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On Some Reachability Problems for Diffusion Processes

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Abstract

The main purpose of this paper is to discuss the minimization of energy spent in order that a controlled diffusion process reaches a given target, a d-dimensional bounded domain. The exterior Dirichlet problem for the Hamilton-Jacobi-Bellman equation is studied for a class of criteria which includes the case of energy. Extensions to diffusion with jumps, examples and some other reachability problems are considered.

1 Introduction

Our purpose here is to study some reachability problems for diffusion processes. Indeed, denote by $x(t)$ a diffusion process in $\mathbb{R}^d$, by $v(t)$ a control acting in the drift term of the state equation, by D a bounded open subset of $\mathbb{R}^d$ and by τ the first time $x(t)$ reaches D, i.e., $\tau = \inf\{t \geq 0 : x(t) \in D\}$. The main problem we will study is the *minimum energy reachability*, namely, the minimization of

$$J_x(v) = E_x^v\{\int_0^\tau |v(t)|^2 \mathrm{d}t\}, \tag{1.1}$$

under the constraint

$$E_x^v\{\tau\} < \infty, \tag{1.2}$$

and assuming that there exists a control satisfying (1.2). Actually, (1.1) will be treated via a simpler problem

$$J_x^\varepsilon(v) = E_x^v\{\int_0^\tau \big(|v(t)|^2 + \varepsilon\big)\mathrm{d}t\}, \tag{1.3}$$

where $E_x^v\{\cdot\} = E^v\{\cdot \mid x(0) = x\}$. As mentioned later, more general critera can be considered. The condition $E_x\{\tau\} < \infty$ is related to the recurrence of the diffusion process. This recurrence property has been studied in Bensoussan [2] and Khasminskii [3] (among other) for continuous diffusion, and in Menaldi and Robin [6, 7] for diffusion with jumps. The existence of v such that $E_x\{\tau\} < \infty$ is also a strong controllability

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condition and we refer to Arapostathis et al. [1], Zabczyk [10] and the references therein for the controllability aspect.

The paper is organized as follows: Section 2 deals with the problem (1.1) above and the related Hamilton-Jacobi-Bellman equation. Examples are given in Section 3 and extensions to diffusion processes with jumps are briefly described in Section 4. Finally, Section 5 contains a few other control problems related to reachability.

2 Minimum Energy Problem

2.1 Assumptions

Let V be a compact metric space, and set $\Omega = C([0,\infty),\mathbb{R}^d)$ the canonical space, $x(t,\omega) = \omega(t)$ the canonical process and $F_t = \sigma(x(s) : s \le t)$, $F = F_\infty$ the filtration used in probability. Let $a(x) = [a_{ij}(x)]$ be a symmetric matrix for each x such that

$$a_{ij} \in W^{1,\infty}(\mathbb{R}^d), \qquad c_0 \le \sum_{ij} a_{ij}(x)\xi_i\xi_j \le c_0^{-1}|\xi|^2, \qquad \forall\, x,\xi \in \mathbb{R}^d. \tag{2.1}$$

Let $g(x,v) : R^d \times V \to \mathbb{R}^d$ and $f(x,v) : R^d \times V \to \mathbb{R}$ be Borel functions, continuous in v and such that

$$|g(x,v)| \le C_1(1+|x|), \qquad \forall\, x \in \mathbb{R}^d,\ v \in V, \tag{2.2}$$

$$0 \le f(x,v) \le C_2, \qquad \forall\, x \in \mathbb{R}^d,\ v \in V. \tag{2.3}$$

A control $v(t)$ is a F_t–adapted process with values in V. Under the above assumptions (e.g. see Stroock and Varadhan [9]), for each control v there is a unique probability P_x^v such that for any φ in $C_b^2(\mathbb{R}^d)$, the process

$$\varphi_t = \varphi(x(t)) - \varphi(x) - \int_0^t \nabla\varphi(x(s))\cdot g(x(s),v(s))\mathrm{d}s - \int_0^t A\varphi(x(s))\mathrm{d}s, \tag{2.4}$$

is a (P_x^v, F_t)–martingale, where ∇ is the gradient operator and

$$A\varphi(x) = \frac{1}{2}\sum_{ij} a_{ij}(x)\frac{\partial^2\varphi}{\partial x_i\partial x_j}(x).$$

Let D be a smooth bounded domain in $\mathbb{R}^d$, and

$$\tau = \inf\{t \ge 0 : x(t) \in D\}. \tag{2.5}$$

We then consider the first problem **(P1)**: to minimize

$$J_x(v) = E_x\Big\{\int_0^\tau \big(f(x(t),v(t))+1\big)\mathrm{d}t\Big\}, \tag{2.6}$$

over $\mathcal{V}$ the set of control processes such that $E_x^v\{\tau\} < \infty$. Thus, we denote by u the optimal cost function, i.e.,

$$u(x) = \inf\{J_x(v) : v \in \mathcal{V}\}. \tag{2.7}$$

We will use the additional assumption: there exists a measurable feedback $v_0 = v_0(x)$ (i.e., a Borel measurable function from $\mathbb{R}^d$ into V) with a corresponding sub-solution u_0, i.e., a function satisfying

$$\begin{cases} 0 \le u_0(x) \le C_0(1+|x|), \quad \forall x \in \mathbb{R}^d, & \lim_{|x|\to\infty} u_0(x) = +\infty, \\ u_0 \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d), \qquad L(v_0)u_0 + 1 \le 0 & \text{in } \mathbb{R}^d \setminus D, \end{cases} \tag{2.8}$$

where

$$L(v)\varphi = \nabla\varphi \cdot g(\cdot, v) + A\varphi. \tag{2.9}$$

From [7] one can see that assumption (2.8) ensures $E^v_x\{\tau\} < \infty$, and in fact the finite expectation of the reaching time of any bounded open set. In the sense of Zabczyk [10], the system corresponding to (2.4) is strongly controllable.

By means of problems of the type (P1), we will study the problem **(P2)**: to minimize

$$J_x(v) = E_x\Big\{ \int_0^\tau f(x(t), v(t))\mathrm{d}t \Big\}, \tag{2.10}$$

over $\mathcal{V}$.

Notice that we are interested in the case where $f(x, 0) \equiv 0$, so if the process corresponding to the constant feedback $v(x) = 0$ belongs to V (i.e., it satisfies $E^0_x\{\tau\} < \infty$), then $v = 0$ is optimal for (2.10). Also, if $f = 0$ then P1 is the minimum time problem. Finally, since V is bounded, the minimum energy problem is a particular case of (2.10).

2.2 HJB Equation for P1

Let us first state a result on the exterior Dirichlet problem derived form assumption (2.8).

Proposition 2.1. *Let the assumptions of Section 2.1 hold, and let h be in $L^\infty(\mathbb{R}^d \setminus D)$. Then the exterior Dirichlet problem*

$$L(v_0)\bar{u} + h = 0 \quad \text{in} \quad \mathbb{R}^d \setminus D, \qquad \bar{u} = 0 \quad \text{on} \quad \partial D, \tag{2.11}$$

has a unique solution $\bar{u}$ in $W^{2,p}_{loc}(\mathbb{R}^d \setminus D)$ for any $p < \infty$ and such that $\bar{u}/u_0$ is bounded, and

$$\bar{u}(x) = E^{v_0}_x\Big\{ \int_0^\tau h(x(t))\mathrm{d}t \Big\}, \tag{2.12}$$

for any x in $\mathbb{R}^d \setminus D$. □

This is an extension of a result in Bensoussan [2] to unbounded g. Notice that condition (2.8) implies

$$E^{v_0}_x\{\tau\} < \infty. \tag{2.13}$$

The HJB equation for (2.7) is then

$$Au + \inf_{v\in V}\{\nabla u \cdot g(\cdot, v) + f(\cdot, v)\} + 1 = 0 \quad \text{in } \mathbb{R}^d \setminus D, \qquad u = 0 \quad \text{in } \partial D. \tag{2.14}$$

Theorem 2.2. *Let the assumptions of Section 2.1 hold. Then* (2.14) *has a unique solution u in $W^{2,p}_{loc}(\mathbb{R}^d \setminus D)$ for any $p < \infty$ and such that u/u_0 is bounded. Any measurable selection*

$$\hat{v}(x) \in Arg\min_{v \in V}\{\nabla u(x) \cdot g(x,v) + f(x,v)\}$$

is an optimal feedback control and

$$u(x) = \inf\{J_x(v) : v \in \mathcal{V}\},$$

for any x in $\mathbb{R}^d \setminus D$.

Proof. We use the policy iteration method based on Proposition 2.1. Let $v_0(x)$ and $u_0(x)$ as in assumption (2.8). Define u_1 as the solution of the linear equation

$$L(v_0)u_1 + f(\cdot, v_0) + 1 = 0 \quad \text{in} \quad \mathbb{R}^d \setminus D, \qquad u_1 = 0 \quad \text{on} \quad \partial D, \tag{2.15}$$

which has a solution according to Proposition 2.1. Then, let $v_1(x)$ be defined as

$$v_1(x) \in \operatorname{Arg}\min_{V}\{\nabla u_1(x) \cdot g(x,v) + f(x,v)\}, \quad \forall x \in \mathbb{R}^d. \tag{2.16}$$

By definition

$$\nabla u_1 \cdot g(\cdot, v_1) + f(\cdot, v_1) \leq \nabla u_1 \cdot g(\cdot, v_0) + f(\cdot, v_0)$$

and therefore

$$L(v_1)u_1 + f(\cdot, v_1) + 1 \leq L(v_0)u_1 + f(\cdot, v_0) + 1 = 0$$

i.e.,

$$L(v_1)u_1 + 1 \leq 0, \quad \text{in } \mathbb{R}^d \setminus D,$$

which means that u_1 is a subsolution for v_1, i.e, condition (2.8) is satisfied with u_0, v_0 replaced by u_1, v_1. Therefore, again by means of Proposition 2.1, v_1 belongs to $\mathcal{V}$ and we can define u_2 as the solution of

$$L(v_1)u_2 + f(\cdot, v_1) + 1 = 0 \quad \text{in} \quad \mathbb{R}^d \setminus D, \qquad u_2 = 0 \quad \text{on} \quad \partial D. \tag{2.17}$$

One can continue the policy iteration process with

$$\begin{cases} v_n(x) \in \operatorname{Arg}\min_{v \in V}\{\nabla u_n(x) \cdot g(x,v) + f(x,v)\}, \quad \forall x \in \mathbb{R}^d, \\ L(v_n)u_{n+1} + f(\cdot, v_n) + 1 = 0 \quad \text{in } \mathbb{R}^d \setminus D, \qquad u_{n+1} = 0 \quad \text{on } \partial D. \end{cases} \tag{2.18}$$

Also we have

$$L(v_1)u_2 + f(\cdot, v_1) + 1 = L(v_0)u_1 + f(\cdot, v_0) + 1 \geq L(v_1)u_1 + f(\cdot, v_1) + 1, \tag{2.19}$$

so $L(v_1)(u_2 - u_1) \geq 0$, and by the maximum principle $u_2 \leq u_1$. More generally, the sequence $\{u_n\}$ is decreasing and positive. Using $W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d \setminus D)$ estimates, we conclude that u_n converges to u weakly in $W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d \setminus D)$ and uniformly (up to the first derivatives) on every compact subset of $\mathbb{R}^d \setminus D$.

Define a measurable selection

$$\hat{v}(x) \in \operatorname{Arg}\min_{v\in V}\{\nabla u(x)\cdot g(x,v)+f(x,v)\}, \quad \forall x \in \mathbb{R}^d. \tag{2.20}$$

Since

$$Au_n + \inf_{v\in V}\{\nabla u_n \cdot g(\cdot,v)+f(\cdot,v)\}+1 \le 0$$

one has

$$Au + \inf_{v\in V}\{\nabla u \cdot g(\cdot,v)+f(\cdot,v)\}+1 \le 0,$$

and therefore $E_x^v\{\tau\} < \infty$, i.e., $\hat{v}$ belongs to $\mathcal{V}$. Moreover, by definition of v_n we have

$$g(\cdot,v_n)\cdot\nabla u_n + f(\cdot,v_n) \le g(\cdot,\hat{v})\cdot\nabla u_n + f(\cdot,\hat{v}),$$

which together with

$$g(\cdot,\hat{v})\cdot\nabla u_n + f(\cdot,\hat{v}) \le g(\cdot,\hat{v})\cdot\nabla u + f(\cdot,\hat{v}) + g(\cdot,\hat{v})\cdot\nabla(u_n - u),$$

and

$$g(\cdot,v_n)\cdot\nabla u_{n+1} + f(\cdot,v_n) \le g(\cdot,v_n)\cdot\nabla u_n + f(\cdot,v_n) + g(\cdot,v_n)\cdot\nabla(u_{n+1} - u_n),$$

yield

$$g(\cdot,v_n)\cdot\nabla u_{n+1} + f(\cdot,v_n) \le g(\cdot,\hat{v})\cdot\nabla u + f(\cdot,\hat{v}) + \\ + \big[\sup_{v\in V}|g(\cdot,v)|\big]\big(|\nabla(u_n - u)| + |\nabla(u_{n+1} - u_n)|\big),$$

i.e.,

$$\big[\sup_{v\in V}|g(\cdot,v)|\big]\big(|\nabla(u_n - u)| + |\nabla(u_{n+1} - u_n)|\big) \le \\ \le Au_{n+1} + g(\cdot,\hat{v})\cdot\nabla u + f(\cdot,\hat{v}) + 1,$$

after using (2.18). Next, because ∇u_n converges uniformly over any compact subset of $\mathbb{R}^d \setminus D$, we get

$$0 \le Au + \nabla u \cdot g(\cdot,\hat{v}) + f(\cdot,\hat{v}) + 1,$$

proving that u solves the HJB equation (2.14). The uniqueness follows from the stochastic interpretation. □

2.3 HJB Equation for P2

The only issue is that one may have $f(x,0) = 0$ and $v \equiv 0$ is in V. Then the HJB equation corresponding to (2.10), namely

$$Au + \inf_{v\in V}\{\nabla u\cdot g(\cdot,v)+f(\cdot,v)\} = 0 \quad \text{in } \mathbb{R}^d\setminus D, \qquad u = 0 \quad \text{in } \partial D, \tag{2.21}$$

has a trivial solution $u \equiv 0$. Thus, if $g(x,0)$ gives a recurrent process, $v \equiv 0$ belongs to $\mathcal{V}$ and is optimal. So the only case to be considered is when $g(x,0)$ does not give a recurrent process.

Let us consider the problem of minimizing

$$J^{\varepsilon}_x(v) = E^v_x\Big\{ \int_0^{\tau} \big(|v(t)|^2 + \varepsilon\big) \mathrm{d}t \Big\}, \qquad \varepsilon > 0.$$

over $\mathcal{V}$, i.e., controls such that $E^v_x\{\tau\} < \infty$, for which we have the HJB equation

$$A u_\varepsilon + \inf_{v \in V} \{\nabla u_\varepsilon \cdot g(\cdot, v) + f(\cdot, v)\} + \varepsilon = 0 \quad \text{in } \mathbb{R}^d \setminus D, \qquad u_\varepsilon = 0 \quad \text{in } \partial D, \tag{2.22}$$

for which Theorem 2.2 applies.

Theorem 2.3. *If the assumptions of Section 2.1 holds then the solution u_ε of* (2.22) *converges weakly in $W^{2,p}_{loc}(\mathbb{R}^d \setminus D)$ for any $p < \infty$ to the* maximum *solution $\hat{u}$ of* (2.21) *and such that $\hat{u}/u_0$ is bounded. Moreover*

$$\hat{u}(x) = \inf\{J_x(v) : v \in \mathcal{V}\}, \tag{2.23}$$

for any x in $\mathbb{R}^d \setminus D$.

Proof (sketch). Clearly u_ε decrease as $\varepsilon \to 0$ and $u_\varepsilon \geq 0$. Then, classical estimates imply that $u_\varepsilon \to \hat{u}$, a solution of (2.22). If u is another solution, then

$$u(x) \leq J_x(v), \quad \forall v \in \mathcal{V}.$$

Hence

$$u(x) \leq J_x(v) + \varepsilon E^v_x\{\tau\}, \quad \forall v \in \mathcal{V},$$

and the result follows. We also have

$$u \leq \inf\{J_x(v) : v \in \mathcal{V}\} \leq u_\varepsilon$$

and (2.3) follows. In order to have the existence of an optimal control, we would need to show that

$$\hat{v}(x) \in \operatorname{Arg}\min_{v \in V} \{\nabla u(x) \cdot g(x, v) + f(x, v)\}, \quad \forall x \in \mathbb{R}^d, \tag{2.24}$$

is such that $E^{\hat{v}}_x\{\tau\} < \infty$, which is not true in general. However if instead of (2.8) we assume that there exists a smooth ψ (Liapunov function) such that

$$\begin{cases} 0 \leq \psi(x) \leq C_0(1 + |x|), \quad \forall x \in \mathbb{R}^d, \qquad \lim_{|x| \to \infty} \psi(x) = +\infty, \\ \psi \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d), \qquad L(v)\psi + 1 \leq 0 \quad \text{in } \mathbb{R}^d \setminus D, \quad \forall v \in V, \end{cases} \tag{2.25}$$

then there is an optimal control given by (2.24). □

Remark 2.4. Clearly, if $f(x, v) \geq \gamma > 0$ in $\mathbb{R}^d \times V$ then Theorem 2.2 applies. □

3 Examples

3.1 Stable system

The case

$$g(x, v) = b(x) + g_0(x, v),$$

with b Lipschitz continuous, $b(x) = 0$, satisfying

$$-\sum_i b_i(x)\, x_i \geq c_0 |x|^2, \quad \forall x \in \mathbb{R}^d,\ |x| \geq r_0,$$

for some constants c_0, $r_0 > 0$, and $g_0(x, v)$ Borel bounded in $R^d \times V$, continuous in v, corresponds to the assumptions in Bensoussan [2]. Then assumption (2.25) is satisfied, and therefore (2.8). See [2] for an example of Liapunov function ψ.

3.2 Wiener and drift

Let us consider a diffusion given by

$$\mathrm{d}x(t) = v(t)\mathrm{d}t + \mathrm{d}w(t), \quad \text{in } \mathbb{R}^d, \tag{3.1}$$

with $|v(t)| \leq 1$ (norm in $\mathbb{R}^d$), where the process does not satisfy condition (2.25). However, the weaker assumption (2.8) holds in this case. Indeed, the result in Morimoto and Okada [5] states that for a given $h \geq 0$, convex, C^1 such that $h(x) \leq C(1 + |x|)$ the problem

$$\lambda = \frac{1}{2}\Delta\varphi + \inf_{|v|\leq 1}\{v \cdot \nabla\varphi + |v|^2\}, \quad \text{in } \mathbb{R}^d, \tag{3.2}$$

has a solution (λ, φ) (unique when imposing $\inf \varphi = 0$), with φ in C^2, convex, with quadratic growth, and $\lambda \geq 0$. Moreover, there is an optimal feedback $\hat{v}(x)$. Therefore, if $D = \{x : h(x) - \lambda \geq 1\}$ then

$$\frac{1}{2}\Delta\varphi + \hat{v} \cdot \nabla\varphi + 1 \leq 0, \quad \text{in } \mathbb{R}^d \setminus D,$$

so that a variant of assumption (2.8) is satisfied.

One concludes that Theorem 2.2 applies. However, when $f(x, v) = |v|^2$, with (3.1), we can conjecture that the trivial solution identically zero is the maximum solution in Theorem 2.3 and there is no optimal control for the limit problem (2.10).

3.3 One dimension Wiener and drift

As in the previous case with $d = 1$,

$$\mathrm{d}x(t) = v(t)\mathrm{d}t + \sqrt{2}\mathrm{d}w(t), \quad \text{in } \mathbb{R},$$

with $D =]-a, +a[$, $-1 \leq v(t) \leq 1$ and

$$J_x^\varepsilon(v) = E_x^v\Big\{\int_0^\tau (|v|^2 + \varepsilon)\mathrm{d}t\Big\}.$$

Then, the HJB equation is

$$u_\varepsilon'' + F(u_\varepsilon') + \varepsilon = 0, \quad \text{for } |x| > a, \qquad u_\varepsilon(\pm a) = 0,$$

with

$$F(p) = \begin{cases} -p^2/4 & \text{for } |p| \leq 2, \\ -|p| + 1 & \text{otherwise.} \end{cases}$$

Direct calculations show that the solution with linear growth is

$$u_\varepsilon(x) = 2\sqrt{\varepsilon}\,(|x| - a)$$

and that $u_\varepsilon \to 0$ as $\varepsilon \to 0$. The limit equation does not provide an optimal control because $v \equiv 0$ is not admissible.

If now we consider

$$\mathrm{d}x(t) = g_0(x(t)) + v(t)\mathrm{d}t + \sqrt{2}\mathrm{d}w(t), \quad \text{in } \mathbb{R},$$

with D, V as above and $g_0(x) = 1/4$ for $x > a$, $g_0(x) = -1/4$ for $x < -a$, and smooth in $[-a, +a]$, then one finds

$$u_\varepsilon(x) = \big(1/2 + 2\sqrt{\varepsilon + 1/16}\big)\,(|x| - a)$$

and the limit problem is well posed. Notice that if $|g_0(x)| > 1$ and the system is unstable as in the previous example, the control $|v| \leq 1$ cannot compensate g_0, and assumption (2.8) is not satisfied.

4 Extension to Diffusions with Jumps

It would be too long to go into details here so we just give a sketch of possible extension. We refer to [6], [7] for a precise construction of the controlled diffusions with jumps. The HJB equation is of the following form:

$$Au+I_0u+\inf_{v\in V}\{\nabla u\cdot g(\cdot,v)+I(v)u+f(\cdot,v)\}=0,\ \text{in } \mathbb{R}^d\backslash D,\quad u=0 \text{ in } D,$$

with A as in Section 2.1,

$$I_0\varphi(x)=\int_{\mathbb{R}^d_*}\big[\varphi(x+z)-\varphi(x)-z\cdot\nabla\varphi(x)\big]M_0(x,\mathrm{d}z),$$
$$I(v)\varphi(x)=\int_{\mathbb{R}^d_*}\big[\varphi(x+z)-\varphi(x)\big]c(x,v,z)M_0(x,\mathrm{d}z),$$

where the Levy kernel $M(x,dz)$ is a Radon measure on $\mathbb{R}^d_*=\mathbb{R}^d\setminus\{0\}$ for any fixed x, and satisfies

$$\int_{|z|<1}|z|^2M_0(x,dz)+\int_{|z|\ge 1}|z|M_0(x,dz)<\infty,\quad \forall x\in\mathbb{R}^d, \tag{4.1}$$

and

$$0\le c(x,v,z)\le C(1\wedge|z|)\quad \forall x\in\mathbb{R}^d,\ v\in V, \tag{4.2}$$

for some constant $C>0$. Condition (4.1) means that the Levy measure $M_0(x,dz)$ may have a singularity of second order at the origin, which is refer to as *jumps of order 2* and translates into the fact that the integro-operator I_0 is well defined for function with compact support and continuous second derivative. Similarly, condition (4.2) makes controllable only the *first order part* of the jump process. Actually, as in Garroni and Menaldi [4], the Levy kernel $M_0(x,dz)$ is assumed to have a particular structure (which makes clear the x-dependency), namely

$$M_0(x,A)=\int_{\{\zeta:j(x,\zeta)\in A\}}m_0(x,\zeta)\pi(d\zeta), \tag{4.3}$$

where $\pi(\cdot)$ is a σ-finite measure on the measurable space $(F,\mathcal{F})$, the functions $j(x,\zeta)$ and $m_0(x,\zeta)$ are measurable for (x,ζ) in $\mathbb{R}^d\times F$, and there exist a measurable and positive function $j_0(\zeta)$ and constants $C_0>0$ such that for every x,ζ and complementing (4.1) we have

$$\begin{cases}|j(x,\zeta)|\le j_0(\zeta), \qquad 0\le m_0(x,\zeta)\le 1,\\ \displaystyle\int_F [j_0(\zeta)]^2(1+j_0(\zeta))^{-1}\pi(d\zeta)\le C_0,\end{cases} \tag{4.4}$$

the function $j(x,\zeta)$ is continuously differentiable in x for any fixed ζ and there exists a constant $c_0>0$ such that for any (x,ζ) we have

$$c_0\le\det(\mathbf{1}+\theta\nabla j(x,\zeta))\le c_0^{-1},\quad \forall\theta\in[0,1], \tag{4.5}$$

where $\mathbf{1}$ denotes the identity matrix in $\mathbb{R}^d$, ∇ is the gradient operator is x, and $\det(\cdot)$ means the determinant of a matrix.

The exterior Dirichlet problem of the form (2.11), as in Proposition 2.1, becomes

$$L(\bar{v_0})\bar{u}+h=0 \quad \text{in} \quad \mathbb{R}^d\setminus D, \qquad \bar{u}=0 \quad \text{on} \quad \overline{D}, \tag{4.6}$$

where the operator $L(\bar{v_0})$ is now given by

$$L(v)\varphi=\nabla\varphi\cdot g(\cdot,v)+I(v)\varphi+A\varphi+I_0\varphi. \tag{4.7}$$

Then, under assumptions similar to those in Section 2.1 and using the results of [6], [7], [8], one can extend Theorems 2.2 and 2.3 to diffusion with jumps. There are technical difficulties which are studied in a similar way as in the ergodic case [8]. This will be the focus of further developments.

5 Other Problems

5.1 Case $\sup P^v_{x,t}\{\tau \leq T\}$

For strongly controllable systems, one can consider the maximization of the probability that $\tau \leq T$. This leads to the evolution problem

$$\begin{cases} \partial_t u + Au + \sup_{v\in V}\{g(\cdot,v)\cdot\nabla u\} = 0, & \text{in } (\mathbb{R}^d\setminus D)\times]0,T[, \\ u(\cdot,t)=1 \quad \text{in } \partial D, \ \forall t\in]0,T]. \end{cases} \tag{5.1}$$

Using, for instance, approximate problem on $B_r\setminus D$, with $B_r=\{x : |x|<r\}$, and $u_r=0$ on ∂B_r, one can show that u_r is increasing to the minimal positive solution u of (5.1), as $r\to\infty$.

If we assume that there exists a smooth Liapunov function

$$\begin{cases} \psi\geq 0, \qquad \lim_{x\to\infty}\psi(x)=\infty, \\ A\psi+g(\cdot,v)\cdot\nabla\psi+1\leq 0 \text{ in } \mathbb{R}^d\setminus D, \ \forall v\in V, \end{cases}$$

then

$$\hat{v}(x)\in \operatorname{Arg}\min_{v\in V}\{g(x,v)\cdot\nabla u\}$$

defines an optimal control and u is the unique positive solution of (5.1), via the stochastic interpretation.

5.2 Case $\sup P^v_x\{\tau<\infty\}$

For *non* controllable systems, one can consider the maximization of $P^v_x\{\tau<\infty\}$, for which the HJB equation is

$$\begin{cases} Au+\sup_{v\in V}\{g(\cdot,v)\cdot\nabla u\}=0, & \text{in } \mathbb{R}^d\setminus D, \\ u=1 \quad \text{in } \partial D. \end{cases}$$

If the system is controllable, then $u\equiv 1$ is the solution. Otherwise, an approximation on $B_r\setminus D$ with $u=0$ on ∂B_r increases to the minimal positive solution.

References

[1] A. Arapostathis, R.K. George, M. K. Ghosh, On the Controllability of a Class of Nonlinear Stochastic Systems, 37th IEEE CDC Conference, Tampa, Florida, USA, December 1998.

[2] A. Bensoussan, *Perturbation methods in optimal control*, Wiley, New York, 1988.

[3] R.Z. Khasminskii, (Hasminskii) *Stochastic Stability of Differential Equations*, Sijthoff and Noordhoff, The Netherlands, 1980.

[4] M.G. Garroni and J.L. Menaldi, *Green functions for second order integral-differential problems*, Pitman Research Notes in Mathematics Series No 275, Longman, Essex, 1992.

[5] H. Morimoto, M. Okada, Some Results on the Bellman Equation of Ergodic Control, *SIAM J. Control Optim.*, **38** (1999), 159–174.

[6] J.L. Menaldi and M. Robin, Ergodic control of reflected diffusions with jumps, *Appl. Math. Optim.*, **35** (1997), 117–137.

[7] J.L. Menaldi and M. Robin, Invariant Measure for Diffusions with Jumps, *Appl. Math. Optim.*, **40** (1999), 105–140.

[8] J.L. Menaldi and M. Robin, On Optimal Ergodic Control of Diffusions With Jumps, in *Stochastic Analysis, Control, Optimization and Applications (A Volume in Honor of W.H. Fleming)*, Eds. W.M. McEneaney, G. Yin, and Q. Zhang, Birkhauser, Boston, 1999, 439–456.

[9] D.W. Stroock and S.R. Varadhan, *Multidimensional diffusion processes*, Springer-Verlag, Berlin, 1979.

[10] J. Zabczyk, Controllability of Stochastic Linear Systems, *Systems & Control Letters*, **1** (1981), 25–30.