1-1-1996

Optimal Starting-Stopping Problems for Markov-Feller Processes

Jose-Luis Menaldi
Wayne State University, menaldi@wayne.edu

Maurice Robin
Centre Europen de Recherche Nucleare

Min Sun
University of Alabama

Recommended Citation

J.-L. Menaldi, M. Robin and M. Sun, Optimal starting-stopping problems for Markov-Feller processes, *Stochastics*, **56** (1996), pp. 17-23. doi: 10.1080/17442509608834033
Available at: http://digitalcommons.wayne.edu/mathfrp/45

Stochastic and Stochastic Reports, 56 (1996), 17-32

Optimal Starting-Stopping Problems for Markov-Feller Processes

Jose-Luis Menaldi* Maurice Robin† Min Sun‡

ABSTRACT

By means of nested inequalities in semigroup form we give a characterization of the value functions of the starting-stopping problem for general Markov-Feller processes. Next, we consider two versions of constrained problems on the final state or on the final time. The plan is as follows:
1. Introduction
2. Nested variational inequalities
3. Solution of optimal starting-stopping problem
4. Problems with constraints
References.

1 Introduction

The optimal stopping problems have been extensively studied for diffusion processes, or other Markov processes, or for more general stochastic processes. We refer to Bensoussan and Lions [2] for a wide bibliography. As an example, a classical stopping problem is to minimize the functional

$$J_x(\tau) = E_x\{\int_0^\tau e^{-\alpha t} L(x_t)dt + e^{-\alpha\tau}\varphi(x_\tau)\}$$

where $(\Omega, F, F_t, x_t, P_x)$ is a Markov process, and τ is an F_t stopping time. The optimal value function

$$\hat{u}(x) = \inf_\tau J_x(\tau)$$

can be characterized as the maximum solution of a set of inequalities involving the semi-group of the Markov process (cf. Bensoussan [1], Bensoussan and Lions [2]).

*Department of Mathematics, Wayne State University, Detroit, Michigan 48202, USA, jlm@math.wayne.edu. This work was supported in part by the NSF grant DMS-9101360.

†Centre Europen de Recherche Nucleare, CH - 1211, Geneve 23, Suisse, maurice_robin@macmail.cern.ch.

‡Department of Mathematics, University of Alabama, Tuscaloosa, Alabama 35487, USA, msun@ua1vm.ua.edu.

Other approaches for the optimal stopping problem are possible, mainly based on the Snell envelop technique, which give characterizations using the so-called *reduite and super-median* functions (cf. Bismut [3], El Karoui et al. [5], Mertens [6], among others). These arguments require a deeper analysis involving the *general theory* of processes, which are not consider here and left for future extensions.

Sun [8] introduced various versions of a starting-stopping problem for diffusion processes where the functional to be minimized is

$$J_x(\tau_1,\tau_2) = E_x\{\int_{\tau_1}^{\tau_2} e^{-\alpha t}L(x_t)dt + \varphi(x_{\tau_1})e^{-\alpha\tau_1} + \psi(x_{\tau_2})e^{-\alpha\tau_2}\} \tag{1.1}$$

over the set of stopping times (τ_1, τ_2) with $\tau_1 \le \tau_2$. There, variational inequalities are used to study this problem.

In the present work we first study the characterization of the value functions of the starting-stopping problems for general Feller-Markov processes. This leads to nested inequalities in semigroup form.

Moreover, we consider two versions of constrained problems, mainly with a constraint on the final state

$$x_{\tau_2} \notin F \tag{1.2}$$

where F is a subset of the state space, or with a constraint of the type

$$E_x\mu(x_{\tau_2})e^{-\alpha\tau_2} \le K. \tag{1.3}$$

In this last case, randomized stopping times have to be used in order to obtain an optimal solution.

The paper is organized as follows: In Section 2 we introduce an abstract system of inequalities which is the semigroup version of the nested variational inequalities associated with (1.1). In Section 3, we give the interpretation of the functions studied in Section 2 as the optimal value function for an optimal starting-stopping problem. In Section 4, we study the constrained problems corresponding to (1.2) and (1.3).

2 Nested Variational Inequalities

First we give the assumptions and statements of our problem. Let E be a Polish space endowed with its Borel σ-algebra $\mathcal{E}$. We denote by B the space of Borel bounded functions on E, and by C the space of uniformly continuous functions on E.

We are given a semigroup of linear operator $\Phi(t)$ satisfying

$$\begin{cases} \Phi(t) : B \to B, \ \ \Phi(0) = I \\ \Phi(t)\Phi(s) = \Phi(t+s) \\ \Phi(t)g \ge 0 \ \text{ if } \ g \ge 0 \ \text{ in } \ B \\ \|\Phi\| \le 1 \ \text{ where } \ \|\cdot\| \ \text{ is the operator norm.} \end{cases} \tag{2.1}$$

Moreover, it is assumed that

$$\begin{cases} \Phi(t) : C \to C \\ \lim_{t\downarrow 0} \Phi(t)f = f \quad \text{in} \quad C \qquad ,\forall f \in C\,. \end{cases} \tag{2.2}$$

Let be given

$$\begin{cases} \psi, \varphi \in C \quad \text{and} \quad L \in B, \ \text{such that} \\ t \to \Phi(t)L \ \text{ is measurable from } \ \mathbf{R}^+ \ \text{ into } \ C\,. \end{cases} \tag{2.3}$$

We can now consider the set of functions (u, v) satisfying

$$\begin{cases} u, v \in C \\ v \leq \psi \\ v \leq e^{-\alpha t}\Phi(t)v + \int_0^t e^{-\alpha s}\Phi(s)Lds \\ u \leq \varphi + v \\ u \leq e^{-\alpha t}\Phi(t)u\,. \end{cases} \tag{2.4}$$

Theorem 2.1 *Under the assumptions (2.1) to (2.3) the set of functions (u, v) satisfying (2.4) has a maximum element $(\hat{u}, \hat{v})$.*

Proof. Let us first consider the set of functions v satisfying

$$\begin{cases} v \in C, \quad v \leq \varphi \\ v \leq e^{-\alpha t}\Phi(t)v + \int_0^t e^{-\alpha s}\Phi(s)Lds. \end{cases} \tag{2.5}$$

Then by Bensoussan [1, Theorem 5.3, p. 316], this set has a maximum element $\hat{v}$. Similarly, consider the set of functions u satisfying

$$\begin{cases} u \in C, \quad u \leq \varphi + \hat{v} \\ u \leq e^{-\alpha t}\Phi(t)u\,. \end{cases} \tag{2.6}$$

This set has a maximum element $\hat{u}$. We claim that $(\hat{u}, \hat{v})$ is the maximum element of problem (2.4).

Actually, let (u, v) satisfy (2.4). Then v satisfies (2.5) and so

$$v \leq \hat{v}\,.$$

Since

$$\begin{cases} u \leq \varphi + v \\ u \leq e^{-\alpha t}\Phi(t)u\,, \end{cases}$$

we deduce $\varphi + v \leq \varphi + \hat{v}$, which implies that

$$u \leq \hat{u}$$

and the conclusion follows. □

Remark 2.2 *We could also consider (2.4) as a quasi-variational inequality (QVI) in semi-group form (See Bensoussan and Lions [2]) with*

$$M\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \varphi + v \\ \psi \end{pmatrix}. \ \square$$

Remark 2.3 *It is not difficult to study a discretized version of (2.4), namely, if*

$$L_h = \frac{1}{h}\int_0^h e^{-\alpha s}\Phi(s)Lds \ ,$$

where h is a parameter which will tend to zero, the discrete version of (2.4) will be

$$\begin{cases} v_h = \min\{\psi \ : \ hL_h + e^{-\alpha h}\Phi(h)v_h\} \\ u_h = \min\{\varphi + v_h \ : \ e^{-\alpha h}\Phi(h)u_h\}. \end{cases} \tag{2.7}$$

Then adapting Bensoussan [1, §5.3], one can show that the unique solution in C of (2.7) converges to $(\hat{u}, \hat{v})$ as h goes to zero. □

3 Solution of the Starting-Stopping Problem

We add some assumptions about the state space E and the semigroup $\Phi(t)$ in order to build a Markov process corresponding to $\Phi(t)$ and to interpret $\hat{u}$ as the value function of a starting-stopping problem.

Let us assume that (following Bensoussan [1]).

$$\begin{cases} E \ \text{ is a locally compact Hausdorff space with countable base} \\ \Phi(t)1 = 1 \end{cases} \tag{3.1}$$

and, if E is not compact, we assume that

$$\begin{cases} \text{(i)} \quad \hat{C} := \{f \in C \ : \quad \forall\varepsilon, \ \ \exists K_\varepsilon \ \text{ compact satisfying} \\ \qquad\qquad\qquad |f(x)| < \varepsilon \ \text{ for any } \ x \notin K_\varepsilon\} \\ \qquad\ \text{is a closed subspace of } \ C \ . \\ \text{(ii)} \quad \Phi(t)f \to f \ \text{ in } \ \hat{C} \ \text{ as } \ t \to 0, \ \ \forall f \in \hat{C} \ . \end{cases} \tag{3.2}$$

The Markov process associated with $\Phi(t)$ is defined as follows. Let

$$P(x, t, \Gamma) = \Phi(t)\chi_\Gamma(x)$$

for any Borel set Γ of E, where χ_Γ is the characteristic function of Γ. Consider the canonical space $\Omega_0 = D(R^+, E)$, the space of functions $\omega(\cdot)$ continuous from the right and having left limits, with $F_0 = \sigma(x(t), t \geq 0)$, $x(t, \omega) = \omega(t)$, and $F_t = \sigma(x(s), 0 \leq s \leq t)$.

From the general theory of Markov processes (cf. Dynkin [4]) there exists a unique probability measure P_x on (Ω_0, F_0) such that,

$$\text{if } \ \bar{F}_t = F_{t+} \ \text{ completed, and } \ \bar{F}_0 = F_{0+} \ \text{ completed,}$$

then the process

$$(\Omega_0, \bar{F}_0, \bar{F}_t, P_x, x(t))$$

is a right continuous, quasi–left continuous, strong Markov process and

$$P_x(x(0) = x) = 1 \ .$$

Define now the cost functional

$$\left\{ \begin{array}{lcl} J_x(\tau_1,\tau_2) & = & E_x\{\int_{\tau_1}^{\tau_2} e^{-\alpha t} L(x(t))dt + \varphi(x(\tau_1))e^{-\alpha\tau_1} \\ & & \qquad +\psi(x(\tau_2))e^{-\alpha\tau_2}\} \end{array} \right. \tag{3.3}$$

for any pair of $\bar{F}_t$ stopping times $(\tau_1,\tau_2), \tau_1 \le \tau_2$. The main result is the following

Theorem 3.1 *Under the assumptions of Theorem 2.1 and (3.1), (3.2)*

$$\hat{u}(x) = \inf_{(\tau_1,\tau_2)} J_x(\tau_1,\tau_2) \tag{3.4}$$

where $\hat{u}$ (and $\hat{v}$) is defined in Section 2. Moreover, there exists an optimal control $(\hat{\tau}_1,\hat{\tau}_2)$ given by

$$\hat{\tau}_1 = \left\{ \begin{array}{ll} \inf\{t \ge 0 \ : \ \hat{u}(x(t)) = \varphi(x(t)) + \hat{v}(x(t))\} & \textit{if finite}\,, \\ +\infty & \textit{otherwise}\,, \end{array} \right. \tag{3.5}$$

and

$$\hat{\tau}_2 = \hat{\tau}_1 + \hat{\tau} \circ \theta_{\hat{\tau}_1}\,, \tag{3.6}$$

where θ is the shift operator associated to the space Ω_0 (e.g. Dynkin [4]) and

$$\hat{\tau} = \inf\{t \ge 0 \ : \ \hat{v}(x(t)) = \psi(t))\}\ .$$

Proof: Using the fact that $\hat{v}$ satisfies (2.6) and by means of the Markov property, one deduces that $\hat{u}(x(s))e^{-\alpha s}$ is an $(\bar{F}_t,\ P_x)$ submartingale. Therefore, for any control (τ_1,τ_2)

$$\hat{u}(x) \le E_x e^{-\alpha\tau_1}\hat{u}(x_{\tau_1})$$

and since $\hat{u} \le \varphi + \hat{v}$,

$$\hat{u}(x) \le E_x\{e^{-\alpha\tau_1}\varphi(x(\tau_1)) + e^{-\alpha\tau_1}\hat{v}(x(\tau_1))\}\ . \tag{3.7}$$

Using (2.5) and the submartingale property for

$$e^{-\alpha t}\hat{v}(x(t)) + \int_0^t e^{-\alpha s}L(x(s))ds$$

we also have

$$E_x\{\hat{v}(x(\tau_1))e^{-\alpha\tau_1}\} \le E_x\{\hat{v}(x(\tau_2))e^{-\alpha\tau_2} + \int_{\tau_1}^{\tau_2} e^{-\alpha s}L(x(s))ds\}\ .$$

Since $\hat{v} \le \psi$, we deduce

$$\begin{array}{lcl} \hat{u}(x) & \le & E_x\{\int_{\tau_1}^{\tau_2} e^{-\alpha s}L(x(s))ds + \varphi(x(\tau_1))e^{-\alpha\tau_1} + \psi(x(\tau_2))e^{-\alpha\tau_2}\} = \\ & = & J_x(\tau_1,\tau_2)\ . \end{array}$$

If now $(\hat{\tau}_1,\hat{\tau}_2)$ is given by (3.5) and (3.6), we see that $\hat{\tau}_1$ is an optimal stopping time for the stopping problem associated to the cost function

$$I_x(\tau) = E_x\{e^{-\alpha\tau}[\varphi(x(\tau)) + \hat{v}(x(\tau))]\}\ ,$$

i.e.

$$\hat{u}(x) = \inf_{\tau} I_x(\tau) = I_x(\hat{\tau}_1) \ . \tag{3.8}$$

That is, (3.7) is replaced by an equality.

Now, we have

$$E_x\{\int_{\hat{\tau}_1}^{\hat{\tau}_2} e^{-\alpha s} L(x(s))ds + \psi(x(\hat{\tau}_2))e^{-\alpha\hat{\tau}_2} | \bar{F}_{\hat{\tau}_1}\} = e^{-\alpha\hat{\tau}_1} E_x\{Z \circ \theta_{\hat{\tau}_1} | \bar{F}_{\hat{\tau}_1}\} \ ,$$

with

$$Z = \int_0^{\hat{\tau}} e^{-\alpha s} L(x(s))ds + \psi(x(\hat{\tau}))e^{-\alpha\hat{\tau}} \ .$$

By means of the strong Markov property,

$$E_x\{Z \circ \theta_{\hat{\tau}_1} | \bar{F}_{\hat{\tau}_1}\} = E_{x(\hat{\tau}_1)} Z \ ,$$

but

$$\hat{v}(x) = E_x(Z) \ ,$$

therefore

$$E_x\{\int_{\hat{\tau}_1}^{\hat{\tau}_2} e^{-\alpha s} L(x(s))ds + \psi(x(\hat{\tau}_2))e^{-\alpha\hat{\tau}_2} \mid \bar{F}_{\hat{\tau}_1}\} = e^{-\alpha\hat{\tau}_1} v(x(\hat{\tau}_1)) \ , \tag{3.9}$$

and after using (3.9) in (3.8) , we obtain

$$\hat{u}(x) = J_x(\hat{\tau}_1, \hat{\tau}_2)$$

which completes the proof. □

4 Problems with Constraints

Optimal starting-stopping problems with constraints are considered in this section within the framework of general Markov-Feller processes. The constraints we are interested in are of two typical kinds. One is on the final state, and the other on the stopping times. For simplicity of notation, we now assume that $F_t = F_{t+}$ for any $t \geq 0$. As mentioned in Section 1, approaches different from the so-called QVI are possible, but we do not discuss them herein.

4.1 Constraint on the Final State

We consider the problem of Section 3 with for instance a constraint on the final state, namely

$$x(\tau_2) \notin F \text{ where } F \text{ is a given open set in } E. \tag{4.1}$$

The classical stopping problem with such a constraint was studied in Bensoussan and Lions [2] for diffusion processes and in Robin [7] for Feller-Markov processes with suitable assumptions. The key argument is to reduce the constrained problem to an unconstrained one. We use the same method here.

Let us consider the problem of minimization of

$$J_x(\tau_1, \tau_2) = E_x\{\int_{\tau_1}^{\tau_2} e^{-\alpha t} L(x(t))dt + \varphi(x(\tau_1)e^{-\alpha\tau_1} + \psi(x(\tau_2))e^{-\alpha\tau_2}\} , \tag{4.2}$$

on

$$V_{ad} = \{(\tau_1, \tau_2) \; : \; \tau_1 \leq \tau_2 \;, \;\; x(\tau_2) \notin F\} \tag{4.3}$$

and define

$$\hat{u}(x) = \inf\{J_x(\tau_1, \tau_2), (\tau_1, \tau_2) \in V_{ad}\} \; . \tag{4.4}$$

Define

$$g(x) = E_x\{\int_0^T e^{-\alpha t} L(x(t))dt + e^{-\alpha T}\psi(x(T))\} \tag{4.5}$$

with

$$T = \inf\{s \geq 0, \;\; x(s) \notin F\} \tag{4.6}$$

and let us consider the problem of minimization of

$$\tilde{J}_x(\tau_1, \tau_2) = E_x\{\int_{\tau_1}^{\tau_2} e^{-\alpha t} L(x(t))dt + \varphi(x(\tau_1))e^{-\alpha\tau_1} + g(x(\tau_2))e^{-\alpha\tau_2}\} \tag{4.7}$$

over any stopping times (τ_1, τ_2) with $\tau_1 \leq \tau_2$, and define

$$w(x) = \inf \tilde{J}_x(\tau_1, \tau_2) \; . \tag{4.8}$$

Theorem 4.1 *Under the assumptions of Theorem 3.1 $\hat{u} = w$.*

Proof: Assume $(\tau_1, \tau_2) \in V_{ad}$. Then (τ_1, τ_2) is admissible for (4.8) and since $x(\tau_2) \notin F$, we have $g(x(\tau_2)) = \psi(x(\tau_2))$. Therefore

$$\tilde{J}_x(\tau_1, \tau_2) = J_x(\tau_1, \tau_2) \; .$$

Hence

$$\begin{aligned} w(x) &\leq \inf\{\tilde{J}_x(\tau_1, \tau_2) \; : \; (\tau_1, \tau_2) \in V_{ad}\} = \\ &= \inf\{J_x(\tau_1, \tau_2) \; : \; (\tau_1, \tau_2) \in V_{ad}\} = \\ &= \hat{u}(x). \end{aligned}$$

Let now (τ_1, τ_2) be any pair of stopping times such that $\tau_1 \le \tau_2$. We have

$$\begin{aligned} E_x e^{-\alpha\tau_2} g(x(\tau_2)) &= E_x e^{-\alpha\tau_2} \{E_{x(\tau_2)} \int_0^T e^{-\alpha t} L(x(t))dt + e^{\alpha T}\psi(x(T))\} = \\ &= E_x\{\int_{\tau_2}^{\tilde\tau_2} e^{-\alpha t} L(x(t))dt + e^{\alpha\tilde\tau_2}\psi(x(\tilde\tau_2))\} \end{aligned}$$

where

$$\tilde\tau_2 = \tau_2 + T \circ \theta_{\tau_2} \ .$$

Therefore,

$$\begin{aligned} \tilde J_x(\tau_1, \tau_2) &= E_x\{\int_{\tau_1}^{\tau_2} e^{-\alpha t} L(x(t))dt + \varphi(x(\tau_1))e^{-\alpha\tau_1} + \int_{\tau_2}^{\tilde\tau_2} e^{-\alpha t} L(x(t))dt + \\ &+ e^{\alpha\tilde\tau_2}\psi(x(\tilde\tau_2))\} \ = \ J_x(\tau_1, \tilde\tau_2) \ . \end{aligned}$$

Hence $w \ge \hat u$ which completes the proof. □

Theorem 4.2 *Assume $g \in C$. Then*
(i) *The function u is characterized as in Theorem 2.1, namely, there exists a maximum element $(\hat u, \hat v)$ for the set of functions (u, v) satisfying*

$$\begin{cases} u, v \in C \\ v \le g \\ v \le e^{-\alpha t}\Phi(t)v + \int_0^t e^{-\alpha s}\Phi(s)Lds \\ u \le e^{-\alpha t}\Phi(t)u \\ u \le \varphi + v \ . \end{cases} \tag{4.9}$$

(ii) *Let (T_1, T_2) be optimal for the unconstrained problem (4.8), then*

$$\begin{cases} \hat\tau_1 = T_1 \\ \hat\tau_2 = T_2 + T \circ \theta_{T_2} \end{cases} \tag{4.10}$$

is optimal for (4.4).

Proof. **(i)** The pair $(w, \hat v)$ for the unconstrained problem is characterized as in Theorem 2.1 as the maximum solution of (4.9). Since $\hat u = w$ the result follows.

(ii) By the optimality of (T_1, T_2) we have

$$\tilde J_x(T_1, T_2) = w(x) \ .$$

But, as in the proof of Theorem 4.1.

$$\tilde J_x(T_1, T_2) = J_x(T_1, \hat\tau_2)$$

and because $w = \hat u$, we have

$$J_x(T_1, \hat\tau_2) = \hat u(x) \ .$$

Since $(T_1, \hat\tau_2)$ is in V_{ad}, it is optimal for (4.4). □

Remark 4.3 *One could obtain more general results, namely without assuming $g \in C$. This means that one may work with bounded measurable functions on E and some* good *theorems about optimal stopping problems (cf. references in Section 1).* $\square$

Remark 4.4 *The assumption $g \in C$ is realized for diffusion processes and for some other Markov processes such as diffusions with jumps under suitable hypothesis on the coefficients. Actually it depends on the regularity of the solution of the Dirichlet problem*

$$\begin{cases} Ag - \alpha g &= L \ \textit{inside} \ F \\ g &= \psi \ \textit{outside} \ F. \ \square \end{cases}$$

4.2 Constraint on the Stopping Times.

One can consider several kinds of constraints on the control namely the stopping times (τ_1, τ_2): for instance a constraint like $\tau_1 \leq T \leq \tau_2$ ($T > 0$ given), as in Sun [8]. This can be extended to general Markov-Feller process without serious difficulties.

Another kind of constrained problem arises when for example τ_2 has to satisfy

$$E_x(e^{-\alpha\tau_2}) \leq \theta, \quad \text{for some } \ \theta \ \text{ given in } \ [0,1]\,, \tag{4.11}$$

which may be interpreted in several ways, e.g. as an upper bound on the average of discount $e^{-\alpha\tau_2}$, or as a version of constraint on the lower bound of time the game may be stopped. Such a problem is the simplest version of the situation where we want to minimize a cost $J_x(\tau_1, \tau_2)$ under a constraint $I_x(\tau_1, \tau_2) \leq 0$ where I_x is a functional similar to J_x in (4.2), say with $L'()$, $\varphi'()$ $\psi'()$ instead of $L()$, $\varphi()$ $\psi()$. For instance, this type of formulation is involved when J_x represents the degree of realization of a technical objective and I_x the corresponding cost. It is then natural to optimize J_x under a constraint on the cost I_x, or the converse.

The following stopping problem:

$$\text{minimize} \quad J_x(\tau) = E_x\{\int_0^\tau e^{-\alpha t} L(x_t)dt + e^{-\alpha t}\psi(x_\tau)\} \tag{4.12}$$

under the constraint

$$E_x e^{-\alpha\tau} \leq \theta \tag{4.13}$$

was studied in Robin [7].

In general, there is no "pure" optimal control. We have to enlarge the admissible controls to randomized stopping times. We are going to show briefly how to adapt those results to the problem of the minimization of (3.3) with the constraint (4.12).

Firstly, we will formulate the problem with randomized stopping τ_2. Similarly, a formulation with a randomized starting time τ_1 (or both simultaneously) can be presented. Let (Ω', F', P') be a probability space on which we assume that variables $\{\xi_p\}_{p\in[0,1]}$, are defined such that ξ_p takes the value 1 with probability p and 2 with probability $1 - p$.

Define

$$\Omega_1 = \Omega \times \Omega', \quad F^1_t = F_t \otimes F', \quad P^1_x = P_x \otimes P'\,.$$

Then $x^1 = (\Omega_1, F_t^1, x_t, P_x^1)$ is still a Markov process with the semigroup $\Phi(t)$.

Given τ_2', τ_2'', F_t-stopping times, one can define

$$\tau_2 = \tau_2' \chi_{\{\xi_p=1\}} + \tau_2'' \chi_{\{\xi_p=2\}} \ ,$$

where χ_A is the indicator function of the set A. Then τ_2 is an F_t^1-stopping time.

The problem we address is now P_0: to minimize

$$J_x(\tau_1, \tau_2) = E_x^1 \{ \int_{\tau_1}^{\tau_2} e^{-\alpha t} L(x_t) dt + \varphi(x_{\tau_1}) e^{-\alpha \tau_1} + \psi(x_{\tau_2}) e^{-\alpha \tau_2} \} \tag{4.14}$$

with

$$\begin{cases} \tau_1 \leq \tau_2 \\ E_x^1 e^{-\alpha \tau_2} \leq \theta \,. \end{cases} \tag{4.15}$$

Denote by $(\hat{\tau}_1, \hat{\tau}_2)$ the optimal solution of the problem of the Section 3 (unconstrained). In order to consider a non trivial problem, we assume

$$E_x^1 e^{-\alpha \hat{\tau}_2} > \theta \ , \tag{4.16}$$

otherwise, $(\hat{\tau}_1, \hat{\tau}_2)$ is obviously a solution of P_0. We introduce the problem P_λ: (with $\lambda \geq 0$) to minimize

$$J_x^\lambda(\tau_1, \tau_2) = J_x(\tau_1, \tau_2) + \lambda E_x^1 e^{-\alpha \tau_2} \ , \tag{4.17}$$

which is related to the usual Lagrange function for constrained optimization.

According to the Sections 2 and 3, this problem has an optimal solution denoted by

$$(\tau_1^\lambda, \tau_2^\lambda)$$

(which is not randomized) and let (u^λ, v^λ) be the corresponding (maximum) solutions of (2.4).

Notice that in particular, v^λ is the maximum solution of

$$\begin{cases} v & \leq & \psi + \lambda \\ v & \leq & e^{-\alpha t} \Phi(t) v + \displaystyle\int_0^t e^{-\alpha s} \Phi(s) L \, ds \,. \end{cases} \tag{4.18}$$

We have the following

Lemma 4.5 *If $E_x^1\{e^{-\alpha \tau_2^\lambda}\} = \theta$, for some λ (which is actually a stronger version of the usual Kuhn-Tucker optimality condition), then $(\tau_1^\lambda, \tau_2^\lambda)$ is optimal for P_0.*

Proof: From the optimality of $(\tau_1^\lambda, \tau_2^\lambda)$ we have

$$J_x^\lambda(\tau_1^\lambda, \tau_2^\lambda) \leq J_x^\lambda(\tau_1, \tau_2) \ \text{ for any } \ (\tau_1, \tau_2).$$

The condition $E_x^1(e^{-\alpha \tau_2^\lambda}) = \theta$ gives

$$J_x(\tau_1^\lambda, \tau_2^\lambda) \leq J_x(\tau_1, \tau_2) + \lambda E_x^1(e^{-\alpha \tau_2} - \theta).$$

Therefore, for any (τ_1, τ_2) such that $E_x^1 e^{-\alpha \tau_2} \leq \theta$ we get $J_x(\tau_1^\lambda, \tau_2^\lambda) \leq J_x(\tau_1, \tau_2)$. $\square$

Lemma 4.6 *Denote the supremum norm by* $\|\cdot\|$ *(i.e. the norm in* C*). We have*

$$\begin{array}{ll} \textbf{(I)} & \|u^\lambda - u^{\lambda'}\| \leq |\lambda - \lambda'| \\ & \|v^\lambda - v^{\lambda'}\| \leq |\lambda - \lambda'| \\ \textbf{(II)} & \text{if } \lambda \uparrow \lambda' \quad \text{then} \quad \tau_i^\lambda \uparrow \tau_i^{\lambda'} \quad i = 1, 2\,. \end{array}$$

Proof: (I) is obvious, so we will prove only (II).

Firstly, for the stopping problem corresponding to (4.19), the property (II), for τ^λ given by

$$\tau^\lambda = \inf\{t \geq 0 \;:\; v^\lambda(x_t) = \psi(x_t) + \lambda\} \tag{4.19}$$

is proved in Robin [7]. Let us show that $\lambda \to \tau_1^\lambda$ is increasing. For that purpose, we will show that, if $\mu > 0$,

$$\{x \;:\; u^\lambda < \varphi + v^\lambda\} \subset \{u^{\lambda+\mu} < \varphi + v^{\lambda+\mu}\}. \tag{4.20}$$

It is enough to prove that

$$w^\lambda = u^\lambda - v^\lambda \quad \text{is decreasing w.r.t.} \quad \lambda\,. \tag{4.21}$$

We first consider the approximation of v^λ by the penalized problem

$$\left\{ \begin{array}{lcl} v_\varepsilon^\lambda & = & \displaystyle\int_0^\infty e^{-\alpha t}\phi(t)[L - \frac{1}{\varepsilon}(v_\varepsilon^\lambda - \psi - \lambda)^+]dt \\ v_\varepsilon^\lambda & \in & C\,. \end{array} \right. \tag{4.22}$$

It is known that $v_\varepsilon^\lambda \searrow v^\lambda$ uniformly as $\varepsilon \searrow 0$ (cf. Bensoussan and Lions [2] for instance). Associated with (4.23) one can define the corresponding approximation of u^λ, namely u_ε^λ, the maximum solution of

$$\left\{ \begin{array}{lcl} u_\varepsilon^\lambda & \leq & \varphi + v_\varepsilon^\lambda \\ u_\varepsilon^\lambda & \leq & e^{-\alpha t}\Phi(t)u_\varepsilon^\lambda \end{array} \right. \tag{4.23}$$

and since $v_\varepsilon^\lambda \to v^\lambda$ uniformly, the same property holds for u_ε^λ and u^λ (cf. Bensoussan and Lions [2] p.320). Using (4.23) and (4.24), we deduce that $w_\varepsilon^\lambda = u_\varepsilon^\lambda - v_\varepsilon^\lambda$ is the maximum solution of

$$\left\{ \begin{array}{lcl} w_\varepsilon^\lambda & \leq & \varphi \\ w_\varepsilon^\lambda & \leq & e^{-\alpha t}\Phi(t)w_\varepsilon^\lambda + \displaystyle\int_0^t e^{-\alpha s}\phi(s)[-L + \frac{1}{\varepsilon}(v_\varepsilon^\lambda - \psi - \lambda)^+]ds \\ w_\varepsilon^\lambda & \in & C\,. \end{array} \right. \tag{4.24}$$

Defining

$$\xi_\varepsilon^\lambda = v_\varepsilon^\lambda - \lambda$$

one has

$$\xi_\varepsilon^\lambda = \int_0^\infty e^{-\alpha t}\Phi(t)[L - \alpha\lambda - \frac{1}{\varepsilon}(\xi_\varepsilon^\lambda - \psi)^+]dt\,.$$

One easily checks that $\lambda' \geq \lambda$ implies $\xi^{\lambda'} \leq \xi^{\lambda}$. Rewriting (4.25) as

$$\begin{cases} w_\varepsilon^\lambda & \leq \varphi \\ w_\varepsilon^\lambda & \leq e^{-\alpha t}\Phi(t) w_\varepsilon^\lambda + \int_0^t e^{-\alpha s}\Phi(s)\tilde{L}(\lambda,\varepsilon) ds \,, \end{cases} \tag{4.25}$$

where

$$\tilde{L}(\lambda,\varepsilon) = \frac{1}{\varepsilon}(\xi_\varepsilon^\lambda - \psi)^+ - L$$

which is decreasing w.r.t. λ. Since the maximum solution of (4.25) is increasing w.r.t. $\tilde{L}$, we deduce that

$$w_\varepsilon^{\lambda+\mu} \leq w_\varepsilon^\lambda \quad \text{for} \quad \nu \geq 0.$$

When ε goes to zero we obtain (4.22) and therefore (4.21). The proof that $\lim_{\lambda\uparrow\lambda'} \tau^\lambda = \tau^{\lambda'}$ is identical to the one in Robin [7].

Condition (II) for τ_2^λ is a consequence of the definition of τ_2^λ

$$\tau_2^\lambda = \tau_1^\lambda + \tau^\lambda \circ \theta_{\tau_1^\lambda}$$

and the same property (II) for τ^λ and τ_1^λ. $\square$

Theorem 4.7 *There exists an optimal solution of the problem P_0 (among the randomized controls).*

Proof: Define $h(\lambda) = E_x^1\{e^{-\alpha\tau_2^\lambda}\}$ and

$$\lambda_0 = \sup\{\lambda \geq 0 \ : \ h(\lambda) > \theta\}.$$

Notice that $h(\lambda) \leq \theta$ when λ is large enough. Moreover, if λ_0 is a point of continuity of $h(\lambda)$, we have a λ_0 such that $h(\lambda_0) = \theta$ and therefore, Lemma 4.1 gives the result (with non randomized control). But in general, $h(\lambda)$ is not continuous because, even for "regular" processes, τ_i^λ is not continuous from the right. Generally speaking, when $\varepsilon \searrow 0$, $\tau_i^{\lambda+\varepsilon} \searrow \bar{\tau}_i \geq \tau_i^\lambda$ and

$$P_x\{\bar{\tau}_i > \tau_i^\lambda\} > 0\,.$$

However, consider $\lambda_0 + \varepsilon$, $\varepsilon > 0$

$$u^{\lambda_0+\varepsilon} = J_x^{\lambda_0+\varepsilon}(\tau_1^{\lambda_0+\varepsilon}, \tau_2^{\lambda_0+\varepsilon}).$$

one has

$$\tau_i^{\lambda_0+\varepsilon} \searrow \bar{\tau}_i \text{ as } \varepsilon \searrow 0,; \quad i = 1,2,$$

where $\bar{\tau}_i$ is an F_t^1 stopping time since F_t^1 is continuous from the right.

Moreover $u^{\lambda_0+\varepsilon} \to u^{\lambda_0}$ uniformly as ε goes to 0. Therefore, as $\varepsilon \searrow 0$, we obtain

$$u^{\lambda_0} = J_x^{\lambda_0}(\bar{\tau}_1, \bar{\tau}_2)\ ,$$

meaning that $(\bar\tau_1, \bar\tau_2)$ is also an optimal control for P_{λ_0}.

One can choose $p \in [0,1]$ such that

$$\tau_2^\star = \tau_2^{\lambda_0} \chi_{\{\xi_p=1\}} + \bar\tau_2 \chi_{\{\xi_p=2\}}$$

satisfies

$$E_x^1 e^{-\alpha \tau_2^\star} = \theta \,.$$

Defining τ_1'' with the same randomization, we obtain a control satisfying Lemma 4.1. □

Remark 4.8 *The previous results are easily extended to a constraint of the form*

$$E_x\{\mu(x_{\tau_2}) e^{-\alpha\tau_2}\} \le K$$

where μ is a positive continuous function. The same kind of method can be also used for a constraint like

$$E_x \int_{\tau_1}^{\tau_2} e^{-\alpha t} h(x_t) dt \le K \,.$$

This requires some adaptation: for instance if $h \ge 0$, v^λ and u^λ will be increasing, but

$$\tau^\lambda = \inf\{t \ge 0 \;:\; v^\lambda(x_t) = \psi(x_t)\}$$

is now decreasing w.r.t. λ. □

To complete the above theorem, we will describe an algorithm to approximate (i) the value of λ_0 as defined in the proof of Theorem 4.8, (ii) the value of $u_0(x)$ defined as the minimum of the cost functional (4.15) subject to (2.16) over all the randomized policies (τ_1, τ_2), (iii) an ε-optimal randomized control policy (τ_1, τ_2).

To describe the algorithm, let us introduce the Lagrange functional associated with the current constrained problem

$$L_x^\lambda(\tau_1, \tau_2) = J_x(\tau_1, \tau_2) - \lambda[\theta - E e^{-\alpha\tau_2}] \,.$$

Algorithm:

Step 0. Set $\lambda = 0$.

Step 1. Minimize $L_x^\lambda(\tau_1, \tau_2)$ over F_t^1-policies without constraint (2.16) to get a non randomized optimal policy $(\tau_1^\lambda, \tau_2^\lambda)$ and the value functions (u^λ, v^λ) as described in Section 3.

Step 1.1 If $E e^{-\alpha\tau_2} \le \theta$ then $(\tau_1^\lambda, \tau_2^\lambda)$ is the desired optimal policy and **stop**.

Step 1.2 If $E e^{-\alpha\tau_2} > \theta$ then set $\lambda^+ = \lambda$, and update $\lambda = \lambda + [E e^{-\alpha\tau_2} - \theta]$.

Step 2. Minimize $L_x^\lambda(\tau_1, \tau_2)$ over F_t^1-policies without constraint (2.16) to get a non randomized optimal policy $(\tau_1^\lambda, \tau_2^\lambda)$ and the value functions (u^λ, v^λ) as in **Step 1**.

Step 2.1 If $E e^{-\alpha\tau_2} = \theta$ then $(\tau_1^\lambda, \tau_2^\lambda)$ is the desired optimal policy and **stop**.

Step 2.2 If $Ee^{-\alpha\tau_2} < \theta$ then set $\lambda^- = \lambda$, and update $\lambda = (\lambda^+ + \lambda^-)/2$.

Step 2.3 If $Ee^{-\alpha\tau_2} > \theta$ then set $\lambda^+ = \lambda$, and

Step 2.3.1 If λ^- has not been defined then update $\lambda = \lambda + [Ee^{-\alpha\tau_2} - \theta]$

Step 2.3.2 If λ^- has been defined then update $\lambda = (\lambda^+ + \lambda^-)/2$.

Step 3 Repeat **Step 2** until a prescribed stopping criterion is satisfied. □

Let us analyze the above procedure. If this algorithm terminates in a finite number of steps, then it reaches an optimal non randomized control policy. Otherwise, it generates one of the following two sequences

$$\textbf{either} \quad \{\lambda_n^+, \lambda_n, \tau_{1,n}, \tau_{2,n}, u_n, v_n\} \qquad \textbf{or} \quad \{\lambda_n^+, \lambda_n^-, \tau_{1,n}, \tau_{2,n}, u_n, v_n\} ,$$

where

$$\tau_{i,n} = \tau_i^{\lambda_n} , \quad i = 1, 2 , \qquad\qquad w_n = w^{\lambda_n} , \quad w = u, v .$$

We describe below how to construct a near optimal solution in those two cases.

* **Case 1.** The algorithm generates the sequence $\{\lambda_n^+, \lambda_n, \tau_{1,n}, \tau_{2,n}, u_n, v_n\}$. We specifically note that both λ_n and $\tau_{2,n}$ increase and that $Ee^{-\alpha\tau_2^\lambda} < \theta$ for large λ. Thus we have

$$\lambda_n^+ = \lambda_n \nearrow \lambda^* , \qquad \tau_{2,n} \nearrow \tau^* , \qquad Ee^{-\alpha\tau_{2,n}} \nearrow Ee^{-\alpha\tau^*} = \theta .$$

By means of Lemma 4.7 (I), we deduce

$$\tau^* = \tau_2^{\lambda^*} .$$

Therefore $(\tau_1^{\lambda^*}, \tau_2^{\lambda^*})$ is an optimal non randomized policy for the constrained problem. Thus, for any $\varepsilon > 0$ there is some n such that $(\tau_{1,n}, \tau_{2,n})$ satisfies (with the notation of Theorem 4.8)

$$\begin{aligned} &|u_0(x) - u_n(x)| \le \varepsilon , \qquad |\lambda_n^+ - \lambda_0| \le \varepsilon \\ &J_x(\tau_{1,n}, \tau_{2,n}) \le u_0(x) + \varepsilon , \qquad Ee^{-\alpha\tau_{2,n}} \le \theta + \varepsilon . \end{aligned}$$

* **Case 2.** The algorithm generates the sequence $\{\lambda_n^+, \lambda_n^-, \tau_{1,n}, \tau_{2,n}, u_n, v_n\}$. Here, we let

$$\begin{aligned} &\theta_n^+ = Ee^{-\alpha\tau_2^{\lambda_n^+}} , \qquad \theta_n^- = Ee^{-\alpha\tau_2^{\lambda_n^-}} , \qquad p_n = \frac{\theta - \theta_n^-}{\theta_n^+ - \theta_n^-} , \\ &\tau_{1,n}^* = \tau_1^{\lambda_n^+} \chi\{\xi_{p_n} = 1\} + \tau_1^{\lambda_n^-} \chi\{\xi_{p_n} = 2\} , \\ &\tau_{2,n}^* = \tau_2^{\lambda_n^+} \chi\{\xi_{p_n} = 1\} + \tau_2^{\lambda_n^-} \chi\{\xi_{p_n} = 2\} . \end{aligned}$$

It is not hard to show that for any $\varepsilon > 0$, there is some n such that $(\tau_{1,n}^*, \tau_{2,n}^*)$ is an ε-optimal randomized control policy for the constrained problem, i.e.,

$$|u_0(x) - [u_n(x) - \lambda_n\theta]| \le \varepsilon , \qquad |\lambda_n^+ - \lambda_0| \le \varepsilon .$$

Remark 4.9 *If the updating formula* $\lambda = \lambda + [Ee^{-\alpha\tau_2} - \theta]$ *in our algorithm is replaced by*

$$\lambda = \lambda + \max\{[Ee^{-\alpha\tau_2} - \theta], \delta\} \ , \tag{4.26}$$

for some prescribed $\delta > 0$*, then* **Case 1** *is easily avoided. Nevertheless, the updating formula*

$$\lambda = \lambda + [Ee^{-\alpha\tau_2} - \theta] \tag{4.27}$$

is more popular in the nonlinear programming literature. Note that in **Case 1** *our algorithm gives us an approximate non randomized policy, but with the constant* θ *being slightly perturbed.* □

Reference

1. A. Bensoussan, *Stochastic Control by Functional Analysis Methods*, North Holland, Amsterdam, 1982.

2. A. Bensoussan and J.L. Lions, *Applications des inequations variationnelles en controle stochastique*, Dunod, Paris 1978. [English Version North Holland, Amsterdam, 1982].

3. J.M. Bismut, Temps d'arrët optimal, quasi-temps d'arrët et retournement du temps, *Ann. Probab.*, **7** (1979), pp. 993–964.

4. E.B. Dynkin, *Markov Processes*, Vols. 1 and 2, Springer-Verlag, Berlin, 1965.

5. N. El Karoui, J.L. Lepeltier and A. Millet, A Probabilistic Approach to the Reduite in Optimal Stopping, *Probab. Math. Stat.*, **13** (1992), pp. 97–121.

6. J.F. Mertens, Strongly Supermedian Functions and Optimal Stopping, *Z. Wahrsch. und Verw.*, **22** (1972), pp. 45–68.

7. M. Robin, An Optimal Stochastic Control Problems with Constraints, in *Game Theory and Related Topics*, O. Moeschlin and D. Pallaschke, Eds., North Holland, Amsterdam, 1979.

8. M. Sun, Nested Variational Inequalities and Related Starting-Stopping Problems, *J. Appl. Probab.*, **29** (1992), pp. 104–115.