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REMARKS ON ESTIMATES FOR THE GREEN FUNCTION

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Introduction

The main purpose of this paper is to give a comprehensive motivation, with most of the key ideas used in the recent work by Garroni and Menaldi [10], on the construction of the Green function $G(x,t,\xi,\tau)$ for an integro-differential initial boundary value problem, when the differential operator is a second-order parabolic operator not in divergence form

$$\begin{aligned} \partial_t\varphi(x,t) &+ A_0\varphi(x,t) + D\varphi(x,t), \\ A_0\varphi(x,t) &= -\sum_{i,j=1}^{d} a_{ij}(x,t)\partial_{ij}^x\varphi(x,t), \\ D\varphi(x,t) &= \sum_{i=1}^{d} a_i(x,t)\partial_i^x\varphi(x,t) + a_0(x,t)\varphi(x,t), \end{aligned}$$

where ∂_t, ∂_i^x and ∂_{ij}^x denote the partial derivatives in t, x_i and x_i, x_j, respectively. The boundary operator has only Hölder continuous coefficients

$$B\varphi(x,t) = \sum_{i=1}^{d} b_i(x,t)\partial_i^x\varphi(x,t) + b_0(x,t)\varphi(x,t),$$

and the integral operator is

$$I\varphi(x,t) = \int_E [\varphi(x+\gamma(x,t,\zeta),t) - \varphi(x,t)]\beta(x,t,\zeta)\pi(\zeta),$$

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where the Radon measure $\pi(\zeta)$ is singular when $\gamma = 0$.

Integro-differential operator of this type arise in control (or game) problems with state processes modelled by a diffusion with jumps. Some fundamental studies on this subject can be found in Bensoussan and Lions [3], Gikhman and Skorokhod [12], Komatsu [15], Lepeltier and Marchal [16], Stroock [18], among others.

The diffusion processes with jumps and reflected on the boundary have been studied in Anulova [1,2], Chaleyat-Maurel et al. [5], Menaldi and Robin [17] under regularity hypotheses on the coefficients.

The Green function is constructed as a solution of the Volterra integral equation

$$G(x,t,\xi,\tau) = G_0(x,t,\xi,\tau) + \int_\tau^t ds \int_\Omega G_0(x,t,s,y,)(I-D)G(y,s,\xi,\tau)dy,$$

where G_0 is the Green function for the initial boundary value problem associated with A_0, B as constructed in Garroni and Solonnikov [11]. The solution G is expressed in the form of a series

$$G(x,t,\xi,\tau) = \sum_{k=0}^{\infty} G_k(x,t,\xi,\tau),$$

where G_k is the iterated kernel

$$G_k(x,t,\xi,\tau) = \int_\tau^t ds \int_\Omega G_0(x,t,y,s)(I-D)G_{k-1}(y,s,\xi,\tau)dy.$$

As far as we know the Green function has been constructed for differential problem (e.g. Eidel'man [7], Friedman [9], Ir'in et al [13], Ladyzhenskay et al. [14]), only recently some attention has been addressed to integro-differential problems (cf. Garroni and Menaldi [10]). Usually in these problems the starting point is the green function and/or the fundamental solutions for simple problems (e.g. constant coefficients).

The most delicate point of this construction is the exact evaluation of punctual and integral estimates for the iterate kernel G_k. There, the exponential heat-kernel

$$\exp(-\frac{|x-\xi|^2}{t-\tau}),$$

or a suitable transformation of it, plays an essential role.

Let us point out to this respect that the heat-kernel used to estimate G_0 disappears during the iterations, due to the presence of the nonlocal integral operator I. This suggests the use of convenient function spaces in which the integral operator work and to which the iterate kernel G_k belongs.

The paper is organized as follows. In Section 1 we give an explicit calculation of the fundamental solution associated with the integro-differential operator

$$\partial_t\varphi(x,t) - \frac{1}{2}\sum_{i=1}^{n}\partial_{ij}^{x}\varphi(x,t) - \lambda[\varphi(x+\gamma,t) - \varphi(x,t)],$$

where λ and γ are constants. Then, in Section 2 we point out the essential properties that will allow us to build the fundamental solution for an integral operator of the type

$$\begin{aligned} I\varphi(x,t) &= \int_E [\varphi(x+\gamma(\zeta),t) - \varphi(x,t)]\pi(d\zeta), \\ \pi(E) &= \int_E \pi(d\zeta) < \infty. \end{aligned}$$

However, in Section 3 we study the Green functions for Dirichlet and Neumann boundary conditions. We notice there that even for the simplest case (i.e. Wiener and Poisson processes) only estimates up to the first derivatives of the kernel are obtained. To end this motivation, in Section 4 we present a quick connection with the probabilistic counterpart. Next, in Section 5 we state the assumptions and the main results relative to the Green function for parabolic second order integro-differential operator with oblique boundary conditions on a bounded domain. Finally, in Section 6 we state the main results corresponding to the asymptotic-elliptic case.

1. Wiener and Poisson processes

Let us consider two particular Markov-Feller processes in the space $\Re^n$, $(w(t), t \geq 0)$ and $(p(t), t \geq 0)$. The first is a standard Wiener process on the canonical space $C([0,\infty),\Re^n)$ and the second is standard Poisson process on the canonical space $D([0,\infty),\Re^n)$. Denote by P the probability measure on product sample space $C([0,\infty),\Re^n) \times D([0,\infty),\Re^n)$ which makes the processes independent each of other.

Recall that $D([0,\infty),\Re^n)$ (*resp.* $C([0,\infty,\Re^n))$ denotes the space of right continuous functions ω from $[0,\infty)$ into $\Re^n$ having left-hand limits (resp. continuous functions). Notice that any function in $D([0,\infty),\Re^n)$ is locally bounded and has at most countable many points of discontinuity. The space $C([0,\infty),\Re^n)$ (*resp.* $D([0,\infty),\Re^n)$) endowed with the local uniform convergence (resp. with the Skorokhod topology) becomes a complete separable metric space.

If we denote by $E\{\cdot\}$ the mathematical expectation w.r.t. the canonical probability measure P we have

$$E\{f(x+w(t))\} = \int_{\Re^n} \Gamma_0(x-y,\ t)f(y)dy,\quad t>0, x\in\Re^n$$

(1.1) $\Gamma_0(x,t) = (2\pi t)^{-\frac{n}{2}} \exp(-\frac{|x|^2}{2t}),\quad t>0, x\in\Re$

and

$$E\{f(x+p(t))\} = e^{-\lambda t}\sum_{k=0}^{\infty}\frac{(\lambda t)^k}{k!}f(x+k\gamma),\quad t>0, x\in\Re^n$$

where $\lambda > 0$, $\gamma\in\Re^n$ are the characteristic parameters of the Poisson process.

Because the processes are independent,

$$E\{f(x+w(t)+p(t))\} = \sum_{k=0}^{\infty} E\{f(x+w(t)+p(t))|p(t)=k\gamma\}\times$$
$$\times P\{p(t)=k\gamma\} = \sum_{k=0}^{\infty}(\int_{\Re^N}\Gamma_0(x+k\gamma-y,t)f(y)dy)e^{-\lambda t}\frac{(\lambda t)^k}{k!}.$$

This proves that the transition density function corresponding to the Feller-Markov process $(w(t)+p(t), t\geq 0)$ is

(1.2) $\Gamma(x,t) = e^{-\lambda t}\sum_{k=0}^{\infty}\frac{(\lambda t)^k}{k!}\Gamma_0(x+k\gamma,t),\quad t>0, x\in\Re^n.$

However, as soon as we pretend to repeat this simple computations for some small variants of the standard Wiener and Poisson processes we get troubles. For instance, replace $(w(t), t\geq 0)$ by a standard reflected Wiener process in a half-space, or add a stopping time at the exist of a domain Ω, or allow $(p(t), t\geq 0)$ to have a Levy measure different from the Dirac measure, or even think how to deal with a general jump diffusion process.

From this, we have at least an explicit expression for the transition density functions of a simple jumps diffusion process, namely, a standard Wiener-Poisson process.

The equivalent analytic counterpart will give us some more information.

Denote by Δ the Laplacian operator, in $\Re^n$ by I the following jump operator

(1.3) $I\varphi(x) = \lambda[\varphi(x+\gamma)-\varphi(x)],\quad \lambda>0, \gamma\in\Re^n.$

It is clear that $\frac{1}{2}\Delta$ and I are the infinitesimal generators of the above Wiener and Poisson processes.

The function $\Gamma(x,t)$ given by (1.2) solves the equation

$$\text{(1.4)}\quad (\partial_t - \tfrac{1}{2}\Delta - I)\Gamma = \delta \text{ in } \Re^n \times (-\infty, +\infty),$$
$$\Gamma = 0 \text{ in } \Re^n \times (-\infty, 0],$$

where δ is the Dirac measure at the origin in $\Re^n \times [0,\infty)$. Thus, the function Γ is a fundamental solution relative to the operator $\partial_t - \frac{1}{2}\Delta - I$.

In order to solve this problem, we may use the fact that Γ_0, given by (1.1), satisfies the equation

$$\text{(1.5)}\quad (\partial_t - \tfrac{1}{2}\Delta)\Gamma_0 = \delta \text{ in } \Re^n \times (-\infty, +\infty),$$
$$\Gamma_0 = 0 \text{ in } \Re^n \times (-\infty, 0].$$

A classic argument is to propose

$$\text{(1.6)}\quad \Gamma = \Gamma_0 + \Gamma_0 \star F\ ,$$

where $\star$ denotes the convolution in $\Re^n \times [0,\infty)$, i.e.

$$(\Gamma_0 \star F)(x,t) = \int_0^t ds \int_{\mathcal{R}^n} \Gamma_0(x-y, t-s) F(y,s) dy.$$

In (1.6), the function F is unknown and so is Γ. We compute

$$(\partial_t - \frac{1}{2}\Delta)\Gamma = \delta + F, \text{ i.e. } F = I\Gamma$$

which together with (1.6) provide an integral equation of Volterra type for the function Γ, namely

$$\text{(1.7)}\quad \Gamma = \Gamma_0 + \Gamma_0 \star I\Gamma\ .$$

Similarly, we compute

$$(\partial_t - \frac{1}{2}\Delta - I)\Gamma = \delta - I\Gamma_0 + F - I\Gamma_0 \star F$$

to get the equation

(1.8) $F = I\Gamma_0 + I\Gamma_0 \star F.$

To solve (1.7) and (1.8) we set

$$\begin{aligned} \Gamma &= \Gamma_0 + \Gamma_1 + \ldots + \Gamma_k + \ldots, \\ \Gamma_{k+1} &= \Gamma_0 \star I\Gamma_k, \quad k = 0, 1, \ldots \end{aligned}$$

and

$$\begin{aligned} F &= F_0 + F_1 + \ldots + F_k + \ldots, \\ F &= I\Gamma_0, \quad F_{k+1} = I\Gamma_0 \star F_k, \quad k = 0, 1, \ldots \end{aligned}$$

Computing

$$\begin{aligned} \Gamma_1 &= tI\Gamma_0, \quad \Gamma_2 = \frac{t^2}{2!} I^2 \Gamma_0, \ldots, \quad \Gamma_k = \frac{t^k}{k!} I^k \Gamma_0, \\ I^k \varphi(x) &= \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} \varphi(x + i\gamma) \lambda^k \end{aligned}$$

Hence

(1.9) $\Gamma(x,t) = \sum_{k=0}^{\infty} \sum_{i=0}^{k} \left(\binom{k}{i}\right)(-1)^{k-i} \frac{(\lambda t)^k}{k!} \Gamma_0(x + i\gamma, t),$

which reproduces (1.2).

This calculation does not go further but, it gives us a way of approaching the problem by studying the structure of the function Γ in formula (1.2).

2. Essential Properties

Let us recall some properties of the fundamental solution $\Gamma_0(x,t)$ given by (1.1). To the purpose, we consider the function

(2.1) $e(x,t,r,c) = t^{-\frac{r}{2}} \exp(-c\frac{|x|^2}{t}), \ \forall t, r, c > 0, x \in \Re^n.$

Based on the elementary inequality for any $r, c, \varepsilon > 0$ there exists $c > 0$ such that

(2.2) $|x|^r e(x,t,r,c) \leq Ce(x,t,0,c-\varepsilon), \ \forall t > 0, x \in \Re^n,$

we obtain for any $i = 0, 1, 2, 3, 4$ and some $C_0, c_0 > 0$, (2.3) $|\nabla_x^i \Gamma_0(x,t)| \leq C_0 e(x,t,n+i,c_0),\ \forall t>0, x\in\Re^n,$

where ∇_x^i represents all the derivatives in x of the i-order. Also, we can replace ∇_x^{2i} by ∂_t^i, i.e. two differentations in x is equivalent to one differentation in t. Moreover, by means of expressions of the type

$$\Gamma_0(x,t)-\Gamma_0(x',t)=\int_0^1 (x-x')\cdot\nabla\Gamma_0(x+\theta(x-x'),t)d\theta$$

and

$$|\Gamma_0(x,t)-\Gamma_0(x',t)|\leq(|\Gamma_0(x,t)|+|\Gamma_0(x',t)|)^{1-\alpha}|\Gamma_0(x,t)-\Gamma_0(x',t)|^\alpha,$$

for $0<\alpha<1$, we deduce for any $i=0,1,2$ and some $M_0, m_0>0$

(2.4) $|\nabla^i\Gamma_0(x,t)-\nabla^i\Gamma_0(x',t')|\leq M_0(|x-x'|^\alpha+|t-t'|^{\alpha/2})\times$
$\times[e(x,t,n+i+\alpha,m_0)+e(x',t',n+i+\alpha,m_0)],\ \forall t,t'>0,\ x,x'\in\Re^n,$

where actually either $x=x'$ or $t=t'$. Again ∇^2 can be replaced by ∂_t.

We would like to generalize the analytic computation of the fundamental solution $\Gamma(x,t)$. So, let us look at each term of the series (1.9),

$$\Gamma_k(x,t)=\sum_{j=0}^k(\binom{k}{j})(-1)^{k-j}\frac{(\lambda t)^k}{k!}\Gamma_0(x+j\gamma,t)$$

This function has several singular points, namely $x=-j\gamma,\ t=0$ for $j=0,1,\ldots,k$. Then we cannot expect to have bounds in term of the function $e(x,t,r,c)$ given by (2.1). Since the propagation of singularities is only in the variable x, we should seek for properties which are not pointwise in the variable x. For instance,

$$|\nabla^i\Gamma_k(x,t)|\leq\frac{\lambda^k}{k!}\sum_{j=0}^k(\binom{k}{i})t^k|\nabla^i\Gamma_0(x+j\gamma,t)|$$

gives two estimates

(2.5) $|\nabla^i\Gamma_k(x,t)|\leq\dfrac{(2\lambda)^k}{k!}C_0t^{-\frac{n+i-2k}{2}},\ \forall t>0, x\in\Re^n,$

(2.6) $\displaystyle\int_{\Re^n}|\nabla^i\Gamma_k(x,t)|dx\leq\frac{(2\lambda)^k}{k!}K_0t^{-\frac{i-2k}{2}},\ \forall t>0, x\in\Re^n,$

where $i = 0,1,2,3,4$ and

$$K_0 = C_0 \int_{\Re^n} \exp(-c_0|\eta|^2)d\eta.$$

and C_0, c_0 are the same constants of (2.3). Notice the change of variable $t\eta = x$ used to obtain (2.6). Similarly from (2.4) we obtain the estimates

$$\begin{aligned}
&(2.7)\ |\nabla^i\Gamma_k(x,t) - \nabla^i\Gamma_k(x',t')| \le \frac{(2\lambda)^k}{k!} M_0(|x-x'|^\alpha + |t-t'|^{\alpha/2} \times \\
&\quad \times (t^{-\frac{n+i+\alpha-2k}{2}} + t'^{-\frac{n+i+\alpha-2k}{2}}), \ \forall t,t' > 0, x,x' \in \Re^n, \\
&\quad \int_{\Re^n} |\nabla^i\Gamma_k(x-y,t) - \nabla^i\Gamma_k(x'-y,t')|dy \le \frac{(2\lambda)^k}{k!} N_0(|x-x'|^\alpha + |t-t'|^{\alpha/2}) \\
&\quad \times (t^{-\frac{i+\alpha-2k}{2}} + t'^{-\frac{i+\alpha-2k}{2}}), \ \forall t,t' > 0, x,x' \in \Re^n,
\end{aligned}$$

where

$$N_0 = M_0 \int_{\Re^n} \exp(-m_0|\eta|^2)d\eta$$

and M_0, m_0 are the same constants of (2.4). Notice that the integration is $\Re^n$ help us to cancel a singularity of the type $t^{-\frac{n}{2}}$.

Now, consider an integral operator of the form

$$(2.9)\quad I\varphi(x) = \int_E [\varphi(x+\gamma(\zeta)) - \varphi(x)]\pi(d\zeta),$$

where π is a finite measure on E and $\gamma(z)$ is measurable, i.e.

$$(2.10)\quad \pi(E) < \infty$$

We should study how the estimates (2.5),...,(2.8) changes when an integral operator of the form (2.9) is used. To that purpose, let us denote by $C(\varphi,2k), K(\varphi,2k), M(\varphi,2k)$ and $N(\varphi,2k)$ the infimum of the multiplicative constants that can be used in the left-hand side of the estimates (2.5),...,(2.8) when φ replaces $\nabla^i\Gamma_k$. For instance $C(\varphi,k)$ is the infimum of all constants $C > 0$ satisfying

$$|\varphi(x,t)| \le Ct^{-\frac{n-2k}{2}}, \ \forall t > 0, x \in \Re^n.$$

It is clear that from (2.5),..., (2.8) we have

$$C(\nabla^i\Gamma_k, 2k-i) \le \frac{(2\lambda)^k}{k!} C_0, \quad K(\nabla^i\Gamma_k, 2k-i) \le \frac{(2\lambda)^k}{k!} K_0,$$

$$M(\nabla^i\Gamma_k, 2k-i) \le \frac{(2\lambda)^k}{k!} M_0, \quad N(\nabla^i\Gamma_k, 2k-i) \le \frac{(2\lambda)^k}{k!} N_0.$$

Notice that n and α are fixed.

In order to find a fundamental solution for the operator $\partial_t - \frac{1}{2}\Delta - I$, where I is now given by (2.9), we may solve the Volterra equation

$$\Gamma = \Gamma_0 + \Gamma_0 \star I\Gamma$$

for the unknown Γ. To that end, we should evaluate the iterate terms

$$\Gamma_{k+1} = \Gamma_0 \star I\Gamma_k.$$

For instance

$$\Gamma_1(x,t) = \int_0^t ds \int_{\Re^n} \Gamma_0(x-y,t-s)dy \int_E [\Gamma_0(y+\gamma(\zeta),s) - \Gamma_0(y,s)]\pi(\zeta),$$

we exchange the order of the integrals in $\Re^n$ and E to obtain

$$\Gamma_1(x,t) = \int_0^t ds \int_E [\Gamma_0(x+\gamma(\zeta),t) - \Gamma_0(x,t)]\pi(d\zeta),$$

after using the equality

$$\Gamma_0(x+z,t) = \int_{\Re^n} \Gamma_0(x-y,t-s)\Gamma_0(y+z,s)dy.$$

This is

$$\Gamma_1(x,t) = tI\Gamma_0(x,t).$$

For $k \geq 2$, we use the fact that I and $\star$ commute,

$$\Gamma_0 \star I\Gamma_k = I(\Gamma_0 \star \Gamma_k).$$

Then

$$(\Gamma_0 \star \Gamma_1)(x,t) = \int_0^t sds \int_{\Re^n} \Gamma_0(x-y,t-s)dy \int_E [\Gamma_0(y+\gamma(\zeta),s) - \Gamma_0(y,s)]\pi(d\zeta)$$

which gives

$$\Gamma_2(x,t) = \frac{t^2}{2!} I^2 \Gamma_0(x,t)$$

and in general

(2.11) $\Gamma_k(x,t) = \frac{t^k}{k!} I^k \Gamma_0(x,t) \quad ,\forall t > 0, x \in \Re^n,$

where I^k denotes the k-iteration (power) of the integral operator I defined by (2.9).

On the other hand, it is easy to check that

$$\begin{aligned} C(I\varphi,j) &\le 2\pi(E)C(\varphi,j), \\ K(I\varphi,j) &\le 2\pi(E)K(\varphi,j), \\ M(I\varphi,j) &\le 2\pi(E)M(\varphi,j), \\ N(I\varphi,j) &\le 2\pi(E)N(\varphi,j), \ \forall j \end{aligned}$$

Hence, we deduce

$$\begin{aligned} C(\nabla^i\Gamma_k, 2k-i) &\le \frac{[2\pi(E)]^k}{k!}C(\nabla^i\Gamma_0,-i), \\ K(\nabla^i\Gamma_k, 2k-i) &\le \frac{[2\pi(E)]^k}{k!}K(\nabla^i\Gamma_0,-i), \\ M(\nabla^i\Gamma_k, 2k-i) &\le \frac{[2\pi(E)]^k}{k!}M(\nabla^i\Gamma_0,-i), \\ N(\nabla^i\Gamma_k, 2k-i) &\le \frac{[2\pi(E)]^k}{k!}N(\nabla^i\Gamma_0,-i), \end{aligned}$$

which proves, for instance, the convergence of the series

$$\Gamma(x,t) = \sum_{k=0}^{\infty} \Gamma_k(x,t)$$

and the estimates

$$\begin{aligned} C(\nabla^i\Gamma,-i) &\le \exp(2\pi(E))C(\nabla^i\Gamma_0 - i) \\ K(\nabla^i\Gamma,-i) &\le \exp(2\pi(E))K(\nabla^i\Gamma_0,-i) \end{aligned}$$

and similarly for $M(\nabla^i\Gamma,-i)$, $N(\nabla^i\Gamma,-i)$. Notice that $\lambda = \pi(E)$ when π is a Dirac measure.

Until now, we have construct a fundamental solution for the operator $\partial_t - \frac{1}{2}\Delta - I$, with the I given by (2.9). Estimates of the type (2.5), ..., (2.8) still hold true, with λ replaced by $\pi(E)$. However, we have not seen precise use of the estimate (2.7), (2.8) for $i = 0,1,2$. This is because they are consequence of (2.5), (2.6) for $i = 1,2,3,4$. Remark that ∇^{2i} can be replaced by ∂_t^i. We will see the utility of (2.7), (2.8) as soon as we pretend to use this approach for variable coefficients in the integro-differential operator. At least for the integral operator I, it will be necessary to include the case of variable coefficients $\gamma = \gamma(x, t, z)$ in (2.9) if we want to consider bounded domain. An assumption of the type

$$x + \gamma(x, t, \zeta) \in \bar{\Omega}, \ \ \forall x \in \bar{\Omega}, t > 0, \zeta \in E$$

is necessary to localize the integral operator I to a region $\bar{\Omega}$ of $\Re^n$.

Remark that the way in which $\Gamma(x, t)$ converges to δ as t goes to zero has not been yet specified. We may consider the convergence to take place in the Schwartz distribution sense of $\mathcal{D}'(\Re^n)$, i.e.

$$(2.12) \quad \int_{\Re^n} \Gamma(x, t)\varphi(x)ds \to \varphi(0) \ \text{ as } \ t \to 0$$

for any test function φ.

3. Dirichlet and Neumann problems

In a half-space $\Re^n_+ = \Re^{n-1} \times (0, \infty)$ let us consider the integro-differential operator $\partial_t - \frac{1}{2}\Delta - I$, where I is given by (2.9) with the coefficient $\gamma(\zeta)$ satisfying

$$(3.1) \quad \gamma(\zeta) = (\gamma_1(\zeta), \ldots, \gamma_n(\zeta)), \ \gamma_n(\zeta) \geq 0, \ \ \forall \zeta \in E.$$

Under this conditions, the integral operator I is meaningful for function defined only on $\Re^n_+$

The Green function $G(\tilde{x}, \ x_n, t, \xi_n), x = (\tilde{x}, x_n)$ in $\Re^n_+, \ t, \xi_n > 0$ solves the equation

$$(3.2) \quad (\partial_t - \tfrac{1}{2}\Delta - I)G(\tilde{x}, x_n, t, \xi_n) = \delta(\tilde{x})\delta(x_n - \xi_n)\delta(t) \ \text{ in } \ \Re^n_+, t > 0$$
$$G(\cdot, \cdot, t, \xi_n) = 0 \ \text{ for } \ t < 0,$$

where again δ denotes the Dirac measure at the origin, and a boundary condition, either the Dirichlet boundary condition

$$(3.3) \quad G(\cdot, x_n, \cdot, \xi_n) \to 0 \ \text{ as } \ x_n \to 0$$

or the Neumann boundary condition

$$(3.4) \quad \partial_n G(\cdot, x_n, \cdot, \xi_n) \to 0 \ \text{ as } \ x_n \to 0 \, .$$

Notice that the operator $\frac{1}{2}\Delta + I$ is acting on the first variable, i.e. $x = (\tilde{x}, x_n)$, and ∂_n means the partial derivative w.r.t. the variable x_n. The variable ξ_n is only a parameter for the equation. When necessary, we will denote the Green functions G^D and G^N for Dirichlet and Neumann boundary conditions, respectively.

An usual way of constructing the Green function is to express it as the sum of a fundamental solution plus something else. To do so, it is necessary to find the so-called Poisson function. The Poisson function $P(\tilde{x}, x_n, t), x = (\tilde{x}, x_n)$ in $\Re^n, t > 0$, solves the equation

$$\begin{aligned} (3.5)\ & (\partial_t - \tfrac{1}{2}\Delta - I)\, P\, (\tilde{x}, x_n, t) = 0 \ \text{ in } \ \Re^n_+,\ t > 0, \\ & P(\cdot, \cdot, t) = 0 \ \text{ for } \ t < 0 \end{aligned}$$

and either the Dirichlet boundary condition

$$(3.6)\ \ P^D(\tilde{x}, x_n, t) \to \delta(\tilde{x})\delta(t) \ \text{ as } \ x_n \to 0$$

or the Neumann boundary condition

$$(3.7)\ \ -\partial_n P^N(\tilde{x}, x_n, t) \to \delta(\tilde{x})\delta(t) \ \text{ as } \ x_n \to 0$$

Again, we will distinguish P^D and P^N when necessary. Because the equation (3.5) has constant coefficients, it is clear that

$$P^D = -\partial_n P^N$$

and that one should have

$$G^N(\cdot, \cdot, \cdot, \xi_n) \to P^N \ \text{ as } \ \xi_n \to 0.$$

Let us recall this procedure for the Laplacian operator. The reflection principle allow us to construct directly the Green function for the differential operator $\partial_t - \frac{1}{2}\Delta$ in $\Re^n_+$ with Neumann boundary conditions. That principle states that a one dimensional Wiener process starting from zero, has the same probability of becoming positive as becoming negative. Analytically, that means that the fundamental solution is a even function, as easily checked from the explicit expression (1.1). This amounts to show that

(3.8) $G_0^N(\tilde{x}, x_n, t, \xi_n) = \Gamma_0(\tilde{x}, x_n - \xi_n, t) + \Gamma_0(\tilde{x}, x_n + \xi_n, t),$

where $\Gamma_0(\tilde{x}, x_n, t) = \Gamma_0(x, t)$ given by (1.1), satisfies the properties

(3.9) $(\partial_t - \frac{1}{2}\Delta)G_0^N(\tilde{x}, x_n, t, \xi_n) = \delta(\tilde{x})\delta(x_n - \xi_n)\delta(t)$ in $\Re_+^n, t > 0$

$G_0^N(\cdot, \cdot, t, \xi_n) = 0$ for $t < 0$,

$\partial_n G_0^N(\cdot, x_n, \cdot, \xi_n) \to 0$ as $x_n \to 0$.

Then

(3.10) $P_0^N(\tilde{x}, x_n, t) = 2\Gamma_0(\tilde{x}, x_n, t)$

and

(3.11) $P_0^D(\tilde{x}, x_n, t) = 2x_n t^{-1}\Gamma_0(\tilde{x}, x_n, t).$

Notice that $\tilde{x} \in \Re^{n-1}, x_n, \xi_n, t > 0$.

In order to obtain the Green function for the Dirichlet boundary conditions, we notice that

$$(\partial_t - \frac{1}{2}\Delta)\Gamma_0(\tilde{x}, x_n + \xi_n, t) = 0 \text{ in } \Re_+^n, t > 0.$$

Hence, the function

(3.12) $G_0^D(\tilde{x}, x_n, t, \xi_n) = \Gamma_0(\tilde{x}, x_n - \xi_n, t) - \Gamma_0(\tilde{x}, x_n, \xi_n, t)$

satisfies the equation

(3.13) $(\partial_t - \frac{1}{2}\Delta)G_0^D(\tilde{x}, x_n, t, \xi_n) = \delta(\tilde{x})\delta(x_n - \xi_n)\delta(t)$ in $\Re_+^n, t > 0$

$G_0^D(\cdot, \cdot, t, \xi_n) = 0$ for $t < 0$,

$G_0^D(\cdot, x_n, \cdot, \xi_n) \to 0$ as $x_n \to 0$.

The Green function $G_0(\tilde{x}, x_n, t, \xi_n)$ for Dirichlet as well as for Neumann boundary conditions possesses singularities similar to those of the fundamental solution $\Gamma_0(x, t)$. For any $i = 0, 1, 2, 3, 4$ and some $C_0, c_0 > 0$

(3.14) $|\nabla_x^i G_0(\tilde{x} - \tilde{\xi}, x_n, t, \xi_n)| \le C_0\, e(x - \xi, t, n + i, c_0),$

$\forall t, x_n, \xi_n > 0, \quad \tilde{x}, \tilde{\xi} \in \Re^{n-1},$

where $e(\cdot,\cdot,\cdot,\cdot)$ is the function (2.1), $x = (\tilde{x}, x_n), \xi = (\tilde{\xi}, \xi_n)$, and again ∇_x^{2i} can be replaced by ∂_t^i. Estimates similar to (2.4) also hold for G_0. In view of (3.10) and (3.11) we deduce the corresponding estimates for the Poisson function.

Actually to prove the estimate (3.14) for the Green function, we notice that

$$|x_n + \xi_n| \geq |x_n - \xi_n|, \quad \forall x_n, \xi_n > 0$$

and the equalities (3.8), (3.12) together with the estimates (2.3) for $\Gamma_0(x,t)$, give (3.14).

Notice that

$$\text{(3.15)} \quad G_0^D(\tilde{x}, x_n, t, \xi_n) > 0, \quad \forall \tilde{x} \in \Re^{n-1}, x_n \geq 0, t, \xi_n > 0,$$

and

$$\text{(3.16)} \quad G_0^N(\tilde{x}, x_n, t, \xi_n) \geq \Gamma_0(\tilde{x}, x_n - \xi_n, t), \quad \forall \tilde{x} \in \Re^{n-1}, x_n, \xi_n, t > 0$$

To construct the Green function $G(\tilde{x}, x_n, t, \xi_n)$ corresponding to the integro-differential operator $\partial_t - \frac{1}{2}\Delta - I$, with I being given by (2.9) under the assumptions (2.10) and (3.1), we propose

$$\text{(3.17)} \quad G(\cdot, x_n, \cdot, \xi_n) = G_0(\cdot, x_n, \cdot, \xi_n) + \int_0^\infty G_0(\cdot, x_n, \cdot, \lambda) \star F(\cdot, \lambda, \cdot, \xi_n) d\lambda,$$

where F is an unknown function and $\star$ denotes the convolution in the $\tilde{x}, t$ variables, i.e.

$$G_0(\tilde{x}, \cdot, t, \cdot) \star F(\tilde{x}, \cdot, t, \cdot) = \int_0^t ds \int_{\Re^{n-1}} G_0(\tilde{x} - \tilde{y}, \cdot, t - s, \cdot) F(\tilde{y}, \cdot, s, \cdot) d\tilde{y}.$$

This gives the following Volterra type equation for the unknown Green Function G,

$$\text{(3.18)} \quad G(\cdot, x_n, \cdot, \xi_n) = G_0(\cdot, x_n, \cdot \xi_n) + \int_0^\infty G_0(\cdot, x_n, \cdot, \lambda) \star IG(\cdot, \lambda, \cdot, \xi_n) d\lambda,$$

i.e.

$$\text{(3.19)} \quad G = G_0 + G_1 + \cdots + G_k + \cdots, \text{ and for } k = 0, 1, \ldots,$$
$$G_{k+1}(\cdot, x_n, \cdot, \xi_n) = \int_0^\infty G_0(\cdot, x_n, \cdot, \lambda) \star IG_k(\cdot, \lambda, \cdot, \xi_n) d\lambda.$$

For the sake of simplicity, we consider the one dimensional case, $n = 1$, $x = x_n$, $\xi = \xi_n$. Then, the iterate G_k takes the form

(3.20) $G_{k+1}(x,t,\xi)=\int_0^t ds\int_0^\infty G_0(x,t-s,y)IG_k(y,s,\xi)dy$

Our purpose is to show the convergence of the series (3.19). Notice that an explicit expression like (2.11) does not hold true now. However, the technique can be used.

Let us defined the space of continuous functions $\varphi(x,t,\xi)$, x,t,ξ in $(0,\infty)$ satisfying (recall $n=1$)

(3.21) $|\varphi(x,t,\xi)|\le C_0t^{-\frac{1}{2}+k}$, $\forall x,t,\xi$,

(3.22) $\int_0^\infty|\varphi(x,t,y)|dy+\int_0^\infty|\varphi(y,t,\xi)|dy\le K_0t^k$, $\forall x,t,\xi$,

for some constants $C_0,K_0>0$. Denote by $S_k, k=-1,0,\ldots$, such a space and by $C(\varphi,2k),K(\varphi,2k)$ the infimum of constants C_0,K_0 that make (3.21), (3.22) to be satisfied. It is easy to check that

(3.23) $C(I\varphi,k)\le 2\pi(E)C(\varphi,k)$, $\forall\varphi,k$,
$K(I\varphi,k)\le 2\pi(E)K(\varphi,k)$, $\forall\varphi,k$,

and that G_0 belongs to S_0. To show that G_k belongs to S_k we proceed as follows:

$$\begin{aligned}G_{k+1}(x,t,\xi) &= \int_0^{\frac{t}{2}}ds\int_0^\infty G_0(x,t-s,y)IG_k(y,s,\xi)dy+\\ &+ \int_{\frac{t}{2}}^t ds\int_0^\infty G_0(x,t-s),y)IG_k(y,s,\xi)dy=I+II,\\ |I| &\le C(G_0,0)K(IG_k,k)\int_0^{\frac{t}{2}}(t-s)^{-\frac{1}{2}}s^kds,\\ |II| &\le K(G_0,0)C(IG_k,k)\int_{\frac{t}{2}}^t(t-s)^{-\frac{1}{2}}s^kds\end{aligned}$$

and by means of the changes of variables $s=t\theta$ we get

$$\begin{aligned}|I| &\le t^{-\frac{1}{2}+(k+1)}C(G_0,0)K(IG_k,k)\int_0^{\frac{1}{2}}(1-\theta)^{-\frac{1}{2}}\theta^kd\theta\\ |II| &\le t^{-\frac{1}{2}+(k+1)}K(G_0,0)C(IG_k,k)\int_{\frac{1}{2}}^1\theta^{-\frac{1}{2}+k}d\theta,\end{aligned}$$

which give

(3.24) $|G_{k+1}(c,t,\xi)|\le C(G_{k+1},k+1)t^{-\frac{1}{2}+(k+1)}$,
$C(G_{k+1},2k+2)\le\frac{2}{k}\max\{C(G_0,0)K(IG_k,2k),K(G_0,0)C(IG_k,2k)\}$

Similarly, if we integrate in x or ξ then

(3.25) $K(G_{k+1}, 2k+2) \leq \frac{2}{k} K(G_0, 0) K(IG_k, 2k).$

Combining (3.23), ..., (3.25) we deduce

$$\begin{aligned} C(G_{k+1}, 2k+2) &+ K(G_{k+1}, 2k+2) \leq \\ &\leq \frac{r}{k}[C(G_k, 2k) + K(G_k, 2k)], \ \forall k, \end{aligned}$$

where

$$r = 4\pi(E)[C(G_0, 0) + K(G_0, 0)],$$

i.e.

(3.26) $C(G_{k+1}, 2k+2) + K(G_{k+1}, 2k+2) \leq \frac{r^k}{k!}, \ \forall k = 0, 1, \ldots$

It is clear that the until now, we know that the series (3.18) converges. Moreover each term G_k belongs to the space S_k and estimate (3.26) holds. However, no convergence of the derivative has been yet established. The above procedure go through for the first derivative in x, the difference is that (3.26) becomes

(3.27) $C(\nabla G_{k+1}, 2k+1) + K(\nabla G_{k+1}, 2k+1) \leq a_0 a_1 \ldots a_k, \forall k,$

where

$$a_k = 4\pi(E)[C(\nabla G_0, -1) + K(\nabla G_0, -1)] \int_0^1 (1-\theta)^{-\frac{1}{2}} \theta^k d\theta.$$

This ensure also the convergence of the first derivative in x.

4. Stochastic Representation

Let $(w(t), t \geq 0)$ and $(p(t, \cdot), t \geq 0)$ be a standard Brownian motion on the canonical space $C([0, \infty), \Re^n)$ and a Poisson (random) measure with Levy measure $m(\cdot)$ gives by

(4.1) $m(A) = \pi(\{\zeta \in E : \gamma(\zeta) \in A\}),$

where π is a finite measure on E and $\gamma(\zeta)$ is measurable. The Poisson measure $(p(t, \cdot), t \geq 0)$ defines a Markov-Feller process with paths in $D([0, \infty), \Re^n)$. Denote by $\tilde{P}$ the probability measure on the product sample spaces $C([0, \infty(, \Re^n) \times D([0, \infty), \Re^n)$ which makes the standard Wiener process $(w(t), t \geq 0)$ independent of the Poisson measure $(p(t, \cdot), t \geq 0)$.

Setting

(4.2) $\tilde{X}(t) = x + w(t) + \int_0^t \int_E \gamma(\zeta) dp(s,\zeta),\ t \geq 0$

we have constructed a Markov-Feller process $(X(t), t \geq 0)$ with right continuous (having left limits) paths under the probability measure P on $D([0,\infty), \Re^n)$, where P is the image probability measure of $\tilde{P}$ through $\tilde{X}(\cdot)$ and $X(t) = w(t)$, i.e. the identity mapping.

This Markov-Feller process $(X(t), t \leq 0)$ has the transition density function

(4.3) $P(X(t) \in B | X(s) = x) = \int_B \Gamma(x - y, t - s) dy,$
$\forall x \in \Re^n, 0 < s < t, B \in \mathcal{B}(\Re^n),$

where $\Gamma(x,t)$ is the fundamental solution of Section 2, i.e.

(4.4) $\Gamma(x,t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} I^k \Gamma_0(x,t),$

with $\Gamma_0(x,t)$ being the Gauss kernel (1.1) and I^k the k-iteration of the integral operator

(4.5) $I\varphi(x) = \int_E [\varphi(x + \gamma(\zeta)) - \varphi(x)] \pi(d\zeta).$

The associated semigroup, $(\Phi(t), t \geq 0)$ is given by

(4.7) $L = \frac{1}{2}\Delta + I,$

where Δ is the Laplacian.

In order to interpret the Dirichlet problem in $\Re^n_+$, we consider the stopping time

(4.8) $\tau = \inf\{t \geq 0 : X(t) \notin \Re^n_+\}.$

The new Markov-Feller process $(X^D(t), t \geq 0)$ obtained by stopping $X(t)$ at the time τ, i.e.

(4.9) $X^D(L) = \begin{cases} X(t) & \text{if } 0 \leq t \leq \tau \\ X(t) & \text{if } t \geq \tau, \end{cases}$

gives a semigroup

(4.10) $\Phi^D(t) f(x) = E\{f(X(t \wedge \tau)) | X(0) = x\}.$

The infinitesimal generator still being (4.7) but the transition density function is the Green function with Dirichlet boundary condition, i.e.

(4.11) $\Phi^D(t)f(x) = \int_{\Re_+^N}[\Gamma(\tilde{x}-\tilde{y}, x_n - y_n, t) - \Gamma(\tilde{x}-\tilde{y}, x_n + y_n, t)]f(y)dy,$

where $x = (\tilde{x}, x_n), y = (\tilde{x}, y_n)$. Remark that we have assumed (3.1), i.e. $\gamma_n(\zeta) \geq 0$.

For the Neumann boundary conditions, we need to construct another process $(\tilde{X} \equiv N(t), t \geq 0)$ as follows:

(4.12) $\tilde{X}_i^N(t) = \tilde{X}_i(t)$ for $i = 1, 2, \ldots, n-1$
$\tilde{X}_n^N(t) = \tilde{X}_n(t) + \sup\{[\tilde{X}_n(s)]^- : 0 \leq s \leq t\}.$

This is a reflected Poisson-Wiener process. The transition density function is given by

(4.13) $P(X^N(t) \in b | X^N(s) = x) = \int_B [\Gamma(\tilde{x}-\tilde{x}, x_n - y_n, t-s) +$
$+\Gamma(\tilde{x}-\tilde{y}, x_n + y_n, t-s)]dy, \ \forall x \in \Re_+^n, 0 < s < t,$
$B \in \mathcal{B}(\Re_+^n), x = (\tilde{x}, x_n), y = (\tilde{y}, y_n).$

The probability measure P associated with either $(X^N(t), t \geq 0)$ or $(X^D(t), t \geq 0)$ is actually defined on $D([0,\infty), \Re_+^n)$. The infinitesimal generator still being (4.7), but it is clear that the domain of the infinitesimal generator contains the boundary conditions.

For example, we refer to the books of Dynkin [6], Ethier and Kurtz [8], among others, for a background on Markov-Feller processes.

5. General Results

Let Ω be a bounded open subset of $\Re^d$ having smooth boundary $\partial\Omega$. Consider the integral operator

(5.1) $I\varphi(x,t) = \int_E [\varphi(x + \gamma(x,t,\zeta), t) - \varphi(x,t)]\beta(x,t,\zeta)\pi(d\zeta),$

where $\pi(\cdot)$ is a σ-finite measure on the measurable space $(E, \mathcal{E})$, and the differential operators

(5.2) $D\varphi(x,t) = \sum_{i=1}^{d} a_i(x,t)\partial_i^x \varphi(x,t) + a_0(x,t)\varphi(x,t),$

(5.3) $A_0\varphi(x,t) = -\sum_{i,j=1}^{d} a_{ij}(x,t)\partial_{ij}^x \varphi(x,t),$

(5.4) $B\varphi(x,t) = \sum_{i=1}^{d} b_i(x,t)\partial_i^x \varphi(x,t) + b_0(r,t)\varphi(x,t),$

where $\partial_i^x = \partial_i$ and $\partial_{ij}^x = \partial_{ij}$ denote the first and the second partial derivatives with respect to the first variable x_i and x_i, x_j.

The assumptions on the coefficients are as follows. For some exponent $0 < \alpha < 1$ we have:

(A.1) the functions $\gamma(x,t,\zeta), \beta(x,t,\zeta)$ are continuous
for (x,t) in $\bar{\Omega} \times [0,1]$ and $\mathcal{E}$ – measurable in ζ, and
there exist, a $\mathcal{E}$ – measurable function $\gamma_0(\zeta)$
and a constant $C_0 > 0$ satisfying for every
x, x' in $\bar{\Omega}, t, t'$ in $[0,1]$
$0 < |\gamma(x,t,\zeta)| \leq \gamma_0(\zeta), \quad \int_E \gamma_0(\zeta)\pi(d\zeta) = C_0,$
$|\gamma(x,t,\zeta) - \gamma(x',t',\zeta)| \leq \gamma_0(\zeta)(|x-x'|^\alpha + |t-t'|^{\alpha/2}),$
$|\beta(x,t,\zeta) - \beta(x',t',\zeta)| \leq |x-x'|^\alpha + |t-t'|^{\alpha/2},$
$0 \leq \beta(x,t,\zeta) \leq 1$

(A.2) the function $\gamma(x,t,\zeta)$ is continuously differentiable
in x and there is a constant $C_0 > 0$ satisfying
for every x, x' $\Re^d, t, t'$ in $[0,1], \zeta$ in E,
$|x-x'| \leq C_0|(x-x') + t'(\gamma(x,t,\zeta) - \gamma(x',t,\zeta)]),$
$|\gamma(x,t,\zeta) - \gamma(x',t',\zeta)| \leq C_0(|x-x'| + |t-t'|^{\frac{1}{2}}),$

(A.3) for any (x,t,ζ) $\bar{\Omega} \times [0,1] \times E$ such that
$\beta(x,t,\zeta) \neq 0$, the segment $[x, x+\gamma(x,t,\zeta)]$
is included in $\bar{\Omega}$,

(A.4) there exist constants $C_0 \geq c_0 > 0$ satisfying for
every x, x' in $\bar{\Omega}, t, t'$ in $[0,1]$,
$\sum_{i=0}^{d} |a_i(x,t)| + \sum_{i,j=1}^{d} |a_{ij}(x,t)| \leq C_0$,

$$\sum_{i=0}^{d} |a_i(x,t) - a_i(x',t')| + \sum_{i,j=1}^{d} |a_{ij}(x,t) - a_{ij}(x',t')| \leq$$
$$\leq C_0(|x-x'|^\alpha + |t-t'|^{\alpha/2}),$$
$$\sum_{i,j=1}^{d} a_{ij}(x,t)\xi_i\xi_j \geq c_0|\xi|^2, \forall \xi = (\xi_1, \ldots \xi_d) \in \Re^d,$$

(A.5) there exist constants $C_0 \geq c_0 > 0$ satisfying

for every x, x' in $\partial\Omega, t, t'$ in $[0,1]$,

$$\sum_{i=0}^{d} |b_i(x,t)| \leq C_0,$$
$$\sum_{i=0}^{d} |b_i(x,t) - b_i(x',t')| \leq C_0(|x-x'| + |t-t'|^{1/2}, \text{ [1]}$$
$$\sum_{i=1}^{d} b_i(x,t)n_i(x) \geq c_0,$$

where $|\cdot|$ denotes the appropriate Euclidian norm either on $\Re$ or $\Re^d$ and $n(x) = (n_1(x), \ldots, n_d(x))$ is the outward unit normal to the boundary $\partial\Omega$ at the point x.

Notice that condition (A.2) implies that the change of variables $X = x_d + t\gamma(x,t,\zeta)$ is a diffeomorphism of class C^1 in $\Re^d$ for any fixed t, t' in $[0,1], \zeta$ in E. Assumption (A.3) states that the integral operator $I\varphi$ involves only the values of φ on $\bar{\Omega}$, moreover

$$\text{(5.5)} \quad I\varphi(x,t) = \int_0^1 d\theta \int_E \gamma(x,t,\zeta) \cdot \nabla\varphi(x + \theta\gamma(x,t,\zeta), t)\beta(x,t,\zeta)\pi(d\zeta),$$

where $\nabla\varphi$ denotes the gradient of φ with respect to x. Clearly, hypothesis (A. 1) ensures the meaning of the integral operator (5.1), (5.5) for all Lipschitz functions $\varphi(x,t)$ in x.

Let us denote by $\mathcal{G}_\alpha^k$, or $\mathcal{G}_\alpha^k(\bar{\Omega} \times [0,1], \Re^n)$ when necessary, $k \geq 0,\ n \geq 1,\ 0 < \alpha < 1$, the space of all continuous functions $\varphi(x,t,\xi,\tau)$ defined for x, ξ in $\bar{\Omega} \subset \Re^d$ and $0 \leq \tau < t \leq 1$, with values in $\Re^n$ and such that the following define (5.6),..., (5.16) are finite.

$$\text{(5.6)} \quad C(\varphi,k) = \inf\{c \geq 0 : |\varphi(x,t,\xi,\tau)| \leq C(t-s)^{-1+\frac{k-d}{2}},\ \forall x,t,\xi,\tau\},$$

$$\text{(5.7)} \quad K(\varphi,k) = \inf\{K \geq 0 : \int_\Omega [|\varphi(x,t,\xi,\tau)| + |\varphi(z,t,\xi,\tau)|]dz \leq$$
$$\leq K(t-s)^{1+\frac{k}{2}}, \forall x,t,\xi,\tau\},$$

$$\text{(5.8)} \quad M(\varphi,k,\alpha) = M_1(\varphi,k,\alpha) + M_2(\varphi,k,\alpha),$$

[1]If b_i are only Hölder continuous then a singularity on the boundary $\partial\Omega$ occurs for the second derivatives.

(5.9) $M_1(\varphi,k,\alpha)=\inf\{M_1\geq 0:|\varphi(x,t,\xi,\tau)-\varphi(x',t',\xi,\tau)|\leq$
$\leq M_1(|x-x'|^\alpha+|t-t'|^{\frac{\alpha}{2}})[(t-\tau)^{-1+\frac{k-d-\alpha}{2}}\wedge$
$\wedge(t'-\tau)^{-1+\frac{k-d-\alpha}{2}};\ \forall x,x',t,t',\xi,\tau\},$

(5.10) $M_2(\varphi,k,\alpha)=\inf\{M_2\geq 0:|\varphi(x,t,\xi,\tau)-\varphi(x,t,\xi',\tau')|\leq$
$\leq M_2(|\xi-\xi'|^\alpha+|\tau-\xi'|^{\frac{\alpha}{2}})(t-\tau)^{-1+\frac{k-d-\alpha}{2}}\wedge$
$\wedge(t-\tau')^{-1+\frac{k-d-\alpha}{2}},\ \forall x,t,\xi,\xi',\tau,\tau'\},$

(5.11) $N(\varphi,k,\alpha)=N_1(\varphi,k,\alpha)+N_2(\varphi,k,\alpha),$

(5.12) $N_1(\varphi,k,\alpha)=\inf\{N_1\geq 0:\int_\Omega[|\varphi(x,t,z,\tau)-\varphi(x',t',z,\tau)|+$
$+|\varphi(z,t,\xi,\tau)-\varphi(z,t',\xi,\tau)|]dz\leq$
$\leq N_1(|x-x'|^\alpha+|t-t'|^{\frac{\alpha}{2}})[(t-\tau)^{-1+\frac{k-\alpha}{2}}\wedge$
$\wedge(t'-\tau)^{-1+\frac{k-\alpha}{2}},\ \forall x,x',t,t',\xi,\tau\},$

(5.13) $N_2(\varphi,k,\alpha)=\inf\{N_2\geq 0:\int_\Omega[|\varphi(x,t,z,\tau)-\varphi(x,t,z,\tau')|+$
$+\varphi(z,t,\xi,\tau)-\varphi(z,t,\xi',\tau')|dz\leq$
$\leq N_2(|\xi-\xi'|^\alpha+|\tau-\tau'|^{\frac{\alpha}{2}})[(t-\tau)^{-1+\frac{k-\alpha}{2}}\wedge$
$\wedge(t-\tau')^{-1+\frac{k-\alpha}{2}},\ \forall x,t,\xi,\xi',\tau,\tau'\},$

(5.14) $R(\varphi,k,\alpha)=R_1(\varphi,k,\alpha)+R_2(\varphi,k,\alpha),$

(5.15) $R_1(\varphi,k,\alpha)=\inf\{R_1\geq 0:\int_\Omega|\varphi(Z,t,\xi,\tau)-\varphi(Z',t,\xi,\tau)|$
$J_\eta(Z,Z')dz\leq R_1\eta^\alpha(t-\tau)^{-1+\frac{k-\alpha}{2}},\ \forall Z,Z',t,\xi,\tau \text{ and } \eta>0\},$

(5.16) $R_2(\varphi,k,\alpha)=\inf\{R_2\geq 0:\int_\Omega|\varphi(x,t,Z,\tau)-\varphi(x,t,Z')\tau)|$
$J_\eta(Z,Z')dz\leq R_2\eta^\alpha(t-\tau)^{-1+\frac{k-\alpha}{2}},\ \forall x,t,Z,Z',\tau \text{ and } \eta>0\},$

where the change of variables $Z(z)$ and $Z'(z)$ are diffeomorphisms of class C^1 in $\Re^d$, and the Jacobian

(5.17) $$J_\eta(Z,Z')=\begin{cases}|\det(\nabla Z)|\wedge|\det(\nabla Z')| & \text{if } |Z-Z'|\leq\eta \text{ and } Z,Z'\in\bar\Omega,\\ 0 & \text{otherwise,}\end{cases}$$

det$(\cdot)$ means the determinant of a $d \times d$ matrix, $\nabla Z, \nabla Z'$ stand for the matrices of the first partial derivatives of $Z(z), Z'(z)$ with respect to the variable z, and $\wedge, \vee$ denote the minimum, maximum (resp.) between two real numbers.

Let $\varphi(x,t,\xi,\tau)$ be a Heat kernel type function, i.e. a continuous function defined for x, ξ in $\bar{\Omega} \subset \Re^d$ and $0 \le \tau < t \le 1$, and such that for every $x, x', t, t', \xi, \xi', \tau, \tau'$ we have

(5.18) $|\varphi(x,t,\xi,\tau)| \le \Lambda(t-\tau)^{-1+\frac{k-d}{2}} \exp[-\lambda|x-\xi|^2(t-\tau)^{-1}]$,

(5.19) $|\varphi(x,t,\xi,\tau) - \varphi(x',t',\xi,\tau)| \le \Theta(|x-x'|^\alpha + |t-t'|^{\frac{\alpha}{2}}) \times$
$\times \{(t-\tau)^{-1+\frac{k-d-\alpha}{2}} \exp[-\theta|x-\xi|^2(t-\tau)^{-1}] \vee$
$\vee (t'-\tau)^{-1+\frac{k-d-\alpha}{2}} \exp[-\theta|x'-\xi|^2(t'-\tau)^{-1}]\}$,

(5.20) $|\varphi(x,t,\xi,\tau) - \varphi(x,t,\xi',\tau')| \le \Theta(|\xi-\xi'|^\alpha + |\tau-\tau'|^{\frac{\alpha}{2}}) \times$
$\times \{(t-\tau)^{-1+\frac{k-d-\alpha}{2}} \exp[-\theta|x-\xi|^2(t-\tau)^{-1}] \vee$
$\vee (t-\tau')^{-1+\frac{k-d-\alpha}{2}} \exp[-\theta|x-\xi'|^2(t-\tau')^{-1}]\}$,

for some constants $\Lambda \ge \lambda > 0, \Theta \ge \theta > 0, 0 < \alpha < 1, k \ge 0$.

(5.21) $C(\varphi,k) \le \Lambda, \quad K(\varphi,k) \le 2\pi^{\frac{d}{2}} \Lambda \lambda^{-\frac{d}{2}}$,

Then $\varphi(x,t,y,s)$ belongs to the Green space $\mathcal{G}_\alpha^k$, i.e. the infima (5.6),...,(5.16) are finite. Moreover, we have the relation:

(5.22) $M(\varphi,k,\alpha) \le \Theta, \quad N(\varphi,k,\alpha) \le 2\pi^{\frac{d}{2}} \Theta \theta^{-\frac{d}{2}}$,

(5.23) $R(\varphi,k,\alpha) \le 2\pi^{\frac{d}{2}} \Theta \theta^{-\frac{d}{2}}$.

For a given function φ in $\mathcal{G}_\alpha^k$, we denote by $[\varphi]_{k,\alpha}$ the minimum of the values $C(\varphi,k)$, $K(\varphi,k)$, $M(\varphi,k,\alpha)$, $N(\varphi,k,\alpha)$, $R(\varphi,k,\alpha)$ defined by (5.6),..., (5.16). It is clear that $[\cdot]_{k,\alpha}$ provides a norm for $\mathcal{G}_\alpha^k$, which becomes a Banach space. Notice that from any sequence of functions $\{\varphi_n, n = 1,2,\ldots\}$ in $\mathcal{G}_\alpha^k$ such that

$$C(\varphi_n,k) \le C_0 \text{ and } M(\varphi_n,k,\alpha) \le M_0, \quad \forall n = 1,2,\ldots$$

we can extract an uniformly convergence subsequence on x, ξ belonging to $\bar{\Omega}$ and $0 \le \tau,\ t \le 1,\ t-\tau \ge \varepsilon$, for any fixed $\varepsilon > 0$. Then $\mathcal{G}_\alpha^k$ has a compact inclusion on the Frechet space C^0 of all continuous functions $\varphi(x,t,\xi,\tau)$ is x, ξ belonging to $\bar{\Omega}$, $0 \le \tau < t \le 1$.

If $p(x,t,\xi,\tau)$ is a continuous function, defined for x,ξ, in $\bar{\Omega}$ and $0 \le \tau < t \le 1$, such that for any $x,x',t,t',\xi,\xi',\tau,\tau'$,

(5.24) $|p(x,t,\xi,\tau)| \le p_0$,

(5.25) $|p(x,t,\xi,\tau) - p(x',t',\xi,\tau)| \le p_1(|x-x'|^\alpha + |t-t'|^{\frac{\alpha}{2}})\times$
$\times[(t-\tau)^{-\frac{\alpha}{2}} \vee (t'-\tau)^{-\frac{\alpha}{2}}]$,

(5.26) $|p(x,t,\xi,\tau) - p(x,t,\xi',\tau')| \le p_1(|\xi-\xi'|^\alpha + |\tau-\tau'|^{\frac{\alpha}{2}})\times$
$\times[(t-\tau)^{-\frac{\alpha}{2}} \vee (t-\tau')^{-\frac{\alpha}{2}}]$;

then the product by the function p on the space $\mathcal{G}_\alpha^k$ is a continuous operation, more precisely we have

(5.27) $C(p,\varphi,k) \le p_0 C(\varphi,k)$,

(5.28) $K(p\varphi,k) \le p_0 K(\varphi,k)$,

(5.29) $M(p\varphi,k,\alpha) \le p_0 M(\varphi,k,\alpha) + p_1 C(\varphi,k)$,

(5.30) $N(p\varphi,k,\alpha) \le p_0 N(\varphi,k,\alpha) + 2p_1 K(\varphi,k)$,

(5.31) $R(p\varphi,k,\alpha) \le p_0 R(\varphi,k,\alpha) + p_1 K(\varphi,k)$,

for any φ in G_α^k. □

Let $\psi(x,t,\xi,\tau)$ be a function belonging to $\mathcal{G}_\alpha^r$, for some fixed $0 < \alpha < 1$, $\alpha < r \le 2$,, i.e.

(5.32) $[\psi]_{r,\alpha} < \infty$,

the infima (5.6),..., (5.16) are finite for $\varphi = \psi$ and $k = r$. We consider the integral transformation

(5.33) $T\varphi(x,t,y,s) = \int_s^t d\tau \int_\Omega \psi(x,t,\xi,\tau)\varphi(\xi,\tau,y,s)d\xi$,

for φ in $\mathcal{G}_\alpha^k$.

Under the condition (5.32), the operator T maps the space $\mathcal{G}_\alpha^k$ into $\mathcal{G}_\alpha^{k+r}$, for any $k > \alpha$, more precisely we have the estimates

(5.34) $C(T\varphi, k+r) \leq 2^{d+2}(r^{-1}+k^{-1})[C(\psi,r)K(\varphi,k)+K(\psi,r)C(\varphi,k)]$,
for $0<k\leq d$,
$C(T\varphi,k+r) \leq \beta(\frac{r}{2},\frac{k-d}{2})K(\psi,r)C(\varphi,k)$, for $k>d$,

(5.35) $K(T\varphi,k+r) \leq \beta(\frac{r}{2},\frac{k}{2})K(\psi,r)K(\varphi,k)$, for $k>0$,

(5.36) $M(T\varphi,k+r) \leq 2^{d+2}[(r-\alpha)^{-1}+(k-\alpha)^{-1}[C(\psi,r)+K(\psi,r)+$
$+M(\psi,r,\alpha)+N_1(\psi,r,\alpha)][C(\varphi,k)+K(\varphi,k+M_2(\varphi,k,\alpha)+$
$N_2(\varphi_1,k,\alpha)]$, for $0<k<d-\alpha$,
$M(T,\varphi,k+r,\alpha) \leq \beta(\frac{r-\alpha}{2},\frac{k-d}{2}-\alpha)[N_1(\psi,r,\alpha)+K(\psi,r)]\times$
$\times[C(\varphi,k)+M_2(\varphi,k,\alpha)]$, for $k>d-\alpha$,

(5.37) $N(T\varphi,k+r,\alpha) \leq \beta(\frac{r-\alpha}{2},\frac{k-\alpha}{2})[N_1(\psi,r,\alpha)+K(\psi,r)]\times$
$\times[K(\varphi,k)+N_2(\varphi,k,\alpha)]$, for $k>\alpha$,

(5.38) $R(T\varphi,k+r+\alpha) \leq \beta(\frac{r-\alpha}{2},\frac{k-\alpha}{2})[R_1(\psi,r,\alpha)+K(\psi,r)]\times$
$\times[K(\varphi,k)+R_2(\varphi,k,\alpha)]$, for $k>\alpha$,

where $\beta(p,q)$ is the Beta function, i.e.

(5.39) $\beta(p,q)=\int_0^1(1-\theta)^{p-1}\theta^{q-1}d\theta,\ \ p,q>0.$

The operator (5.1) and (5.2) are always considered as acting on the first variable x in $\bar{\Omega}$. In view of the properties (5.24),...,(5.24), it is clear that the operator D maps $\mathcal{G}_\alpha^k$ into $\mathcal{G}_\alpha^{k-1}$. It is also possible to prove that there exists a constant $C_1>0$ such that

(5.40) $C(I\varphi,k) \leq C_1C(\nabla\varphi,k)$,

(5.41) $K(I\varphi,k) \leq C_1K(\nabla\varphi,k)$,

(5.42) $M(I\varphi,k) \leq C_1[C\nabla\varphi,k)+M(\nabla\varphi,k)]$,

(5.43) $N(I\varphi,k) \leq C_1[K(\nabla\varphi,k)+N(\nabla\varphi,k)+R(\nabla\varphi,k)]$,

(5.44) $R(I\varphi,k) \leq C_1[K(\nabla\varphi,k)+R(\nabla\varphi,k)]$, for any $k \geq 0$.

Let $G_0(x,t,\xi,\tau)$ be the Green function associated with the parabolic second order differential operator $\partial_t + A_0$ and the boundary operator B. As proved in Garroni and Solonnikov [11], such a function exists and satisfies the estimates (5.18),...,(5.19) for $\psi = G_0, k = 2$ and $\psi = \nabla G_0, k = 1$.

Theorem 1.

Under the assumptions (A.1),...,(A.5) the Volterra integral equation

$$(5.45)\quad G(x,t,\xi,\tau) = G_0(x,t,\xi,\tau) + \int_\tau^t ds \int_\Omega G_0(x,t,x,s)(I-D)G(z,s,\xi,\tau)ds$$

possesses one and only one solution G in the Green space $\mathcal{G}_\alpha^0$. Moreover we have

$$(5.46)\quad G(x,t,\xi,\tau) = \sum_{k=0}^{\infty} G_k(x,t,\xi,\tau),$$
$$G_k(x,t,\xi,\tau) = \int_\tau^t ds \int_\Omega G_0(x,t,x,s)(I-D)G_{k-1}(z,s,\xi,\tau)dz,$$

where the convergence is uniform over compact subsets of the domain for G and ∇G. Furthermore, G has the semigroup properties

$$(5.47)\quad G(x,t,\xi,\tau) = \int_\Omega G(x,t,z,s)G(s,z,\xi,\tau)dz, \forall s \in (\tau,t),$$

$$(5.48)\quad \int_\Omega G(x,t,\xi,\tau)\varphi(\xi)d\xi \to \varphi(x) \text{ as } (t-\tau)\to 0,$$
uniformly in x, for continuous function φ.

also, if $a_0 = 0, b_0 = 0$ then

$$(5.49)\quad \int_\Omega G(x,t,\xi,\tau)d\xi = 1. \quad \square$$

In order to estimate the second order derivatives we go back to the operator T given by (5.33). We assume that ψ satisfies (5.18),...,(5.20) for $\varphi = \psi, k = 0$, instead of (5.32), i.e. essentially $\psi = \nabla^2 G_0$. Also ψ satisfies the following "cancellation" properties:

$$(5.50)\quad \left|\int_\Omega \psi(x,t,z,\tau)dz\right| + \left|\int_\Omega \psi(z,t,\xi,\tau)dz\right| \leq K_0(t-\tau)^{\alpha/2-1},$$

$$(5.51)\ \left|\int_\Omega[\psi(x,t,z,\tau)-\psi(x',t',\tau)]dz\right|+$$
$$+\left|\int_\Omega[\psi(z,t,\xi,\tau)-\psi(z,t,\xi',\tau')]dz\right|\le$$
$$\le N_0(t-\tau)^{\varepsilon/2-1}(|x-x'|^\gamma+|t-t'|^{\gamma/2}+|\xi-\xi'|^\gamma+|\tau-\tau'|^{\gamma/2}),$$

for every $0\le\tau'\le\tau<t\le t'\le 1$, x,ξ in $\bar\Omega$, $\alpha=\gamma+\varepsilon$, $\gamma,\varepsilon>0$, and some suitable constants $K_0>0$, $N_0=N_0(\varepsilon)>0$. Under these conditions, there exist constants C_1, K_1, M_1, N_1 and R_1 depending on ψ, ε such that

$$(5.52)\ C(\nabla^2T\varphi,1)\le C_1[C(\varphi,1)+K(\varphi,1)+M(\varphi,1)],$$

$$(5.53)\ K(\nabla^2T\varphi,1)\le K_1[K(\varphi,1)+N(\varphi,1)+R(\varphi,1)],$$

$$(5.54)\ M(\nabla^2T\varphi,1,\gamma)\le M_1[C(\varphi,1)+K(\varphi,1)+M(\varphi,1,\alpha)+$$
$$+N(\varphi,1,\gamma)],$$

$$(5.55)\ N(\nabla^2T\varphi,1,\gamma)\le N_1[K(\varphi,1)+N(\varphi,1,\alpha)+R(\varphi,1,\alpha)],$$

$$(5.56)\ R(\nabla^2T\varphi,1,\gamma)\le R_1[K(\varphi,1)+N(\varphi,1,\alpha)+R(\varphi,1,\alpha)],$$

where $M(\cdot,\cdot,\gamma)$, $N(\cdot,\cdot,\gamma)$, $R(\cdot,\cdot,\gamma)$ denote the infima (5.8), (5.11), (5.14) with γ in lieu of α, and $\alpha=\gamma+\varepsilon$, $\gamma,\varepsilon>0$.

Because in the assumption (A.5) we took Lipschitz coefficients we have no singularity on the boundary for ∇^2G_0, ∂_tG_0 as proved in Garroni and Solonnikov [11]. Thus for $\psi=\nabla^2G_0$ or $\psi=\partial_tG_0$ we have (5.18),...,(5.20) with $\varphi=\psi$, $k=0$, as well as the cancellation properties (5.50) , (5.51).

In view of the equation (5.45) and the fact that $(I-D)G$ belongs to $\mathcal{G}^1_\alpha$, we deduce from the estimates (5.52),...,(5.56) that

$$(5.57)\ G=G_0+G^1,\quad G^1\in\mathcal{G}^3_\alpha,\ \nabla G^1\in\mathcal{G}^2_\alpha,\ \nabla^2G^1,\partial_tG^1\in\mathcal{G}^1_\gamma,$$

for any $0<\gamma<\alpha$. Hence, G is indeed a "classic" Green function since G^1 satisfies the equation

$$(5.58)\ \partial_tG^1(\cdot,\cdot,\xi,\tau)+AG^1(\cdot,\cdot,\xi,\tau)=F_n,\ \text{ in } \Omega\times(\tau,1],$$
$$G^1(t,\cdot,\xi,\tau)\to 0 \text{ as } t\to\tau, \text{ in } \Omega,$$
$$BG^1(\cdot,\cdot,\xi,\tau)=0 \text{ in } \partial\Omega\times(0,1],$$

for any fixed ξ,τ and $F_0(x,t,\xi,\tau)=(I-D)G_0(x,t,\xi,\tau)$. From this, we can obtain the representation formula

$$(5.59)\quad u(x,t)=\int_0^t d\tau\int_\Omega G(x,t,\xi,\tau)f(\xi,\tau)d\xi+\int_\Omega G(x,t,\xi,0)u_0(\xi)d\xi,$$

where u is the solution of the equation

$$(5.60)\quad \partial_t u-Au=f,\ \text{ in }\ \Omega\times(0,1]\,,$$
$$u(\cdot,0)=u_0,\ \text{ in }\ \Omega,\quad Bu=0,\ \text{ on }\ \partial\Omega\times(0,1],$$

in the appropriated sense according to the degree of regularity of the data $f,\ u_0$.

When the coefficients of the boundary operator B are only Hölder continuous, we need some more analysis (cf. Garroni and Menaldi [10]).

These results are extended to boundary conditions of Dirichlet type, i.e. "$BG=0$" is replaced by "$G=0$" on the boundary $\partial\Omega\times(0,1]$. It is clear that any interval $(0,T]$ may be used in lieu of $(0,1]$.

6. Asymptotic Results

Let Ω be a bounded open subset of $\Re^d$ having smooth boundary $\partial\Omega$. In this section we assume that the coefficients of the operators A and B are independent of $t\geq 0$, i.e.

$$(6.1)\quad A\varphi(x)=-\sum_{i,j=1}^d a_{ij}(x)\partial^x_{ij}\varphi(x)+\sum_{i=1}^d a_i(x)\partial^x_i\varphi(x)-$$
$$-\int_E[\varphi(x+\gamma(x,\zeta))-\varphi(x)]\beta(x,\zeta)\pi(d\xi),$$

$$(6.2)\quad B\varphi(x)=\sum_{i=1}^d b_i(x)\partial^x_i\varphi(x),$$

where for some exponent $0<\alpha<1$ the assumptions are:

(B.1) the function $\gamma(x,\zeta),\beta(x,\zeta)$ are continuous for x in $\bar\Omega$ and $\mathcal{E}-$measurable in ζ, and there exist a $\mathcal{E}-$measurable function $\gamma_0(\zeta)$ and a constant $C_0>0$ satisfying for every x,x' in $\bar\Omega$

$$0<|\gamma(x,\zeta)|\leq\gamma_0(\zeta),\quad \int_E\gamma_0(\zeta)\pi(d\zeta)=C_0,$$
$$|\gamma(x,\zeta)-\gamma(x',\zeta)|\leq\gamma_0(\zeta)|x-x'|^\alpha,$$
$$|\beta(x,\zeta)-\beta(x',\zeta)|\leq|x-x'|^\alpha,\quad 0\leq\beta(x,\zeta)\leq 1,$$

(B.2) the function $\gamma(x,\zeta)$ is continuously differentiable

in x and there is a constant $C_0 > 0$

satisfying for every x, x' in $\bar{\Omega}$, ζ in E, $0 \leq \theta \leq 1$

$|x - x'| \leq C_0|(x, -x') + \theta[\gamma|(x,\zeta) - \gamma(x',\zeta)]|$,

$|\gamma(x,\zeta) - \gamma(x',\zeta)| \leq C_0|x - x'|$,

(B.3) for any (x,ζ) in $\bar{\Omega} \times E$ such that $\beta(x,\zeta) \neq 0$, the

segment $[x, x + \gamma(x,\zeta)]$ is included in $\bar{\Omega}$,

(B.4) there exist constants $C_0 \geq c_0 > 0$ satisfying for every x, x' in $\bar{\Omega}$

$$\sum_{i=1}^{d} |a_i(x)| + \sum_{i,j=1}^{d} |a_{ij}(x)| \leq C_0,$$

$$\sum_{i=1}^{d} |a_i(x) - a_i(x') + \sum_{i,j=1}^{d} |a_{ij}(x) - a_{ij}(x')| \leq C_0|x - x'|^{\alpha},$$

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geq c_0|\xi|^2, \forall \xi = (\xi_1, \ldots, \xi_d) \in \Re^d,$$

(B.5) there exist constants $C_0 \geq c_0 > 0$ satisfying for

every x, x' in $\partial\Omega$

$$\sum_{i=1}^{d} |b_i(x)| \leq C_0, \quad \sum_{i=1}^{d} b_i(x) n_i(x) \geq c_0,$$

$$\sum_{i=1}^{d} |b_i(x) - b_i(x')| \leq C_0|x - x'|,$$

where $|\cdot|$ denotes the appropriate Euclidian norm either in $\Re$ or $\Re^d$ and $n(x) = (n_1(x), \ldots, n_d(x))$ is the outward unit normal to the boundary $\partial\Omega$ at the point x.

We want to consider the following three problems:

(6.3) $\partial_t u + Au = f$ in $\Omega \times (0,\infty)$,

$u(\cdot,0) =$ in Ω, $Bu = 0$ on $\partial\Omega \times [0,\infty)$,

(6.4) $Au_\lambda + \lambda u_\lambda = g$ in Ω, $Bu_\lambda = 0$ on $\partial\Omega$

and

(6.5) $Au_0 = g$ in Ω, $Bu_0 = 0$ on $\partial\Omega$

where ∂_t denotes the partial derivative with respect to the second variable, say t, the functions $f = f(x,t)$, $g = g(x)$, x in $\bar{\Omega}$, t in $[0,\infty)$ and the constant $\lambda > 0$ are given. We wish to know under which circumstances the problem (6.5) possesses one or more solutions and when we have

$$\text{(6.6)} \quad u(\cdot,t) \to u_0 \text{ as } t \to \infty, \text{ and } u_\lambda \to u_0 \text{ as } \lambda \to 0.$$

It is clear that these questions lead to the study of the invariant measures of the homogeneous semigroup generated by the integro-differential operator (6.1) and the differential boundary operator (6.2).

By means of the previous section (cf. Garroni and Menaldi [10]) we can construct the Green function $G(x,t,\xi), x, y$ in $\bar{\Omega}, t > 0$, which solves the problem

$$\begin{aligned}\text{(6.7)} \quad & \partial_t G(x,t,\xi) + A_x G(x,t,\xi) = \delta(x-\xi)\delta(t) \text{ in } \Omega \times (0,\infty),\\ & B_x G(x,t,\xi) = 0 \text{ on } \partial\Omega \times [0,\infty),\\ & G(x,t,\xi) \to \delta(x-\xi) \text{ as } t \to 0,\end{aligned}$$

in an appropriate sense. Such a function admits the representation

$$\text{(6.8)} \quad G(x,t,\xi) = G_0(x,t,\xi) + G^1(x,t,\xi), \ \forall x, \xi \in \bar{\Omega}, t > 0,$$

where G_0 is the Green function corresponding to the principal part of A (the second order differential terms) and the boundary operator B. The function G^1 belongs to $\mathcal{G}^3_\alpha$, ∇G^1 belongs to $\mathcal{G}^2_\alpha$ and $\nabla^2 G^1$, $\partial_t G^1$ belongs to $\mathcal{G}^1_\gamma$ for any $0 < \gamma < \alpha$, but only locally in t, i.e. valid on every bounded set $\bar{\Omega} \times [0,T]$, for any fixed $T > 0$.

From the results in Garroni and Solonnikov [11], we deduce that for $i = 0,1,2$,

$$\text{(6.9)} \quad |\nabla^i G_0(x,t,\xi)| \le C_0 (t \wedge 1)^{-\frac{d+i}{2}} \exp(-c_0 \frac{|x-\xi|^2}{t}),$$

$$\begin{aligned}\text{(6.10)} \quad & |\nabla^i G_0(x,t,\xi) - \nabla^i G_0(x',t',\xi')| \le M_0(|x-x'|^\alpha + |t-t'|^{\alpha/2} +\\ & +|\xi-\xi'|^\alpha)[(t\wedge 1)^{-\frac{d+\alpha+i}{2}} \exp(-m_0 \frac{|x-\xi|^2}{t}) +\\ & +(t'\wedge 1)^{-\frac{d+\alpha+i}{2}} \exp(-m_0 \frac{|x'-\xi'|^2}{t'}),\end{aligned}$$

for every x, x', ξ, ξ' in $\bar{\Omega}$, t, t' in $(0,\infty)$ and some constants $C_0, c_0, M_0, m_0 > 0$. Moreover, Calderon and Zygmund type estimate hold, i.e. for any $1 < p < \infty$ there exists a constant

C_p depending only on Γ_0 and p, but not on $T \geq 0$ such that for any measurable function $f(\xi,\tau) \geq 0, \xi$ in $\Omega, \tau \geq 0$ we have

$$\text{(6.11)} \quad \int_0^T dt \int_\Omega |\int_0^t d\tau \int_\Omega \nabla^2 G_0(x, t-\tau, \xi) f(\xi,\tau) d\xi|^p dx \leq \\ \leq C_p \int_0^T d\tau \int_\Omega |f(\xi,\tau)|^p d\xi,$$

where the hessian ∇^2 can be replaced by ∂_t in (6.9), (6.10), (6.11), and $\wedge$ denotes the minimum between real numbers.

Let us redefine the function spaces $\mathcal{G}_\alpha^k$, or $\mathcal{G}_\alpha^k(\bar\Omega \times [0,\infty), \Re^n)$ when necessary, $k \geq 0$, $n \geq 1$, $0 < \alpha < 1$, as the space of all continuous functions $\varphi(x,t,\xi)$ defined for x, ξ in $\bar\Omega$, $t > 0$, with values in $\Re^n$, and such that the following infima (6.12),...,(6.16) are finite.

$$\text{(6.12)} \quad C(\varphi,k) = \inf\{C \geq 0 : |\varphi(x,t,\xi)| \leq C(t \wedge 1)^{-1+\frac{k-d}{2}}, \forall x,t,\xi\}$$

$$\text{(6.13)} \quad K(\varphi,k) = \inf\{K \geq 0 \int_\Omega [|\varphi(x,t,z)| + |\varphi(z,t,\xi)| dz \leq \\ \leq K(t \wedge 1)^{-1+\frac{k}{2}}, \ \forall x,t,\xi\},$$

$$\text{(6.14)} \quad M(\varphi,k,\alpha) = \inf\{M \geq 0 : |\varphi(x,t,\xi) - \varphi(x',t',\xi')| \leq \\ \leq M(|x-x'|^\alpha + |t-t'|^{\frac{\alpha}{2}} + |\xi - \xi'|^\alpha) \times \\ \times [(t \wedge 1)^{-1+\frac{k-d-\alpha}{2}} + (t' \wedge 1)^{-1+\frac{k-d-\alpha}{2}}, \ \forall x,x',t,t',\xi,\xi'\}$$

$$\text{(6.15)} \quad N(\varphi,k,\alpha) = \inf\{N \geq 0 : \int_\Omega [|\varphi(x,t,z) - \varphi(x',t',z)| + \\ + |\varphi(z,t,\xi) - \varphi(z,t',\xi')|] \leq N(|x-x'|^\alpha + |t-t'|^{\frac{\alpha}{2}} + \\ + |\xi-\xi'|^\alpha)[(t \wedge 1)^{-1+\frac{k-\alpha}{2}} + (t' \wedge 1)^{-1+\frac{k-\alpha}{2}}], \ \forall x,x',t,t',\xi,\xi'\},$$

$$\text{(6.16)} \quad R(\varphi,k,\alpha) = \inf\{R \geq 0 : \int_\Omega [|\varphi(Z,t,\xi) - \varphi(Z',t,\xi)| + \\ + |\varphi(x,t,Z) - \varphi(x,t,Z')|] J_\eta(Z,Z') dz \leq \\ \leq R\eta^\alpha (t \wedge 1)^{-1+\frac{k-\alpha}{2}}, \ \forall x,t,Z,Z' \text{ and } \eta > 0\}$$

where the change of variables $Z(z)$ and $Z'(z)$ are diffeomorphisms of class C^1 in $\Re^d$, and the Jacobian

$$\text{(6.17)} \quad J_\eta(Z,Z') = \begin{cases} |\det(\nabla Z)| \wedge |\det(\nabla Z') & \text{if } |Z - Z'| \leq \eta \text{ and } Z, Z' \in \bar\Omega, \\ 0 & \text{otherwise.} \end{cases}$$

Based on the semigroup properties

(6.18) $G(x,t+s,\xi) = \int_\Omega G(x,t,z)G(z,s,\xi)dz$,
for every x, y in $\bar\Omega, t, s > 0$,

(6.19) $\varphi(x) = \lim_{t\to 0} \int_\Omega G(x,t,\xi)\varphi(\xi)d\xi$,
uniformly in x belonging to $\bar\Omega$ and for any continuous function φ in $\bar\Omega$,

(6.20) $\int_\Omega G(x,t,\xi)d\xi = 1$, for every x in $\bar\Omega, t > 0$,

we can prove the

Theorem 6.1

Under the assumption (B.1),...,(B.5) the function G^1 in (6.8) is such that

(6.21) $G^1 \in \mathcal{G}^3_\alpha,\ \nabla G^1 \in \mathcal{G}^2_\alpha,\ \nabla^2 G^1,\ \partial_t G^1 \in \mathcal{G}^1_\gamma,\ \forall\gamma < \alpha$,

i.e. the infima $C(\nabla^i G^1, 3-i)$, $K(\nabla^i G^1, 3-i)$, $M(\nabla^i G^1, 3-i, \alpha)$, $N(\nabla^i G^1, 3-i, \alpha)$, $R(\nabla^i G^1, 3-i, \alpha), i = 0, 1$, and $C(\nabla^2 G^1, 1)$, $C(\partial_t G^1, 1)$, $K(\nabla^2 G^1, 1)$, $K(\partial, G^1, 1)$, $M(\nabla^2 G^1, 1, \gamma)$, $M(\partial_t, G^1, 1, \gamma)$, $N(\nabla^2 G^1, 1, \gamma)$, $N(\partial_t G^1, 1, \gamma)$, $R(\nabla^2 G^1, 1, \gamma)$, $R(\partial_t G^1, 1, \gamma)$, $0 < \gamma < \alpha$, are finite. The Green function (6.8) is positive, precisely, there exist two constant $0 < \eta, \delta \le 1$ such that

(6.22) $G(x,t,\xi) \ge \eta > 0,\ \forall x, \xi \in \bar\Omega,\ t \ge \delta > 0.$

If the oblique boundary condition given by the operator B are changed into homogeneous Dirichlet boundary conditions then (6.22) holds only for x in Ω, dist $(x, \partial\Omega) \ge \varepsilon > 0$, for any $\varepsilon > 0$. □

Therefore, the solution of the parabolic equation (6.3) is given by

(6.23) $u(x,t) = \int_0^t d\tau \int_\Omega G(x, t-\tau, \xi) f(\xi, \tau) d\xi$,

according to the previous section. Then, for (6.4) we should purpose

(6.24) $u_\lambda(x) = \int_0^\infty e^{-\lambda t} dt \int_\Omega G(x,t,\xi) g(\xi) d\xi,\ \ \lambda > 0.$

Let us denote by

(6.25) $[f]_{\alpha,T} = \inf\{C \geq 0 : |f(x,t) - f(x',t')| \leq C(|x-x'|^{\alpha} + |t-t'|^{\alpha/2})$
and $|f(x,t)| \leq C, \forall x, x' \in \bar{\Omega}, t, t' \in [0,T]\}$,

(6.26) $[g]_{\alpha} = \inf\{C \geq 0 : |g(x) - g(x')| \leq C|x-x'|^{\alpha}$
and $|g(x)| \leq C, \ \forall x, x' \in \bar{\Omega}\}$,

(6.27) $\|f\|_{p,T} = (\int_0^T dt \int_{\Omega} |f(x,t)|^p dx)^{1/p}$,

(6.28) $\|g\|_p = (\int_{\Omega} |g(x)|^p dx)^{1/p}$

and

(6.29) $\|f\|^{*}_{p,T} = (\int_0^{T\wedge 1} dt \int_{\Omega} |f(x,t)|^p dx)^{1/p} +$
$+ \sup\{(\int_0^1 ds \int_{\Omega} |f(x,t-s)|^p dx)^{1/p} \ : \ 1 < t \leq T\}$,

for any $t > 0$ and $1 \leq p < \infty$.

Theorem 6.2

Let the assumptions (B. 1),...,(B.5) hold. Then the function (6.23) and (6.24) are the solutions of the equations (6.3) and (6.4) respectively. Either classic or weak solution, according to the hypotheses made on f and g. Moreover, we have the following estimates:

(6.30) $(T\wedge 1)^{-2}[u]_{\alpha,T} + (T\wedge 1)^{-1}[\nabla u]_{\alpha,T} \leq C([f]_{0,T} + \|f\|_{1,T})$,

(6.31) $[\nabla^2 u]_{\gamma,T} + [\partial_t, u]_{\gamma,T} \leq C_{\gamma}([f]_{\alpha,T} + \|f\|_{1,T})$,

(6.32) $[u_{\lambda}]_{\alpha} + |\nabla u_{\lambda}]_{\alpha} \leq C([g]_0 + \lambda^{-1}\|g\|_1$,

(6.33) $[\nabla^2 u_{\lambda}]_{\gamma} \leq C_{\gamma}([g]_{\alpha} + \lambda^{-1}\|g\|_1)$,

for any $0 < \gamma < \alpha$, $t > 0$, $1 \geq \lambda > 0$ and some constants C, C_{γ} independent of f, T and g, λ;

(6.34) $(T\wedge 1)^{-2}\|u(\cdot,T)\|_p + (T\wedge 1)^{-1}\|\nabla u(\cdot,T)\|_p \leq C_p(\|f\|^{*}_{p,T} + \|f\|_{1,T})$,

(6.35) $\|\nabla^2 u\|_{p,T} + \|\partial_t u\|_{p,T} \leq M_p(\|f\|^{*}_{p,T} + \|f\|_{1,T})$, (6.36) $\|u_\lambda\|_p + \|\nabla u_\lambda\|_p \leq C_p(\|g\|_p + \lambda^{-1}\|g\|_1)$,

(6.37) $\|\nabla^2 u_\lambda\|_p \leq M_p(\|g\|_p + \lambda^{-1}\|g\|_1)$,

where the constants C_p, M_p are independent of f, T and g, λ; and

(6.38) $\eta\|f\|_{1,T} \leq |u(x,T)| \leq T[f]_{0,T}, \ \forall T \geq 1$,

(6.39) $\eta\|g\|_1 \leq \lambda|u_\lambda(x)| \leq [g]_0, \ \forall 0 < \lambda \leq 1$,

for any x in $\bar{\Omega}$ and some constant $\eta > 0$ independent of f, t and g, λ. □

Let us now present a result relative to the adjoint equation

(6.40) $$A^*m = 0 \text{ in } \Omega, \quad B^*m = 0 \text{ on } \partial\Omega,$$
$$m > 0, \quad \int_\Omega m(x)dx = 1,$$

which is actually *not defined* (cf. Bensoussan et al. [4]).

Theorem 6.3

Let assumptions (B. 1),...,(B.5) hold. Then there exists a unique Hölder continuous function $m(\cdot)$ in $\bar{\Omega}$, with exponent α, such that

(6.41) $$m(\xi) = \lambda \int_0^\infty e^{-\lambda t}dt \int_\Omega G(x,t,\xi)m(x)dx,$$
$$\int_\Omega m(x)dx = 1, \quad \forall \xi \in \bar{\Omega}, \lambda > 0.$$

Moreover, there exists a positive constant ν such that

(6.42) $m(\xi) \geq \nu > 0, \ \forall \xi \in \bar{\Omega}$,

and for every $t \geq 1$, g in $L^1(\Omega)$, x in $\bar{\Omega}$,

(6.43) $$\left|\int_\Omega G(x,t,\xi)g(\xi)d\xi - \int_\Omega g(\xi)m(\xi)d\xi\right| \leq C\exp(-\nu t)\|g\|_1,$$

for some constant C independent of t, g, x. Further more,

(6.44) $m(\xi) = \int_\Omega G(x,t,\xi)m(x)dx, \ \forall \xi \in \bar{\Omega}, t > 0.$ □

Theorem 6.4

Under the assumptions (B.1),...,(B.5) the limit problem (6.5) possesses a solution unique up to an additive constant if and only if

(6.45) $\int_\Omega g(x)m(x)dx = 0,$

where $m(\cdot)$ is the function defined in Theorem 6.3. Moreover, the solution of problem (6.5), satisfying (6.45) with u_0 instead of g, admits the representation

(6.46) $u_0(x) = \int_0^\infty dt \int_\Omega G(x,t,\xi)g(\xi)d\xi, \ \forall x \in \bar{\Omega},$

and

(6.47) $u_0(x) = \lim_{\lambda\to 0} \int_0^\infty e^{-\lambda t}dt \int_\Omega G(x,t,\xi)g(\xi)d\xi, \forall x \in \bar{\Omega}.$

Furthermore, the class of functions to which $u_0(\cdot)$ belongs is determinate by the following estimates according to how smooth is the data $g(\cdot)$,

(6.48) $[u_0]_\alpha + [\nabla u_0]_\alpha \le C[g]_0,$

(6.49) $[\nabla^2 u_0]_\alpha \le C_\gamma [g]_\alpha, \ \forall 0 < \gamma < \alpha,$

(6.50) $\|u_0\|_p + \|\nabla u_0\|_p \le C_p \|g\|_p,$

(6.51) $\|\nabla u_0\|_p \le M_p \|g\|_p, \ \forall 1 < p < \infty,$

where the constants C, C_γ, C_p, M_p are independent of g. □

Instead of the problem (6.5) we consider:

(6.52) find a function $v = v(x)$ and a real number π such that
$$Av + \pi = g \text{ in } \Omega, \quad Bv = 0 \text{ on } \partial\Omega.$$

If u is an integrable function in Ω then we set

(6.53) $m(u) = \int_\Omega u(x)m(x)dx,$

where $m = m(x)$ is the invariant density function given in Theorem 6.3 by (6.41), (6.44).

Theorem 6.5

Let the assumptions (B.1),...,(B.5) and

(6.54) $f(\cdot,t) \to g$ in $L^1(\Omega)$ as $t \to \infty$

hold true. Then the problem (6.52) possesses a solution (v,π) with $\pi = m(g)$ and v unique up to an additive constant, in the class of functions determinate by the estimates (6.48),..,(6.51) for v instead of u_0. Moreover, denoting by $u(\cdot,t)$ and u_λ the solutions of problems (6.3), (6.4), we have

(6.55) $u_\lambda - m(u_\lambda) = w_\lambda \quad \to \quad v_0 = v - m(v)$ as $\lambda \to 0$,

and

(6.56) $u(\cdot,t) - m(u(\cdot,t)) = w(\cdot,t) \quad \to v_0$ as $t \to \infty$,

where the convergence is in the topology derived from the estimates (6.48),...,(6.51) with w_λ in lieu of u_0, uniformly in $0 < \lambda \leq 1$, and

(6.57) $[w(\cdot,t)]_\alpha + [\nabla w(\cdot,t)]_\alpha \leq C \sup\{[f(\cdot,s)]_0 : 0 \leq s \leq t\}$,

(6.58) $[\nabla^2 w(\cdot,t)]_\gamma \leq C_\gamma \sup\{[f(\cdot,s)]_\alpha 0 \leq s \leq t\}$,

for any $t \geq 1, 0 < \gamma < \alpha$ and some constants C, C_γ independent of f and t,

(6.59) $$\|w\cdot,t)\|_p + \|\nabla w(\cdot,t)\|_p + \|\nabla^2 w(\cdot,t)\|_p \leq$$
$$\leq C_p(\|f\|^*_{p,t} + \sup\{\|f(\cdot,s)\|_1 : 0 \leq s \leq t\}),$$

where the constant C_p is independent of f and t, for any $1 < p < \infty$. ◫

The proof of all the above results can be found in Garroni and Menaldi [10].

Notice that as a consequence of the results for the parabolic problem we obtain similar results for the elliptic equation.

The key point in the above theorems is the use of the estimates on the Green function to deduce a priori estimates for the solution of the second order integro-differential equations.

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