Extremes
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Adapting the Hill estimator to distributed inference: dealing with the bias

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Abstract
The distributed Hill estimator is a divide-and-conquer algorithm for estimating the extreme value index when data are stored in multiple machines. In applications, estimates based on the distributed Hill estimator can be sensitive to the choice of the number of the exceedance ratios used in each machine. Even when choosing the number at a low level, a high asymptotic bias may arise. We overcome this potential drawback by designing a bias correction procedure for the distributed Hill estimator, which adheres to the setup of distributed inference. The asymptotically unbiased distributed estimator we obtained, on the one hand, is applicable to distributed stored data, on the other hand, inherits all known advantages of bias correction methods in extreme value statistics.

Keywords Extreme value index · Distributed inference · Bias correction

1 Introduction

Consider a distribution function F which belongs to the maximum domain of attraction of an extreme value distribution with a positive *extreme value index* $\gamma > 0$, that is,

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$$\lim_{t\to\infty}\frac{U(tx)}{U(t)} = x^{\gamma},\ x > 0,$$

where $U(t) := F^{\leftarrow}(1 - 1/t)$ with $t > 1$, and $^{\leftarrow}$ denotes the left-continuous inverse function. Such a distribution is also called a heavy-tailed distribution, where the extreme value index governs the tail of the distribution. Estimating the extreme value index is a key step for making statistical inference on the tail behaviour of F. Various methods have been proposed to estimate the extreme value index, such as the Hill estimator (Hill 1975), the maximum likelihood estimator (Smith 1987; Drees et al. 2004; Zhou 2009) and the moment estimator (Dekkers et al. 1989).

Conducting extreme value analysis often requires large datasets in order to select extreme observations in the tail. Such datasets may be stored in multiple machines and cannot be combined into one dataset due to data privacy issue. For example, datasets collected in industries such as banking and healthcare require high level consumer privacy and cannot be shared across different organizations. Another potential situation is that some massive datasets cannot be processed by a single computer due to internet traffic or memory constraints. *Distributed inference* refers to the statistical problem of analyzing data stored in multiple machines. It often requires a divide-and-conquer (DC) algorithm. In a DC algorithm, one calculates statistical estimators on each machine in parallel and then communicates them to a central machine. The final estimator is obtained on the central machine, often by a simple average; see, for example, Li et al. (2013) for kernel density estimation, Fan et al. (2019) for principal component analysis, Volgushev et al. (2019) for quantile regression.

In this paper, we aim at estimating the extreme value index in the distributed inference context. Assume that independent and identically distributed (i.i.d.) observations $X_1, \dots, X_N$ drawn from F are stored in m machines with n observations on each machine, i.e. $N = mn$. In the context of distributed inference, we assume that only limited (finite) number of results can be transmitted from each machine to the central machine. As a result, we cannot apply statistical procedures to the oracle sample, i.e., the hypothetically combined dataset $\left\{X_1, \dots, X_N\right\}$.

Chen et al. (2021) proposes the distributed Hill estimator to estimate the extreme value index γ. On each machine, the Hill estimator is applied and then transmitted to the central machine. On the central machine, the average of the Hill estimates collected from the m machines are calculated. Let $M_j^{(1)} \geq \cdots \geq M_j^{(n)}$ denote the order statistics of the observations on machine j for $j = 1, \dots, m$. Then the Hill estimator on machine j can be constructed by using the top k exceedance ratios $M_j^{(i)}/M_j^{(k+1)}, i = 1, \dots, k$, as

$$\hat{\gamma}_{j,k} = \frac{1}{k}\sum_{i=1}^{k}\left(\log M_j^{(i)} - \log M_j^{(k+1)}\right), \quad j = 1, \dots, m.$$

The distributed Hill estimator is defined as

$$\hat{\gamma}_{DH,k} := \frac{1}{m}\sum_{j=1}^{m}\hat{\gamma}_{j,k} = \frac{1}{m}\sum_{j=1}^{m}\frac{1}{k}\sum_{i=1}^{k}\left(\log M_j^{(i)} - \log M_j^{(k+1)}\right).$$

Chen et al. (2021) studies the asymptotic behaviour of the distributed Hill estimator and shows sufficient conditions under which the distributed Hill estimator possesses the oracle property: its speed of convergence and asymptotic distribution coincides with the oracle Hill estimator. Here, the oracle Hill estimator is the Hill estimator using the top km exceedance ratios of the oracle sample $\{X_1, \dots, X_N\}$, i.e. $\hat{\gamma} = l^{-1}\sum_{i=1}^{l}\left(\log M^{(i)} - \log M^{(l+1)}\right)$, where $l = km$ and $M^{(1)} \geq \cdots \geq M^{(N)}$ are the order statistics of the oracle sample $\{X_1, \dots, X_N\}$. The choice of $l = km$ is in line with the standard distributed inference literature. Note that the oracle property compares the distributed estimator to the oracle estimator when the two estimators are constructed based on the same sample size. Different from standard statistics, extreme value statistics use observations in the tail only, for example, the Hill estimator is based on the exceedance ratios. Therefore, the oracle property for the Hill estimator is meaningful only if the distributed estimator and the oracle estimator are constructed based on the same number of exceedance ratios.

In applications with finite sample size, one important tuning parameter in the Hill estimator is the number of exceedance ratios l used in the estimation. Recall that the maximum domain of attraction condition is a limiting relation instead of an exact model, it provides only an approximation to the tail of a distribution. Consequently, the number of exceedance ratios used in the estimation, l, is related to the asymptotic bias in the limit distribution of the Hill estimator. This differs from classical statistics where bias often vanishes sufficiently fast as sample size tending to infinity. More specifically, the choice leads to a bias-variance tradeoff: with a low level of l, the estimation variance is at a high level; by increasing the level of l, the estimation variance is reduced but the estimation bias may arise. For the distributed Hill estimator $\hat{\gamma}_{DH,k}$, this issue is regarding the choice of k on each machine. One needs to balance the number of exceedance ratios (k) with the number of machines (m), in order to control the total bias in the distributed estimator. In addition, recall that the effective number of exceedance ratios involved in $\hat{\gamma}_{DH,k}$ is km. As k increases by 1, the effective number of exceedance ratios will increase by m. Thus, the performance of $\hat{\gamma}_{DH,k}$ is very sensitive to the choice of k. If m is large, with even a low level of k, the asymptotic bias may be at a high level which may not be acceptable in applications.

In existing extreme value statistics literature, there are two types of solutions for selecting the number of exceedance ratios in the estimation. The first stream of literature aims at finding the optimal level that balances the asymptotic bias and variance, see e.g. Danielsson et al. (2001) and Guillou and Hall (2001). The second stream of literature corrects the bias and eventually allows for choosing a high level of the number of exceedance ratios, see e.g. Gomes et al. (2008) and de Haan et al. (2016). In applications, if the sample size is large, the bias correction methods are preferred since they possess at least two advantages. First, bias correction methods allow for choosing a higher level of the number of exceedance ratios than that used for the original estimator, which results in also a lower level of variance. Second, bias correction methods lead to estimates that are less sensitive to the choice of the number of exceedance ratios.

In this paper, we shall adapt the distributed Hill estimator such that it is suitable for finite sample applications. More specifically, we introduce a bias correction procedure for estimating the extreme value index, without compromising the distributed inference setup. Notice that existing bias correction methods often rely on estimating a second order parameter and a second order scale function as given in (1) below. Such an estimation again requires the oracle sample which is infeasible in the context of distributed inference. Therefore, we resort to a different approach, sticking to the requirement that only limited (fixed) number of results can be transmitted from each machine to the central machine. In such a way, the resulting estimator is not only asymptotically unbiased, but also in the same spirit of a DC algorithm. We name it as "asymptotically unbiased distributed estimator" for the extreme value index. The asymptotically unbiased distributed estimator, on the one hand, is applicable to distributed stored data, on the other hand, inherits the advantages of bias correction methods in extreme value statistics.

We remark that the requirement of transmitting limited (fixed) number of results from each machine to the central machine is in line with the privacy concern in practice. Consider a practical example where various insurance companies would not like to share their client level claim data, but would nevertheless be willing to collaborate with each other such that they can obtain a more accurate estimation for the tail risk of a certain type of insurance claims. They are willing to share some estimation results provided that other companies cannot infer client level data from the shared results. Given the sensitivity of the data, insurance companies would like to share as few results as possible. The less results transmitted and shared, the less likely that client level data can be recovered. In the proposed asymptotically unbiased distributed estimator, we require that each machine transmit *five* results to the central machine. We nevertheless consider other alternatives when further limitations on the number of results transmitted are imposed. We compare their performance by an extensive simulation study.

The rest of the paper is organized as follows. Section 2 presents the idea for bias correction. Section 3 proposes a DC algorithm for estimating the second order parameter, defines the asymptotically unbiased distributed estimator for the extreme value index and shows the main theoretical results. Section 4 provides a simulation study to confirm that the asymptotically unbiased distributed estimator exhibits superior performance compared to the distributed Hill estimator. We discuss some extensions of our results in Sect. 5. The proofs are given in the Appendix.

Throughout the paper, $a(t) \asymp b(t)$ means that both $|a(t)/b(t)|$ and $|b(t)/a(t)|$ are $O(1)$ as $t \to \infty$.

2 Bias correction methodology

To obtain the asymptotic normality of the distributed Hill estimator $\hat{\gamma}_{DH,k}$, Chen et al. (2021) assumes the following second order condition. Suppose that there exist an eventually positive or negative function A with $\lim_{t\to\infty} A(t) = 0$ and a real number $\rho \le 0$ such that

$$\lim_{t\to\infty} \frac{\frac{U(tx)}{U(t)} - x^\gamma}{A(t)} = x^\gamma \frac{x^\rho - 1}{\rho},$$

for all $x > 0$, which is equivalent to

$$\lim_{t\to\infty} \frac{\log U(tx) - \log U(t) - \gamma \log x}{A(t)} = \frac{x^\rho - 1}{\rho}. \tag{1}$$

In addition, assume that as $N \to \infty$,

$$m = m(N) \to \infty, \quad n = n(N) \to \infty, \quad n/\log m \to \infty, \tag{2}$$

and k is either a fixed integer or an intermediate sequence, i.e. $k = k(N) \to \infty$, $k/n \to 0$. Under conditions (1) and (2), Chen et al. (2021) shows that the distributed Hill estimator possesses the following asymptotic expansion:

$$\hat{\gamma}_{DH,k} - \gamma = \frac{\gamma P_N}{\sqrt{km}} + \frac{A(n/k)}{1-\rho} g(k, n, \rho) + \frac{1}{\sqrt{km}} o_P(1),$$

where $P_N \sim N(0, 1)$ and

$$g(k, n, \rho) := \left(\frac{k}{n}\right)^\rho \frac{\Gamma(n+1)\Gamma(k-\rho+1)}{\Gamma(n-\rho+1)\Gamma(k+1)}, \tag{3}$$

with Γ denoting the gamma function. By Lemma 2 (see below), we have that, if k is a fixed integer, then $g(k, n, \rho) \to k^\rho \Gamma(k-\rho+1)/\Gamma(k+1)$, as $N \to \infty$. If k is an intermediate sequence, then $g(k, n, \rho) \to 1$, as $N \to \infty$.

Since the bias term of the distributed Hill estimator is an explicit function $(1-\rho)^{-1}A(n/k)g(k, n, \rho)$, we shall estimate the bias, subtract it from the original distributed Hill estimator, which leads to the asymptotically unbiased distributed estimator.

The estimation of the bias term requires estimating the second order parameter ρ and the second order scale function A in condition (1). For simplicity, we follow the bias correction literature to assume that $\rho < 0$, see e.g. de Haan et al. (2016) and Gomes and Pestana (2007). In order to obtain the asymptotic behavior of the estimator for ρ, a third order condition is often assumed. We invoke the third order condition in Alves et al. (2003) as follows. Suppose that there exist an eventually positive or negative function B with $\lim_{t\to\infty} B(t) = 0$ and a real number $\tilde{\rho} \le 0$ such that

$$\lim_{t\to\infty} \frac{1}{B(t)} \left\{ \frac{\log U(tx) - \log U(t) - \gamma \log x}{A(t)} - \frac{x^\rho - 1}{\rho} \right\} = \frac{1}{\tilde{\rho}} \left(\frac{x^{\rho+\tilde{\rho}} - 1}{\rho + \tilde{\rho}} - \frac{x^\rho - 1}{\rho} \right). \tag{4}$$

Lastly, following Cai et al. (2012) and de Haan et al. (2016), we use a higher intermediate sequence k_ρ for estimating the second order parameter ρ. Assume that as $N \to \infty$, $k_\rho = k_\rho(N) \to \infty$, $k_\rho/n \to 0$, and

$$\sqrt{k_\rho m} A(n/k_\rho) \to \infty, \sqrt{k_\rho m} A^2(n/k_\rho) \to \lambda_1 \in \mathbb{R}, \sqrt{k_\rho m} A(n/k_\rho) B(n/k_\rho) \to \lambda_2 \in \mathbb{R}. \tag{5}$$

Similar to de Haan et al. (2016), in the eventual asymptotically unbiased distributed estimator for the extreme value index, one can choose a higher number of exceedance ratios than that used in the distributed Hill estimator. In our context, we choose a sequence k_n such that, as $N \to \infty$, $k_n/k_\rho \to 0$ and

$$\sqrt{k_n m}A(n/k_n) \to \infty, \sqrt{k_n m}A^2(n/k_n) \to 0, \sqrt{k_n m}A(n/k_n)B(n/k_n) \to 0. \tag{6}$$

Here, similar to the distributed Hill estimator, k_n can be either a fixed integer or an intermediate sequence.

3 Main results

We first introduce the estimator for the second order parameter ρ in the distributed inference setup and study its asymptotic behavior. Then we define the asymptotically unbiased distributed estimator for the extreme value index and show its asymptotic behavior.

3.1 Estimating the second order parameter

If the oracle sample can be used, then there are several estimators for the second order parameter ρ, see e.g. Alves et al. (2003) and Gomes et al. (2002). However, since we cannot apply a statistical procedure to the oracle sample, we need to develop a DC algorithm for estimating ρ. Consider the following statistics computed based on observations on machine j,

$$R_{j,k}^{(\alpha)} := \frac{1}{k}\sum_{i=1}^{k}\left\{ \log M_j^{(i)} - \log M_j^{(k+1)} \right\}^\alpha, \quad \alpha = 1, 2, 3.$$

We request that each machine sends the values $R_{j,k}^{(\alpha)}, \alpha = 1, 2, 3$ to the central machine. On the central machine, we take the average of the $R_{j,k}^{(\alpha)}$ statistics to obtain

$$R_k^{(\alpha)} = \frac{1}{m}\sum_{j=1}^{m} R_{j,k}^{(\alpha)}, \quad \alpha = 1, 2, 3.$$

Motivated by Alves et al. (2003), we define the estimator for the second order parameter ρ as

$$\hat{\rho}_{k,\tau} := -3\left|\frac{T_{k,\tau} - 1}{T_{k,\tau} - 3}\right|, \tag{7}$$

where

$$T_{k,\tau} := \frac{\left(R_k^{(1)}\right)^{\tau} - \left(R_k^{(2)}/2\right)^{\tau/2}}{\left(R_k^{(2)}/2\right)^{\tau/2} - \left(R_k^{(3)}/6\right)^{\tau/3}},$$

and $\tau \geq 0$ is a tuning parameter. For $\tau = 0$, $T_{k,\tau}$ is defined by continuity. In practice, it is suggested to choose $\tau \in [0, 1]$, see e.g. Gomes and Pestana (2007) and Gomes et al. (2008).

Before studying the asymptotics of $\hat{\rho}_{k,\tau}$, we first establish that for $R_k^{(\alpha)}$ in the following proposition. Note that in this proposition, we use a general sequence k. Nevertheless, the proposition will be applied both for $k = k_n$ and $k = k_\rho$, see Sect. 3.2.

Proposition 1 *Assume that the distribution function F satisfies the third order condition (4) with parameters $\gamma > 0, \rho < 0$ and $\tilde{\rho} \leq 0$, and condition (2) holds. In addition, suppose that an intermediate sequence k satisfies that as $N \to \infty$, $k/n \to 0$ and $\sqrt{kmA(n/k)B(n/k)} = O(1)$, $\sqrt{km}A^2(n/k) = O(1)$. Then for suitable versions of the functions A and B, denoted as A_0 and B_0 (see Lemma 4 below), we have that as $N \to \infty$,*

(i)

$$\sqrt{km}\left(R_k^{(1)} - \gamma\right) - \gamma P_N^{(1)} - \frac{g(k,n,\rho)}{1-\rho}\sqrt{km}A_0(n/k) - \frac{g(k,n,\rho+\tilde{\rho})}{1-\rho-\tilde{\rho}}$$
$$\sqrt{km}A_0(n/k)B_0(n/k) = o_p(1),$$

(ii)

$$\begin{aligned}\sqrt{km}\left(R_k^{(2)} - 2\gamma^2\right) &- \gamma^2 P_N^{(2)} - 2\gamma\sqrt{km}A_0(n/k)\frac{g(k,n,\rho)}{\rho}\left\{\frac{1}{(1-\rho)^2} - 1\right\}\\ &- \sqrt{km}A_0^2(n/k)\frac{g(k,n,2\rho)}{\rho^2}\left(\frac{1}{1-2\rho} - \frac{2}{1-\rho} + 1\right)\\ &- 2\gamma\sqrt{km}A_0(n/k)B_0(n/k)\frac{g(k,n,\rho+\tilde{\rho})}{\rho+\tilde{\rho}}\left\{\frac{1}{(1-\rho-\tilde{\rho})^2} - 1\right\} = o_p(1),\end{aligned}$$

(iii)

$$\begin{aligned}\sqrt{km}\left(R_k^{(3)} - 6\gamma^3\right) &- \gamma^3 P_N^{(3)} - 6\gamma^2\sqrt{km}A_0(n/k)\frac{g(k,n,\rho)}{\rho}\left\{\frac{1}{(1-\rho)^3} - 1\right\}\\ &- 3\gamma\sqrt{km}A_0^2(n/k)\frac{g(k,n,2\rho)}{\rho^2}\left\{\frac{1}{(1-2\rho)^2} - \frac{2}{(1-\rho)^2} + 1\right\}\\ &- 6\gamma^2\sqrt{km}A_0(n/k)B_0(n/k)\frac{g(k,n,\rho+\tilde{\rho})}{\rho+\tilde{\rho}}\left\{\frac{1}{(1-\rho-\tilde{\rho})^3} - 1\right\} = o_P(1),\end{aligned}$$

where $(P_N^{(1)}, P_N^{(2)}, P_N^{(3)})^T \sim N(\mathbf{0}, \mathbf{\Sigma})$ with

$$\boldsymbol{\Sigma} = \begin{pmatrix} 1 & 4 & 18 \\ 4 & 20 & 98 \\ 18 & 98 & 684 \end{pmatrix}.$$

Applying Proposition 1 leads to the asymptotic behavior of $\hat{\rho}_{k,\tau}$ as follows.

Theorem 1 *Assume that the distribution function F satisfies the third order condition (4) with parameters $\gamma > 0, \rho < 0$ and $\tilde{\rho} \leq 0$, and condition (2) holds. Suppose that the intermediate sequence k_ρ satisfies condition (5). Then as $N \to \infty$, for each $\tau \geq 0$,*

$$\sqrt{k_\rho m} A_0(n/k_\rho)(\hat{\rho}_{k_\rho,\tau} - \rho) = O_P(1),$$

where $\hat{\rho}_{k_\rho,\tau}$ is defined in (7).

3.2 Asymptotically unbiased distributed estimator for the extreme value index

Motived by de Haan et al. (2016), we define the following estimator as the asymptotically unbiased distributed estimator for the extreme value index:

$$\tilde{\gamma}_{k_n,k_\rho,\tau} := R_{k_n}^{(1)} - \frac{R_{k_n}^{(2)} - 2\left(R_{k_n}^{(1)}\right)^2}{2R_{k_n}^{(1)} \hat{\rho}_{k_\rho,\tau}(1 - \hat{\rho}_{k_\rho,\tau})^{-1}}, \tag{8}$$

where $\tau \geq 0$ is a tuning parameter. Notice that the estimator $\tilde{\gamma}_{k_n,k_\rho,\tau}$ in (8) adheres to a DC algorithm since each machine only sends five values $\left\{ R_{j,k_n}^{(1)}, R_{j,k_n}^{(2)}, R_{j,k_\rho}^{(1)}, R_{j,k_\rho}^{(2)}, R_{j,k_\rho}^{(3)} \right\}$ to the central machine.

Remark 1 The statistic $R_{k_n}^{(1)}$ is the original distributed Hill estimator $\hat{\gamma}_{DH,k_n}$.

The following theorem shows the asymptotic normality of the asymptotically unbiased distributed estimator.

Theorem 2 *Assume that the distribution function F satisfies the third order condition (4) with parameters $\gamma > 0, \rho < 0$ and $\tilde{\rho} \leq 0$, and condition (2) holds. Suppose that k_ρ, k_n satisfy conditions (5) and (6) respectively. Then as $N \to \infty$, for each $\tau \geq 0$,*

$$\sqrt{k_n m}\left(\tilde{\gamma}_{k_n,k_\rho,\tau} - \gamma\right) \xrightarrow{d} N\left[0, \gamma^2\left\{1 + \left(\rho^{-1} - 1\right)^2\right\}\right].$$

Remark 2 We investigate the conditions in Theorem 2 to determine the range of m (and k) such that the oracle property holds. The last statement in Condition (2), $n/\log m \to \infty$ as $N \to \infty$, provides an upper bound for m as $m = o(N/\log N)$ as

$N \to \infty$. Condition (6) leads to an upper bound for $k_n m$: based on the second order condition (1), we need to have $k_n m = O(N^{\xi})$ with $\xi < 1$. Clearly, for the number of machine m, the second upper bound is a stricter requirement than the first.

Remark 3 The limit distribution in Theorem 2 is the same as that of the bias corrected Hill estimator based on the oracle sample, see for example de Haan et al. (2016). In other words, the asymptotically unbiased distributed estimator achieves the oracle property regardless whether k_n is a fixed integer or an intermediate sequence. Chen et al. (2021) shows that when k_n is a fixed integer, the distributed Hill estimator may possess a higher bias than that of the oracle Hill estimator. Consequently, the distributed Hill estimator achieves the oracle property only if additional conditions are assumed, see Corollary 1 therein. If the additional conditions fail, the violation of the oracle property is due to the difference in the asymptotic biases of the two estimators. By contrast, the asymptotically unbiased distributed estimator achieves the oracle property without any additional assumption when k_n is a fixed integer. This is due to the fact that the asymptotic bias was corrected.

Nevertheless, if Condition (6) is violated in the following sense: as $N \to \infty$, $\sqrt{k_n m} A^2(n/k_n) \to \lambda_3$ and $\sqrt{k_n m} A(n/k_n) B(n/k_n) \to \lambda_4$ where $\lambda_3 \neq 0$ or $\lambda_4 \neq 0$, then the oracle bias corrected estimator will possess a non-zero asymptotic bias. In this case, the asymptotically unbiased distributed estimator may not possess the oracle property.

Remark 4 We investigate the optimal choice for k_n in terms of the level of the asymptotic root mean squared error (RMSE). We first consider the asymptotically unbiased distributed estimator. To simplify the discussion, we focus on the case $A(t) \asymp t^{\rho}, B(t) \asymp t^{\tilde{\rho}}$ as $t \to \infty$. The best attainable rate of convergence is achieved when squared bias and variance are of the same order, that is, when

$$\frac{1}{\sqrt{k_n m}} \asymp A(n/k_n)\left\{A(n/k_n) + B(n/k_n)\right\},$$

as $N \to \infty$. Solving k_n yields that $k_n^{DC} \asymp N^{-2\rho^*/(1-2\rho^*)} m^{-1}$ as $N \to \infty$, where $\rho^* = \rho + \max(\rho, \tilde{\rho})$.

Similarly, we obtain the optimal choice of k_n in a single machine as $k_n^{Single} \asymp n^{-2\rho^*/(1-2\rho^*)}$. Note that, as $N \to \infty$, $k_n^{DC}/k_n^{Single} \asymp m^{-1/(1-2\rho^*)} \to 0$. We conclude that the two optimal choices do not match each other: the optimal choice of k_n at each individual machine is too high for optimally using the asymptotically unbiased distributed estimator. In practice, for example, in the insurance claim example, to make use of the asymptotically unbiased distributed estimator, one needs to coordinate the choice of k_n at all insurance companies instead of allowing each insurance company to choose the optimal level of k_n based on their own data.

4 Simulation study

4.1 Comparison with the original distributed Hill estimator

In this subsection, we conduct a simulation study to demonstrate the finite sample performance of the asymptotically unbiased distributed estimator for the extreme value index. Data are simulated from three distributions: the Fréchet distribution, $F(x) = \exp\left(-x^{-1}\right), x > 0$; the Burr distribution, $F(x) = 1 - (1 + x^{1/2})^{-2}, x > 0$; and the absolute Cauchy distribution with the density function $f(x) = 2/\left\{\pi(1 + x^2)\right\}$, $x > 0$. The first, second and third order indices of the three distributions are listed in Table 1. We generate $r = 1000$ samples with sample size $N = 10000$. The value of k_ρ is chosen to be $[n^{0.98}]$ as suggested by Cai et al. (2012), where $[x]$ denotes the largest integer less than or equal to x.

To apply the asymptotically unbiased distributed estimator, we use the following procedure:

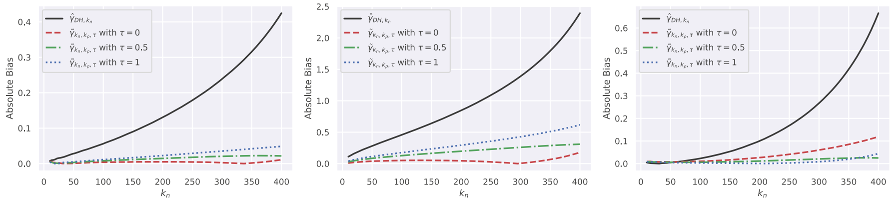
1. On each machine j, we calculate $R^{(1)}_{j,k_n}, R^{(2)}_{j,k_n}, R^{(1)}_{j,k_\rho}, R^{(2)}_{j,k_\rho}, R^{(3)}_{j,k_\rho}$ and transmit them to the central machine.
2. On the central machine, we take the average of the $R^{(1)}_{j,k_n}, R^{(2)}_{j,k_n}, R^{(1)}_{j,k_\rho}, R^{(2)}_{j,k_\rho}, R^{(3)}_{j,k_\rho}$ statistics collected from the m machines to obtain $R^{(1)}_{k_n}, R^{(2)}_{k_n}, R^{(1)}_{k_\rho}, R^{(2)}_{k_\rho}, R^{(3)}_{k_\rho}$.
3. On the central machine, we estimate the second order parameter ρ by (7) with $k = k_\rho$. The value of the tuning parameter τ is set at 0, 0.5 and 1.
4. On the central machine, we estimate the extreme value index by (8) for various values of k_n, using $\hat{\rho}_{k_\rho,\tau}$.

We assume that the $N = 10000$ observations are stored in $m = 1, 20, 100$ machines with $n = N/m$ observations each. Note that the case $m = 1$ corresponds to applying the statistical procedure to the oracle sample directly. The corresponding estimator is therefore the oracle estimator.

Figure 1 shows the absolute bias against various levels of k_n for the three distributions with $m = 20$. The results for other values of m show similar patterns and are thus omitted. We observe that, the asymptotically unbiased distributed estimator $\tilde{\gamma}_{k_n,k_\rho,\tau}$ generally has superior performance compared to the original distributed Hill estimator $\hat{\gamma}_{DH,k_n}$. As k_n increases, the bias of the distributed Hill estimator increases, while the asymptotically unbiased distributed estimator has almost zero bias except for very high level of k_n. This is in line with the asymptotic theory. In addition, the

Table 1 The first, second and third order indicies for the distributions

	Fréchet	Burr	Absolute Cauchy
γ	1	1	1
ρ	−1	−1/2	−2
$\tilde{\rho}$	−1	−1/2	−4



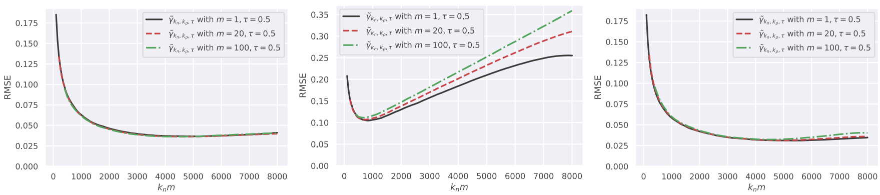
(a) Fréchet **(b)** Burr **(c)** Absolute Cauchy

Fig. 1 Absolute bias for different levels of k_n with $m = 20$

choice of τ affects the performance of the asymptotically unbiased distributed estimator. When $\rho < -1$ (absolute Cauchy distribution), $\tau = 1$ is a better choice than $\tau = 0$. When $\rho \geq -1$ (Fréchet distribution and Burr distribution), $\tau = 0$ is a better choice than $\tau = 1$. This is in line with the findings in Alves et al. (2003).

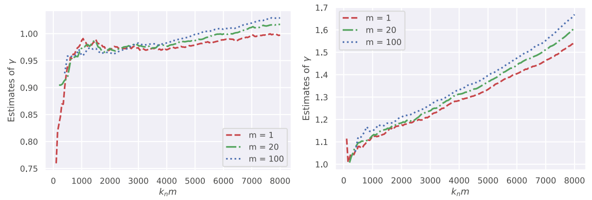
Next, we compare the performance of the asymptotically unbiased distributed estimator for different values of m. In this comparison, we fix $\tau = 0.5$. We plot the RMSE of the estimators against various levels of $k_n m$ in Fig. 2. For the Fréchet distribution and the absolute Cauchy distribution, the performance of the asymptotically unbiased distributed estimator is generally not sensitive to the variation in m. The performance across different values of m is comparable to the case $m = 1$, i.e., the oracle property holds. For the Burr distribution, the oracle property only holds when $k_n m$ is low. When $k_n m$ is high, the oracle bias corrected estimator fails to correct the bias and the RMSE for the distributed estimator is higher than that of the oracle estimator. This observation is in line with the theoretical discussion in Remark 3.

One important advantage of bias correction method in extreme value statistics is that the bias corrected estimator is relatively insensitive to the number of tail observations used in estimation, when applying it to a single sample. This advantage might be less pronounced for the distributed estimator since increasing k_n by 1 will effectively lead to an increase of the number of tail observations by m. To examine this effect, we compare the single sample performance of the asymptotically unbiased distributed estimator with different values of m. Figure 3 shows the plot of the estimates against various levels of $k_n m$ based on one single sample consisting of 10000 observations. We observe that the path of the asymptotically unbiased distributed estimator across different values of m is comparable to the case $m = 1$. In other



(a) Fréchet **(b)** Burr **(c)** Absolute Cauchy

Fig. 2 RMSE for different levels of $k_n m$



(a) Fréchet **(b)** Burr **(c)** Absolute Cauchy

Fig. 3 Single sample performance

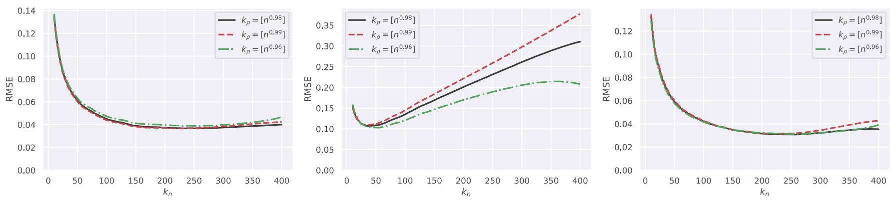
words, the asymptotically unbiased distributed estimator inherits the advantage of the bias correction estimator: it stabilizes the performance over a broader range of $k_n m$.

Finally, we examine the impact of choosing k_ρ. In this comparison, we fix $m = 20$ and $\tau = 0.5$, and consider three choices of $k_\rho = [n^{0.96}], [n^{0.98}], [n^{0.99}]$. Figure 4 shows the plots of the RMSE against various levels of k_n. For the Fréchet and the absolute Cauchy distribution, the asymptotically unbiased distributed estimator is not sensitive to the choice of k_ρ, while $k_\rho = [n^{0.98}]$ performing slight better for high level of k_n. For the Burr distribution, $k_\rho = [n^{0.96}]$ yields slightly better performance. Nevertheless, the RMSEs for the three choices of k_ρ are still comparable when k_n is low.

4.2 Further limitation for transmission

Recall that for the asymptotically unbiased distributed estimator, we need to transmit five statistics from each of the m machines to the central machine. If there are further limitations on the number of results that can be transmitted, such as only three, or even one statistic can be transmitted, the estimation procedure in Sect. 4.1 will not be applicable. In this subsection, we consider two alternative procedures for bias correction in the distributed inference setup with fewer number of transmissions.

Firstly, we consider a bias correction procedure if only three statistics can be transmitted. We can estimate the second order parameter ρ on each machine and transmit the estimates for ρ to the central machine. The detailed procedures are given as follows:



(a) Fréchet **(b)** Burr **(c)** Absolute Cauchy

Fig. 4 Performance for different choices of k_ρ

- On each machine j, we calculate $R^{(1)}_{j,k_n}, R^{(2)}_{j,k_n}, R^{(1)}_{j,k_\rho}, R^{(2)}_{j,k_\rho}, R^{(3)}_{j,k_\rho}$.
- On each machine j, we estimate the second order parameter ρ by

$$\hat{\rho}_{j,k_\rho,\tau} := -3\left|\frac{T_{j,k_\rho,\tau}-1}{T_{j,k_\rho,\tau}-3}\right|, \tag{9}$$

  with

$$T_{j,k_\rho,\tau} := \frac{\left(R^{(1)}_{j,k_\rho}\right)^\tau - \left(R^{(2)}_{j,k_\rho}/2\right)^{\tau/2}}{\left(R^{(2)}_{j,k_\rho}/2\right)^{\tau/2} - \left(R^{(3)}_{j,k_\rho}/6\right)^{\tau/3}},$$

  and transmit $\hat{\rho}_{j,k_\rho,\tau}, R^{(1)}_{j,k_n}, R^{(2)}_{j,k_n}$ to the central machine.
- On the central machine, we take the average of the $\hat{\rho}_{j,k_\rho,\tau}, R^{(1)}_{j,k_n}, R^{(2)}_{j,k_n}$ to obtain

$$\tilde{\rho}_{k_\rho,\tau} = \frac{1}{m}\sum_{j=1}^{m}\hat{\rho}_{j,k_\rho,\tau},\ R^{(1)}_{k_n} = \frac{1}{m}\sum_{j=1}^{m}R^{(1)}_{j,k_n},\ R^{(2)}_{k_n} = \frac{1}{m}\sum_{j=1}^{m}R^{(2)}_{j,k_n}.$$

- On the central machine, we estimate the extreme value index by

$$\tilde{\gamma}^{(2)}_{k_n,k_\rho,\tau} := R^{(1)}_{k_n} - \frac{R^{(2)}_{k_n} - 2\left(R^{(1)}_{k_n}\right)^2}{2R^{(1)}_{k_n}\tilde{\rho}_{k_\rho,\tau}(1-\tilde{\rho}_{k_\rho,\tau})^{-1}}.$$

Secondly, we consider a bias correction procedure if only one statistic can be transmitted. We can conduct bias correction on each machine and transmit the estimates using the bias-corrected Hill estimator to the central machine. Then we take the average of these estimates on the central machine. In this procedure, each machine only sends one statistic to the central machine. The detailed procedures are as follows:

- On each machine j, we calculate $R^{(1)}_{j,k_n}, R^{(2)}_{j,k_n}, R^{(1)}_{j,k_\rho}, R^{(2)}_{j,k_\rho}, R^{(3)}_{j,k_\rho}$ and estimate the second order parameter ρ by (9).
- On each machine j, we estimate the extreme value index by

$$\tilde{\gamma}_{j,k_n,k_\rho,\tau} := R^{(1)}_{j,k_n} - \frac{R^{(2)}_{j,k_n} - 2\left(R^{(1)}_{j,k_n}\right)^2}{2R^{(1)}_{j,k_n}\hat{\rho}_{j,k_\rho,\tau}(1-\hat{\rho}_{j,k_\rho,\tau})^{-1}},$$

  and transmit the estimates $\tilde{\gamma}_{j,k_n,k_\rho,\tau}$ to the central machine.
- On the central machine, we take the average of these estimates by

$$\tilde{\gamma}^{(3)}_{k_n,k_\rho,\tau} := \frac{1}{m}\sum_{j=1}^{m}\tilde{\gamma}_{j,k_n,k_\rho,\tau}.$$

The asymptotic theories of these two estimators $\tilde{\gamma}^{(2)}_{k_n,k_\rho,\tau}$ and $\tilde{\gamma}^{(3)}_{k_n,k_\rho,\tau}$ are left for further study. We only provide a finite sample comparison between the proposed estimator and these two estimators.

In this comparison, we fix $\tau = 0.5$. Figure 5 shows the RMSE for the Fréchet distribution. The figures for the Burr distribution and the absolute Cauchy distribution have similar patterns and are therefore omitted. We observe that all three bias corrected estimators $\tilde{\gamma}_{k_n,k_\rho,\tau}$, $\tilde{\gamma}^{(2)}_{k_n,k_\rho,\tau}$ and $\tilde{\gamma}^{(3)}_{k_n,k_\rho,\tau}$ generally perform better than the original distributed Hill estimator. In addition, $\tilde{\gamma}_{k_n,k_\rho,\tau}$ and $\tilde{\gamma}^{(2)}_{k_n,k_\rho,\tau}$ have similar performance for all three values of m with $\tilde{\gamma}_{k_n,k_\rho,\tau}$ performing slightly better for the Fréchet distribution and $\tilde{\gamma}^{(2)}_{k_n,k_\rho,\tau}$ performing slightly better for the absolute Cauchy distribution.

The performance of $\tilde{\gamma}^{(3)}_{k_n,k_\rho,\tau}$ is unstable when m is at a high level. In this case, n is at a low level. Therefore, conducting bias correction on each machine is suboptimal since the bias correction procedure requires a relatively large sample size.

5 Discussion

In this section, we discuss three extensions of our main results. The first two considers relaxing some technical assumptions in the current framework. The last one extends our result to estimating high quantiles.

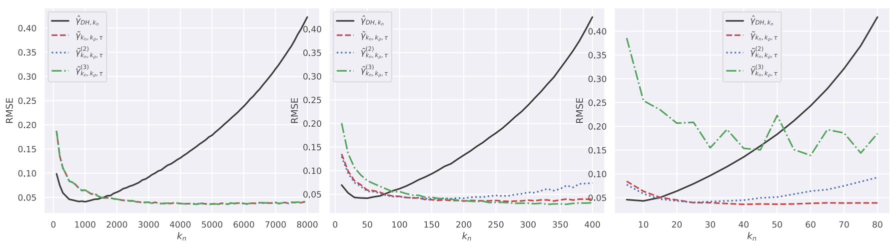
First, we relax the assumption that the sample sizes on all machines are equal. Assume that N observations are distributed stored in m machines with $n_j = n_j(N)$ observations in machine j, $j = 1, 2, \ldots, m$, i.e. $N = \sum_{j=1}^m n_j$. We assume that all $n_j, j = 1, 2, \ldots, m$ diverge in the same order. Mathematically, there exist positive constants c_1 and c_2, such that for all $N \geq 1$,

$$c_1 \leq \min_{1\leq j\leq m} n_j m/N \leq \max_{1\leq j\leq m} n_j m/N \leq c_2.$$

We choose $k_j, j = 1, 2, \ldots, m$ such that the ratios k_j/n_j are homogenous across all the m machines, i.e.,

$$k_1/n_1 = k_2/n_2 = \cdots = k_m/n_m =: k/n,$$

where $k = m^{-1}\sum_{j=1}^m k_j$ and $n = N/m$. Define



(a) m=1 **(b)** m=20 **(c)** m=100

Fig. 5 RMSE for the Fréchet distribution

$$R_k^{(\alpha)} := \sum_{j=1}^{m} \frac{n_j}{N} R_{j,k}^{(\alpha)}, \quad \alpha = 1, 2, 3.$$

Under the same conditions as in Proposition 1, by following similar steps as in the proof of the proposition, we can obtain that, as $N \to \infty$,

$$\sqrt{km}\left(R_k^{(1)} - \gamma\right) = \gamma P_N^{(1)} + \sqrt{km} A_0(n/k) \frac{1}{m} \sum_{j=1}^{m} \frac{g(k_j, n_j, \rho)}{1-\rho} + \sqrt{km} A_0(n/k) B_0(n/k)$$

$$\frac{1}{m} \sum_{j=1}^{m} \frac{g(k_j, n_j, \rho + \tilde{\rho})}{1 - \rho - \tilde{\rho}} + o_P(1).$$

Similar results hold for $R_k^{(2)}$ and $R_k^{(3)}$.

Then, with defining the asymptotically unbiased distributed estimator for the extreme value index as

$$\tilde{\gamma}_{k_n, k_\rho, \tau} := R_{k_n}^{(1)} - \frac{R_{k_n}^{(2)} - 2\left(R_{k_n}^{(1)}\right)^2}{2R_{k_n}^{(1)} \hat{\rho}_{k_\rho, \tau} (1 - \hat{\rho}_{k_\rho, \tau})^{-1}},$$

Theorem 2 still holds.

Second, we relax the assumption that all the data are drawn from the same distribution. We maintain the assumption that observations on the same machine follow the same distribution, but assume that observations across machines are not identically distributed. More specifically, denote the common distribution function of the observations in machine j as $F_{m,j}, j = 1, 2, \ldots, m$. We assume the heteroscedastic extreme model in Einmahl et al. (2016) holds for $F_{m,j}, j = 1, 2, \ldots, m$: there exists a continous distribution function F such that

$$\lim_{x \to \infty} \frac{1 - F_{m,j}(x)}{1 - F(x)} = c_{m,j}, \tag{10}$$

uniformly for all $1 \le j \le m$ and all $m \in \mathbb{N}$ with $c_{m,j}$ uniformly bounded away from 0 and ∞.

Under this heteroscedastic extremes setup, the first order parameters γ for all $F_{m,j}$, $j = 1, 2, \ldots, m$ are the same. This heteroscedastic extreme setup is similar to the setup in Sect. 3 in Chen et al. (2021). Its practical relevance can be again illustrated by the example of estimating tail risks in insurance claims. For a given type of insurance, claims in different insurance companies may not follow the same distribution due to the fact that different companies may be specialized in different segments of the market. Nevertheless, they may share the same shape parameter of the tail due to the underlying nature of the insured risk.

Chen et al. (2021) introduces additional assumptions to ensure that the heteroscedastic extremes assumption does not introduce an additional bias; see assumptions in Theorem 4 therein, particularly Condition D. Under the same assumption, by following similar techniques in the proof, we can show that the heteroscedastic extremes setup does not affect the statement in Theorem 2.

Third, we discuss how to obtain the asymptotically unbiased distributed estimator for the high quantile $x(p_N) := U(1/p_N)$, where $p_N = O(1/N)$ as $N \to \infty$. Motivated by de Haan et al. (2016), we define the asymptotically unbiased distributed estimator for high quantile as

$$\hat{x}_{k_n,k_\rho,\tau}(p_N) := \frac{1}{m}\sum_{j=1}^{m} M_j^{(k_n+1)}\left(\frac{k}{np_N}\right)^{\hat{\gamma}_{k_n,k_\rho,\tau}}\left(1 - \frac{\left(R_{k_n}^{(2)} - \left(R_{k_n}^{(1)}\right)^2\right)\left(1-\hat{\rho}_{k_\rho,\tau}\right)^2}{2R_{k,n}^{(1)}\left(\hat{\rho}_{k_\rho,\tau}\right)^2}\right).$$

Note that, the estimator $\hat{x}_{k_n,k_\rho,\tau}$ also adheres to a DC algorithm since each machine only sends *six* values $\left\{R_{j,k_n}^{(1)}, R_{j,k_n}^{(2)}, R_{j,k_\rho}^{(1)}, R_{j,k_\rho}^{(2)}, R_{j,k_\rho}^{(3)}, M_j^{(k_n+1)}\right\}$ to the central machine. Since $\hat{x}_{k_n,k_\rho,\tau}(p_N)$ are constructed by $R_k^{(\alpha)}$ ($k = k_n$ and k_ρ, $\alpha = 1, 2, 3$) and $m^{-1}\sum_{j=1}^{m} M_j^{(k_n+1)}$, the asymptotic theory of $\hat{x}_{k_n,k_\rho,\tau}(p_N)$ can be established using similar techniques as in the proof of Theorem 4.2 in de Haan et al. (2016). We leave the details to the readers.

Appendix

Proofs

Preliminary

Lemma 1 *Let $Y, Y_1, \dots, Y_n$ be i.i.d. Pareto* (1) *random variables with distribution function* $1 - 1/y$, $y \ge 1$. *Let $Y^{(1)} \ge \cdots \ge Y^{(n)}$ be the order statistics of $\left\{Y_1, \dots, Y_n\right\}$. Let f be a function such that* $\mathrm{Var}\{f(Y)\} < \infty$. *Then for any $k \ge 1$,*

$$\frac{1}{k}\sum_{i=1}^{k} f\left(\frac{Y^{(i)}}{Y^{(k+1)}}\right) \stackrel{d}{=} \frac{1}{k}\sum_{i=1}^{k} f(Y_i^*),$$

where $Y_1^*, Y_2^*, \dots, Y_k^*$ are i.i.d. Pareto (1) *random variables. Moreover,*

$$\sqrt{k}\left\{\frac{1}{k}\sum_{i=1}^{k} f\left(\frac{Y^{(i)}}{Y^{(k+1)}}\right) - \mathbb{E}f(Y)\right\}$$

is independent of $Y^{(k+1)}$ and asymptotically normally distributed with mean zero and variance $\mathrm{Var}\{f(Y)\}$ *as $n \to \infty$, provided that $k = k(n) \to \infty$ and $k/n \to 0$.*

Proof of Lemma 1 This Lemma follows directly from Lemma 3.2.3 in de Haan and Ferreira (2006) with the fact that $\log Y$ follows a standard exponential distribution.

□

Lemma 2 *Let $Y_1, \dots, Y_n$ be i.i.d. Pareto (1) random variables and $Y^{(1)} \geq \cdots \geq Y^{(n)}$ be the order statistics of $\{Y_1, \dots, Y_n\}$. Then for any $\rho < 0$,*

$$\mathbb{E}\left\{ \left(\frac{k}{n} Y^{(k+1)} \right)^{\rho} \right\} = g(k, n, \rho),$$

where $g(k, n, \rho)$ is defined in (3). Moreover, if k is a fixed integer, then $g(k, n, \rho) \to k^{\rho}\Gamma(k - \rho + 1)/\Gamma(k + 1)$ as $n \to \infty$. If k is an intermediate sequence, i.e. $k \to \infty, k/n \to 0$ as $n \to \infty$, then,

$$g(k, n, \rho) = 1 + \frac{1}{2}(\rho^2 - \rho)k^{-1} - \frac{1}{2}(\rho^2 - \rho)(n - \rho)^{-1} + O(k^{-2}).$$

Proof of Lemma 2

$$\begin{aligned}
\mathbb{E}\left\{ \left(\frac{k}{n} Y^{(k+1)} \right)^{\rho} \right\} &= \frac{n!}{(n-k-1)!k!} \int_1^{\infty} \left(1 - \frac{1}{y}\right)^{n-k-1} \left(\frac{1}{y}\right)^{k+2} \left(\frac{k}{n} y\right)^{\rho} dy \\
&= \left(\frac{k}{n}\right)^{\rho} \frac{n!}{(n-k-1)!k!} \int_1^{\infty} \left(1 - \frac{1}{y}\right)^{n-k-1} \left(\frac{1}{y}\right)^{k+2-\rho} dy \\
&= \left(\frac{k}{n}\right)^{\rho} \frac{\Gamma(n+1)\Gamma(k-\rho+1)}{\Gamma(n-\rho+1)\Gamma(k+1)} \\
&= g(k, n, \rho).
\end{aligned}$$

We first handle the case when k is a fixed integer. By the Stirling's formula,

$$\Gamma(x) = \sqrt{2\pi(x-1)} \left\{ e^{-1}(x-1) \right\}^{x-1} \left\{ 1 + (x-1)^{-1}/12 + O(1/x^2) \right\}$$

as $x \to \infty$, we have that, as $n \to \infty$,

$$\Gamma(n+1) \sim (2\pi n)^{1/2} \left(\frac{n}{e}\right)^n, \quad \Gamma(n-\rho+1) \sim \{2\pi(n-\rho)\}^{1/2} \left(\frac{n-\rho}{\rho}\right)^{n-\rho},$$

which leads to

$$g(k, n, \rho) \to k^{\rho} \frac{\Gamma(k-\rho+1)}{\Gamma(k+1)}.$$

Next, we handle the case when k is an intermediate sequence. By the Stirling's formula, we have that, as $n \to \infty$,

$$\begin{aligned}
g(k, n, \rho) &= \left(1 - \frac{\rho}{k}\right)^{k-\rho+1/2} \left(1 + \frac{\rho}{n-\rho}\right)^{n-\rho+1/2} \frac{1 + n^{-1}/12 + O(n^{-2})}{1 + (n-\rho)^{-1}/12 + O(n^{-2})} \frac{1 + (k-\rho)^{-1}/12 + O(k^{-2})}{1 + k^{-1}/12 + O(k^{-2})} \\
&= \left(1 - \frac{\rho}{k}\right)^{k-\rho+1/2} \left(1 + \frac{\rho}{n-\rho}\right)^{n-\rho+1/2} \left\{1 + O(n^{-2})\right\}\left\{1 + O(k^{-2})\right\}.
\end{aligned}$$

By the Taylor's formula and some direct calculation, we obtain that, as $n \to \infty$,

$$\left(1-\frac{\rho}{k}\right)^{k-\rho+1/2}=e^{-\rho}\left\{1+\frac{1}{2}(\rho^2-\rho)k^{-1}+O\left(k^{-2}\right)\right\},$$

and

$$\left(1+\frac{\rho}{n-\rho}\right)^{n-\rho+1/2}=e^{\rho}\left\{1-\frac{1}{2}\left(\rho^2-\rho\right)(n-\rho)^{-1}+O\left(n^{-2}\right)\right\}.$$

It follows that, as $n\to\infty$,

$$g(k,n,\rho)=1+\frac{1}{2}(\rho^2-\rho)k^{-1}-\frac{1}{2}(\rho^2-\rho)(n-\rho)^{-1}+O(k^{-2}).$$

□

Lemma 3 *Let $Y_1,\dots,Y_n$ be i.i.d. Pareto* (1) *random variables and $Y^{(1)}\geq\cdots\geq Y^{(n)}$ be the order statistics of $\left\{Y_1,\dots,Y_n\right\}$. Define for $\rho<0$,*

$$Z_k=\frac{1}{k}\sum_{i=1}^{k}\frac{\left(Y^{(i)}/Y^{(k+1)}\right)^{\rho}-1}{\rho}.$$

Then, the following results hold.

(i) *For fixed k, $\mathbb{E}(Z_k^a)<\infty$, for $a=1,2,3,4$. Moreover, $\mathbb{E}\left(Z_k^2\right)-\left\{\mathbb{E}\left(Z_k\right)\right\}^2>0$.*
(ii) *For intermediate k, i.e., $k=k(n)\to\infty$, $k/n\to 0$ as $n\to\infty$, and $a=1,2,3,4$,*

$$\mathbb{E}\left(Z_k^a\right)=\frac{1}{(1-\rho)^a}\left\{1+\frac{a(a-1)}{2(1-2\rho)}\frac{1}{k}+O\left(k^{-2}\right)\right\}.$$

Proof of Lemma 3 By Lemma 1, we have that,

$$Z_k\overset{d}{=}\frac{1}{k}\sum_{i=1}^{k}\frac{\left(Y_i^*\right)^{\rho}-1}{\rho},$$

where $Y_1^*,\dots,Y_k^*$ are i.i.d. Pareto (1) random variables. Denote $T_i=\left\{(Y_i^*)^{\rho}-1\right\}/\rho$, for $i=1,\dots,k$ and $Z_k=k^{-1}\sum_{i=1}^{k}T_i$. Then, $T_i, i=1,\dots,k$ follows the generalized Pareto distribution with the cumulative distribution function $F(t)=1-(1+\rho t)^{-1/\rho}$. Thus, we have that for $a=1,2,3,4$,

$$\mathbb{E}(T_i^a)=\frac{a!}{(1-a\rho)\cdots(1-\rho)}.$$

First, we handle the case when k is fixed. The result is obvious since kZ_k is a finite sum of i.i.d. generalized Pareto random variables with shape parameter $\rho<0$.

Next, we handle the case when k is an intermediate sequence. For $a=1$, we have that, $E(Z_k)=E(T_i)=(1-\rho)^{-1}$.

For $a = 2$, we have that,

$$\begin{aligned}\mathbb{E}\left(Z_k^2\right) &= \frac{1}{k^2}\left\{\sum_{i=1}^{k} E\left(T_i^2\right) + \sum_{i\neq j}\mathbb{E}\left(T_i\right)\mathbb{E}\left(T_j\right)\right\}\\ &= \frac{1}{k^2}\left[kE\left(T_i^2\right) + k(k-1)\left\{\mathbb{E}\left(T_i\right)\right\}^2\right]\\ &= \frac{1}{(1-\rho)^2} + \frac{1}{k}\frac{1}{(1-2\rho)(1-\rho)^2}.\end{aligned}$$

For $a = 3$, we have that

$$\begin{aligned}\mathbb{E}\left(Z_k^3\right) &= \frac{1}{k^2}\left\{\sum_{i=1}^{k}\mathbb{E}\left(T_i^3\right) + \sum_{i=j\neq l}\mathbb{E}\left(T_iT_j\right)\mathbb{E}\left(T_l\right) + \sum_{i\neq j\neq l}\mathbb{E}\left(T_i\right)\mathbb{E}\left(T_j\right)\mathbb{E}\left(T_l\right)\right\}\\ &= \frac{1}{k^3}\left[k\mathbb{E}\left(T_i^3\right) + 3k(k-1)\mathbb{E}\left(T_i^2\right)\mathbb{E}\left(T_i\right) + k(k-1)(k-2)\left\{\mathbb{E}(T_i)\right\}^3\right]\\ &= \frac{1}{(1-\rho)^3} + \frac{1}{k}\frac{3}{(1-2\rho)(1-\rho)^3} + O(k^{-2}).\end{aligned}$$

The term $\mathbb{E}\left(Z_k^4\right)$ can be handled in a similar way as that for handling $\mathbb{E}\left(Z_k^3\right)$. □

Lemma 4 *Assume that the distribution function F satisfies the third order condition* (4). *Then there exist two functions $A_0(t) \sim A(t)$ and $B_0(t) = O\{B(t)\}$ as $t \to \infty$, such that for any $\delta > 0$, there exists a $t_0 = t_0(\delta) > 0$, for all $t \geq t_0$ and $tx \geq t_0$,*

$$\left|\frac{\frac{\log U(tx)-\log U(t)-\gamma\log x}{A_0(t)} - \frac{x^{\rho}-1}{\rho}}{B_0(t)} - \frac{x^{\rho+\tilde{\rho}}-1}{\rho+\tilde{\rho}}\right| \leq \delta x^{\rho+\tilde{\rho}}\max(x^{\delta}, x^{-\delta}).$$

***Proof of Lemma* 4** This lemma follows from applying Theorem B.3.10 in de Haan and Ferreira (2006) to the function $f(t) := \log U(t) - \gamma\log t$. □

Proofs for Section 3

Recall that $U = \{1/(1-F)\}^{\leftarrow}$. Then $X\stackrel{d}{=}U(Y)$, where Y follows the Pareto (1) distribution. Since we have i.i.d. observations $\left\{X_1, \dots, X_N\right\}$, we can write $X_i\stackrel{d}{=}U(Y_i)$, where $\left\{Y_1, \dots, Y_N\right\}$ is a random sample of Y. Recall that the N observations are stored in m machines with n observations each. For machine j, let $Y_j^{(1)} \geq \cdots \geq Y_j^{(n)}$ denote the order statistics of the n Pareto (1) distributed variables corresponding to the n observations in this machine. Then $M_j^{(i)}\stackrel{d}{=}U(Y_j^{(i)}), i = 1, \dots, n, j = 1, \dots, m$.

***Proof of Proposition* 1** We intend to replace t and tx in Lemma 4 by n/k and $Y_j^{(i)}, i = 1, \dots, k+1, j = 1, \dots, m$, respectively. For this purpose, we introduce the set

$$\mathcal{F}_{t_0} := \left\{Y_j^{(k+1)} \geq t_0, \textit{ for all } 1 \leq j \leq m\right\}.$$

By Lemma S.2 in the supplementary material of Chen et al. (2021), we have that for any $t_0 > 1$, if condition (2) holds, then $\lim_{N\to\infty} \mathbb{P}\left(\mathcal{F}_{t_0}\right) = 1$. Then, we can apply the intended replacement to get that, as $N \to \infty$,

$$\begin{aligned}\log U\left(Y_j^{(i)}\right) - \log U(n/k) = -\gamma \log\left(kY_j^{(i)}/n\right) - A_0(n/k)\left\{\left(kY_j^{(i)}/n\right)^{\rho} - 1\right\}/\rho \\ + A_0(n/k)B_0(n/k)\left\{\left(kY_j^{(i)}/n\right)^{\rho+\tilde{\rho}} - 1\right\}/(\rho+\tilde{\rho}) \\ + o_P(1)A_0(n/k)B_0(n/k)\left(kY_j^{(i)}/n\right)^{\rho+\tilde{\rho}\pm\delta},\end{aligned} \tag{11}$$

where the $o_P(1)$ term is uniform for all $1 \le i \le k+1$ and $1 \le j \le m$. By applying (11) twice for a general i and $i = k+1$ and the inequality $x^{\rho\pm\delta}/y^{\rho\pm\delta} \le (x/y)^{\rho\pm\delta}$ for any $x, y > 0$, we get that as $N \to \infty$,

$$\begin{aligned}\log U\left(Y_j^{(i)}\right) - \log U\left(Y_j^{(k+1)}\right) = \gamma\left(\log Y_j^{(i)} - \log Y_j^{(k+1)}\right) \\ + A_0(n/k)\left(kY_j^{(k+1)}/n\right)^{\rho}\left\{\left(Y_j^{(i)}/Y_j^{(k+1)}\right)^{\rho} - 1\right\}/\rho \\ + A_0(n/k)B_0(n/k)\left(kY_j^{(k+1)}/n\right)^{\rho+\tilde{\rho}}\left\{\left(Y_j^{(i)}/Y_j^{(k+1)}\right)^{\rho+\tilde{\rho}} - 1\right\}/(\rho+\tilde{\rho}) \\ + o_P(1)A_0(n/k)B_0(n/k)\left(kY_j^{(k+1)}/n\right)^{\rho+\tilde{\rho}\pm\delta}\left\{\left(Y_j^{(i)}/Y_j^{(k+1)}\right)^{\rho+\tilde{\rho}\pm\delta} + 1\right\}.\end{aligned} \tag{12}$$

By taking the average across i and j, we obtain that

$$\begin{aligned}\sqrt{km}\left(R_k^{(1)} - \gamma\right) &= \gamma\sqrt{km}\frac{1}{m}\frac{1}{k}\sum_{j=1}^{m}\sum_{i=1}^{k}\left\{\log\left(Y_j^{(i)}/Y_j^{(k+1)}\right) - \gamma\right\} \\ &+ \sqrt{km}A_0(n/k)\frac{1}{m}\sum_{j=1}^{m}\left(kY_j^{(k+1)}/n\right)^{\rho}\rho^{-1}\frac{1}{k}\sum_{i=1}^{k}\left\{\left(Y_j^{(i)}/Y_j^{(k+1)}\right)^{\rho} - 1\right\} \\ &+ \sqrt{km}A_0(n/k)B_0(n/k)\frac{1}{m}\sum_{j=1}^{m}\left(kY_j^{(k+1)}/n\right)^{\rho+\tilde{\rho}}(\rho+\tilde{\rho})^{-1}\frac{1}{k}\sum_{i=1}^{k}\left\{\left(Y_j^{(i)}/Y_j^{(k+1)}\right)^{\rho+\tilde{\rho}} - 1\right\} \\ &+ o_P(1)\sqrt{km}A_0(n/k)B_0(n/k)\frac{1}{m}\sum_{j=1}^{m}\left(kY_j^{(k+1)}/n\right)^{\rho+\tilde{\rho}\pm\delta}\frac{1}{k}\sum_{i=1}^{k}\left\{\left(Y_j^{(i)}/Y_j^{(k+1)}\right)^{\rho+\tilde{\rho}\pm\delta} + 1\right\} \\ &=: I_1 + I_2 + I_3 + I_4.\end{aligned}$$

Firstly, we handle I_1. By Lemma 1, we have that,

$$I_1 \stackrel{d}{=} \gamma\sqrt{km}\left(\frac{1}{km}\sum_{j=1}^{m}\sum_{i=1}^{k}\log Y_i^{j,*} - 1\right),$$

where $Y_i^{j,*}, i = 1, \dots, k, j = 1, \dots, m$ are independent and identically distributed Pareto (1) random variables. The central limit theorem yields that as $N \to \infty$, $I_1 = \gamma P_N^{(1)} + o_P(1)$, where $P_N^{(1)} \sim N(0, 1)$.

For I_2, write $\delta_{j,n} = \left(kY_j^{(k+1)}/n\right)^{\rho}(k\rho)^{-1}\sum_{i=1}^{k}\left\{\left(Y_j^{(i)}/Y_j^{(k+1)}\right)^{\rho} - 1\right\}$. Then we have that $I_2 = \sqrt{km}A_0(n/k)m^{-1}\sum_{j=1}^{m}\delta_{j,n}$, where $\delta_{j,n}, j = 1, \dots, m$ are i.i.d. random variables.

We are going to show that, as $N \to \infty$,

$$\sqrt{km}\left\{ \frac{1}{m} \sum_{j=1}^{m} \delta_{j,n} - \mathbb{E}\left(\delta_{j,n}\right) \right\} = O_P(1). \tag{13}$$

If k is fixed, (13) follows directly from Lemma 3 (i) and the Lyapunov central limit theorem for triangular array.

Next, we handle the case when k is an intermediate sequence. In this case, in order to apply the Lyapunov central limit theorem with 4-th moment, we need to calculate $\mathrm{Var}\left(\delta_{j,n}\right)$ and $\mathbb{E}[\left\{\delta_{j,n} - \mathbb{E}\left(\delta_{j,n}\right)\right\}^4]$. Denote $m_n^{(a)} := \mathbb{E}\left\{\left(\delta_{j,n}\right)^a\right\}$, $a = 1, 2, 3, 4$. By Lemma 1, we have that,

$$m_n^{(a)} = g(k, n, a\rho)\mathbb{E}\left[\left\{ \frac{1}{k} \sum_{i=1}^{k} \frac{\left(Y_j^{(i)} / Y_j^{(k+1)}\right)^\rho - 1}{\rho} \right\}^a\right].$$

First, we calculate $\mathrm{Var}\left(\delta_{j,n}\right)$. By Lemma 3, we have that,

$$\begin{aligned}
\mathrm{Var}(\delta_{j,n}) &= m_n^{(2)} - \left(m_n^{(1)}\right)^2 \\
&= g(k, n, 2\rho)\left\{ \frac{1}{(1-\rho)^2} + \frac{1}{k} \frac{1}{(1-2\rho)(1-\rho)^2} + O(k^{-2}) \right\} - \{g(k, n, \rho)\}^2 \left\{ \frac{1}{(1-\rho)^2} + O(k^{-2}) \right\} \\
&= \frac{1}{k} g(k, n, 2\rho) \frac{1}{(1-2\rho)(1-\rho)^2} + \left[g(k, n, 2\rho) - \{g(k, n, \rho)\}^2\right] \frac{1}{(1-\rho)^2} + O(k^{-2}),
\end{aligned}$$

here in the last step, we used the fact that as $n \to \infty$, $g(k, n, \rho) \to 1$ and $g(k, n, 2\rho) \to 1$. By Lemma 2, we have that, as $n \to \infty$,

$$\begin{aligned}
g(k, n, 2\rho) - \{g(k, n, \rho)\}^2 &= 1 + \frac{1}{2}\left(4\rho^2 - 2\rho\right)\frac{1}{k} + o(k^{-1}) - \left\{ 1 + \frac{1}{2}\left(\rho^2 - \rho\right)\frac{1}{k} + o(k^{-1}) \right\}^2 \\
&= \frac{1}{k}\rho^2 + o(k^{-1}).
\end{aligned}$$

Hence, as $n \to \infty$, $\mathrm{Var}\left(\delta_{j,n}\right) = k^{-1}(1-\rho)^{-2}\left((1-2\rho)^{-1} + \rho^2\right) + o(k^{-1})$.

Next, we calculate $\mathbb{E}[\left\{\delta_{j,n} - \mathbb{E}\left(\delta_{j,n}\right)\right\}^4]$. By Lemma 2 and Lemma 3, we have that, for $a = 3, 4$, as $N \to \infty$,

$$\begin{aligned}
m_n^{(a)} = (1-\rho)^{-a} &\left\{ 1 + \frac{1}{2}\frac{1}{k}\frac{a(a-1)}{1-2\rho} + O\left(k^{-2}\right) \right\} \left\{ 1 + \frac{1}{2}\left(a^2\rho^2 - a\rho\right)k^{-1} \right. \\
&\left. - \frac{1}{2}\left(a^2\rho^2 - a\rho\right)(n - a\rho)^{-1} + O(k^{-2}) \right\} \\
= (1-\rho)^{-a} &\left\{ 1 + k^{-1}\frac{1}{2}\frac{a(a-1)}{1-2\rho} + \frac{1}{2}\left(a^2\rho^2 - a\rho\right)k^{-1} \right. \\
&\left. - \frac{1}{2}\left(a^2\rho^2 - a\rho\right)(n - a\rho)^{-1} + O\left(k^{-2}\right) \right\}.
\end{aligned}$$

Note that,

$$\mathbb{E}\left[\left\{(\delta_{j,n}-\mathbb{E}\left(\delta_{j,n}\right)\right\}^4\right]=m_n^{(4)}-4m_n^{(3)}m_n^{(1)}+6m_n^{(2)}\left(m_n^{(1)}\right)^2-3\left(m_n^{(1)}\right)^4.$$

By some direct calculation, all terms of order k^{-1} and n^{-1} are cancelled out. Thus, as $N\to\infty,\mathbb{E}[\left\{(\delta_{j,n}-\mathbb{E}\left(\delta_{j,n}\right)\right\}^4]=O(k^{-2})$. Combining $\mathrm{Var}(\delta_{j,n})$ and $\mathbb{E}[\left\{\delta_{j,n}-\mathbb{E}\left(\delta_{j,n}\right)\right\}^4]$, we conclude that the sequences $\left\{\delta_{j,n}\right\}_{j=1}^m$ satisfy the Lyapunov's condition. Then, (13) follows by the central limit theorem. Applying (13), we obtain that, as $N\to\infty$,

$$I_2=\sqrt{km}A_0(n/k)\left\{\mathbb{E}\left(\delta_{j,n}\right)+O_P\left(1/\sqrt{km}\right)\right\}=\frac{g(k,n,\rho)}{1-\rho}\sqrt{km}A_0(n/k)+o_P(1).$$

For I_3, by using the weak law of large numbers for triangular array, we have that, as $N\to\infty$,

$$\begin{aligned}I_3&=\frac{\sqrt{km}A_0(n/k)B_0(n/k)}{1-\rho-\tilde{\rho}}\mathbb{E}\left\{\left(kY_1^{(k+1)}/n\right)^{\rho+\tilde{\rho}}\right\}\left\{1+o_P(1)\right\}\\&=\sqrt{km}A_0(n/k)B_0(n/k)\frac{g(k,n,\rho+\tilde{\rho})}{1-\rho-\tilde{\rho}}+o_P(1),\end{aligned}$$

where the last equality follows by the condition $\sqrt{km}A(n/k)B(n/k)=O_P(1)$.

For I_4, by similar arguments as for I_3, we obtain that, as $N\to\infty$, $I_4\to0$. Combining I_1,I_2,I_3 and I_4, we have proved (i).

Next, we handle $R_k^{(2)}$. By (12), we obtain that, as $N\to\infty$,

$$\begin{aligned}\sqrt{km}\left(R_k^{(2)}-2\gamma^2\right)=&\gamma^2\frac{1}{mk}\sum_{j=1}^m\sum_{i=1}^k\left\{\log^2\left(Y_j^{(i)}/Y_j^{(k+1)}\right)-2\right\}\\&+2\gamma\sqrt{km}A_0(n/k)\frac{1}{km}\sum_{j=1}^m\left(kY_j^{(k+1)}/n\right)^\rho\sum_{i=1}^k\log\left(Y_j^{(i)}/Y_j^{(k+1)}\right)\\&\left\{\left(Y_j^{(i)}/Y_j^{(k+1)}\right)^\rho-1\right\}/\rho\\&+\sqrt{km}A_0^2(n/k)\frac{1}{km}\sum_{j=1}^m\left(kY_j^{(k+1)}/n\right)^{2\rho}\sum_{i=1}^k\left\{\left(Y_j^{(i)}/Y_j^{(k+1)}\right)^\rho-1\right\}^2/\rho^2\\&+2\gamma\sqrt{km}A_0(n/k)B_0(n/k)\frac{1}{km}\sum_{j=1}^m\left(kY_j^{(k+1)}/n\right)^{\rho+\tilde{\rho}}\\&\sum_{i=1}^k\log\left(Y_j^{(i)}/Y_j^{(k+1)}\right)\frac{\left(Y_j^{(i)}/Y_j^{(k+1)}\right)^{\rho+\tilde{\rho}}-1}{\rho+\tilde{\rho}}\\&+o_P(1)\\=:&\ I_5+I_6+I_7+I_8+o_P(1).\end{aligned}$$

For I_5, by Lemma 1, we have that

$$I_5 \overset{d}{=} \gamma^2 \sqrt{km} \left\{ \frac{1}{km} \sum_{j=1}^{m} \sum_{i=1}^{k} \left(\log Y_i^{j,*} \right)^2 - 2 \right\}.$$

The central limit theorem yields that as $N \to \infty$, $I_5 = \gamma^2 P_N^{(2)} + o_P(1)$, where $P_N^{(2)} \sim N(0, 20)$. In addition, the covariance of $P_N^{(1)}$ and $P_N^{(2)}$ is equal to the covariance of $\log Y_i^{j,*}$ and $\left(\log Y_i^{j,*} \right)^2$, where $Y_i^{j,*}$ follows the Pareto (1) distribution. Hence, $\mathrm{Cov}(P_N^{(1)}, P_N^{(2)}) = 4$.

For I_6, we write $I_6 = 2\sqrt{km} A_0(n/k) m^{-1} \sum_{j=1}^{m} \eta_{j,n}$, where

$$\eta_{j,n} = \left(k Y_j^{(k+1)} / n \right)^{\rho} (k\rho)^{-1} \sum_{i=1}^{k} \log \left(Y_j^{(i)} / Y_j^{(k+1)} \right) \left\{ \left(Y_j^{(i)} / Y_j^{(k+1)} \right)^{\rho} - 1 \right\}$$

are i.i.d. random variables for $j = 1, 2, \ldots, m$. We can verify the Lyapunov's condition for the series $\left\{ \eta_{j,n} \right\}_{j=1}^{m}$ following similar steps as those for $\left\{ \delta_{j,n} \right\}_{j=1}^{m}$. Then by applying the central limit theorem and Lemma 2, we obtain that

$$I_6 = 2\gamma \sqrt{km} A_0(n/k) g(k, n, \rho) \frac{1}{\rho} \left\{ \frac{1}{(1-\rho)^2} - 1 \right\} + o_P(1).$$

By the weak law of large numbers for triangular array, we have that

$$I_7 = \sqrt{km} A_0^2(n/k) \frac{g(k, n, 2\rho)}{\rho^2} \left\{ \frac{1}{1-2\rho} - \frac{2}{1-\rho} + 1 \right\} + o_P(1),$$

and

$$I_8 = 2\gamma \sqrt{km} A_0(n/k) B_0(n/k) \frac{g(k, n, \rho + \tilde{\rho})}{\rho + \tilde{\rho}} \left\{ \frac{1}{(1-\rho-\tilde{\rho})^2} - 1 \right\} + o_P(1).$$

Combining the results for I_5, I_6, I_7 and I_8, we have proved (ii).

Finally, we handle $R_k^{(3)}$. Also, by (12), we have that

$$\sqrt{km}\left(R_k^{(3)} - 6\gamma^3\right) = \gamma^3 \frac{1}{mk} \sum_{j=1}^{m} \sum_{i=1}^{k} \left\{ \log^3 \left(Y_j^{(i)} / Y_j^{(k+1)} \right) - 6 \right\}$$

$$+ 3\gamma^2 \sqrt{km} A_0(n/k) \frac{1}{km} \sum_{j=1}^{m} \left(k Y_j^{(k+1)} / n \right)^{\rho}$$

$$\sum_{i=1}^{k} \left\{ \log \left(Y_j^{(i)} / Y_j^{(k+1)} \right) \right\}^2 \frac{\left(Y_j^{(i)} / Y_j^{(k+1)} \right)^{\rho} - 1}{\rho}$$

$$+ 3\gamma \sqrt{km} A_0^2(n/k) \frac{1}{km} \sum_{j=1}^{m} \left(k Y_j^{(k+1)} / n \right)^{2\rho}$$

$$\sum_{i=1}^{k} \log \left(Y_j^{(i)} / Y_j^{(k+1)} \right) \left\{ \frac{\left(Y_j^{(i)} / Y_j^{(k+1)} \right)^{\rho} - 1}{\rho} \right\}^2$$

$$+ 3\gamma^2 \sqrt{km} A_0(n/k) B_0(n/k) \frac{1}{km} \sum_{j=1}^{m} \left(k Y_j^{(k+1)} / n \right)^{\rho + \tilde{\rho}}$$

$$\sum_{i=1}^{k} \left\{ \log \left(Y_j^{(i)} / Y_j^{(k+1)} \right) \right\}^2 \frac{\left(Y_j^{(i)} / Y_j^{(k+1)} \right)^{\rho + \tilde{\rho}} - 1}{\rho + \tilde{\rho}} + o_P(1)$$

$$=: I_9 + I_{10} + I_{11} + I_{12} + o_P(1).$$

By similar steps as for handling the four items I_5, I_6, I_7 and I_8, we can show that $I_9 = \gamma^3 P_N^{(3)} + o_P(1)$, where $P_N^{(3)} \sim N(0, 684)$ and $\mathrm{Cov}(P_N^{(1)}, P_N^{(3)}) = 18, \mathrm{Cov}(P_N^{(2)}, P_N^{(3)}) = 98$. And

$$I_{10} = 6\gamma^2 \sqrt{km} A_0(n/k) \frac{g(k, n, \rho)}{\rho} \left\{ \frac{1}{(1-\rho)^3} - 1 \right\} + o_P(1),$$

$$I_{11} = 3\gamma \sqrt{km} A_0^2(n/k) \frac{g(k, n, 2\rho)}{\rho^2} \left\{ \frac{1}{(1-2\rho)^2} - \frac{2}{(1-\rho)^2} + 1 \right\} + o_P(1),$$

$$I_{12} = 6\gamma^2 \sqrt{km} A_0(n/k) B_0(n/k) \frac{g(k, n, \rho + \tilde{\rho})}{\rho + \tilde{\rho}} \left\{ \frac{1}{(1-\rho-\tilde{\rho})^3} - 1 \right\} + o_P(1),$$

which yields (iii). □

***Proof of Theorem* 1** Applying Proposition 1 with $k = k_\rho$, we have that, as $N \to \infty$,

$$R_{k_\rho}^{(1)} = \gamma + \frac{\gamma}{\sqrt{k_\rho m}} P_N^{(1)} + \frac{g(k_\rho, n, \rho)}{1-\rho} A_0(n/k_\rho) + \frac{g(k_\rho, n, \rho+\tilde{\rho})}{1-\rho-\tilde{\rho}} A_0(n/k_\rho) B_0(n/k_\rho) + \frac{1}{\sqrt{k_\rho m}} o_P(1),$$

$$\begin{aligned} R_{k_\rho}^{(2)} = {} & 2\gamma^2 + \frac{\gamma^2}{\sqrt{k_\rho m}} P_N^{(2)} + 2\gamma A_0(n/k_\rho) \frac{g(k_\rho, n, \rho)}{\rho} \left\{ \frac{1}{(1-\rho)^2} - 1 \right\} \\ & + A_0^2(n/k_\rho) \frac{g(k_\rho, n, 2\rho)}{\rho^2} \left(\frac{1}{1-2\rho} - \frac{2}{1-\rho} + 1 \right) \\ & + 2\gamma A_0(n/k_\rho) B_0(n/k_\rho) \frac{g(k_\rho, n, \rho+\tilde{\rho})}{\rho+\tilde{\rho}} \left\{ \frac{1}{(1-\rho-\tilde{\rho})^2} - 1 \right\} + \frac{1}{\sqrt{k_\rho m}} o_P(1), \end{aligned}$$

$$\begin{aligned} R_{k_\rho}^{(3)} = {} & 6\gamma^3 + \frac{\gamma^3}{\sqrt{k_\rho m}} P_N^{(3)} + 6\gamma A_0(n/k_\rho) \frac{g(k_\rho, n, \rho)}{\rho} \left\{ \frac{1}{(1-\rho)^3} - 1 \right\} \\ & + 3A_0^2(n/k_\rho) \frac{g(k_\rho, n, 2\rho)}{\rho^2} \left\{ \frac{1}{(1-2\rho)^2} - \frac{2}{(1-\rho)^2} + 1 \right\} \\ & + 6\gamma A_0(n/k_\rho) B_0(n/k_\rho) \frac{g(k_\rho, n, \rho+\tilde{\rho})}{\rho+\tilde{\rho}} \left\{ \frac{1}{(1-\rho-\tilde{\rho})^3} - 1 \right\} + \frac{1}{\sqrt{k_\rho m}} o_P(1). \end{aligned}$$

As a consequence, we have that, as $N \to \infty$,

$$\begin{aligned} \left(R_{k_\rho}^{(1)}\right)^\tau = {} & \gamma^\tau \left\{ 1 + \frac{\tau}{\sqrt{k_\rho m}} P_N^{(1)} + \frac{\tau}{\gamma} \frac{g(k_\rho, n, \rho)}{1-\rho} A_0(n/k_\rho) + \frac{\tau}{\gamma} \frac{g(k_\rho, n, \rho+\tilde{\rho})}{1-\rho-\tilde{\rho}} A_0(n/k_\rho) B_0(n/k_\rho) \right\} \\ & + \frac{1}{\sqrt{k_\rho m}} o_P(1), \end{aligned}$$

$$\begin{aligned} \left(R_{k_\rho}^{(2)}/2\right)^{\tau/2} = {} & \gamma^\tau \left[1 + \frac{\tau}{\sqrt{k_\rho m}} P_N^{(2)} + \frac{\tau}{2\gamma} A_0(n/k_\rho) \frac{g(k_\rho, n, \rho)}{\rho} \left\{ \frac{1}{(1-\rho)^2} - 1 \right\} \right. \\ & + \frac{\tau}{4\gamma} A_0^2(n/k_\rho) \frac{g(k_\rho, n, 2\rho)}{\rho^2} \left(\frac{1}{1-2\rho} - \frac{2}{1-\rho} + 1 \right) \\ & \left. + \frac{\tau}{2\gamma} A_0(n/k_\rho) B_0(n/k_\rho) \frac{g(k_\rho, n, \rho+\tilde{\rho})}{\rho+\tilde{\rho}} \left\{ \frac{1}{(1-\rho-\tilde{\rho})^2} - 1 \right\} \right] + \frac{1}{\sqrt{k_\rho m}} o_P(1), \end{aligned}$$

$$\begin{aligned} \left(R_{k_\rho}^{(3)}/6\right)^{\tau/3} = {} & \gamma^\tau \left[1 + \frac{\tau}{\sqrt{k_\rho m}} P_N^{(3)} + \frac{\tau}{3\gamma} A_0(n/k_\rho) \frac{g(k_\rho, n, \rho)}{\rho} \left\{ \frac{1}{(1-\rho)^3} - 1 \right\} \right. \\ & + \frac{\tau}{6\gamma} A_0^2(n/k_\rho) \frac{g(k_\rho, n, 2\rho)}{\rho^2} \left\{ \frac{1}{(1-2\rho)^2} - \frac{2}{(1-\rho)^2} + 1 \right\} \\ & \left. + \frac{\tau}{3\gamma} A_0(n/k_\rho) B_0(n/k_\rho) \frac{g(k_\rho, n, \rho+\tilde{\rho})}{\rho+\tilde{\rho}} \left\{ \frac{1}{(1-\rho-\tilde{\rho})^3} - 1 \right\} \right] + \frac{1}{\sqrt{k_\rho m}} o_P(1). \end{aligned}$$

It follows that, as $N \to \infty$,

$$\begin{aligned} \gamma^{-\tau} \left\{ \left(R_{k_\rho}^{(1)}\right)^\tau - \left(R_{k_\rho}^{(2)}/2\right)^{\tau/2} \right\} = {} & \frac{\tau}{\sqrt{k_\rho m}} \left(P_N^{(1)} - P_N^{(2)} \right) + \frac{\tau}{\gamma} g(k_\rho, n, \rho) A_0(n/k_\rho) \frac{-\rho}{2(1-\rho)^2} \\ & + A_0^2\big(n/k_\rho\big) O(1) + A_0\big(n/k_\rho\big) B_0\big(n/k_\rho\big) O(1) + \frac{1}{\sqrt{k_\rho m}} o_P(1), \end{aligned}$$

and

$$\gamma^{-\tau}\left\{\left(R_{k_\rho}^{(2)}/2\right)^{\tau/2}-\left(R_{k_\rho}^{(2)}/6\right)^{\tau/3}\right\}=\frac{\tau}{\sqrt{k_\rho m}}\left(P_N^{(2)}-P_N^{(3)}\right)+\frac{\tau}{\gamma}g(k_\rho,n,\rho)A_0(n/k_\rho)\frac{\rho(\rho-3)}{6(1-\rho)^3}$$
$$+A_0^2\left(n/k_\rho\right)O(1)+A_0\left(n/k_\rho\right)B_0\left(n/k_\rho\right)O(1)+\frac{1}{\sqrt{k_\rho m}}o_P(1).$$

By the condition (5), the dominating terms in the two expressions above are

$$\frac{\tau}{\gamma}g(k_\rho,n,\rho)A_0(n/k_\rho)\frac{-\rho}{2(1-\rho)^2}\quad\text{and}\quad\frac{\tau}{\gamma}g(k_\rho,n,\rho)A_0(n/k_\rho)\frac{\rho(\rho-3)}{6(1-\rho)^3},$$

respectively. Therefore, as $N\to\infty$,

$$T_{k_\rho,\tau}=3\frac{\rho-1}{\rho-3}\left\{1+\frac{\gamma}{\sqrt{k_\rho m}}\frac{2(1-\rho)^2}{-\rho A_0(n/k_\rho)}\left(P_N^{(1)}-P_N^{(2)}\right)-\frac{\gamma}{\sqrt{k_\rho m}A_0(n/k_\rho)}\frac{6(1-\rho)^3}{\rho^2-3\rho}\left(P_N^{(2)}-P_N^{(3)}\right)\right\}$$
$$+O_P\left\{A_0\left(n/k_\rho\right)\right\}+O_P\left\{B_0\left(n/k_\rho\right)\right\}+\frac{1}{\sqrt{k_\rho m}A_0(n/k_\rho)}o_P(1).$$

It follows that as $N\to\infty$,

$$\sqrt{k_\rho m}A_0(n/k_\rho)\left(T_{k_\rho,\tau}-3\frac{\rho-1}{\rho-3}\right)=-\gamma\frac{2(1-\rho)^2}{\rho}\left(P_N^{(1)}-P_N^{(2)}\right)$$
$$-\gamma\frac{6(1-\rho)^3}{\rho^2-3\rho}\left(P_N^{(2)}-P_N^{(3)}\right)+O_P(1).$$

Theorem 1 is thus proved by applying the Cramér's delta method. □

Proof of Theorem 2 By Proposition 1, as $N\to\infty$, $R_{k_n}^{(1)}$ has the following asymptotic expansion:

$$\sqrt{k_n m}\left(R_{k_n}^{(1)}-\gamma\right)-\gamma P_N^{(1)}-\frac{g(k_n,n,\rho)}{1-\rho}\sqrt{k_n m}A_0(n/k_n)=o_P(1),$$

which leads to

$$\sqrt{k_n m}\left\{\left(R_{k_n}^{(1)}\right)^2-\gamma^2\right\}-2\gamma^2P_N^{(1)}-2\gamma\frac{g(k_n,n,\rho)}{1-\rho}\sqrt{k_n m}A_0(n/k_n)=o_P(1).$$

Together with the asymptotic expansion of $R_{k_n}^{(2)}$, we have that, as $N\to\infty$,

$$\sqrt{k_n m}\left\{R_{k_n}^{(2)}-2\left(R_{k_n}^{(1)}\right)^2\right\}-\gamma^2\left(P_N^{(2)}-4P_N^{(1)}\right)-\sqrt{k_n m}A_0(n/k_n)g(k_n,n,\rho)\frac{2\gamma\rho}{(1-\rho)^2}=o_P(1).$$

Thus, as $N\to\infty$,

$$\begin{aligned}\sqrt{k_n m}\left(\tilde{\gamma}_{k_n,k_\rho,\tau}-\gamma\right) &= \sqrt{k_n m}\left(R_{k_n}^{(1)}-\gamma\right)-\frac{1}{2R_{k_n}^{(1)}\hat{\rho}_{k_\rho,\tau}(1-\hat{\rho}_{k_\rho,\tau})^{-1}}\sqrt{k_n m}\left\{R_{k_n}^{(2)}-2\left(R_{k_n}^{(1)}\right)^2\right\}\\ &= \gamma P_N^{(1)}+\sqrt{k_n m}A_0(n/k_n)\frac{g(k_n,n,\rho)}{1-\rho}+o_P(1)\\ &\quad -\frac{1}{2R_{k_n}^{(1)}\hat{\rho}_{k_\rho,\tau}(1-\hat{\rho}_{k_\rho,\tau})^{-1}}\left\{\gamma^2\left(P_N^{(2)}-4P_N^{(1)}\right)\right.\\ &\quad \left.+\sqrt{k_n m}A_0(n/k_n)g(k_n,n,\rho)\frac{2\gamma\rho}{(1-\rho)^2}+o_P(1)\right\}\\ &= \gamma P_N^{(1)}-\frac{\gamma^2(1-\hat{\rho}_{k_\rho,\tau})}{R_{k_n}^{(1)}\hat{\rho}_{k_\rho,\tau}}\left(P_N^{(2)}/2-2P_N^{(1)}\right)\\ &\quad +\sqrt{k_n m}A_0(n/k_n)\frac{\rho}{(1-\rho)^2}g(k_n,n,\rho)\left(\frac{1-\rho}{\rho}-\frac{1-\hat{\rho}_{k_\rho,\tau}}{\hat{\rho}_{k_\rho,\tau}}\right)+o_P(1).\end{aligned}$$

The relation $k_n/k_\rho \to 0$ implies that $A(n/k_n)/A(n/k_\rho) \to 0$ as $N \to \infty$. Thus, by Theorem 1, we have that, as $N \to \infty$,

$$\sqrt{k_n m}A_0(n/k_n)\frac{\rho}{(1-\rho)^2}g(k_n,n,\rho)\left(\frac{1-\rho}{\rho}-\frac{1-\hat{\rho}_{k_\rho,\tau}}{\hat{\rho}_{k_\rho,\tau}}\right)=o_P(1).$$

Together with the consistency of $\hat{\rho}_{k_\rho,\tau}$ and $R_{k_n}^{(1)}$, we have that, as $N \to \infty$,

$$\sqrt{k_n m}\left(\tilde{\gamma}_{k_n,k_\rho,\tau}-\gamma\right)=\frac{\gamma}{\rho}\left\{P_N^{(2)}(\rho-1)/2+P_N^{(1)}(2-\rho)\right\}+o_P(1).$$

Combining with Proposition 1, we obtain that, as $N \to \infty$,

$$\sqrt{k_n m}\left(\tilde{\gamma}_{k_n,k_\rho,\tau}-\gamma\right)\xrightarrow{d}N\left[0,\gamma^2\left\{1+\left(\rho^{-1}-1\right)^2\right\}\right].$$

□

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Data availability Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Code availability The code for simulation study is available upon request.

Declarations

Conflicts of interest The authors declare that they have no conflict of interest.

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