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Tabassum Sindhu

Navid Feroze
Government Post Graduate College Muzaffarabad, Azad Kashmir, Pakistan, navidferoz@gmail.com

Muhammad Aslam

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Doubly Censored Data from Two-Component Mixture of Inverse Weibull Distributions: Theory and Applications

Tabassum Naz Sindhu
Quaid-i-Azam University
Islamabad, Pakistan

Navid Feroze
Ripha International University
Islamabad, Pakistan

Muhammad Aslam
Ripha International University
Islamabad, Pakistan

Finite mixture distributions consist of a weighted sum of standard distributions and are a useful tool for reliability analysis of a heterogeneous population. They provide the necessary flexibility to model failure distributions of components with multiple failure models. The analysis of the mixture models under Bayesian framework has received sizable attention in the recent years. However, the Bayesian estimation of the mixture models under doubly censored samples has not yet been introduced in the literature. The main objective of this paper is to discuss the Bayes estimation of the inverse Weibull mixture distributions under doubly censoring. Different priors and loss functions were assumed for the posterior estimation. The performance of the different estimators has been compared in terms of posterior risks.

Keywords: Inverse transformation method, mixture model, doubly censoring, loss functions, Bayes estimator

Introduction

In survival analysis, data are subject to censoring. The most common type of censoring is right censoring, in which the survival time is larger than the observed right censoring time. In some cases, however, data are subject to left as well as right censoring. When left censoring occurs, the only information available to an analyst is that the survival time is less than or equal to the observed left censoring time. A more complex censoring scheme is found when both initial and final times are interval-censored. This situation is referred to as double censoring, and the data with both right and left censored observations are known as doubly censored data.

Tabassum Sindhu is in the Department of Statistics. Email them at: sindhuqau@gmail.com. Navid Feroze is in the Department of Statistics and is a lecturer in the Department of Statistics at the Government Post Graduate College Muzaffarabad, Azad Kashmir, Pakistan. Email them at: navidferoz@gmail.com. Muhammad Aslam is in the Department of Statistics. Email him at: aslamsdqu@yahoo.com.

Analysis of doubly censored data for simple (single) distribution has been studied by many authors. Fernandez (2000) investigated maximum likelihood prediction based on type-II doubly censored exponential data. Fernandez (2006) has discussed Bayesian estimation based on trimmed samples from Pareto populations. Khan, Provost, and Singh (2010) studied predictive inference from a two-parameter Rayleigh life model given a doubly censored sample. Kim and Song (2010) have discussed Bayesian estimation of the parameters of the generalized exponential distribution from doubly censored samples. Khan, Albatineh, Alshahrani, Jenkins, and Ahmed (2011) studied sensitivity analysis of predictive modeling for responses from the three-parameter Weibull model with a follow-up doubly censored sample of cancer patients. Pak, Parham, and Saraj (2013) proposed the estimation of Rayleigh scale parameter under doubly type-II censoring from imprecise data.

A mixture distribution is signified as a convex fusion of other probability distributions. It can be used to model a statistical population with subpopulations, where the constituents of mixture probability densities are the densities of the subpopulations. Mixture distribution may appropriately be used for certain data sets where the subsets of the whole data set possess different properties that can best be modeled separately. They can be more mathematically manageable, because the individual mixture components are dealt with more ease than the overall mixture density. The families of mixture distributions have a wider range of applications in different fields such as fisheries, agriculture, botany, economics, medicine, psychology, electrophoresis, finance, communication theory, geology, and zoology.

Soliman (2006) derived estimators for the finite mixture of Rayleigh model based on progressively censored data. Sultan, Ismail, and Al-Moisheer (2007) have discussed some properties of the mixture of two inverse Weibull distributions. Saleem and Aslam (2008) presented a comparison of the Maximum Likelihood (ML) estimates with the Bayes estimates assuming the Uniform and the Jeffreys priors for the parameters of the Rayleigh mixture. Kundu and Howalder (2010) considered the Bayesian inference and prediction of the inverse Weibull distribution for type-II censored data. Saleem, Aslam, and Economou (2010) considered the Bayesian analysis of the mixture of Power function distribution using the complete and the censored sample. Shi and Yan (2010) studied the case of the two parameter exponential distribution under type-I censoring to get empirical Bayes estimates. Eluebaly and Bouguila (2011) have presented a Bayesian approach to analyze finite generalized Gaussian mixture models which incorporate several standard mixtures, widely used in signal and image processing applications, such as Laplace and Gaussian. Sultan and Al-Moisheer (2012) developed approximate Bayes estimation of the parameters and reliability function of mixture of two inverse Weibull distributions under type-II censoring.

Model and Likelihood Function

If the probability density function (pdf) of the Weibull distribution is

$$\mathrm{f}_i\left(y_{ij},\theta,\tau\right)=\theta_i\tau_i y_{ij}^{\theta_i-1}\exp\left(-\tau_i y_{ij}^{\theta_i}\right)$$

with $y_{ij} > 0$, $i = 1, 2$, and $j = 1, 2,\ldots, n_i$, then the random variable $x_{ij} = 1/y_{ij}$ has the inverse Weibull distribution with pdf

$$\mathrm{f}_i\left(x_{ij},\theta,\tau\right)=\theta_i\tau_i x_{ij}^{-(\theta_i+1)}\exp\left(-\tau_i x_{ij}^{-\theta_i}\right) \tag{1}$$

with $x_{ij} > 0$, $i = 1, 2$, and $j = 1, 2,\ldots, n_i$, and where $\theta_i > 0$ and $\tau_i > 0$ are shape and scale parameters, respectively.

The cumulative distribution function (cdf) of the distribution is

$$\mathrm{F}_i\left(x_{ij},\theta,\tau\right)=\exp\left(-\tau_i x_{ij}^{-\theta_i}\right),\quad x_{ij},\theta_i,\tau_i>0, i=1,2,\ j=1,2,\ldots,n_i \tag{2}$$

A density function for the mixture of two components densities with mixing weights (p_1, $1 - p_1$) is given by

$$\mathrm{f}\left(x\right)=p_1\,\mathrm{f}_1\left(x\right)+\left(1-p_1\right)\mathrm{f}_2\left(x\right),\quad 0<p_1<1 \tag{3}$$

The cdf for the mixture model is:

$$\mathrm{F}\left(x\right)=p_1\,\mathrm{F}_1\left(x\right)+\left(1-p_1\right)\mathrm{F}_2\left(x\right) \tag{4}$$

Consider a random sample of size n from the inverse Weibull distribution, and let $x_r, x_{r+1},\ldots, x_s$ be the ordered observations that can only be observed. The remaining $r - 1$ smallest observations and the $n - s$ largest observations have been assumed to be censored. Now based on causes of failure, the failed items are assumed to come either from subpopulation 1 or from subpopulation 2; so the $x_{1r_1},\ldots x_{1s_1}$ and $x_{2r_2},\ldots x_{2s_2}$ failed items come from first and second subpopulations, respectively.

The rest of the observations which are less than x_r and greater than x_s have been assumed to be censored from each component, where $x_s = \max\left(x_{1,s_1}, x_{2,s_2}\right)$ and $x_r = \min\left(x_{1,r_1}, x_{2,r_2}\right)$. Therefore, $m_1 = s_1 - r_1 + 1$ and $m_2 = s_2 - r_2 + 1$ number of failed items can be observed from first and second subpopulation, respectively. The remaining $n - (s - r + 2)$ items are assumed to be censored observations, and $s - r + 2$ are the uncensored items, where $r = r_1 + r_2$, $s = s_1 + s_2$, and $m = m_1 + m_2$. Then the likelihood function for the type-II doubly censored sample $\mathbf{x} = \left\{\left(x_{1r_1}, \ldots, x_{1s_1}\right), \left(x_{2r_2}, \ldots, x_{2s_2}\right)\right\}$, assuming the causes of the failure of the left censored items are identified, can be written as

$$\begin{aligned} \mathrm{L}\left(\tau_1, \tau_2, p_1 \mid \mathbf{x}\right) \propto & \left\{\mathrm{F}_1\left(x_{(r_1)}, \tau_1\right)\right\}^{r_1-1} \left\{\mathrm{F}\left(x_{(r_2)}, \tau_2\right)\right\}^{r_2-1} \left\{1 - \mathrm{F}\left(x_s, \tau_1, \tau_2\right)\right\}^{n-2} \\ & \times \left\{\prod_{i=r_1}^{s_1} p_1 \mathrm{f}_1\left(x_{1(i)}, \tau_1\right)\right\} \left\{\prod_{i=r_2}^{s_2} \left(1 - p_1\right) \mathrm{f}_2\left(x_{2(i)}, \tau_2\right)\right\} \end{aligned} \tag{5}$$

$$\begin{aligned} \mathrm{L}\left(\tau_1, \tau_2, p_1 \mid \mathbf{x}\right) \propto & \left\{\exp\left(-\tau_1 x_{(r_1)}^{-\theta_1}\right)\right\}^{r_1-1} \left\{\exp\left(-\tau_2 x_{(r_2)}^{-\theta_2}\right)\right\}^{r_2-1} \\ & \times \left\{1 - p_1 \exp\left(-\tau_1 x_{(r_1)}^{-\theta_1}\right) - p_2 \exp\left(-\tau_2 x_{(r_2)}^{-\theta_2}\right)\right\}^{n-2} \\ & \times \left\{\prod_{i=r_1}^{s_1} p_1 \theta_1 \tau_1 x_{1(i)}^{-(\theta_1+1)} \exp\left(-\tau_1 x_{1(i)}^{-\theta_1}\right)\right\} \\ & \times \left\{\prod_{i=r_2}^{s_2} \left(1 - p_1\right) \theta_2 \tau_2 x_{2(i)}^{-(\theta_2+1)} \exp\left(-\tau_2 x_{2(i)}^{-\theta_2}\right)\right\} \end{aligned} \tag{6}$$

Assuming the shape parameter to be known, the likelihood function (6) reduces to

$$\begin{aligned} \mathrm{L}\left(\tau_1, \tau_2, p_1 \mid \mathbf{x}\right) \propto & \sum_{k_1=0}^{n-s} \sum_{k_2=0}^{k_1} (-1)^{k_1} \binom{n-s}{k_1} \binom{k_1}{k_2} p_1^{m_1+k_1-k_2} \left(1 - p_1\right)^{m_2+k_2} \\ & \times \tau_1^{m_1} \tau_2^{m_2} \exp\left\{-\tau_1\left(\gamma_1\left(x_{1j}\right)\right)\right\} \exp\left\{-\tau_2\left(\gamma_2\left(x_{2j}\right)\right)\right\} \end{aligned} \tag{7}$$

where

$$\gamma_1\left(x_{1j}\right)=\sum_{i=r_1}^{s_1} x_{1(i)}^{-\theta_1}+\left(k_1-k_2\right)x_{(s)}^{-\theta_1}+\left(r_1-1\right)x_{(r_1)}^{-\theta_1}$$
$$\gamma_2\left(x_{2j}\right)=\sum_{i=r_2}^{s_2} x_{2(i)}^{-\theta_2}+\left(k_2\right)x_{(s)}^{-\theta_2}+\left(r_2-1\right)x_{(r_2)}^{-\theta_2}$$

Bayes Estimation

The simple estimation of the scale parameter often pre-assumes the knowhow of the shape parameter (for more detail, see Panaitescu, George, Cozma, & Popa, 2010; Zanakis, 1979; Kundu & Howaldar, 2010; Shi & Yan, 2010; etc.). For the Bayesian estimation, let us assume that the parameters τ_i, $i = 1, 2$, and p_1 are independent random variables, and then consider the following priors for different parameters.

Bayesian Estimation using Conjugate Prior

The prior for the rate parameters τ_i for $i = 1, 2$, is assumed to be the gamma distribution, with the hyperparameters a_i and b_i given by

$$\mathrm{f}_{\tau_i}\left(\tau_i\right)=\frac{b_i^{a_i}}{\Gamma\left(a_i\right)}\tau_i^{a_i-1}\exp\left(-\tau_i b_i\right),\quad a_i,b_i>0 \tag{8}$$

The prior for p_1 is the beta distribution, whose density is given by

$$\mathrm{f}_p\left(p_1\right)=\frac{\Gamma\left(c_1+d_1\right)}{\Gamma\left(c_1\right)\Gamma\left(d_1\right)}p_1^{c_1-1}\left(1-p_1\right)^{d_1-1},\quad c_1,d_1>0 \tag{9}$$

From equations (8)-(9), the following joint prior density of the vector $\mathbf{\Theta} = (\tau_1, \tau_2, p_1)$ is proposed:

$$\mathrm{g}\left(\mathbf{\Theta}\right)\propto\tau_i^{a_i-1}\exp\left(-\tau_i b_i\right)p_1^{c_1-1}\left(1-p_1\right)^{d_1-1},\quad 0<p_1<1,a_i,b_i,c_1,d_1>0 \tag{10}$$

By multiplying equation (10) and equation (7), the joint posterior density for the vector $\mathbf{\Theta}$, given the data, becomes

$$\pi(\mathbf{\Theta} \mid x) = \Omega_1^{-1} \sum_{k_1=0}^{n-s} \sum_{k_2=0}^{k_1} (-1)^{k_1} \binom{n-s}{k_1} \binom{k_1}{k_2} p_1^{m_1+k_1-k_2+c_1-1} (1-p_1)^{m_2+k_2+d_1-1} \times \exp\left\{-\tau_i \left(\xi_i \left(x_{ij}\right)\right)\right\} \tag{11}$$

where

$$\Omega_1 = \sum_{k_1=0}^{n-s} \sum_{k_2=0}^{k_1} (-1)^{k_1} \binom{n-s}{k_1} \binom{k_1}{k_2} Beta\left(m_1+k_1-k_2+c_1, m_2+k_2+d_1\right) \times \prod_{i=1}^{2} \frac{\Gamma\left(a_i+m_i\right)}{\left\{\xi_i\left(x_{ij}\right)\right\}^{\left(a_i+m_i\right)}}$$

and $\xi_i(x_{ij}) = \gamma_i(x_{ij}) + b_i$ for $i = 1, 2$. Marginal distributions of τ_i, $i = 1, 2$, and p_1 can be obtained by integrating the nuisance parameters.

Bayesian Estimation using Inverse Levy Prior

The prior for the rate parameters τ_i for $i = 1, 2$, is assumed to be the inverse Levy distribution, with hyperparameter ν_i, given by

$$\mathrm{f}_{\tau_i}\left(\tau_i\right) = \sqrt{\frac{\nu_i}{2\pi}} \tau_i^{-1/2} \exp\left(\frac{-\tau_i \nu_i}{2}\right), \quad \nu_i > 0 \tag{12}$$

The prior for p_1 is the beta distribution, whose density is given by

$$\mathrm{f}_p\left(p_1\right) = \frac{\Gamma\left(c_2+d_2\right)}{\Gamma\left(c_2\right)\Gamma\left(d_2\right)} p_1^{c_2-1} \left(1-p_1\right)^{d_2-1}, \quad c_2, d_2 > 0 \tag{13}$$

From equation (12)-(13), we propose the following joint prior density of the vector $\mathbf{\Theta} = (\tau_1, \tau_2, p_1)$:

$$\mathrm{g}(\mathbf{\Theta}) \propto \tau_i^{-1/2} \exp\left(\frac{-\tau_i \nu_i}{2}\right) p_1^{c_2-1} \left(1-p_1\right)^{d_2-1}, \quad 0 < p_1 < 1, \nu_i, c_2, d_2 > 0 \tag{14}$$

By multiplying equation (14) with equation (7), the joint posterior density for the vector $\mathbf{\Theta}$, given the data, becomes

$$\pi(\mathbf{\Theta} \mid x) = \Omega_2^{-1} \sum_{k_1=0}^{n-s} \sum_{k_2=0}^{k_1} (-1)^{k_1} \binom{n-s}{k_1} \binom{k_1}{k_2} p_1^{m_1+k_1-k_2+c_2-1} (1-p_1)^{m_2+k_2+d_2-1} \times \exp\left\{-\tau_i \left(\psi_i \left(x_{ij}\right)\right)\right\} \tag{15}$$

where

$$\Omega_2 = \sum_{k_1=0}^{n-s} \sum_{k_2=0}^{k_1} (-1)^{k_1} \binom{n-s}{k_1} \binom{k_1}{k_2} Beta\left(m_1+k_1-k_2+c_2, m_2+k_2+d_2\right) \times \prod_{i=1}^{2} \frac{\Gamma\left(m_i+1/2\right)}{\left\{\psi_i\left(x_{ij}\right)\right\}^{(m_i+1/2)}}$$

and $\psi_i(x_{ij}) = \gamma_i(x_{ij}) + v_i/2$. Marginal distributions of τ_i, $i = 1, 2$, and p_1 can be obtained by integrating the nuisance parameters.

Bayes Estimation of the Vector of Parameters Θ

The Bayesian point estimation is connected to a loss function in general, signifying the loss is induced when the estimate $\hat{\theta}$ differs from the true parameter θ. Because there is no specific rule that helps to identify the appropriate loss function to be used, we can use the K-loss function (KLF), which is particularized as

$$l\left(\hat{\theta}, \theta\right) = \frac{\left(\hat{\theta}-\theta\right)^2}{\hat{\theta}\theta}$$

is proposed by Wasan (1970), and is well-fitted for a measure of inaccuracy for an estimator of a scale parameter of a distribution defined on $\mathbb{R}^+ = (0, \infty)$. The Bayes estimator and posterior risk under KLF are $\hat{\theta} = \left\{E(\theta \mid \mathbf{x}) \big/ E\left(\theta^{-1} \mid \mathbf{x}\right)\right\}^{-1/2}$ and $\rho\left(\hat{\theta}\right) = 2\left\{E(\theta \mid \mathbf{x}) E\left(\theta^{-1} \mid \mathbf{x}\right) - 1\right\}$, respectively. In Bayesian analysis, a widely used loss function is the quadratic loss function given by $l\left(\hat{\theta}, \theta\right) = w\left(\hat{\theta}-\theta\right)^2$; if $w = 1$, it reduces to the squared error loss function (SELF) and, for $w = \theta^{-2}$, it becomes $l\left(\hat{\theta}, \theta\right) = \left(\hat{\theta}-\theta\right)^2 \big/ \theta^2$. This is known as the minimum expected loss function

(MELF), and is introduced by Tummala and Sathe (1978) in their study. The Bayes estimator and posterior risk under MELF are $\hat{\theta} = \mathrm{E}\left(\theta^{-1} \mid \mathbf{x}\right) \big/ \mathrm{E}\left(\theta^{-2} \mid \mathbf{x}\right)$ and $\rho\left(\hat{\theta}\right) = 1 - \left\{\mathrm{E}\left(\theta^{-1} \mid \mathbf{x}\right)\right\}^2 \big/ \mathrm{E}\left(\theta^{-2} \mid \mathbf{x}\right)$, respectively.

The respective marginal distribution of each parameter is used to derive the Bayes estimators and posterior risks of τ_1, τ_2, and p_1 under KLF and MELF. The Bayes estimators and their posterior risks of the parameters τ_1, τ_2, and p_1 for the conjugate (gamma and beta) priors using the KLF and MELF functions are given in the Appendix. Thus, expressions for Bayes estimators and their posterior risks under the inverse Levy can be obtained with little alteration.

Elicitation

The elicitation of opinion is a crucial step. It helps to make it easy for us to understand what the experts believe in, and what their opinions are. In statistical inference, the characteristics of a certain predictive distribution proposed by an expert determine the hyperparameters of a prior distribution. In this article, we focused on a method of elicitation based on prior predictive distribution. The elicitation of hyperparameters from the prior p(λ) is a difficult task. The prior predictive distribution is used for the elicitation of the hyperparameters, which are compared with the experts' judgment about this distribution and then the hyperparameters are chosen in such a way so as to make the judgment agree as closely as possible with the given distribution. Readers desiring more detail may refer to: Grimshaw, Collings, Larsen, and Hurt (2001), O'Hagan et al. (2006), Jenkinson (2005) and Leon, Vazquez-Polo, and Gonzalez (2003). According to Aslam (2003), the preferred method of elicitation is to compare the prior predictive distribution with experts' assessment about this distribution, and then to choose the hyperparameters that make the assessment agree closely with the member of the family. The prior predictive distributions under all the priors are derived using the following formula:

$$\mathrm{p}\left(y\right) = \int_{\boldsymbol{\Theta}} \mathrm{p}\left(y \mid \boldsymbol{\Theta}\right) \mathrm{p}\left(\boldsymbol{\Theta}\right) d\boldsymbol{\Theta}$$

Elicitation under Gamma Distribution

The prior predictive distribution using gamma prior is

$$\mathrm{p}(y) = \frac{\theta_1 y^{-(\theta_1+1)} a_1 b_1^{a_1} c_1}{(c_1+d_1)\left(y^{-\theta_1}+b_1\right)^{(a_1+1)}} + \frac{\theta_2 y^{-(\theta_2+1)} a_2 b_2^{a_2} d_1}{(c_1+d_1)\left(y^{-\theta_2}+b_2\right)^{(a_2+1)}}, \quad y > 0 \tag{16}$$

Assume $(\theta_1, \theta_2) = (1, 1)$ for convenience in calculations. For the elicitation of the six hyperparameters, six different intervals are considered. From equation (16), the experts' probabilities/assessments are supposed to be 0.10 for each case. The six integrals for equation (16) are considered with the following limits of the values of random variable Y: (0, 10), (10, 20), (20, 30), (30, 40), (40, 50), and (50, 60) respectively. For the elicitation of hyperparameters, a_1, a_2, b_1, b_2, c_1, and d_1, these six integrals are solved simultaneously through computer program developed in SAS package using the command of PROC SYSLIN. Thus the values of hyperparameters obtained by applying this methodology are: $a_1 = 4.982587$, $a_2 = 3.356211$, $b_1 = 0.987542$, $b_2 = 0.46523$, $c_1 = 1.45987$, and $d_1 = 0.05690$.

Elicitation under Inverse Levy Prior

$$\mathrm{p}(y) = \frac{\theta_1 y^{-(\theta_1+1)} \left(\frac{\nu_1}{2}\right)^{\frac{1}{2}} c_1}{2(c_1+d_1)\left(\frac{y^{-\theta_1}+\nu_1}{2}\right)^{\frac{\nu_1}{2}+1}} + \frac{\theta_2 y^{-(\theta_2+1)} \left(\frac{\nu_2}{2}\right)^{\frac{1}{2}} d_2}{2(c_1+d_1)\left(\frac{y^{-\theta_2}+\nu_2}{2}\right)^{\frac{\nu_2}{2}+1}}, \quad y > 0$$

Now, to elicit four hyperparameters, consider the four integrals. The expert probabilities are assumed to 0.15 for each integral with the following limits of the values of random variable Y: (0, 15), (15, 30), (30, 45), and (45, 60). Using a similar kind of program as discussed above, we have the following values of the hyperparameters: $\nu_1 = 0.062138$, $\nu_2 = 0.19136$, $c_2 = 0.895777$, and $d_2 = 0.63889$.

Simulation Study and Comparisons

A simulation study was conducted to compare the performance of the discussed estimators on the basis of generated samples from the inverse Weibull mixture distribution using doubly censored data. Assume $(\theta_1, \theta_2) = (1, 1)$ for convenience in calculations. Take random samples of sizes $n = 20$, 40, and 80 from the two component mixture of inverse Weibull distributions with following choice of parametric values: $(\tau_1, \tau_2) \in \{(0.1, 0.15), (10, 15), (0.1, 15), (10, 0.15)\}$, $p_1 = 0.45$ and 0.6. To develop a mixture data, we adopt the probabilistic mixing model with probability p_1 and $(1 - p_1)$. A uniform number u is generated n times and, if $u < p_1$, the observation is taken randomly from F_1 (the inverse Weibull distribution with parameter τ_1), and is otherwise taken from F_2 (the inverse Weibull with parameter τ_2). Hence, the parameters to be estimated are known to be (τ_1, τ_2) and p_1. The choice of the censoring time is made in such a way that the censoring rate in the resultant sample is to be approximately 20%. The simulated data sets have been obtained using following steps:

Step 1: Draw samples of size n from the mixture model

Step 2: Generate a uniform random number u for each observation

Step 3: If $u \leq \pi$, take the observation from first subpopulation; otherwise, take the observation from the second subpopulation

Step 4: Determine the test termination points on left and right, that is, determine the values of x_r and x_s

Step 5: The observations which are less than x_r and greater than x_s have been considered to be censored from each component

Step 6: Use the remaining observations from each component for the analysis

To avoid an extreme sample, simulate 10,000 data sets, each of size n. The Bayes estimates and posterior risks (in parenthesis) are computed using Mathematica 8.0. The average of these estimates and corresponding risks are reported in Tables 1-8. The abbreviations used in the tables are: BEs: Bayes estimators; PRs: Posterior risks; GP: Gamma prior; ILP: Inverse Levy prior.

The simulation study has revealed some interesting properties of the Bayes estimates. It is worth mentioning that in each case the posterior risks of estimates of lifetime parameters are decreasing as the sample size increases. The posterior risks of the estimates of τ_1, τ_2 have been assessed to be quite large when the values of the parameters are large, and entirely small for rather smaller values of τ_1, τ_2. Another interesting point regarding the posterior risks of the estimates of parameters τ_1, τ_2 is that by increasing (decreasing) the proportion of the component in mixture reduces (increases) the posterior risk of the concerned τ parameter's estimate.

Table 1. BEs and their PRs under GP for (τ_1, τ_2, p_1) = (0.10, 0.15, 0.45) and (0.10, 0.15, 0.60)

	K-Loss Function					
n	$\hat{\tau}_1$	$\hat{\tau}_2$	$\hat{p}_1$	$\hat{\tau}_1$	$\hat{\tau}_2$	$\hat{p}_1$
20	0.153042	0.217287	0.488886	0.149162	0.243873	0.652455
	(0.161568)	(0.166297)	(0.118884)	(0.127883)	(0.227595)	(0.060474)
40	0.130631	0.181089	0.461140	0.126142	0.188513	0.635182
	(0.101929)	(0.091817)	(0.069768)	(0.076355)	(0.131665)	(0.034437)
80	0.113720	0.171546	0.449263	0.115099	0.182363	0.627186
	(0.074710)	(0.063162)	(0.049079)	(0.054635)	(0.092332)	(0.024224)

	Minimum Expected Loss Function					
n	$\hat{\tau}_1$	$\hat{\tau}_2$	$\hat{p}_1$	$\hat{\tau}_1$	$\hat{\tau}_2$	$\hat{p}_1$
20	0.152631	0.194781	0.445046	0.136376	0.200732	0.621717
	(0.080798)	(0.083651)	(0.066112)	(0.064041)	(0.114609)	(0.033749)
40	0.123116	0.167329	0.447311	0.118357	0.168123	0.618551
	(0.051022)	(0.046079)	(0.036942)	(0.038252)	(0.066219)	(0.018267)
80	0.113790	0.161134	0.447937	0.113935	0.162226	0.610625
	(0.037331)	(0.031706)	(0.025748)	(0.027277)	(0.046496)	(0.012683)

Table 2. BEs and their PRs under GP for (τ_1, τ_2, p_1) = (10, 15, 0.45) and (10, 15, 0.60)

	K-Loss Function					
n	$\hat{\tau}_1$	$\hat{\tau}_2$	$\hat{p}_1$	$\hat{\tau}_1$	$\hat{\tau}_2$	$\hat{p}_1$
20	7.2322300	11.9032000	0.4851680	7.8576700	10.4070000	0.6564870
	(0.1628030)	(0.165620)	(0.1206010)	(0.1274380)	(0.2306880)	(0.0594520)
40	8.0121000	13.7528000	0.4556190	8.7621200	12.0339000	0.6369690
	(0.1029490)	(0.0908861)	(0.0709860)	(0.0763040)	(0.1328160)	(0.0342290)
80	8.4481100	14.0172700	0.4465120	8.7865800	12.9782000	0.6284630
	(0.0750960)	(0.0628280)	(0.0493037)	(0.0546180)	(0.0929030)	(0.0217830)

	Minimum Expected Loss Function					
n	$\hat{\tau}_1$	$\hat{\tau}_2$	$\hat{p}_1$	$\hat{\tau}_1$	$\hat{\tau}_2$	$\hat{p}_1$
20	6.2983400	10.8209000	0.4383870	7.0637800	8.8599400	0.6246390
	(0.0817250)	(0.0830211)	(0.0675390)	(0.0639590)	(0.1158660)	(0.0334390)
40	7.3851200	2.3639000	0.4397130	8.2515200	11.2008100	0.6191910
	(0.0514960)	(0.0456430)	(0.0375050)	(0.0382830)	(0.0665690)	(0.0182460)
80	7.7764800	13.1101000	0.4473950	8.6210200	12.9293400	0.6068140
	(0.0378730)	(0.0316780)	(0.0327710)	(0.0272560)	(0.0466960)	(0.0129840)

Table 3. BEs and their PRs under GP for (τ_1, τ_2, p_1) = (0.10, 15, 0.45) and (0.10, 15, 0.60)

	K-Loss Function					
n	$\hat{\tau}_1$	$\hat{\tau}_2$	$\hat{p}_1$	$\hat{\tau}_1$	$\hat{\tau}_2$	$\hat{p}_1$
20	0.1533410	12.5483000	0.4483640	0.1397540	11.7884000	0.5905870
	(0.1669040)	(0.1504220)	(0.1273990)	(0.1334790)	(0.1951910)	(0.0685210)
40	0.1193940	14.5209000	0.4489700	0.1107460	13.6061000	0.5978991
	(0.1053590)	(0.0823310)	(0.0740830)	(0.0800540)	(0.1096810)	(0.0388500)
80	0.1114640	15.0405000	0.4511250	0.1057960	14.6865000	0.5986610
	(0.0771020)	(0.0565920)	(0.0432290)	(0.0580370)	(0.0777460)	(0.0048650)

	Minimum Expected Loss Function					
n	$\hat{\tau}_1$	$\hat{\tau}_2$	$\hat{p}_1$	$\hat{\tau}_1$	$\hat{\tau}_2$	$\hat{p}_1$
20	0.140090	11.354900	0.404051	0.133802	10.261200	0.567283
	(0.083452)	(0.075235)	(0.070580)	(0.066740)	(0.097732)	(0.037961)
40	0.112806	13.171700	0.419673	0.109543	12.681400	0.567551
	(0.052679)	(0.041176)	(0.039065)	(0.040027)	(0.054874)	(0.020488)
80	0.108045	14.175500	0.429351	0.103915	13.796700	0.587920
	(0.038552)	(0.028369)	(0.031335)	(0.028531)	(0.038394)	(0.022886)

Table 4. BEs and their PRs under GP for (τ_1, τ_2, p_1) = (10, 0.15, 0.45) and (10, 0.15, 0.60)

	K-Loss Function					
n	$\hat{\tau}_1$	$\hat{\tau}_2$	$\hat{p}_1$	$\hat{\tau}_1$	$\hat{\tau}_2$	$\hat{p}_1$
20	7.789440	0.206080	0.544287	8.052190	0.224464	0.695652
	(0.144196)	(0.176093)	(0.086478)	(0.118203)	(0.239330)	(0.044531)
40	8.918560	0.166512	0.522777	8.909610	0.175136	0.681657
	(0.087638)	(0.098245)	(0.049039)	(0.069216)	(0.139309)	(0.024854)
80	9.274560	0.155907	0.515036	9.687610	15.652800	0.652686
	(0.062971)	(0.068130)	(0.033541)	(0.049070)	(0.098428)	(0.001594)

	Minimum Expected Loss Function					
n	$\hat{\tau}_1$	$\hat{\tau}_2$	$\hat{p}_1$	$\hat{\tau}_1$	$\hat{\tau}_2$	$\hat{p}_1$
20	7.007170	0.175346	0.507474	7.329860	0.187362	0.671009
	(0.072105)	(0.088049)	(0.047976)	(0.059115)	(0.119666)	(0.024700)
40	8.392620	0.148824	0.503235	8.052989	0.155773	0.668620
	(0.043817)	(0.049123)	(0.025863)	(0.034608)	(0.069655)	(0.013108)
80	8.850450	0.151859	0.495015	9.424450	0.151359	0.661397
	(0.031496)	(0.034063)	(0.017918)	(0.024634)	(0.049023)	(0.024360)

Table 5. BEs and their PRs under ILP for (τ_1, τ_2, p_1) = (0.10, 0.15, 0.45) and (0.10, 0.15, 0.60)

	K-Loss Function					
n	$\hat{\tau}_1$	$\hat{\tau}_2$	$\hat{p}_1$	$\hat{\tau}_1$	$\hat{\tau}_2$	$\hat{p}_1$
20	0.107446	0.174887	0.454348	0.108630	0.172018	0.620560
	(0.256954)	(0.215458)	(0.136335)	(0.180738)	(0.336696)	(0.069258)
40	0.104352	0.164206	0.441143	0.104008	0.154266	0.618750
	(0.133538)	(0.104548)	(0.075409)	(0.133544)	(0.104585)	(0.036939)
80	0.098973	0.158185	0.436433	0.102810	0.151531	0.617152
	(0.090341)	(0.068909)	(0.045525)	(0.062284)	(0.106620)	(0.025056)

	Minimum Expected Loss Function					
n	$\hat{\tau}_1$	$\hat{\tau}_2$	$\hat{p}_1$	$\hat{\tau}_1$	$\hat{\tau}_2$	$\hat{p}_1$
20	0.092775	0.147587	0.407706	0.097211	0.134592	0.586746
	(0.128863)	(0.108963)	(0.075846)	(0.090791)	(0.171009)	(0.038768)
40	0.096182	0.147725	0.416953	0.099533	0.142006	0.600375
	(0.066809)	(0.052609)	(0.039818)	(0.046432)	(0.081512)	(0.019661)
80	0.096554	0.149812	0.429139	0.102130	0.146210	0.600153
	(0.045175)	(0.034586)	(0.027256)	(0.031126)	(0.053686)	(0.014347)

Table 6. BEs and their PRs under ILP for (τ_1, τ_2, p_1) = (10, 15, 0.45) and (10, 15, 0.60)

	K-Loss Function					
n	$\hat{\tau}_1$	$\hat{\tau}_2$	$\hat{p}_1$	$\hat{\tau}_1$	$\hat{\tau}_2$	$\hat{p}_1$
20	9.985290	14.431300	0.456402	10.696500	13.981300	0.625004
	(0.255990)	(0.216752)	(0.135572)	(0.179341)	(0.341694)	(0.068025)
40	10.643800	14.798400	0.443693	10.480710	14.656400	0.620334
	(0.132816)	(0.105192)	(0.074823)	(0.092251)	(0.163078)	(0.036711)
80	10.122700	14.845100	0.453762	10.174900	14.854300	0.617783
	(0.090007)	(0.069232)	(0.051825)	(0.062220)	(0.106884)	(0.025106)

	Minimum Expected Loss Function					
n	$\hat{\tau}_1$	$\hat{\tau}_2$	$\hat{p}_1$	$\hat{\tau}_1$	$\hat{\tau}_2$	$\hat{p}_1$
20	9.277410	11.474100	0.412225	9.321530	9.642290	0.593112
	(0.127923)	(0.110266)	(0.074962)	(0.089785)	(0.173529)	(0.037767)
40	9.637820	14.223600	0.417554	9.502120	12.713300	0.601931
	(0.066774)	(0.052724)	(0.039748)	(0.046313)	(0.081862)	(0.019531)
80	9.729790	14.560200	0.428610	9.999100	13.616000	0.601586
	(0.045118)	(0.034705)	(0.027784)	(0.031143)	(0.053736)	(0.013452)

Table 7. BEs and their PRs under ILP for (τ_1, τ_2, p_1) = (0.10, 15, 0.45) and (0.10, 15, 0.60)

	K-Loss Function					
n	$\hat{\tau}_1$	$\hat{\tau}_2$	$\hat{p}_1$	$\hat{\tau}_1$	$\hat{\tau}_2$	$\hat{p}_1$
20	0.0995060	15.8176000	0.4191220	0.1019610	16.6349000	0.5696700
	(0.2666630)	(0.1911270)	(0.1435660)	(0.1904710)	(0.2689990)	(0.0772190)
40	0.0957830	15.7349000	0.4315360	0.0972550	15.9842000	0.5698800
	(0.1379300)	(0.0932430)	(0.0788790)	(0.0975590)	(0.1298730)	(0.0413270)
80	0.0925177	15.3503000	0.4450500	0.0931070	15.5486000	0.5765170
	(0.0929980)	(0.0616730)	(0.0448390)	(0.0655840)	(0.0856100)	(0.0268270)

	Minimum Expected Loss Function					
n	$\hat{\tau}_1$	$\hat{\tau}_2$	$\hat{p}_1$	$\hat{\tau}_1$	$\hat{\tau}_2$	$\hat{p}_1$
20	0.078252	14.237100	0.372592	0.089637	12.991000	0.535129
	(0.133332)	(0.095557)	(0.079529)	(0.095236)	(0.134672)	(0.042776)
40	0.084029	14.501970	0.380824	0.090933	14.232200	0.545625
	(0.068965)	(0.046618)	(0.041602)	(0.048779)	(0.064944)	(0.021795)
80	0.086764	14.687900	0.403030	0.091828	15.480600	0.548745
	(0.046508)	(0.030839)	(0.030867)	(0.032782)	(0.042834)	(0.019410)

Table 8. BEs and their PRs under ILP for (τ_1, τ_2, p_1) = (10, 0.15, 0.45) and (10, 0.15, 0.60)

	K-Loss Function					
n	$\hat{\tau}_1$	$\hat{\tau}_2$	$\hat{p}_1$	$\hat{\tau}_1$	$\hat{\tau}_2$	$\hat{p}_1$
20	11.639900	0.143830	0.516704	10.857400	0.143796	0.667442
	(0.212306)	(0.235279)	(0.096294)	(0.160525)	(0.363626)	(0.050583)
40	11.407900	0.144247	0.508186	10.697180	0.148710	0.665794
	(0.108873)	(0.114281)	(0.051895)	(0.081860)	(0.173910)	(0.026553)
80	10.967200	0.143250	0.501879	10.568890	0.149423	0.636676
	(0.073551)	(0.075455)	(0.014772)	(0.054934)	(0.114285)	(0.017568)

	Minimum Expected Loss Function					
n	$\hat{\tau}_1$	$\hat{\tau}_2$	$\hat{p}_1$	$\hat{\tau}_1$	$\hat{\tau}_2$	$\hat{p}_1$
20	10.949000	0.122689	0.477731	9.730160	0.112718	0.640643
	(0.106221)	(0.117641)	(0.053440)	(0.080295)	(0.181814)	(0.028050)
40	1.033170	0.123964	0.468809	10.421800	0.121727	0.653162
	(0.054432)	(0.057141)	(0.027368)	(0.040932)	(0.086955)	(0.014005)
80	10.185800	0.132493	0.450288	10.186800	0.125570	0.650629
	(0.036673)	(0.037766)	(0.027487)	(2.748000)	(0.057149)	(0.011849)

It was observed that for the relatively smaller value of τ, i.e. (0.10, 0.15), the performance of the minimum expected loss function and the gamma prior is better than their counterparts, as the amounts of posterior risks are smaller than those in case of their counterparts. However, the inverse Levy prior produces some closer estimates to the true value of parameters. Estimates of mixing proportion are found to be underestimated using inverse Levy prior when $p_1 = 0.45$, but they are pretty good under gamma prior. When we consider the estimation of comparatively larger value of τ, i.e. (10, 15), again under estimation is observed of the estimates of parameters under both priors and loss functions. But the extent of underestimation is higher under the minimum expected loss function using gamma prior. Nonetheless, this underestimation is due to the random procedure and is tolerable.

Further, this problem can be faced off by using lager sample sizes. As far as the efficiency of the prior is concerned, gamma is found to be the efficient than inverse Levy prior. Moreover, on assessing the behavior of estimates, in the case of the extremely different value of the parameters ($\tau_1 < \tau_2$ and $\tau_1 > \tau_2$) = (0.10, 15 and 10, 0.15), i.e. one is small and other is hundred fold large, it is noticed that the parameters are once again underestimated, and this underestimation is higher at every point using the minimum expected loss function under both priors. However, the use of the K-loss function has exhibited pretty good estimates with few exceptions (in terms of convergence). In general, the estimates under gamma prior using the minimum expected loss function are the best, as the amounts of posterior risks associated with these estimates are the least in almost all cases.

Real Data Analysis

Real data sets are considered to illustrate the methodology discussed in previous sections. In order to show the usefulness of the proposed mixture model, consider survival times (in days) of guinea pigs, injected with different doses of tubercle bacilli, in Table 9. This data set was discussed by Kundu and Howlader (2010). Singh, Singh, and Sharma (2013) also analyzed this data set. The regimen number is the common logarithm of the number of bacillary units in 0.5 mL of challenge solution; e.g., regimen 6.6 corresponds to 4.0 $*10^6$ bacillary units per 0.5 mL. Corresponding to regimen 6.6, there are 72 observations listed below. Further, the Kolmogorov-Smirnov and chi-square tests are used to see if the data follow the inverse Weibull distribution. These tests say that the data follow the inverse Weibull distribution at 5% level of significance with p-values 0.1361 and 0.1290, respectively. We have assumed $(\theta_1, \theta_2) = (1, 1)$ for convenience in calculations.

Table 9. Survival times (in days) of guinea pigs injected with different doses of tubercle bacilli

12	15	22	24	24	32	32	33	34	38	38	43	44	48	52
53	54	54	55	56	57	58	58	59	60	60	60	60	61	62
63	65	65	67	68	70	70	72	73	75	76	76	81	83	84
85	87	91	95	96	98	99	109	110	121	127	129	131	143	146
146	175	211	233	99	258	258	263	297	341	341	376			

Consider the case when the data are doubly Type II censored. Data are randomly grouped into two sets when $p_1 = 0.45$. It is assumed that we observe 33 data points belonging to population I and 39 data points belonging to population II. To implement censored samplings, the $x_{1r_1},\ldots,x_{1s_1}$ and $x_{2r_2},\ldots,x_{2s_2}$ failed items come from the first and second subpopulations, respectively. The rest of the observations, which are less than x_r and greater than x_s, have been assumed to be censored from each component. Here, $m_1 = s_1 - r_1 + 1$ and $m_2 = s_2 - r_2 + 1$ numbers of failed items can be observed from the first and second subpopulations, respectively. The remaining $n - (s - r + 2)$ items are assumed to be censored observations, and $s - r + 2$ are the uncensored items, where $r = r_1 + r_2$, $s = s_1 + s_2$, and $m = m_1 + m_2$. The detail of the censored mixture data can be found in Table 10.

The following characteristics are extracted from the censored data for the analysis of the mixture model:

$$p_1 = 0.45, n = 72, r = 8, r_1 = 4, r_2 = 43, s = 64, s_1 = 29, s_2 = 35, n_1 = 33, n_2 = 39$$
$$\theta_1 = \theta_2 = 0.5, x_{r_1} = 32, x_{s_1} = 233, x_{r_2} = 33, x_{s_2} = 258$$
$$\sum_{i=r_1}^{s_1} x_{1(i)}^{-\theta_1} = 3.21314, \sum_{i=r_2}^{s_2} x_{2(i)}^{-\theta_2} = 3.85409$$

Similar methodology was employed when

$$p_1 = 0.60, n = 72, r = 8, r_1 = 5, r_2 = 3, s = 64, s_1 = 39, s_2 = 25, n_1 = 44, n_2 = 28$$
$$\theta_1 = \theta_2 = 0.5, x_{r_1} = 33, x_{s_1} = 211, x_{r_2} = 32, x_{s_2} = 175$$
$$\sum_{i=r_1}^{s_1} x_{1(i)}^{-\theta_1} = 4.16450, \sum_{i=r_2}^{s_2} x_{2(i)}^{-\theta_2} = 3.21392$$

Table 10. Doubly-censored mixture real life data

Population I							Population II						
61	12	24	60	24	32	65	15	131	87	143	91	95	175
34	68	38	43	67	72	48	110	121	127	297	341	60	62
54	73	76	55	81	83	58	65	63	70	96	211	98	258
84	233	341	263	146	175	129	258	70	75	76	59	60	57
146	109	99	35	376			56	58	53	54	44	52	43
							38	33	32	22			

Table11. BEs and their PRs under minimum expected loss function and K-loss function for the real data set

Priors	K-loss function			Minimum expected loss function		
p_1 = 0.45	$\hat{r}_1$	$\hat{r}_2$	$\hat{p}_1$	$\hat{r}_1$	$\hat{r}_2$	$\hat{p}_1$
Gamma	7.023900	7.914180	0.453725	6.699360	7.600860	0.439455
	(0.062637)	(0.053542)	(0.041482)	(0.031384)	(0.026819)	(0.021459)
Inverse Levy	7.613170	7.918130	0.446087	7.206180	7.583200	0.431593
	(0.072641)	(0.058103)	(0.042864)	(0.036424)	(0.029113)	(0.022179)
p_1 = 0.60	$\hat{r}_1$	$\hat{r}_2$	$\hat{p}_1$	$\hat{r}_1$	$\hat{r}_2$	$\hat{p}_1$
Gamma	7.400650	6.984160	0.610524	7.142880	6.603080	0.600336
	(0.047031)	(0.074187)	(0.021878)	(0.023548)	(0.037188)	(0.011324)
Inverse Levy	7.923470	6.899140	0.602689	7.616030	6.478070	0.592309
	(0.052462)	(0.083158)	(0.022581)	(0.026276)	(0.041710)	(0.011689)

The results in Table 11 indicate that the Bayes estimates under gamma prior are better than those under inverse levy prior under both loss functions. Similarly, in the comparison of the loss functions, it has been assessed that the performance of the minimum expected loss function is better than the K-loss function. The larger values of the mixing parameter (p_1) impose a positive impact on the performance of the estimation of the first component of the mixture. Hence the analysis of real-life data endorsed the findings of the simulation study, suggesting the preference of gamma prior along with minimum expected loss function.

Graphical Representation of Posterior Risks under Different Loss Functions, Various Priors

Risks of the estimators are empirically evaluated based on a Monte-Carlo simulation study of samples. A number of values of unknown parameters are considered. Sample size is varied to observe the effect of small and large samples on the estimators. Different combinations of parameters are considered in studying the change in the estimators and their risks. The results are summarized in Figures 1-4. The risk of the estimators will be a function of sample size, population parameters, and hyperparameters of the prior distribution. After an extensive study of the results, the conclusions are drawn regarding the behavior of the estimators, which are summarized below. (Due to space restrictions, all results are not shown in the graphs.) As sample size increases, the risk of all the estimators decrease, as indicated in Figures 1-4. The effect of variation of parameters on the risks of the estimator has also been studied. The risk of the estimators increases when the value of parameters increases.

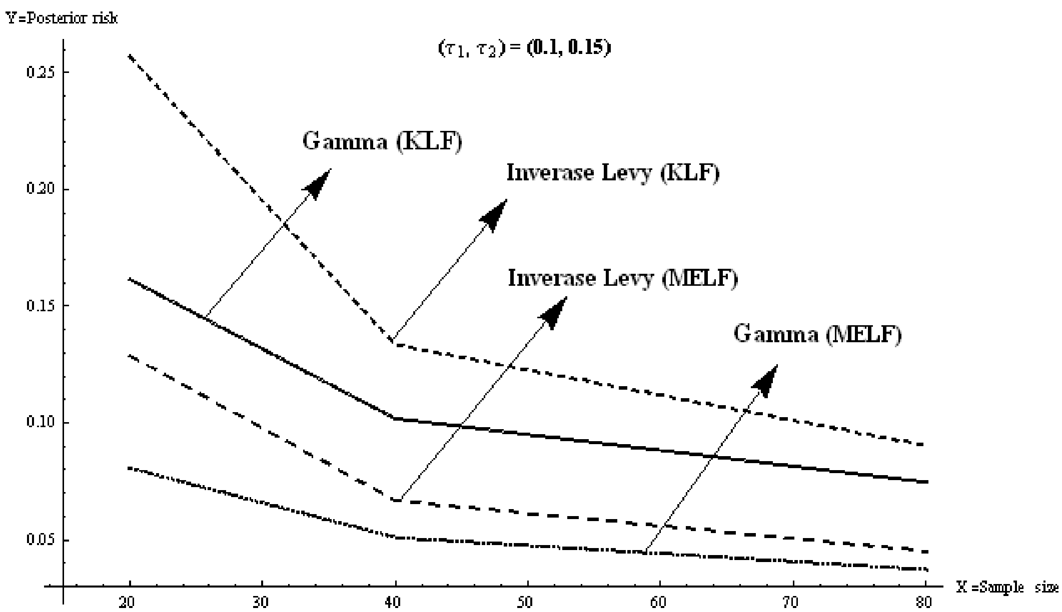

Figure 1. Posterior risks of τ_1 for (τ_1, τ_2, p_1) = (0.10, 0.15, 0.45)

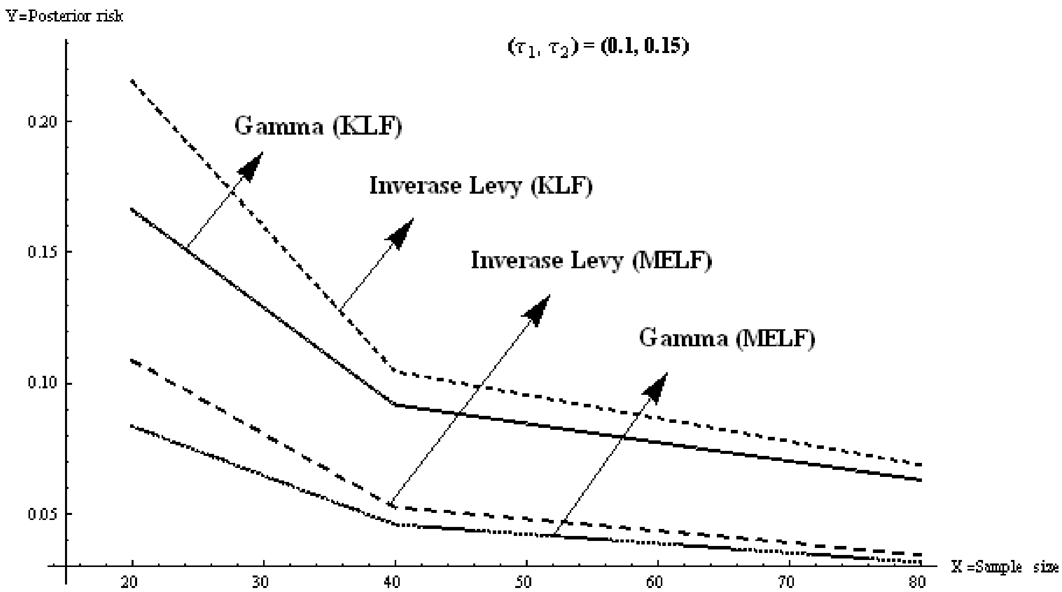

Figure 2. Posterior risks of τ_2 for (τ_1, τ_2, p_1) = (0.10, 0.15, 0.45)

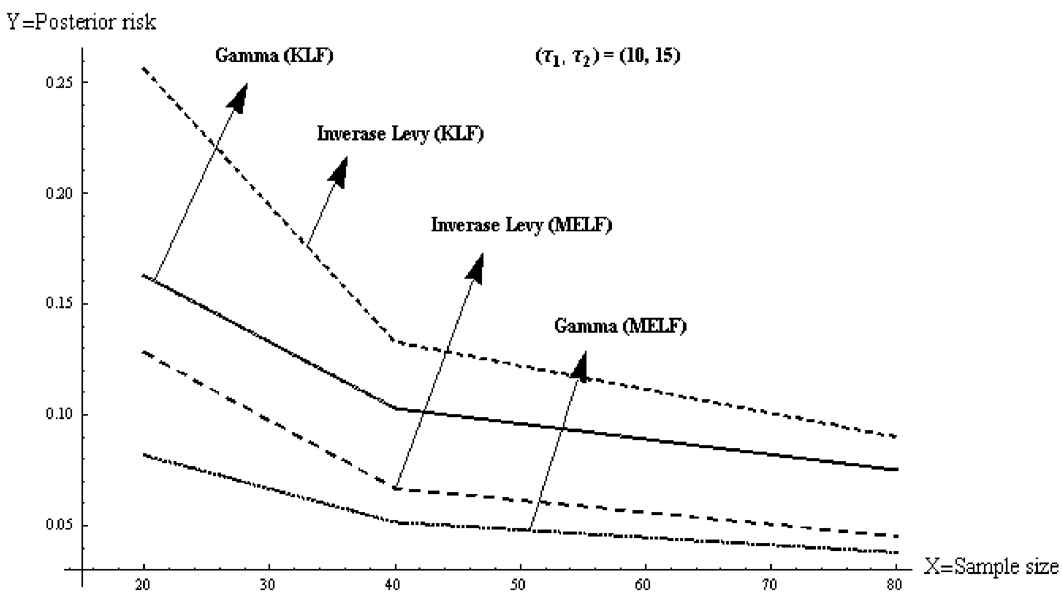

Figure 3. Posterior risks of τ_1 for (τ_1, τ_2, p_1) = (10, 15, 0.45)

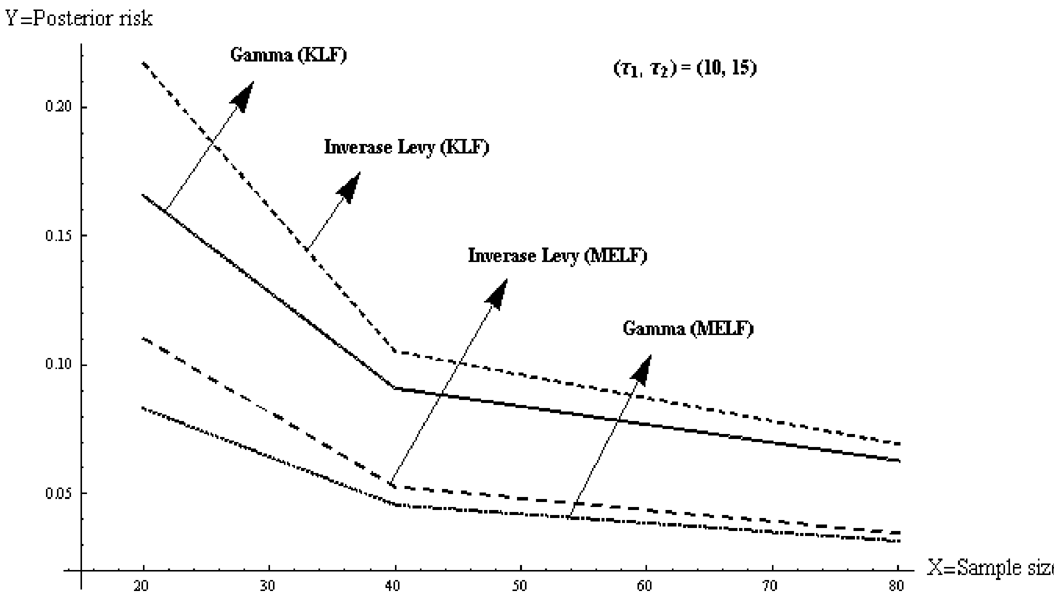

Figure 4. Posterior risks of τ_2 for (τ_1,τ_2, p_1) = (10, 15, 0.45)

Conclusion

The Bayesian inference of inverse Weibull mixture distribution based on doubly type-II censored data was considered. The prior belief of the model is represented by the independent gamma, beta priors and inverse Levy, beta priors on the scale, and mixing proportion parameters. Numerical results of the simulation study presented in Tables 1-8 exposed salient properties of the proposed Bayes estimators. The parameters of the mixture distributions have been over/under estimated in different cases. In general, the larger values of the parameters have been overestimated and smaller values of the parameters have been underestimated in the majority of cases. However, it is nice to observe that the estimated values converge to the true values and the amounts of the posterior risks tend to decrease by increasing the sample size.

This indicates that the proposed estimators are consistent. The smaller (larger) values of the parameter representing one component of the mixture impose a positive (negative) impact on the estimation of the parameter representing the other component of the mixture distribution. The larger values of the mixing parameter (p_1) impose a positive impact on the performance of the estimation of the first component of the mixture. This may be due to the fact that the lager values of the mixing parameter incorporate more values for the analysis of the first component.

Bayes estimators performed better under the minimum expected loss function than under the K-loss function under both priors. In addition, the performance of the estimates under gamma prior is better than those under inverse levy prior using both loss functions. However, in the case of gamma prior, the estimates under both loss functions are comparatively more underestimated, though this problem is less severe in the larger samples. Therefore, on the basis of the above discussion, we can recommend the use of the gamma prior under minimum expected loss function for the analysis of the inverse Weibull mixture distribution under the Bayesian framework.

However, when such a mixture model was used in real-life, the prior may be chosen as well as the loss function according to the need. In case of loss functions, if lower posterior risk is desired than in the present scenario, the minimum expected loss function should be given importance. If compromise on risk is affordable then one can easily select to use the K-loss function. Also, the informative gamma prior can easily be preferred over the other informative prior as shown by results. It may be mentioned here that, because of space restriction, only selected results are included and presented graphically. The findings of real life example are in accordance with the simulation study. The findings of the paper are useful for the analysts (from different fields) in dealing with the Bayesian analysis of the time to failure data when causes of the failure are more than one, and the data is doubly censored.

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Appendix

The Bayes estimators of τ_1, τ_2, and p_1 under KLF, assuming gamma prior are:

$$\hat{\tau}_{1(\mathrm{KLF})}=\left(\frac{\sum_{k_1=0}^{n-s}\sum_{k_2=0}^{k_1}(-1)^{k_1}\binom{n-s}{k_1}\binom{k_1}{k_2}Beta\left(m_1+k_1-k_2+c_1,m_2+k_2+d_1\right)\frac{\Gamma\left(a_1+m_1+1\right)\Gamma\left(a_2+m_2\right)}{\left\{b_1+\gamma_1\left(x_{1j}\right)\right\}^{\left(a_1+m_1+1\right)}\left\{b_2+\gamma_2\left(x_{2j}\right)\right\}^{\left(a_2+m_2\right)}}}{\sum_{k_1=0}^{n-s}\sum_{k_2=0}^{k_1}(-1)^{k_1}\binom{n-s}{k_1}\binom{k_1}{k_2}Beta\left(m_1+k_1-k_2+c_1,m_2+k_2+d_1\right)\frac{\Gamma\left(a_1+m_1-1\right)\Gamma\left(a_2+m_2\right)}{\left\{b_1+\gamma_1\left(x_{1j}\right)\right\}^{\left(a_1+m_1-1\right)}\left\{b_2+\gamma_2\left(x_{2j}\right)\right\}^{\left(a_2+m_2\right)}}}\right)^{\frac{1}{2}}$$

$$\hat{\tau}_{2(\mathrm{KLF})}=\left(\frac{\sum_{k_1=0}^{n-s}\sum_{k_2=0}^{k_1}(-1)^{k_1}\binom{n-s}{k_1}\binom{k_1}{k_2}Beta\left(m_1+k_1-k_2+c_1,m_2+k_2+d_1\right)\frac{\Gamma\left(a_1+m_1\right)\Gamma\left(a_2+m_2+1\right)}{\left\{b_1+\gamma_1\left(x_{1j}\right)\right\}^{\left(a_1+m_1\right)}\left\{b_2+\gamma_2\left(x_{2j}\right)\right\}^{\left(a_2+m_2+1\right)}}}{\sum_{k_1=0}^{n-s}\sum_{k_2=0}^{k_1}(-1)^{k_1}\binom{n-s}{k_1}\binom{k_1}{k_2}Beta\left(m_1+k_1-k_2+c_1,m_2+k_2+d_1\right)\frac{\Gamma\left(a_1+m_1\right)\Gamma\left(a_2+m_2-1\right)}{\left\{b_1+\gamma_1\left(x_{1j}\right)\right\}^{\left(a_1+m_1\right)}\left\{b_2+\gamma_2\left(x_{2j}\right)\right\}^{\left(a_2+m_2-1\right)}}}\right)^{\frac{1}{2}}$$

$$\hat{p}_{1(\mathrm{KLF})}=\left(\frac{\sum_{k_1=0}^{n-s}\sum_{k_2=0}^{k_1}(-1)^{k_1}\binom{n-s}{k_1}\binom{k_1}{k_2}Beta\left(m_1+k_1-k_2+c_1+1,m_2+k_2+d_1\right)\prod_{i=1}^{2}\frac{\Gamma\left(a_i+m_i\right)}{\left\{b_i+\gamma_i\left(x_{ij}\right)\right\}^{\left(a_i+m_i\right)}}}{\sum_{k_1=0}^{n-s}\sum_{k_2=0}^{k_1}(-1)^{k_1}\binom{n-s}{k_1}\binom{k_1}{k_2}Beta\left(m_1+k_1-k_2+c_1-1,m_2+k_2+d_1\right)\prod_{i=1}^{2}\frac{\Gamma\left(a_i+m_i\right)}{\left\{b_i+\gamma_i\left(x_{ij}\right)\right\}^{\left(a_i+m_i\right)}}}\right)^{\frac{1}{2}}$$

The posterior risks of τ_1, τ_2, and p_1 under KLF using gamma prior are:

$$\rho\left(\hat{\tau}_{1(\mathrm{KLF})}\right)=2\Omega_1^{-2}\sum_{k_1=0}^{n-s}\sum_{k_2=0}^{k_1}(-1)^{k_1}\binom{n-s}{k_1}\binom{k_1}{k_2}Beta\left(m_1+k_1-k_2+c_1,m_2+k_2+d_1\right)\frac{\Gamma\left(a_1+m_1+1\right)\Gamma\left(a_2+m_2\right)}{\left\{b_1+\gamma_1\left(x_{1j}\right)\right\}^{(a_1+m_1+1)}\left\{b_2+\gamma_2\left(x_{2j}\right)\right\}^{(a_2+m_2)}}$$
$$\times\sum_{k_1=0}^{n-s}\sum_{k_2=0}^{k_1}(-1)^{k_1}\binom{n-s}{k_1}\binom{k_1}{k_2}Beta\left(m_1+k_1-k_2+c_1,m_2+k_2+d_1\right)\frac{\Gamma\left(a_1+m_1-1\right)\Gamma\left(a_2+m_2\right)}{\left\{b_1+\gamma_1\left(x_{1j}\right)\right\}^{(a_1+m_1-1)}\left\{b_2+\gamma_2\left(x_{2j}\right)\right\}^{(a_2+m_2)}}-1$$

$$\rho\left(\hat{\tau}_{2(\mathrm{KLF})}\right)=2\Omega_1^{-2}\sum_{k_1=0}^{n-s}\sum_{k_2=0}^{k_1}(-1)^{k_1}\binom{n-s}{k_1}\binom{k_1}{k_2}Beta\left(m_1+k_1-k_2+c_1,m_2+k_2+d_1\right)\frac{\Gamma\left(a_1+m_1\right)\Gamma\left(a_2+m_2+1\right)}{\left\{b_1+\gamma_1\left(x_{1j}\right)\right\}^{(a_1+m_1)}\left\{b_2+\gamma_2\left(x_{2j}\right)\right\}^{(a_2+m_2+1)}}$$
$$\times\sum_{k_1=0}^{n-s}\sum_{k_2=0}^{k_1}(-1)^{k_1}\binom{n-s}{k_1}\binom{k_1}{k_2}Beta\left(m_1+k_1-k_2+c_1,m_2+k_2+d_1\right)\frac{\Gamma\left(a_1+m_1\right)\Gamma\left(a_2+m_2-1\right)}{\left\{b_1+\gamma_1\left(x_{1j}\right)\right\}^{(a_1+m_1)}\left\{b_2+\gamma_2\left(x_{2j}\right)\right\}^{(a_2+m_2-1)}}-1$$

$$\rho\left(\hat{\tau}_{2(\mathrm{KLF})}\right)=2\Omega_1^{-2}\sum_{k_1=0}^{n-s}\sum_{k_2=0}^{k_1}(-1)^{k_1}\binom{n-s}{k_1}\binom{k_1}{k_2}Beta\left(m_1+k_1-k_2+c_1,m_2+k_2+d_1\right)\prod_{i=1}^{2}\frac{\Gamma\left(a_i+m_i\right)}{\left\{b_i+\gamma_i\left(x_{ij}\right)\right\}^{(a_i+m_i)}}$$
$$\times\sum_{k_1=0}^{n-s}\sum_{k_2=0}^{k_1}(-1)^{k_1}\binom{n-s}{k_1}\binom{k_1}{k_2}Beta\left(m_1+k_1-k_2+c_1,m_2+k_2+d_1\right)\prod_{i=1}^{2}\frac{\Gamma\left(a_i+m_i\right)}{\left\{b_i+\gamma_i\left(x_{ij}\right)\right\}^{(a_i+m_i)}}-1$$

The Bayes estimators of τ_1, τ_2, and p_1 under MELF, assuming gamma prior are:

$$\hat{\tau}_{1(\mathrm{MELF})}=\left(\frac{\sum_{k_1=0}^{n-s}\sum_{k_2=0}^{k_1}(-1)^{k_1}\binom{n-s}{k_1}\binom{k_1}{k_2}Beta\left(m_1+k_1-k_2+c_1,m_2+k_2+d_1\right)\frac{\Gamma\left(a_1+m_1-1\right)\Gamma\left(a_2+m_2\right)}{\left\{b_1+\gamma_1\left(x_{1j}\right)\right\}^{\left(a_1+m_1-1\right)}\left\{b_2+\gamma_2\left(x_{2j}\right)\right\}^{\left(a_2+m_2\right)}}}{\sum_{k_1=0}^{n-s}\sum_{k_2=0}^{k_1}(-1)^{k_1}\binom{n-s}{k_1}\binom{k_1}{k_2}Beta\left(m_1+k_1-k_2+c_1,m_2+k_2+d_1\right)\frac{\Gamma\left(a_1+m_1-2\right)\Gamma\left(a_2+m_2\right)}{\left\{b_1+\gamma_1\left(x_{1j}\right)\right\}^{\left(a_1+m_1-2\right)}\left\{b_2+\gamma_2\left(x_{2j}\right)\right\}^{\left(a_2+m_2\right)}}}\right)^{\frac{1}{2}}$$

$$\hat{\tau}_{2(\mathrm{MELF})}=\left(\frac{\sum_{k_1=0}^{n-s}\sum_{k_2=0}^{k_1}(-1)^{k_1}\binom{n-s}{k_1}\binom{k_1}{k_2}Beta\left(m_1+k_1-k_2+c_1,m_2+k_2+d_1\right)\frac{\Gamma\left(a_1+m_1\right)\Gamma\left(a_2+m_2-1\right)}{\left\{b_1+\gamma_1\left(x_{1j}\right)\right\}^{\left(a_1+m_1\right)}\left\{b_2+\gamma_2\left(x_{2j}\right)\right\}^{\left(a_2+m_2-1\right)}}}{\sum_{k_1=0}^{n-s}\sum_{k_2=0}^{k_1}(-1)^{k_1}\binom{n-s}{k_1}\binom{k_1}{k_2}Beta\left(m_1+k_1-k_2+c_1,m_2+k_2+d_1\right)\frac{\Gamma\left(a_1+m_1\right)\Gamma\left(a_2+m_2-2\right)}{\left\{b_1+\gamma_1\left(x_{1j}\right)\right\}^{\left(a_1+m_1\right)}\left\{b_2+\gamma_2\left(x_{2j}\right)\right\}^{\left(a_2+m_2-2\right)}}}\right)^{\frac{1}{2}}$$

$$\hat{p}_{1(\mathrm{SELF})}=\left(\frac{\sum_{k_1=0}^{n-s}\sum_{k_2=0}^{k_1}(-1)^{k_1}\binom{n-s}{k_1}\binom{k_1}{k_2}Beta\left(m_1+k_1-k_2+c_1-1,m_2+k_2+d_1\right)\prod_{i=1}^{2}\frac{\Gamma\left(a_i+m_i\right)}{\left\{b_i+\gamma_i\left(x_{ij}\right)\right\}^{\left(a_i+m_i\right)}}}{\sum_{k_1=0}^{n-s}\sum_{k_2=0}^{k_1}(-1)^{k_1}\binom{n-s}{k_1}\binom{k_1}{k_2}Beta\left(m_1+k_1-k_2+c_1-2,m_2+k_2+d_1\right)\prod_{i=1}^{2}\frac{\Gamma\left(a_i+m_i\right)}{\left\{b_i+\gamma_i\left(x_{ij}\right)\right\}^{\left(a_i+m_i\right)}}}\right)^{\frac{1}{2}}$$

The posterior risks of τ_1, τ_2, and p_1 under KLF using gamma prior are:

$$\rho\left(\hat{\tau}_{1(\mathrm{SELF})}\right)=1-\Omega_1^{-1}\frac{\left\{\begin{array}{l}\sum_{k_1=0}^{n-s}\sum_{k_2=0}^{k_1}(-1)^{k_1}\binom{n-s}{k_1}\binom{k_1}{k_2}Beta\left(m_1+k_1-k_2+c_1,m_2+k_2+d_1\right)\\ \qquad\times\frac{\Gamma\left(a_1+m_1-1\right)\Gamma\left(a_2+m_2\right)}{\left\{b_1+\gamma_1\left(x_{1j}\right)\right\}^{\left(a_1+m_1-1\right)}\left\{b_2+\gamma_2\left(x_{2j}\right)\right\}^{\left(a_2+m_2\right)}}\end{array}\right\}^2}{\begin{array}{l}\sum_{k_1=0}^{n-s}\sum_{k_2=0}^{k_1}(-1)^{k_1}\binom{n-s}{k_1}\binom{k_1}{k_2}Beta\left(m_1+k_1-k_2+c_1,m_2+k_2+d_1\right)\\ \qquad\times\frac{\Gamma\left(a_1+m_1-1\right)\Gamma\left(a_2+m_2\right)}{\left\{b_1+\gamma_1\left(x_{1j}\right)\right\}^{\left(a_1+m_1-1\right)}\left\{b_2+\gamma_2\left(x_{2j}\right)\right\}^{\left(a_2+m_2\right)}}\end{array}}$$

$$\rho\left(\hat{\tau}_{2(\mathrm{SELF})}\right)=1-\Omega_1^{-1}\frac{\left\{\begin{array}{l}\sum_{k_1=0}^{n-s}\sum_{k_2=0}^{k_1}(-1)^{k_1}\binom{n-s}{k_1}\binom{k_1}{k_2}Beta\left(m_1+k_1-k_2+c_1,m_2+k_2+d_1\right)\\ \qquad\times\frac{\Gamma\left(a_1+m_1\right)\Gamma\left(a_2+m_2-1\right)}{\left\{b_1+\gamma_1\left(x_{1j}\right)\right\}^{\left(a_1+m_1\right)}\left\{b_2+\gamma_2\left(x_{2j}\right)\right\}^{\left(a_2+m_2-1\right)}}\end{array}\right\}^2}{\begin{array}{l}\sum_{k_1=0}^{n-s}\sum_{k_2=0}^{k_1}(-1)^{k_1}\binom{n-s}{k_1}\binom{k_1}{k_2}Beta\left(m_1+k_1-k_2+c_1,m_2+k_2+d_1\right)\\ \qquad\times\frac{\Gamma\left(a_1+m_1\right)\Gamma\left(a_2+m_2-2\right)}{\left\{b_1+\gamma_1\left(x_{1j}\right)\right\}^{\left(a_1+m_1\right)}\left\{b_2+\gamma_2\left(x_{2j}\right)\right\}^{\left(a_2+m_2-2\right)}}\end{array}}$$

$$\rho\left(\hat{p}_{1(\mathrm{SELF})}\right)=1-\Omega_1^{-1}\frac{\left\{\sum_{k_1=0}^{n-s}\sum_{k_2=0}^{k_1}(-1)^{k_1}\binom{n-s}{k_1}\binom{k_1}{k_2}\mathrm{B}\left(m_1+k_1-k_2+c_1-1,m_2+k_2+d_1\right)\times\prod_{i=1}^{2}\frac{\Gamma\left(a_i+m_i\right)}{\left\{b_i+\gamma_i\left(x_{ij}\right)\right\}^{\left(a_i+m_i\right)}}\right\}^2}{\sum_{k_1=0}^{n-s}\sum_{k_2=0}^{k_1}(-1)^{k_1}\binom{n-s}{k_1}\binom{k_1}{k_2}\mathrm{B}\left(m_1+k_1-k_2+c_1-2,m_2+k_2+d_1\right)\times\prod_{i=1}^{2}\frac{\Gamma\left(a_i+m_i\right)}{\left\{b_i+\gamma_i\left(x_{ij}\right)\right\}^{\left(a_i+m_i\right)}}}$$