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Journal of Modern Applied Statistical Methods

Volume 15 | Issue 2 Article 30

11-1-2016

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Recommended Citation
Makhdoom, Iman; Nasiri, Parviz; and Pak, Abbas (2016) "Estimating the Parameter of Exponential Distribution under Type II Censoring From Fuzzy Data," *Journal of Modern Applied Statistical Methods*: Vol. 15: Iss. 2, Article 30.
DOI: 10.22237/jmasm/1478003280
Available at: http://digitalcommons.wayne.edu/jmasm/vol15/iss2/30

Journal of Modern Applied Statistical Methods
November 2016, Vol. 15, No. 2, 495-509.
doi: 10.22237/jmasm/1478003280

ISSN 1538 − 9472

Estimating the Parameter of Exponential Distribution under Type II Censoring From Fuzzy Data

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The problem of estimating the parameter of Exponential distribution on the basis of type II censoring scheme is considered when the available data are in the form of fuzzy numbers. The Bayes estimate of the unknown parameter is obtained by using the approximation forms of Lindley (1980) and Tierney and Kadane (1986) under the assumption of gamma prior. The highest posterior density (HPD) estimate of the parameter of interest is found. A Monte Carlo simulation is used to compare the performances of the different methods. A real data set is investigated to illustrate the applicability of the proposed methods.

Keywords: Type II censoring, fuzzy lifetime data, exponential distribution, Bayesian estimation

Introduction

In life testing and reliability studies, the experimenter may not always obtain complete information on failure times for all experimental units. Data obtained from such experiments are called censored data. One of the most common censoring scheme is Type II (failure) censoring, where the life testing experiment will be terminated upon the r^{th} (r is pre-fixed) failure. This scheme is often adopted for toxicology experiments and life testing applications by engineers as it has been proven to save time and money. Several authors have addressed inferential issues based on Type II censored samples; for example, Ng, Kundu, and Balakrishnan (2006) discussed point and interval estimation for the two parameter Birnbaum-Saunders distribution base on Type II censored samples. Balakrishnan and Han (2008) considered inference for a simple step-stress model

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from exponential distribution under Type II censoring. Iliopoulos and Balakrishnan (2011) studied likelihood inference for Laplace distribution based on Type II censored samples. Dey and Kuo (1991) obtained a new class of empirical Bayes estimator for exponential distribution parameter from Type II censored data. Singh and Kumar (2007) considered Bayesian estimation of the exponential parameter under a multiply Type II Censoring scheme. Kundu and Raqab (2012) addressed Bayesian inference for Weibull distribution under Type II censoring scheme.

The above research results are based on precise lifetime data. However, in real situations, some collected data might be imprecise quantities. For instance, the lifetime of a battery may be reported as: 'about 1000 *h*', 'approximately 1400 *h*', 'almost between 1000 *h* and 1200 *h*', 'essentially less than 1200 *h*', and so on. The lack of precision of such data can be described using fuzzy sets. The classical statistical estimation methods are not appropriate to deal with such imprecise cases. Therefore, the conventional procedures used for estimating the parameter of Exponential distribution will have to be adapted to the new situation.

In recent years, several researchers considered applying the fuzzy sets to estimation theory. Gertner and Zhu (1996) considered Bayesian estimation in forest surveys when samples or prior information are fuzzy. Huang, Zuo, and Sun (2006) proposed a new method to determine the membership function of the estimates of the parameters and the reliability function of multiparameter lifetime distributions. Coppi, Gil, and Kiers (1991) presented some applications of fuzzy techniques in statistical analysis. Akbari and Rezaei (2007) proposed a new method for uniformly minimum variance unbiased fuzzy point estimation. Pak, Parham, and Saraj (2013, 2014) conducted a series of studies to develop the inferential procedures for the lifetime distributions on the basis of fuzzy numbers. However, there are no reports on estimating the parameter of Exponential parameter from Type II fuzzy censored data. Hence, the purpose of this study is to consider Bayesian estimation of the parameter of Exponential distribution under Type II censoring scheme when the lifetime observations are reported in the form of fuzzy numbers.

Below are the main definitions of fuzzy sets and some of the formula:

Definition 1: Let X be a universe set. A fuzzy set $\tilde{A}$ in X is defined by a membership function $\mu_{\tilde{A}}(x) \rightarrow [0,1]$, where $\mu_{\tilde{A}}(x)$, $\forall x \in X$, indicates the degree of x in A.

Definition 2: A fuzzy subset $\tilde{A}$ of the universe set X is normal iff $\sup_{x\in X} \mu_{\tilde{A}}(x)=1$, where X is the universe set.

Definition 3: A fuzzy subset $\tilde{A}$ of universe set X is convex iff $\mu_{\tilde{A}}(\lambda x+(1-\lambda)y)\geq \min\{\mu_{\tilde{A}}(x),\mu_{\tilde{A}}(y)\}, \forall x,y\in X, \forall\lambda\in[0,1]$.

Definition 4: A fuzzy set $\tilde{x}$ is a fuzzy number iff $\tilde{x}$ be normal and convex on X.

In all of fuzzy types of presentation, *LR*-type fuzzy numbers are most used as in linguistic, decision making, knowledge representation, medical diagnosis, control systems, databases. Therefore, we shall focus on the set of *LR*-type fuzzy numbers.

Suppose that $L:\mathbb{R}^+\to[0,1]$ and $R:\mathbb{R}^+\to[0,1]$ be two continuous functions with the following properties:

1) $L(-x)=L(x),\ \ R(-x)=R(x)$.
2) $L(0)=1,\ \ R(0)=1$.
3) L and R be decreasing in $[0, \infty)$.
4) $\lim_{x\to\infty} L(x)=0,\ \ \lim_{x\to\infty} R(x)=0$

Definition 5: A fuzzy number $\tilde{x}$ is said to be an *LR*-type fuzzy number if

$$\mu_{\tilde{x}}(x)=\begin{cases} L\left(\dfrac{m-x}{\alpha}\right) & x\leq m \\ R\left(\dfrac{x-m}{\beta}\right) & x\geq m \end{cases}$$

where m characterizes the mean value of $\tilde{x}$, while α and β are the left and the right coefficient of fuzziness, respectively. Symbolically, the *LR*-type fuzzy number is denoted by $\tilde{x}=(\alpha,m,\beta)$.

Data, likelihood and parameter estimation

Consider a generalization of the likelihood function based on Type II censoring when the lifetime observations are reported in the form of *LR*-type fuzzy numbers. The Bayes estimate of the unknown parameter will then be obtained using suitable conjugate prior of the unknown parameter, and the highest posterior density estimation will be discussed.

Fuzzy lifetime data and the likelihood function

Suppose that n independent units are placed on a life test with the corresponding lifetimes $X_1,\ldots, X_n$. It is assumed that these variables are independent and identically distributed as Exponential $E(\lambda)$, with probability density function (pdf)

$$f(x;\lambda) = \lambda \exp(-\lambda x),\ x > o,\ \lambda > 0. \tag{1}$$

Prior to the experiment, a number $r < n$ is determined and the experiment is terminated after the r^{th} failure. Now consider the problem where under the Type II censoring scheme, failure times are not observed precisely and only partial information about them are available in the form of fuzzy numbers $\tilde{x}_i = (\alpha_i, m_i, \beta_i)$,$I = 1,\ldots, r$, with the corresponding membership functions $\mu_{\tilde{x}_1}(x_1),\ldots,\mu_{\tilde{x}_r}(x_r)$. Let the maximum value of the means of these fuzzy numbers to be $m_{(r)}$. The lifetime of $n - r$ surviving units, which are removed from the test after the m^{th} failure, can be encoded as fuzzy numbers $\tilde{x}_{r+1},\ldots,\tilde{x}_n$ with the membership functions

$$\mu_{\tilde{x}_j}(x) = \begin{cases} 0 & x \le m_{(r)} \\ 1 & x > m_{(r)} \end{cases},\quad j = r+1,\ldots,n.$$

The fuzzy data $\tilde{\mathbf{x}} = (\tilde{x}_1,\ldots,\tilde{x}_n)$ is thus the vector of observed lifetimes. Then, by using Zadeh's definition of the probability of a fuzzy event (Zadeh, 1968), the corresponding observed-data likelihood function can be obtained as

$$\ell(\tilde{\mathbf{x}};\lambda) = \lambda^r \exp\left[-(n-r)\lambda m_{(r)}\right]\prod_{i=1}^{r}\int \exp(-\lambda x)\mu_{\tilde{x}_i}(x)\,dx. \tag{2}$$

Bayesian estimation

In recent decades, the Bayes viewpoint, as a powerful and valid alternative to traditional statistical perspectives, has received frequent attention for statistical inference. Consider the Bayesian estimation of the unknown parameter λ. As conjugate prior for λ, we take the *Gamma* (a, b) density with pdf given by

$$\pi(\lambda)=\frac{b^a}{\Gamma(a)}\lambda^{a-1}\exp(-\lambda b),\;\; \lambda>0, \tag{3}$$

where $a > 0$ and $b > 0$. Based on this prior, the posterior density function of λ given the data can be written as follows:

$$\pi(\lambda|\tilde{\mathbf{x}})=\frac{\lambda^{r+a-1}\exp\left[-\lambda(n-r)m_{(r)}+b\right]\prod_{i=1}^{r}\int\exp(-\lambda x)\mu_{\tilde{x}_i}(x)dx}{\int_0^\infty\lambda^{r+a-1}\exp\left[-\lambda(n-r)m_{(r)}+b\right]\prod_{i=1}^{r}\left(\int\exp(-\lambda x)\mu_{\tilde{x}_i}(x)dx\right)d\lambda}, \tag{4}$$

Then, under a squared error loss function, the Bayes estimate of any function of λ, say $g(\lambda)$, is

$$E\left(h(\lambda)|\tilde{\mathbf{x}}\right)=\frac{\int_0^\infty h(\lambda)\lambda^{r+a-1}\exp\left[-\lambda(n-r)m_{(r)}+b\right]\prod_{i=1}^{r}\left(\int\exp(-\lambda x)\mu_{\tilde{x}_i}(x)dx\right)d\lambda}{\int_0^\infty\lambda^{r+a-1}\exp\left[-\lambda(n-r)m_{(r)}+b\right]\prod_{i=1}^{r}\left(\int\exp(-\lambda x)\mu_{\tilde{x}_i}(x)dx\right)d\lambda} \tag{5}$$

Note that (5) can not be obtained analytically; therefore, adopt two approximations-Lindley's approximation and Tierney and Kadane's approximation for computing the Bayes estimate.

Lindley's approximation

Setting $F(\lambda)=\ln\pi(\lambda)+\ln\ell(\tilde{\mathbf{x}};\lambda)\equiv\vartheta(\lambda)+L(\lambda)$, (5) can be rewritten as

$$E\left(h(\lambda)\middle|\tilde{\mathbf{x}}\right)=\frac{\int_0^\infty h(\lambda)e^{F(\lambda)}d\lambda}{\int_0^\infty e^{F(\lambda)}d\lambda}. \tag{6}$$

Then, by using Lindley's approximation (see Lindley, 1980), the ratio of the two integrals in (6) can be obtained as

$$h(\lambda)+\frac{1}{2}h_{11}\delta_{11}+\vartheta_1 h_1\delta_{11}+\frac{1}{2}F_3\delta_{11}^2 h_1, \tag{7}$$

where

$$h_1=\frac{dh(\lambda)}{d\lambda},\;\; h_{11}=\frac{d^2h(\lambda)}{d\lambda^2},\;\; \vartheta_1=\frac{d\vartheta(\lambda)}{d\lambda}$$

$$F_3=\frac{\partial^3F(\lambda)}{\partial\lambda^3},\;\; \delta_{11}=\left[-\frac{\partial^2F(\lambda)}{\partial\lambda^2}\right]^{-1}.$$

Evaluating all the expressions in (7) at the maximum likelihood estimate (MLE) of λ produces the approximation $\hat{h}_B$ to (6). In this case,

$$F(\lambda)=r\log\lambda+\sum_{i=1}^{r}\log\int\exp(-\lambda x)\mu_{\tilde{x}_i}(x)dx-(n-r)\lambda m_{(r)}.$$

The MLE of λ, say $\hat{\lambda}$, is the solution of the equation

$$\frac{\partial F(\lambda)}{\partial\lambda}=\frac{r}{\lambda}-\sum_{i=1}^{r}\frac{\int x\exp(-\lambda x)\mu_{\tilde{x}_i}(x)dx}{\int\exp(-\lambda x)\mu_{\tilde{x}_i}(x)dx}-(n-r)m_{(r)}=0.$$

Now, to apply Lindley's form in (7), first obtain

$$\delta_{11}=-\frac{r}{\hat{\lambda}^2}+\sum_{i=1}^{r}\left\{\frac{\int x^2\exp\left(-\hat{\lambda}x\right)\mu_{\tilde{x}_i}(x)dx}{\int\exp\left(-\hat{\lambda}x\right)\mu_{\tilde{x}_i}(x)dx}-\left[\frac{\int x\exp\left(-\hat{\lambda}x\right)\mu_{\tilde{x}_i}(x)dx}{\int\exp\left(-\hat{\lambda}x\right)\mu_{\tilde{x}_i}(x)dx}\right]^2\right\}$$

$$F_3 = \frac{2r}{\hat{\lambda}^3} - \sum_{i=1}^{r} \left\{ \begin{array}{l} \dfrac{\int x^3 \exp\left(-\hat{\lambda}x\right) \mu_{\tilde{x}_i}(x)\,dx}{\int \exp\left(-\hat{\lambda}x\right) \mu_{\tilde{x}_i}(x)\,dx} \\ -\dfrac{\left[\int x^2 \exp\left(-\hat{\lambda}x\right) \mu_{\tilde{x}_i}(x)\,dx\right]\left[x \exp\left(-\hat{\lambda}x\right) \mu_{\tilde{x}_i}(x)\,dx\right]}{\left[\int \exp\left(-\hat{\lambda}x\right) \mu_{\tilde{x}_i}(x)\,dx\right]} \end{array} \right\}$$

$$+2\sum_{i=1}^{r} \left\{ \frac{\int x \exp\left(-\hat{\lambda}x\right) \mu_{\tilde{x}_i}(x)\,dx}{\int \exp\left(-\hat{\lambda}x\right) \mu_{\tilde{x}_i}(x)\,dx} \times \left(\begin{array}{l} \dfrac{\int x^2 \exp\left(-\hat{\lambda}x\right) \mu_{\tilde{x}_i}(x)\,dx}{\int \exp\left(-\hat{\lambda}x\right) \mu_{\tilde{x}_i}(x)\,dx} \\ -\left[\dfrac{\int x \exp\left(-\hat{\lambda}x\right) \mu_{\tilde{x}_i}(x)\,dx}{\int \exp\left(-\hat{\lambda}x\right) \mu_{\tilde{x}_i}(x)\,dx}\right]^2 \end{array} \right) \right\}$$

The approximate Bayes of λ, say $\hat{\lambda}_B$, for the squared error loss function is the posterior mean of $h(\lambda) = \lambda$, which is by (7) as follows.

$$\hat{\lambda}_B = \hat{\lambda} + \left[\frac{a-1}{\hat{\lambda}} - b\right]\delta_{11} + \frac{1}{2} F_3 \delta_{11}^2. \tag{8}$$

Tierney and Kadane's approximation

Setting $W(\lambda) = L(\lambda) / n$ and $W^*(\lambda) = [\ln h(\lambda) + L(\lambda)] / n$, the expression in (6) can be re-expressed as

$$E\left(h\left(\hat{\lambda}\right)\middle|\tilde{\mathbf{x}}\right) = \frac{\int_0^\infty h(\lambda) e^{nW^*(\lambda)} d\lambda}{\int_0^\infty e^{nW(\lambda)} d\lambda}. \tag{9}$$

Following Tierney and Kadane (1986), (9) can be approximated as the following form:

$$\hat{g}_{BT}(\lambda) = \left(\frac{|\phi^*|}{|\phi|}\right)^{\frac{1}{2}} \exp\left\{n\left[W^*\left(\tilde{\lambda}^*\right) - W\left(\tilde{\lambda}\right)\right]\right\}, \tag{10}$$

where $\tilde{\lambda}^*$ and $\tilde{\lambda}$ maximize $W^*(\lambda)$ and $W(\lambda)$, respectively, and ϕ^* and ϕ are minus the inverse of the second derivatives of $W^*(\lambda)$ and $W(\lambda)$ at $\tilde{\lambda}^*$ and $\tilde{\lambda}$, respectively.

In this case,

$$W(\lambda) = \frac{1}{n}\left\{\begin{array}{l} k + (r+a-1)\log\lambda - \lambda\left[b + (n-r)m_{(r)}\right] \\ +\sum_{i=1}^{r}\log\int\exp(-\lambda x)\mu_{\tilde{x}_i}(x)\,dx \end{array}\right\}, \tag{11}$$

where k is a constant, and

$$W^*(\lambda) = H(\lambda) + \frac{1}{n}\ln\lambda. \tag{12}$$

Substituting for (11) and (12) in (10), the Bayes estimate $\hat{\lambda}_{BT}$ of a function $h(\lambda) = \lambda$ under squared error loss can then be obtained straightforwardly.

HPD estimation

The highest posterior density (HPD) estimation is another popular method used by the Bayesian perspective. This method is based on the maximum likelihood principle; hence, it leads to the mode of the posterior density. The HPD estimate, $\hat{\lambda}_H$, of λ is obtained by solving the equation $\frac{\partial\pi(\lambda|\tilde{\mathbf{x}})}{\partial\lambda} = 0$ where

$$\frac{\partial\pi(\lambda|\tilde{\mathbf{x}})}{\partial\lambda} = \frac{r+a-1}{\lambda} - \left[b + (n-r)m_{(r)}\right] - \sum_{i=1}^{r}\frac{\int x\exp(-\lambda x)\mu_{\tilde{x}_i}(x)\,dx}{\int\exp(-\lambda x)\mu_{\tilde{x}_i}(x)\,dx}. \tag{13}$$

However, the solution cannot be obtained explicitly. In the following, Theorem 1 discusses the existence and uniqueness of the HPD estimate of λ.

Theorem 1. Let $g(\lambda)$ denote the function on the right-hand side of the expression in (13). Then the root of the equation $g(\lambda) = 0$ exists and is unique.

Proof. From (13) it is easily seen that $\lim_{\lambda\to\infty} g(\lambda) = \infty$. Also, note that $g(\lambda) < \frac{r+a-1}{\lambda}$, $\forall \lambda \in (0,\infty)$, and consequently

$$\lim_{\lambda\to\infty} g(\lambda) < \lim_{\lambda\to\infty} \frac{r+a-1}{\lambda} = 0 \quad \forall \lambda \in (0,\infty)$$

Therefore, the equation $g(\lambda) = 0$ has at least one root in $(0, \infty)$. To prove that the root is unique, we consider the first derivative of g, $g('\lambda)$, given by

$$g('\lambda) = -\frac{r+a-1}{\lambda^2} + \sum_{i=1}^{r} \frac{\partial^2}{\partial\lambda^2} \log \int \exp(-\lambda x)\, \mu_{\tilde{x}_i}(x)\, dx$$

Let $u(\lambda) = \exp(-\lambda x)$ and $v_i(\lambda) = \int \exp(-\lambda x)\, \mu_{\tilde{x}_i}(x)\, dx$. Then $g('\lambda)$ can be written as

$$g('\lambda) = -\frac{r+a-1}{\lambda^2} + \sum_{i=1}^{r} \frac{\partial^2}{\partial\lambda^2} \log v_i(\lambda)$$

It is clearly that $u(\lambda)$ is a log-concave function of λ, and by the Prekopa-Leindler inequality (see Gardner, 2002) $v_i(\lambda)$, $i = 1,\ldots, m$, are also log-concave in λ. It follows that g is a strictly decreasing function w.r.t. λ and hence the equation $g(\lambda) = 0$ has exactly one solution.

Because there is no closed form of the solution to the equation (13), an iterative numerical search such as Newton-Raphson method can be used to obtain the HPD estimate of λ. The second-order derivative form required for proceeding with the Newton-Raphson method, is obtained as follows.

$$\frac{\partial^2 \pi(\lambda|\tilde{\mathbf{x}})}{\partial\lambda^2} = -\frac{(r+a-1)}{\lambda^2} + \sum_{i=1}^{r} \left\{ \frac{\int x^2 \exp(-\lambda x)\, \mu_{\tilde{x}_i}(x)\, dx}{\int \exp(-\lambda x)\, \mu_{\tilde{x}_i}(x)\, dx} - \left(\frac{\int x \exp(-\lambda x)\, \mu_{\tilde{x}_i}(x)\, dx}{\int \exp(-\lambda x)\, \mu_{\tilde{x}_i}(x)\, dx} \right)^2 \right\}.$$

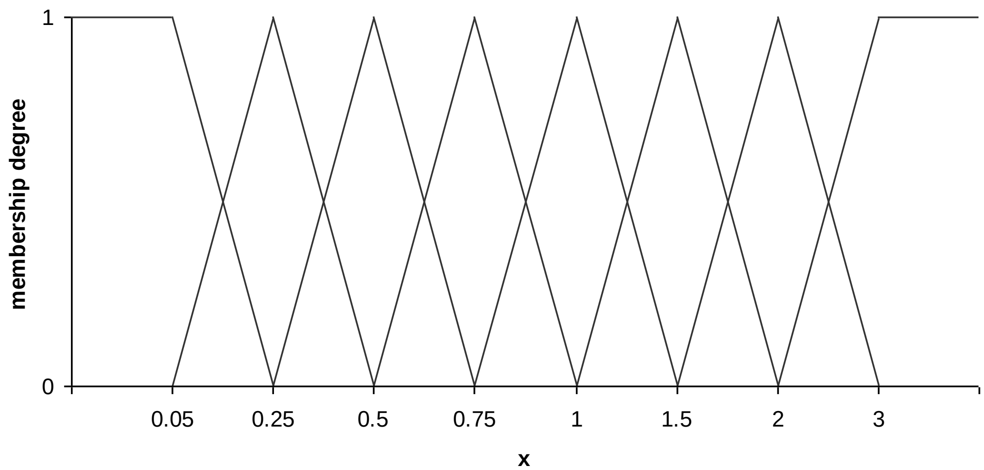

Figure 1. Fuzzy information system used to encode the simulated data

Numerical Study

A Monte Carlo simulation study and one example are presented to illustrate the methods of inference developed in this paper. First, for fixed $\theta = 1$ and different choices of n and r, generated Type II censored samples were generated, say $\mathbf{x} = (x_1,\ldots, x_r)$, from the exponential distribution using the method proposed by Aggarwala and Balakrishnan (1998). Each realization of $\mathbf{x}$ was fuzzified using the fuzzy information system (see Pak et al., 2014) shown in Figure 1, corresponding to the membership functions

$$\mu_{\tilde{x}_1}(x) = \begin{cases} 1 & x \le 0.25, \\ \dfrac{0.5 - x}{0.25} & 0.25 \le x \le 0.5, \\ 0 & \text{otherwise,} \end{cases} \qquad \mu_{\tilde{x}_2}(x) = \begin{cases} \dfrac{x - 0.25}{0.25} & 0.25 \le x \le 0.5, \\ \dfrac{0.75 - x}{0.25} & 0.5 \le x \le 0.75, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mu_{\tilde{x}_3}(x)=\begin{cases}\frac{x-0.5}{0.25} & 0.5\le x\le 0.75,\\ \frac{1-x}{0.25} & 0.75\le x\le 1,\\ 0 & \text{otherwise,}\end{cases}\qquad \mu_{\tilde{x}_4}(x)=\begin{cases}\frac{x-0.75}{0.25} & 0.75\le x\le 1,\\ \frac{1.25-x}{0.25} & 1\le x\le 1.25,\\ 0 & \text{otherwise,}\end{cases}$$

$$\mu_{\tilde{x}_5}(x)=\begin{cases}\frac{x-1}{0.25} & 1\le x\le 1.25,\\ \frac{1.5-x}{0.25} & 1.25\le x\le 1.5,\\ 0 & \text{otherwise,}\end{cases}\qquad \mu_{\tilde{x}_6}(x)=\begin{cases}\frac{x-1.25}{0.25} & 1.25\le x\le 1.5,\\ \frac{1.75-x}{0.25} & 1.5\le x\le 1.75,\\ 0 & \text{otherwise,}\end{cases}$$

$$\mu_{\tilde{x}_7}(x)=\begin{cases}\frac{x-1.5}{0.25} & 1.5\le x\le 1.75,\\ \frac{2-x}{0.25} & 1.75\le x\le 2,\\ 0 & \text{otherwise,}\end{cases}\qquad \mu_{\tilde{x}_8}(x)=\begin{cases}x-1.75 & 1.75\le x\le 2,\\ 1 & x\ge 2,\\ 0 & \text{otherwise.}\end{cases}$$

Then, the approximate Bayes estimates (via Lindley approximation or Tierney and Kadane approximation) and the HPD estimates of λ for the fuzzy sample were computed under the assumption that λ has *Gamma* (a, b) prior, including the non-informative gamma prior, i.e. $a = b = 0$, and informative gamma prior, i.e. $a = b = 2$. The average values and mean squared errors of the estimates, computed based on 1000 replication, are presented in Tables 1 and 2.

In viewing the tables, using Lindley approximation or Tierney and Kadane approximation for the computation of Bayes estimates gave similar estimation results. The performance of HPD estimates are better than the Bayes estimates in terms of MSE. Also, the approximate Bayes estimates based on informative prior are uniformly better than that of non-informative prior. In all the cases, it was observed that as the effective sample size m increases the performances in terms of MSE become better.

Table 1. Average values (AV) and mean squared errors (MSE) of the Bayes and HPD estimates of *λ* based on non-informative prior (*a* = *b* = 0) and for different sample sizes.

n	*r*	$\hat{\lambda}_B$		$\hat{\lambda}_{BT}$		$\hat{\lambda}_H$	
		AV	MSE	AV	MSE	AV	MSE
20	10	1.081	0.175	1.082	0.175	1.061	0.147
20	12	1.078	0.156	1.078	0.155	1.057	0.131
20	15	1.055	0.124	1.030	0.123	1.045	0.115
30	15	1.092	0.104	1.091	0.104	1.075	0.085
30	20	1.065	0.096	1.065	0.096	1.052	0.066
30	25	1.040	0.071	1.041	0.071	1.028	0.048
50	20	1.051	0.098	1.050	0.098	1.040	0.073
50	25	1.034	0.055	1.034	0.054	1.026	0.037
50	35	1.021	0.037	1.021	0.037	1.018	0.029

Table 2. Average values (AV) and mean squared errors (MSE) of the Bayes and HPD estimates of *λ* based on informative prior (*a* = *b* = 2) and for different sample sizes.

n	*r*	$\hat{\lambda}_B$		$\hat{\lambda}_{BT}$		$\hat{\lambda}_H$	
		AV	MSE	AV	MSE	AV	MSE
20	10	1.069	0.151	1.068	0.152	1.047	0.129
20	12	1.059	0.133	1.059	0.132	1.036	0.117
20	15	1.038	0.105	1.038	0.105	1.030	0.092
30	15	1.077	0.081	1.076	0.080	1.056	0.070
30	20	1.051	0.067	1.051	0.067	1.041	0.051
30	25	1.024	0.052	1.024	0.053	1.017	0.033
50	20	1.040	0.079	1.041	0.078	1.028	0.056
50	25	1.019	0.041	1.018	0.041	1.015	0.025
50	35	1.012	0.020	1.012	0.020	1.007	0.014

Application example

To demonstrate the application of the proposed methods to real data, consider the following life-testing experiment in which $n = 22$ identical valves are placed on test. The unknown lifetime x_i of valve i may be regarded as a realization of a random variable X_i, induced by random sampling from a total population of valves, which is distributed as Exponential by an unknown parameter of λ. A tested valve may be considered as failed, or -strictly speaking- as nonconforming, when at least one value of its parameters falls beyond specification limits. In practice, however, there isn’t the possibility to measure all parameters and are not able to define precisely the moment of a failure. So, the observed failure times (in $100h$) are reported in the form of lower and upper bounds, as well as a point estimate which are as follows.

Data Set:

(20.68, 25.86, 29.73), (22.72, 28.41, 32.67), (24.61, 30.77, 35.38), (26.43, 33.04, 37.99), (28.15, 35.19, 40.46), (30.29, 37.87, 43.55), (34.32, 42.91, 49.34), (35.51, 44.39, 51.04), (37.80, 47.25, 54.33), (41.16, 51.45, 59.16), (42.52, 53.16, 61.13), (43.97, 54.97, 63.21), (44.31, 55.39, 63.69), (46.75, 58.44, 67.20), (47.69, 59.62, 68.56), (48.09, 60.12, 69.13), (52.27, 65.34, 75.14), (53.65, 67.07, 77.13), (60.72, 75.91, 87.29), (63.45, 79.32, 91.21), (65.69, 82.12, 94.43), (73.48, 91.86, 105.63).

Each triple is modeled by a triangular fuzzy number $\tilde{x}_i$, and is interpreted as a possibility distribution related to an unknown value x_i, itself a realization of a random variable X_i. Randomness arises from the selection of objects from the total population of batteries. In contrast, fuzziness arises from the limited ability of the observer to describe the moment of a failure using numbers, which is not influenced by random factors. Consider Type II censored samples of size r = 12, 15, 20 from the above data and compute the estimate of λ using the Bayes and HPD procedures under the assumption of non-informative and informative priors. All the results are summarized in Table 3.

Table 3. Bayes and HPD estimates for application example.

r	$a = b = 0$			$a = b = 2$		
	$\hat{\lambda}_B$	$\hat{\lambda}_{BT}$	$\hat{\lambda}_H$	$\hat{\lambda}_B$	$\hat{\lambda}_{BT}$	$\hat{\lambda}_H$
12	0.0118	0.0117	0.0107	0.0136	0.0135	0.0126
15	0.0141	0.0140	0.0131	0.0158	0.0159	0.0152
20	0.0163	0.0162	0.0154	0.0181	0.0181	0.0172

Conclusion

Statistical analysis of exponential distribution under Type II censoring is based on precise lifetime data. Precisely reported lifetimes are common when data comes from specially designed life tests. In such a case a failure should be precisely defined, and all tested items should be continuously monitored. However, in real situations these test requirements might not be fulfilled. In these cases, it is sometimes impossible to obtain exact observations of lifetime. The obtained lifetime data may be imprecise most of the time. Therefore, a suitable statistical methodology is needed to handle these data as well.

The Bayesian inference for the exponential distribution parameter under Type II censoring was addressed when the lifetime observations are fuzzy numbers. Based on the results of the simulation study, the HPD procedure produces the estimates with smaller MSE than the Bayes estimates. Using the informative prior for computing the approximate Bayes estimates provides an improvement in the estimates in terms of MSE.

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