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The discrete skewed Laplace distribution is a flexible distribution with integer domain and simple closed form that can be applied to model count data. Parameters are estimated under empirical Bayes (EB) analysis and comparison are made between the Bayesian parameter estimation and classical parameter estimation, i.e. the maximum likelihood (ML) approach. The results show that the Bayesian parameter estimations are preferable.

Keywords: Empirical Bayes, discrete skewed Laplace distribution, Bayesian parameter

Introduction

Skewed distributions are a non-normality of interest (Azzalini, 1985). For example, discrete skewed distributions could be used to model count data. One way to produce discrete skewed distribution is based on the survival function to corresponded continuous case (Roy & Dasgupta, 2001; Roy, 2003). The discretization for continuous distributions based on the positive real numbers produce discreet distributions on the positive integer numbers, such as the discrete Gamma, Weibull, and negative binomial distributions in Chakraborty and Chakravarty (2012). Roy (2004) presented the discrete Rayleigh distribution and Krishna and Sing (2009) investigated discrete Burr and Pareto. One of the flexible discrete skewed distributions that is defined by Barbiero (2014) on the whole integer numbers is the discrete skewed Laplace distribution. A main advantage for this distribution is the closed and simple forms of its probability function, distribution function, mathematical expectation, and variance. The purpose of this study is to present empirical Bayesian analysis for the parameter estimation.

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Discrete Skewed Laplace Distribution

Yu and Zhang (2005) and Kozubowski and Nadarajah (2010) defined different forms for the discrete skewed Laplace distribution. Here, a simple closed form based of the difference between survival functions is used as a way to create a discrete distribution based on the continuous one (Barbiero, 2014).

So, let the continuous skewed Laplace distribution be as follows:

$$\mathrm{f}\left(x;p,q\right)=\frac{-\log\left(p\right)\log\left(q\right)}{\log\left(p\right)}\begin{cases}p^{x} & x\geq 0\\ q^{-x} & x<0\end{cases}$$

such that $0 < p,\ q < 1$ are unknown parameters. The survival function for this distribution is defined as

$$\mathrm{S}\left(x;p,q\right)=\frac{1}{\log\left(pq\right)}\begin{cases}\dfrac{1}{\log\left(p\right)}q^{-x} & x=\ldots,-2,-1\\ \dfrac{1}{\log\left(q\right)}p^{x} & x=0,1,2,\ldots\end{cases}$$

Now, using the instruction rule to construct a discrete distribution based on the differences between survival functions of the continuous one, i.e.

$$\varnothing\left(x\right)=\begin{cases}\mathrm{S}\left(x\right)-\mathrm{S}\left(x+1\right) & x\in\mathbb{Z}\\ 0 & \text{o.w.}\end{cases}$$

we have

$$\begin{aligned}\varnothing\left(x;p,q\right)&=\frac{1}{\log\left(pq\right)}\begin{cases}\dfrac{1}{\log\left(p\right)}\left[q^{-(x+1)}-q^{-x}\right]\\ \dfrac{1}{\log\left(q\right)}\left[p^{x}-p^{x+1}\right]\end{cases}\\ &=\frac{1}{\log\left(pq\right)}\begin{cases}\log\left(p\right)\left[q^{-(x+1)}\left(1-q\right)\right] & x=\ldots,-2,-1\\ \log\left(q\right)\left[p^{x}\left(1-p\right)\right] & x=0,1,2,\ldots\end{cases}\end{aligned}$$

This is the simple closed form for the discrete skewed Laplace distribution with $0 < p, q < 1$ that known as

$$\mathbf{X}_d \sim \text{ADSLaplace}(p,q)$$

Now, using the iid sample $\mathbf{X} = (X_1,\ldots, X_n)$, the maximum likelihood (ML) function is defined as

$$\begin{aligned}
\mathrm{l}(p,q;\mathbf{X}) &= \log\prod_{i=1}^{n}\varnothing(p,q;X_i) \\
&= s^{-}\log\left(\frac{\log(p)}{\log(pq)}\right) + s^{+}\log\left(\frac{\log(q)}{\log(pq)}\right) - \left(s^{-} + \sum_{X_i<0} X_i\right)\log(q) \\
&\quad + s^{-}\log(1-q) + \sum_{X_i<0} X_i \log(p) + s^{+}\log(1-p)
\end{aligned}$$

such that s^- and s^+ are defined as number of the negative and positive samples, respectively, i.e. as

$$s^{-} = \sum_{i=1}^{n} 1_{x_i<0}, \quad s^{+} = \sum_{i=1}^{n} 1_{x_i\geq 0}$$

Now, using first order derivative of the likelihood function, the ML estimation for the desired parameters are the solutions to the following equations:

$$\begin{aligned}
\frac{\partial \mathrm{l}(p,q;\mathbf{x})}{\partial p} &= s^{-}\frac{\log(pq)}{log(p)}\frac{1}{p}\frac{1}{\left(log(pq)\right)^2} - s^{+}\frac{\log(pq)}{log(q)}\frac{1}{p}\frac{\log q}{\left(log(pq)\right)^2} \\
&\quad + \sum_{x_i\geq 0}\frac{x_i}{p} - \frac{s^{+}}{1-p} \\
&= \frac{s^{-}}{p}\frac{\log(q)}{\log(p)}\frac{1}{\log(pq)} - \frac{s^{+}}{p}\frac{1}{\log(pq)} + \sum_{x_i\geq 0}\frac{x_i}{p} - \frac{s^{+}}{1-p} \\
&= \frac{s^{-}\log q + s^{-}\log p - s^{-}\log p - s^{+}\log p}{p\log p\log(pq)} + \sum_{x_i\geq 0}\frac{x_i}{p} - \frac{s^{+}}{1-p}
\end{aligned}$$

$$= \frac{s^-}{p\log(p)} - \frac{n}{p\log(pq)} + \sum_{x_i \ge 0} \frac{x_i}{p} - \frac{s^+}{1-p}$$

So we have

$$\frac{\partial \mathrm{l}(p,q;\mathbf{x})}{\partial p} = s^- \frac{1}{p\log(p)} - n\frac{1}{p\log(pq)} + \sum_{x_i \ge 0} \frac{x_i}{p} - \frac{s^+}{1-p}$$

In a similar way,

$$\frac{\partial \mathrm{l}(p,q;\mathbf{x})}{\partial q} = s^+ \frac{1}{q\log(q)} - n\frac{1}{q\log(pq)} - \sum_{x_i < 0} \frac{x_i}{q} - \frac{s^-}{q(1-q)}$$

So the solutions of these equations lead to the $(\hat{p}_{ML}, \hat{q}_{ML})$ such that we did not have closed form and should solve analytically.

Bayesian Analysis

Let $\boldsymbol{\theta} = (p, q)$ be the parameters of the discrete skewed Laplace distribution with the prior distribution $\pi(\boldsymbol{\theta}) = \pi(p)\pi(q \mid p)$. Note that we assume p and q are independent, so $\pi(\boldsymbol{\theta}) = \pi(p)\pi(q)$. Also, the prior distribution for p and q are the non-informative prior U(0, 1), the uniform distribution. If $\mathrm{f}(\mathbf{x} \mid \boldsymbol{\theta})$ is the desired distribution, then the posterior distribution of $\boldsymbol{\theta}$ given $\mathbf{x}$ is as follows:

$$\pi(\boldsymbol{\theta} \mid \mathbf{x}) = \frac{\mathrm{f}(\mathbf{x} \mid \boldsymbol{\theta})}{\int_{\boldsymbol{\theta}} \mathrm{f}(\mathbf{x} \mid \boldsymbol{\theta})\, d\boldsymbol{\theta}}$$

Note that, under the square integrable loss function, the Bayes estimator for $\boldsymbol{\theta} = (p, q)$ is as follows:

$$\boldsymbol{\theta}^{\mathrm{B}} = \frac{\int_{\boldsymbol{\theta}} \boldsymbol{\theta} \prod_{i=1}^{n} \mathrm{f}(x_i \mid \boldsymbol{\theta})\pi(\boldsymbol{\theta})\, d\boldsymbol{\theta}}{\int_{\boldsymbol{\theta}} \prod_{i=1}^{n} \mathrm{f}(x_i \mid \boldsymbol{\theta})\pi(\boldsymbol{\theta})\, d\boldsymbol{\theta}}$$

That leads to

$$\boldsymbol{\theta}^{\mathrm{B}} = \frac{\mathrm{E}\left[\boldsymbol{\theta}\prod_{i=1}^{n}\mathrm{f}\left(x_i \mid \boldsymbol{\theta}\right)\pi\left(\boldsymbol{\theta}\right)\right]}{\mathrm{E}\left[\prod_{i=1}^{n}\mathrm{f}\left(x_i \mid \boldsymbol{\theta}\right)\pi\left(\boldsymbol{\theta}\right)\right]}$$

Now, if $\theta_1, \theta_2, \ldots, \theta_m$ are m iid samples from the prior distribution $\pi(\boldsymbol{\theta})$, we have

$$\frac{1}{m}\sum_{j=1}^{m}\theta_j\prod_{i=1}^{n}\mathrm{f}\left(x_i \mid \theta_j\right)\xrightarrow{\text{a.s.}}\mathrm{E}\left[\boldsymbol{\theta}\prod_{i=1}^{n}\mathrm{f}\left(x_i \mid \boldsymbol{\theta}\right)\right], \quad m \to \infty$$

and

$$\frac{1}{m}\sum_{j=1}^{m}\prod_{i=1}^{n}\mathrm{f}\left(x_i \mid \theta_j\right)\xrightarrow{\text{a.s.}}\mathrm{E}\left[\prod_{i=1}^{n}\mathrm{f}\left(x_i \mid \boldsymbol{\theta}\right)\right], \quad m \to \infty$$

So the empirical Bayes (EB) estimator $\boldsymbol{\theta}^{\mathrm{EB}}$ is as follows:

$$\boldsymbol{\theta}^{\mathrm{EB}} = \frac{\frac{1}{m}\sum_{j=1}^{m}\theta_j\prod_{i=1}^{n}\mathrm{f}\left(x_i \mid \boldsymbol{\theta}\right)}{\frac{1}{m}\sum_{j=1}^{m}\prod_{i=1}^{n}\mathrm{f}\left(x_i \mid \boldsymbol{\theta}\right)}$$

and finally the Bayes estimators $\hat{p}_{\mathrm{EB}}$ and $\hat{q}_{\mathrm{EB}}$ can be easily found.

Simulation Study

Now, to validate our estimation method presented in this paper, we simulate 1000 samples for different combinations of the parameters (p, q) and compare the ML estimator with the EB method. Note that this can be easily achieved through the R package DiscreteLaplace (Barbiero, 2014; Barbiero & Inchingolo, 2016). As Table 1 shows, the differences between the considered values for (p, q) and their EB estimators are less than that of the estimators provided by the ML method.

Table 1. Simulation study for empirical Bayes and maximum likelihood methods

(p, q)	$\hat{p}_{ML}$	$\hat{q}_{ML}$	$\hat{p}_{EB}$	$\hat{q}_{EB}$
(0.25, 0.25)	0.211	0.229	0.231	0.238
(0.25, 0.50)	0.221	0.472	0.231	0.495
(0.50, 0.25)	0.482	0.232	0.491	0.239
(0.75, 0.75)	0.724	0.722	0.731	0.739

Numerical Example

Kappenman (1975) and later Barbiero (2014) considered the following data:

1.96, 1.96, 3.60, 3.80, 4.79, 5.66, 5.76, 5.78, 6.27, 6.30, 6.76, 7.65,
7.84, 7.99, 8.51, 9.18, 10.13, 10.24, 10.25, 10.43, 11.45, 11.48, 11.75,
11.81, 12.34, 12.78, 13.06, 13.29, 13.98, 14.18, 14.40, 16.22, 17.06

which are assumed to represent a random sample of size n = 33 from a symmetrical Laplace distribution with a location parameter. Before employing these data for our purposes, we transform them by subtracting their median, 10.13, and then take its integer part. We expect that these final values can be modeled through our proposed discrete distribution. We then apply our estimation methods discussed above and compare these estimators to those which the maximum likelihood method provides: $\hat{p}_{\text{ML}} = 0.7165$ and $\hat{q}_{\text{ML}} = 0.7657$, while $\hat{p}_{\text{EB}} = 0.7005$ and $\hat{q}_{\text{EB}} = 0.7511$. The Bayesian information criteria (BIC) for the ML estimation are -211.75, while this criteria for empirical Bayes method is -236.54; this shows that the EB estimators are more efficient than classical estimators.

Conclusion

The presented paper investigates Bayesian analysis for the discrete skewed Laplace distribution and compares it to the classical estimation method, the maximum likelihood estimator. The BIC criteria show that the empirical Bayes estimators are preferable.

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