9-1-2000

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Recommended Citation

A. Bensoussan and J.-L. Menaldi, Stochastic hybrid control, *Journal of Mathematical Analysis and Applications*, **249** (2000), 261-288.
doi: 10.1006/jmaa.2000.7102
Available at: http://digitalcommons.wayne.edu/mathfrp/40

Journal of Mathematical Analysis and Applications, 249 (2000), 261-288

Stochastic Hybrid Control

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Introduction

The objective of this paper is to study the stochastic version of a previous paper of the authors, in which hybrid control for deterministic systems was considered. The modelling is quite similar to the deterministic case. We have a system whose state is composed of a continuous part and a discrete part. They are affected by a continuous type control and an impulse control. The dynamics is moreover perturbed by noise, also a continuous and a discrete noise process. The Markovian character of the state process is preserved. We develop the model and show how the dynamic programming approach leads to some involved quasi-variational inequality.

1 Stochastic Hybrid Model

In this section we present first a formal description and next an abstract model of hybrid control for diffusion processes. The continuous and the discrete parts of the state variable have their own natural evolution, but the main point is how they interact. Our model uses a "set-interface" (set-of-discontinuity or set-of-marked-states) and "time-interface" (or time-of discontinuity) to describe the analog/digital interface. We conclude this section with a quick summary of the main points and assumptions of our abstract model.

1.1 Formal Description

The time t is measured continuously. The state of the system is represented by a continuous variable x and a discrete variable n. Similarly the control has two parts, a continuous-type control v and a discrete-type (or impulse) control k. The intrinsic difference between the discrete and continuous variable is not merely the fact that the former can assume only a countable number of values, but also the way how they involve through time. A stochastic differential equation models the continuous evolution, which affects only the continuous state variable x. The discrete dynamics produces transitions in both (continuous and discrete) state variables x, n. Thus, a sample trajectory has the form $(x(t), n_i, t \geq 0, i = 0, 1, \ldots)$, where $(x(t), t \geq 0)$ is piecewise continuous. Let $\{0 \leq t_0 \leq t_1 \leq \ldots \leq t_i \leq t_{i+1} \leq \ldots\}$ be the sequence of times at which the continuous and the discrete part of the system interact or exchange information (and therefore the discrete dynamics is activated). This sequence (called "time-interface") may or may not be part of the control. This sequence of time-interface is generated when the state of the system passes through a set of "marked states", denoted by D and referred to as the "set-interface".

The dynamics of the system can be formally characterized as follows: **(i)** if (x, n) is not in D then the variable x follows a stochastic differential equation and the variable n remains unchanged, **(ii)** if (x, n) belongs to D then a discrete transition takes place instantaneously, **(iii)** the continuous-type control v acts only on the continuous transition (of the continuous state variable x) and the impulse-type control k acts only on the discrete transition (of the joint state variables x, n).

Assuming that the discrete transition dynamics (i) takes place instantaneously is not a restriction on the model. Actually this allows us to describe in more detail the behavior of the system during a "waiting time", i.e. the period of time where the system is waiting (or delayed) for and impulse action to be executed.

To preserve the Markovian character it is necessary to include all dynamic information on the state variable (x, n) in such a way to satisfy condition (iii) on the controllable parameters of the system. This is not a restriction on the model, it is only a convenient normalization for modeling.

If the sequence of time-interface is not part of the control then the set-interface D is given a priori. Otherwise, the set-interface is part of the control and subject to various constrains.

Since the discrete transitions are instantaneous, it is natural (in most cases) to assume that a continuous transition must follow any discrete transition. In this way, the sequence of time-interface satisfies $\{0 \leq t_0 < t_1 < \ldots < t_i < t_{i+1} < \ldots\}$, c.f. Remark 1.1. Thus, a sample trajectory has the form $(x(t), n(t), t \geq 0)$, where $x(t)$ is piecewise continuous and $n(t)$ is piecewise constant.

1.2 Abstract Model

The dynamics of the system is continuously observed at any time t in $[0, +\infty)$ and the state variable (x, n) belongs to a subset S of $\mathbb{R}^d \times \mathbb{R}^m$. Trajectories are piecewise continuous for the continuous variable x, piecewise constant for the discrete variable n and they are normalized to be right-continuous with left-hand limits, and to each trajectory $\{(x(t), n(t)),\, t \geq 0\}$ it is associated a unique sequence $\{0 \leq t_0 < t_1 < \ldots, < t_i < t_{i+1} \ldots\}$ of times where the function $t \mapsto (x(t), n(t))$ is discontinuous. This is also referred to as the sequences of "impulse-times". There is a closed set $D \subset S$ of "marked states" where discontinuities of trajectories are produced, called set-interface.

Remark 1.1. Notice that two simultaneous impulses (or discrete transitions) are not allowed. This may look as a restriction, however the number of impulses in a bounded time-interval is usually a priori finite (otherwise a finite terminal-time would exist). In our model, an impulse can be decomposed into several simultaneous impulses-per-coordinate. Thus, by adding more state variables, we can include in our model the case where a finite number of simultaneous impulses is allowed. $\Box$

A continuous-type control is a measurable stochastic process $v(t)$ taking values in a compact subset V of $\mathbb{R}^p$; and an impulse-type control is a sequence $\{k_0, k_1, \ldots, k_i, k_{i+1}, \ldots\}$ of random variables with values in a compact subset K of $\mathbb{R}^q$. The impulse-type control satisfy some "compatibility" restrictions imposed by the set-interface D, which we will discuss later.

To describe the dynamics, we need to consider a d_1-dimensional Wiener process $(w(t),\, t \geq 0)$ in a complete probability space (Ω, F, P), and a sequence $(\zeta_1, \zeta_2, \ldots)$ of i.i.d. m_1-dimensional random variables, which are independent of the Wiener process. On D the discrete transition is used,

$$\begin{cases} x(t_i) = X(x(t_i-), n(t_i-), k_i, \zeta_{i+1}), \\ n(t_i) = N(x(t_i-), n(t_i-), k_i, \zeta_{i+1}), \end{cases} \tag{1.1}$$

where the notation (t_i-) means the left-hand limit and the transition functions X, N satisfy

$$(X, N) : D \times K \times \mathbb{R}^{m_1} \to S \setminus D, \text{ continuous.} \tag{1.2}$$

While in $S \setminus D$, the continuous evolution is activated for $t > t_i$,

$$\begin{cases} dx(t) = g(x(t), n(t), v(t))dt + \sigma(x(t), n(t))dw(t), \\ \;\; n(t) = n(t_i), \end{cases} \tag{1.3}$$

until the hitting time of D

$$t_{i+1} = T(t_i, x(t_i), n(t_i), v(\cdot)) \tag{1.4}$$

defined by

$$T(t_i, x(t_i), n(t_i), v(\cdot)) = \inf \big\{ t > t_i ; (x(t-), n(t-)) \in D \big\} \tag{1.5}$$

which is set equal to $+\infty$ if the process never hits the target D. Thus, by induction on (1.1), (1.3) and (1.4) we construct the process $(x(t), n(t),\, t \geq 0)$.

It is clear that we must assume that the continuous-type control $(v(t),\, t \geq 0)$ is non-anticipative w.r.t. the Wiener process, i.e.

$$v(t) \text{ is independent of } w(s) - w(t), \quad \forall s > t. \tag{1.6}$$

Similarly, the impulse-type control is non-anticipative w.r.t. the sequence of i.i.d. random variables, i.e.

$$k_i \text{ is independent of } \zeta_j, \quad \forall j > i. \tag{1.7}$$

The controlled drift g and diffusion matrix σ are continuous,

$$g : (S \setminus D) \times V \longrightarrow \mathbb{R}^d, \qquad \sigma : S \setminus D \longrightarrow \mathbb{R}^d \times \mathbb{R}^{d_1}, \tag{1.8}$$

plus some conditions (to be discussed later) to ensure a proper continuous evolution on $S \setminus D$.

In order to generate (by means of the above induction procedure) a trajectory defined for every $t \geq 0$, we need to know that $t_i \to +\infty$ *a.s.* as i$\to \infty$. This issue will be considered later.

If the controller has total access to the set-interface (i.e., the action of switching from the continuous dynamics to the discrete dynamics is always an option) then the time-interface are part of the control instead of being defined by (1.4). The other extreme situation is when the controller cannot access the set-interface (i.e., the set D is given a priori). Other cases are discussed in the next section. In short we have

$$\begin{cases} dx(t) = \{g(x(t), n(t), v(t)) + \sum_{i=0}^{\infty} \big[X(x(t_i-), n(t_i-), k_i, \zeta_{i+1}) - \\ \qquad - x(t_i-)\big]\delta(t - t_i)\}dt + \sigma(x(t), n(t))dw(t), \\ n(t) = \sum_{i=0}^{\infty} \big[N(x(t_i-), n(t_i-), k_i, \zeta_{i+1}) - n(t_i-)\big]\mathbf{1}_{(t_i \leq t)}, \end{cases} \tag{1.9}$$

where δ is the Dirac measure.

1.3 Model Summary

The key point is the set-interface D, of which only the boundary ∂D is really used. We suppose that there are a minimal and a maximal set-interface. When the state reaches the minimal set, a mandatory impulse (jump or switch) takes place. While the state belongs to the maximal set, an optional impulse (jump or switch) may be applied, upon decision of the controller.

Summing up, the data are as follows:

$$\begin{cases} \text{state-spare } S \subset \mathbb{R}^d \times \mathbb{R}^m, \text{ open or closed,} \\ \text{minimal set-interface } D^{\wedge} \subset S, \text{ closed,} \\ \text{maximal set-interface } D^{\vee} \subset S, \text{ closed, } D^{\wedge} \subset D^{\vee}, \\ \text{control-spaces } V \subset \mathbb{R}^p,\ K \subset \mathbb{R}^q, \text{ compact.} \end{cases} \tag{1.10}$$

Sometimes $V \times K$ may not be assumed compact, but then some assumptions on the performance index (such as coercitivity) may be needed to insure the existence of an optimal control. The discrete and continuous transitions are governed by

$$(X, N) : D^{\vee} \times K \times \mathbb{R}^{m_1} \to S \backslash D^{\wedge}, \text{ uniformly continuous,} \tag{1.11}$$

and

$$\begin{cases} g : (S \backslash D^{\wedge}) \times V \to \mathbb{R}^d, \ \sigma : S \backslash D^{\wedge} \to \mathbb{R}^d \times \mathbb{R}^{d_1}, \\ \text{uniformly continuous, bounded and such that} \\ |g(x,n,v) - g(x',n,v)| + |\sigma(x,n) - \sigma(x',n)| \leq M \ |x - x'|, \end{cases} \tag{1.12}$$

for any x, x', n, v and some constant $M > 0$.

The trajectories $(x(\cdot), n(\cdot))$ are normalized to be right continuous with left-hand limits, the continuous component $x(\cdot)$ is piecewise continuous and the discrete component $n(\cdot)$ is piecewise constant, and the dynamics follows the rule

$$\begin{cases} \textbf{(i)} \text{ a mandatory impulse is applied on } D^{\wedge}, \\ \textbf{(ii)} \text{ a continuous evolution takes place on } S \setminus D^{\vee}, \\ \textbf{(iii)} \text{ the controller chooses either (i) or (ii) on } D^{\vee} \setminus D^{\wedge}. \end{cases} \tag{1.13}$$

Without any loss of generality the impulses (jumps or switching) are implemented "instantaneously." When the state of the system hits the set $D^{\wedge}$ or belongs to the set $D^{\vee} \backslash D^{\wedge}$ and the controller chooses to switch (from continuous to discrete dynamics) an impulse is produced following the discrete transition (1.1). The continuous evolution (1.3) takes place on $S \setminus D^{\vee}$ and also on $D^{\vee} \backslash D^{\wedge}$, at the controller option. It is implicitly understood that the continuous evolution takes place in some

$$S_n = \{x \in \mathbb{R}^d : (x,n) \in S\} \tag{1.14}$$

If S_n is not the whole space $\mathbb{R}^d$ then some assumptions on the controlled drift g, diffusion matrix σ and domains S_n,

$$D_n^{\wedge} = \{x \in \mathbb{R}^d : (x,n) \in D^{\wedge}\} \tag{1.15}$$

are needed.

Let (Ω, F, P) be a complete probability space; $(w(t),\, t \geq 0)$ be a d_1 dimensional Wiener space; and $(\zeta_0, \zeta_1, \ldots, \zeta_i, \zeta_{i+1}, \ldots)$ be a sequence of *i.i.d.* m_1-dimensional random variables, which are independent of the Wiener process.

To properly define the control process, we consider the family of σ-algebras $\{F_i^t,\, t \geq 0,\, i = 1, 2, \ldots\}$ generated by $(w(s), \zeta_j,\, s \leq t,\, j \leq i)$, complete and right-continuous, i.e. the smallest family of complete σ-algebras satisfying

$$\begin{cases} w(s) \text{ and } \zeta_j \text{ are } F_i^t\text{-measurables}, \quad \forall s \leq t,\ j \leq i, \\ \bigcap_{s>t} F_i^t = F_i^t, \quad \forall t \geq 0,\ i = 1, 2, \ldots \end{cases} \tag{1.16}$$

A continuous-type control is a measurable stochastic process $(v(t), t \geq 0)$ with values in V, and an impulse-type control is a sequence of random variables $(k_i, i = 0, 1, \ldots)$ with values in K. Moreover, there are adapted to the family of σ-algebras $(F_i^t, t \geq 0, i = 1, 2, \ldots)$. For a given initial state (x, n) and set-interface D we proceed by induction as follows:

$$\begin{cases} \text{if} \quad (x, n) \in S \setminus D \quad \text{then set} \quad x(0) = x, n(0) = n, \\ \text{if} \quad (x, n) \in D \quad \text{then set} \quad x(0-) = x, n(0-) = n. \end{cases} \tag{1.17}$$

Now, set $t_0 = 0$ and use either

$$\begin{cases} x(t_i) = X(x(t_i-), n(t_i-), k_i, \zeta_{i+1}), \\ n(t_i) = N(x(t_i-), n(t_i-), k_i, \zeta_{i+1}), \end{cases} \tag{1.18}$$

or

$$\begin{cases} dx(t) = g(x(t), n(t_i), v(t))dt + \sigma(x(t), n(t_i))dw(t), \\ \ n(t) = n(t_i), \quad t_i \leq t < t_{i+1}, \end{cases} \tag{1.19}$$

and

$$t_{i+1} = \inf\big\{t > t_i \ :\ (x(t-), n(t-) \in D\big\} \tag{1.20}$$

Notice that we start with (1.18) if (x, n) is in D and we use (1.19) otherwise. In (1.18) the notation (t_i-) means the left-hand limit.

The control satisfies

$$\begin{cases} v(t)\, \mathbf{1}(t \leq t_i) \quad \text{is adapted to} \quad F_i^t, \quad t \geq 0,\ i = 1, 2, \ldots, \\ k_i \quad \text{is measurable w.r.t.} \quad F_i^{t_i},\ i = 0, 1, \ldots, \end{cases} \tag{1.21}$$

Another way of expression is condition is to consider a sequence of controls $\{v_i(t) : i = 1, 2, \ldots\}$, where v_i is adapted to $F_i^{t_i}$ and it is used only after t_i, i.e., the global control $v(t)$ is equal to $v_i(t)$ for t in (t_i, t_{i+1}).

Notice that the initial impulse-type control k_0 (deterministic) is only used when (x, n) is in D. We have

$$\big(x(t_i-), n(t_i-)\big) \in D \quad \text{on} \quad [t_i < \infty], \quad \forall i = 1, 2, \ldots. \tag{1.22}$$

The sequence of time-interface$\{0 < t_i < t_2 < \ldots\}$ defined by (1.10) is subordinate to the set-interface (it depends also on $v(t), k_i, x, n$). Thus if $D^{\wedge} = D = D^{\vee}$ [automation case] this time-interface is not directly accessible to the controller. However, the extreme case where $D^{\wedge} = \phi$ and $D^{\vee} = S$ (therefore D is any closed set in S) classic impulse or switching control), the sequence of time-interface is completely part of the control, we may replace (1.20) by the condition

$$\begin{cases} t_i \quad \text{stopping time w.r.t.} \quad F_i^t, \\ t_i < t_{i+1} \quad \text{on} \quad [t_{i+1} < \infty]. \end{cases} \tag{1.23}$$

Definition 1.2 (admissible control). A control process $(v(t), k_i,\ t \geq 0,\ i = 0, 1, \ldots)$ is called admissible w.r.t. a prescribed set-interface D and a given initial state (x, n) if (1.21) and

$$t_i \to \infty \quad \text{as} \quad i \to \infty. \tag{1.24}$$

are satisfied. □

Thus, an admissible control allows us to define the controlled process $(x(t), n(t))$ for any $t \geq 0$. If we fix a feedback function (v, k),

$$\begin{cases} v : (S \setminus D^{\wedge}) \longrightarrow V, \qquad k : D^{\vee} \longrightarrow K, \\ v(t) = v(x(t), n(t)), \qquad k_i = k(x(t_i-), n(t_i-)) \end{cases} \tag{1.25}$$

then we obtain a Markov process.

It may happen that the impulsive dynamic has to be used several times before switching to the continuous evolution. In this case we use the convention

$$n_{i+1} = N(x(t_i-), n_i, k_i, \zeta_{i+1}), \qquad n(t_i-) = n_i, \tag{1.26}$$

so that essentially we keep record of the transitions $(n_1, n_2, \ldots)$, but we denote by $n(t_i-)$ the state n_i. Thus, the actual evolution of the state of system is $(x(t), n_i,\ t \geq 0,\ i = 0, 1, \ldots)$.

As discussed later, a natural condition to obtain admissible controls is to impose the condition

$$\begin{cases} \exists\, c > 0 \quad \text{such that} \quad \forall z,\ \forall k \in K,\ \forall (x,n), (\xi,\eta) \in D^{\wedge} \\ |\xi - X(x,n,k,\zeta)| + |\eta - N(x,n,k,\xi)| \geq c. \end{cases} \tag{1.27}$$

Even more general, if we allow up to r discrete transitions before going into the continuous evolution, we replace X and N by their r-power X^r and N^r in the condition (1.27).

2 Controlled Hybrid Process

2.1 Autonomous Switching or Jumps

Suppose that the set interface D is given a priori and contains all states where a switching or jump is enabled. Any trajectory (or path) of the controlled hybrid process has discontinuities when hitting the set D. These switching and jumps are autonomous, and the set D is not part of the control, i.e. the controller cannot modify the set $D = D^{\wedge} = D^{\vee}$.

The controller selects a dynamics by means of the discrete variable n. We assume that

$$\begin{cases} S_n = \{x \in \mathbb{R}^d \ :\ (x,n) \in S\} \quad \text{is open,} \\ D_n = \{x \in \mathbb{R}^d \ :\ (x,n) \in D\} \quad \text{is closed} \end{cases} \tag{2.1}$$

(the set S_n may also be the closure of an open set) and usually require that

$$N = \{n \in \mathbb{R}^m \ :\ S_n \neq \phi\} \quad \text{is countable,} \tag{2.2}$$

and

$$\text{the boundary} \quad \partial S_n \quad \text{is piecewise smooth.} \tag{2.3}$$

Thus the continuous evolution is well defined until the first exit time of $(x(t-), n(t-))$ from S. It is clear that the discrete (or impulse) transition X and N may be called the jump and switching (respect.) transition functions. Some natural examples may be represented by the case where

$$D_n = \partial S_n, \tag{2.4}$$

and the values of the discrete variable n (i.e., the set N) need not be countable.

For the non-degenerate case (i.e. $\sigma(x,n)$ is invertible) the continuous evolution will leave any bounded region S_n. Thus a controller/modeler decision should be taken on the boundary ∂S_n. For instance, we may stop the process at the first exit time from S_n; or we may produce a reflection on the boundary; or even a jump so that ∂S_n is a part of D_n. These situations will be discussed, in some detail later.

Notice that this model can be regarded as a “stochastic differential automation” following the approach in Tavernini [27] and more general as in Back et al. [3], Branicky et al. [12].

In order to be sure that at least for some control we do not have infinitely many impulses simultaneously, we need the following assumption:

$$\sup_{k\in K} \inf_{(x,n),(\xi,\eta)\in D} \big\{|\xi - X(x,n,k,\zeta)| + |\eta - N(x,n,k,\zeta)|\big\} \geq c > 0. \tag{2.5}$$

On the other hand, if we do not allow simultaneous impulses for any control then we may impose

$$\inf_{k\in K} \inf_{(x,n),(\xi,\eta)\in D} \big\{|\xi - X(x,n,k,\zeta)| + |\eta - N(x,n,k,\zeta)|\big\} \geq c > 0, \tag{2.6}$$

in lieu of (2.5). Both for any ζ in $\mathbb{R}^{m_1}$.

As mentioned above, to check that the trajectories $(x(t), n(t))$ are well defined we need to know that $t_{i+1} > t_i$ and if possible that $t_i \to \infty$ or $i \to \infty$.

Assumptions of the type (2.5) or (2.6) give reason to the so-called “impulse with state-delay”.

Definition 2.1 (state-delay). The dynamic of a system, has a deterministic state-delay $\delta > 0$ if for any two consecutive impulses t_i and t_{i+1} we have $|x(t_{i+1}-) - x(t_i)| \geq \delta$, with probability 1. □

Consider the stopping time

$$\begin{cases} T(x,n,t,v) = \inf\big\{s > t \ : \ (y(s),n) \in D\big\}, \\ y(t) = x, \qquad dy(s) = g(y(s),n,v(s))ds + \sigma(y(s),n)dw(s). \end{cases} \tag{2.7}$$

By construction we have

$$t_{i+1} - t_i = T(x(t_i), n(t_i), t_i, v(\cdot)), \quad \forall i. \tag{2.8}$$

So that, if we define

$$\begin{cases} \tau(x,n,t,v) = \inf\big\{ s > t \ : \ |y(s) - x| \geq \delta \big\}, \\ y(t) = x, \quad dy(s) = g(y(s),n,v(s))ds + \sigma(y(s),n)dw(s), \end{cases} \tag{2.9}$$

we see that condition (2.6) implies

$$T(x,n,t,v) \geq \tau(x,n,t,v), \quad \forall x,n,t,v(\cdot) \tag{2.10}$$

for $\delta = c$. Thus, under assumption (2.6) our system has a deterministic state-delay for any choice of controls. Similarly, condition (2.5) provides a state-delay for certain choice of controls.

Theorem 2.2. *Under the assumptions* (1.11) *and* (1.12), *any state-delay system has its trajectories well defined, i.e.,* $t_{i+1} > t_i \to +\infty$. *Moreover, we have the estimate*

$$E\{e^{-\alpha\tau}\} \leq \frac{||g||^2 + ||\sigma||^2}{\alpha\delta + ||g||^2 + ||\sigma||^2}, \quad \forall x,n,t,v(\cdot), \tag{2.11}$$

where τ, δ *are as in* (2.9), $\alpha \geq 1$ *and* $||\cdot||$ *denotes the supremum norm.*

Proof. Since

$$-|g(y,n,v)|^2 + 2(g(y,n,v), y-x) - \alpha|y-x|^2 \leq 0$$

if $\alpha \geq 1$, we have

$$2(g(y,n,v), y-x) + \mathrm{tr}[\sigma\sigma^*(y,n)] - \alpha|y-x|^2 \leq \\ \leq \|g\|^2 + \|\sigma\|^2.$$

Thus form Itô's formula we deduce

$$E\{|y(\tau) - x|^2 e^{-\alpha\tau}\} \leq (\|g\|^2 + \|\sigma\|^2)\, E\{\frac{1 - e^{-\alpha\tau}}{\alpha}\},$$

where τ and $y(\cdot)$ are given by (2.9). Hence

$$\big[\delta + \frac{1}{\alpha}(\|g\|^2 + \|\sigma\|^2)\big]\, E\big\{e^{-\alpha\tau}\big\} \leq \frac{1}{\alpha}(\|g\|^2 + \|\sigma\|^2)$$

which implies (2.11) and the desired result. □

Remark 2.3. If we assume (2.6) then the dynamic system has a (deterministic) state-delay $\delta > 0$ and Theorem 2.2 can be applied. On the other hand, if we only assume (2.5) then Theorem 2.2 will apply only for controls satisfying (2.5). □

Let us mention that to study the continuity of the trajectory with respect to the initial state is a delicate issue, which is not discussed here.

2.2 Totally Controlled Switching or Jumps

The big difference with the previous section is the fact that the set-interface D is now part of the control. The qualifier "totally" refer to the case where $D^{\wedge} = \emptyset$, i.e., there are not mandatory switching or jumps but in the region $S \setminus D^{\vee}$, only continuous evolution is allowed. In this case, we extend the definition of the function (X, N) as follows

$$\big(X(x, n, \cdot, \cdot), N(x, n, \cdot, \cdot)\big) = (x, n), \quad \forall (x, n) \in S \backslash D^{\vee}. \tag{2.12}$$

This extension does not preserve the uniform continuity of the discrete transition function (X, N), but it is used only formally. Thus, on the region $S \setminus D^{\vee}$ we allow impulsion, but they do not actually modify the continuous evolution.

In this case, a continuous-type control is a sequence of measurable stochastic processes $(v(t),\, t \geq 0$ with values in V, and an impulse-type control is a sequence $(t_i, k_i,\, i = 1, 2, \ldots)$ of times $t_i < t_{i+1}$, for any i, and random variables k_i with values in K. Moreover, they are adapted to the family of σ–algebras $(F_i^t,\, t \geq 0,\, i = 1, 2, \ldots)$, defined by (1.16). This means that

$$\begin{cases} \textbf{(a)}\ t_i \quad \text{is a stopping time w.r.t.} \quad F_{i+1}^t, \quad t_0 = 0 \\ \textbf{(b)}\ k_i \quad \text{is measurable w.r.t.} \quad F_i^{t_i}, \\ \textbf{(c)}\ v(t)\, \mathbf{1}_{(t \leq t_i)} \quad \text{is adapted to} \quad F_i^t, \end{cases} \tag{2.13}$$

for any $i = 1, 2, \ldots$, cf (1.21). Because the times (t_i) are totally part of the control we may allow $t_i \leq t_{i+1}$, for all i. In any case, also we impose that $t_i \to +\infty$ as $i \to \infty$ to have an admissible control, cf. Definition1.2.

To define the controlled process $(x(t), n(t))$ with initial state (x, n) and controls $(v(t), t \geq 0)$, $(t_i, k_i, i = 1, 2, ...)$ we proceed as follows by induction. Set $t_0 = 0$, and

$$\begin{cases} \text{if} \quad t_i > 0 \quad \text{then set} \quad x(t_0) = x, \quad n(t_0) = n, \\ \text{if} \quad t_1 = 0 \text{ then set} \quad x(t_0-) = x, \quad n(t_0-) = n. \end{cases} \tag{2.14}$$

Now, use either the discrete transition

$$\begin{cases} x(t_i) = X(x(t_i-), \quad n(t_i-), k_i, \zeta_{i+1}), \\ n(t_i) = N(x(t_i-), n(t_i-), k_i, \zeta_{i+1}), \end{cases} \tag{2.15}$$

or the continuous transition (evolution)

$$\begin{cases} dx(t) = g(x(t), n(t_i), v(t))dt + \sigma(x(t), n(t_i))dw(t), \\ n(t) = n(t_i), \quad t_i \le t < t_{i+1}. \end{cases} \tag{2.16}$$

Comparing with the construction in the previous section, we notice that the exit time expression by (1.20) is necessary. Also, recall that the non impulsion restriction on the region $S - D^{\vee}$ is enforced by the "singular" extension (2.12).

There is a vast bibliography on this class of problems, but without the discrete variable n (i.e. $S \subset \mathbb{R}^d$, all continuous-type variables) and with $S = D^{\vee}$ (cf. Bensoussan and Lions [7]).

Note that in this open-loop setting (2.14),…, (2.16), the set-interface has no reference, even no reference to $D^{\vee}$. However, for a feedback formulation we need to use a set-interface $D \subset D^{\vee}$, a continuous-type feedback function

$$v : (S \backslash D) \longrightarrow V, \qquad \text{Borel measurable} \tag{2.17}$$

and an impulse-type feedback function

$$k : D \longrightarrow K, \qquad \text{Borel measurable.} \tag{2.18}$$

We proceed as in (1.17),…, (1.20) with

$$\begin{cases} k_i = k(x(t_i-), n(t_i-)), \quad i = 0, 1, ... \\ v(t) = v(x(t), n(t)), \quad t_i < t < t_{i+1}. \end{cases} \tag{2.19}$$

For the open-loop controls we can study the continuity w.r.t. the initial data. Let us assume that the coefficients are bounded, Lipschitz continuous in the state variable (on $S \setminus D^{\vee}$) and uniformly continuous in the control variable i.e., for some constants $C, M > 0$ we have

$$\begin{cases} |g(x, n, v)| + |\sigma(x, n)| + |x - \bar{X}(x, n, k)| + \\ \qquad + |n - \bar{N}(x, n, k)| \le C, \quad \forall (x, n, k) \in S \times K, \end{cases} \tag{2.20}$$

$$\begin{cases} |g(x, n, v) - g(x', n', v')| + |\sigma(x, n) - \sigma(x', n'| \le \\ \qquad \le M\big[|x - x'| + |n - n'| + \rho(|v - v'|)\big], \\ \qquad\qquad \forall (x, n), (x', n') \in S, \ v, v' \in V, \end{cases} \tag{2.21}$$

$$\begin{cases} |\bar{X}(x,n,k) - \bar{X}(x',n',k')| + |\bar{N}(x,n,k) - \bar{N}(x',n',k')| \leq \\ \qquad \leq M\big[|x - x'| + |n - n'| + \rho(|k - k'|)\big], \\ \qquad\quad \forall (x,n),(x',n') \in S \setminus D^{\vee},\ k,k' \in K, \end{cases} \tag{2.22}$$

where $\rho(\cdot)$ is a modulus of continuity i.e., is positive, increasing and vanishing at zero and $\bar{X}(x,n,k) = E\{x(x,n,k,z_1)\}$, $\bar{N}(x,n,k) = E\{N(x,n,k,z_1)\}$. Notice that (X,N) has been extended to the whole space S, but usually (2.22) does not hold on $S \times K$. Moreover, if the discrete state-space or the impulse control-space is discrete (i.e., composed of only isolated points in the Euclidian space) then all functions are continuous (or Lipschitz continuous) in that variable.

Let $(t_i, k_i,\ i = 0,1,\dots)$ and $(t'_i, k'_i,\ i = 0,1,\dots)$ be two admissible impulse control such that

$$\begin{cases} t_i = t'_i, \quad \forall i = 1,2,\dots, \quad t_0 \leq t'_0, \\ (t_i) \quad \text{increases (a.s.) to} \quad +\infty, \end{cases} \tag{2.23}$$

i.e., they have the same impulse times after the initial impulse. The number of impulses up to the time $t > 0$ is given by the counting process

$$\sum_{i=1}^{\infty} \mathbf{1}_{(t_i \leq t < t_{i+1})} = i \quad \text{if} \quad t \in [t_i, t_{i+1}). \tag{2.24}$$

We define the following (exponentially) decay process

$$r(t) = \exp\big[-\alpha t - (\ln \beta) \sum_{i=1}^{\infty} \mathbf{1}_{(t_i \leq t < t_{i+1})}\big], \quad t \geq 0, \tag{2.25}$$

which is adapted to the filtration $F_i t_v t_i$. Notice that the constant M in assumptions (2.20), (2.21) and (2.22) will be used to choose $\alpha, \beta > 0$. For any initial states (x,n), (x',n') and continuous type controls $v(t), v'(t)$, we consider the corresponding state processes $(x(t), n(t))$ and $(x'(t), n'(t))$.

Theorem 2.4. *Let's assume* $D^{\wedge} = \emptyset$, $D^{\wedge} \subset S$, (2.20), (2.21) *and* (2.22)*. Then with the above notation, we can choose constants* $\alpha, \beta > 0$ *and* M

(depending only on the bounds in the assumptions) so that

$$\begin{cases} E\big\{|x(t)-x'(t)|r(t)+|n(t)-n'(t)|r(t)\big\} \le M\big[|x-x'|+ \\ \qquad +|n-n'|+|t_0-t_0'|^{1/2}\big] + E\big\{\sum_{i=0}^{\infty}\rho(|k_i-k_i'|)\,r(t_i)\big\}+ \\ \qquad + E\big\{\int_0^t \rho(|v(s)-v'(s)|)\,r(s)ds\big\}, \end{cases} \tag{2.26}$$

where x, x', n, n', t_0, t_0' *are deterministic values.*

Proof. Based on assumption (2.22) and the fact that the sequence $(z_i, i = 0,1,...)$ is i.i.d., we have

$$\begin{cases} E\big\{\big[|x(t_i)-x'(t_i)|+|n_{i+1}-n'_{i+1}|\big]r(t_i)\big\} \le \\ \qquad \le M\,E\big\{\big[|x_i-x_i'|+|n_i-n_i'|+\rho(|k_i-k_i'|)|\big]r(t_i)\big\}, \end{cases} \tag{2.27}$$

for any $i = 1,2,...$

Now, we use Itô's formula for the process $z(t) = [x(t)-x'(t), n(t)-n'(t)]$ and the function

$$t \mapsto \big(\varepsilon+|z|^2\big)^{1/2} e^{-\alpha t}\beta^{-i}, \quad \text{between} \quad t_i \quad \text{and} \quad t_{i+1},$$
$$d\big(\varepsilon+|z(t)|^2\big)^{1/2} e^{-\alpha t}\beta^{-i} = \ell(t)\,dt + dM_t,$$

where (M_t) is an Itô's integral and

$$\begin{aligned} \ell(t) = {} & \big(\varepsilon+|z(t)|^2\big)^{-1/2}\big(x(t)-x'(t)\big)\cdot\big(g(x(t),n(t),v(t))- \\ & -(g(x'(t),n'(t),v'(t)\big) + \big(\varepsilon+|z(t)|^2\big)^{-1/2}\frac{1}{2}\mathrm{tr}\big[(\sigma(x(t),n(t))- \\ & -\sigma(x'(t),n'(t)))(\sigma(x(t),n(t))-\sigma(x'(t),n'(t)))^*\big]- \\ & -\big(\varepsilon+|z(t)|^2\big)^{-3/2}\frac{1}{2}\mathrm{tr}\big[(\sigma(x(t),n(t))-\sigma(x'(t),n'(t)))\times \\ & \times(x(t)-x'(t))\,(x(t),x'(t))^*(\sigma(x(t),n(t))-\sigma(x'(t),n'(t)))^*\big]. \end{aligned}$$

Thus, for α large enough we have

$$|\ell(t)| \le M\,\rho(|v(t)-v'(t))|)\,e^{-\alpha t}\beta^{-i}$$

and

$$E\big\{\big[|x(t)-x'(t)|\mathbf{1}_{(t_i\le t<t_{i+1})}+|n_i-n_i'|\big]\,e^{-\alpha t}\,\beta^{-i}\big\}\le$$
$$\le E\big\{M\int_{t_i}^{t\wedge t_{i+1}}\rho(|v(s)-v'(s)|)\,r(s)ds\big\}+$$
$$+E\big\{\big[|x(t_i)-x'(t_i)|+|n_i-n_i'|\big]\,e^{-\alpha t_i}\,\beta^{i}\big\}.$$

Hence

$$\begin{cases} E\big\{\big[|x_{i+1}-x_{i+1}'|+|n_i-n_i'|\big]\,r(t_{i+1})\big\}\le \\ \qquad \le \dfrac{1}{\beta}\,E\big\{\big[|x(t_i)-x'(t_i)|+|n_i-n_i'|\big]\,r(t_i)\big\}+ \\ \qquad\quad +\dfrac{M}{\beta}\,E\big\{\displaystyle\int_{t_i}^{t^{i+1}}\rho(|v(s)-v'(s)|)\,r(s)ds\big\}. \end{cases} \tag{2.28}$$

By means of (2.27) we get

$$E\big\{|x_{i+1}-x_{i+1}'|\,r(t_{i+1})\big\}\le\frac{M}{\beta}\,E\big\{|x_i-x_i'|\,r(t_{i+1})\big\}+$$
$$+\frac{M+1}{\beta}E\big\{|n_i-n_i'|\,r(t_i)\big\}+\frac{M}{\beta}E\big\{\rho(|k_i-k_i'|)\,r(t_i)\big\}+$$
$$+\frac{M}{\beta}E\big\{\int_{t_i}^{t_{i+1}}\rho(|v(s)-v'(s)|)\,r(s)ds\big\}$$

and again, going back to (2.27) we have

$$E\big\{|n_{i+1}-n_{i+1}'|\,r(t_{i+1})\big\}\le\frac{M}{\beta}E\{|x_i-x_i'|\,r(t_i)\big\}+$$
$$+\frac{M}{\beta}E\big\{|n_i-n_i'|\,r(t_i)\big\}+\frac{M}{\beta}E\big\{\rho(|k_i-k_i'|)\,r(t_i)\big\}.$$

Now, take $\beta\ge 2(M+1)$ to deduce

$$E\big\{\big[|x_{i+1}-x_{i+1}'|+|n_{i+1}-n_{i+1}'|\big]\,r(t_{i+1})\big\}\le$$
$$\le E\big\{\big[|x_i-x_i'|+|n_i-n_i'|\big]\,r(t_i)\big\}+E\big\{\rho(|k_i-k_i'|)\,r(t_{i+1})\big\}+$$
$$+E\big\{\int_{t_i}^{t_{i+1}}\rho(|v(s)-v'(s)|)\,r(s)ds\big\},$$

which implies

$$\begin{cases} E\big\{\big[|x_i - x_i'| + |n_i - n_i'|\big]\, r(t_i)\big\} \le E\big\{\big[|x_1 - x_1'| + \\ \qquad + |n_1 - n_1'|\big]\, r(t_1)\big\} + E\big\{\displaystyle\sum_{i=1}^{\infty} \rho(|k_i - k_i'|)\, r(t_i)\big\} + \\ \qquad + E\big\{\displaystyle\int_{t_1}^{\infty} \rho(|v(t) - v'(t)|)\, r(t)dt\big\}. \end{cases} \tag{2.29}$$

On the stochastic interval $[0, t_1]$ we have only two possible impulses at $t_0 \le t_0'$. So we can obtain

$$E\big\{\big[|x_1 - x_1'| + |n_1 - n_1'|\big]\, r(t_1)\big\} \le M\big[|x - x'| + |n - n'| + \\ + |t_0 - t_0'|^{1/2} + \rho(|k_0 - k_0'|)\big] + E\big\{\int_0^{t_1} \rho(|v|t) - v'(t)|)\, r(t)dt\big\},$$

from which (2.26) follows. □

3 Dynamic programming

First we add a performance index in the form of a cost, next we study the volume function and finally we discuss the quasi-variational inequality.

3.1 Performance index

In order to set up a control problem we need to compare control policies. The orientation used is given through a certain performance index. In our model, we normalize the problem to the minimization case. Therefore, we refer to a cost to be minimized.

Our control policy affects the state of the system in several ways. A cost is associated with each intervention.

The continuous part of control is active in the region $S \setminus D$, and a discounted marginal cost.

$$f(x(t), n(t), v(t)) \exp\big(-\int_0^t c(x(s), n(s), v(s))ds\big) \tag{3.1}$$

is paid. Usually, the discount factor is constant, i.e.

$$\exp\big(-\int_0^t c(x(s), n(s), v(s))ds\big) = e^{-ct} \tag{3.2}$$

and because of our infinite horizon setting, we need either to assume $c > 0$ or to stop the system evolution after a finite time. Thus, on a single period of continuous controlling, say $[t_i, t_{i+1}]$, we incur a "running" cost of

$$\int_{t_i}^{t_{i+1}} f(x(t), n(t), r(t)) \exp\big(-\int_0^t c(x(s), n(s), v(s))ds\big)\, dt. \tag{3.3}$$

The impulsive part of the control is active in the region D, and a discounted cost-per-impulse

$$\ell(x(t_i-), n(t_i-), k_i) \exp\big(-\int_0^{t_i} c(x(s), n(s), v(s))ds\big) \tag{3.4}$$

must be paid. Notice that in order to make evident the fact that the optional decision of switching from continuous to discrete evolution has some cost, we need to impose

$$\ell(x, n, k) \geq \ell_0 > 0, \quad \forall (x, n) \in D^{\vee} \backslash D^{\wedge},\ \forall k \in K. \tag{3.5}$$

Actually, this condition establishes the key condition between a continuous-type control and an impulse-type control.

For a given control policy $v = v(\cdot)$, $k = k(\cdot)$ we have a total "expected" cost given by

$$\begin{cases} J_{x,n}(v, k) = E\big\{ \displaystyle\int_0^{+\infty} f(x(t), n(t), v(t)) e_{v,k}(t) dt + \\ \qquad\qquad + \displaystyle\sum_{i=0}^{\infty} \ell(x(t_i-), n(t_i-), k(t_i-)) e_{v,k}(t_i) \big\} \end{cases} \tag{3.6}$$

where $e_{v,k}(t) = \exp\big(-\int_0^t c(x(s), n(s), v(s))ds\big)$, and (x, n) is the initial state.

Therefore, the data are the following functions

$$\begin{cases} f,\, c : S \times V \longrightarrow [0 + \infty), & \text{uniformly continuous} \\ \ell : D^{\wedge} \times K \longrightarrow [0 + \infty), & \text{uniformly continuous} \end{cases} \tag{3.7}$$

and all three functions are bounded and (3.5) holds. Sometime, we only have

$$f,\, c : (S \backslash D^{\wedge}) \times V \longrightarrow [0 + \infty) \tag{3.8}$$

be bounded and uniformly continuous and the restriction of ℓ to $(D^{\vee} \backslash D^{\wedge}) \times K$ and to $(D^{\wedge}) \times K$ are bounded and uniformly continuous.

3.2 Value function

Let us denote by P the set of admissible control policies, i.e. the set of measurable stochastic processes $(v(t), t \geq 0)$ with values in V, the sequence of random variables $(k_0, k_1, \ldots)$ with values in K and a sequence of times $(t_0, t_1, \ldots)$ satisfying (2.13) and such that

$$0 = t_0 \leq t_1 \leq ... \leq t_i \leq t_{i+1} \leq ..., t_i \to +\infty \quad \text{as} \quad i \to +\infty \tag{3.9}$$

and

$$t_{i+1} = \inf \big\{ t \geq t_i \ : \ (x(t-), n(t-)) \in D \big\} \tag{3.10}$$

for some closed set D satisfying

$$D^{\wedge} \subset D \subset D^{\wedge}. \tag{3.11}$$

The value function or optimal (minimal) cost is given by

$$v(x,n) = \inf \big\{ J_{x,n}(v,k) \ : \ (v,k) \in P \big\}. \tag{3.12}$$

To discuss some "implementable" approximation of the value function we consider two relatively trivial approximation methods for the trajectory. Given an admissible impulse control policy k, we denote by k_ε and k^ε, for $\varepsilon > 0$, the two impulse control policies constructed as follows:

$$\begin{cases} k_\varepsilon \quad \text{is identical to} \quad k \quad \text{up to the first} \quad [1/\varepsilon] \\ \text{(mandatory or optional) impulses, and} \\ \text{afterward, only mandatory impulses are applied} \end{cases} \tag{3.13}$$

and

$$\begin{cases} k^\varepsilon \quad \text{is identical to} \quad k \quad \text{up to the first} \quad [1/\varepsilon] \\ \text{(mandatory or optional) impulses, and} \\ \text{no more impulses are applied,} \end{cases} \tag{3.14}$$

where $[1/\varepsilon]$ denotes the largest integer number inferior or equal to $1/\varepsilon$. Notice that in our terminology, the impulse control input k_ε is admissible, but k^ε may not be so. If we represent the policy k by the sequence $(t_0, t_1, \ldots)$ and $(k_0, k_1, \ldots)$ then we see that the policy k_ε is represented by sequences

$(t_0, t_1, \ldots, t_i, t^\varepsilon_{i+1}, t^\varepsilon_{i+2} \ldots)$, and $(k_0, k_1, \ldots)$, where $[1/\varepsilon] = i + 1$ are mandatory or optional impulses and $(t^\varepsilon_{i+1}, t^\varepsilon_{i+2}, \ldots)$ are mandatory impulses only. Similarly, the policy k^ε is represented by finite sequences $(t_0, t_1, ..., t_i)$ and $(k_0, k_1, \ldots, k_i)$, where again i is equal to $[1/\varepsilon] - 1$. For the sake of simplicity with notation, we identify any finite sequence, e.g. $(t_0, t_1, \ldots, t_i)$, with a sequence with infinite symbols, i.e. with $(t_0, t_1, \ldots, t_i, \infty, \infty, \ldots)$. This is necessary for the sequence of time-interfaces, but useless for the sequence of impulses.

It is clear that, for a given admissible control policy (v, k), the construction of Section 1 allows us to define trajectories $(x(\cdot), n(\cdot))$, $(x_\varepsilon(\cdot), n_\varepsilon(\cdot))$ and $(x^\varepsilon(\cdot), n^\varepsilon(\cdot))$ associated with the control inputs $(v, k), (v, k_\varepsilon)$ and (v, k^ε), respectively. Moreover

$$\begin{cases} (x(t), n(t)) = (x_\varepsilon(t), n_\varepsilon(t)), & \forall t \in [0, \tau_\varepsilon), \\ (x(t), n(t)) = (x^\varepsilon(t), n^\varepsilon(t)), & \forall t \in [0, \tau^\varepsilon), \end{cases} \tag{3.15}$$

where both times satisfy

$$\tau_\varepsilon, \tau^\varepsilon \to +\infty \quad \text{as} \quad \varepsilon \to 0. \tag{3.16}$$

Recall that in order to avoid an undesirable accumulation of little jumps for autonomous jump/switching mechanism we need to assume (2.5) or (2.6). On the other hand, for an infinite horizon we need to assume that

$$c(x, n, v) \geq c_0 > 0, \quad \forall (x, n) \in S, \ \forall v \in V. \tag{3.17}$$

Otherwise, we need to work on a finite horizon i.e. to assume that one of the continuous state variables, say the $\sigma_1(x, n) = 0$, first one, is the time, so for any $x = (t, x'), n, v, k$ we have

$$g_1(x, n, v) = 1, \quad \sigma(x, n), \quad X_1(x, n, k) = t, \tag{3.18}$$

$$f(x, n, v) = \ell(x, n, k) = 0 \ \text{ if } \ t \geq \tau, \tag{3.19}$$

for some finite time $\tau > 0$. Clearly, another alternative may be also considered. Notice that under (3.18) and (3.19), we count only impulses which are strictly before the final time τ. In the discounted cost-per-impulse (3.4) we assumed implicitly $c(\cdot) > 0$, otherwise we need to add a factor which is equal to 1 only if $t_i < +\infty$ (or $t_i < \tau$) and it is equal to 0 otherwise. This resolves the incompatibility of (3.5) and (3.19) for a continuous data ℓ.

Denote by P_ε and P^ε the set of all control policies constructed as in (3.13) and (3.14).

$$u_\varepsilon(x,n) = \inf\big\{J_{x,n}(v,k_\varepsilon)\ :\ (v,k_\varepsilon)\in P_\varepsilon\big\}, \tag{3.20}$$

and

$$u_\varepsilon(x,n) = \inf\big\{J_{x,n}(v,k^\varepsilon)\ :\ (v,k_\varepsilon)\in P_\varepsilon\big\}, \tag{3.21}$$

where the cost is given again by (3.6). Since $P_\varepsilon \subset P$ we have u, which is not true (in general) for u^ε instead of u_ε.

Contrary to the deterministic case, we do not have a (deterministic) time-delay, i.e. some $k_0 > 0$ such that $t_{i+1} \geq t_i + k_0$ for two consecutive mandatory impulses. Under the condition (2.5) or (2.6) we only have a so-called state-delay, of Definition 2.1.

Theorem 3.1. *Let the assumptions* (1.10), (1.11), (1.12), (1.27), (3.5), (3.8), (3.17) [or (3.9) and (3.19)]*, and* (3.23) *be satisfied. Then all (admissible) control policies in P_ε have finite costs (which is almost obvious for policies in P^ε). Moreover we have the estimates*

$$0 \leq u_\varepsilon(x,n) - u(x,n) \leq C\varepsilon^r, \quad \forall (x,n)\in S, \tag{3.22}$$

$$|u^\varepsilon(x,n) - u(x,n)| \leq C\varepsilon^r, \quad \forall (x,n)\in S, \tag{3.23}$$

for the values functions (3.12), (3.12), (3.20), (3.21) *and for some positive constants C, r depending only on the various hypotheses.* □

At this point, we may proceed as in the deterministic case. Most of the results remain true, however, some more details are necessary.

3.3 Quasi-variational inequalities

Either the dynamic programming principle (e.g. Bellman [5]) or the (open-loop) maximum principle (Pontryagin et al. [25]) formally express the fact that a global (in the time horizon) optimal trajectory is also locally (in the time horizon) optimal. In another way, the optimality is a local property and therefore the feedback control policy should be optimal all the time. In our case, a feedback control policy is as follows:
(i) on $D^\wedge$ we must apply an impulse (mandatory, impulsive control),
(ii) on $D^\vee \setminus D^\wedge$ we may apply an impulse (optional impulsive control),
(iii) if an optional impulse is not applied, then a continuous control is used.

Since these three actions are mutually exclusive, an optimal feedback control should optimize all of them. Thus, the value function (3.13) should satisfy (in some appropriate sense!) the conditions:

$$u(x,n) \le \ell(x,n,k) + u(X(x,n,k), N(x,n,k)), \tag{3.24}$$

$\forall (x,n) \in D^\wedge, \forall k \in K$,

$$u(x,n) \le \ell(x,n,k) + u(X(x,n,k), N(x,n,k)), \tag{3.25}$$

$\forall (x,n) \in D^\vee \setminus D^\wedge, \forall k \in K$, and

$$c(x,n,v)u(x,n) \le f(x,n,v) + g(x,n,v) \cdot \frac{\partial u}{\partial x}(x,n), \tag{3.26}$$

$\forall (x,n) \in S \setminus D^\wedge, \forall v \in V$, and the optimal (feedback) control policy should be such that the equality holds (i.e. the inequality is tight) at least for one of the inequalities (3.24), (3.25) or (3.26) at any time (i.e. for any states x, n). This set of conditions was referred to as "quasi-variational inequalities" (QVI) under other assumptions. Without the discrete state variable n (and all its consequences) we can find several references, e.g. the books of Bensoussan and Lions [7] for stochastic diffusion processes, Davis [16] for piecewise deterministic process, and e.g. the papers by Menaldi [22, 23] for degenerate dynamics, among others. Once the model has been properly set, most of the techniques of the above references can be adapted to this new situation. Clearly, new difficulties and challenges need to be considered.

To properly phrase the dynamic programming principle we need some notation:

$$\begin{aligned} M\varphi(x,n) &= \inf\big\{E[\varphi(X(x,n,k,\zeta_1), N(x,n,k,\zeta_1))] + \\ &\qquad +\ell(x,n,k) : k \in K\big\} \end{aligned} \tag{3.27}$$

defined for bounded functions, and the continuous part of the Hamiltonian operator

$$\begin{aligned} H\varphi(x,n) &= \min\Big\{(1/c(x,n,v))\big(f(x,n,v) + g(x,n,v) \cdot \nabla_x\varphi(x,n) + \\ &\qquad +(1/2)\mathrm{tr}[\sigma(x,n)\sigma^*(x,n)\nabla_x^2\varphi(x,n)]\big) : v \in V\Big\}, \end{aligned} \tag{3.28}$$

where $\nabla_x\varphi = \frac{\partial\varphi}{\partial x}$ is the gradient in the first variable x and $\nabla_x^2\varphi$ is the hessian. Thus we can rewrite the QVI as

$$u(x,n) = Mu(x,n), \qquad \forall (x,n) \in D^\wedge \tag{3.29}$$

$$u(x,n) = \min\{Hu(x,n), Mu(x,n)\} \qquad \forall (x,n) \in D^\vee \setminus D^\wedge, \tag{3.30}$$

$$u(x,n) = Hu(x,n) \qquad \forall (x,n) \in S \setminus D^\vee, \tag{3.31}$$

with the above notation. Similarly to the case deterministic, we can obtain

Theorem 3.2. *Let us assume* (1.10), (1.11), (1.12), (3.5), (3.7), (3.17) *[or* (3.18) *and* (3.19)*. Then the following versions of the dynamic programming principle hold true*

$$\begin{aligned} u(x,n) &= \inf_{(v,k)} E\Big\{ \int_0^{t_0} f(x(t),n(t),v(t)) e_{v,k}(t) dt + \\ &\qquad + Mu(x(t_0-), n(t_0-)) e_{v,k}(t_0) \Big\}, \end{aligned} \tag{3.32}$$

and

$$\begin{cases} u(x,n) = \inf_{(v,k)} E\Big\{ \int_0^{\tau} f(x(t),n(t),v(t)) e_{v,k}(t) dt + \\ \quad + \sum_{i=0}^{\infty} \ell(x(t_i-), n(t_i-), k_i) e_{v,k}(t_i) 1_{(t_i<\tau)} + \\ \quad + u(x(\tau-), n(\tau-)) e_{v,k}(\tau) Big\}, \end{cases} \tag{3.33}$$

where t_0 is the first impulse-time of the admissible control policy (v,k), $\tau \geq 0$ is an arbitrary time [which may depend on the policy v,k],

$$e_{v,k}(t) = \exp(-\int_0^t c(x(s),n(s),v(s)) ds) \tag{3.34}$$

is the discount factor (or rate) and $u(x,n)$ is the value function defined by (3.13). □

At this point, we do not have a tool to show that the value function $u(x,n)$ given by (3.32) is differentiable everywhere. However, under the assumptions of the previous theorem we have the equality (3.29). Moreover, if the value function u is continuous, and twice differentiable in the first variable at the point (x,n) then we have either (3.30) or (3.31) depending on where the point (x,n) belong.

The above analysis can be extended to the approximate value functions (3.20) and (3.21). This requires the use of "variational inequalities".

3.4 Viscosity solutions

If the system is non-degenerate, i.e., $\sigma^{-1}(x,n)$ exists and is bounded, then the classic treatment as in Bensoussan and Lions [7] can be used. For the degenerate case, we may us the semigroup formulation and the concept of *maximum solution*, e.g., Bensousaan [6], Menaldi [22, 23]. An alternative powerful and elegant way is to use the so-called viscosity solutions, e.g. see Crandall et al. [15], Fleming and Soner [18], Lions [20].

It may be important to realize at this moment that the discrete state variable n plays the role of a parameter in the Hamiltonian (3.28). Its only active role is within the non-local operator (3.27). Thus, our "hybrid" QVI or Hamilton-Jacobi-Bellman (HJB) equation can be viewed as a system of HJB equation in the continuous state variable x, indexed by the discrete state variable n, and couples through the infimum-type operator M defined by (3.27).

The Hamiltonian (3.28) was written as a minimum [as well as condition (3.30)] to emphasize the fact that originally, the performance criterion was to minimize a cost functional [instead of maximizing the utility, for instance]. However, to deal with viscosity arguments and agree with standard notation, we need to rewrite the Hamiltonian (3.28) as a maximum, i.e.

$$H(x,n,r,p) = \max\{c(x,n,v)r - g(x,n,v)\cdot p - f(x,n,v) : v \in V\}, \tag{3.35}$$

and the HJB equation is then

$$u(x,n) = Mu(x,n), \quad \forall (x,n) \in D^{\wedge} \tag{3.36}$$

$$\begin{cases} \max\{-\frac{1}{2}\mathrm{tr}(\sigma\sigma^*(x,n)\frac{\partial^2 u}{\partial x}(x,n)) + H(x,n,u(x,n),\frac{\partial u}{\partial x}(x,n)), \\ \quad , u(x,n) - Mu(x,n)\} = 0, \quad \forall (x,n) \in D^{\vee} \setminus D^{\wedge} \end{cases} \tag{3.37}$$

$$\begin{cases} -\frac{1}{2}\mathrm{tr}(\sigma\sigma^*(x,n)\frac{\partial^2 u}{\partial x}(x,n) + H(x,n,u(x,n),\frac{\partial u}{\partial x}(x,n)) = 0, \\ \quad \forall (x,n) \in S \setminus D^{\vee}. \end{cases} \tag{3.38}$$

Sometimes, one may want to include the second order derivative into the Hamiltonian (3.35), i.e.,

$$\begin{cases} \bar{H}(x,n,r,p,q) = \max\{c(x,n,v)r - g(x,n,v)\cdot p - \\ \qquad -\frac{1}{2}\mathrm{tr}(\sigma\sigma^*(x,n)q - f(x,n,v) : v \in V\}, \end{cases} \tag{3.39}$$

which becomes relevant when the diffusion coefficient σ is allowed to depend on the continuous-type control v. For the sake of simplicity, we do not include this case, even if the viscosity technique is very well adapted to this situation.

Let us recall one of the several equivalent ways of defining what is a continuous viscosity solution.

Definition 3.3. Denote by $\mathrm{BUC}(S)$ the space of bounded and uniformly continuous function in S. We say that a function w is a viscosity sub-(resp. super-) solution of the HJB equation (3.36), (3.37) and (3.38) if for any smooth function φ [e.g. bounded, with continuous and bounded second order derivative, $\varphi = \varphi(x)$] the following property holds. At each local maximum (resp. minimum) point (x_0, n) of $w(x, n) - \varphi(x)$ in $S_n \setminus D_n^\wedge$ we have:

$$\begin{cases} \textbf{either } (x_0, n) \in S_n \setminus D_n^\vee \text{ and} \\ \text{(i) } \bar{H}(x_0, n, w(x_0, n), \nabla_x \varphi(x_0), \nabla_x^2 \varphi(x_0)) \le 0 \text{ (resp. } \ge 0), \\ \textbf{or } (x_0, n) \in D_n^\vee \setminus D_n^\wedge \text{ and} \\ \text{(ii) } \max\{\bar{H}(x_0, n, w(x_0, n), \nabla_x \varphi(x_0), \nabla_x^2 \varphi(x_0)),\, w(x_0, n) - \\ \qquad - M w(x_0, n)\} \le 0 \text{ (resp. } \ge 0). \end{cases} \tag{3.40}$$

Notice the fact that n is an "index parameter"[1] for the property (3.40). A viscosity solution is a sub- and super-solution simultaneously. □

Other equivalent definitions may be used, e.g. we may replace the "local" character in (3.40) with "global", and in that case we may even replace Mw by $M\varphi$ for (3.40).

Noticing that only continuity and boundness in $S \setminus D^\wedge$ is used for the viscosity definition, we can use the same technique of Theorem 3.2 to prove the following result.

Theorem 3.4. *Under the assumptions of* Theorem 3.2 *and if u_ε is continuous in $S \setminus D^\wedge$, we deduce that the value function u, given by* (3.12)*, is a continuous [may not be uniformly continuous] viscosity solution of* (3.37) *and* (3.38). □

In order to incorporate the boundary condition (3.36) into the (viscosity) QVI (3.37) and (3.38), we need to discuss more details on the continuity assumption. First, it is clear that only the boundary points of $D^\wedge$ will play some active role. So, we assume that $D^\wedge$ is a piecewise smooth boundary

[1] recall that viscosity solutions for system of equations have not yet been considered, here the system is coupled only through the "infimum-type" operator M

(of some dimension strictly inferior to d, the dimension of the continuous state variable x), i.e. for some function $\rho(x,n)$, piecewise twice-continuously differentiable in x and continuous in n we have the representation

$$\begin{cases} D^\wedge = \{(x,n) \in S : \rho(x,n) = 0\}, \\ |\nabla_x \rho(x,n)| \geq \rho_0 > 0, \quad \forall (x,n) \in D^\wedge. \end{cases} \tag{3.41}$$

In most of the cases, we state that the continuous evolution would transverse this boundary $D^\wedge$ if the automata jump/switch were not present. Then, it is natural to assume that for some constant $c_0 > 0$

$$\begin{cases} |(1/2)\mathrm{tr}(\sigma\sigma^*(x,n)\nabla_x^2 \rho(x,n) + \nabla_x \rho(x,n) \cdot g(x,n,v)| \geq c_0 > 0, \\ \quad \forall (x,n) \in D^\wedge, \ \forall v \in V. \end{cases} \tag{3.42}$$

Sometimes, a weaker version of 3.42, namely

$$\begin{cases} \forall (x,n) \in D^\wedge \ \exists v \in V \ \text{ such that} \\ |(1/2)\mathrm{tr}(\sigma\sigma^*(x,n)\nabla_x^2 \rho(x,n) + \nabla_x \rho(x,n) \cdot g(x,n,v)| \geq c_0 > 0 \end{cases} \tag{3.43}$$

may suffice. The analysis is however easier under 3.42. By continuity, the inequality 3.42 holds in a neighborhood of $D^\wedge$, denoted by

$$S_\varepsilon = \{(x,n) \in S : \mathrm{dist}((x,n), D^\wedge) < \varepsilon\}, \tag{3.44}$$

for some $\varepsilon > 0$. It makes sense to define

$$\begin{cases} S_\varepsilon^+ \;=\; \big\{(x,n) \in S_\varepsilon \setminus D^\wedge : (1/2)\mathrm{tr}(\sigma\sigma^*(x,n)\nabla_x^2 \rho(x,n) + \\ \qquad + \nabla_x \rho(x,n) \cdot g(x,n,\cdot) \geq c_0/2, \ \text{ if } \ \rho(x,n) < 0, \\ \qquad \text{or } \ (1/2)\mathrm{tr}(\sigma\sigma^*(x,n)\nabla_x^2 \rho(x,n) + \\ \qquad + \nabla_x \rho(x,n) \cdot g(x,n,\cdot) \leq -c_0/2 \ \text{ if } \ \rho(x,n) > 0\big\} \end{cases} \tag{3.45}$$

and its complement $S_\varepsilon^- = S_\varepsilon \setminus (S_\varepsilon^+ \cup D^\wedge)$.

We see that under the continuous evolution, points in S_ε^+ are attracted (directed toward) by $D^\wedge$, but, points in S_ε^- are repelled (directed backward) by $D^\wedge$. Hence, if we can approach $D^\wedge$ by points in S_ε^-, then we see a “natural” discontinuity. Any trajectory, where the jump-transition X produces jumps on $D^\wedge$, is going to be discontinuous across $D^\wedge$ if we are coming from S_ε^-. It is clear that this discontinuity is passed to the value function 3.12. Viscosity solution can be discontinuous, but its treatment is more delicate. To avoid this situation, we will assume that

$$X(x,n,k,\zeta) = x, \qquad \forall x \in \overline{S_\varepsilon^-} \cap D^\wedge, \tag{3.46}$$

which is somehow equivalent to assume that S_ε^- is empty.

The boundary condition (3.36) is then translated to

$$\max\big\{\bar{H}(x,n,u(x,n),\nabla_x u(x,n),\nabla_x^2 u(x,n)), \\ , u(x,n) - Mu(x,n)\big\} = 0, \quad \forall (x,n) \in \overline{S}_\varepsilon^- \cap D^\wedge \tag{3.47}$$

and

$$u(x,n) = Mu(x,n), \quad \forall (x,n) \in D^\wedge \setminus \overline{S}_\varepsilon^- . \tag{3.48}$$

It is clear that (3.47) is going to be understood in the viscosity sense [like (3.40)] and that (3.48) makes sense in view of the continuity across $D^\wedge \setminus \overline{S}_\varepsilon^-$.

In order to simplify this presentation, we will assume that

$$D^\wedge = \emptyset \quad \text{and} \quad \{x \in \mathbb{R}^d : (x,n) \in S\} = \mathbb{R}^d \tag{3.49}$$

so that no boundaries are considered. Without (3.49), the discussion is more complicate and a more fine analysis is necessary.

Theorem 3.5. *Let the assumptions of* Theorem3.2 *and condition* (3.49) *hold true. Then the value function* (3.12) *is the unique bounded and uniformly continuous viscosity solution of the QVI* (3.47), (3.48), (3.37) *and* (3.38). □

The proof is very similar to the one in [8] and the guidelines in Crandall et al. [15], Fleming and Soner [18], Lions [20]. For convenient to the reader, several references regarding hybrid (deterministic) control problems and related subject have been added. Specific comments were made in [8].

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