochastic* $(N\times N)$*-matrices, the following properties are equivalent:*

(a) There is an integer $\nu \geq 1$ *such that each word* $\mathbf{P}$ *of length* $k \geq \nu$ *from the family* $\mathcal{P}$ *is scrambling.*

(b) There is an integer $\mu \geq 1$ such that each word $\mathbf{P}$ of length $k \geq \mu$ from the family $\mathcal{P}$ has a strictly positive column.

(c) Each word $\mathbf{P}$ from $\mathcal{P}$ is SIA, i.e., we have convergence of type (4.2) for some probability vector $\mathbf{p}^T = (p_1, \cdots, p_N)$, depending on the word $\mathbf{P}$.

For the discussion of these properties, we refer to the introductory remarks in [2], who refer to [35]. But the essential ideas are already in [16]. It is clear that (a) implies (b) with $\mu = (N-1)\nu$, while (b) trivially implies (a) with $\nu = \mu$. More interesting is the fact that -given one of these equivalent conditions -we can bound the numbers ν and μ. This was already observed by [35] for the scrambling property, while [2] mention this for the positive column property. We follow here an argument given by [26] in the proof of their Theorem 2.1; see also the proof of Lemma 2.2 in [21] dealing with words from a family of column stochastic matrices.

Lemma 4.8. *If condition (c) in Lemma 4.7 holds true, then conditions (a) and (b) hold for some numbers ν and μ satisfying $\nu \leq \mu \leq k_n := 2^{N^2}$.*

Proof. For any word $\mathbf{P} = (p_{i,j})_{i,j=1}^N$ we can check its signum matrix (or its pattern, as Hajnal calls this)

$$\sigma(\mathbf{P}) = (\sigma(p_{i,j}))_{i,j=1}^N,$$

obtained from $\mathbf{P}$ by replacing each non-zero coefficient by 1. It is clear from the dimension of the matrices that we have strictly less than 2^{N^2} different sign patterns for the set of row stochastic $(N \times N)$-matrices. So if we have 2^{N^2} words, at least two of them must have the same pattern.

Now assume that property (c) holds, and that we can find some word

$$\mathbf{P}_{k_n,1} = \mathbf{P}_{k_n}\mathbf{P}_{k_n-1}\cdots\mathbf{P}_1,$$

of length $k = k_n$, which does not have a positive column. Then all words $\mathbf{P}_{\ell,1} = \mathbf{P}_\ell\mathbf{P}_{\ell-1}\cdots\mathbf{P}_1, \ell = 1, \dots, 2^{N^2}$, cannot have a positive column either, and among them two words must have an identical sign pattern. *I.e.*, there exist indices $1 \leq \ell_1 < \ell_2 \leq 2^{N^2}$ such that

$$\sigma(\mathbf{P}_{\ell_1,1}) = \sigma(\mathbf{P}\mathbf{P}_{\ell_1,1}), \qquad \mathbf{P} = \mathbf{P}_{\ell_2}\cdots\mathbf{P}_{\ell_1+1}.$$

Using the identity $\sigma(\mathbf{AB}) = \sigma(\sigma(\mathbf{A})\sigma(\mathbf{B}))$ for the product of row stochastic matrices, we get by induction that

$$\sigma(\mathbf{P}_{\ell_1,1}) = \sigma(\mathbf{P}^k\mathbf{P}_{\ell_1,1}), \qquad k = 0, 1, \dots$$

telling that none of the matrices $\mathbf{P}^k\mathbf{P}_{\ell_1,1}$ has a positive column. Since $\mathbf{P}$ is SIA, the limit as $k \to \infty$ exists as in (4.2), and

$$\lim_{k\to\infty} \mathbf{P}^k\mathbf{P}_{\ell_1,1} = (1, \cdots, 1)^T\ (p_1, \cdots, p_N)\ \mathbf{P}_{\ell_1,1}$$

is a row stochastic matrix with all row identical. Thus the limit must have a strictly positive column. This is a contradiction, and the lemma is proved. □

In order to complete the proof of Lemma 4.7, we have to show that condition (a) implies condition (c) which we will verify in the proof of Theorem 4.1; see the remarks after that theorem. We also quote an extension of Lemma 3 in [35] which is mentioned in the introduction of [2]:

Lemma 4.9. *If $\mathbf{P}_1, \mathbf{P}_2$ are row stochastic matrices satisfying $\sigma(\mathbf{P}_1\mathbf{P}_2) = \sigma(\mathbf{P}_1)$, then the SIA property for the right-hand factor $\mathbf{P}_2$ implies that the left-hand factor $\mathbf{P}_1$ has a positive column, hence has property SIA as well.*

Proof. As before, we find inductively that $\sigma(\mathbf{P}_1) = \sigma(\mathbf{P}_1\mathbf{P}_2^k)$ for $k = 0, 1, \ldots$. For large enough k, since $\lim_{k\to\infty} \mathbf{P}_2^k$ is a row stochastic matrix with identical rows, the power $\mathbf{P}_2^k$ and hence $\mathbf{P}_1\mathbf{P}_2^k$ must have a positive column. □

Remark 4.3.
1. The sign pattern of a non-negative matrix relates the class of matrices with the same pattern to its directed graph. The estimate $\nu \leq \mu \leq k_n := 2^{N^2}$ given in Lemma 4.8 is rather rough, but sufficient to our analysis.

2. In the theory of homogeneous Markov processes, the usual assumption on the row stochastic system matrix $\mathbf{P}$ is its primitivity, meaning that some power $\mathbf{P}^k$ is strictly positive. The least number for which this property holds, is sometimes called the index of primitivity, see Section 2.2 in [33].

3. The scrambling index of a primitive matrix $\mathbf{P}$ is the least positive integer k such that $\mathbf{P}^k$ is a scrambling matrix in [1].

4. A directed graph G is called primitive if for some positive integer k there is a path of length exactly k from each vertex u to each vertex v; the scrambling index of a primitive directed graph G is the least positive integer k such that for every pair of vertices u and v, we can get to a vertex w from both u and v in the directed graph G by directed paths of length k in [1].

4.6 The Anthonisse – Tijms result on left convergence of products of row stochastic matrices

The left convergence of products of row stochastic matrices was studied in subdivision, or in multiresolution analysis using different means. Micchelli and Prautzsch [26] use the 'gain inequality' (4.8) without refering to the corresponding result in [31]. Daubechies and Lagarias [6] use the notion of joint spectral radius. We are going to use the following result from nonhomogeneous finite Markov processes, which will give us the possibility to define the limit function ϕ of our subdivision scheme pointwise, as well as to show that this limit function is Hölder continuous.

Theorem 4.1. *(see [2], Theorem 1) Let $\mathcal{P}$ be a finite set of row stochastic $(N \times N)$-matrices, such that each word from $\mathcal{P}$ is SIA. Then there is an integer $\nu \geq 1$ and a number c with $0 \leq c < 1$, such that the following holds true: For any sequence*

$\mathbf{P}_1, \mathbf{P}_2, \ldots$ of elements from $\mathcal{P}$ there is a probability vector $\pi = (\pi_1, \cdots, \pi_N)$ such that, for all $k \geq 1$ and all $i, j = 1, \ldots, N$:

$$|(\mathbf{P}_k \mathbf{P}_{k-1} \cdots \mathbf{P}_1)_{i,j} - \pi_j| \leq c^{\lfloor k/\nu \rfloor}.$$

Here, $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

Proof. For completeness, we present the proof. By Lemma 4.8 (see also Lemma 4 in [35]), under the given assumptions on the family $\mathcal{P}$ of matrices, there exists a number $\nu \leq 2^{N^2}$ such that each word of length ν is scrambling. If $\mathbf{P}$ is such a word, its ergodic coefficient satisfies $\tau(\mathbf{P}) < 1$, and since there are only finitely many words of length ν,

$$c := \max\{\tau(\mathbf{P}) \ : \ \mathbf{P} \text{ is a word of length } \nu\} < 1. \tag{4.9}$$

As we will see, c serves as a contraction coefficient for the family $\mathcal{P}$.

Let us consider words of length k of type

$$\mathbf{P}_{k,1} := \mathbf{P}_k \cdots \mathbf{P}_1 = (p_{i,j}^{(k)})_{i,j=1}^N.$$

Refering to the j-th column of $\mathbf{P}_{k,1}$, we put

$$M_j^{(k)} = \max_{i=1,\ldots,N} p_{i,j}^{(k)} \quad \text{and} \quad m_j^{(k)} = \min_{i=1,\ldots,N} p_{i,j}^{(k)}, \qquad j = 1, \ldots, N.$$

Since the values $p_{i,j}^{(k)}$ are computed as convex combinations of the values in the j-th column of $\mathbf{P}_{k-1,1}$, we see that, for fixed j, the sequence $M_j^{(k)}$ is nonincreasing, the sequence $m_j^{(k)}$ is nondecreasing, and

$$M_j^{(k)} - m_j^{(k)} \leq M_j^{(1)} - m_j^{(1)} \leq 1 \qquad \text{for all } k.$$

Here, we have used Lemma 4.4 in a weak sense by refering to $\tau(\mathbf{P}_\kappa) \leq 1$, for $\kappa = 1, \ldots, k$. But the gain inequality (4.8) shows more, if applied to prefixes $\mathbf{P} = \mathbf{P}_k \mathbf{P}_{k-1} \cdots \mathbf{P}_{k-\nu+1}$ of length ν: Since $\tau(\mathbf{P}) \leq c$, we have

$$M_j^{(k)} - m_j^{(k)} \leq c \left(M_j^{(k-\nu)} - m_j^{(k-\nu)} \right) \quad \text{for} \quad k > \nu,$$

and, for

$$k = \alpha\nu + \beta \quad \text{with} \quad \alpha, \beta \in \mathbb{N},\ 0 \leq \beta < \nu,$$

we get

$$M_j^{(k)} - m_j^{(k)} \leq c^\alpha \left(M_j^{(\beta)} - m_j^{(\beta)} \right) \leq c^\alpha.$$

From this the estimate of the theorem follows, by chosing

$$\pi_j = \lim_{k\to\infty} M_j^{(k)} = \lim_{k\to\infty} m_j^{(k)}, \quad j = 1, \ldots, N.$$

We have the rank-one convergence

$$\lim_{k\to\infty} \mathbf{P}_k \mathbf{P}_{k-1} \cdots \mathbf{P}_1 = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} (\pi_1, \cdots, \pi_N),$$

hence $\pi = (\pi_1, \cdots, \pi_N)$ must be a probability vector. □

Remark 4.4.
1. It is clear that the converse of the statement of the theorem is true as well. *So the property 'Each word from $\mathcal{P}$ is SIA' is equivalent to the convergence at a geometric rate as stated in the theorem.*

2. The proof shows that, in Lemma 4.7, condition (a) implies condition (c). Connecting this with the proof of Lemma 4.8 yields the full proof of Lemma 4.7

3. The geometric convergence result remains to hold true for infinite families, if eq. (4.9) is satisfied - with max replaced by sup.

4. Concerning the convergence of inhomogenous Markov chains, we also refer to the papers by Hartfiel and Rothblum [19] and Neumann and Schneider [28], and to the references given there.

Chapter 5

Uniform convergence of non-negative subdivision

In this chapter, we will discuss uniform convergence of non-negative subdivision in the univariate and in the bivariate case; and in Section 3 of this chapter, we will give some sufficient conditions for uniform convergence of non-negative bivariate subdivision.

5.1 The univariate case

Recall that the mask is non-negative,

$$a(\alpha) \geq 0, \qquad \alpha \in \mathbb{Z},$$

as well as

$$\text{supp}\, a := \{\alpha \in \mathbb{Z} : a(\alpha) \neq 0\} \subset \{0, 1, \ldots, N\}, \quad \text{and} \quad a(0)a(N) \neq 0, \tag{5.1}$$

for some integer $N \geq 1$. We also assume that the mask satisfies the necessary condition (3.2).

In non-negative and univariate subdivision, our alphabet is the pair of $(N \times N)$-matrices

$$\mathcal{A} = \{\mathbf{A_0}, \mathbf{A_1}\}$$

from (3.4) and (3.5), and we are dealing with sequences $\mathbf{A}_{\epsilon_1}, \mathbf{A}_{\epsilon_2}, \ldots$, where the indices refer to the bits in the dyadic expansion of points $x \in [0, 1]$,

$$x = \sum_{i=1}^{\infty} \epsilon_i \frac{1}{2^i} \quad \text{with} \quad \epsilon_i \in \{0, 1\}, \quad i = 1, 2, \ldots. \tag{5.2}$$

We note that here, $x = 1$ is allowed, and we also allow ambiguity for the bits in this expansion which appears when x is a dyadic point.

We will assume here that

$$0 < a(0),\ a(N) < 1. \tag{5.3}$$

This implies $N \geq 2$ since for $N = 1$, the row stochasticity just means $a(0) = a(1) = 1$. (5.3) is a necessary condition for uniform convergence, since *e.g.*, $a(0) = 1$ implies $\phi(0) = 1$, hence ϕ cannot be continuous at $x = 0$.

5.1.1 Pointwise definition of the basic limit function

Assuming the scrambling property for words of length ν, for some $\nu \leq 2^{N^2}$, the probability vector from Theorem 4.1 is given by

$$\pi_\beta = \pi_\beta(x) =: \phi(x+\beta), \qquad \beta = 0, 1, \ldots, N-1.$$

In order to see that ϕ is unambigously defined, we have to overcome the ambiguity for representing dyadic numbers. We start with a useful lemma.

Lemma 5.1. *If the family $\mathcal{A} = \{\mathbf{A}_0, \mathbf{A}_1\}$ satisfies the scrambling property for words of length ν, for some $\nu \leq 2^{N^2}$, and if*

$$0 < a(0) < 1 \quad \text{and} \quad 0 < a(N) < 1,$$

then there is a probability vector $\mathbf{p}^T = (p_1, \cdots, p_{N-1})$ such that

$$\lim_{k\to\infty} \mathbf{A}_0^k = \begin{pmatrix} 1 \\ \mathbf{e} \end{pmatrix} \begin{pmatrix} 0 \mid \mathbf{p}^T \end{pmatrix} \quad \text{and} \quad \lim_{k\to\infty} \mathbf{A}_1^k = \begin{pmatrix} \mathbf{e} \\ 1 \end{pmatrix} \begin{pmatrix} \mathbf{p}^T \mid 0 \end{pmatrix} \tag{5.4}$$

Here, $\mathbf{e}^T = (1, \cdots, 1)$ is an $(N-1)$-vector. In particular

$$\begin{pmatrix} 0 \mid \mathbf{p}^T \end{pmatrix} \mathbf{A}_0 = \begin{pmatrix} 0 \mid \mathbf{p}^T \end{pmatrix}, \qquad \begin{pmatrix} \mathbf{p}^T \mid 0 \end{pmatrix} \mathbf{A}_1 = \begin{pmatrix} \mathbf{p}^T \mid 0 \end{pmatrix}, \tag{5.5}$$

and the following compatibility condition holds:

$$\begin{pmatrix} 0 \mid \mathbf{p}^T \end{pmatrix} \mathbf{A}_1 = \begin{pmatrix} \mathbf{p}^T \mid 0 \end{pmatrix} \mathbf{A}_0. \tag{5.6}$$

Proof. The convergence of the powers of $\mathbf{A}_0$ and $\mathbf{A}_1$ can be put in a compact form. From the definition of the matrix $\mathbf{A}$, we get that

$$\mathbf{A} = \left(\begin{array}{c|c} \mathbf{A}_0 & \mathbf{0} \\ \hline * & a(N) \end{array}\right) \quad \text{and} \quad \mathbf{A} = \left(\begin{array}{c|c} a(0) & * \\ \hline \mathbf{0} & \mathbf{A}_1 \end{array}\right),$$

whence

$$\mathbf{A}^k = \left(\begin{array}{c|c} \mathbf{A}_0^k & \mathbf{0} \\ \hline * & a^k(N) \end{array}\right) \quad \text{and} \quad \mathbf{A}^k = \left(\begin{array}{c|c} a^k(0) & * \\ \hline \mathbf{0} & \mathbf{A}_1^k \end{array}\right).$$

Using the above convergence theorem, Theorem 4.1, and the fact that $\lim_{k\to\infty} a^k(0) = \lim_{k\to\infty} a^k(N) = 0$, we see that

$$\lim_{k\to\infty} \mathbf{A}^k = \begin{pmatrix} 1 \\ \mathbf{e} \\ 1 \end{pmatrix} \begin{pmatrix} 0 \mid \mathbf{p}^T \mid 0 \end{pmatrix},$$

for some probability vector $\mathbf{p}$ with $N-1$ components, and correspondingly we get eq. (5.4). Since

$$\lim_{k\to\infty} \mathbf{A}_0^k = \lim_{k\to\infty} \mathbf{A}_0^{k+1} = \left(\lim_{k\to\infty} \mathbf{A}_0^k\right) \cdot \mathbf{A}_0,$$

$$\left(1 \mid \mathbf{e}^T\right) \begin{pmatrix} 1 \\ \mathbf{e} \end{pmatrix} \left(0 \mid \mathbf{p}^T\right) = \left(1 \mid \mathbf{e}^T\right) \begin{pmatrix} 1 \\ \mathbf{e} \end{pmatrix} \left(0 \mid \mathbf{p}^T\right) \mathbf{A}_0,$$

and similarly also for the matrix $\mathbf{A}_1$, we get eq. (5.5).

The compatibility condition in eq. (5.6) follows from the fact that

$$\begin{aligned}
\left(0 \mid \mathbf{p}^T\right) \mathbf{A}_1 &= \left(0 \mid \mathbf{p}^T\right) \begin{pmatrix} a(1) & a(3) & a(5) & \cdots & 0 \\ a(0) & a(2) & a(4) & \cdots & 0 \\ 0 & a(1) & a(3) & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & a(N-3) & a(N-1) & 0 \\ 0 & \cdots & a(N-4) & a(N-2) & a(N) \end{pmatrix} \\
&= \mathbf{p}^T \begin{pmatrix} a(0) & a(2) & a(4) & \cdots & 0 \\ 0 & a(1) & a(3) & \cdots & 0 \\ 0 & a(0) & a(2) & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & a(N-3) & a(N-1) & 0 \\ 0 & \cdots & a(N-4) & a(N-2) & a(N) \end{pmatrix} \\
&= \left(\mathbf{p}^T \mid 0\right) \begin{pmatrix} a(0) & a(2) & a(4) & \cdots & 0 \\ 0 & a(1) & a(3) & \cdots & 0 \\ 0 & a(0) & a(2) & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & a(N-4) & a(N-2) & a(N) \\ 0 & \cdots & a(N-5) & a(N-3) & a(N-1) \end{pmatrix} \\
&= \left(\mathbf{p}^T \mid 0\right) \mathbf{A}_0.
\end{aligned}$$

□

Lemma 5.1 describes the left eigenspace for both matrices, for the dominant eigenvalue $\lambda = 1$, the right eigenspace being spanned by the N-vector with all components equal to 1. The result could also be described by the inner block

$$\mathbf{B} = \left(a(-\alpha + 2\beta)\right)_{\alpha,\beta=1}^{N-1}$$

of $\mathbf{A}$, which is an SIA matrix again satisfying

$$\lim_{k\to\infty} \mathbf{B}^k = \mathbf{e}\, \mathbf{p}^T. \tag{5.7}$$

Remark 5.1. The vector $\mathbf{p}^T$ is non-negative, and the sum of the entries is 1. There is the special case where all but one entry vanish. Here, $\mathbf{p}$ is a canonical $(N-1)$-unit vector. The position of the entry $p_\beta = 1$ is connected with $\mathbf{B}$ via $\mathbf{p}^T\mathbf{B} = \mathbf{p}^T$ telling that the β-th row of $\mathbf{B}$ equals $\mathbf{p}^T$. In this case, the mask satisfies $a(\beta) = 1$, and from the row stochasticity we conclude that β must be odd, while N must be even.

The last equation of Lemma 5.1 refers to the compatibility condition used in [26], Theorem 3.1, in case $p = 2$. Using that result (or repeating the easy proof given there), we arrive at the following definition of the basic limit function, as a consequence of Theorem 4.1:

Proposition 5.1. *Suppose $N \geq 2$ and $0 < a(0), a(N) < 1$. If the family $\mathcal{A} = \{\mathbf{A_0}, \mathbf{A}_1\}$ satisfies the scrambling property for words of length ν, for some $\nu \leq 2^{N^2}$, then for given*

$$x = \sum_{i=1}^{\infty} \epsilon_i \frac{1}{2^i} \in [0, 1]$$

we have

$$\lim_{k \to \infty} \mathbf{A}_{\epsilon_k} \mathbf{A}_{\epsilon_{k-1}} \cdots \mathbf{A}_{\epsilon_1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \pi(x) \tag{5.8}$$

with

$$\pi(x) = (\phi(x), \phi(x+1), \cdots, \phi(x+N-1)).$$

This defines the limit function pointwise (and unambigously) on the interval $[0, N]$, while $\phi(x) = 0$ outside this interval.

Proof. Let

$$x + \beta = \frac{\alpha}{2^{k_0}} + \beta, \qquad 0 \leq \beta < N,$$

with $0 \leq \alpha < 2^{k_0}$, where

$$\begin{aligned} \alpha &= \alpha_1 + 2\alpha_2 + \cdots + 2^{k_0-1}\alpha_{k_0} \\ &= \epsilon_{k_0} + 2\epsilon_{k_0-1} + \cdots + 2^{k_0-1}\epsilon_1, \end{aligned}$$

with $\alpha_1 = \epsilon_{k_0} = 1$. For $x = \frac{\alpha}{2^{k_0}}$, since $\epsilon_{k_0} = 1$, we have

$$x = \sum_{i=1}^{k_0} \frac{\epsilon_i}{2^i} = \sum_{i=1}^{k_0-1} \frac{\epsilon_i}{2^i} + \sum_{i=k_0+1}^{\infty} \frac{1}{2^i},$$

and correspondingly, we have two dyadic representations, namely

$$x = \sum_{i=1}^{\infty} \frac{\epsilon_i}{2^i}, \quad \text{with } \epsilon_{k_0} = 1 \text{ and } \epsilon_i = 0, i = k_0+1, k_0+2, \ldots \tag{5.9}$$

and

$$x = \sum_{i=1}^{\infty} \frac{\tilde{\epsilon}_i}{2^i}, \quad \text{with} \begin{cases} \tilde{\epsilon}_i = \epsilon_i, & i = 1, \ldots, k_0 - 1, \\ \tilde{\epsilon}_{k_0} = 0, & i = k_0, \\ \tilde{\epsilon}_i = 1, & i = k_0+1, k_0+2, \ldots. \end{cases} \tag{5.10}$$

For $k > k_0$, since

$$\mathbf{A}_{\epsilon_k} \cdots \mathbf{A}_{\epsilon_2} \mathbf{A}_{\epsilon_1} = \mathbf{A}_0 \cdots \mathbf{A}_0 \mathbf{A}_1 \mathbf{A}_{\epsilon_{k_0-1}} \cdots \mathbf{A}_{\epsilon_1}$$

and

$$\mathbf{A}_{\tilde{\epsilon}_k} \cdots \mathbf{A}_{\tilde{\epsilon}_2} \mathbf{A}_{\tilde{\epsilon}_1} = \mathbf{A}_1 \cdots \mathbf{A}_1 \mathbf{A}_0 \mathbf{A}_{\epsilon_{k_0-1}} \cdots \mathbf{A}_{\epsilon_1},$$

by eq. (5.4), we have

$$\lim_{k\to\infty} \mathbf{A}_{\epsilon_k} \mathbf{A}_{\epsilon_{k-1}} \cdots \mathbf{A}_{\epsilon_1} = \begin{pmatrix} 1 \\ \mathbf{e} \end{pmatrix} \left(0 \mid \mathbf{p}^T\right) \mathbf{A}_1 \mathbf{A}_{\epsilon_{k_0-1}} \cdots \mathbf{A}_{\epsilon_1}$$

and

$$\lim_{k\to\infty} \mathbf{A}_{\tilde{\epsilon}_k} \mathbf{A}_{\tilde{\epsilon}_{k-1}} \cdots \mathbf{A}_{\tilde{\epsilon}_1} = \begin{pmatrix} \mathbf{e} \\ 1 \end{pmatrix} \left(\mathbf{p}^T \mid 0\right) \mathbf{A}_0 \mathbf{A}_{\epsilon_{k_0-1}} \cdots \mathbf{A}_{\epsilon_1},$$

the compatibility condition (5.6) implies that the above two limits are the same. By eq. (3.11) and (3.12) again, we get that eq. (5.8) is established and the limit function is pointwise and unambigously on the interval $[0, N]$. □

We see, in particular, that the probability vector of Lemma 5.1 is given by

$$\left(0 \mid \mathbf{p}^T \mid 0\right) = (\phi(0) \mid \phi(1), \cdots, \phi(N-1) \mid \phi(N)).$$

The special case of $\mathbf{p}^T$ being a canonical unit vector (which we have addressed in the remark following Lemma 5.1) shows the interpolating property of the basic limit function: It vanishes on the set of all integers except at some β with $0 < \beta < N$.

5.1.2 Hölder continuity of the basic limit function

Proposition 5.1 shows that Theorem 3.1 of [26] can be derived from the convergence theorem of [2] by relatively simple arguments, using properties of stochastic matrices. However, the valuable continuity results from Micchelli and Prautzsch for the basic limit function need further discussion. Let us show that - with a more careful technique, based on Theorem 4.1 - we can even show Hölder continuity of the basic limit function. The Hölder exponent will refer to the contraction coefficient c from eq. (4.9)

Proposition 5.2. *Under the assumptions of Proposition 5.1, the limit function ϕ is Hölder continuous. More precisely, we have*

$$\|\pi(x) - \pi(y)\| \leq C|x-y|^\gamma \quad \textit{with} \quad \gamma = -\frac{1}{\nu}\log_2(c)$$

and c the contraction constant from Theorem 4.1 for $\mathcal{P} = \mathcal{A}$. The constant C only depends on c and ν, and on the chosen vector norm.

Proof. We can assume that $|x-y|<1$, since for $|x-y|\geq 1$, we find

$$\|\pi(x)-\pi(y)\|_\infty \leq 2 \leq C|x-y|^\gamma$$

for any constant γ and $C\geq 2$. This follows from the fact that the limit function is non-negative and bounded by one. Therefore, assume now that $i_0>0$ is chosen to satisfy

$$\frac{1}{2}\frac{1}{2^{i_0}} < |x-y| \leq \frac{1}{2^{i_0}}.$$

We may also assume that $0\leq x<y\leq N$. Putting

$$x=\alpha+\sum_{i=1}^{\infty}\epsilon_i\frac{1}{2^i} \quad \text{and} \quad y=\beta+\sum_{i=1}^{\infty}\eta_i\frac{1}{2^i}, \qquad 0\leq\alpha,\beta\leq N-1,$$

then $\alpha\leq\beta$ and in case $\alpha=\beta$, $\epsilon_i\leq\eta_i, i=1,2,\ldots$. If we assume that for the dyadic number x the bits $\epsilon_1,\epsilon_2,\ldots$ end with a string of zeros, while for the dyadic number y the bits $\eta_1,\eta_2,\ldots$ end with a string of ones. We will show the following: If $\frac{1}{2}\frac{1}{2^{i_0}}\leq|x-y|\leq\frac{1}{2^{i_0}}$ for some integer $i_0>0$, then

$$\|\pi(x)-\pi(y)\| \leq C(c,\nu)\left(c^{1/\nu}\right)^{i_0} \tag{5.11}$$

with the constant not depending on i_0. Since $c^{1/\nu}=2^{-\gamma}$ we have

$$\left(c^{1/\nu}\right)^{i_0}=2^{-\gamma i_0}\leq(2|x-y|)^\gamma=c^{-1/\nu}|x-y|^\gamma.$$

In order to verify (5.11), we shall involve Theorem 4.1 in the following form:

$$\left\|\mathbf{e}_1^T\mathbf{A}_{\epsilon_k}\mathbf{A}_{\epsilon_{k-1}}\cdots\mathbf{A}_{\epsilon_1}-\pi(x)\right\|\leq c^{\lfloor k/\nu\rfloor}\leq c^{k/\nu-1}, \qquad k=1,2,\ldots \tag{5.12}$$

with $\mathbf{e}_1^T=(1,0,\cdots,0)$. This holds in view of Proposition 5.1, and for $\|\cdot\|$ the max norm. We will also refer to i-cells, which are the intervals of type $\left[\frac{\alpha}{2^i},\frac{\alpha+1}{2^i}\right],\alpha=0,1,\ldots,2^i-1$, for $i=0,1,\ldots$.

We first observe that the assumption $|x-y|\leq 2^{-i_0}$ implies that x and y either belong to a common i_0-cell, or to two adjacent i_0-cells. In the first case we have the situation

$$\xi_\alpha=\frac{\alpha}{2^{i_0}}\leq x<y\leq\frac{\alpha+1}{2^{i_0}}=\xi_{\alpha+1} \tag{5.13}$$

for some $\alpha=0,\ldots,2^{i_0}-1$, while in the second case we have

$$\xi_\alpha\leq x<\xi_{\alpha+1}<y\leq\xi_{\alpha+2} \tag{5.14}$$

for some $\alpha=0,\ldots,2^{i_0}-2$. Thus, if we can show that assuming (5.13) for some i_0 will imply estimate (5.11), we are done. Namely, in case (5.14), estimate (5.11) will then hold with the constant $C(c,\nu)$ replaced by $2\cdot C(c,\nu)$. In that case,

$$\|\pi(x)-\pi(y)\|\leq\|\pi(x)-\pi(\xi_{\alpha+1})\|+\|\pi(\xi_{\alpha+1})-\pi(y)\|,$$

where x and $\xi_{\alpha+1}$, as well as $\xi_{\alpha+1}$ and y, belong to a common i_0-cell.

Let us consider the situation (5.13). If both, x and y agree with the endpoints of the cell, then for the bits of the binary expansions we have $\epsilon_i = \eta_i$ for $i = 1, \dots, i_0$, while $\epsilon_i = 0$ and $\eta_i = 1$ for $i > i_0$. Using estimate (5.12) and the fact that $\mathbf{A}_{\epsilon_{i_0}} \cdots \mathbf{A}_{\epsilon_1} = \mathbf{A}_{\eta_{i_0}} \cdots \mathbf{A}_{\eta_1}$, we find

$$\begin{aligned}\|\pi(x) - \pi(y)\| &\leq \left\|\mathbf{e}_1^T \mathbf{A}_{\epsilon_{i_0}} \cdots \mathbf{A}_{\epsilon_1} - \pi(x)\right\| + \left\|\mathbf{e}_1^T \mathbf{A}_{\eta_{i_0}} \cdots \mathbf{A}_{\eta_1} - \pi(y)\right\| \\ &\leq 2c^{\lfloor i_0/\nu \rfloor} \leq \frac{2}{c} \left(c^{1/\nu}\right)^{i_0}.\end{aligned}$$

If $x = \xi_\alpha$ but y is arbitrary we have, as before, $\epsilon_i = \eta_i$ for $i = 1, \dots, i_0$, and $\epsilon_i = 0$ for $i > i_0$. Putting $x = x_0$ and $x_k = \sum_{i=1}^{i_0+k} \eta_i \frac{1}{2^i}$ for $k = 1, 2, \dots$, we have $y = \lim_{k \to \infty} x_k$. Moreover, for $k = 1, 2, \dots$, the points x_{k-1} and x_k are either identical (case $\eta_{i_0+k} = 0$), or are the endpoints of a common $(i_0 + k)$-cell (case $\eta_{i_0+k} = 1$). Thus

$$\begin{aligned}\|\pi(x) - \pi(y)\| &\leq \sum_{k=1}^{\infty} \|\pi(x_{k-1}) - \pi(x_k)\| \leq \sum_{k=1}^{\infty} \eta_{i_0+k} 2c^{\lfloor (i_0+k)/\nu \rfloor} \\ &\leq \frac{2}{c} \sum_{k=1}^{\infty} \left(c^{1/\nu}\right)^{i_0+k} = \frac{2}{c} \frac{c^{1/\nu}}{1 - c^{1/\nu}} \left(c^{1/\nu}\right)^{i_0}.\end{aligned}$$

The same estimate is obtained if x is arbitrary, but $y = \xi_{\alpha+1}$. The general case of arbitrary x and y from a common cell (5.13) can be now reduced to these cases using the triangle inequality in the form

$$\|\pi(x) - \pi(y)\| \leq \|\pi(x) - \pi(\xi_\alpha)\| + \|\pi(\xi_\alpha) - \pi(y)\|.$$

This shows: In the situation of (5.13) we have estimate (5.11) with

$$C(c, \nu) =: \frac{4}{c} \frac{c^{1/\nu}}{1 - c^{1/\nu}},$$

while in the situation of (5.14) we have estimate (5.11) with $C(c, \nu)$ replaced by $2 \cdot C(c, \nu)$. This completes the proof. □

5.1.3 Characterization of convergence

We are now ready to present our main theorem.

Theorem 5.1. *Given a non-negative univariate subdivision scheme with mask* $a = (a(\alpha))_{\alpha \in \mathbb{Z}}$ *satisfying supp* $a \subset \{0, 1, \dots, N\}, 0 < a(0), a(N) < 1$, *and*

$$\sum_{\alpha \in \mathbb{Z}} a(2\alpha) = \sum_{\alpha \in \mathbb{Z}} a(2\alpha + 1) = 1,$$

the following are equivalent:

1. *The subdivision scheme converges in the sense of Definition 2.1.*
2. *For the family* $\mathcal{A} = \{\mathbf{A}_0, \mathbf{A}_1\}$ *from eq.* (3.4) *and* (3.5) *one of the following equivalent properties holds:*

(a) There is an integer $\nu \geq 1$ such that each word of length $k \geq \nu$ from the family $\mathcal{A}$ is scrambling.

(b) There is an integer $\mu \geq 1$ such that each word of length $k \geq \mu$ from the family $\mathcal{A}$ has a strictly positive column.

(c) Each word $\mathbf{P}$ from $\mathcal{A}$ is SIA.

If (a) or (b) hold, then the properties must hold already for some ν and μ satisfying $\nu \leq \mu \leq 2^{N^2}$.

In case of convergence, the limit function is Hölder continuous with Hölder exponent as in Proposition 5.2.

Proof. The equivalence of conditions $2.(a) - 2.(c)$ follows from Lemma 4.7 and Lemma 4.8.

In order to see that 2 implies 1, we first use the pointwise convergence from Proposition 5.1 and the continuity of the basic limit function from Proposition 5.2. The uniform convergence result (3.11) then again follows from eq. (3.12) via Theorem 4.1. This proves convergence for the starting sequence $c^{(0)} = \delta$.

For general $c = c^{(0)} = (c(\beta))_{\beta\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z})$, we observe that $c^{(k)}(\alpha) = \sum_{\beta\in\mathbb{Z}} c(\beta)a^{(k)}(\alpha - 2^k\beta)$. If we put $f_c(x) = \sum_{\beta\in\mathbb{Z}} c(\beta)\phi(x-\beta)$ then, since ϕ has compact support, f_c is Hölder continuous of the same type as ϕ. For $\alpha \in \mathbb{Z}$ and $k = 1, 2, \ldots$ we have

$$\begin{aligned}\left|f_c\left(\frac{\alpha}{2^k}\right) - c^{(k)}(\alpha)\right| &\leq \|c\|_\infty \sum_{\beta\in\mathbb{Z}} \left|\phi\left(\frac{\alpha}{2^k}-\beta\right) - a^{(k)}(\alpha - 2^k\beta)\right| \\ &\leq (N+1)\sup_{\ell\in\mathbb{Z}} \left|\phi\left(\frac{\ell}{2^k}\right) - a^{(k)}(\ell)\right|,\end{aligned}$$

since the width of the support of ϕ and of $a^{(k)}$ is N and $(2^k-1)N$, respectively. Taking the sup on the left-hand side with respect to i, the limit vanishes as $k \to \infty$.

Conversely, let us show that 1 implies 2. Let $\mathbf{P} = \mathbf{A}_{\epsilon_\ell}\mathbf{A}_{\epsilon_{\ell-1}}\cdots\mathbf{A}_{\epsilon_1}$ be a word of length ℓ, for some positive integer ℓ. We have to show that the powers of $\mathbf{P}$ converge to a rank one matrix. To this end, we consider the periodically extended bit sequence $\epsilon_i = \epsilon_{i-\ell}, i > \ell$, and put $x = \sum_{i=1}^\infty \epsilon_i \frac{1}{2^i} \in [0,1]$. From Lemma 3.1 we conclude, as in (3.12), that

$$a^{(k)}\left(-m + 2^k \sum_{i=1}^k \epsilon_i \frac{1}{2^i} + 2^k n\right) = \mathbf{A}_{\epsilon_k}\mathbf{A}_{\epsilon_{k-1}}\cdots\mathbf{A}_{\epsilon_1}(m,n)$$

for $0 \leq m, n \leq N-1$ and $k = 1, 2, \ldots$. If we fix m, and scale

$$\alpha = -m + 2^k \sum_{i=1}^k \epsilon_i \frac{1}{2^i} + 2^k n$$

to get

$$x_k = \frac{\alpha}{2^k} = -\frac{m}{2^k} + \sum_{i=1}^{k} \epsilon_i \frac{1}{2^i} + n,$$

we see that the convergence assumption and the uniform continuity of ϕ imply that

$$\lim_{k\to\infty} \mathbf{A}_{\epsilon_k}\mathbf{A}_{\epsilon_{k-1}}\cdots\mathbf{A}_{\epsilon_1} = \begin{pmatrix} \phi(x) & \phi(x+1) & \cdots & \phi(x+N-1) \\ \vdots & \vdots & \ddots & \vdots \\ \phi(x) & \phi(x+1) & \cdots & \phi(x+N-1) \end{pmatrix}.$$

In particular, going through an appropriate subsequence, the powers of the word **P** converge to the same rank one matrix.

The theorem is proved. □

Remark 5.2. The assumptions imply that $N \geq 2$. Assumption $0 < a(0), a(N) < 1$ excludes those critical situations where the limit function would be defined pointwise inside the open interval $(0, N)$, but would be discontinuous at one or both endpoints. In that case, a modified type of convergence must be considered.

5.1.4 The SIA property of the master matrix A

Under the uniform convergence of univariate subdivision scheme, by Theorem 5.1, we get each word **P** from the family $\mathcal{A} = \{\mathbf{A}_0, \mathbf{A}_1\}$ is SIA, which is not quite independent with the assumption of **A** being SIA. We have the following that:

Theorem 5.2. *In case $N \geq 2$, the properties that each word* **P** *from the family $\mathcal{A} = \{\mathbf{A}_0, \mathbf{A}_1\}$ is SIA implies that* **A** *is SIA.*

Proof. Under the assumption, by Lemma 4.7, there exists some positive integer k such that column j_0 of $\mathbf{A}_0^k$ and column j_1 of $\mathbf{A}_1^k$ are strictly positive, respectively. By the definition of $\mathbf{A}_0$ and $\mathbf{A}_1$ from eq. (3.4) and (3.5), we have

$$\mathbf{A}_0^k = \left(\begin{array}{c|c} a^k(0) & * \\ \hline \mathbf{0} & \mathbf{B}^k \end{array}\right) \quad \text{and} \quad \mathbf{A}_1^k = \left(\begin{array}{c|c} \mathbf{B}^k & \mathbf{0} \\ \hline * & a^k(N) \end{array}\right),$$

with $\mathbf{B} = (a(-\alpha+2\beta))_{\alpha,\beta=1}^{N-1}$ the inner block submatrix of **A**. Correspondingly, we have

$$\mathbf{A}^k = \left(\begin{array}{c|c|c} a^k(0) & * & 0 \\ \hline \mathbf{0} & \mathbf{B}^k & \mathbf{0} \\ \hline 0 & * & a^k(N) \end{array}\right). \tag{5.15}$$

Since the inner block submatrix **B** is not empty ($N \geq 2$), we have that $\mathbf{A}^{2k}$ has a strictly positive column $j = \max\{j_0, j_1 + 1\}$. Hence, we have the SIA property for **A**. □

5.1.5 The GCD-condition

Theorem 5.3. *For $I := supp\ a \subset \{0, 1, \ldots, N\}$, assume that the mask a satisfies*

$$\sum_{\alpha\in\mathbb{Z}} a(2\alpha) = \sum_{\alpha\in\mathbb{Z}} a(2\alpha+1) = 1 \quad \textit{and} \quad a(0), a(N) \neq 0. \tag{5.16}$$

For the greatest common divisor

$$d := GCD\{\gamma \mid \gamma \in I\},$$

we have the following :

(a) d must be odd.

(b) If $d > 1$, then there exists some permutation matrix $\mathbf{P}$ such that

$$\mathbf{B} = (a(-\alpha+2\beta))_{\alpha,\beta=1}^{N-1}$$

can be written as a block diagonal matrix

$$\mathbf{P}^T\mathbf{B}\mathbf{P} = \left(\begin{array}{c|c} \mathbf{B}_1 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{B}_2 \end{array}\right).$$

Proof. (a) If d is even, then $I \subseteq 2\mathbb{Z}$ and $\sum_{\alpha\in\mathbb{Z}} a(2\alpha+1) = 0$ contradicting (5.16).

(b) Let

$$V_1 = \{1, \ldots, N-1\} \quad \text{and} \quad V_d = \{0, d, 2d, \ldots, \lfloor \frac{N}{d} \rfloor d\}.$$

Then

$$I \subseteq V_d \quad \text{and} \quad N \equiv 0 \pmod{d},$$

as well as

$$I \cap V_1 \subseteq V_d \quad \text{and} \quad V_1 \setminus V_d \neq \emptyset.$$

For $\alpha, \beta = 1, \ldots, N-1$, we order the rows and columns of $\mathbf{B}$ according to whether $\alpha, \beta \in V_1 \cap V_d$ or not.

If $\alpha \in V_1 \cap V_d$ and $\beta \in V_1 \setminus V_d$, then $a(-\alpha+2\beta) = 0$, since otherwise $-\alpha+2\beta \in I \subseteq V_d$, which implies that d divides $-\alpha+2\beta$. This is a contradiction to $d \mid \alpha$ and $d \nmid \beta$.

On the other hand, if $\alpha \in V_1 \setminus V_d$ and $\beta \in V_1 \cap V_d$, then $a(-\alpha+2\beta) = 0$ again, since otherwise $d \mid -\alpha+2\beta$, whence $d \mid \alpha$, a contradiction.

Therefore, we can find some permutation matrix $\mathbf{P}$ such that $\mathbf{B}$ can be written as a block diagonal matrix

$$\mathbf{P}^T\mathbf{B}\mathbf{P} = \left(\begin{array}{c|c} \mathbf{B}_1 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{B}_2 \end{array}\right).$$

□

5.1.6 Melkman's univariate string condition

Melkman [27] introduces the

Univariate string condition at level k**:** $I^{(k)}$ *contains a string (i.e., a sequence of consecutive integers) of length* $2^k + N - 1$.

In our approach, we will show the following:

Theorem 5.4. *If the univariate string condition at level* k *is satisfied, then each word of length* k *from the family* $\mathcal{A} = \{\mathbf{A}_0, \mathbf{A}_1\}$ *has a strictly positive column.*

Proof. By Lemma 3.1, for any sequence $\delta_i \in \{0,1\}, i = 1, \dots, k$, we have

$$\mathbf{A}_{\delta_1}\mathbf{A}_{\delta_2}\cdots\mathbf{A}_{\delta_k}(\alpha,\beta) = a^{(k)}(-\alpha + \lambda + 2^k\beta), \qquad 0 \le \alpha, \beta \le N-1, \tag{5.17}$$

where $\lambda = \delta_1 + 2\delta_2 + \cdots + 2^{k-1}\delta_k$. It is clear that $0 \le \lambda \le 2^k - 1$. According to the assumption, we let this string be

$$I' = \{j, j+1, \dots, j + 2^k + N - 2\}, \qquad \text{for some } j \ge 0.$$

Without loss of generality, we let

$$j + (2^k + N - 2) = 2^k\beta' + j'$$

with $\beta' \ge 1$ and $0 \le j' \le 2^k - 1$, then this string is given by

$$I' = \{2^k\beta' + j' - (2^k + N - 2), 2^k\beta' + j' - (2^k + N - 3), \dots, 2^k\beta' + j'\}.$$

We are now ready to discuss the following two cases, respectively.

(a) $\lambda \le j'$.

In this case, we have

$$\max_{0\le\alpha\le N-1}\{-\alpha + \lambda + 2^k\beta'\} \le \lambda + 2^k\beta' \le j' + 2^k\beta';$$

since $0 \le j' \le 2^k - 1$, we also have

$$\begin{aligned}\min_{0\le\alpha\le N-1}\{-\alpha + \lambda + 2^k\beta'\} &\ge -(N-1) + \lambda + 2^k\beta' \\ &\ge -(N-1) + j' - (2^k - 1) + 2^k\beta' \\ &= 2^k\beta' + j' - (2^k + N - 2).\end{aligned}$$

Hence, for any $0 \le \alpha \le N-1$, we have

$$-\alpha + \lambda + 2^k\beta' \in I' \subseteq I^{(k)}.$$

Therefore, column β' of $\mathbf{A}_{\delta_1}\mathbf{A}_{\delta_2}\cdots\mathbf{A}_{\delta_k}$ is strictly positive.

(b) $j' < \lambda \le 2^k - 1$.

In this case, we have

$$\max_{0\le\alpha\le N-1}\{-\alpha+\lambda+2^k(\beta'-1)\} \le (\lambda-2^k)+2^k\beta' < 2^k\beta' \le j'+2^k\beta';$$

and

$$\begin{aligned}\min_{0\le\alpha\le N-1}\{-\alpha+\lambda+2^k(\beta'-1)\} &\ge -(N-1)+(j'+1)+2^k\beta'-2^k\\ &= 2^k\beta'+j'-(2^k+N-2).\end{aligned}$$

Hence, for any $0 \le \alpha \le N-1$, we also have

$$-\alpha+\lambda+2^k(\beta'-1) \in I' \subseteq I^{(k)},$$

telling that column $(\beta'-1)$ of $\mathbf{A}_{\delta_1}\mathbf{A}_{\delta_2}\cdots\mathbf{A}_{\delta_k}$ is strictly positive.

This completes the proof. □

Melkman [27] has proved that univariate non-negative subdivision scheme is uniformly convergent, if

$$\{0, \ell, \ell+1, N\} \subset \text{supp } a \qquad \text{for some } 0 \le \ell < N. \tag{5.18}$$

The following Theorem 5.5 is based on the Theorem 8 in [27], but we should note that Melkman's considerations are based on the bi-infinite matrix

$$\mathbf{A} = (a(i-2j))_{i,j\in\mathbb{Z}},$$

while we deal with the finite $(N+1)\times(N+1)$-matrix

$$\mathbf{A} = (a(-\alpha+2\beta))_{\alpha,\beta=0}^{N}$$

and the $N \times N$-submatrices $\mathbf{A}_0$, $\mathbf{A}_1$ from eq. (3.4) and (3.5). The approach of the following Theorem 5.5 originates from the proof of Theorem 8 in [27], but we still make some modifications.

Theorem 5.5. *For $I =: \text{supp } a \subset \{0, 1, \ldots, N\}$, assume that the mask a satisfies*

$$\sum_{\alpha\in\mathbb{Z}} a(2\alpha) = \sum_{\alpha\in\mathbb{Z}} a(2\alpha+1) = 1 \quad \text{and} \quad a(0), a(N) \ne 0. \tag{5.19}$$

If I contains two successive integers ℓ and $\ell+1$ satisfying

$$0 < a(\ell), a(\ell+1) < 1,$$

then there is an i such that for any $k \ge i$, the support $I^{(k)}$ of the iterated mask $a^{(k)}(\alpha)$ contains a string of length at least $2^k + 2^{k-i} - 1$. Hence, for $2^{k-i} \ge N$, the univariate string condition at level k is satisfied.

Proof. According to the assumptions, we will discuss the following five cases.
(1). ℓ is even and $0 < 2q = \ell \le N-2$. Let $I_1 = I - \{\ell\}$ and we have

$$I_1 \supseteq \{-2q, 0, 1, N-2q\};$$

(2). ℓ is odd and $0 < 2p-1 = \ell \le N-2$. Let $I_2 = \{\ell+1\} - I$, and we get

$$I_2 \supseteq \{0, 1, 2p\};$$

(3). $\ell = 0$. From eq. (5.19), we know that I contains at least one even integer $2p, p > 0$, in this case, let $I_1 = I - \{\ell\} = I - \{0\}$, and we get

$$I_1 \supseteq \{0, 1, 2p\};$$

(4). $\ell = N-1$ and N is even. Let $N = 2p > 0$, we have

$$I_2 = \{\ell+1\} - I = \{N\} - I \supseteq \{0, 1, 2p\};$$

(5). $\ell = N-1$ and N is odd. From eq. (5.19), we know that I contains at least one odd integer N_1 satisfying $N - N_1 = 2p > 0$, and we also have

$$I_2 = \{\ell+1\} - I = \{N\} - I \supseteq \{0, 1, 2p\}.$$

In summary,

$$I_1 \supseteq \{0, 1, 2p\} \quad \text{and} \quad I_2 \supseteq \{0, 1, 2p\},$$

where except in case (1) when N is odd; in case (1), when N is odd, we have

$$I_1 \supseteq \{-2q, 0, 1, 2r+1\}.$$

Furthermore, for some integer $k > 0$, we have

(a)

$$\begin{aligned} I_1^{(k)} &= I_1 + 2I_1 + \cdots + 2^{k-1} I_1 \\ &= (I - \{\ell\}) + 2\,(I - \{\ell\}) + \cdots + 2^{k-1}\,(I - \{\ell\}) \\ &= I + 2I + \cdots + 2^{k-1} I - \{\ell + 2\ell + \cdots + 2^{k-1}\ell\} \\ &= I^{(k)} - \{(2^k - 1)\ell\}; \end{aligned}$$

(b)

$$\begin{aligned} I_2^{(k)} &= I_2 + 2I_2 + \cdots + 2^{k-1} I_2 \\ &= (\{\ell+1\} - I) + 2\,(\{\ell+1\} - I) + \cdots + 2^{k-1}\,(\{\ell+1\} - I) \\ &= \{(\ell+1) + 2(\ell+1) + \cdots + 2^{k-1}(\ell+1)\} - (I + 2I + \cdots + 2^{k-1} I) \\ &= \{(2^k - 1)(\ell+1)\} - I^{(k)}. \end{aligned}$$

In the above two situations, we have

$$\{0, 1, \ldots, 2^k - 1\} \subseteq I_1^{(k)} \quad \text{and} \quad \{0, 1, \ldots, 2^k - 1\} \subseteq I_2^{(k)},$$

because $\{0,1\} \subset I_1$ and $\{0,1\} \subset I_2$.

We are now ready to prove that

$$2^i \in I_1^{(i)} \quad \text{and} \quad 2^i \in I_2^{(i)} \qquad \text{for some integer } i \geq 1.$$

Here, our proof repeatedly use the property that if there exists some integer $a \notin I^{(k)}$, then $\frac{1}{2}(a-b) \notin I^{(k-1)}$ for any integer $b \in I$, because $I^{(k)} = I + 2I^{(k-1)}$.

(I) $I_1 \supseteq \{0,1,2p\}$ and $I_2 \supseteq \{0,1,2p\}$.

Let i be such that $2^i \geq 2p$. Then $2^i \in I_1^{(i)}$ and $2^i \in I_2^{(i)}$. Otherwise, the contrary that $2^i \notin I_1^{(i)}$ and $2^i \notin I_2^{(i)}$, then

$$2^{i-1} - p \notin I_1^{(i-1)} \quad \text{and} \quad 2^{i-1} - p \notin I_2^{(i-1)},$$

contradicting

$$I_1^{(i-1)} \supseteq \{0,1,\ldots,2^{i-1}-1\} \quad \text{and} \quad I_2^{(i-1)} \supseteq \{0,1,\ldots,2^{i-1}-1\},$$

since $p \geq 1$.

(II) $I_1 \supseteq \{-2q,0,1,2r+1\}$.

First we show that $2^i \in I_1^{(i)}$ for some $i \geq 1$ when $r \geq q > 0$. Let $q = 2^m(2n+1)$ and let i be such that $2^{i-m-2} \geq r-n$. Then $2^i \in I_1^{(i)}$. For otherwise $2^{i-1}+q \notin I_1^{(i-1)}$, and thus also $2^{-m}(2^{i-1}+q) \notin I_1^{(i-1-m)}$. Since the latter is an odd integer we conclude that $2^{i-m-2}+n-r \notin I_1^{(i-m-2)}$, in contradiction to

$$\{0,1,\ldots,2^{i-m-2}-1\} \subseteq I_1^{(i-m-2)},$$

since $0 \leq 2^{i-m-2}+n-r \leq 2^{i-m-2}-1$. When $q > r \geq 0$, let $I_3 = -I_1 + \{1\} = \{-2r,0,1,2q+1\}$, similarly, we have $\{0,1,\ldots,2^k-1\} \subseteq I_3^{(k)}$ and $2^i \in I_3^{(i)}$ for some $i \geq 1$.

For, from

$$I_1^{(k)} = I_1^{(k-i)} + 2^{k-i}I_1^{(i)} \quad \text{and} \quad I_2^{(k)} = I_2^{(k-i)} + 2^{k-i}I_2^{(i)},$$

it then follows that

$$I_1^{(k)} \supseteq \{0,1,\ldots,2^{k-i}-1\} + \{2^k\} = \{2^k, 2^k+1, \ldots, 2^k+2^{k-i}-1\}$$

and

$$I_2^{(k)} \supseteq \{0,1,\ldots,2^{k-i}-1\} + \{2^k\} = \{2^k, 2^k+1, \ldots, 2^k+2^{k-i}-1\}.$$

Hence,

$$I_1^{(k)} \supseteq \{0,1,\ldots,2^k+2^{k-i}-1\} \quad \text{and} \quad I_2^{(k)} \supseteq \{0,1,\ldots,2^k+2^{k-i}-1\}.$$

Similarly, we also have

$$I_3^{(k)} \supseteq \{0,1,\ldots,2^k+2^{k-i}-1\}.$$

In addition,

$$\begin{aligned}I_3^{(k)} &= I_3 + 2I_3 + \cdots + 2^{k-1}I_3 \\ &= (-I_1 + \{1\}) + 2(-I_1 + \{1\}) + \cdots + 2^{k-1}(-I_1 + \{1\}) \\ &= -I^{(k)} + \{(2^k - 1)\ell + 2^k - 1\}.\end{aligned}$$

Choosing k large enough such that $2^{k-i} \geq N$, therefore, $I^{(k)}$ also contains a univariate string of length at least $2^k + N - 1$. This completes the proof. □

5.2 The bivariate case

Our application of the geometric convergence result of Anthonisse and Tijms, Theorem 4.1, refers now to our family

$$\mathcal{P} = \mathcal{A} = \{\mathbf{A}_\epsilon : \epsilon \in E\}, \qquad E := \{0,1\}^2 = \{(0,0),(1,0),(0,1),(1,1)\}$$

of $(N \times N)$-matrices, with $N = N_1 \cdot N_2$, defined in (3.16) and (3.17). We assume throughout this section that the family has the following property:

Assumption A: *Each word from the family $\mathcal{A}$ is SIA.*

Equivalently, by Lemma 4.7 and Lemma 4.8, the family has the scrambling property (or the positive column property) for words of length ν (μ, respectively) for some $\nu \leq \mu \leq 2^{N^2}$.

We point to the fact that rows and columns of vectors and matrices are now, again, double-indexed using indices

$$\alpha = (\alpha_1, \alpha_2) \in R_{N_1-1,N_2-1}.$$

To this end, we must assume that these points have been ordered somehow.

5.2.1 The SIA property in bivariate subdivision

Our first statement is just an immediate consequence of Theorem 4.1.

Proposition 5.3. *Assume that Assumption A holds. Then, for any sequence $\epsilon_1, \epsilon_2, \cdots$, with $\epsilon_i \in E, i = 1, 2, \ldots,$ there is a probability row vector $\mathbf{p} = (p_\beta)_{\beta \in R_{N_1-1,N_2-1}}$, depending on the sequence, such that*

$$\lim_{k\to\infty} \mathbf{A}_{\epsilon_k}\mathbf{A}_{\epsilon_k-1}\cdots\mathbf{A}_{\epsilon_1} = \mathbf{e}\,\mathbf{p},$$

with the N-vector $\mathbf{e} = (1, \cdots, 1)^T$.

Specializing this result to constant sequences $\epsilon_i = \epsilon, i = 1, 2, \cdots$, for some fixed $\epsilon \in E$, we define the probability vectors $\pi^{(\epsilon)}$ by

$$\lim_{k\to\infty} \mathbf{A}_\epsilon^k = \mathbf{e}\pi^{(\epsilon)}, \qquad \epsilon \in E. \tag{5.20}$$

Thus, $\pi^{(\epsilon)}$ is the left eigenvector of $\mathbf{A}_\epsilon$, with respect to the eigenvalue $\lambda = 1$, uniquely defined through its normalization as a probability vector.

These results can be given a more compact form, subject to an additional condition for the matrix $\mathbf{A}$ in (3.15). By reordering rows and columns, choosing an appropriate permutation matrix $\mathbf{P}_\epsilon$, we look at the block structure

$$\mathbf{P}_\epsilon^T \mathbf{A} \mathbf{P}_\epsilon = \left(\begin{array}{c|c} \mathbf{A}_\epsilon & \mathbf{B}_\epsilon \\ \hline \mathbf{C}_\epsilon & \mathbf{D}_\epsilon \end{array} \right), \qquad \epsilon \in E. \tag{5.21}$$

The choice of $\mathbf{P}_\epsilon$ leaves some freedom; but this will be not important, as we shall see. By

$$\partial R_{N_1,N_2} = \{\alpha = (\alpha_1, \alpha_2) \in R_{N_1,N_2} \mid \alpha_1 \in \{0, N_1\} \text{ or } \alpha_2 \in \{0, N_2\}\}$$

and $R^0_{N_1,N_2} := R_{N_1,N_2} \setminus \partial R_{N_1,N_2}$ we denote the boundary, and the interior of the discrete rectangle R_{N_1,N_2}, and we note that

$$R^0_{N_1,N_2} = \{1, \cdots, N_1 - 1\} \times \{1, \cdots, N_2 - 1\}. \tag{5.22}$$

Proposition 5.4. *Assume that $\mathcal{A}$ satisfies Assumption A.*

(a) The block $\mathbf{B}_\epsilon$ is always a zero block, whence $\mathbf{P}_\epsilon^T \mathbf{A} \mathbf{P}_\epsilon$ is block lower triangular.

(b) The matrix $\mathbf{A}$ is SIA if and only if

$$\lim_{k\to\infty} \mathbf{D}_\epsilon^k = \mathbf{0}, \qquad \epsilon \in E. \tag{5.23}$$

(c) In case $\mathbf{A}$ is SIA, we have

$$\lim_{k\to\infty} \mathbf{A}^k = \tilde{\mathbf{e}}\, \pi, \tag{5.24}$$

with $\tilde{\mathbf{e}} = (1, \cdots, 1)^T$ a column vector with $(N_1+1)(N_2+1) = N + N_1 + N_2 + 1$ entries, and $\pi = (\pi_\alpha)_{\alpha \in R_{N_1,N_2}}$ a probability row vector supported in $R^0_{N_1,N_2}$, i.e.,

$$\pi_\alpha = 0, \qquad \alpha \in \partial R_{N_1,N_2}. \tag{5.25}$$

It satisfies the compatibility conditions

$$\pi \mathbf{P}_\epsilon = \left(\pi^{(\epsilon)} \mid \mathbf{0}\right), \qquad \epsilon \in E. \tag{5.26}$$

Proof. (*a*) Since

$$\mathbf{A}_\epsilon(\alpha, \beta) = a(-(\alpha + \epsilon) + 2(\beta + \epsilon)), \qquad \alpha, \beta \in R_{N_1-1,N_2-1}$$

and

$$\mathbf{A}(\alpha', \beta') = a(-\alpha' + 2\beta'), \qquad \alpha', \beta' \in R_{N_1,N_2},$$

an element $a(-\alpha' + 2\beta')$ in the block $\mathbf{B}_\epsilon$ refers to a row index $\alpha' = \alpha + \epsilon \in \epsilon + R_{N_1-1,N_2-1}$ and to a column index $\beta' \in R_{N_1,N_2} \setminus (\epsilon + R_{N_1-1,N_2-1})$. The assumption $-\alpha' + 2\beta' \in \operatorname{supp} a \subset R_{N_1,N_2}$ leads to

$$2\beta' \in \alpha' + R_{N_1,N_2} \subset \epsilon + R_{2N_1-1,2N_2-1} = \epsilon + \bigcup_{\delta \in E} (\delta + 2 \cdot R_{N_1-1,N_2-1}),$$

whence $\beta' \in \epsilon + R_{N_1-1,N_2-1}$, a contradiction. So $a(-\alpha' + 2\beta') = 0$.

(b) If $\mathbf{A}$ is SIA, then (5.23) holds trivially, since in the limit the columns of $\mathbf{A}^k$ converge to columns with constant entries. Conversely, if (5.23) holds, in addition to Assumption A, then the spectrum of $\mathbf{D}_\epsilon$ is a subset of the open unit disk, whence the eigenvalue $\lambda = 1$ of $\mathbf{A}_\epsilon$ is a simple eigenvalue for $\mathbf{A}$, and is dominant. This implies the rank-one convergence of type (5.24).

(c) We have seen that Assumption A in connection with the SIA property for $\mathbf{A}$ implies that $R^0_{N_1,N_2} \neq \emptyset$. Furthermore, if $R^0_{N_1,N_2} \neq \emptyset$, by choosing an appropriate permutation matrix $\mathbf{P}$ we have

$$\mathbf{P}^T \mathbf{A} \mathbf{P} = \left(\begin{array}{c|c} \tilde{\mathbf{A}} & \tilde{\mathbf{B}} \\ \hline \tilde{\mathbf{C}} & \tilde{\mathbf{D}} \end{array} \right), \qquad \tilde{\mathbf{A}}(\alpha, \beta) = a(-\alpha + 2\beta)_{\alpha,\beta \in R^0_{N_1,N_2}}, \tag{5.27}$$

with a non-empty block $\tilde{\mathbf{A}}$. As above we conclude that $\tilde{\mathbf{B}} = \mathbf{0}$, since otherwise $-\alpha + 2\beta \in R_{N_1,N_2}$ for some $\alpha \in R^0_{N_1,N_2}$ and $\beta \in \partial R_{N_1,N_2}$, a contradiction. The SIA property of $\mathbf{A}$ is now equivalent to

$$\lim_{k\to\infty} \tilde{\mathbf{D}}^k \to \mathbf{0} \text{ as } k \to \infty, \quad \text{and} \quad \tilde{\mathbf{A}} \text{ being SIA}.$$

According to Theorem 4.1 again, we get eq. (5.24).

Eq. (5.21) gives

$$\mathbf{P}_\epsilon^T \mathbf{A} \mathbf{P}_\epsilon = \left(\begin{array}{c|c} \mathbf{A}_\epsilon & \mathbf{0} \\ \hline \mathbf{C}_\epsilon & \mathbf{D}_\epsilon \end{array} \right), \qquad \epsilon \in E.$$

Correspondingly we have that

$$\mathbf{P}_\epsilon^T \mathbf{A}^k \mathbf{P}_\epsilon = \left(\begin{array}{c|c} \mathbf{A}_\epsilon^k & \mathbf{0} \\ \hline \mathbf{C}_\epsilon^{(k)} & \mathbf{D}_\epsilon^k \end{array} \right), \qquad \epsilon \in E.$$

Hence,

$$\lim_{k\to\infty} \mathbf{P}_\epsilon^T \mathbf{A}^k \mathbf{P}_\epsilon = \lim_{k\to\infty} \left(\begin{array}{c|c} \mathbf{A}_\epsilon^k & \mathbf{0} \\ \hline \mathbf{C}_\epsilon^{(k)} & \mathbf{D}_\epsilon^k \end{array} \right) = \left(\begin{array}{c|c} \mathbf{e}\pi^{(\epsilon)} & \mathbf{0} \\ \hline * & \mathbf{0} \end{array} \right).$$

Since

$$\lim_{k\to\infty} \mathbf{P}_\epsilon^T \mathbf{A}^k \mathbf{P}_\epsilon = \mathbf{P}_\epsilon^T \left(\lim_{k\to\infty} \mathbf{A}^k \right) \mathbf{P}_\epsilon = \mathbf{P}_\epsilon^T (\tilde{\mathbf{e}}\pi) \mathbf{P}_\epsilon = \tilde{\mathbf{e}}\pi \mathbf{P}_\epsilon,$$

this implies the compatibility conditions

$$\pi \mathbf{P}_\epsilon = \left(\pi^{(\epsilon)} \mid \mathbf{0} \right), \qquad \epsilon \in E.$$

$\square$

Remark 5.3.

1. It is sufficient to require that $\lim_{k\to\infty} \mathbf{D}_\epsilon^k = \mathbf{0}$ for some $\epsilon \in E$, since the property of the spectrum of this block $\mathbf{D}_\epsilon$ being a subset of the open unit disk, in view of Assumption A, transforms to the same property of the spectrum of $\mathbf{D}_\eta$, for each $\eta \in E$.

2. If $\mathbf{A}$ is SIA and $R^0_{N_1,N_2} \neq \emptyset$, we have

$$0 \leq a(\alpha) < 1, \qquad \alpha \in \partial R_{N_1,N_2}.$$

This follows directly from the decomposition in (5.27). If $a(\alpha) = 1$ for some $\alpha \in \partial R_{N_1,N_2}$, then $a(\alpha) = a(-\alpha + 2\alpha) = 1$ would refer to a diagonal entry in block $\tilde{\mathbf{D}}$. So all other entries in the corresponding row of $\mathbf{P}^T\mathbf{A}\mathbf{P}$ must vanish. We see that $\lambda = 1$ belongs to the spectrum of $\tilde{\mathbf{D}}$, whence $\mathbf{A}$ cannot be SIA, if $\tilde{\mathbf{A}}$ is so.

3. In case $\mathbf{A}$ is SIA, the compatibility condition (5.26) tells that the non-zero part of the left eigenvectors $\pi^{(\epsilon)}$ of the matrices $\mathbf{A}_\epsilon$, for all $\epsilon \in E$, are common components of the left eigenvector $\tilde{\pi}$ of the row stochastic matrix $\tilde{\mathbf{A}}$, for the dominant eigenvalue $\lambda = 1$. Moreover, the left eigenvector π of $\mathbf{A}$ in (5.24) originates from $\tilde{\pi}$ through padding with zeroes, and rearranging the components appropriately: We have

$$\pi^{(\epsilon)}(\alpha) = \pi(\alpha + \epsilon), \quad \alpha \in R_{N_1-1,N_2-1},\ \epsilon \in E. \tag{5.28}$$

This extends the condition given by Micchelli and Prautzsch [26], at least for binary subdivision; see the condition (a) given in their Theorem 5.1.

We have seen that the Assumption A and the assumption of $\mathbf{A}$ being SIA are not quite independent. The following holds true:

Lemma 5.2. *In case $R^0_{N_1,N_2} \neq \emptyset$, Assumption A implies that $\mathbf{A}$ is SIA.*

Proof. We consider the block decompositions in (5.21)) and (5.27) with zero blocks $\mathbf{B}_\epsilon$ and $\tilde{\mathbf{B}}$, and we note that $\tilde{\mathbf{A}}$ is a submatrix of each $\mathbf{A}_\epsilon$. Looking at powers of $\mathbf{A}$, we have

$$\mathbf{P}_\epsilon^T \mathbf{A}^k \mathbf{P}_\epsilon = \left(\begin{array}{c|c} \mathbf{A}_\epsilon^k & \mathbf{0} \\ \hline \mathbf{C}^{(k)} & \mathbf{D}_\epsilon^k \end{array}\right), \qquad \epsilon \in E,$$

and

$$\mathbf{P}^T \mathbf{A}^k \mathbf{P} = \left(\begin{array}{c|c} \tilde{\mathbf{A}}^k & \mathbf{0} \\ \hline \tilde{\mathbf{C}}^{(k)} & \tilde{\mathbf{D}}^k \end{array}\right).$$

The structure of the left-bottom blocks could be detailed, but is not so important. However, we use the fact that the block $\tilde{\mathbf{A}}^k$ in the second decomposition appears as a subblock in $\mathbf{A}_\epsilon^k$.

Now, Assumption A implies by Lemma 4.7 and Lemma 4.8 that we can find k such that each matrix $\mathbf{A}_\epsilon^k$ is scrambling (or has a positive column). We shall show that this implies that in each row of the block $\tilde{\mathbf{C}}^{(k)}$ there is a non-zero entry, whence the row sums of the block $\tilde{\mathbf{D}}^k$ are all less than one. This yields that the spectrum of $\tilde{\mathbf{D}}^k$, and hence of $\tilde{\mathbf{D}}$, is a subset of the open unit disk, which proves our claim.

In order to complete the proof, choose two indices α_1 and α_2 refering to two rows of $\mathbf{P}^T\mathbf{A}^k\mathbf{P}$, with α_1 refering to a row of $(\tilde{\mathbf{A}}^k\ \mathbf{0})$, and α_2 refering to a row of $(\tilde{\mathbf{C}}^{(k)}\ \tilde{\mathbf{D}}^k)$. The first row will then also appear in each matrix $(\mathbf{A}_\epsilon^k\ \mathbf{0})$, maybe in different order, and we can choose $\epsilon \in E$ such that the second row will appear in the same matrix, in a rearranged way. The two rows have a common column index such that the corresponding entries are both non-zero. And this property, in turn, shows that row α_2 of block $\tilde{\mathbf{C}}^{(k)}$ has a non-zero entry. The lemma is proved. □

5.2.2 Pointwise definition of the basic limit function

The definition of a limit function starts with relating the sequence in Proposition 5.3 to the dyadic expansion of $x \in [0,1]^2$ as in (3.22). This idea is due to Micchelli and Prautzsch [26]. We thus assume that

$$x = (x_1, x_2) = \sum_{i=1}^{\infty} \frac{\epsilon_i}{2^i}, \qquad \text{with} \quad \epsilon_i = (\epsilon_i^{(1)}, \epsilon_i^{(2)}) \in E, \quad i = 1, 2, \ldots \tag{5.29}$$

We shall show that the probability vector in Proposition 5.3 is given by

$$\mathbf{p} = \Phi(x) = (\phi(x+\beta))_{\beta \in R_{N_1-1,N_2-1}}, \tag{5.30}$$

see eq. (3.24). This is a pointwise definition of the basic limit function ϕ, subject the ambiguity in the dyadic expansion has been considered appropriately.

Proposition 5.5. *For given $x = (x_1, x_2) \in [0,1]^2$ and the corresponding dyadic expansion of eq.* (5.29)*, the limit of Proposition 5.3 with the probability vector $\mathbf{p} = \Phi(x)$ as in eq.* (5.30) *defines the function ϕ at the points $x + \beta \in [0, N_1] \times [0, N_2]$, with $\beta \in R_{N_1-1,N_2-1}$, subject that the limit does not depend on the ambiguity of the representation for x, in case x_1 or x_2 is a dyadic number.*

Now, for given $j = 1, 2$, the component $x_j \in [0,1]$ of x,

$$x_j = \sum_{i=1}^{\infty} \frac{\epsilon_i^{(j)}}{2^i}, \qquad \text{with} \quad \epsilon_i^{(j)} \in \{0, 1\}$$

is a dyadic number if either $x_j = 0$ (whence $\epsilon_i^{(j)} = 0, j = 1, 2$), or $x_j = 1$ (whence $\epsilon_i^{(j)} = 1, j = 1, 2$), or $0 < x_j < 1$ and

$$\sum_{i=1}^{\infty} \frac{\epsilon_i^{(j)}}{2^i} = \frac{k}{2^\kappa}, \qquad 0 < k < \kappa, \quad \kappa \in \mathbb{N},$$

where k and κ depend on j, of course. Writing $k = \eta_\kappa^{(j)} + 2 \cdot \eta_{\kappa-1}^{(j)} + \cdots + 2^{\kappa-1} \cdot \eta_1^{(j)}$ with $\eta_\ell^{(j)} \in \{0, 1\}$, we see that at least one of the η's in this representation for k

does not vanish. Take the maximal index $\bar{\ell}$, say, among $\{1, \dots \kappa\}$ such that $\eta_{\bar{\ell}}^{(j)} = 1$. Then the two possible representations for x_j are given by

$$\epsilon_i^{(j)} = \begin{cases} \eta_i^{(j)}, & i = 1, \dots, \bar{\ell} - 1, \\ 1, & i = \bar{\ell}, \\ 0, & i > \bar{\ell}, \end{cases} \qquad \text{and} \quad \tilde{\epsilon}_i^{(j)} = \begin{cases} \eta_i^{(j)}, & i = 1, \dots, \bar{\ell} - 1, \\ 0, & i = \bar{\ell}, \\ 1, & i > \bar{\ell}. \end{cases}$$

Let us denote these the upper and the lower representation, respectively, of the dyadic number $x_j \in (0, 1)$.

Theorem 5.6. *If Assumption A holds and if, in addition,* $\mathbf{A}$ *is SIA, then Proposition 5.3 with* $\mathbf{p} = \Phi(x)$ *defines the limit function at each point* $y = x + \beta \in [0, N_1] \times [0, N_2]$*, with* $\beta \in R_{N_1-1,N_2-1}$*, where the components of* x *are either both dyadic, or both non-dyadic. In this case,*

$$\lim_{k\to\infty} \mathbf{A}^k = (1, \cdots, 1)^T (\phi(\alpha))_{\alpha \in R_{N_1,N_2}}.$$

Proof. For the non-dyadic case, the dyadic expansion of x is unique. So there is nothing to prove. In the dyadic case, however, we have four possible representations for $x = (x_1, x_2)$, since for both components we have the lower and the upper dyadic representation.

Let us look first at the case where $x = (\frac{1}{2}, \frac{1}{2})$. The four dyadic expansions for x depend on the choice of $\epsilon = \epsilon_1 \in E$, since this choice determines the other ϵ_i's uniquely: For $i > 1$, ϵ_i does not depend on i, and is uniquely defined by

$$\epsilon_i + \epsilon_1 = (1, 1).$$

We put $\tilde{\epsilon} = (1, 1) - \epsilon, \epsilon \in E$, and we have to show that

$$\lim_{k\to\infty} \mathbf{A}_\epsilon^k \mathbf{A}_{\tilde{\epsilon}}, \qquad \epsilon \in E,$$

does not depend on ϵ. In view of (5.20), we have to show that the vectors $\pi^{(\epsilon)} \mathbf{A}_{\tilde{\epsilon}}$ do not depend on $\epsilon \in E$. But this is easy to see, since for $\epsilon \in E$ and $\beta \in R_{N_1-1,N_2-1}$, by (5.28) and (3.17):

$$\begin{aligned} \left(\pi^{(\epsilon)} \mathbf{A}_{\tilde{\epsilon}}\right)(\beta) &= \sum_{\alpha \in R_{N_1-1,N_2-1}} \pi(\alpha + \epsilon) a(-\alpha + \tilde{\epsilon} + 2\beta) \\ &= \sum_{\alpha' \in \mathbb{Z}} \pi(\alpha') a(-\alpha' + \epsilon + \tilde{\epsilon} + 2\beta). \end{aligned}$$

Here, we put $\pi(\alpha') = 0$ for $\alpha' \notin R_{N_1,N_2}$. The right-hand expression is independent of ϵ, since $\epsilon + \tilde{\epsilon}$ is constant.

The general case where both components of x are dyadic, follows from this. The limits of type

$$\lim_{k\to\infty} \mathbf{A}_\epsilon^k \mathbf{A}_{\tilde{\epsilon}} \mathbf{A}_{\epsilon_{\bar{\ell}-1}} \cdots \mathbf{A}_{\epsilon_1}, \qquad \epsilon \in E,$$

for some appropriate integer $\bar{\ell}$, do not depend on the the choice of $\epsilon \in E$, if the bits $\epsilon_1, \cdots, \epsilon_{\bar{\ell}-1}$ are fixed. □

Remark 5.4. The theorem does not cover the situations where one of the components of x is dyadic, while the other one is not. However, as we shall see below, also in this case the ambiguity of the dyadic expansion will have no effect on the limit in Proposition 5.3. This will define our limit function ϕ at all points in the closed rectangle $[0, N_1] \times [0, N_2]$, and ϕ vanishes outside this rectangle.

5.2.3 Hölder continuity of the basic limit function

We will show that ϕ restricted to the set of dyadic points, is Hölder continuous. Thus it has a unique continuous extension $\bar{\phi}$ which vanishes outside the rectangle $[0, N_1] \times [0, N_2]$, and is Hölder continuous with the same Hölder exponent. In addition, at non-dyadic points from the rectangle of type $x+\beta$, for $x \in [0,1]^2$ and $\beta \in R_{N_1-1,N_2-1}$, the value $\bar{\phi}(x+\beta)$ must be the limit of Proposition 5.3. Thus in the case of ambiguity of the dyadic expansion of x, this limit must be the same.

Lemma 5.3. *At dyadic points $\xi, \zeta \in \mathbb{R}^2$, the limit function ϕ is pointwise uniquely defined, and is Hölder continuous*

$$|\phi(\xi) - \phi(\zeta)| \leq \frac{2^{2+\gamma}}{c} \|\xi - \zeta\|_\infty^\gamma . \tag{5.31}$$

Here, $\gamma = -\frac{1}{\nu} \log_2 c$, with the parameters ν and c as in Theorem 4.1, when applied to the family $\mathcal{P} = \mathcal{A}$.

Proof. Since ϕ is non-negative, and since the integer-translates sum to 1, we have $|\phi(\xi) - \phi(\zeta)| \leq 1$, for any dyadic points $\xi, \zeta \in \mathbb{R}^2$. We may therefore assume that $\|\xi - \zeta\|_\infty \leq 1$. In addition, we can assume that $\xi, \zeta \in [0, N_1] \times [0, N_2]$, since otherwise $|\phi(\xi) - \phi(\zeta)| = 0$, or $|\phi(\xi) - \phi(\zeta)| = |\phi(\tilde{\xi}) - \phi(\tilde{\zeta})|$ for some $\tilde{\xi}, \tilde{\zeta} \in [0, N_1] \times [0, N_2]$ with $\|\tilde{\xi} - \tilde{\zeta}\|_\infty \leq \|\xi - \zeta\|_\infty$.

We let $\xi = x + \alpha, \zeta = y + \beta$ with $x, y \in [0,1]^2$ and $\alpha, \beta \in R_{N_1-1,N_2-1}$. The dyadic expansions of x and y are assumed to be

$$x = \sum_{i=1}^{\infty} \frac{\epsilon_i}{2^i} \quad \text{and} \quad y = \sum_{i=1}^{\infty} \frac{\eta_i}{2^i}, \qquad \epsilon_i, \eta_i \in E.$$

In addition, for given ξ and ζ, the integer $\kappa_0 \geq 0$ is determined by

$$\frac{1}{2} \frac{1}{2^{\kappa_0}} < \|\xi - \zeta\|_\infty \leq \frac{1}{2^{\kappa_0}}. \tag{5.32}$$

We look at cells of level κ which are given by squares originating from the scaled grid $\frac{1}{2^\kappa}\mathbb{Z}^2$, with sidelength $h_\kappa = \frac{1}{2^\kappa}$. The upper estimate in assumption (5.32) then tells that ξ and ζ are either from a common κ_0-cell, or belong to two adjacent κ_0-cells.

First assume that both ξ and ζ are from a common κ_0-cell. Then $\xi = x + \beta$ and $\zeta = y + \beta$ for the same $\beta \in R_{N_1-1,N_2-1}$, and we can choose the dyadic expansions such that $\epsilon_i = \eta_i, i = 1, \ldots, \kappa_0$, hence $\mathbf{A}_{\epsilon_k} \cdots \mathbf{A}_{\epsilon_1} = \mathbf{A}_{\eta_k} \cdots \mathbf{A}_{\eta_1}$ for $k = \kappa_0$. By Theorem 4.1 we have

$$\begin{aligned} |\phi(\xi) - \phi(\zeta)| &\leq |\mathbf{A}_{\epsilon_{\kappa_0}} \cdots \mathbf{A}_{\epsilon_1}(1, \beta) - \phi(x+\beta)| + \\ &\qquad |\mathbf{A}_{\eta_{\kappa_0}} \cdots \mathbf{A}_{\eta_1}(1, \beta) - \phi(y+\beta)| \\ &\leq 2c^{\lfloor \kappa_0/\nu \rfloor} \leq \frac{2}{c} c^{\kappa_0/\nu} = \frac{2}{c}(2^{-\gamma})^{\kappa_0}, \end{aligned}$$

with $c^{1/\nu} = 2^{-\gamma}$. If ξ and ζ belong to adjacent κ_0-cells, then we can find $\theta \in \frac{1}{2^{\kappa_0}}\mathbb{Z}^2$ such that ξ and θ, as well as ζ and θ, are in a common κ_0-cell. Whence,

$$|\phi(\xi) - \phi(\zeta)| \leq \frac{4}{c}(2^{-\gamma})^{\kappa_0}.$$

In either case, using the lower estimate in (5.32),

$$|\phi(\xi) - \phi(\zeta)| \leq \frac{4}{c}(2^{-\kappa_0})^{\gamma} \leq \frac{2^{2+\gamma}}{c} \left\| \xi - \zeta \right\|_{\infty}^{\gamma}.$$

This proves the lemma. ☐

Lemma 5.4. *The function ϕ which is defined at dyadic points by Theorem 5.6, has a continuous extension $\bar{\phi}$, say, to all of $\mathbb{R}^2$, with support inside the rectangle $[0, N_1] \times [0, N_2]$. This extension is Hölder continuous with the same Hölder exponent as in* (5.31)*.*

This is clear from a general extension result using uniform continuity, and from a continuity argument in (5.31).

Lemma 5.5. *At each non-dyadic point $x + \beta$, with $x \in [0, 1]^2$ and $\beta \in R_{N_1-1,N_2-1}$, the value $\bar{\phi}(x + \beta)$ coincides with the limit $\phi(x + \beta)$ in* (5.30)*, and this limit is independent of the chosen dyadic expansion for x.*

Proof. If $x = \sum_{i=1}^{\infty} \frac{\epsilon_i}{2^i}$ is non-dyadic, we can approximate it by dyadic points $x^{(k)} = \sum_{i=1}^{k} \frac{\epsilon_i}{2^i}, k = 1, 2, \ldots$. Then, by the continuity of $\bar{\phi}$,

$$\lim_{k \to \infty} \phi(x^{(k)} + \beta) = \lim_{k \to \infty} \bar{\phi}(x^{(k)} + \beta) = \bar{\phi}(x + \beta).$$

Now the left-hand limit is given by

$$\lim_{k \to \infty} \left\{ \lim_{\ell \to \infty} \mathbf{A}_{\epsilon}^{\ell} \mathbf{A}_{\epsilon_k} \cdots \mathbf{A}_{\epsilon_1} \right\} (1, \beta),$$

for $\epsilon = (0, 0)$. But $\lim_{\ell \to \infty} \mathbf{A}_{\epsilon}^{\ell} = \mathbf{e}\pi^{(\epsilon)}$, by eq. (5.20), whence the limit exists as

$$\left\{ \mathbf{e}\pi^{(\epsilon)} \lim_{k \to \infty} \mathbf{A}_{\epsilon_k} \cdots \mathbf{A}_{\epsilon_1} \right\} (1, \beta).$$

In addition, if x has different dyadic expansions, the limit must be the same in either case.

In conclusion, we have the following stronger version of Theorem 5.6:

Theorem 5.7. *If Assumption A holds and if, in addition,* $\mathbf{A}$ *is SIA, then Proposition 5.3 defines the fundamental limit function* ϕ *at each point* $\xi = x+\beta \in [0, N_1]\times[0, N_2]$*, with* $x \in [0, 1]^2$ *and* $\beta \in R_{N_1-1,N_2-1}$*, in the following way: If* x *has the binary expansion*

$$x = \sum_{i=1}^{\infty} \frac{\epsilon_i}{2^i}$$

then

$$\lim_{k\to\infty} \mathbf{A}_{\epsilon_k} \cdots \mathbf{A}_{\epsilon_1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} (\phi(x+\beta))_{\beta\in R_{N_1-1,N_2-1}},$$

and the convergence is geometric as in Theorem 4.1. This limit function is Hölder continuous of type (5.31)*, and* ϕ *is supported inside the rectangle* $[0, N_1] \times [0, N_2]$*.*

5.2.4 Uniform convergence

A combination of Theorem 5.7, Lemma 3.2 and Lemma 5.2 shows that Assumption A is sufficient for convergence of the subdivision scheme, in the sense of Definition (2.1), subject that $R^0_{N_1,N_2} \neq \emptyset$. Conversely, if the scheme is convergent, then we have rank-one convergence as in eqs. (3.23), (3.24) and this implies that each word from the family $\mathcal{A}$ is SIA.

Theorem 5.8. *For a non-negative bivariate subdivision scheme satisfying the necessary condition* (3.20) *the following are equivalent, subject that* $R^0_{N_1,N_2} \neq \emptyset$*, i.e.,* $N_1 > 1$ *and* $N_2 > 1$*:*

(a) The scheme is uniformly convergent in the sense of Definition (2.1)*.*

(b) Assumption A is satisfied.

If one of these conditions holds, then the fundamental limit function is Hölder continuous.

Remark 5.5.
1. We point once more to the fact that condition (b) is equivalent, by Lemma 4.7 and Lemma 4.8, to the scrambling property, or the positive column property, for words from the alphabet $\mathcal{A}$ of length 2^{N^2}. In this sense, Theorem 5.8 combines the statements of several authors, and improves on them by relating the Hölder continuity to the contraction coefficient c from eq. (4.9). We refer to Theorem 2.1 in [26], Theorem 6.1 in [6] (with statement C2 removed, since this is stated incorrectly), Theorem 1.1 in [21], and the remarkable papers by X.-L. Zhou [38, 36, 37] on univariate subdivision with non-negative masks.

2. In addition, Proposition 4.1 connects Assumption A with a corresponding property of the graph $G(\mathbf{P})$, for any word $\mathbf{P}$ from the family $\mathcal{A}$.

3. If $N_1 = 1$, *e.g.*, then the block decomposition of (5.21) for $\epsilon = (0, 0)$ shows the following: For α and β both in R_{N_1-1,N_2-1}, the non-zero entries of type $a(-\alpha + 2\beta)$

all appear in the block $\mathbf{A}_{(0,0)}$, while the non-zero entries of type $a(-(\alpha+\epsilon)+2(\beta+\epsilon))$, for all $(0,0) \neq \epsilon \in E$ appear in the block $\mathbf{D}_{(0,0)}$. This shows that the matrices $\mathbf{A}_\epsilon$ for $\epsilon \neq (0,0)$ are submatrices of $\mathbf{D}_{(0,0)}$, subject to a reordering of rows and columns, and $\mathbf{A}$ cannot be SIA, by Proposition 5.4. In this case, the matrix $\mathbf{A}$ decomposes into several components which are SIA.

Now, assume that Assumption A is still satisfied. Assuming that the fundamental limit function is continuous, it must vanish along the lines $x_1 = 0$ and $x_1 = 1$. But this contradicts Proposition 5.3, which we see from (5.20) when choosing $\epsilon = (0,0)$ since all non-zero components of the probability vector $\pi^{(0,0)}$ refer to arguments from these two lines.

The case $N_2 = 1$ is treated in a similar way. This shows that the additional assumption $R^0_{N_1,N_2} \neq 0$ is also necessary if we deal with uniform convergence of subdivision. For weaker type of convergence, this additional assumption is not necessarily indispensable.

5.3 Sufficient conditions for uniform convergence of non-negative bivariate subdivision

In this section we will discuss some sufficient conditions for uniform convergence of non-negative bivariate subdivision.

5.3.1 Masks with scrambling matrices $\mathbf{A}_\epsilon$

Scrambling families $\mathcal{A}$ give rise to convergent subdivision schemes, since here Assumption A is trivially satisfied (see Lemma 4.5):

Theorem 5.9. *Assume that the non-negative mask satisfies the necessary condition for convergence. If all matrices $\mathbf{A}_\epsilon, \epsilon \in E$, are scrambling, and if $R^0_{N_1,N_2} \neq \emptyset$, then the subdivision scheme is uniformly convergent, in the sense of Definition 2.1.*

In order to relate the assumption here to a geometric property of the support of the mask, we put again $I := \text{supp}\ a$. Since

$$\mathbf{A}_\epsilon = (a(-\alpha + \epsilon + 2\beta))_{\alpha,\beta \in R_{N_1-1,N_2-1}}, \qquad \epsilon \in E,$$

the scrambling property for $\mathbf{A}_\epsilon$ reads as follows: *For any $\alpha, \alpha' \in R_{N_1-1,N_2-1}, \alpha \neq \alpha'$, there is an index $\beta \in R_{N_1-1,N_2-1}$ such that*

$$\epsilon + 2\beta \in (\alpha + I) \cap (\alpha' + I).$$

Since the sets $\epsilon + 2R_{N_1-1,N_2-1}, \epsilon \in E$, form a disjoint covering of the set $R_{2N_1-1,2N_2-1}$, the scrambling property for the matrices $\mathbf{A}_\epsilon$ can be thus expressed as the following condition.

Assumption S: For each pair $\alpha, \alpha' \in R_{N_1-1,N_2-1}, \alpha \neq \alpha'$, and each $\epsilon \in E$, the set $(\alpha + I) \cap (\alpha' + I)$ has a common point with the index set $R_\epsilon := \epsilon + 2R_{N_1-1,N_2-1}$.

Convergence results can now be obtained from Theorem 5.8 by using the implication

$$\text{Assumption S} \implies \text{Assumption A}$$

for the family $\mathcal{A}$.

5.3.2 The bivariate string condition

Definition 5.1. *We say that a bivariate non-negative mask a with support $I \subset R_{N_1,N_2}$ and $N_1, N_2 > 1$ satisfies a rectangular bivariate string condition at level k if there exist two strings of integers I_1 and I_2, of length $2^k + N_1 - 1$ and $2^k + N_2 - 1$, respectively, such that*

$$I_1 \times I_2 \subset I^{(k)} = \textit{supp } a^{(k)}.$$

The bivariate string is an extension of Melkman's univariate string condition, and we conclude that there exists some relationship between the bivariate string and the positive column property for the family $\mathcal{A}$.

Theorem 5.10. *If the rectangular bivariate string condition at level k is satisfied, then each word of length k from the family $\mathcal{A}$ has a strictly positive column.*

Proof. For any $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2) \in R_{N_1-1,N_2-1}$, we must assume that these points have been ordered somehow. By Lemma 3.2, for any sequence $\delta_i \in E, i = 1, \ldots, k$, we have

$$\mathbf{A}_{\delta_1}\mathbf{A}_{\delta_2}\cdots\mathbf{A}_{\delta_k}(\alpha,\beta) = a^{(k)}(-\alpha + \lambda + 2^k\beta), \qquad \alpha, \beta \in R_{N_1-1,N_2-1}, \tag{5.33}$$

where $\lambda = \delta_1 + 2\delta_2 + \cdots + 2^{k-1}\delta_k$. It is clear that $\lambda = (\lambda_1, \lambda_2)$ satisfying $0 \leq \lambda_1 \leq 2^k - 1$ and $0 \leq \lambda_2 \leq 2^k - 1$. According to the definition of the bivariate string, we let

$$I_1 = \{j_1, j_1 + 1, \cdots, j_1 + 2^k + N_1 - 2\} \qquad \text{for some } j_1 \geq 0$$

and

$$I_2 = \{j_2, j_2 + 1, \cdots, j_2 + 2^k + N_2 - 2\} \qquad \text{for some } j_2 \geq 0$$

respectively, such that

$$I_1 \times I_2 \subset I^{(k)} = \text{supp } a^{(k)}.$$

Without loss of generality we let

$$j_1 + 2^k + N_1 - 2 = 2^k\beta'_1 + j'_1 \quad \text{and} \quad j_2 + 2^k + N_2 - 2 = 2^k\beta'_2 + j'_2$$

with $\beta'_1, \beta'_2 \geq 1$ and $0 \leq j'_1, j'_2 \leq 2^k - 1$, then the strings I_1 and I_2 are given by

$$I_1 = \{2^k\beta'_1 + j'_1 - (2^k + N_1 - 2), 2^k\beta'_1 + j'_1 - (2^k + N_1 - 3), \cdots, 2^k\beta'_1 + j'_1\}$$

and

$$I_2 = \{2^k\beta_2' + j_2' - (2^k + N_2 - 2), 2^k\beta_2' + j_2' - (2^k + N_2 - 3), \cdots, 2^k\beta_2' + j_2'\},$$

respectively.

We are now ready to discuss the following four cases, respectively.

(1) $\lambda_1 \le j_1'$ and $\lambda_2 \le j_2'$.

In this case, we have

$$\max_{0\le\alpha_1\le N_1-1}\{-\alpha_1 + \lambda_1 + 2^k\beta_1'\} \le \lambda_1 + 2^k\beta_1' \le j_1' + 2^k\beta_1'$$

and

$$\max_{0\le\alpha_2\le N_2-1}\{-\alpha_2 + \lambda_2 + 2^k\beta_2'\} \le \lambda_2 + 2^k\beta_2' \le j_2' + 2^k\beta_2';$$

since $0 \le j_1' \le 2^k - 1$ and $0 \le j_2' \le 2^k - 1$, so that

$$\begin{aligned}\min_{0\le\alpha_1\le N_1-1}\{-\alpha_1 + \lambda_1 + 2^k\beta_1'\} &\ge -(N_1 - 1) + \lambda_1 + 2^k\beta_1' \\ &\ge -(N_1 - 1) + j_1' - (2^k - 1) + 2^k\beta_1' \\ &= 2^k\beta_1' + j_1' - (2^k + N_1 - 2)\end{aligned}$$

and

$$\begin{aligned}\min_{0\le\alpha_2\le N_2-1}\{-\alpha_2 + \lambda_2 + 2^k\beta_2'\} &\ge -(N_2 - 1) + \lambda_2 + 2^k\beta_2' \\ &\ge -(N_2 - 1) + j_2' - (2^k - 1) + 2^k\beta_2' \\ &= 2^k\beta_i + j_2' - (2^k + N_2 - 2).\end{aligned}$$

Hence, for any $0 \le \alpha_1 \le N_1 - 1$ and $0 \le \alpha_2 \le N_2 - 1$, we have

$$-(\alpha_1, \alpha_2) + (\lambda_1, \lambda_2) + 2^k(\beta_1', \beta_2') \in I_1 \times I_2 \subseteq I^{(k)}.$$

Therefore, column (β_1', β_2') of $\mathbf{A}_{\delta_1}\mathbf{A}_{\delta_2}\cdots\mathbf{A}_{\delta_k}$ is strictly positive.

(2) $\lambda_1 \le j_1'$ and $j_2' < \lambda_2 \le 2^k - 1$.

We only need to check the case $j_2' < \lambda_2 \le 2^k - 1$. In this case,

$$\max_{0\le\alpha_2\le N_2-1}\{-\alpha_2 + \lambda_2 + 2^k(\beta_2' - 1)\} \le (\lambda_2 - 2^k) + 2^k\beta_2' < 2^k\beta_2' \le j_2' + 2^k\beta_2'$$

and

$$\begin{aligned}\min_{0\le\alpha_2\le N_2-1}\{-\alpha_2 + \lambda_2 + 2^k(\beta_2' - 1)\} &\ge -(N_2 - 1) + (j_2' + 1) + 2^k\beta_2' - 2^k \\ &= 2^k\beta_2' + j_2' - (2^k + N_2 - 2).\end{aligned}$$

Hence, for any $0 \le \alpha_1 \le N_1 - 1$ and $0 \le \alpha_2 \le N_2 - 1$, we have

$$-(\alpha_1, \alpha_2) + (\lambda_1, \lambda_2) + 2^k(\beta_1', \beta_2' - 1) \in I_1 \times I_2 \subseteq I^{(k)}.$$

Therefore, column $(\beta_1', \beta_2' - 1)$ of $\mathbf{A}_{\delta_1}\mathbf{A}_{\delta_2}\cdots\mathbf{A}_{\delta_k}$ is also strictly positive.

(3) $\lambda_2 \leq j_2'$ and $j_1' < \lambda_1 \leq 2^k - 1$.

In this case, similarly, we get from case (1) and case (2) that for any $0 \leq \alpha_1 \leq N_1 - 1$ and $0 \leq \alpha_2 \leq N_2 - 1$,

$$-(\alpha_1, \alpha_2) + (\lambda_1, \lambda_2) + 2^k(\beta_1' - 1, \beta_2') \in I_1 \times I_2 \subseteq I^{(k)},$$

telling that column $(\beta_1' - 1, \beta_2')$ of $\mathbf{A}_{\delta_1}\mathbf{A}_{\delta_2} \cdots \mathbf{A}_{\delta_k}$ is strictly positive.

(4) $j_1' < \lambda_1 \leq 2^k - 1$ and $j_2' < \lambda_2 \leq 2^k - 1$.

In this case, according to the above case (1), (2) and (3), similarly, it is easy to shown that for any $0 \leq \alpha_1 \leq N_1 - 1$ and $0 \leq \alpha_2 \leq N_2 - 1$,

$$-(\alpha_1, \alpha_2) + (\lambda_1, \lambda_2) + 2^k(\beta_1' - 1, \beta_2' - 1) \in I_1 \times I_2 \subseteq I^{(k)},$$

telling that column $(\beta_1' - 1, \beta_2' - 1)$ of $\mathbf{A}_{\delta_1}\mathbf{A}_{\delta_2} \cdots \mathbf{A}_{\delta_k}$ is also strictly positive.

This completes the proof. □

5.4 Uniform convergence is a support property of the mask

It has been observed by several authors that the convergence of non-negative subdivision only depends on the support of the mask, and not on the actual values of the mask coefficients. Slightly more general is the following statement.

Theorem 5.11. *If a and b are two non-negative masks satisfying the necessary conditions for convergence, and if*

$$\text{supp } a \subset \text{supp } b,$$

then uniform convergence of S_a implies uniform convergence of S_b.

Proof. We may assume that supp $a \subset$ supp $b \subset R_{N_1,N_2}$, with $N_1 > 1$ and $N_2 > 1$, and we consider the families

$$\mathcal{A} = \{\mathbf{A}_\epsilon : \epsilon \in E\} \quad \text{and} \quad \mathcal{B} = \{\mathbf{B}_\epsilon : \epsilon \in E\}$$

of $(N \times N)$-matrices, with $N = N_1 \cdot N_2$. By our assumption on the support of the masks, the sign pattern of the matrices changes monotonely, *i.e.*,

$$\sigma(\mathbf{A}_\epsilon) \leq \sigma(\mathbf{B}_\epsilon), \qquad \epsilon \in E.$$

Now assume that $\mathcal{A}$ satisfies assumption A. Since

$$\sigma(\mathbf{A}_{\epsilon_k} \cdots \mathbf{A}_{\epsilon_1}) \leq \sigma(\mathbf{B}_{\epsilon_k} \cdots \mathbf{B}_{\epsilon_1})$$

for any sequence $\epsilon_1, \ldots, \epsilon_k \in E$, we see that $\mathcal{B}$ satisfies Assumption A as well. Thus, our result follows from Theorem 5.8. □

5.5 Convex combinations of non-negative masks

Convergence of subdivision schemes is maintained by taking convex combination of the masks.

Corollary 5.1. *If* $a^{(j)}, j = 1, \ldots, n$, *are non-negative masks satisfying the necessary conditions for convergence, and if*

$$b = \sum_{j=1}^{n} \gamma_j a^{(j)}, \qquad \gamma_j > 0, j = 1, \ldots, n, \quad \sum_{j=1}^{n} \gamma_j = 1,$$

then the convergence of just one of the subdivision schemes $\mathcal{S}_{a^{(j)}}$ *implies convergence of* $\mathcal{S}_b$.

Proof. We may assume that

$$\text{supp } a^{(j)} \subset R_{N_1,N_2}, \qquad j = 1, \ldots, n,$$

for a common discrete rectangle, with $N_1 > 1$ and $N_2 > 1$. If $\mathcal{S}_{a^{(1)}}$, say, is convergent, we have supp $a^{(1)} \subset$ supp $b \subset R_{N_1,N_2}$, and the result follows from Theorem 5.11. □

Chapter 6

Tensor product subdivision schemes

6.1 Preliminaries

Tensor product subdivision schemes refer to two univariate subdivision schemes, with masks

$$b = (b(\alpha_1))_{\alpha_1 \in \mathbb{Z}} \quad \text{and} \quad c = (c(\alpha_2))_{\alpha_2 \in \mathbb{Z}} \tag{6.1}$$

supported on

$$R_{N_1} = \{0, 1, \ldots, N_1\} \quad \text{and} \quad R_{N_2} = \{0, 1, \ldots, N_2\},$$

respectively. The tensor product mask is then given by

$$\begin{aligned} a(\alpha_1, \alpha_2) &= (b \otimes c)(\alpha_1, \alpha_2) \\ &= b(\alpha_1) \cdot c(\alpha_2), \quad \alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2. \end{aligned} \tag{6.2}$$

Lemma 6.1. *Suppose the corresponding two univariate subdivision scheme are uniformly convergent, respectively, then*

$$\sum_{\beta \in \mathbb{Z}^2} a(\epsilon + 2\beta) = 1, \tag{6.3}$$

where $\epsilon = (\epsilon_1, \epsilon_2) \in E = \{0, 1\}^2$ *and* $\beta = (\beta_1, \beta_2) \in \mathbb{Z}^2$.

Proof. By Proposition 2.2, we know that

$$\sum_{\beta_1 \in \mathbb{Z}} b(\epsilon_1 + 2\beta_1) = \sum_{\beta_2 \in \mathbb{Z}} c(\epsilon_2 + 2\beta_2) = 1, \qquad \epsilon_1, \epsilon_2 \in \{0, 1\}.$$

Hence,

$$\begin{aligned}\sum_{\beta\in\mathbb{Z}^2} a(\epsilon+2\beta) &= \sum_{\beta\in\mathbb{Z}^2} (b\otimes c)\,(\epsilon+2\beta)\\ &= \sum_{\beta_1,\beta_2\in\mathbb{Z}} b(\epsilon_1+2\beta_1)\cdot c(\epsilon_2+2\beta_2)\\ &= \left(\sum_{\beta_1\in\mathbb{Z}} b(\epsilon_1+2\beta_1)\right)\cdot\left(\sum_{\beta_2\in\mathbb{Z}} c(\epsilon_2+2\beta_2)\right)\\ &= 1.\end{aligned}$$

□

For the iterated mask, we have the following:

Lemma 6.2. *The iterated mask of a tensor product subdivision schemes has the following form in terms of the univariate iterated masks:*

$$\begin{aligned} a^{(k)}(\alpha) &= (b\otimes c)^{(k)}\,(\alpha_1,\alpha_2)\\ &= b^{(k)}(\alpha_1)\cdot c^{(k)}(\alpha_2). \end{aligned} \tag{6.4}$$

Proof. It proceeds by induction on k. For $k = 1$, the statement is just the definition given in (6.2). Suppose $k > 1$, and assume that the lemma has been verified for any positive integer less than k. Then, we have

$$\begin{aligned} a^{(k)}(\alpha) &= (b\otimes c)^{(k)}\,(\alpha_1,\alpha_2)\\ &= \sum_{\beta_1,\beta_2\in\mathbb{Z}} (b\otimes c)\,(\alpha_1-2\beta_1,\alpha_2-2\beta_2)\cdot(b\otimes c)^{(k-1)}\,(\beta_1,\beta_2)\\ &= \sum_{\beta_1,\beta_2\in\mathbb{Z}} b(\alpha_1-2\beta_1)\cdot c(\alpha_2-2\beta_2)\cdot b^{(k-1)}(\beta_1)\cdot c^{(k-1)}(\beta_2)\\ &= \left(\sum_{\beta_1\in\mathbb{Z}} b(\alpha_1-2\beta_1)\cdot b^{(k-1)}(\beta_1)\right)\cdot\left(\sum_{\beta_2\in\mathbb{Z}} c(\alpha_2-2\beta_2)\cdot c^{(k-1)}(\beta_2)\right)\\ &= b^{(k)}(\alpha_1)\cdot c^{(k)}(\alpha_2). \end{aligned}$$

This completes our proof. □

We are now in this position to state the uniform convergence of the tensor product of two uniformly convergent subdivision schemes.

Proposition 6.1. *(see [4], Proposition 2.4) Suppose the two univariate subdivision schemes with masks in* (6.1) *respectively are uniformly convergent, then the tensor product subdivision scheme generated by mask in* (6.2) *is uniformly convergent. If* ϕ_b *and* ϕ_c *are the fundamental univariate limit functions, then the fundamental limit function for the tensor product scheme is given by*

$$\phi(x_1,x_2) := (\phi_b\otimes\phi_c)(x_1,x_2) = \phi_b(x_1)\cdot\phi_c(x_2). \tag{6.5}$$

Proof. If the two univariate subdivision schemes with masks in (6.1) respectively are uniformly convergent, by Definition 2.1, we have

$$\lim_{k\to\infty} \sup_{\alpha_1\in\mathbb{Z}} \left|\phi_b\left(\frac{\alpha_1}{2^k}\right) - a^{(k)}(\alpha_1)\right| = 0$$

and

$$\lim_{k\to\infty} \sup_{\alpha_2\in\mathbb{Z}} \left|\phi_c\left(\frac{\alpha_2}{2^k}\right) - a^{(k)}(\alpha_2)\right| = 0.$$

Since ϕ_b and ϕ_c are compactly supported and continuous, for any $\alpha_1, \alpha_2 \in \mathbb{Z}$, we have

$$\left|\phi_b\left(\frac{\alpha_1}{2^k}\right)\right| \le \|\phi_b\|_\infty \quad \text{and} \quad \left|a^{(k)}(\alpha_1)\right| \le \|\phi_b\|_\infty + C_1$$

and

$$\left|\phi_c\left(\frac{\alpha_2}{2^k}\right)\right| \le \|\phi_c\|_\infty \quad \text{and} \quad \left|a^{(k)}(\alpha_2)\right| \le \|\phi_c\|_\infty + C_2,$$

where C_1 and C_2 are positive constants. Furthermore,

$$\begin{aligned}\left|\phi\left(\frac{\alpha}{2^k}\right) - a^{(k)}(\alpha)\right| &= \left|(\phi_b\otimes\phi_c)\left(\frac{\alpha_1}{2^k},\frac{\alpha_2}{2^k}\right) - (b\otimes c)^{(k)}(\alpha_1,\alpha_2)\right| \\ &= \left|\phi_b\left(\frac{\alpha_1}{2^k}\right)\cdot\phi_c\left(\frac{\alpha_2}{2^k}\right) - b^{(k)}(\alpha_1)\cdot c^{(k)}(\alpha_2)\right| \\ &\le \left|\phi_b\left(\frac{\alpha_1}{2^k}\right) - b^{(k)}(\alpha_1)\right|\cdot\left|\phi_c\left(\frac{\alpha_2}{2^k}\right)\right| \\ &\quad + \left|b^{(k)}(\alpha_1)\right|\cdot\left|\phi_c\left(\frac{\alpha_2}{2^k}\right) - c^{(k)}(\alpha_2)\right|.\end{aligned}$$

Therefore,

$$\begin{aligned}\sup_{\alpha\in\mathbb{Z}^2}\left|\phi\left(\frac{\alpha}{2^k}\right) - a^{(k)}(\alpha)\right| &\le \|\phi_c\|_\infty\cdot\left(\sup_{\alpha_1\in\mathbb{Z}}\left|\phi_b\left(\frac{\alpha_1}{2^k}\right) - b^{(k)}(\alpha_1)\right|\right) \\ &\quad + (\|\phi_b\|_\infty + C_1)\cdot\left(\sup_{\alpha_2\in\mathbb{Z}}\left|\phi_c\left(\frac{\alpha_2}{2^k}\right) - c^{(k)}(\alpha_2)\right|\right),\end{aligned}$$

Taking the limit on the both-hand sides with respect to k, the limit on the left-hand side vanishes as $k\to\infty$. This completes the proof. □

6.2 Application

In an attempt to extend the univariate string condition to the bivariate case, we make the following:

Theorem 6.1. *For $I_1 =: supp\ b \subseteq \{0,1,\ldots,N_1\}$ and $I_2 =: supp\ c \subseteq \{0,1,\ldots,N_2\}$, assume that the univariate non-negative masks b and c satisfy*

$$\sum_{\beta_1\in\mathbb{Z}} b(2\beta_1) = \sum_{\beta_1\in\mathbb{Z}} b(2\beta_2+1) = 1, \qquad b(0), b(N_1) \ne 0$$

and

$$\sum_{\beta_2\in\mathbb{Z}} c(2\beta_2) = \sum_{\beta_2\in\mathbb{Z}} c(2\beta_2+1) = 1, \qquad c(0), c(N_2) \ne 0,$$

respectively. If I_1 and I_2 contain two successive integers $\ell_1, \ell_1 + 1$ and $\ell_2, \ell_2 + 1$, satisfying

$$0 < b(\ell_1), b(\ell_1 + 1) < 1 \quad \textit{and} \quad 0 < c(\ell_2), c(\ell_2 + 1) < 1,$$

respectively, then the tensor product mask in (6.2) *satisfies a rectangular bivariate string condition at level k in Definition 5.1.*

Proof. Under the assumption, by Theorem 5.5, there exist some positive integer k_1 and k_2 such that $I_1^{(k_1)}$ and $I_2^{(k_2)}$ contain a string of length $2^{k_1}+N_1-1$ and $2^{k_2}+N_2-1$, respectively. Furthermore, from Theorem 5.5, we know that the property that the non-negative mask satisfies a univariate string condition at level k implies that the non-negative mask also satisfies a univariate string condition at level $k+1$. Therefore, let $k = \max(k_1, k_2)$, we have that $I_1^{(k)}$ and $I_2^{(k)}$ contain a string I_1' and I_2' of length $2^k + N_1 - 1$ and $2^k + N_2 - 1$, respectively. From eq. (6.4), we get

$$I_1^{(k)} \otimes I_2^{(k)} = I^{(k)},$$

where $I^{(k)} := \text{supp } a^{(k)}$, correspondingly, the two strings I_1' and I_2' also satisfy

$$I_1' \otimes I_2' \subseteq I^{(k)},$$

i.e., the tensor product mask in (6.2) satisfies a rectangular bivariate string condition at level k in Definition 5.1 The theorem is proved. □

Chapter 7

Extensions

7.1 Zonotopes and box spline subdivision

Non-negative subdivision refering to centered zonotopes has been considered [4], while the general case was dealt with by Jia and Zhou in [21] under an additional condition which they call unimodularity. In the bivariate case, we are dealing here with a set

$$\mathbf{\Theta} = \{\theta_j \in \mathbb{Z}^2,\ j = 1, \ldots, n\}$$

of directional vectors $\mathbf{0} \neq \theta_j, j = 1, \ldots, n$. We may think of the set $\mathbf{\Theta}$ also in terms of an integer $(2 \times n)$-matrix, with columns θ_j. The zonotope related to this directional matrix is just the image of the $n-$ dimensional unit cube $[0,1]^n$ under the linear map

$$L_{\mathbf{\Theta}} : \mathbb{R}^n \to \mathbb{R}^2,\ \mathbf{x} \longmapsto \mathbf{\Theta}\, \mathbf{x},$$

while the corresponding discrete zonotope is the intersection of the zonotope with $\mathbb{Z}^2$,

$$D_{\mathbf{\Theta}} := L_{\mathbf{\Theta}}([0,1]^n) \cap \mathbb{Z}^2. \tag{7.1}$$

An important example of non-negative subdivision with masks a satisfying

$$\text{supp } a = D_{\mathbf{\Theta}},$$

is box spline subdivision, in case the directional matrix $\mathbf{\Theta}$ has the property that each (2×2)-submatrix has determinant $0, 1$ or -1. However, box spline subdivision convergences under more general assumptions; it depends on two properties:

Assumption BS: The directional matrix $\mathbf{\Theta}$ satisfies $L_{\mathbf{\Theta}}(\mathbb{Z}^2) = \mathbb{Z}^2$, and for any column θ_j the remaining set of columns $\mathbf{\Theta} \setminus \{\theta_j\}, j = 1, \ldots, n$, is still a spanning set for $\mathbb{R}^2$.

The first assumption is a lattice property for $\mathbb{Z}^2$ which is satisfied if and only if the greatest common divisor of the (absolute value of the) determinants of each regular (2×2)-submatrix of $\mathbf{\Theta}$ is 1. The second property refers to the fact that the corresponding box spline is continuous. Bivariate box spline subdivision now refers to the mask symbol

$$a_{\boldsymbol{\Theta}}(\mathbf{z}) = 4\prod_{j=1}^{n} \frac{1+\mathbf{z}^{\theta_j}}{2}, \qquad \boldsymbol{\Theta} = (\theta_1, \cdots, \theta_n),$$

and by Theorem 23 in [3], Section VII, the following result holds true:

Theorem 7.1. *Bivariate box spline subdivision with directional matrix $\boldsymbol{\Theta}$ is convergent, if Assumption BS is satisfied.*

Theorem 5.11 allows to reduce the assumptions in this theorem to a great amount. First, we may shift the mask to the positive orthant, by multiplying the mask symbol with some power $\mathbf{z}^{\beta}$, for some $\beta \in \mathbb{Z}^2$. Second, we may reduce the number of directional vectors to less vectors as long as the reduced directional matrix $\tilde{\boldsymbol{\Theta}}$ still satisfies Assumption BS. In some cases, we may even end up with 3 directional vectors only.

A combination of this convergence result with the monotonicity theorem implies uniform convergence for a big class of non-negative subdivision schemes.

Example: In case of the directional matrix

$$\boldsymbol{\Theta} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

Assumption BS is apparently satisfied, since each (2×2)-submatrix is unimodular (*i.e.*, has determinant $+1$ or -1). Box spline subdivision, in this case, refers to the mask

$$a = \frac{1}{2}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix};$$

here, we only display the coefficients with index $\alpha \in R_{2,2}$. We order these indices from left to right starting with the bottom three, and ending with the top three. With this ordering, lines 1 and 3 of $\mathbf{A}$ are given by $1/2\times$

$$\begin{aligned} \alpha = (0,0): &\quad \to \ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0 \\ \alpha = (2,0): &\quad \to \ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0 \end{aligned}$$

showing that $\mathbf{A}$ is not scrambling. Concerning the family $\mathcal{A}$, it is easy to verify that the matrices $\mathbf{A}_{(0,0)}$ and $\mathbf{A}_{(1,1)}$ both have a positive column, while $\mathbf{A}_{(1,0)}$ and $\mathbf{A}_{(0,1)}$ are not even scrambling. However, in this case each word of length 2 has the positive column property. So Assumption A is satisfied, and the results of Section 5.2 may be applied. The scheme is uniformly convergent to its limit function, which is the Courant element.

The monotonicity theorem now tells the following: *If we add any further directional vectors to the three ones from $\boldsymbol{\Theta}$ of the example, the corresponding box spline subdivision will be uniformly convergent.*

The result may be extended using as starting vectors any three vectors $\theta_1, \theta_2, \theta_3 \in \mathbb{Z}^2$ such that each (2×2)-submatrix of $\mathbf{\Theta} = (\theta_1\ \theta_2\ \theta_3)$ is unimodular. This will give rise to an approach of a convergence theorem which includes the one given in [21], Example 4.3.

7.2 Joint spectral radius

The Lemma 4.6 can be applied to any word $\mathbf{P}$ from our family $\mathcal{A}$. The norm can be related to the length of the word, using again Theorem 4.1. If $\mathbf{P}_1, \mathbf{P}_2, \ldots$ is any sequence from the alphabet $\mathcal{A}$ and if $\mathbf{P} = \mathbf{P}_k\mathbf{P}_{k-1}\cdots\mathbf{P}_1$, then for $\mathbf{x} \in V$:

$$\mathbf{x}\,\mathbf{P} = \mathbf{x}\,(\mathbf{P} - \mathbf{e}\pi),$$

since $\mathbf{x}\,\mathbf{e} = 0$. From this we get the following result:

Proposition 7.1. *If the alphabet $\mathcal{A}$ satisfies Assumption A, then for any word $\mathbf{P}$ of length k the restricted norm can be bounded as*

$$\|\mathbf{P}\|_V \leq c^{\lfloor k/\nu \rfloor}.$$

Here, the contraction coefficient c from eq. (4.9) and the parameter ν depend on the family $\mathcal{A}$ of course. We know that $\nu \leq 2^{N^2}$. *This shows that the restricted joint spectral radius*

$$\rho(\mathcal{A}) = \lim_{k\to\infty} (\ \max\{\|\mathbf{P}\|_V : \mathbf{P} \textit{ is a word of length } k\}\ )^{1/k}$$

for the family $\mathcal{A}$ can be bounded by $c^{1/\nu}$. For $c = 0$ this is obvious, and for $0 < c < 1$ we use $\rho(\mathcal{A}) \leq \lim_{k\to\infty} \left(c^{k/\nu - 1}\right)^{1/k} \leq c^{1/\nu}$.

Chapter 8

Appendix

There exist some relationships between non-negative matrix and graph theory. In this chapter, we give some definitions in matrix and graph theory; and we introduce some basic lemmas and theorems about matrix and graph theory.

The following basic graph theory are obtained from [14, 20]:

A directed graph G consists of a vertex set

$$V(G) = \{1, 2, \ldots, N\} \tag{8.1}$$

and an edge set

$$E(G) \subset \{(i, j) : i, j \in V(G)\}, \tag{8.2}$$

where an edge is an ordered pair of vertices in $V(G)$. Here, we allow for self-loops, namely, the edges with the same vertices.

If (i, j) is an edge of G, i and j are defined as the parent and child vertices, respectively.

A path in a directed graph G is a sequence $i_1, \ldots, i_k$ of vertices such that

$$(i_\ell, i_{\ell+1}) \in V(G) \qquad \text{for} \quad \ell = 1, \ldots, k-1. \tag{8.3}$$

A directed graph G is connected if there is a vertex $i \in V(G)$ such that for any $j \in V(G) \setminus \{i\}$ there is a path that begins at i and ends at j (*i.e.*, from i to j). Furthermore, a directed graph G is strongly connected if between any pair of distinct vertices i, j in G, there is a path that begins at i and ends at j (*i.e.*, from i to j).

A subgraph G_s of a directed graph G is a directed graph such that the vertex set $V(G_s) \subset V(G)$ and the edge set $E(G_s) \subset E(G)$. If $V(G_s) = V(G)$, we call G_s a spanning subgraph of G.

A directed tree is a directed graph, where every vertex, except the root, has exactly one parent.

A spanning tree of a directed graph G is a directed tree formed by graph edges that connect all the vertices of G.

We say that a graph has (or contains) a spanning tree if a subset of the edges forms a spanning tree.

Given an $(N \times N)$ non-negative matrix $\mathbf{P} = (p_{i,j})_{i,j=1}^N$, the associated directed graph of $\mathbf{P}$, denoted by $G(\mathbf{P})$, is a directed graph G such that

$$(i, j) \in E(G(\mathbf{P})) \tag{8.4}$$

if and only if $p_{j,i} > 0$.

Definition 8.1. *(see [33], Definition 1.15) For $N \geq 2$, an $(N \times N)$ complex matrix $\mathbf{P}$ is reducible if there exists an $(N \times N)$ permutation matrix $\mathbf{P}_0$ such that*

$$\mathbf{P}_0^T \mathbf{P} \mathbf{P}_0 = \left(\begin{array}{c|c} \mathbf{P}_{1,1} & \mathbf{0} \\ \hline \mathbf{P}_{2,1} & \mathbf{P}_{2,2} \end{array} \right), \tag{8.5}$$

where $\mathbf{P}_{1,1}$ is an $(r \times r)$ submatrix and $\mathbf{P}_{2,2}$ is an $((N-r) \times (N-r))$ submatrix, where $1 \leq r < N$. If no such permutation matrix exists, then $\mathbf{P}$ is irreducible. If $\mathbf{P}$ is a 1×1 complex matrix, then $\mathbf{P}$ is irreducible if its single entry is non-zero, and reducible otherwise.

Theorem 8.1. *(see [33], Theorem 1.17) An $(N \times N)$ non-negative matrix $\mathbf{P}$ is irreducible if and only if its directed graph $G(\mathbf{P})$ is strongly connected.*

Gerschgorin Theorem *(see [33], Theorem 1.11) Let $\mathbf{P} = (p_{i,j})_{i,j=1}^N$ be an arbitrary $(N \times N)$ complex matrix, and let*

$$\mathbf{\Lambda}_i := \sum_{i \neq j = 1}^{N} |p_{i,j}|, \qquad 1 \leq i \leq N, \tag{8.6}$$

where $\mathbf{\Lambda}_1 := 0$ if $N = 1$. If λ is an eigenvalue of $\mathbf{P}$, then there is a positive integer r, with $1 \leq r \leq N$, such that

$$|\lambda - p_{r,r}| \leq \mathbf{\Lambda}_r.$$

Hence, all eigenvalues λ of $\mathbf{P}$ lie in the union of the disks

$$|z - p_{i,i}| \leq \mathbf{\Lambda}_i, \qquad 1 \leq i \leq N.$$

Perron-Frobenius Theorem *(see [33], Theorem 2.7) Let $\mathbf{P} = (p_{i,j})_{i,j=1}^N$ be an $(N \times N)$ non-negative irreducible matrix. Then,*

(1) $\mathbf{P}$ has a positive real eigenvalue equal to its spectral radius.

(2) To $\rho(\mathbf{P})$ there corresponds an eigenvector $\mathbf{x} > \mathbf{0}$.

(3) $\rho(\mathbf{P})$ increases when any entry of $\mathbf{P}$ increases.

(4) $\rho(\mathbf{P})$ is a simple eigenvalue of $\mathbf{P}$.

Furthermore, we give the following generalization of Perron-Frobenius Theorem:

Theorem 8.2. *(see [33], Theorem 2.20) Let* $\mathbf{P} = (p_{i,j})_{i,j=1}^N$ *be an* $(N \times N)$ *non-negative matrix. Then,*

(1) $\mathbf{P}$ *has a non-negative real eigenvalue equal to its spectral radius. Moreover, this eigenvalue is positive unless* $\mathbf{P}$ *is reducible.*

(2) To $\rho(\mathbf{P})$ *there corresponds a non-zero eigenvector* $\mathbf{x} \geq \mathbf{0}$*.*

(3) $\rho(\mathbf{P})$ *does not decrease when any entry of* $\mathbf{P}$ *is increased.*

Definition 8.2. *(see [33], Definition 2.10) Let* $\mathbf{P} = (p_{i,j})_{i,j=1}^N \geq 0$ *be an* $(N \times N)$ *irreducible matrix, and let* k *be the number of eigenvalues of* $\mathbf{P}$ *of maximum modulus* $\rho(\mathbf{P})$*. If* $k = 1$*, then* $\mathbf{P}$ *is primitive. If* $k > 1$*, then* $\mathbf{P}$ *is cyclic of index* k*.*

The following lemma was proved in [30].

Theorem 8.3. *(see [30], Lemma 3.3) Given an* $(N \times N)$*-real matrix* $\mathbf{P} = (p_{i,j})_{i,j=1}^N$*, where* $p_{i,i} \leq 0$*,* $p_{i,j} \geq 0$ *for any* $i \neq j$*, and*

$$\sum_{j=1}^{N} p_{i,j} = 0, \qquad i = 1, \ldots, N. \tag{8.7}$$

Then $\lambda = 0$ *is an eigenvalue of* $\mathbf{P}$ *and all of the non-zero eigenvalues are in the open left-half plane. Furthermore,* $\lambda = 0$ *is algebraically simple if and only if the directed graph* $G(\mathbf{P})$ *associated with* $\mathbf{P}$ *has a spanning tree.*

Theorem 8.4. *(see [30], Lemma 3.4) Let* $\mathbf{P} = (p_{i,j})_{i,j=1}^N$ *be an* $(N \times N)$*- row stochastic matrix. Then the associated directed graph* $G(\mathbf{P})$ *has a spanning tree if and only if* $\lambda = 1$ *is an algebraically simple eigenvalue of* $\mathbf{P}$*.*

Lemma 8.1. *(see [33], pp. 48, Exercises 1) Let* $\mathbf{P} = (p_{i,j})_{i,j=1}^N \geq 0$ *be an* $(N \times N)$ *irreducible matrix. If* $\mathbf{P}$ *has exactly* $d \geq 1$ *diagonal entries positive, then* $\mathbf{P}$ *is primitive.*

Lemma 8.2. *(see [20], Lemma 8.2.7) Let* $\mathbf{P} = (p_{i,j})_{i,j=1}^N$ *be an* $(N \times N)$*-real matrix, let* $\lambda \in \mathbb{C}$ *be given, where* $\mathbb{C}$ *denotes a set of the complex numbers, and suppose* $\mathbf{x}$ *and* $\mathbf{y}$ *are vectors such that*

(1) $\mathbf{P}\mathbf{x} = \lambda \mathbf{x}$*;*

(2) $\mathbf{P}^T\mathbf{y} = \lambda \mathbf{y}$*;*

(3) $\mathbf{x}^T\mathbf{y} = 1$*;*

(4) λ *is an eigenvalue of* $\mathbf{P}$ *with geometric multiplicity* 1*;*

(5) $|\lambda| = \rho(\mathbf{P}) > 0$*; and*

(6) λ *is the only eigenvalue of* $\mathbf{P}$ *with modulus* $\rho(\mathbf{P})$*,*

where $\rho(\mathbf{P})$ *is the spectral radius of* $\mathbf{P}$*. Define* $\mathbf{L} = \mathbf{x}\mathbf{y}^T$*. Then*

$$(\lambda^{-1}\mathbf{P})^k = \mathbf{L} + (\lambda^{-1}\mathbf{P} - \mathbf{L})^k \to \mathbf{L} \qquad as \quad k \to \infty.$$

Bibliography

[1] M. Akelbek and S. Kirkland, Coefficients of ergodicity and the scrambling index, *Lin. Alg. Appl.* **430** (2009), 1111–1130.

[2] J. M. Anthonisse and H. Tijms, Exponential convergence of products of stochastic matrices, *J. Math. Anal. Appl.* **59** (1977), 360–364.

[3] C. de Boor, K. Höllig, and S. D. Riemenschneider, *Box Splines*, Applied Math. Sciences, Vol. 98, Springer-Verlag, New York, 1993.

[4] A. S. Cavaretta, W. Dahmen, and C. A. Micchelli, *Stationary Subdivision*, Memoirs Amer. Math. Soc., Vol. 93, No. 453, 1991.

[5] G. M. Chaikin, An algorithm for high speed curve generation, *Comput. Graph. Image Process.* **3** (1974), 346-349.

[6] I. Daubechies and J. C. Lagarias, Sets of matrices all infinite products of which converge, *Lin. Alg. Appl.* **161** (1992), 227–263, Corrigendum/Addendum *Lin. Alg. Appl.* **327** (2001), 69–83.

[7] I. Daubechies and J. C. Lagarias, Two-scale difference equations I. Existence and global regularity of solutions, *SIAM J. Math. Anal.* **22** (1991), 1388-1410.

[8] I. Daubechies and J. C. Lagarias, Two-scale difference equations II. Local regularity, infinite products of matrices and fractals, *SIAM J. Math. Anal.* **23** (1992), 1031-1079.

[9] J. L. Doob, *Stochastic Processes*, John Wiley and Sons, New York, 1953.

[10] N. Dyn, J. Gregory and D. Levin, Analysis of linear binary subdivision schemes for curve design, *Constr. Approx.* **7** (1991), 127-147.

[11] N. Dyn, Subdivision schemes in Computer-Aided Geometric Design, in: *Advances in Numerical Analysis, vol. II: Wavelets, Subdivision Algorithms and Radial Basis Functions* (W. A. Light, Ed.), pp. 36–104, University Press, Oxford, 1992.

[12] N. Dyn and D. Levin, Subdivision schemes in geometric modelling, Acta Numerica, **11** (2002), 73-144.

[13] F. J. Fritz, B. Huppert and W. Willems, *Stochastische Matrizen*, Springer-Verlag, Berlin, 1979.

[14] C. Godsil and G. Royle, *Algebraic Graph Theory*, Graduate Texts in Mathematics, Vol. 207, Springer-Verlag, New York, 2001.

[15] D. E. Gonsor, Subdivision algorithms with non-negative masks generally converge, *Adv. Comput. Math.* **1** (1993), 215-221.

[16] J. Hajnal, Weak ergodicity in nonhomogeneous Markov-chains, *Proc. Cambridge Philos. Soc.* **54** (1958), 233–246.

[17] B. Han and R.-Q. Jia, Multivariate refinement equations and convergence of subdivision schemes, *SIAM Journal on Mathematical Analysis,* **29** (1998), 1177-1199.

[18] S. Harizanov, Stability of nonlinear subdivision schemes and multiresolutions, Master's Thesis, Jacobs University, Bremen, 2007.

[19] D. J. Hartfiel and Uriel G. Rothblum, Convergence of inhomogenous products of matrices and coefficients of ergodicity, *Lin. Alg. Appl.* **277** (1998), 1–9.

[20] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, New York, 1985.

[21] R.-J. Jia and D.-X. Zhou, Convergence of subdivision schemes associated with nonnegative masks, *SIAM J. Matrix Anal. Appl.* **21** (1999), 418–430.

[22] R. M. Jungers, *Infinite Matrix Products. From the joint spectral radius to combinatorics,* PhD thesis, Université catholique de Louvain, 2008.

[23] J. G. Kemeny and J. L. Snell, *Finite Markov Chains*, Springer-Verlag, Heidelberg, 1976.

[24] S. Li and X.-L. Zhou, Multivariate refinement equation with nonnegative masks, *Sci. China Ser. A Math.* **52** (2006), 239-247.

[25] C. A. Micchelli, *Mathematical Aspects of Geometric Modeling,* CBMS-NSF Regional Conf. Ser. in Appl. Math. 65, SIAM, Philadelphia, 1995.

[26] C. A. Micchelli and H. Prautzsch, Uniform refinement of curves, *Lin. Alg. Appl.* **114/115** (1989), 841–870.

[27] A. A. Melkman, Subdivision schemes with non-negative masks converge always – unless they obviously cannot?, *Ann. Numer. Math.* **4** (1997), 451–460.

[28] M. Neumann and H. Schneider, The convergence of general products of matrices and the weak ergodicity of Markov chains, *Lin. Alg. Appl.* **287** (1999), 307–314.

[29] A. Paz, Definite and quasidefinite sets of stochastic matrices, *Proc. Amer. Math. Soc.* **16** (1965), 634-641.

[30] W. Ren and R. W. Beard, Consensus seeking in multiagent systems under dynamically changing interaction topologies, *IEEE. Trans. Automat. Contr.* **50** (2005), 655–661.

[31] E. Seneta, *Non-negative Matrices*, Allen and Unwin, London, 1973.

[32] J. Theys, *Joint Spectral Radius: Theory and approximations*, PhD thesis, Université catholique de Louvain, 2005.

[33] R. S. Varga, *Matrix Iterative Analysis*, 2nd ed., Springer-Verlag, Berlin Heidelberg, 2000.

[34] Y. Wang, Subdivision schemes and refinement equations with non-negative masks, *J. Approx. Theory* **113** (2001), 207-220.

[35] J. Wolfowitz, Products of indecomposable, aperiodic, stochastic matrices, *Proc. Amer. Math. Soc.* **14** (1963), 733–737.

[36] X.-L. Zhou, Subdivision schemes with nonnegative masks, *Math. Comp.* **47** (2005), 819–839.

[37] X.-L. Zhou, On multivariate subdivision schems with nonnegative masks, *Proc. Amer. Math. Soc.* **134** (2006), 859–869.

[38] X.-L. Zhou, Positivity of refinable functions defined by nonnegative finite mask, *Appl. Comput. Harmonic Analysis* **27** (2009), 133–156.

Zusammenfassung

Subdivision ist ein iterativer Prozess zur Generierung von Kurven und Flächen aus einer Anzahl von diskreten Daten, die rekursiv unter Verwendung von einer Familie von Subdivisionsregeln (einem sog. Subdivisionsschema) verfeinert werden. Im Grenzfall, wenn die Anzahl der Subdivisionsschritte anwächst, versucht man so stetige oder sogar glatte Grenzfunktionen zu erzeugen.

Wenn man in jedem Schritt die gleichen Unterteilungsregeln verwendet, spricht man von stationärer Subdivision. Die Theorie solcher stationärer Subdivisionsschemata wurde durch die grundlegende Arbeit von Micchelli und Prautzsch [26] beeinflusst. Diese verwendet Eigenschaften von nicht-negativen Matrizen, schöpft diese aber nicht voll aus. Bei unserem Studium der Arbeiten [6] und [21], die den Begriff des 'joint spectral radius' von Matrixfamilien verwenden, wurde unser Interesse an dem Zusammenhang zwischen stationärer Subdivision und endlichen inhomogenen Markov-Prozessen geweckt. Dies war der Ausgangspunkt für unsere Forschung auf diesem Gebiet.

Aus der Numerischen Linearen Algebra wissen wir, dass die Potenzen einer zeilen-stochastischen Matrix $\mathbf{A}$ genau dann gegen eine Matrix vom Rang Eins konvergieren, wenn der Eigenwert $\lambda = 1$ strikt dominant ist. Die sich im Grenzfalle ergebende Matrix ist dann eine zeilen-stochastische Matrix mit gleichen Zeilen, und der gemeinsame Zeilenvektor ist ein Linkseigenvektor für $\mathbf{A}$ zum dominanten Eigenwert. In der Theorie endlicher Markov-Prozesse beschreiben die Einträge von $\mathbf{A}$ Übergangswahrscheinlichkeiten zwischen den einzelnen Zuständen in diskreten Zeitschritten. Für solche Anwendungen wurde von Wolfowitz [35] die zur strikten Dominanz von $\lambda = 1$ äquivalente Notation einer SIA-Matrix eingeführt, die auf die Eigenschaften stochastic, indecomposable und aperiodic Bezug nimmt. Diese Eigenschaften erwiesen sich auch beim Studium endlicher Familien von zeilen-stochastischen Matrizen als hilfreich, die bei der Untersuchung von inhomogenen Markov-Prozessen eine Rolle spielen. Hier fordert man, dass jeder endliche Abschnitt des Produktes, als eine zeilen-stochastische Matrix, wieder diese SIA-Eigenschaften hat. Auch die Interpretation der SIA-Eigenschaft als eine Zusammenhangseigenschaft des gerichteten Graphen einer nicht-negativen Matrix erwies sich als nützlich beim Studium dieser Rang-Eins-Konvergenz [30].

Die Dissertation behandelt den Zusammenhang zwischen SIA-Matrizen und nicht-negativer Subdivision. Sie ist folgendermaßen aufgebaut: Nach einem einleitenden Kapitel wird in Kapitel 2 die grundlegende Notation bereit gestellt.

Anschließend beschreiben wir in Kapitel 3 zunächst den formalen Zusammenhang zwischen nicht-negativer Subdivision und einem hierzu gehörenden Markov-Prozess. Wir führen dazu eine Familie $\mathcal{A}$ von Matrizen ein, die aus der Maske des Subdivisionsschemas aufgebaut werden. Unter anderem beschreiben Lemma 3.1 and Lemma 3.2, wie die Koeffizienten der iterierten Masken sich durch Matrix-Produkte aus der Familie $\mathcal{A}$ deuten lassen. Im Grenzfall ergeben sich so die Funktionswerte der Fundamentalfunktion des Subdivisionsprozesses aus den Einträgen eines unendlichen Matrix-Produktes.

Die Kapitel 4 und 5 stellen den zentralen Beitrag dieser Dissertation dar. Zunächst geben wir dort einen Überblick über Spektraleigenschaften von zeilenstochastischen Matrizen und Eigenschaften ihrer gerichteten Graphen, wobei die SIA-Eigenschaft wieder im Vordergrund steht. Wir verweisen auf den Begriff der 'scrambling power', eingeführt von Hajnal [16], und den zugehörigen ergodischen Koeffizienten. Hinsichtlich der Eigenschaften gerichteter Graphen von SIA-Matrizen verbessern wir eine Aussage von Ren und Beard [30]. Anschließend studieren wir Familien von SIA-Matrizen, deren Indikator-Matrizen und die Zusammenhangseigenschaften der betreffenden gerichteten Graphen. Wir glauben, dass dies einen wichtigen Beitrag zur Theorie nicht-negativer Subdivision darstellt, weil dieser Hintergrund nunmehr eine Anwendung des Konvergenzsatzes von Anthonisse und Tijms [2] zulässt. Diesen Konvergenzsatz greifen wir in Abschnitt 4.6 auf. Er beschreibt die Rang-Eins-Konvergenz ohne Bezug auf den 'joint spectral radius', sondern verwendet hierzu den (äquivalenten) Begriff des ergodischen Koeffizienten. Eine Reihe von Eigenschaften, die zur SIA-Eigenschaft äquivalent sind, werden in Lemma 4.7 und dem anschließenden Lemma 4.8 aufgelistet; diese nehmen Bezug auf Eigenschaften (scrambling property, positive column property), wie sie in der bisherigen Literatur zur Konvergenz nicht-negativer Subdivision auftauchen.

Kapitel 5 enthält einen vollständigen Beweis der Charakterisierung gleichmäßiger Konvergenz für nicht-negative Subdivision, im Fall einer und zweier Veränderlichen, wobei letzterer Fall repräsentativ ist für den Fall mehrerer Variablen. Er benutzt die punktweise Definition der Grenzfunktion in dyadischen Punkten - wobei auf die Binärentwicklung reeller Vektoren aus dem Einheitswürfel Bezug genommen wird - unter Bezug auf den Konvergenzsatz von Anthonisse-Tijms. Eine geeignete Verallgemeinerung der Kompatibilitätsbedingung von Micchelli und Prautzsch berücksichtigt hierbei die Mehrdeutigkeit der Binärentwicklung in dyadischen Punkten. Als Folge hiervon lässt sich der Hölder-Exponent der Fundamentalfunktion durch den ergodischen Koeffizienten der Familie $\mathcal{A}$ ausdrücken. Unsere Ergebnisse zur Konvergenz, in Theorem 5.1 und Theorem 5.8, fassen die existierenden Ergebnisse zur nicht-negativen Subdivision zusammen. Ausgenommen ist hiervon die GCD-Bedingung, die offensichtlich einen Spezialfall darstellt, der sich auf den Fall einer einzigen Variablen bezieht.

Kapitel 5 enthält auch einige Ansätze zu unserem Versuch, hinreichende Bedingungen zur gleichmäßigen Konvergenz, die im Fall einer Variablen bekannt sind, auf den Fall zweier oder mehrerer Variablen zu verallgemeinern. Ein Analogon zu Melkmans [27] univariater 'string condition' ist unsere 'rectangular string condition' für den bivariaten Fall. Das Kapitel schließt mit einem Hinweis auf die Tatsache, dass die Eigenschaft der gleichmäßigen Konvergenz tatsächlich allein eine Träger-Eigenschaft der Maske ist, modulo offensichtlicher notwendiger Zusatzeigenschaften wie z. B. die 'sum rule'. Eine typische Anwendung dieser Trägereigenschaft liefert die Charakterisierung der gleichmäßigen Konvergenz bei Masken, die sich als Konvex-Kombinationen einfacherer Masken deuten lassen.

Die Dissertation schließt mit zwei kurzen Kapiteln zur Tensorprodukt- und zur Box-Spline-Subdivision, sowie einem Anhang, in dem Definitionen und nützliche Hilfsergebnisse und Theoreme zur Theorie von Matrizen und deren Graphen ohne Beweise aufgeführt sind.

Summary

Subdivision is a process for the design of curves and surfaces starting from a discrete set of data, which are then recursively refined using a set of subdivision rules (called a subdivision scheme). In the limit, as the number of subdivision steps increases, one aims at generating a continuous or even smooth limit function.

If the same rules are repeated in each step, the subdivision scheme is called stationary. The theory of such subdivision schemes was very much influenced by the seminal paper of Micchelli and Prautzsch [26]. It uses properties of non-negative matrices, although it does not really exploit those in full detail. When studying [6] and [21], who use properties of the joint spectral radius of a family of matrices, we got interested in the connection of stationary subdivision to finite non-homogeneous Markov processes which was mentioned there. This was the starting point for our research on this topic, leading to the results presented here.

From Numerical Linear Algebra we know that the powers of a row stochastic matrix $\mathbf{A}$ converge to a rank one matrix, if and only if its eigenvalue $\lambda = 1$ is strictly dominant, and then the limit is a row stochastic matrix with equal rows, the common row vector being a left eigenvector of $\mathbf{A}$ for this dominant eigenvalue. In the theory of finite Markov processes – where $\mathbf{A}$ describes the transition probabilities between states as the process develops – the equivalent notion of an SIA matrix (refering to the properties of being stochastic, indecomposable, and aperiodic) was introduced by Wolfowitz [35]. It has proved to be useful for the study of infinite products of a finite family of row stochastic matrices as well, thus for the study of a non-homogeneous Markov process. Here, one has to require that each finite section of the product, as a corresponding row stochastic matrix, has this SIA property. Also, the interpretation of the SIA property in terms of the directed graph of a non-negative matrix has proved to be useful for studying this rank one convergence [30].

This dissertation is concerned with SIA matrices and non-negative stationary subdivision, and is organized as follows: After an introducing chapter where some basic notation is given we describe, in Chapter 3, how non-negative subdivision is connected to a corresponding non-homogenous Markov process. The family of matrices $\mathcal{A}$, built from the mask of the subdivision scheme, is introduced. Among other results, Lemma 3.1 and Lemma 3.2 relate the coefficients of the iterated masks to matrix products from the family $\mathcal{A}$, and in the limiting case the values of the basic limit function are found from the entries in an infinite product of matrices.

Chapter 4 and Chapter 5 are the core of this dissertation. In Chapter 4, we first review some spectral and graph properties of row-stochastic matrices and, in particular, of SIA matrices. We point to the notion of scrambling power, introduced by Hajnal [16], and of the related coefficient of ergodicity. We also consider the directed graph of such matrices, and we improve upon a condition given by Ren and Beard in [30]. Then we study finite families of SIA matrices, the properties of their indicator matrices and the connectivity of their directed graphs. We consider this chapter to be an important contribution to the theory of non-negative subdivision, since it explains the background in order to apply the convergence result of Anthonisse and Tijms [2], which we reprove in Section 4.6, to rank one convergence of infinite products of row stochastic matrices. It does not use the notion of joint spectral radius but the (equivalent) coefficient of ergodicity. Properties equivalent to SIA are listed in Lemma 4.7 and in the subsequent Lemma 4.8; they connect the SIA property to equivalent conditions (scrambling property, positive column property) as they appear in the existing literature dealing with convergence of non-negative subdivision.

The fifth chapter of the dissertation contains the full proof of the characterization of uniform convergence for non-negative subdivision, for the univariate and bivariate case, the latter one being a representative for multivariate aspects. It uses the pointwise definition of the limit function at dyadic points - refering to the dyadic expansion of real vectors from the unit cube - using the Anthonisse-Tijms pointwise convergence result, and employs the proper extension of the Micchelli-Prautzsch compatibility condition to the multivariate case, taking care of the ambiguity of representation of dyadic points. As a consequence, the Hölder exponent of the basic limit function can be expressed in terms of the coefficient of ergodicity of the family $\mathcal{A}$. Our convergence theorems, in Theorem 5.1 and Theorem 5.8, include the existing characterizations of uniform convergence for non-negative univariate and bivariate subdivision from the literature, except for the GCD condition, which seems to be a condition applicable to univariate subdivision only.

Chapter 5 also reports on some further attempts where we have tried to extend conditions from univariate subdivision, which are sufficient for convergence, to the bivariate case. We could find a bivariate analogue of Melkman's univariate string condition, which we call - in the bivariate case - a rectangular string condition. The chapter concludes with stating the fact that uniform convergence of non-negative stationary subdivision is a property of the support of the mask alone, modulo some apparent necessary conditions such as the sum rules. A typical application of this support property characterizes uniform convergence in the case where the mask is a convex combination of other masks.

The dissertation ends with two short chapters on tensor product and box spline subdivision, and an appendix where some definitions and useful lemmas and theorems about matrix and graph theory are stated without proofs.

Acknowledgements

First of all, I would like to express my sincere and deep thanks to my supervisor Professor Dr. Kurt Jetter, for introducing me to the beauty of subdivision schemes and SIA matrices and for his supervision of my research work, his support and patience. During my PhD study, he carefully supervised my research progress, and provided clear research guidance. His solid fundamental research skills and inspiration have had a great impact on my mathematical research and my scientific development. His teaching has made the research subjects interesting to me. He has helped me to straighten several sections from earlier drafts of this dissertation, and without his help this dissertation would have never been finished.

I am also very grateful to Professor Dr. Georg Zimmermann for his very valuable suggestions, and especially for his help for recovery of my files when my computer crashed. I would also like to express my sincere gratitude to Professor Dr. Uwe Jensen for his helpful advices concerning my research work and my life in Germany. My sincere thanks go to the staff of the Institut für Angewandte Mathematik und Statistik, Dr. Maik Döring, Dr. Elena Berdysheva, Mr. Christian Wollmann, Ms. Yan Chu, Ms. Fabienne Bufler and Ms. Paola Ferrario for their encouragement and their help in my work and life here. Deep thanks go to all my friends here, who have made my life in Germany more enjoyable.

Finally, I wish to express my deep gratitude to my parents, for their long support and encouragement in my life, before and after I came to Germany. Their endless love has been the source of my courage and of my strength, and has been the basis to overcome difficulties and to pursue my PhD study and dreams.

Curriculum Vitae

Personal Information

Name:	Xianjun Li
Date of birth:	14. Sep 1982
Place of birth:	Liaocheng, Shandong, V R China
Nationality:	Chinese
Address:	Adornostrasse 35, 70599 Stuttgart

Education

11.2008 -	Universität Hohenheim PhD Student
09.2005 - 07.2008	Shanghai University, China Master of Science in Mathematics
09.2001 - 07.2005	Shanxi University, China Bachelor of Science in Information and Computing Science

Work and Research Experience

11.2008 -	PhD Research on SIA Matrices and Non-Negative Stationary Subdivision, Universität Hohenheim
09.2005 - 07.2008	Master Research on Wavelets and Filters, Shanghai University, China
09.2005 - 06.2007	Teaching Assistant, Shanghai University, China

Erklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit selbständig angefertigt habe, nur die angegebenen Quellen und Hilfsmittel benutzt und wörtlich oder inhaltlich übernommene Stellen als solche gekennzeichnet habe.

Hohenheim, 2012 Xianjun Li