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Birkhoff Normal Forms, KAM Theory and Symmetries for Certain Second Order Rational Difference Equation with Quadratic Term

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Abstract

By using the KAM theory, we investigate the stability of the equilibrium solution of a certain difference equation. We also use the symmetries to find effectively the periodic solutions with feasible periods. The specific feature of this difference equation is the fact that we were not able to use the invariant to prove stability or to find feasible periods of the solutions.

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1 Introduction and Preliminaries

We consider the dynamics of the following equation

$$x_{n+1} = \frac{Ax_n^2 + F}{ex_{n-1}}, \quad n = 0, 1, \ldots \tag{1.1}$$

where $A, F, e > 0$ and the initial conditions $x_{-1}, x_0 > 0$. Equation (1.1) is a special case of the more general equation

$$x_{n+1} = \frac{Ax_n^2 + Ex_{n-1} + F}{ax_n^2 + ex_{n-1} + f}, \quad n = 0, 1, \ldots \tag{1.2}$$

where all parameters and the initial conditions are nonnegative and such that $A + F + E > 0, ax_n^2 + ex_{n-1} + f > 0$ for $n = 0, 1, \ldots$. Equation (1.2) has very reach dynamics and it can exhibit different types of bifurcations such as the period doubling, as well as very simple behavior such as the global asymptotic stability of the unique equilibrium which happens in the case of equation

$$x_{n+1} = \frac{x_{n-1}}{ax_n^2 + ex_{n-1} + f}, \quad n = 0, 1, \ldots$$

when $f \geq 1$. Some special cases of (1.2) possess an exact solution such as the Riccati equation obtained for $A = a = 0$.

Equation (1.1) has very specific dynamics since it can be transformed into an equation for which the corresponding map is an area preserving with two complex conjugate roots which belong to the unit disk. This means that the KAM theory is the appropriate tool to investigate the dynamics of (1.1). In that respect this equation is similar to Lyness' equation or Gumowski–Mira equation considered in [1, 2, 4, 12, 14, 16, 17, 19], which was considered by either the KAM theory as in [9, 17–19] or combination of algebraic and geometric techniques as in [1–4, 22]. The second technique was always based on the existence of invariants which analysis lead to the properties of the solutions and in particular, to the results on feasible periods, chaotic solutions etc. For instance the Lyness' equation

$$x_{n+1} = \frac{x_n + F}{x_{n-1}}, \quad n = 0, 1, \ldots \tag{1.3}$$

introduced in [13] and first studied systematically in [22] has an invariant of the form

$$I(x_n, x_{n-1}) = \left(1 + \frac{1}{x_n}\right)\left(1 + \frac{1}{x_{n-1}}\right)(F + x_n + x_{n-1}), \tag{1.4}$$

for $n = 0, 1, \ldots$ with the property that $I(x_{n+1}, x_n) = I(x_n, x_{n-1}), n = 0, 1, \ldots$. The algebraic and geometric analysis of invariant (1.4) initiated in [22] has provided precise description of all feasible periods of (1.3) and the chaotic solutions of (1.3). See [1–5, 22]. These techniques were successfully applied to the corresponding equation with

the periodic coefficients with some periods that allow applications of such techniques. See [5, 7, 16]. In the case when we can not find the invariant of an equation which generates the area preserving map the only available technique seems to be the KAM theory. As it was shown in [6], (1.1) does not possess an algebraic invariant, which is indicated by the simulations and visualizations of the orbits of this equation. See Figures 2.1–2.3.

The rest of this section presents the basic results about Birkhoff normal forms and the KAM theory, see [9, 11, 20, 21]. In Section 2, we apply this theory to (1.1) in order to compute its Birkhoff normal forms and use it to check that the equilibrium solution is stable. In Section 3, we use symmetries to find effective computational method for calculation of the periodic solutions with feasible periods.

By substituting

$$x_n = \sqrt{\frac{F}{e}} t_n$$

in (1.1), we obtain

$$t_{n+1} = \frac{\frac{A}{e} t_n^2 + 1}{t_{n-1}}.$$

If we put $\alpha = \dfrac{A}{e}$, then we get the equation

$$t_{n+1} = \frac{\alpha t_n^2 + 1}{t_{n-1}}, \tag{1.5}$$

which has the unique positive equilibrium point $\bar{t} = \dfrac{1}{\sqrt{1-\alpha}}$ if $\alpha \in (0, 1)$. The next result gives normal form of an equation with an elliptic fixed point, see [9,11,15,19,21].

Theorem 1.1 (Birkhoff Normal Form)**.** *Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be an area-preserving C^n map (n-times continuously differentiable) with a fixed point at the origin whose complex-conjugate eigenvalues λ and $\overline{\lambda}$ are on the unit disk (elliptic fixed point). Suppose there exists an integer l such that*

$$4 \leq l \leq n + 1$$

and suppose that the eigenvalues satisfy

$$\lambda^k \neq 1 \text{ for } k = 3, 4, \ldots l$$

Let $r = [\dfrac{l}{2}]$ be the integer part of $\dfrac{l}{2}$. Then there exists a smooth function $g(z, \overline{z})$ that vanishes with its derivatives up to order $r-1$ at $z = 0$, and there exists a real polynomial

$$\alpha(\omega) = \alpha_1\omega + \alpha_2\omega^2 + \ldots + \alpha_r\omega^r$$

such that the map F can be reduced to the normal form by suitable change of complex coordinates

$$z \to F(z, \overline{z}) = \lambda z e^{i\alpha(z\overline{z})} + g(z, \overline{z}).$$

In other words the corresponding system of difference equations

$$x_{n+1} = F(x_n)$$

can be reduced to the form

$$\begin{pmatrix} r_{n+1} \\ s_{n+1} \end{pmatrix} = \begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix} \begin{pmatrix} r_n \\ s_n \end{pmatrix} + \begin{pmatrix} O_l \\ O_l \end{pmatrix} \tag{1.6}$$

where

$$\omega = \sum_{k=0}^{M} \gamma_k (r_n^2 + s_n^2)^k, \quad M = \left[\frac{l}{2}\right] - 1. \tag{1.7}$$

Here O_l denotes a convergent power series in r_n and s_n with terms of order greater than or equal to l which vanishes at the origin and $[x]$ denotes the least integer greater than or equal to x.

The numbers $\gamma_1, \ldots, \gamma_k$ are called twist coefficients. Using Theorem 1.1, we can state the main stability result for an elliptic fixed point, known as the KAM Theorem (or Kolmogorov–Arnold–Moser theorem), see [9, 11, 15, 21].

Theorem 1.2 (KAM Theorem)**.** *Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be an area-preserving map with an elliptic fixed point at the origin satisfying the conditions of Theorem 1.1. If the polynomial $\alpha(|z|^2)$ is not identically zero, then the origin is a stable equilibrium point. In other words if for some $k \in \{1, \ldots, M\}$ we have $\gamma_k \neq 0$ in* (1.7)*, then the origin is a stable equilibrium point.*

Remark 1.3. Consider an invariant annulus $A_\varepsilon = \{z : \varepsilon < |z| < 2\varepsilon\}$ in a neighborhood of the elliptic fixed point, for ε a sufficiently small positive number. Note that the linear part of normal form approximation leaves invariant all circles. The motion restricted to each of these circles is a rotation by some angle, see [15, Theorem 2.28]. Also note that if at least one of the twist coefficients γ_k is non-zero, the angle of rotation will vary from circle to circle. A radial line through the fixed point will undergo twisting under the mapping. The KAM theorem says that, under the addition of the remainder term, most of these invariant circles will survive as invariant closed curves under the original map [9, 11, 15]. Precisely, the following result holds, see [9, 11, 15].

Theorem 1.4. *Assuming that $\alpha(z\overline{z})$ is not identically zero and ε is sufficiently small, then the map F has a set of invariant closed curves of positive Lebesgue measure close to the original invariant circles. Moreover, the relative measure of the set of surviving invariant curves approaches full measure as ε approaches 0. The surviving invariant closed curves are filled with dense irrational orbits.*

The following is a consequence of Moser's twist map theorem [9, 11, 21].

Theorem 1.5. *Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be an area-preserving diffeomorphism, and $(\overline{x}, \overline{y})$ a nondegenerate elliptic fixed point. There exist periodic points with arbitrarily large period in every neighbourhood of $(\overline{x}, \overline{y})$.*

Indeed the theorem implies that arbitrarily close to the fixed point there are always infinitely many gaps between consecutive invariant curves that contain periodic points. Within these gaps, one finds, in general, orbits of hyperbolic and elliptic periodic points. These facts cannot be deduced from computer pictures. The linearized part of (1.6) represent a rotation for angle ω and so if ω is rational multiple of π every solution is periodic with same period while if ω is irrational multiple of π there will exist chaotic solutions. In this paper, we will not go into detailed study of these behaviors, as we were not able to find any continuous invariant for (1.5).

2 KAM Theory Applied to Equation (1.5) for $\alpha \in (0, 1)$

For $\bar{t} = \dfrac{1}{\sqrt{1-\alpha}}$, $0 < \alpha < 1$, we use the substitution

$$\begin{aligned} x_n &= \ln \frac{t_n}{\bar{t}}, \\ y_n &= x_{n-1}, \end{aligned}$$

to transform (1.5) into the system

$$\left.\begin{aligned} x_{n+1} &= -y_n + \ln\left(\alpha \bar{t}^2 e^{2x_n} + 1\right) - 2\ln \bar{t} \\ y_{n+1} &= x_n. \end{aligned}\right\} \tag{2.1}$$

The equilibrium point $\bar{t}$ is then transformed into $(0, 0)$.

The map T associated to the system (2.1) is of the form

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y + \ln\left(\alpha \bar{t}^2 e^{2x} + 1\right) - 2\ln \bar{t} \\ x \end{pmatrix}.$$

The Jacobian matrix of the map T at the point (x, y) is of the form

$$J_T(x, y) = \begin{pmatrix} \dfrac{2\alpha \bar{t}^2 e^{2x}}{\alpha e^{2x} \bar{t}^2 + 1} & -1 \\ 1 & 0 \end{pmatrix}.$$

It is easy to see that

$$\det J_T(x, y) = 1$$

and

$$J_0 = J_T(0, 0) = \begin{pmatrix} 2\alpha & -1 \\ 1 & 0 \end{pmatrix}.$$

The characteristic equation at $(0,0)$ is

$$\lambda^2 - 2\alpha\lambda + 1 = 0,$$

with the characteristic roots

$$\begin{aligned} \lambda &= \alpha + i\sqrt{1-\alpha^2} \\ \overline{\lambda} &= \alpha - i\sqrt{1-\alpha^2}. \end{aligned}$$

A straightforward calculation gives the following expressions for second, third and fourth power of the characteristic root

$$\begin{aligned} \lambda^2 &= 2\alpha^2 - 1 + 2\alpha i\sqrt{1-\alpha^2} \\ \lambda^3 &= \alpha\left(4\alpha^2 - 3\right) + i\left(4\alpha^2 - 1\right)\sqrt{1-\alpha^2} \end{aligned}$$

and

$$\lambda^4 = 8\alpha^4 - 8\alpha^2 + 1 + 4i\alpha\left(2\alpha^2 - 1\right)\sqrt{1-\alpha^2}.$$

Clearly $|\lambda| = 1, \lambda^3 \neq 1, \lambda^4 \neq 1$ for $\alpha \in (0,1)$. Thus the assumptions of Theorem 1.1 are satisfied for $l = 4$ and we will find the Birkhoff normal form of (2.1) by using the sequence of transformations described in Section 1.

2.1 First Transformation

Notice that the matrix of the linearized system at the origin is given as

$$J_0 = \begin{pmatrix} 2\alpha & -1 \\ 1 & 0 \end{pmatrix}.$$

A straightforward calculation shows that the matrix of the corresponding eigenvectors which correspond to λ and $\overline{\lambda}$ of J_0 is

$$P = \begin{pmatrix} 1 & 1 \\ \overline{\lambda} & \lambda \end{pmatrix}.$$

In order to obtain the Birkhoff normal form of system (2.1), we will expand the right hand sides of the equations of system (2.1) at the equilibrium point $(0,0)$ up to the order $l - 1 = 3$. We obtain

$$\left.\begin{aligned} x_{n+1} &= 2\alpha x_n - y_n + \frac{2\alpha}{\bar{t}^2}\left(x_n^2 + \frac{2\left(2-\bar{t}^2\right)}{3\bar{t}^2}x_n^3\right) + O_4 \\ y_{n+1} &= x_n. \end{aligned}\right\} \tag{2.2}$$

Now the change of variables

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = P \begin{bmatrix} u_n \\ v_n \end{bmatrix} = \begin{bmatrix} u_n + v_n \\ \overline{\lambda}u_n + \lambda v_n \end{bmatrix}$$

transforms system (2.2) into

$$\begin{aligned} u_{n+1} &= \lambda u_n + \sigma\left((u_n+v_n)^2 + \tfrac{2\left(2-\bar{t}^2\right)}{3\bar{t}^2}(u_n+v_n)^3\right) + O_4, \\ v_{n+1} &= \overline{\lambda} v_n + \overline{\sigma}\left((u_n+v_n)^2 + \tfrac{2\left(2-\bar{t}^2\right)}{3\bar{t}^2}(u_n+v_n)^3\right) + O_4, \end{aligned} \tag{2.3}$$

where

$$\sigma = \frac{\lambda}{\lambda-\overline{\lambda}} \cdot \frac{2\alpha}{\bar{t}^2}.$$

2.2 Second Transformation

The objective of second transformation is to obtain the nonlinear terms up to order $l-1$ in normal form. The change of variables

$$\begin{aligned} u_n = \xi_n &+ \left(a_{20}\xi_n^2 + a_{21}\xi_n\eta_n + a_{22}\eta_n^2\right) \\ &+ \left(a_{30}\xi_n^3 + a_{31}\xi_n^2\eta_n + a_{32}\xi_n\eta_n^2 + a_{33}\eta_n^3\right), \end{aligned} \tag{2.4}$$

$$\begin{aligned} v_n = \eta_n &+ \left(\overline{a_{20}}\eta_n^2 + \overline{a_{21}}\xi_n\eta_n + \overline{a_{22}}\xi_n^2\right) \\ &+ \left(\overline{a_{30}}\eta_n^3 + \overline{a_{31}}\xi_n\eta_n^2 + \overline{a_{32}}\xi_n^2\eta_n + \overline{a_{33}}\xi_n^3\right), \end{aligned} \tag{2.5}$$

where

$$\left.\begin{aligned} u_n^2 &= \xi_n^2 + 2a_{20}\xi_n^3 + 2a_{21}\xi_n^2\eta_n + 2a_{22}\xi_n\eta_n^2 + O_4 \\ v_n^2 &= \eta_n^2 + 2\overline{a_{20}}\eta_n^3 + 2\overline{a_{21}}\xi_n\eta_n^2 + 2\overline{a_{22}}\xi_n^2\eta_n + O_4 \\ u_n^3 &= \xi_n^3 + O_4 \\ v_n^3 &= \eta_n^3 + O_4 \\ u_n^2 v_n &= \xi_n^2\eta_n + O_4 \\ u_n v_n^2 &= \xi_n\eta_n^2 + O_4 \\ u_n v_n &= \overline{a_{22}}\xi_n^3 + \left(a_{20}+\overline{a_{21}}\right)\xi_n^2\eta_n \\ &\quad + \left(a_{21}+\overline{a_{20}}\right)\xi_n\eta_n^2 + \xi_n\eta_n + a_{22}\eta_n^3 + O_4 \end{aligned}\right\}$$

yields

$$\begin{aligned} (u_n+v_n)^2 = \xi_n^2 &+ 2a_{20}\xi_n^3 + 2a_{21}\xi_n^2\eta_n + 2a_{22}\xi_n\eta_n^2 \\ &+ 2\left(\overline{a_{22}}\xi_n^3 + \left(a_{20}+\overline{a_{21}}\right)\xi_n^2\eta_n\right. \\ &\left. + \left(a_{21}+\overline{a_{20}}\right)\xi_n\eta_n^2 + \xi_n\eta_n + a_{22}\eta_n^3\right) + \eta_n^2 \\ &+ 2\overline{a_{20}}\eta_n^3 + 2\overline{a_{21}}\xi_n\eta_n^2 + 2\overline{a_{22}}\xi_n^2\eta_n + O_4 \end{aligned} \tag{2.6}$$

$$(u_n+v_n)^3 = \xi_n^3 + 3\xi_n^2\eta_n + 3\xi_n\eta_n^2 + \eta_n^3 + O_4. \tag{2.7}$$

The formulas (2.4)–(2.7) reduces system (2.3) to the form

$$\left.\begin{aligned}\xi_{n+1} &= \left(\lambda\xi_n + \alpha_2\xi_n^2\eta_n\right) + O_4,\\ \eta_{n+1} &= \left(\overline{\lambda}\eta_n + \overline{\alpha_2}\xi_n\eta_n^2\right) + O_4\end{aligned}\right\}. \tag{2.8}$$

By using (2.8) in (2.4) and (2.5) and by replacing n with $n+1$, we have

$$\begin{aligned}u_{n+1} &= \lambda\xi_n + a_{20}\lambda^2\xi_n^2 + a_{30}\lambda^3\xi_n^3 + a_{21}\lambda\overline{\lambda}\xi_n\eta_n\\ &\quad + \left(\alpha_2 + a_{31}\lambda^2\overline{\lambda}\right)\xi_n^2\eta_n + a_{32}\lambda\overline{\lambda}^2\xi_n\eta_n^2\\ &\quad + a_{22}\overline{\lambda}^2\eta_n^2 + a_{33}\overline{\lambda}^3\eta_n^3,\\ v_{n+1} &= \overline{\lambda}\eta_n + \overline{a_{20}}\overline{\lambda}^2\eta_n^2 + \overline{a_{30}}\overline{\lambda}^3\eta_n^3 + \overline{a_{21}}\lambda\overline{\lambda}\xi_n\eta_n\\ &\quad + \left(\overline{\alpha_2} + \overline{a_{31}}\overline{\lambda}^2\lambda\right)\xi_n\eta_n^2 + \overline{a_{32}}\lambda^2\overline{\lambda}\xi_n^2\eta_n\\ &\quad + \overline{a_{22}}\overline{\lambda}^2\xi_n^2 + \overline{a_{33}}\lambda^3\xi_n^3.\end{aligned} \tag{2.9}$$

By using (2.9) in the left-hand side and (2.4), (2.6) and (2.7) in the right-hand side of (2.3), we obtain

$$\begin{aligned}&\lambda\xi_n + a_{20}\lambda^2\xi_n^2 + a_{30}\lambda^3\xi_n^3\\ &+a_{21}\lambda\overline{\lambda}\xi_n\eta_n + \left(\alpha_2 + a_{31}\lambda^2\overline{\lambda}\right)\xi_n^2\eta_n\\ &+a_{32}\lambda\overline{\lambda}^2\xi_n\eta_n^2 + a_{22}\overline{\lambda}^2\eta_n^2 + a_{33}\overline{\lambda}^3\eta_n^3\\ =\ &\lambda\left(\xi_n + \left(a_{20}\xi_n^2 + a_{21}\xi_n\eta_n + a_{22}\eta_n^2\right)\right.\\ &\left.+\left(a_{30}\xi_n^3 + a_{31}\xi_n^2\eta_n + a_{32}\xi_n\eta_n^2 + a_{33}\eta_n^3\right)\right)\\ &+\sigma\begin{pmatrix}\xi_n^2 + 2a_{20}\xi_n^3 + 2a_{21}\xi_n^2\eta_n + 2a_{22}\xi_n\eta_n^2\\ +2\left(\overline{a_{22}}\xi_n^3 + \left(a_{20} + \overline{a_{21}}\right)\xi_n^2\eta_n\right.\\ \left.+\left(a_{21} + \overline{a_{20}}\right)\xi_n\eta_n^2 + \xi_n\eta_n + a_{22}\eta_n^3\right)\\ +\eta_n^2 + 2\overline{a_{20}}\eta_n^3 + 2\overline{a_{21}}\xi_n\eta_n^2 + 2\overline{a_{22}}\xi_n^2\eta_n\\ +\dfrac{2\left(1-\alpha z^2\right)}{z^2}\left(\xi_n^3 + 3\xi_n^2\eta_n + 3\xi_n\eta_n^2 + \eta_n^3\right)\end{pmatrix} + O_4.\end{aligned}$$

The last relation holds if the corresponding coefficients are equal, which leads to the following set of equalities:

$$\begin{aligned}
\xi_n^2: &\quad a_{20}\lambda^2 = \lambda a_{20} + \sigma,\\
\xi_n^3: &\quad a_{30}\lambda^3 = \lambda a_{30} + 2a_{20}\sigma + \sigma\left(2\overline{a_{22}} + \frac{2\left(2-\bar{t}^2\right)}{3\bar{t}^2}\right),\\
\xi_n\eta_n: &\quad a_{21} = \lambda a_{21} + 2\sigma,\\
\xi_n^2\eta_n: &\quad \alpha_2 + a_{31}\lambda = \lambda a_{31} + 2a_{21}\sigma + 2\left(a_{20} + \overline{a_{21}}\right)\sigma\\
&\quad + 2\overline{a_{22}}\sigma + \frac{2\left(2-\bar{t}^2\right)}{\bar{t}^2}\sigma,\\
\xi_n\eta_n^2: &\quad a_{32}\overline{\lambda} = \lambda a_{32} + 2a_{22}\sigma + 2\left(a_{21} + \overline{a_{20}}\right)\sigma\\
&\quad + 2\overline{a_{21}}\sigma + \frac{2\left(2-\bar{t}^2\right)}{\bar{t}^2}\sigma,\\
\eta_n^2: &\quad a_{22}\overline{\lambda}^2 = \lambda a_{22} + \sigma,\\
\eta_n^3: &\quad a_{33}\overline{\lambda}^3 = \lambda a_{33} + 2a_{22}\sigma + 2\overline{a_{20}}\sigma + \frac{2\left(2-\bar{t}^2\right)}{3\bar{t}^2}\sigma,
\end{aligned}$$

$$\begin{aligned}
\alpha_2 &= 2\left(a_{21} + \overline{a_{21}}\right)\sigma + 2\left(a_{20} + \overline{a_{22}}\right)\sigma + \frac{2\left(2-\bar{t}^2\right)}{\bar{t}^2}\sigma\\
&= 4Re\left(a_{21}\right)\sigma + 2\left(a_{20} + \overline{a_{22}}\right)\sigma + \frac{2\left(2-\bar{t}^2\right)}{\bar{t}^2}\sigma,\\
a_{22} &= \frac{\sigma}{\overline{\lambda}^2 - \lambda} = \frac{\alpha}{2\bar{t}^2\left(1-\alpha^2\right)(2\alpha+1)}\\
&\quad \left((2\alpha-1)(\alpha+1) + i\left(2\alpha+1\right)\sqrt{1-\alpha^2}\right)\\
a_{21} &= \frac{2\sigma}{1-\lambda},\\
\sigma &= \frac{\alpha}{\bar{t}^2}\left(1 - \alpha i\frac{\sqrt{1-\alpha^2}}{1-\alpha^2}\right).
\end{aligned}$$

Furthermore

$$\begin{aligned}
\overline{a_{22}} &= \frac{\alpha}{2\bar{t}^2\left(1-\alpha^2\right)(2\alpha+1)}\\
&\quad \left((2\alpha-1)(\alpha+1) - i\left(2\alpha+1\right)\sqrt{1-\alpha^2}\right),
\end{aligned}$$

$$a_{20} = \frac{\sigma}{\lambda(\lambda-1)} = \frac{-\alpha}{2\bar{t}^2\left(1-\alpha^2\right)}\left(\alpha + 1 - i\sqrt{-\alpha^2+1}\right)$$

$$a_{20} + \overline{a_{22}} = \frac{1}{\bar{t}^2}\frac{\alpha}{(2\alpha+1)(\alpha-1)}$$

$$\overline{a_{21}} = \frac{\alpha}{(1-\alpha)\,\bar{t}^2\sqrt{1-\alpha^2}}\left(\sqrt{1-\alpha^2}-i\,(1-\alpha)\right),$$

and

$$\begin{aligned}\alpha_2 &= \left(4Re\,(a_{21})+2\,(a_{20}+\overline{a_{22}})+\frac{2\,(2-\bar{t}^2)}{\bar{t}^2}\right)\sigma\\ &= -2\frac{3\alpha+2\bar{t}^2\alpha^2-\bar{t}^2\alpha-\bar{t}^2+2}{\bar{t}^2\,(2\alpha+1)\,(\alpha-1)}\frac{\alpha}{\bar{t}^2}\\ &\left(1-\alpha i\frac{\sqrt{1-\alpha^2}}{(1-\alpha^2)}\right).\end{aligned}$$

Thus

$$Re\,(\alpha_2) = -2\frac{3\alpha+2\bar{t}^2\alpha^2-\bar{t}^2\alpha-\bar{t}^2+2}{\bar{t}^2\,(2\alpha+1)\,(\alpha-1)}\frac{\alpha}{\bar{t}^2}.$$

Now for $\bar{t}=\dfrac{1}{\sqrt{1-\alpha}}$, $0<\alpha<1$, we obtain

$$Re\,(\alpha_2) = \frac{2\alpha\,(1-\alpha^2)}{2\alpha+1}.$$

2.3 Third Transformation

The objective of third transformation consists in expressing the terms in (2.8) as real values. This is achieved by using the transformation

$$\xi_n = r_n + is_n$$

$$\eta_n = r_n - is_n.$$

Comparing the system obtained with (1.6) and using (1.7) for $l=4$, we determine the twist coefficients γ_0 and γ_1. We have

$$\cos\gamma_0 = Re\,(\lambda) = \alpha \in (0,1) \quad \text{and} \quad \gamma_1 = -\frac{Re\,(\alpha_2)}{\sin\gamma_0}$$

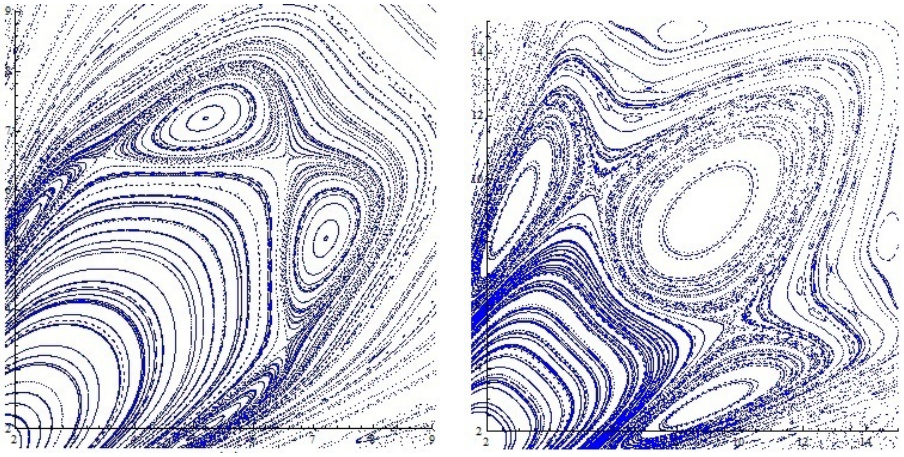
i.e.,

$$\begin{aligned}\cos\gamma_0 &= \alpha,\\ \gamma_1 &= -\frac{2\alpha\,(1-\alpha^2)}{(2\alpha+1)\,\sqrt{1-\alpha^2}}.\end{aligned}$$

Since $\alpha\in(0,1)$, this implies $\gamma_1\neq 0$.

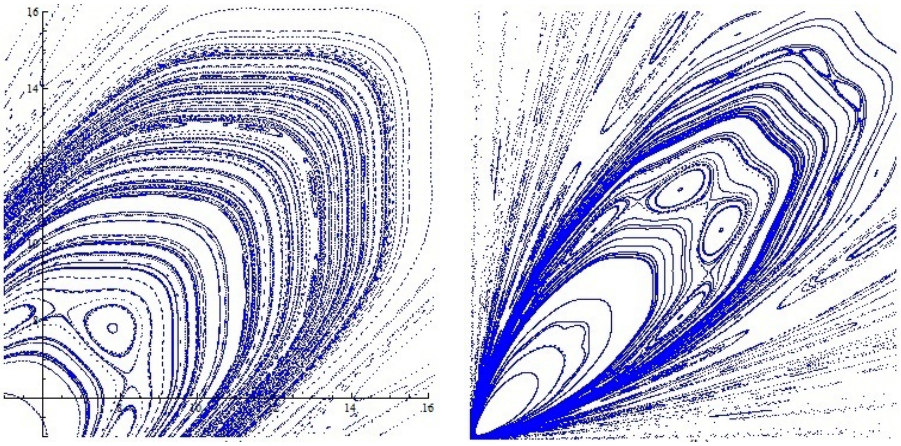
Thus we have proved the following result.

Theorem 2.1. *The positive equilibrium solution $\bar{t}$ of* (1.5) *is stable for* $\alpha\in(0,1)$.



(a) (b)

Figure 2.1: a) Some orbits of the map T for $\alpha = 0.3$. b) Some orbits of the map T for $\alpha = 0.5$.

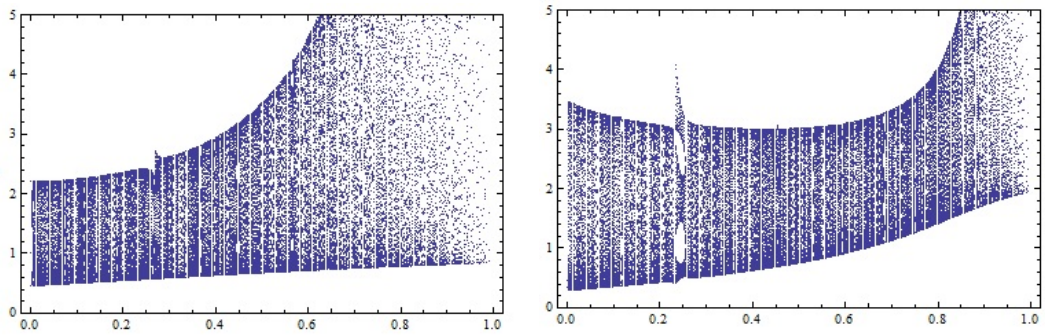


(a) (b)

Figure 2.2: a) Some orbits of the map T for $\alpha = 0.6$. b) Some orbits of the map T for $\alpha = 0.7$.

Figures 2.1–2.3 show phase portraits of the orbits of the map T associated with (1.5) for the values of parameters α equal $0.3, 0.5, 0.6, 0.7$ and the bifurcation diagrams. Neither of these two plots show any self-similarity character. We were not able to find any rational invariant of (1.5), by using software program such as *Dynamica*, [15] and in fact one can prove rigorously that a rational invariant of (1.5) does not exist, see [6]. The existence of transcendental invariant is not excluded but our simulation indicate that such invariant does not exist.

Remark 2.2. The eigenvalues λ and $\overline{\lambda}$ at the elliptic fixed point are of the form $\lambda = e^{i\theta}$ with $\theta = \arccos \alpha$ and $0 < \theta < \pi/2$ Thus the period of the motion around the fixed



(a) (b)

Figure 2.3: Bifurcation diagram for map T.

point must be $q > 2\pi/\theta = 4$; so, in general, the map T cannot have an orbit of period less than or equal to 4 in a neighborhood of the elliptic fixed point $(0, 0)$. If $\alpha = 0.8$, for example, $2\pi/\theta \approx 9.76406$; so the minimal possible period for a periodic orbit in a neighborhood of the elliptic fixed point is 10.

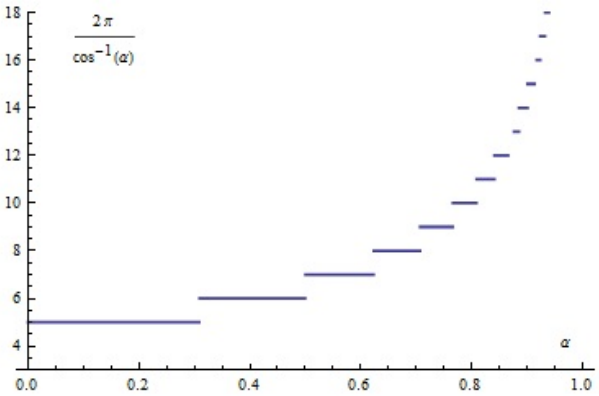


Figure 2.4: Minimal possible period for a periodic orbit in a neighbourhood of the elliptic fixed point $(0, 0)$ for the map T.

The proof of this statement is straightforward and uses the Birkhoff normal form. However, the proof only apply in a small neighborhood of the elliptic fixed point, and thus do not show that smaller period orbits cannot exist outside of this small neighbourhood.

Remark 2.3. In the special case when $\alpha = 1$, (1.5) admits an invariant of the form

$$I(x_n, x_{n-1}) = \frac{1}{x_n x_{n-1}} + \frac{x_n}{x_{n-1}} + \frac{x_{n-1}}{x_n}, \quad n = 0, 1, \ldots, \tag{2.10}$$

which implies that

$$I(x_n, x_{n-1}) = I(x_0, x_{-1}) = I_0, \quad n = 0, 1, \ldots,$$

see [15, pp. 252–256] and [10].

The corresponding curve $I(x,y) = I_0$ can be rewritten as

$$x^2 + y^2 - I_0 xy + 1 = 0,$$

which after the orthonormal substitution $x = \frac{1}{\sqrt{2}}(u+v), y = \frac{1}{\sqrt{2}}(u-v)$ takes the form

$$\left(1 + \frac{I_0}{2}\right) v^2 - \left(\frac{I_0}{2} - 1\right) u^2 + 1 = 0,$$

which is the graph of a hyperbola. Indeed, $I_0 > 2$ since

$$\frac{1 + x_0^2 + x_{-1}^2}{x_0 x_{-1}} > 2 \iff 1 + (x_0 - x_{-1})^2 > 0.$$

Remark 2.4. In the special case when $\alpha > 1$ one can show that every solution $\{x_n\}_{n=-1}^{\infty}$ of (1.5) satisfies

$$\lim_{n\to\infty} x_n = \infty.$$

Indeed in this case (1.5) implies $\frac{x_{n+1}}{x_n} > \alpha \frac{x_n}{x_{n-1}}$ for every n and so $\frac{x_{n+1}}{x_n} > \alpha^{n+1} \frac{x_0}{x_{-1}} = L\alpha^{n+1}$, where $\frac{x_0}{x_{-1}} = L$. This implies $x_n > L\alpha^n x_{n-1}, n = 1, 2, \ldots$ and consequently

$$x_n > L^n \alpha^{\frac{n(n+1)}{2}} x_0 = \left(L\alpha^{\frac{(n+1)}{2}}\right)^n x_0, \quad n = 1, 2, \ldots.$$

Now, we have $L\alpha^{\frac{(n+1)}{2}} > 1 \iff x_{-1} < \alpha^{\frac{(n+1)}{2}} x_0$ for n large enough, which yields $\lim_{n\to\infty} x_n = \infty$.

3 Symmetries

In the study of area-preserving maps, symmetries play an important role since they yield special dynamic behavior. A transformation R of the plane is said to be a *time reversal symmetry* for T if $R^{-1} \circ T \circ R = T^{-1}$, meaning that applying the transformation R to the map T is equivalent to iterating the map backwards in time, see [8, 9]. If the time reversal symmetry R is an involution, i.e., $R^2 = id$, then the time reversal symmetry condition is equivalent to $R \circ T \circ R = T^{-1}$, and T can be written as the composition of two involutions $T = I_1 \circ I_0$, with $I_0 = R$ and $I_1 = T \circ R$. Note that if $I_0 = R$ is a reversor, then so is $I_1 = T \circ R$. Also, the jth involution, defined as $I_j := T^j \circ R$, is also a reversor.

The invariant sets of the involution maps,

$$S_{0,1} = \{(x,y) | I_{0,1}(x,y) = (x,y)\},$$

are one-dimensional sets called the symmetry lines of the map. Once the sets $S_{0,1}$ are known, the search for periodic orbits can be reduced to a one-dimensional root finding problem using the following result, see [8, 9].

Theorem 3.1. *If $(x, y) \in S_{0,1}$, then $T^n(x, y) = (x, y)$ if and only if*

$$\begin{cases} T^{n/2}(x, y) \in S_{0,1}, & \text{for } n \text{ even;} \\ T^{(n\pm1)/2}(x, y) \in S_{1,0}, & \text{for } n \text{ odd.} \end{cases}$$

That is, according to this result, periodic orbits can be found by searching in the one-dimensional sets $S_{0,1}$, rather than in the whole domain. Periodic orbits of different orders can then be found at the intersection of the symmetry lines S_j $j = 1, 2, \ldots$ associated to the jth involution; for example, if $(x, y) \in S_j \cap S_k$, then $T^{j-k}(x, y) = (x, y)$. Also the symmetry lines are related to each other by the following relations: $S_{2j+i} = T^j(S_i), \quad S_{2j-i} = I_j(S_i),$ for all i, j. For example, for $\alpha = 0.3$, in Fig.3.2, we have an intersection between the symmetry lines S_0 and $S_{10} = T^2(S_0)$, $S_4 = T^2(S_0)$ and $S_{10} = T^5(S_0)$ and S_0 and $S_{16} = T^8(S_0)$ of the map T. The intersection points of this lines correspond to the periodic orbits of period 10, 6 and 16 respectively.

See Figure 3.1 for the first nine iterations of the symmetry lines S_0 and S_1 for $\alpha = 0.3$. See Table 3 and Figure 3.2 for numerical examples of periodic orbits of periods $5, 6, 8$ and 16.

For $0 < \alpha < 1$, we use the substitution $x_n = t_n$ and $y_n = t_{n-1}$ to transform (1.5) into

$$\begin{aligned} x_{n+1} &= \frac{\alpha x_n^2 + 1}{y_n} \\ y_{n+1} &= x_n. \end{aligned} \tag{3.1}$$

The map T associated to the system (3.1) is

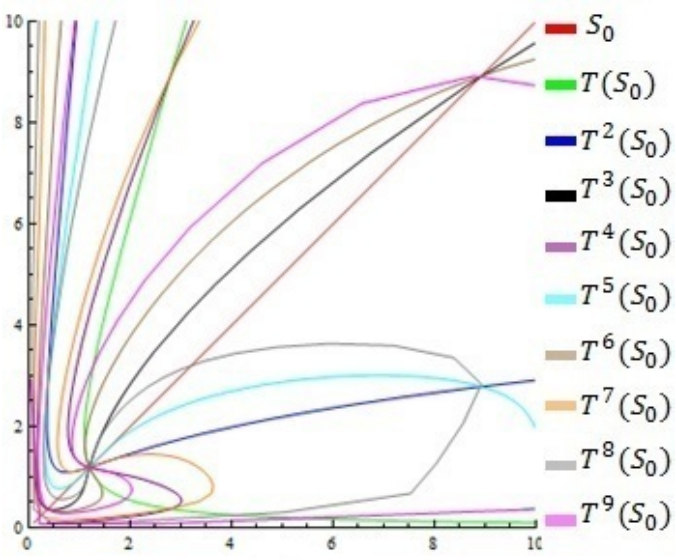
$$T(x, y) = \left(\frac{\alpha x^2 + 1}{y}, x \right)$$

which is defined on the positive quadrant Q in $\mathbb{R}^2$. The inverse of the map (3.1) is the map

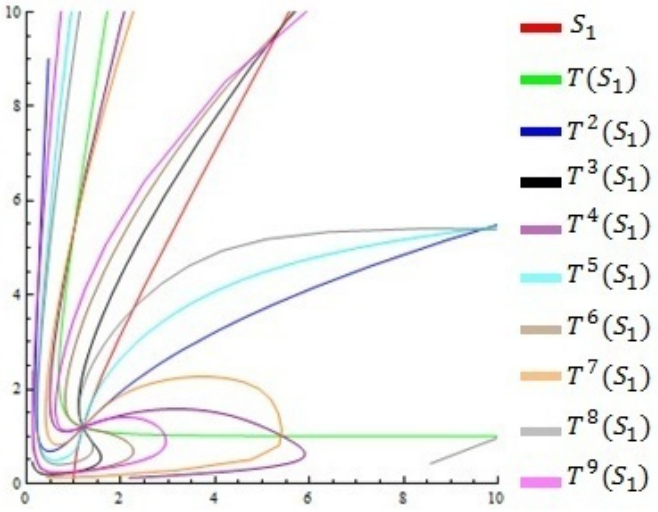
$$T^{-1}(x, y) = \left(y, \frac{\alpha y^2 + 1}{x} \right).$$

The involution $R(x, y) = (y, x)$ is a reversor for (3.1). Indeed,

$$\begin{aligned} (R \circ T \circ R)(x, y) = (R \circ T)(y, x) &= R\left(\frac{\alpha y^2 + 1}{x}, y \right) \\ &= \left(y, \frac{\alpha y^2 + 1}{x} \right) = T^{-1}(x, y). \end{aligned}$$



(a)



(b)

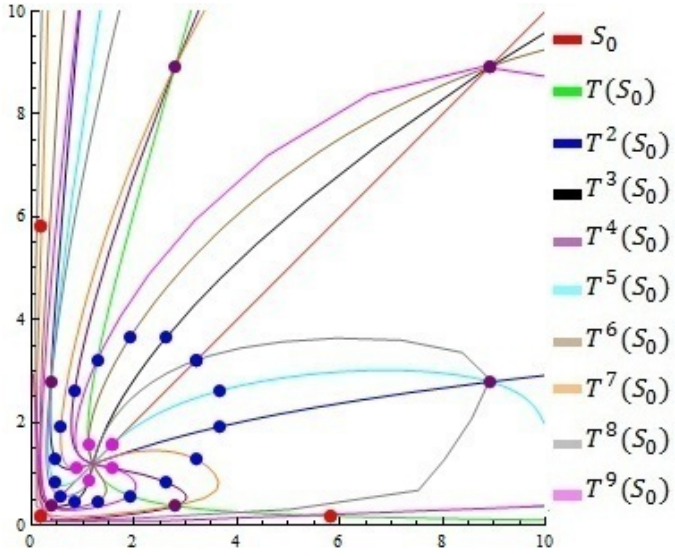
Figure 3.1: a) The first nine iterations of the symmetry line S_0 for $\alpha = 0.3$. b) The first nine iterations of the symmetry line S_1 for $\alpha = 0.3$.

Thus $T = I_1 \circ I_0$ where $I_0(x, y) = R(x, y) = (y, x)$ and

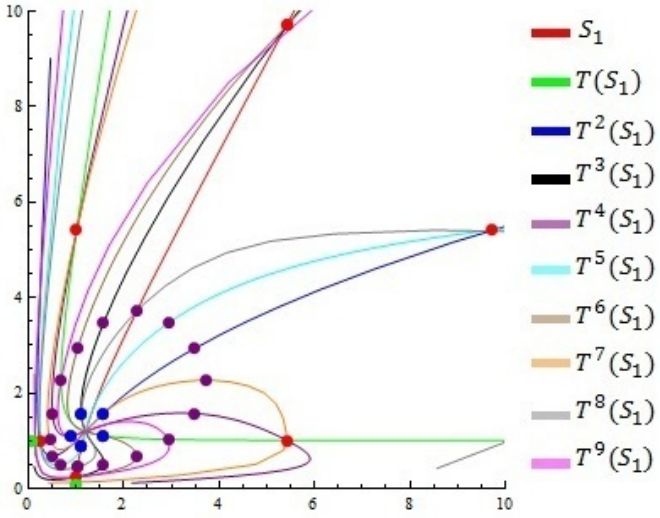
$$I_1(x, y) = T \circ R = \left(\frac{\alpha y^2 + 1}{x}, y \right).$$

The symmetry lines corresponding to I_0 and I_1 are

$$S_0 = \{(x, y) : x = y\},\ S_1 = \{(x, y) : x^2 = 1 + \alpha y^2\}.$$



(a)



(b)

Figure 3.2: a) The periodic orbits of period 6 (red), 5 (blue), 8 (green), and 16 (purple) for $\alpha = 0.3$. b) The periodic orbits of period 15 (red), 6 (purple), 16 (blue), and 5 (magenta) for $\alpha = 0.3$.

Periodic orbits on the symmetry line S_0 with even period n are searched for by starting with points $(x_0, x_0) \in S_0$ and imposing that $(x_{n/2}, y_{n/2}) \in S_0$, where

$$(x_{n/2}, y_{n/2}) = T^{n/2}(x_0, x_0).$$

This reduces to a one-dimensional root finding for the equation $x_{n/2} = y_{n/2}$, where the unknown is x_0. Also, periodic orbits on S_0 with odd period n are obtained by solving

Period	Solution
6	$\{(5.41784, 9.72162), (1.00867, 5.41784), (0.240912, 1.00867), (1.00867, 0.240912), (5.41784, 1.00867), (9.72162, 5.41784)\}$
5	$\{(1.10753, 0.869131), (1.57397, 1.10753), (1.57397, 1.57397), (1.10753, 1.57397), (0.869131, 1.10753)\}$
8	$\{(1.00063, 0.0646056), (20.1279, 1.00063), (122.463, 20.1279), (223.579, 122.463), (122.463, 223.579), (20.1279, 122.463), (1.00063, 20.1279), (0.0646056, 1.00063)\}$
16	$\{(1.03, 0.450577), (2.92574, 1.03), (3.46406, 2.92574), (1.57222, 3.46406), (0.502752, 1.57222), (0.684273, 0.502752), (2.26845, 0.684273), (3.71747, 2.26845), (2.26845, 3.71747), (0.684273, 2.26845), (0.502752, 0.684273), (1.57222, 0.502752), (3.46406, 1.57222), (2.92574, 3.46406), (1.03, 2.92574), (0.450577, 1.03)\}$

Table 1: Some periodic solutions for $\alpha = 0.3$.

for x_0 the equation $x^2_{(n+1)/2} = 1 + \alpha y^2_{(n+1)/2}$, where

$$(x_{(n+1)/2}, y_{(n+1)/2}) = T^{(n+1)/2}(x_0, x_0).$$

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