8-1-1997

Infinite-Dimensional Hamilton-Jacobi-Bellman Equations in Gauss-Sobolev Spaces

Pao-Liu Chow
University of Maryland, College Park, plchow@math.wayne.edu

Jose-Luis Menaldi
Wayne State University, menaldi@wayne.edu

Nonlinear Analysis: Theory, Methods and Applications, 29 (1997), 415-426

INFINITE-DIMENSIONAL HAMILTON-JACOBI-BELLMAN EQUATIONS IN GAUSS-SOBOLEV SPACES*

Pao-Liu Chow† and Jose-Luis Menaldi‡
Department of Mathematics, Wayne State University
Detroit, Michigan 48202, USA

Abstract

We consider the strong solution of a semi linear HJB equation associated with a stochastic optimal control in a Hilbert space H. By strong solution we mean a solution in a $L^2(\mu, H)$-Sobolev space setting. Within this framework, the present problem can be treated in a similar fashion to that of a finite-dimensional case. Of independent interest, a related linear problem with unbounded coefficient is studied and an application to the stochastic control of a reaction-diffusion equation will be given.

Key words and phrases: Dynamic programming, Gaussian and invariant measures, coercivity, monotone operator.

*This work was supported in part by the NSF grant DMS-91-03160.

†e-mail: plchow@cms.cc.wayne.edu

‡e-mail: jlm@math.wayne.edu

1 Introduction

Consider the optimal control problem with the state equation in a Hilbert space H:

$$\begin{cases} du_t &= \{Au_t + B(u_t, \alpha_t)\}dt + dW_t, \\ u_0 &= v \in H, \end{cases} \tag{1.1}$$

where A is an unbounded linear operator, $B(\cdot,\cdot)$ is a, generally, nonlinear operator depending on the control $\alpha_t \in K$ and W_t is a H-valued Wiener process with covariance operator R. We are interested in finding, from the set $\mathcal{K}$ of admissible controls $\alpha.$ the optimal control $\alpha^*_\cdot$ that minimizes the cost function:

$$J_v(\alpha.) = E\int_0^\infty e^{-\Lambda_t} F(u_t, \alpha_t)dt, \tag{1.2}$$

where $F : H \times K \to \mathbb{R}^+$ is the running cost function with

$$\Lambda_t = \int_0^t \lambda(u_s)ds, \tag{1.3}$$

and λ is the discount rate function. Denote the optimal cost or the value function by Φ defined by

$$\Phi(v) = \inf_{\alpha.\in\mathcal{K}} J_v(\alpha.) = J_v(\alpha^*_\cdot). \tag{1.4}$$

Then, by formally applying the dynamic programming principle, we deduce that Φ satisfies the (stationary) Hamilton-Jacobi-Bellman (HJB) equation:

$$\frac{1}{2}Tr.[RD^2\Phi(v)] + (Av, D\Phi(v)) - \lambda(v)\Phi(v) + \mathcal{B}(\Phi)(v) = 0, \tag{1.5}$$

where $D\Phi$ and $D^2\Phi$ denote the first two Fréchet derivatives, Tr. means the trace, $(\cdot,\cdot)$ is the inner produce in H and

$$\mathcal{B}(\Phi) = \inf_{\alpha\in K}\{(B(\cdot,\alpha), D\Phi) + F(\cdot,\alpha)\}. \tag{1.6}$$

Even at the formal level, the equation (1.5) makes sense only when v belongs to the domain $\mathcal{D}(A)$ of an unbounded operator A. But, with respect to the Wiener measure, the set $\mathcal{D}(A)$ may be negligible. However, as shown in the linear case [1], it is possible to define the equation in H almost everywhere with respect to the invariant measure μ for Eq. (1.1) with $B \equiv 0$.

The paper is mainly concerned with the strong solution of the HJB equation (1.5), interpreted properly, in an $L^2(\mu, H)$-Sobolev space setting. Within this framework, the present problem can be treated in a similar fashion to that of a finite-dimensional case. Of independent interest, a related linear problem with unbounded coefficient is studied and an application to the stochastic control of a reaction-diffusion equation will be given.

This work was inspired by an interesting paper [2] of DaPrato, who studied a special form of Eq. (1.4). In contrast with the L^2-theory, he considered a mild solution in a certain Banach space of continuously differentiable functions with sup-norm. Since then several papers have been written by him and his associates on this subject (see, e.g. [3], [4] and [5]). When A is bounded and $\mathcal{B}(\Phi) = \frac{1}{2}(D\Phi, D\Phi)$ in (1.5), this special case was treated by Havarneanu [6] in an Abstract Wiener space setting. However his approach cannot be applied to the general case (1.5). Along an entirely different direction, full nonlinear HJB equations were studied by P.L. Lions [7] in the sense of viscosity solutions.

2 Preliminaries

Let H be a real separable Hilbert space with inner product $(\cdot, \cdot)$ and norm $|\cdot|$. Let $V \subset H$ be a reflexive Banach space with norm $\|\cdot\|$. Denote the dual space of V by V' and the duality pairing by $\langle \cdot, \cdot \rangle$. Assume that the inclusions: $V \subset H \subset V'$ are dense and continuous.

Let $A : V \to V'$ be a continuous linear operator and let W_t be a H-valued Wiener process with covariance operator R. Consider the linear stochastic equation in V':

$$\begin{cases} du_t &= Au_t dt + dW_t, \\ u_0 &= h \in H. \end{cases} \tag{2.1}$$

We suppose that the following conditions hold:

(C.1) Let $A : V \to V'$ be a self-adjoint operator whose normalized eigenfunctions $e_k' s \in V$ and corresponding eigenvalues $\mu_k' s$ are strictly negative with $0 > \mu_1 > \mu_2 > \cdots > \mu_k$ and $\mu_k \to -\infty$ as $k \to \infty$.

(C.2) The resolvent operator $R_\gamma(A)$ of A commutes with covariance operator R.

(C.3) $R : H \to H$ is bounded such that $Tr.A^{-1}R < \infty$.

Then, by a direct computation or by applying a general theorem of invariant measures [8,9], we can claim that

Lemma 2.1 *Under conditions (C.1), (C.2) and (C.3), the stochastic equation (2.1) has a unique invariant measure μ on H, which is a centered Gaussian measure supported in V with covariance operator $\Gamma = -\frac{1}{2}A^{-1}R$.* □

Let $\mathcal{H} = L^2(\mu, H)$ with norm $|||\cdot|||$ defined by

$$|||\Phi||| = \{\int |\Phi(v)|^2 \mu(dv)\}^{1/2}, \tag{2.2}$$

and inner product $[\cdot,\cdot]$ given by

$$[\Phi, \Psi] = \int \Phi(v)\Psi(v)\mu(dv) \quad \text{for} \quad \Phi, \Psi \in \mathcal{H}, \tag{2.3}$$

where the integration is over H (or V).

Let $n = (n_1, n_2, \ldots, n_k, \ldots)$, where n_k is a nonnegative integer and $n_k = 0$ except for a finite number of k's. For $v \in H$, define the Hermite (polynomial) functional of degree n by

$$H_n(v) = \Pi_{k=1}^{\infty} h_{n_k}[\ell_k(v)], \tag{2.4}$$

where $h_j(x)$ is the standard one-dimensional Hermite polynomial of degree j and $\ell_k(v) = (v, \Gamma^{-\frac{1}{2}} e_k)$. For a smooth functional Φ, let $D\Phi$ and $D^2\Phi$ denote the Fréchet derivatives of first and second orders, respectively. Introduce the differential operator

$$\mathcal{A}\Phi(v) = \frac{1}{2} Tr.[RD^2\Phi(v)] + \langle Av, D\Phi(v)\rangle, \tag{2.5}$$

which is defined for a polynomial functional Φ with $D\Phi(v)$ lies in the domain $\mathcal{D}(A)$. It can be shown that [1]

Lemma 2.2 *The set of all Hermite functionals $\{H_n\}$ formed a complete orthonormal system (CONS) in $\mathcal{H}$. Furthermore we have*

$$\mathcal{A}H_n(v) = -\Lambda_n H_n(v), \tag{2.6}$$

where $\Lambda_n = \sum_k n_k \mu_k$ and the summation is over the finite number of nonzero n_k's. □

Definition 2.3 *Let $\mathcal{H}_k$ be the Gauss-Sobolev space of order k defined by*

$$\mathcal{H}_k = \{\Phi \in \mathcal{H} : \|\Phi\|_k < \infty\} \text{ for } k > 0,$$

and $\mathcal{H}_0 = \mathcal{H}$, where

$$\|\Phi\|_k = \|(I - \mathcal{A})^{k/2}\Phi\| = \{\sum_n (1 + \Lambda_n)^k |\Phi_n|^2\}^{1/2}, \tag{2.7}$$

with I being the identity operator and $\Phi_n = [\Phi, H_n]$. Let $\mathcal{H}_{-k}$ denote the dual space of $\mathcal{H}_k$, and the duality between $\mathcal{H}_k$ and $\mathcal{H}_{-k}$ will be denoted by $\langle\langle \cdot, \cdot \rangle\rangle$. □

Clearly, by identifying $\mathcal{H}$ with its dual $\mathcal{H}'$, we have

$$\mathcal{H}_k \subset \mathcal{H} \subset \mathcal{H}_{-k}, k > 0$$

and the inclusions are dense and continuous.

Similar to the Laplacian operator in $\mathbb{R}^d$, the following properties of $\mathcal{A}$ are crucial in the subsequent analysis.

Lemma 2.4 *The operator $\mathcal{A}$ can be defined as a self-adjoint linear operator in $\mathcal{H}$ with domain $\mathcal{D}(\mathcal{A}) \supset \mathcal{H}_2$. Moreover the following integral identity holds:*

$$\int (\mathcal{A}\Phi)\Psi d\mu = -\frac{1}{2} \int (RD\Phi, D\Psi) d\mu, \tag{2.8}$$

for $\Phi, \Psi \in \mathcal{H}_2$. The above identity can be extended to yield a linear operator $A : \mathcal{H}_1 \to \mathcal{H}_{-1}$ defined by

$$\langle\langle \mathcal{A}\Phi, \Psi \rangle\rangle = -\frac{1}{2}[RD\Phi, D\Psi], \quad \forall \Phi, \Psi \in \mathcal{H}_1. \Box \tag{2.9}$$

We see that, with respect to the invariant measure μ, $\mathcal{A}$ can be defined μ - a.e. in H and it behaves like the Laplace operator in $\mathbb{R}^d$.

3 Linear Equation with Unbounded Coefficient

Suppose that $B(v, \alpha) = B(v)$ in Eq. (1.5). Consider an associated linear elliptic problem of the form:

$$\{\mathcal{A} - \lambda(v)I\}\Phi + \mathcal{B}_0\Phi = F(v), \quad \mu - \text{a.e. } v \in H, \tag{3.1}$$

where $\mathcal{A} : \mathcal{H}^2 \to \mathcal{H}$ is defined as in Lemma 2.4,

$$\mathcal{B}_0\Phi(v) = (B(v), D\Phi(v))$$

and λ, F are given functions to be specified. If the coefficient $B(v)$ is bounded and λ is a constant, given $F \in \mathcal{H}_{-1}$, it was proved in [1] that there exists a positive constant λ_0 such that the problem (3.1) has a unique strong solution $\Phi \in \mathcal{H}_1$ for any $\lambda > \lambda_0$. Now we deal with the case of unbounded B satisfying the following growth condition:

(A.1) $B(\cdot) : V \to H_0 = \overline{R^{1/2}(H)}$ is continuous such that

$$|B(v)|_0 = |R^{-1/2}B(v)| \leq b_0(1 + \|v\|^2)^{m/2}, \ \ \forall v \in V,$$

for some $b_0 > 0$ and $m \geq 2$.

For reason which will become clear later, we assume that

(A.2) $\lambda(\cdot) : V \to \mathbb{R}^+$ satisfies the growth condition:

$$\lambda_0(1 + \|v\|^2)^m \leq \lambda(v) \leq \lambda_1(1 + \|v\|^2)^m, \ \ \forall v \in V,$$

for some positive constants $\lambda_0 \leq \lambda_1$.

To control the unbounded coefficient, we need to introduce the space $\mathcal{H}_{0,m}$ defined as follows.

Definition 3.1 *Let $\mathcal{H}_{0,m}$ be a Hilbert subspace of $\mathcal{H}$ defined by*

$$\mathcal{H}_{0,m} = \{\Phi \in \mathcal{H} : \|\Phi\|_{0,m} < \infty\}, m > 0,$$

where the norm is given by

$$\|\Phi\|_{0,m} \ = \ \{\int \Phi^2(v)\rho_m(v)\mu(dv)\}^{1/2} = \|\rho_m^{1/2}\Phi\|,$$

and

$$\rho_m(v) = (1 + \|v\|^2)^m. \square$$

Under conditions (A.1) and (A.2), new function spaces $\mathcal{H}_{k,m}, k = 1, 2, \ldots$ and $m > 0$, need to be introduced.

Definition 3.2 *For $k = 1, 2, \dots$ and $m > 0$, define*

$$\mathcal{H}_{k,m} = \mathcal{H}_k \cap \mathcal{H}_{0,m},$$

with norm $\|\cdot\|_{k,m} = \{|\!|\!|\cdot|\!|\!|_k^2 + \|\cdot\|_{0,m}^2\}^{1/2}$, where $|\!|\!|\cdot|\!|\!|_k$ is the k-th order Gauss-Sobolev norm in Def. 2.3. By convention, we set $\mathcal{H}_{k,0} = \mathcal{H}_k$ and $\mathcal{H}_{0,0} = \mathcal{H}$. □

Clearly, by identifying $\mathcal{H}$ with its dual $\mathcal{H}'$, we have the following inclusions:

$$\mathcal{H}_{k,m} \subset \mathcal{H}_{0,m} \subset \mathcal{H} \cong \mathcal{H}' \subset \mathcal{H}'_{0,m} \subset \mathcal{H}'_{k,m},$$

and $\mathcal{H}_{k',m'} \subset \mathcal{H}_{k,m}$ if $k' \geq k$ and $m' \geq m$, where the inclusions are dense and continuous and the duality pairing between $\mathcal{H}_{k,m}$ and $\mathcal{H}'_{k,m}$ will be denoted by $\langle \cdot, \cdot \rangle_{k,m}$. Note that, in view of the formula (2.8),

$$\begin{aligned} \|\Phi\|_{1,m}^2 &= \frac{1}{2}\int (RD\Phi, D\Phi) d\mu + \int \Phi^2 \rho_m d\mu \\ &= \int \{\frac{1}{2}|R^{1/2} D\Phi|^2 + |\rho_m^{1/2}\Phi|^2\} d\mu \end{aligned} \tag{3.2}$$

Lemma 3.3 *Let $\mathcal{L}_\lambda = (\mathcal{A} - \lambda I) + \mathcal{B}_0$. Then, under conditions (A.1) and (A.2), the linear operator $\mathcal{L}_\lambda : \mathcal{H}_{1,m} \to \mathcal{H}'_{1,m}$ is well defined and bounded such that*

$$|\langle \mathcal{L}_\lambda \Phi, \Psi)\rangle_{1,m}| \leq C \|\Phi\|_{1,m} \|\Psi\|_{1,m}, \quad \forall \Phi, \Psi \in \mathcal{H}_{1,m}, \quad \textit{for some} \;\; C > 0. \tag{3.3}$$

(Proof.) For $\Phi, \Psi \in \mathcal{H}_{1,m}$, let β_λ be a bilinear form on $\mathcal{H}_{1,m} \times \mathcal{H}_{1,m}$ defined by

$$\beta_\lambda(\Phi, \Psi) = \int \{\frac{1}{2}(RD\Phi, D\Psi) + (\lambda\Phi, \Psi) - (\mathcal{B}_0 \Phi, \Psi)\} d\mu \tag{3.4}$$

which defines uniquely a linear operator $\mathcal{L}_\lambda : \mathcal{H}_{1,m} \to \mathcal{H}'_{1,m}$ by setting

$$\langle \mathcal{L}_\lambda \Phi, \Psi \rangle_{1,m} = -\beta_\lambda(\Phi, \Psi). \tag{3.5}$$

To show the boundedness of $\mathcal{L}_\lambda$, it suffices to prove that the inequality (3.3) holds. By conditions (A.1) and (A.2), this follows easily from the estimate

$$\begin{aligned} |\beta_\lambda(\Phi, \Psi)| \;\; &\leq \;\; \int \{\frac{1}{2}|R^{1/2} D\Phi||(R^{1/2} D\Psi| + \lambda_1 |\rho_m^{1/2}\Phi||\rho_m^{1/2}\Psi| + \\ &\qquad + b_0 |R^{1/2} D\Phi||\rho_m^{1/2}\Psi|\} d\mu \leq \\ &\leq \;\; C||\Phi\|_{1,m} \|\Psi\|_{1,m}, \;\; \text{for some} \;\; C > 0, \end{aligned}$$

by applying the Cauchy-Schwarz inequality and noting (3.2). □

Next we introduce the notion of a strong solution.

Definition 3.4 *Given $F \in \mathcal{H}'_{1,m}$, a function Φ on H is said to be a* <u>*strong solution*</u> *of Eq. (3.1) if $\Phi \in \mathcal{H}_{1,m}$ satisfies the following equation*

$$\langle \mathcal{L}_\lambda \Phi, \Psi \rangle_{1,m} = \langle F, \Psi \rangle_{1,m}, \quad \forall \Psi \in \mathcal{H}_{1,m}. \square \tag{3.6}$$

Now we are ready to state and prove the following existence theorem.

Theorem 3.5 *Let the conditions (A.1) and (A.2) hold. Then for given $F \in \mathcal{H}'_{1,m}$, the elliptic problem has a unique strong solution, provided that $\lambda_0 > b_0^2/2$.*

(Proof). The key to the existence proof is to establish the coercivity property of $(-\mathcal{L}_\lambda)$:

$$\exists \delta > 0 \ni \quad \langle -\mathcal{L}_\lambda \Phi, \Phi \rangle_{1,m} = \beta_\lambda(\Phi, \Phi) \geq \delta \|\Phi\|^2_{1,m}, \quad \forall \Phi \in \mathcal{H}_{1,m}. \tag{3.7}$$

To this end, we note, by (3.4) and conditions (A.1) and (A.2), that

$$\begin{aligned} \beta_\lambda(\Phi, \Phi) &= \int \{ \frac{1}{2}(RD\Phi, D\Phi) + (\lambda\Phi, \Phi) - (\mathcal{B}_0 \Phi, \Phi) \} d\mu \\ &\geq \int \{ \frac{1}{2} |R^{1/2} D\Phi|^2 + \lambda_0 |\rho_m^{1/2} \Phi|^2 - b_0 |R^{1/2} D\Phi| |\rho_m^{1/2} \Phi| \} d\mu \\ &\geq \int \{ \frac{1}{2}(1-\varepsilon) |R^{1/2} D\Phi|^2 + (\lambda_0 - \frac{b_0^2}{2\varepsilon}) |\rho_m^{1/2} \Phi|^2 \} d\mu, \end{aligned}$$

for any $\varepsilon > 0$. Therefore, by choosing $\varepsilon < 1$ so that $\lambda_0 > b_0^2/2\varepsilon$, the inequality (3.7) holds with $\delta = \min\{\frac{1}{2}(1-\varepsilon), (\lambda_0 - b_0^2/2\varepsilon)\}$.

By Lemma 3.3, β_λ is a bounded bilinear form on $\mathcal{H}_{1,m} \times \mathcal{H}_{1,m}$, where $\mathcal{H}_{1,m} \subset \mathcal{H}$ is a Hilbert space. If follows immediately from the Lax-Milgram theorem [10] that the equation (3.36) has a unique solution Φ, which, by Definition 3.3, is the desired strong solution. $\square$

4 Hamilton-Jacobi-Bellman Equations

Now we consider the nonlinear elliptic problem arising from the controlled stochastic PDE (1.5). Recall that K denotes an admissible set and the nonlinear operator $\mathcal{B}$ is defined as

$$\mathcal{B}(\Phi)(v) = \inf_{\alpha \in K} \{ (B(v, \alpha), D\Phi(v)) + F(v, \alpha) \}, \tag{4.1}$$

on which we impose the following conditions:

(B.1) $B(\cdot,\cdot) : V \times K \to H_0$ satisfies the condition:

$$|B(v,\alpha)|_0 = |R^{-1/2}B(v,\alpha)| \le b_0(1+\|v\|^2)^{m/2}, \text{ for } m \ge 2, b_0 > 0, \forall \alpha \in K.$$

(B.2) Same as (A.2), let $\lambda(\cdot) = V \to \mathbb{R}^+$ be bounded so that

$$\lambda_0(1+\|v\|^2)^m \le \lambda(v) \le \lambda_1(1+\|v\|^2)^m, \ \forall \alpha \in K,$$

for some positive constants $\lambda_0 \le \lambda_1$.

(B.3) There exists a constant $f_0 > 0$ such that $F(\cdot,\cdot) : V \times K \to \mathbb{R}^+$ has the following bound:

$$|F(v,\alpha)| \le f_0(1+\|v\|^2)^{m/2}, \ \forall v \in V, \alpha \in K.$$

Let $\mathcal{M}_\lambda$ be defined by

$$\mathcal{M}_\lambda(\Phi) = -(\mathcal{A} - \lambda I)\Phi - \mathcal{B}(\Phi). \tag{4.2}$$

Then the Hamilton-Jacobi-Bellman equation (1.5) can be written as

$$\mathcal{M}_\lambda(\Phi) = 0. \tag{4.3}$$

Before presenting an existence theorem, we will prove two technical lemmas.

Lemma 4.1 *Under conditions (B.1) – (B.3), the nonlinear operator $\mathcal{M}_\lambda : \mathcal{H}_{1,m} \to \mathcal{H}'_{1,m}$ is locally bounded and Lipschitz continuous.*

(Proof). For $\Phi, \Psi \in \mathcal{H}_{1,n}$, we have

$$-\langle \mathcal{M}_\lambda(\Phi), \Psi \rangle_{1,m} = \langle\langle (\mathcal{A} - \lambda I)\Phi, \Psi \rangle\rangle + [\mathcal{B}(\Phi), \Psi], \tag{4.4}$$

Clearly, by (2.8) and (B.2),

$$\begin{aligned} |\langle\langle (\mathcal{A} - \lambda I)\Phi, \Psi \rangle\rangle| &\le \frac{1}{2}|[RD\Phi, D\Psi]] + [\lambda\Phi, \Psi]| \\ &\le \|\Phi\|_1 \|\Psi\|_1 + \lambda_1 \|\Phi\|_{0,m} \|\Psi\|_{0,m} \\ &\le (1+\lambda_1)\|\Phi\|_{1,m}\|\Psi\|_{1,m}. \end{aligned} \tag{4.5}$$

By (4.1) and the assumptions,

$$\begin{aligned} |[\mathcal{B}(\Phi), \Psi]| &\leq \int \{|(B(v,\alpha), D\Phi(v))||\Psi(v)| + \\ &\qquad + |F(v,\alpha)||\Psi(v)|\}\mu(dv) \\ &\leq \sqrt{2} b_0 \|\Phi\|_{1,0} \|\Psi\|_{0,m} + f_0 \|\Psi\|_{0,m}. \end{aligned} \tag{4.6}$$

In view of (4.4), (4.5) and (4.6), there exists $b_1 > 0$ such that

$$|\langle \mathcal{M}_\lambda(\Phi), \Psi \rangle_{1,m}| \leq b_1(1 + \|\Phi\|_{1,m}) \|\Psi\|_{1,m},$$

for some $b_1 > 0$, or $\mathcal{M}_\lambda$ is locally bounded.

To show the Lipschitz condition, it suffices to deal with the nonlinear operator $\mathcal{B}$. Let Φ, Φ' and $\Psi \in \mathcal{H}_{1,m}$. Then, by noting conditions (B.1) and (B.2),

$$\begin{aligned} &|\langle \mathcal{B}(\Phi) - \mathcal{B}(\Phi'), \Psi \rangle_{1,m}| = \\ &\quad = |[\mathcal{B}(\Phi) - \mathcal{B}(\Phi'), \Psi]| \\ &\quad \leq \int | \inf_{\alpha \in K} \{(B(v,\alpha), D\Phi(v)) + F(v,\alpha)\} \\ &\qquad\qquad - \inf_{\alpha \in K} \{(B(v,\alpha), D\Phi'(v)) + F(v,\alpha)\}||\Psi(v)|\mu(dv) \\ &\quad \leq \int \sup_{\alpha \in K} \{|(B(v,\alpha), D\Phi(v) - D\Phi'(v))|\}|\Psi(v)|\mu(dv) \\ &\quad \leq b_0 \int |R^{1/2} D(\Phi - \Phi')||\rho_m^{1/2} \Psi| d\mu \\ &\quad \leq \sqrt{2} b_0 \|\Phi - \Phi'\|_{1,0} \|\Psi\|_{0,m}, \end{aligned} \tag{4.7}$$

which shows the desired continuity. □

Lemma 4.2 *Let conditions (B.1) – (B.3) hold. Then, if* $\lambda_0 > b_0^2/2$*, the operator* $\mathcal{M}_\lambda(\cdot) : \mathcal{H}_{1,m} \to \mathcal{H}'_{1,m}$ *is monotone, or there exists* $\delta > 0$ *such that*

$$\langle \mathcal{M}_\lambda(\Phi) - \mathcal{M}_\lambda(\Psi), \Phi - \Psi \rangle_{1,m} \geq \delta \|\Phi - \Psi\|^2_{1,m}, \quad \forall \Phi, \Psi \in \mathcal{H}_{1,}. \tag{4.8}$$

(Proof). By (4.2), we have

$$\begin{aligned} &\langle \mathcal{M}_\lambda(\Phi) - \mathcal{M}_\lambda(\Psi), \Phi - \Psi \rangle_{1,m} \\ &\quad = -\langle (\mathcal{A} - \lambda I)(\Phi - \Psi), \Phi - \Psi \rangle_{1,m} - \langle \mathcal{B}(\Phi) - \mathcal{B}(\Psi), \Phi - \Psi \rangle_{1,m} \end{aligned}$$

$$\geq \frac{1}{2}\int |R^{1/2}D(\Phi-\Psi)|^2 d\mu + \int \lambda(\Phi-\Psi)^2 d\nu \qquad (4.9)$$
$$-|[\mathcal{B}(\Phi)-\mathcal{B}(\Psi),\Phi-\Psi]|$$
$$\geq \frac{1}{2}\int |R^{1/2}D(\Phi-\Psi)|^2 d\mu + \lambda_0||\Phi-\Psi||^2_{0,m} - |[\mathcal{B}(\Phi)-\mathcal{B}(\Psi),\Phi-\Psi]|$$

Similar to (4.7), we get

$$|[\mathcal{B}(\Phi)-\mathcal{B}(\Psi),\Phi-\Psi]| \leq$$
$$\leq b_0\int |R^{1/2}D(\Phi-\Psi)||\rho_m^{1/2}(\Phi-\Psi)|d\mu \qquad (4.10)$$
$$\leq \frac{1}{2}\{\varepsilon\int |R^{1/2}D(\Phi-\Psi)|^2 d\mu + \frac{b_0^2}{\varepsilon}|\Phi-\Psi|^2\rho_m d\mu\}.$$

By invoking (4.10), the inequality (4.9) yields, for any $\varepsilon > 0$,

$$\langle \mathcal{M}_\lambda(\Phi)-\mathcal{M}_\lambda(\Psi),\Phi-\Psi\rangle_{1,m}$$
$$\geq \frac{1}{2}(1-\varepsilon)\int |R^{1/2}D(\Phi-\Psi)|^2 d\mu + (\lambda_0 - \frac{b_0^2}{2\varepsilon})\int |\Phi-\Psi|^2\rho_m d\mu$$

which gives rise to the desired inequality (4.8) for $\lambda_0 > b_0^2/2$, if we choose $\varepsilon < 1$, but sufficiently close to 1.□

Similar to the linear problem, $\Phi \in \mathcal{H}_{1,m}$ is said to be a strong solution of Eq. (4.3) if the following holds:

$$\langle \mathcal{M}_\lambda(\Phi),\Psi\rangle_{1,m} = 0, \quad \forall\Psi \in \mathcal{H}_{1,m}. \qquad (4.11)$$

With the aid of the above lemmas, the existence theorem can be proved easily.

Theorem 4.3 *Let the conditions (B.1), (B.2) and (B.3) hold. Then, if $\lambda_0 > b_0^2/2$, the Hamilton-Jacobi-Bellman equation (4.3) has a unique strong solution Φ and, in fact, $\Phi \in \mathcal{H}_{2,m}$.*

(Proof.) By Lemma 4.1 and Lemma 4.2, we know that $\mathcal{M}_\lambda : \mathcal{H}_{1,m} \to \mathcal{H}'_{1,m}$ is a locally bounded, Lipschitz continuous and monotone operator on a Hilbert space. Note that, by (4.1) and condition (B.3),

$$|\mathcal{M}_\lambda(0)\Phi|_{1,m} \leq f_0\|\Phi\|_{0,m}. \qquad (4.12)$$

It follows from (4.8) and (4.12) that

$$\begin{aligned} \langle \mathcal{M}_\lambda(\Phi), \Phi\rangle_{1,m}/||\Phi||_{1,m} &\geq \{\delta\|\Phi\|_{1,m}^2 - f_0\|\Phi\|_{0,m}\}/|\Phi\|_{1,m} \\ &\to \infty \text{ as } \|\Phi\|_{1,m} \to \infty. \end{aligned}$$

Therefore, by applying a theorem for monotone operator in Lions (p. 171, [11]), the equation (4.3) has a unique strong solution $\Phi \in \mathcal{H}_{1,m}$ satisfying Eq. (4.11). Now, from Eq. (4.11) and estimate (4.6) we have

$$\begin{aligned} |\langle\!\langle(\mathcal{A}-\lambda I)\Phi, \Psi\rangle\!\rangle| &= |[\mathcal{B}(\Phi), \Phi)]| \leq \\ &\leq b_1(1+\|\Phi\|_{1,m})\|\Psi\|_{0,m}, \end{aligned}$$

so that $(\mathcal{A}-\lambda I)\Phi \in \mathcal{H}_{0,m}$ hence $\Phi \in \mathcal{H}_{2,m}$ as claimed. □

Remark 4.4 *Instead of (B.2), the rate function λ may be allowed to depend on the control α so that*

$$\lambda_0(1+\|v\|^2)^m \leq \lambda(v,\alpha) \leq \lambda_1(1+\|v\|^2)^m, \quad \forall v \in V.$$

The same results in Thm. 4.3 hold true. □

Remark 4.5 *A similar approach can be adopted to prove the existence of strong solutions to the corresponding time-dependent HJB equations.* □

5 Example

Consider the stochastic control of the reaction-diffusion equation in one space-dimension:

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \dfrac{\partial^2 u(t,x)}{\partial x^2} + b(u,u_x,\alpha)(t,x) + \dot{W}(t,x), \quad t>0,\ 0<x<1, \\ u(0,x) = v(x), \\ u(t,0) = u(t,1) = 0, \end{cases} \tag{5.1}$$

where $u_x = \frac{\partial u}{\partial x}$, $b(u,u_x,\alpha)(t,x) = b[u(t,x), u_x(t,x), \alpha(t,x)]$, $\alpha(t,x)$ is the control, $\dot{W}(t,x) = \frac{\partial}{\partial t}W(t,x)$ with $W(t,\cdot)$ being a Wiener process in $L^2(0,1)$, and $v \in L^2(0,1)$. Let $r(x,y)$ denote the covariance function, the kernel of the covariance operator R. Let $H = L^2(0,1), V = H_0^1(0,1)$: the first-order Sobolev space $H^1(0,1)$ of functions on $(0,1)$ vanishing at $x = 0,1$, and $A = \frac{\partial^2}{\partial x^2} : V \to V' = H^{-1}(0,1)$. The normalized eigenfunctions e_k of A and the corresponding eigenvalues μ_k are given by

$$e_k(x) = \sqrt{2}\sin k\pi x \ \text{ and } \ \mu_k = -(k\pi)^2, k = 1, 2, \ldots, \tag{5.2}$$

With respect to the basis $\{e_k\}$, the following representation holds:

$$W(t,\cdot) = \sum_{k=1}^{\infty} \sqrt{\gamma_k} b_t^k e_k, \quad \text{a.s.},$$

where b_t^k's are i.i.d. Brownian motions in $\mathbb{R}^1$, and γ_k's are the eigenvalues of R so that

$$(Re_k)(x) = \int_0^1 r(x,y)e_k(y)dy = \gamma_k e_k(x), \quad k = 1, 2, \ldots.$$

or

$$r(x,y) = \sum_{k=1}^{\infty} \gamma_k e_k(x) e_k(y) \tag{5.3}$$

in an L^2-sense. Suppose that

$$\sum_{k=1}^{\infty} \gamma_k / k^2 < \infty \tag{5.4}$$

which implies that

$$Tr.\{(-A)^{-1}R\} = -\sum_{k=1}^{\infty} (A^{-1}Re_k, e_k) = \frac{1}{\pi^2} \sum_{k=1}^{\infty} \gamma_k / k^2 < \infty.$$

In view of (5.2), (5.3) and (5.4), the conditions (C.1), (C.2) and (C.3) are met. In the value function J_v, for simplicity, we assume that both F and λ are of quadratic form:

$$\lambda(v) = \lambda_0\{1 + \|v\|^2\} \tag{5.5}$$

and

$$F(v,\alpha) = f_0\{|\alpha|^2 + \|v\|^2\}^{1/2}, \tag{5.6}$$

where λ_0 and f_0 are positive constants and $\alpha \in K$ with K being a compact subset of H. Then the conditions (B.2) and (B.3) are trivially satisfied with $m = 1$. To apply the existence Thm. 4.3 to the associated HJB equation, we need to check the condition (B.1). This will be done for two special cases according to a finite or infinite $Tr.R$.

(Case 1). Suppose that $Tr.R = \sum_{k=1}^{\infty} \gamma_k = \infty$ and the inverse R^{-1} exists and bounded. In this case we get

$$B(v, \alpha) = b(v, v_x, \alpha)(\cdot),$$

and impose the condition:

$$\begin{aligned} &|b(x, y, z)|^2 \le b_1^2\{1 + |x|^2 + |y|^2\}, \\ &\forall x, y, z \in \mathbb{R}^1, \ \text{ for some } \ b_1 > 0. \end{aligned} \tag{5.7}$$

Then, for the operator norm $\|R^{-1}\| \le c^2$, we have

$$\begin{aligned} |R^{-1/2} B(v, \alpha)|^2 &\le c^2 |b(v, v_x, \alpha)(\cdot)|^2 \\ &= c^2 \int_0^1 |b[v(x), v_x(x), \alpha(x)]|^2 dx \\ &\le c^2 b_1^2 \int_0^1 \{1 + |v(x)|^2 + |v_x(x)|^2\} dx \\ &= c^2 b_1^2 (1 + \|v\|^2), \end{aligned}$$

so that condition (B.1) holds with $b_0 = b_1 c$ and $m = 1$. Therefore, by Thm. 4.3, if the conditions (5.6), (5.7) and (5.8) hold with $\lambda_0 > \frac{1}{2}(b_1 c)$, the HJB equation for this case has a unique strong solution $\Phi \in \mathcal{H}_{2,1}$.

(Case 2). Suppose that $Tr.R = \sum_{k=1}^{\infty} \gamma_k < \infty$ and

$$\inf_k \{-\mu_k \gamma_k\} \ge \delta, \ \text{ for some } \ \delta > 0,$$

which implies that $\mathcal{D}(R^{-1/2}) \subset \mathcal{D}\{(-A)^{1/2}\} = V$ and

$$|R^{-1/2} v|^2 \le \frac{1}{\delta} \langle -Av, v \rangle. \tag{5.8}$$

In this case we have to impose some more stringent conditions:

Let $B(v, \alpha) = b(v, \alpha)(\cdot)$ be independent of v_x such that $b(0, 0) = 0$ and

$$\left| \frac{\partial b(x, y)}{\partial x} \right|^2 + \left| \frac{\partial b(x, y)}{\partial y} \right|^2 \le b_2^2, \ \ \forall x, y \in \mathbb{R}^1, \tag{5.9}$$

for some $b_2 > 0$, and K is a bounded set in $H_0^1(0, 1)$. Then we have

$$|R^{-1/2} B(v, \alpha)|^2 \ \le \ \frac{1}{\delta} < -Ab(v, \alpha), b(v, \alpha) >$$

$$\begin{aligned}
&= -\frac{1}{\delta}\int_0^1 [\frac{\partial^2}{\partial x^2} b(v,\alpha)] b(v,\alpha)dx \\
&= \frac{1}{\delta}\int_0^1 \left(\frac{\partial b(v,\alpha)}{\partial v}\cdot v_x + \frac{\partial b(v,\alpha)}{\partial \alpha}\alpha_x\right)^2 dx \\
&\leq 2\frac{b_2^2}{\delta}\int_0^1 \{v_x^2 + \alpha_x^2\} dx \\
&\leq 2\frac{a^2 b_2^2}{\delta}(1+\|v\|^2).
\end{aligned}$$

where $a^2 = \max\{1, a_0^2\}$ and $a_0^2 = \max_{\alpha\in K}\int_0^1 \alpha_x^2 dx$. The above verifies condition (B.1) with $b_0 = \sqrt{2}(ab_2)/\sqrt{\delta}, m = 1$. Therefore, under the conditions (5.6), (5.7) and (5.10), the HJB equation for this case has a unique solution $\Phi \in \mathcal{H}_{2,1}$, by Thm. 4.3, if $\lambda_0 > \frac{1}{\sqrt{2}}(ab_2)/\sqrt{\delta}$.

Remark 5.1 *If R has a finite range, i.e. $\gamma_k = 0$ for $k \geq (k_0+1)$, the Wiener process becomes a k_0-dimensional Brownian motion and the operator $B(v,\alpha)$ needs to have a finite range.* □

References

1 Chow, P.L., Infinite-dimensional Kolmogorov equations in Gauss-Sobolev Spaces, J. Stoch. Analy. and Appl. (to appear).

2 Da Prato, G., Some results of Bellman equation in Hilbert spaces. SIAM J.Control and Optim., 23, 61–71 (1985).

3 Cannarsa P. & Da Prato G., Second Order Hamilton-Jacobi equations in infinite dimensions, SIAM J. Control and Optim., 29, 474–492. (1991).

4 Cannarsa, P. & Da Prato G., Direct solutions of second order Hamilton-Jacobi equation in Hilbert spaces, Stochastic Partial Differential Equations and Applications, G. DaPrato and L. Tubaro, Editors, Pitman Research Notes in Math. 268, 72–85, Longman, New York (1992).

5 Gozzi, F., Regularity of solutions of a second order Hamilton-Jacobi equation and application to a control problem, Preprints di Matematica # 09, Seuola Normale Superiore, Pisa (1994).

6 HAVARNEAU, T., Existence for the dynamic programming equation of control diffusion process in Hilbert space, Nonlinear Analysis. Theory, Methods & Applications, 9 619–628 (1985).

7 LIONS, P.L., Viscosity solutions of fully nonlinear second-order equations and optimal stochastic control in infinite dimensions. Part III: Uniqueness of viscosity solutions for general second-order equations. J. Funct. Analy. 86, 1–18 (1989).

8 DA PRATO G., & ZABCZYK J., Stochastic Equations in Infinite Dimensions, Cambridge Univ. Press, Cambridge, UK (1992).

9 CHOW, P.L. & KHASMINSKII R.Z., Stationary solutions of nonlinear stochastic evolution equations, J. Stoch. Analy. & Applic. (to appear).

10 YOSIDA K., Functional Analysis, 2nd Ed., Springer-Verlag, Heidelberg-Berlin (1968).

11 LIONS, J.L., Equations Differentielles Operationnelles et Problems aux Limites, Springer-Verlag, Berlin (1961).