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In this paper, hierarchical Bayes approach is presented for estimation and prediction of reliability indexes and remaining lifetimes of a cold standby series system under general progressive Type II censoring scheme. A simulation study has been carried out for comparison purpose. The study will help reliability engineers in various industrial series system setups.

Keywords: Cold standby series system, general progressive Type II censoring, hierarchical Bayes estimation, Monte Carlo simulation

Introduction

A cold standby series system is widely applied to achieve high reliability in various engineering systems used in space exploration and satellite, textile manufacturing and carbon recovery systems. In such a series system, some units are placed in working mode while the rest in cold standby mode. When any unit in the working mode fails, it is replaced by any of the standby units in negligible time to survive the engineering system. The standby system becomes invalid when all standby units are used up, and one of the working units becomes unusable.

Mei, Liao, and Sun (1992) discussed the point estimation of reliability indexes by assuming that the life units in the series system have identical exponential distribution, and the failure rate is a known constant. Under the assumption that the failure rate is a random variable, Su and Gu (2003) derived the Bayes estimates while Bai, Yu, and Hu (1998) derived the multiple Bayes estimates of reliability indexes for the series system. Pham and Turkkan (1994) studied the reliability of

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the series system with Beta distribution component live. Willits (1997) studied reliability estimation of the series system using small binomial samples. Xu, Kang, and Shi (2002) discussed Bayesian and multiple Bayesian analysis of reliability performances for the series system. Barot and Patel (2014) derived the exact confidence limits of the reliability indexes for a cold standby series system under general progressive Type II censoring scheme using an empirical Bayesian approach.

In a life testing experiment, a censoring scheme that can balance between total times spent, number of units used and efficiency of statistical inference based on the results of an experiment is desirable. For this reason a more general censoring scheme called, general progressive Type II censoring scheme, has received a significant importance in the last few decades. This censoring scheme is extremely useful in both industrial life testing and clinical settings. The numerous articles dealing with inference procedures under this censoring scheme have been found in the journals (e.g., Balakrishnan & Sandhu, 1996; Fernández, 2004; Kim & Han, 2009; Barot & Patel, 2014).

In Bayes approach, the posterior distribution of the parameters of interest given the data is obtained by assuming that the model hyper-parameter is known and then inferences are considered based on this distribution. However, when the information regarding the model hyper-parameter is unknown, empirical Bayes or hierarchical Bayes approaches are used to handle the super parameter structure for the estimation and prediction. In the empirical Bayes approach, the posterior distribution of the parameter of interest given the data is first obtained, assuming that the model hyper-parameters are known. The hyper-parameter is estimated from the marginal distribution of the data, and inferences are then based on the estimated posterior distribution.

However, in the case of non-availability of empirical data, estimates of parameters can be obtained through only an expert consulting. In such situations, hierarchical Bayes approach is more preferable than empirical Bayes approach. In hierarchical Bayes approach, a prior distribution of the hyper-parameter is specified according to expert's opinions, and then the posterior distribution of the parameter of interest is obtained. A parameter of interest is then estimated by its posterior mean and its precision is measured by its posterior variance. The hierarchical Bayes approach is straightforward and clear-cut, but computationally intensive, often involving high dimensional integration. It looks promising, but caution should be exercised in applying this approach. It has been described and applied extensively for various statistical inferences in literature (e.g., Han, 1998; Lehmann & Casella, 1998; Papadopoulos, Tiwari, & Zalkikar, 1996; Younes, Delampady, MacGibbon, & Cherkaoui, 2007).

Statistical prediction was the most prevalent form of statistical inference, which is very important in a variety of disciplines such as medicine, engineering, and business. Various authors have studied the prediction problems in reliability and life testing problems (e.g., Dunsmore, 1974; Chhikara & Guttman, 1982; Ali Mousa, 2001; Ali Mousa & Jaheen, 2002).

Most of the research on a cold standby series system has focused on the usual Bayes approach. The objective of the present paper is to investigate estimation and prediction of reliability indexes and remaining lifetimes of the series system using a hierarchical Bayes approach under general progressive Type II censoring scheme.

Bayes Estimation of Reliability Indexes

In reliability and life testing studies, an exponential distribution is one of the most widely used lifetime models, and inference based on this distribution can be used quite effectively. A number of lifetime data have been analyzed, and it was observed that in most of the cases an exponential distribution provides a good fit. This distribution has been used to describe the life span of many items such as electronic tubes, light bulbs and mechanical components.

Suppose that a cold standby series system has $(k + n - 1)$ identical units comprising a series of k working units $U_1, U_2, \ldots, U_k$ being in an operational state and $(n - 1)$ standby units $S_1, S_2, \ldots, S_{(n-1)}$ connected in a series. When any unit of the series of k working units fails, any unit of $(n - 1)$ standby units replaces it immediately through an alternation switch in negligible time, so that the series system stays operational. Figure 1 shows a functional diagram of the series system. Barot and Patel (2014) have considered such a series system and placed it on a life testing experiment under general progressive Type II censoring scheme by assuming that every unit has the failure rate $k\lambda$ with the probability density and cumulative distribution functions, respectively, as

$$\mathrm{f}\left(x \mid \lambda\right) = \lambda k \mathrm{e}^{-\lambda k x}, \quad \lambda, x, k > 0 \tag{1}$$

and

$$\mathrm{F}\left(x \mid \lambda\right) = 1 - \mathrm{e}^{-\lambda k x} \tag{2}$$

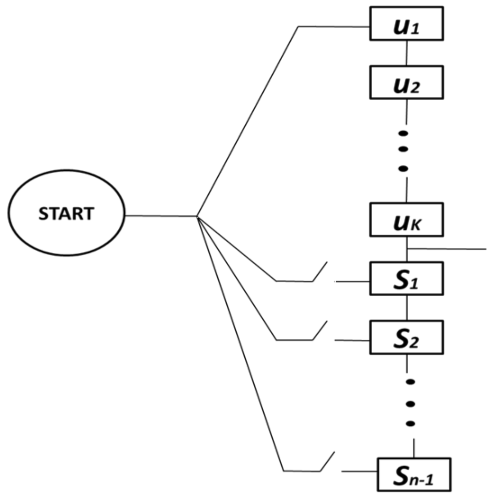

Figure 1. Cold standby series system with (k + n – 1) identical units

According to Cao and Cheng (1986), the reliability R(t) and average life *MTTF* of the series system are strictly monotonic decreasing functions with respect to and can be given, respectively, by

$$\mathrm{R}\left(t\right)=\sum_{i=0}^{n-1}\frac{\mathrm{e}^{-\lambda kt}\left(\lambda kt\right)^{i}}{i!},\quad MTTF=\frac{n}{\lambda k} \tag{3}$$

Under the general progressive Type II scheme, the lifetimes of the first s units, i.e., $x_{(1)}, x_{(2)},\ldots, x_{(s)}$ are not observed, and then the lifetimes until the m^{th} failure, i.e., $x_{(s+1)}, x_{(s+2)},\ldots, x_{(m)}$ are completely observed. At the time of every i^{th} failure, r_i units are randomly removed from the remaining (n – s – 1) standby units (i = s + 1, s + 2,…, m – 1). Instead of continuing the test until the entire standby units are used up, the test is terminated at the time of the m^{th} failure ($m < n$), and all the remaining r_m standby units are removed from the test, where r_m is given by

$$r_m=n-m-1-\sum_{i=s+1}^{m-1}r_i$$

Following Barot and Patel (2014), the likelihood function based on the general progressive Type II sample $\mathbf{x} = (x_{(s+1)}, x_{(s+2)},\ldots, x_{(m)})$ can be written as

$$\mathrm{L}(\mathbf{x} \mid \lambda) = A\lambda^{m-s} k^{m-s} \left(1 - \mathrm{e}^{-\lambda k x_{(s+1)}}\right)^{s} \mathrm{e}^{-\lambda k w} \tag{4}$$

where

$$A = \binom{n-1}{s}(n-s-2)\prod_{j=s+2}^{m}\left(n-j-1-\sum_{i=s+1}^{j-1} r_i\right), \quad w = \sum_{i=s+1}^{m}(1+r_i)\,x_{(i)}$$

The concern in Bayesian estimation is an appropriate choice of a prior distribution for a parameter to consider subjective information from experienced experts. An exponential distribution is one of most prominent random probability distributions, and its good mathematical properties facilitate insight and computational reduction. In reliability analysis and life testing, it is preferred over many other distributions due to its richness, computational ease, better fit to the failure data, analytical tractability, and easy interpretability. To ease the computational burden and get computable closed form expression for the posterior distribution, it is assumed that the unknown failure rate λ is the realization of a random variable and follows an exponential prior with the probability density function

$$\pi(\lambda \mid \beta) = \beta \mathrm{e}^{-\beta\lambda}, \quad \lambda > 0 \tag{5}$$

The likelihood function (4) and prior distribution (5) can be easily combined to form a posterior distribution that represents total knowledge about the parameter λ after the data have been observed. It is

$$\pi(\lambda \mid \mathbf{x}) = \frac{\lambda^{m-s}\left[1 - \mathrm{e}^{-\lambda k x_{(s+1)}}\right]^{2} \mathrm{e}^{-\lambda(kw+\beta)}}{\Gamma(m-s+1)\,D_s} \tag{6}$$

where

$$D_s = \sum_{j=0}^{s}\binom{s}{j}(-1)^j \frac{1}{\left[kw_j + \beta\right]^{m-s+1}}; \quad w_j = w + x_{(s+1)}\,j$$

In Bayesian analysis, a loss function must be specified in order to obtain Bayes estimates. The loss function is a non-negative function of the distance between the estimate and the true value. When decisions become gradually more damaging for large errors, the use of squared error loss function, $\mathrm{L}\left(\hat{\lambda},\lambda\right)=\left(\hat{\lambda}-\lambda\right)^2$, is more appropriate because of its analytical tractability. The Bayes estimate of parameter λ, reliability R(t) and *MTTF* can be obtained under the squared error loss function, respectively, as

$$\hat{\lambda}_B=\frac{\left(m-s+1\right)D_{S1}}{D_S} \tag{7}$$

$$\hat{\mathrm{R}}\left(t\right)_B=\frac{1}{\Gamma\left(m-s+1\right)D_S}\sum_{i=0}^{n-1}\frac{\left(kt\right)^i\,\Gamma\left(m-2+i+1\right)}{i!}D_{S2(i)} \tag{8}$$

$$MTTF_B=\frac{nD_{S3}}{k\left(m-s\right)D_S} \tag{9}$$

where

$$D_{S1}=\sum_{j=0}^{s}\binom{s}{j}\left(-1\right)^j\frac{1}{\left[kw_j+\beta\right]^{m-s+2}};\quad D_{S2(i)}=\sum_{j=0}^{s}\binom{s}{j}\left(-1\right)^j\frac{1}{\left(kt+kw_j+\beta\right)^{m-s+i+1}};$$

$$D_{S3}=\sum_{j=0}^{s}\binom{s}{j}\left(-1\right)^j\frac{1}{\left(kw_j+\beta\right)^{m-s}}$$

Hierarchical Bayesian Analysis

The idea in a Bayesian model is that when you look at a likelihood function and decide right priors for parameters. Instead, it may be more appropriate to use priors depending on other parameters those are not mentioned in a likelihood function. These parameters themselves will require priors and can depend on new ones. This can continue in a hierarchical framework until there are no more parameters to incorporate in the model. In this section, hierarchical Bayes estimates of reliability indexes of the series system are constructed.

Due to the complicity of practical problems and uncertainty about the true level of an expert, it is quite difficult to give the exact estimate of a super parameter β. However, the value of β can be obtained in an approximate interval denoted by (a, b) through an expert consulting. As there is no other information on the parameter β, it is assumed that it has uniform distribution on (a, b) with the probability density function

$$\mathrm{U}(\beta \mid a,b) = \frac{1}{b-a}$$

given in Xu et al. (2002). In order to obtain the posterior density of β given $\mathbf{x}$,

$$\int_a^b \mathrm{U}(\beta \mid a,b)\,\mathrm{m}(\mathbf{x} \mid \beta)\,d\beta = \frac{\Gamma(m-s+1)}{b-a} A k^{m-s} \sum_{j=0}^{s} \binom{s}{j} (-1)^j \int_a^b \frac{\beta}{\left(kw_j + \beta\right)^{m-s+1}}\,d\beta$$
$$= \frac{A k^{m-s}\Gamma(m-s-1)}{a-b} D_{S4}$$

where

$$D_{S4} = \sum_{j=0}^{s} \binom{s}{j} (-1)^j \left[\left(kw_j + b\right)^{-m+s}\left(kw_j + bm - bs\right) - \left(kw_j + a\right)^{-m+s}\left(kw_j + am - as\right)\right]$$

From Bayes theorem, the posterior density of β given $\mathbf{x}$ can be obtained as

$$\mathrm{h}(\beta \mid \mathbf{x}) = \frac{\mathrm{U}(\beta \mid a,b)\,\mathrm{m}(\mathbf{x} \mid \beta)}{\int_a^b \mathrm{U}(\beta \mid a,b)\,\mathrm{m}(\mathbf{x} \mid \beta)\,d\beta} = \frac{-\beta(m-s)(m-s-1) D_S}{D_{S4}}$$

Under the squared error loss function, the Bayes estimate of β can be given by

$$\hat{\beta}_B = \mathrm{E}^h(\beta \mid \mathbf{x}) = \int_a^b \beta\,\mathrm{h}(\beta \mid \mathbf{x})\,d\beta = \frac{D_{S5} - D_{S6}}{(m-s-2) D_{S4}} \tag{10}$$

where

$$D_{S5}=\sum_{j=0}^{s}\binom{s}{j}(-1)^{j}\left\{\frac{2k^{2}w_{j}^{2}+2bk(m-s)w_{j}+(m-s)(m-s-1)b^{2}}{(kw_{j}+b)^{m-s}}\right\}$$

$$D_{S6}=\sum_{j=0}^{s}\binom{s}{j}(-1)^{j}\left\{\frac{2k^{2}w_{j}^{2}+2ak(m-s)w_{j}+(m-s)(m-s-1)a^{2}}{(kw_{j}+a)^{m-s}}\right\}$$

Using (10) in (7), (8), and (9), the hierarchical Bayes estimates of λ, R(t) and *MTTF* under the squared error loss function can be obtained as follows:

$$\hat{\lambda}_{HB}=\frac{(m-s+1)D_{S7}}{D_{S'}} \tag{11}$$

$$\hat{\mathrm{R}}(t)_{HB}=\frac{1}{\Gamma(m-s+1)D_{S'}}\sum_{i=0}^{n-1}\frac{(kt)^{i}\,\Gamma(m-s+i+1)}{i!}D_{S8(i)} \tag{12}$$

$$MTTF_{HB}=\frac{nD_{S9}}{k(m-s)D_{S'}} \tag{13}$$

where

$$D_{S'}=\sum_{j=0}^{s}\binom{s}{j}(-1)^{j}\frac{1}{\left[kw_{j}+\hat{\beta}_{B}\right]^{m-s+1}};\quad D_{S7}=\sum_{j=0}^{s}\binom{s}{j}(-1)^{j}\frac{1}{\left[kw_{j}+\hat{\beta}_{B}\right]^{m-s+2}};$$

$$D_{S8(i)}=\sum_{j=0}^{s}\binom{s}{j}(-1)^{j}\frac{1}{\left(kt+kw_{j}+\hat{\beta}_{B}\right)^{m-s+i+1}};\quad D_{S9}=\sum_{j=0}^{s}\binom{s}{j}(-1)^{j}\frac{1}{\left[kw_{j}+\hat{\beta}_{B}\right]^{m-s}};$$

In Bayesian inference, the 100(1 – α)% highest probability density (HPD) interval of the parameter of interest is the shortest interval in parameter space that contains 100(1 – α)% of the probable values of the parameter. It is one of the most useful tools to measure posterior uncertainty that includes more probable values and excludes the least probable values of the parameter. Since the posterior distribution (6) is unimodal, the 100(1 – α)% Bayes HPD-interval (p_1, p_2) for λ must simultaneously satisfy the equations

$$\pi^*\left(p_1 \mid \mathbf{x}\right) = \pi^*\left(p_2 \mid \mathbf{x}\right) \tag{14}$$

$$\int_{p_1}^{p_2} \pi^*\left(\lambda \mid \mathbf{x}\right) d\lambda = 1-\alpha \tag{15}$$

After tedious algebra, the equations (14) and (15) can be written in the form

$$\left(\frac{p_1}{p_2}\right)^{m-s} \mathrm{e}^{\left(p_2-p_1\right)\left(kw+\beta\right)} = \left(\frac{1-\mathrm{e}^{-p_2 k x_{(s+1)}}}{1-\mathrm{e}^{-p_1 k x_{(s+1)}}}\right) \tag{16}$$

and

$$\frac{1}{D_S}\sum_{j=0}^{s}\binom{s}{j}(-1)^j c_j^{-m+s-1}\left[\mathrm{e}^{-c_j p_1}\sum_{j_1=0}^{m-s}\frac{\left(c_j p_1\right)^{j_1}}{j_1!} - \mathrm{e}^{-c_j p_2}\sum_{j_1=0}^{m-s}\frac{\left(c_j p_2\right)^{j_1}}{j_1!}\right] = 1-\alpha \tag{17}$$

where $c_j = kw_j + \beta$.

The 100(1 – α)% Bayes HPD-intervals of R(t) and *MTTF* can be obtained from (8) and (9). When the super parameter β is unknown, the 100(1 – α)% hierarchical Bayes HPD-intervals of reliability indexes can be obtained by using the estimate $\hat{\beta}_B$ for β.

Prediction of Remaining Lifetimes Truncated at $x_{(m)}$

The prediction of remaining lifetimes, based on a current available sample, known as an informative sample, is an important feature in Bayesian analysis. Howlader (1985) presented HPD-prediction intervals for the z^{th} order statistic of a future sample. Fernández (2000) considered the problem of predicting an independent future sample from the Rayleigh distribution under doubly Type II censoring scheme. Raqab and Madi (2002) considered an estimation of the predictive distribution of the total time on a test up to certain failures in a future sample, as well as that of the remaining testing time until all the units in the original sample have failed.

Let

$$x_{(l)}, \quad m+1 \le l \le n_1,\, n_1 = n - \sum_{i=s+1}^{m-1} r_i$$

denote the lifetime of the l^{th} unit to fail. The conditional probability density function of $y = x_{(l)} - x_{(m)}$ from the probability density function truncated at $x_{(m)}$ is given by

$$\mathrm{f}_1\left(y \mid \lambda\right) = \frac{\left[\mathrm{F}\left(y \mid \lambda\right)\right]^{l-m-1}\left[1-\mathrm{F}\left(y \mid \lambda\right)\right]^{n_1-l}\mathrm{f}\left(y \mid \lambda\right)}{\mathrm{B}\left(l-m, n_1-l+1\right)}, \quad y \ge 0$$

From (1) and (2), the function $\mathrm{f}_1 = (y \mid \lambda)$ can be obtained as

$$\mathrm{f}_1\left(y \mid \lambda\right) = \frac{\lambda k \mathrm{e}^{-\lambda k y\left(n_1-l+1\right)}\left(1-\mathrm{e}^{-\lambda k y}\right)^{l-m-1}}{\mathrm{B}\left(l-m, n_1-l+1\right)}$$

Based on the general progressive Type II censored sample **x**, the conditional joint probability density function of y and λ can be written as

$$\begin{aligned}\mathrm{g}_1\left(y, \lambda \mid \mathbf{x}\right) &= \mathrm{f}_1\left(y \mid \lambda\right)\pi^*\left(\lambda \mid \mathbf{x}\right) \\ &= \frac{k\lambda^{m-s+1}\mathrm{e}^{-\lambda\left[ky\left(n_1-l+1\right)+kw+\beta\right]}\left(1-\mathrm{e}^{-\lambda k y}\right)^{l-m+1}\left(1-\mathrm{e}^{-\lambda k x_{(s+1)}}\right)^{s}}{\Gamma\left(m-s+1\right)\mathrm{B}\left(l-m, n_1-l+1\right)D_S}\end{aligned}$$

The Bayes predictive density function of y can be obtained as

$$\begin{aligned}\mathrm{p}\left(y \mid \mathbf{x}\right) &= \int_0^\infty \mathrm{g}_1\left(y, \lambda \mid \mathbf{x}\right) d\lambda \\ &= k_1\Gamma\left(m-s+2\right)\sum_{p_1=0}^{l-m-1}\sum_{q_1=0}^{s}\frac{\binom{l-m-1}{p_1}\binom{s}{q_1}\left(-1\right)^{p_1+q_1}}{w_{p_1 q_1}^{m-s+2}}\end{aligned} \tag{18}$$

where

$$k_1 = \frac{k}{\Gamma(m-s+1)\mathrm{B}(l-m, n_1-l+1)D_S}$$

$$w_{p_1q_1} = k(n_1-l+1+p_1)y + kw - kx_{(s+1)}q_1 + \beta$$

Under the squared error loss function, the Bayes predictive estimate of y can be obtained as

$$y^* = \frac{\sum_{p_1=0}^{l-m-1}\sum_{q_1=0}^{s}\binom{l-m-1}{p_1}\binom{s}{q_1}(-1)^{p_1+q_1}\frac{\left(kw+kx_{(s+1)}q_1+\beta\right)^{-m+s}}{\left(n_1-l+1+p_1\right)^2}}{k(m-s)\mathrm{B}(l-m, n_1-l+1)D_S}$$

Thus, the Bayes predictive estimate of $x_{(l)}$ can be given by

$$x_{(l)}^* = x_{(m)} + y^* \tag{19}$$

Moreover, the 100(1 – α)% Bayes HPD-prediction interval of y^* is given by (h_1, h_2), where h_1 and h_2 are solutions of the equations

$$\mathrm{p}(h_1 \mid \mathbf{x}) = \mathrm{p}(h_2 \mid \mathbf{x}) \tag{20}$$

and

$$\mathrm{p}(h_1 < y < h_2) = 1-\alpha \tag{21}$$

Using (19) in (20) and (21), after tedious algebra, we have

$$\sum_{p_1=0}^{l-m-1}\sum_{q_1=0}^{s}\frac{\binom{l-m-1}{p_1}\binom{s}{q_1}(-1)^{p_1+q_1}}{\delta_{p_1q_1}^{m-s+2}} = \sum_{p_1=0}^{l-m-1}\sum_{q_1=0}^{s}\frac{\binom{l-m-1}{p_1}\binom{s}{q_1}(-1)^{p_1+q_1}}{\gamma_{p_1q_1}^{m-s+2}} \tag{22}$$

and

$$\frac{-\sum_{p_1=0}^{l-m-1}\sum_{q_1=0}^{s}\binom{l-m-1}{p_1}\binom{s}{q_1}(-1)^{p_1+q_1}\frac{\left(\gamma_{p_1q_1}^{-m+s-1}-\delta_{p_1q_1}^{-m+s-1}\right)}{n_1-l+p_1+1}}{\mathrm{B}\left(l-m,n_1-l+1\right)D_S}=1-\alpha \tag{23}$$

where

$$\delta_{p_1q_1}=k\left(n_1-l+p_1+1\right)h_1+kw+kx_{(s+1)}q_1+\beta$$

$$\gamma_{p_1q_1}=k\left(n_1-l+p_1+1\right)h_2+kw+kx_{(s+1)}q_1+\beta$$

Hence, the 100(1 – α)% Bayes HPD-prediction interval for $x_{(l)}$ is

$$\left(x_{(m)}+h_1, x_{(m)}+h_2\right) \tag{24}$$

When the super parameter β is unknown, the hierarchical Bayes predictive estimates $x_{(l)}$ and the corresponding 100(1 – α)% hierarchical HPD-prediction interval of can be obtained by using the estimate $\hat{\beta}_B$ for β in (19) and (24).

Simulation Study

An extensive Monte Carlo simulation study was carried out to illustrate and compare the performance of hierarchical Bayes estimates of reliability indexes of the system with series of k units in working mode and (n – 1) units in cold standby mode. The performance is evaluated based on estimated risks and biases for different combinations of sample size (n), effective sample size (m – s), and general progressive Type II censoring scheme $\mathbf{r} = (r_{s+1}, r_{s+2},\ldots, r_m)$. The different censoring schemes applied in the simulation study are summarized in Table 1.

For given values a = 0, b = 1 and 100,00,000 generated uniform numbers, two values of β, one is the true value $\beta_T = 0.5002$ and another is the expert value $\beta_E = 0.4999$ were obtained by the Monte Carlo means. The corresponding $\lambda = 2.0008$ is brought from the prior (5) and the expert value β_E. Using the generated value of λ, we have generated a general progressive Type II censored sample $\mathbf{x} = (x_{(s+1)}, x_{(s+2)},\ldots, x_{(m)})$ with the censoring scheme $\mathbf{r}$ from the exponential distribution according to the algorithm presented in Balakrishnan and Sandhu (1996) that involves the following steps:

1. Generate V_m from the Beta distribution with parameters $(n - s)$ and $(s + 1)$
2. Independently generate Z_{s+i} from U(0, 1) and then set $V_{s+i} = Z_{s+i}^{\frac{1}{a_{s+i}}}$, $a_{s+i} = i + \sum_{j=m-i+1}^{m} r_j$ for i = 1, 2,…, $(m - s - 1)$
3. Set $U_{s+1} = 1 - V_m$ and $U_{s+i} = 1 - (V_{m-i+1}V_{m-i+2}\ldots V_m)$, $i = 2, 3,\ldots, (m - s)$
4. For the generated value of λ and given k, $x_{(s+i)} = -\frac{1}{\lambda k}\ln\left(1 - U_{s+i}\right)$, $i = 1, 2,\ldots, (m - s)$ is the required general progressive Type II censored sample of size $(m - s)$ from the exponential distribution

The Bayes estimates, hierarchical Bayes estimates, and the corresponding estimated risks were computed by averaging over 100,000 simulations, and are reported, respectively, in Tables 2-6. From the simulation results, the following points can be drawn:

1) For the fixed sample size n and initial s unobserved failures, as the predetermined number of failures m increases, the estimated risks of estimates of reliability indexes decrease, that is, the performance becomes better in terms of the estimated risks. (Refer to Tables 2-4)
2) For the fixed effective sample size $(m - s)$, the estimated risks of estimates of failure rate λ and reliability $R(t)$ decrease while that of *MTTF* increase with the increasing sample size n. (Refer to Tables 2-4)
3) For the fixed sample size n and predetermined number of failures m, the estimated risks of estimates of failure rate λ and reliability $R(t)$ increase while that of *MTTF* decrease with the increasing number of initial s unobserved failures. (Refer to Tables 2-4)
4) For the fixed sample size n and effective sample size $(m - s)$, the estimated risks of the estimates of *MTTF* decrease while that of reliability $R(t)$ decrease for small sample size and increase for moderate and large sample sizes with increasing number of working units k. (Refer to Table 6)
5) It is noted that an increase in k does not have any dampening effect on the estimated risk of failure rate λ. (Refer to Table 6)

6) The estimated risks of the Bayes estimates of reliability indexes are smaller than the corresponding hierarchical Bayes estimates for all the considered cases. This indicates that Bayes estimates outperform the hierarchical Bayes estimates. (Refer to Tables 2-4)
7) For the fixed effective sample size ($m - s$), as the sample size n increases, the Bayes and hierarchical Bayes estimates of failure rate λ decrease while reliability R(t) and *MTTF* increase, i.e., the series system survives for a long period. (Refer to Tables 2-4)
8) For the fixed sample size n and effective sample size ($m - s$), as the number of working units k increases, the Bayes and hierarchical Bayes estimates of reliability R(t) and *MTTF* decrease, i.e., the series system fails frequently. (Refer to Table 5)

Table 1. Progressive Type II censoring schemes (CS) applied to the simulation study

n	*m*	*s*	CS No.	r = ($r_{s+1}, r_{s+2}, \ldots, r_m$)	*n*	*m*	*s*	CS No.	r = ($r_{s+1}, r_{s+2}, \ldots, r_m$)
20	8	3	[1]	(1, 0, 4, 1, 6)	50	10	3	[19]	(6, 8, 10, 4, 3, 7, 2)
			[2]	(0, 0, 0, 0, 12)				[20]	(0, 0, 0, 0, 0, 0, 40)
			[3]	(12, 0, 0, 0, 0)				[21]	(40, 0, 0, 0, 0, 0, 0)
		4	[4]	(2, 0, 4, 6)			4	[22]	(6, 8, 10, 4, 5, 7)
			[5]	(0, 0, 0, 12)				[23]	(0, 0, 0, 0, 0, 40)
			[6]	(12, 0, 0, 0)				[24]	(40, 0, 0, 0, 0, 0)
	10	3	[7]	(2, 0, 3, 0, 1, 2, 2)	100	8	3	[25]	(16, 12, 20, 14, 30)
			[8]	(0, 0, 0, 0, 0, 0, 10)				[26]	(0, 0, 0, 0, 92)
			[9]	(10, 0, 0, 0, 0, 0, 0)				[27]	(92, 0, 0, 0, 0)
		4	[10]	(3, 0, 2, 1, 0, 4)			4	[28]	(28, 25,17, 22)
			[11]	(0, 0, 0, 0, 0, 10)				[29]	(0, 0, 0, 92)
			[12]	(10, 0, 0, 0, 0, 0)				[30]	(92, 0, 0, 0)
50	8	3	[13]	(6, 12, 11, 4, 9)		10	3	[31]	(6, 13, 15, 14, 8, 12, 22)
			[14]	(0, 0, 0, 0, 42)				[32]	(0, 0, 0, 0, 0, 0, 90)
			[15]	(42, 0, 0, 0, 0)				[33]	(90, 0, 0, 0, 0, 0, 0)
		4	[16]	(8, 15, 7, 12)			4	[34]	(16, 18, 15, 14, 15, 12)
			[17]	(0, 0, 0, 42)				[35]	(0, 0, 0, 0, 0, 90)
			[18]	(42, 0, 0, 0)				[36]	90, 0, 0, 0, 0, 0

Table 2. Estimates of failure rate of λ and their estimated risks

CS	$\hat{\beta}$	$\hat{\lambda}_B$	$\hat{\lambda}_{HB}$	$ER(\hat{\lambda}_B)$	$ER(\hat{\lambda}_{HB})$
1	0.5257	2.2951	2.3031	0.6755	0.7867
2	0.5273	2.2706	2.2772	0.6508	0.7554
3	0.5078	2.5724	2.6029	1.0682	1.3812
4	0.5229	2.3396	2.3504	0.7288	0.8545
5	0.5250	2.3066	2.3154	0.6917	0.8070
6	0.5005	2.6863	2.7278	1.2986	1.6107
7	0.5234	2.3322	2.3365	0.6178	0.6879
8	0.5280	2.2616	2.2629	0.5479	0.6060
9	0.5098	2.5440	2.5598	0.8976	1.0207
10	0.5192	2.3984	2.4059	0.6934	0.7772
11	0.5246	2.3146	2.3181	0.5986	0.6655
12	0.4985	2.7215	2.7505	1.2238	1.4206
13	0.5279	2.2612	2.2676	0.6451	0.7537
14	0.5300	2.2301	2.2348	0.6168	0.7176
15	0.5078	2.5717	2.6025	1.0667	1.2980
16	0.5269	2.2779	2.2850	0.6593	0.7669
17	0.5292	2.2422	2.2473	0.6250	0.7233
18	0.5006	2.6853	2.7265	1.2945	1.6056
19	0.5269	2.2790	2.2808	0.5607	0.6197
20	0.5317	2.2033	2.2042	0.4952	0.5432
21	0.5098	2.5438	2.5593	0.8921	1.0120
22	0.5265	2.2845	2.2865	0.5639	0.6223
23	0.5307	2.2195	2.2199	0.5060	0.5548
24	0.4985	2.7209	2.7495	1.2191	1.4100
25	0.5299	2.2310	2.2358	0.6177	0.7183
26	0.5305	2.2219	2.2262	0.6038	0.6983
27	0.5076	2.5706	2.6017	1.0657	1.2978
28	0.5286	2.2511	2.2564	0.6259	0.7249
29	0.5301	2.2282	2.2323	0.6054	0.6989
30	0.5004	2.6824	2.7254	1.2902	1.5969
31	0.5317	2.2031	2.2040	0.4944	0.5427
32	0.5326	2.1878	2.1883	0.4845	0.5312
33	0.5098	2.5424	2.5588	0.8907	1.0105
34	0.5296	2.2362	2.2365	0.5266	0.5806
35	0.5321	2.1971	2.1970	0.4951	0.5437
36	0.4985	2.7203	2.7403	1.2161	1.4044

Table 3. Estimates of reliability R(*t*) and their estimated risks

CS	R(*t*)	$\hat{R}(t)_B$	$\hat{R}(t)_{HB}$	$ER(\hat{R}(t)_B)$	$ER(\hat{R}(t)_{HB})$
1	0.4695	0.4554	0.4600	0.0598	0.0623
2		0.4638	0.4686	0.0601	0.0617
3		0.3692	0.3717	0.0646	0.0661
4		0.4410	0.4453	0.0600	0.0625
5		0.4520	0.4564	0.0603	0.0626
6		0.3410	0.3428	0.0789	0.0815
7		0.4249	0.4289	0.0586	0.0604
8		0.4505	0.4549	0.0583	0.0601
9		0.3552	0.3578	0.0642	0.0659
10		0.4019	0.4055	0.0596	0.0613
11		0.4311	0.4352	0.0584	0.0602
12		0.3060	0.3075	0.0734	0.0752
13	0.9999	0.9643	0.9676	0.0069	0.0092
14		0.9665	0.9696	0.0063	0.0085
15		0.9361	0.9429	0.0144	0.0195
16		0.9631	0.9666	0.0071	0.0095
17		0.9657	0.9689	0.0065	0.0087
18		0.9219	0.9307	0.0190	0.0259
19		0.9722	0.9745	0.0048	0.0063
20		0.9768	0.9786	0.0039	0.0050
21		0.9513	0.9558	0.0101	0.0131
22		0.9719	0.9742	0.0049	0.0064
23		0.9759	0.9778	0.0040	0.0052
24		0.9318	0.9387	0.0159	0.0209
25	1.0000	0.9993	0.9997	1.136×10^{-5}	1.105×10^{-5}
26		0.9993	0.9997	1.079×10^{-5}	1.172×10^{-5}
27		0.9981	0.9991	6.165×10^{-5}	3.532×10^{-5}
28		0.9993	0.9996	1.934×10^{-5}	1.141×10^{-5}
29		0.9993	0.9997	1.798×10^{-5}	1.262×10^{-5}
30		0.9973	0.9988	8.788×10^{-5}	5.091×10^{-5}
31		0.9997	0.9999	6.051×10^{-5}	3.037×10^{-5}
32		0.9997	0.9999	5.034×10^{-5}	2.081×10^{-5}
33		0.9993	0.9996	2.221×10^{-5}	1.098×10^{-5}
34		0.9997	0.9998	6.072×10^{-5}	3.081×10^{-5}
35		0.9997	0.9998	5.092×10^{-5}	2.594×10^{-5}
36		0.9986	0.9993	4.714×10^{-5}	2.338×10^{-5}

Table 4. Estimates of *MTTF* and their estimated risks

CS	$MTTF$	$MTTF_B$	$MTTF_{HB}$	$\mathrm{ER}\left(MTTF_B\right)$	$\mathrm{ER}\left(MTTF_{HB}\right)$
1	1.9991	2.1663	2.1792	0.4857	0.5213
2		2.1901	2.2039	0.5051	0.5417
3		1.9329	1.9369	0.3798	0.4136
4		2.1270	2.1386	0.4620	0.4965
5		2.1576	2.1702	0.4834	0.5190
6		1.8599	1.8602	0.3788	0.4116
7		2.0519	2.0613	0.3475	0.3687
8		2.1164	2.1276	0.3809	0.4039
9		1.8815	1.8855	0.3062	0.3255
10		1.9954	2.0032	0.3271	0.3473
11		2.0677	2.0776	0.3552	0.3769
12		1.7626	1.7630	0.3027	0.3239
13	4.9979	5.5000	5.5351	3.2263	3.4580
14		5.5787	5.6162	3.4049	3.6458
15		4.8297	4.8395	2.3392	2.5313
16		5.4589	5.4926	3.1398	3.3672
17		5.5472	5.5838	3.3299	3.5669
18		4.6484	4.6492	2.3318	2.5252
19		5.2490	5.2760	2.3295	2.4697
20		5.4308	5.4626	2.6177	2.7724
21		4.7025	4.7124	1.9184	2.2388
22		5.2357	5.2623	2.3140	2.4531
23		5.3902	5.4209	2.5499	2.7009
24		4.4066	4.4072	1.8230	2.1551
25	9.9958	11.1494	11.2244	13.5400	14.4996
26		11.1964	11.2729	13.7659	14.7371
27		9.6457	9.6649	9.2994	10.1653
28		11.0364	11.1081	12.9450	13.8712
29		11.1528	11.2282	13.4762	14.4297
30		9.2739	9.2750	9.2443	10.1482
31		10.8628	10.9265	10.5038	11.1230
32		10.9300	10.9954	10.7570	11.3888
33		9.4016	9.4214	7.6860	8.1665
34		10.7068	10.7662	10.0085	10.6043
35		10.8999	10.9644	10.6881	11.3177
36		8.8144	8.8156	7.0966	7.6261

Table 5. The effect of *k* on the estimates of reliability indexes

CS	k	$\hat{\lambda}_B$	$\hat{\lambda}_{HB}$	$\hat{R}(t)_B$	$\hat{R}(t)_{HB}$	$MTTF_B$	$MTTF_{HB}$
1	4	2.2951	2.3031	0.6261	0.6282	2.7078	2.7240
	8			0.1543	0.1587	1.3539	1.3620
	12			0.0371	0.0388	0.9026	0.9080
4	4	2.3396	2.3504	0.6118	0.6136	2.6588	2.6732
	8			0.1464	0.1506	1.3294	1.3366
	12			0.0346	0.0362	0.8863	0.8911
7	4	2.3322	2.3365	0.6095	0.6115	2.5648	2.3365
	8			0.1210	0.1244	1.2824	1.2883
	12			0.0225	0.0236	0.8549	0.8589
10	4	2.3984	2.4059	0.5866	0.5881	2.4943	2.5040
	8			0.1099	0.1129	1.2471	1.2520
	12			0.0198	0.0206	0.8314	0.8346
13	4	2.2612	2.2675	0.9896	0.9873	6.8750	6.9188
	8			0.8067	0.8056	3.4375	3.4594
	12			0.5095	0.5144	2.2917	2.3063
16	4	2.2779	2.2850	0.9893	0.9869	6.8236	6.8658
	8			0.8022	0.8010	3.4118	3.4329
	12			0.5033	0.5080	2.2745	2.2886
19	4	2.2790	2.2808	0.9930	0.9916	6.5613	6.5950
	8			0.8123	0.8115	3.2806	3.2975
	12			0.4877	0.4923	2.1871	2.1983
22	4	2.2845	2.2865	0.9929	0.9915	6.5446	6.5779
	8			0.8108	0.8100	3.2723	3.2889
	12			0.4853	0.4898	2.1815	2.1926
25	4	2.2310	2.2358	0.9999	0.9998	13.9367	14.0305
	8			0.9919	0.9897	6.9684	7.0152
	12			0.9382	0.9348	4.6456	4.6768
28	4	2.2511	2.2564	0.9999	0.9998	13.7955	13.8852
	8			0.9917	0.9896	6.8977	6.9426
	12			0.9362	0.9327	4.5985	4.6284
31	4	2.2031	2.2020	0.9999	0.9999	13.5785	13.6581
	8			0.9955	0.9945	6.7892	6.8290
	12			0.9527	0.9505	4.5261	4.5527
34	4	2.2362	2.2365	0.9999	0.9999	13.3835	13.4578
	8			0.9949	0.9937	6.6917	6.7289
	12			0.9489	0.9465	4.4611	4.4859

Table 6. The effect of *k* on the estimated risks of estimates of reliability indexes

CS	k	$\mathrm{ER}(\hat{\lambda}_B)$	$\mathrm{ER}(\hat{\lambda}_{HB})$	$\mathrm{ER}(\hat{R}(t)_B)$	$\mathrm{ER}(\hat{R}(t)_{HB})$	$\mathrm{ER}(MTTF_B)$	$\mathrm{ER}(MTTF_{HB})$
1	4	0.6755	0.7867	0.0927	0.0950	0.7590	0.8146
	8			0.0428	0.0454	0.1897	0.2036
	12			0.0043	0.0047	0.0843	0.0905
4	4	0.7288	0.8545	0.0993	0.1018	0.7218	0.7758
	8			0.0394	0.0418	0.1804	0.1939
	12			0.0039	0.0042	0.0802	0.0862
7	4	0.6178	0.6879	0.0985	0.1002	0.5429	0.5761
	8			0.0284	0.0300	0.1357	0.1440
	12			0.0019	0.0021	0.0603	0.0640
10	4	0.6934	0.7772	0.1092	0.1112	0.5111	0.5427
	8			0.0242	0.0256	0.1278	0.1357
	12			0.0015	0.0017	0.0568	0.0603
13	4	0.6451	0.7537	0.0015	0.0026	5.0411	5.4031
	8			0.0790	0.0829	1.2603	1.3508
	12			0.0871	0.0891	0.5601	0.6003
16	4	0.6593	0.7669	0.0015	0.0027	4.9060	5.2612
	8			0.0815	0.0856	1.2265	1.3153
	12			0.0882	0.0902	0.5451	0.5846
19	4	0.5607	0.6197	0.0009	0.0014	3.6399	3.8589
	8			0.0752	0.0782	0.9100	0.9647
	12			0.0804	0.0818	0.4044	0.4288
22	4	0.5640	0.6223	0.0009	0.0014	3.6156	3.8330
	8			0.0759	0.0790	0.9039	0.9582
	12			0.0808	0.0821	0.4017	0.4259
25	4	0.6177	0.7183	2.071×10^{-6}	3.023×10^{-5}	21.1563	22.6557
	8			0.0012	0.0023	5.2891	5.6639
	12			0.0176	0.0210	2.3507	2.5173
28	4	0.6259	0.7249	1.565×10^{-6}	2.568×10^{-5}	20.2266	21.6737
	8			0.0012	0.0022	5.0566	5.4184
	12			0.0180	0.0214	2.2474	2.4082
31	4	0.4944	0.5427	4.014×10^{-7}	5.444×10^{-6}	16.4122	17.3797
	8			0.0006	0.0010	4.1030	4.3449
	12			0.0121	0.0141	1.8236	1.9311
34	4	0.5266	0.5806	3.553×10^{-7}	4.667×10^{-6}	15.6383	16.5692
	8			0.0007	0.0011	3.9096	4.1423
	12			0.0136	0.0159	1.7376	1.8410

Numerical Examples

Two numerical examples are presented to illustrate how the data support the developed model and how to employ the proposed method for estimation of reliability indexes of the series system. Examples 1 and 2 consider the artificial general progressive Type II censored samples generated from the real data set provided by Nelson (1982) and the computer simulation, respectively.

Example 1. Real Life Data

As a numerical illustration, a system comprising a series of 2 working units and 18 cold standby units was considered. This series system is equivalent to a cold standby series system of 19 identical and independent units. The lifetimes of such 19 units were observed until failure during the life test experiment in which specimens of a type of electrical insulating fluid were subject to a constant voltage stress (34 KV/minutes). The 19 failure times were obtained as follows:

0.19	0.78	0.96	1.31	2.78	3.16	4.15	4.67	4.85	6.50
7.35	8.01	8.27	12.06	31.75	32.52	33.94	36.71	72.89	

Asgharzadeh and Valiollahi (2009) checked the validity of an exponential model with mean = 14.2857 and indicated that the exponential model is adequate for this data set. To generate an artificial general progressive Type II censored sample from the given real data set, it is assumed that the lifetimes of the first two failures are lost without observation, and then lifetimes were observed until the eighth failure. At each failure from 3rd failure to 8th failure, units were randomly withdrawn according to the general progressive Type II censoring scheme $\mathbf{r} = (r_3, r_4, \ldots, r_8) = (2, 0, 1, 2, 1, 5)$. The life test was terminated at the eighth failure, and the vector of observed lifetimes was found to be $\mathbf{x} = (x_{(3)}, x_{(4)}, \ldots, x_{(8)}) = (0.96, 1.31, 2.78, 4.85, 6.50, 8.01)$.

Table 7. Estimates of reliability indexes and their (1 – *α*)% HPD-intervals for Example 1

	Parameter	Estimate	95% HPD-interval	99% HPD-interval
Bayes Estimation		0.0519	(0.0209, 0.0866)	(0.0107, 0.1239)
	R(t)	0.9415	(0.6257, 0.9999)	(0.0992, 0.9999)
	MTTF	205.5657	(109.7102, 454.5454)	(76.6798, 887.8505)
Hierarchical Bayes Estimation		0.0519	(0.0203, 0.0880)	(0.0120, 0.1172)
	R(t)	0.9418	(0.5988, 0.9999)	(0.1526, 0.9999)
	MTTF	205.7603	(107.9023, 467.9803)	(81.0286, 791.6666)

Table 8. Predictive estimates of the remaining lifetimes and their (1 – *α*)% HPD-prediction intervals for Example 1

	l	$x_{(l)}$	**95% HPD-interval**	**99% HPD-interval**
Bayes Estimation	9	10.1738	(8.0262, 15.3444)	(8.0206, 22.2503)
	10	12.8786	(8.0218, 20.7168)	(8.0202, 27.9367)
	11	16.4851	(8.0418, 28.2399)	(8.0251, 38.7317)
	12	21.8947	(8.1446, 39.6961)	(8.0996, 55.4946)
	13	32.7139	(8.0212, 46.9190)	(8.0201, 61.6929)
Hierarchical Bayes Estimation	9	10.1758	(8.0470, 16.1734)	(8.0244, 24.3241)
	10	12.8833	(8.0267, 20.7303)	(8.0377, 27.9939)
	11	16.4931	(8.0355, 28.2591)	(8.0298, 38.7608)
	12	21.9078	(8.0907, 39.7260)	(8.0971, 55.5395)
	13	32.7373	(8.0261, 46.9559)	(8.0241, 61.7342)

The Bayes and hierarchical Bayes estimates of failure rate λ, reliability R(t), and *MTTF* and the corresponding HPD-intervals at $t = 100$ have been computed, and are reported in Table 7. The 95% and 99% Bayes and hierarchical Bayes predictive estimates and the corresponding HPD-prediction intervals for the each of the remaining l lifetimes ($9 \leq l \leq$) have also been computed, and are reported in Table 8.

Example 2. Simulated Data

As a numerical illustration, a system initiated with the series of 5 working units being in an operational state is placed on a life test along with the other 19 standby units connected in a series. This series system is equivalent to a cold standby series system of 20 identical and independent units. Under a general progressive Type II censoring scheme, the lifetimes of the first two failures are not observed and then the lifetimes are completely observed until the eighth failure. Using the algorithm presented in the previous section, the general progressive Type II censored sample $\mathbf{x} = (0.01250, 0.01531, 0.02063, 0.02679, 0.03062, 0.05251)$ has been generated with the censoring scheme $\mathbf{r} = (1, 0, 2, 1, 2, 6)$. For this sample, Bayes and hierarchical Bayes estimates of failure rate λ, reliability R(t), and *MTTF*, and the corresponding HPD intervals at $t = 2$, have been computed and are reported in Table 9. Moreover, the 95% and 99% Bayes and hierarchical Bayes predictive estimates and the corresponding HPD-prediction intervals for each of the remaining l lifetimes ($9 \leq l \leq$) have also been computed, and are reported in Table 10.

Table 9. Estimates of reliability indexes and their (1 – *α*)% HPD-intervals for Example 2

	Parameter	Estimate	95% HPD-interval	99% HPD-interval
Bayes Estimation	λ	2.4747	(0.9950, 4.1210)	(0.7440, 4.8827)
	R(*t*)	0.3201	(9.053×10^{-5}, 0.9967)	(9.9×10^{-7}, 0.9999)
	MTTF	1.8184	(0.9706, 4.0201)	(0.8192, 5.3763)
Hierarchical Bayes Estimation	λ	2.4650	(0.9910, 4.1050)	(0.74800, 4.8390)
	R(*t*)	0.3235	(9.895×10^{-5}, 0.9968)	(1.31×10^{-6}, 0.9999)
	MTTF	1.8256	(0.9744, 4.0363)	(0.82661, 5.3476)

Table 10. Predictive estimates of the remaining lifetimes and their (1 – *α*)% HPD-prediction intervals for Example 2

	l	$x_{(l)}$	95% HPD-interval	99% HPD-interval
Bayes Estimation	9	0.0676	(0.0527, 0.1079)	(0.0526, 0.1661)
	10	0.0858	(0.0526, 0.1394)	(0.0526, 0.1888)
	11	0.1086	(0.0530, 0.1859)	(0.0526, 0.2546)
	12	0.1389	(0.0526, 0.2477)	(0.0526, 0.3431)
	13	0.1843	(0.0562, 0.3429)	(0.0526, 0.4821)
Hierarchical Bayes Estimation	9	0.0677	(0.0529, 0.1201)	(0.0526, 0.1662)
	10	0.0860	(0.0527, 0.1398)	(0.0526, 0.1893)
	11	0.1088	(0.0526, 0.1864)	(0.0526, 0.2554)
	12	0.1392	(0.0531, 0.2485)	(0.0526, 0.3442)
	13	0.1848	(0.0526, 0.3440)	(0.0526, 0.4838)

From the results presented in Tables 7-10, it is observed that the hierarchical Bayes estimates and predictors are very close to the Bayes estimates and predictors for both the considered real and simulated data. Furthermore, the Bayes and hierarchical Bayes predictive estimates and the length of the HPD-prediction interval increases as *l* increases. This implies that the prediction is less precise as a large *l* is considered.

Conclusion

This purpose of this study was to study hierarchical Bayes estimation and prediction of reliability indexes and remaining lifetimes of a cold standby series system consisting a series of *k* working units and (*n* – 1) cold standby units under general progressive Type II censoring scheme. The Bayes and hierarchical Bayes estimates as well as an HPD interval for reliability indexes of the series system are derived. In addition, we have derived the Bayes and hierarchical Bayes predictive estimates, and HPD-prediction interval for the remaining lifetimes based on an informative sample. We have presented two numerical examples to illustrate the proposed estimation and prediction methods. The Monte Carlo simulation study is carried out to examine and compare the performance of the Bayes and hierarchical Bayes estimates. The simulation results indicated Bayes estimation should be preferred over the hierarchical Bayes estimation for estimation of reliability indexes of the series system. Furthermore, the number of components in the working condition should be less and the number of components in the cold standby mode should be large to run the series system for a long period.

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