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Remarks on Risk-Sensitive Control Problems

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Abstract

The main purpose of this paper is to investigate the asymptotic behavior of the discounted risk-sensitive control problem for periodic diffusion processes when the discount factor α goes to zero. If $u_\alpha(\theta, x)$ denotes the optimal cost function, θ being the risk factor, then it is shown that $\lim_{\alpha\to 0} \alpha u_\alpha(\theta, x) = \xi(\theta)$ where $\xi(\theta)$ is the average on $]0, \theta[$ of the optimal cost of the (usual) infinite horizon risk-sensitive control problem.

1 Introduction

Let us consider a simple stochastic control model given by the following Itô equation

$$\mathrm{d}x_t = b(x_t, v_t)\mathrm{d}t + \sqrt{2}\,\mathrm{d}B_t, \qquad x_0 = x, \tag{1.1}$$

where x is the state of the system in $\mathbb{R}^d$ and v is the control in $\mathbb{R}^m$. For a parameter $\theta \neq 0$, the functional cost is

$$I_\alpha(\theta, x, v) = \frac{1}{\theta} \ln \Big(\mathbb{E}\Big\{ \exp \Big[\theta \int_0^\infty e^{-\alpha t} \varphi(x_t, v_t)\mathrm{d}t \Big] \Big\} \Big), \tag{1.2}$$

and the value function is, for $\theta > 0$,

$$u_\alpha(\theta, x) = \inf_v I_\alpha(\theta, x, v), \tag{1.3}$$

and we exchange inf with the sup for $\theta < 0$. However, in the sequel, we consider only $\theta > 0$ for the sake of simplicity.

The aim of this paper is to investigate the asymptotic behavior of αu_α when α goes to zero.

Nagai [10] studied the asymptotic behavior of the finite horizon risk-sensitive control problem, namely,

$$J(T, x, v) = \frac{1}{\theta} \ln \Big(\mathbb{E}\Big\{ \exp \Big[\theta \int_0^T \varphi(x_t, v_t)\mathrm{d}t \Big] \Big\} \Big) \tag{1.4}$$

and shows that if θ is fixed and

$$u_T(t,x) = \inf_v J(T-t,x,v) \tag{1.5}$$

then

$$\lim_{T\to\infty} \frac{1}{T} u_T(T,x) = \chi, \text{ (constant)},$$

and

$$\lim_{T\to\infty} \big[u_T(T,x) - u_T(0,x)\big] = z(x), \text{ (function)},$$

where the couple (χ, z) satisfies the equation

$$\chi = \Delta z + \theta |Dz|^2 + \inf_v \big\{\varphi + b\cdot\nabla z\big\}. \tag{1.6}$$

Clearly, (χ, z) may depends on θ.

We will see in Section 2, that the HJB equation for (1.3) is

$$-\alpha\big(u_\alpha + \theta\frac{\partial u_\alpha}{\partial\theta}\big) + \Delta u_\alpha + \theta|\nabla u_\alpha|^2 + \inf_v \big\{\varphi + b\cdot\nabla u_\alpha\big\} = 0. \tag{1.7}$$

Comparing (1.6) and (1.7), we can anticipate that

$$\alpha\big(u_\alpha + \theta\frac{\partial u_\alpha}{\partial\theta}\big) \to \chi(\theta), \quad \text{as} \quad \alpha\to 0. \tag{1.8}$$

In other words, assume that there exists $\xi(\theta)$ (independent of x) such that

$$\alpha u_\alpha(\theta,x) \to \xi(\theta) \;\text{ and }\; \alpha\frac{\partial u_\alpha}{\partial\theta}(\theta,x) \to \frac{\mathrm{d}\xi(\theta)}{\mathrm{d}\theta},$$

as $\alpha\to 0$, we would have, by (1.8),

$$\chi(\theta) = \xi(\theta) + \theta\frac{\mathrm{d}\xi(\theta)}{\mathrm{d}\theta} = \frac{\mathrm{d}}{\mathrm{d}\theta}\big[\theta\,\xi(\theta)\big]$$

and

$$\xi(\theta) = \frac{1}{\theta}\int_0^\theta \chi(r)\mathrm{d}r = \lim_\alpha \alpha u_\alpha(\theta,x). \tag{1.9}$$

Notice that when $\theta = 0$, the equation (1.7) corresponds to the usual discounted control, e.g., see Bensoussan [1]. Condition (1.9) is precisely the result we will obtain here for the case of periodic diffusion (or reflected diffusions on a bounded region of $\mathbb{R}^d$).

The risk-sensitive control problem for diffusion processes (in various cases) has been studied by several authors, particularly in connection with robust control and differential games, for instance, we refer to Jacobson [7], Bensoussan and Van Schuppen [4], Whittle [12], Fleming and McEneaney [6], McEneaney [8], Nagai [9, 10], Runolfsson [11].

In Section 2, we obtain formally the HJB-equation for (1.3), and a *verification theorem.* In Section 3, we study the discounted risk-sensitive problem, and in Section 4, we consider the asymptotic behavior when the discount factor goes to zero.

2 Formal Derivation of the HJB Equation

We start with

$$w_\alpha(\theta,x) = \inf_v \exp\big[\theta I_\alpha(\theta,x,v)\big]. \tag{2.1}$$

Formally, for any $T>0$ and for any Markov control $v_t = v(x_t)$, we argue as follows

$$\begin{aligned} w_\alpha(\theta,x) &= \inf_v \mathbb{E}_x\Big\{ \exp\Big[\theta\int_0^T e^{-\alpha t}\varphi((x_t,v_t)\mathrm{d}t + \\ &\qquad +\theta\int_T^\infty e^{-\alpha t}\varphi((x_t,v_t)\mathrm{d}t\Big]\Big\} = \\ &= \inf_v \mathbb{E}_x\Big\{ \exp\Big[\theta\int_0^T e^{-\alpha t}\varphi((x_t,v_t)\mathrm{d}t\Big] \times \\ &\qquad \times \mathbb{E}_{x_T}\Big\{ \exp\Big[\theta e^{-\alpha T}\int_0^\infty e^{-\alpha t}\varphi((x_t,v_t)\mathrm{d}t\Big]\Big\}\Big\}. \end{aligned}$$

Therefore (formally)

$$w_\alpha(\theta,x) = \inf_{v/[0,T]} \mathbb{E}_x\Big\{ \exp\Big[\theta\int_0^T e^{-\alpha t}\varphi(x_t,v_t)\mathrm{d}t\Big] w_\alpha(\theta e^{-\alpha T}, x_T)\Big\}.$$

Using Itô's formula for $w_\alpha(\theta e^{-\alpha T}, x_T)$, and taking $T>0$ small, we obtain

$$-\alpha\theta\frac{\partial w_\alpha}{\partial\theta} + \Delta w_\alpha + \inf_v \big\{\theta\varphi w_\alpha + b\cdot\nabla w_\alpha\big\} = 0, \tag{2.2}$$

and clearly $w_\alpha(0,x) = 1$.

Next, we set $w_\alpha = \exp(\theta u_\alpha)$ to deduce

$$-\alpha\big(u_\alpha + \theta\frac{\partial u_\alpha}{\partial\theta}\big) + \Delta u_\alpha + \theta|\nabla u_\alpha|^2 + \inf_v \big\{\varphi + b\cdot\nabla u_\alpha\big\} = 0. \tag{2.3}$$

Remark that one should take

$$u_\alpha(0,x) = \inf_v \mathbb{E}_x\Big\{\int_0^\infty e^{-\alpha t}\varphi(x_t,v_t)\mathrm{d}t\Big\}, \tag{2.4}$$

since, when θ is small in (1.2) we have

$$I_\alpha(\theta,x,v) = \mathbb{E}_x\Phi + \theta\mathbb{E}_x\Phi^2 + O(\theta^2),$$

where

$$\Phi = \int_0^\infty e^{-\alpha t}\varphi(x_t,v_t)\mathrm{d}t.$$

Theorem 2.1 (implicit assumptions)**.** *Let us assume that there exists a smooth function $W(\theta, x)$ such that*

$$-\alpha\theta\frac{\partial W}{\partial\theta} + \Delta W + \inf_v \big\{\theta\varphi W + b\cdot\nabla W\big\} = 0, \tag{2.5}$$

and $W(\theta, x) \to 1$ as $\theta \to 0$, locally uniform in x. Also assume that there exists an optimal control v^*. Then

$$W(\theta, x) = w_\alpha(\theta, x). \tag{2.6}$$

Proof. To see this, introduce θ_t defined by

$$\frac{\mathrm{d}\theta_t}{\mathrm{d}t} = -\alpha\theta_t, \qquad \theta_0 = \theta$$

and

$$\psi_T = \exp\Big\{\int_0^T \theta_t\varphi(x_t, v_t)\mathrm{d}t\Big\},$$

for an arbitrary control v_s. By means of Feynman-Kac formula we get

$$\mathbb{E}_x\big\{\psi_T W(\theta_T, x_T)\big\} = W(\theta, x) + \\ +\mathbb{E}_x\Big\{\int_0^T \psi_t\Big[-\alpha\theta\frac{\partial W}{\partial\theta} + \Delta W + \theta\varphi W + b\cdot\nabla W\Big]\mathrm{d}t\Big\}.$$

From the equation for W the last term is nonnegative, and therefore

$$W(\theta, x) \le \mathbb{E}_x\Big\{W(\theta_T, x_T)\exp\Big[\theta\int_0^T e^{-\alpha t}\varphi(x_t, v_t)\mathrm{d}t\Big]\Big\}.$$

Hence, because $\theta_T \to 0$ as $T \to \infty$ and $W(\theta_T, x_T) \to 1$ (locally uniform in x_T) as $\theta \to 0$ we deduce

$$W(\theta, x) \le \mathbb{E}_x\Big\{\exp\Big[\theta\int_0^\infty e^{-\alpha t}\varphi(x_t, v_t)\mathrm{d}t\Big]\Big\},$$

i.e., $W(\theta, x) \le w_\alpha(\theta, x)$.

Similarly, using the optimal control v^* we obtain the equality. □

Clearly, as a Corollary, using U defined by $W = \exp(\theta U)$ we obtain $U = u_\alpha$.

3 Discounted Risk-sensitive Problem

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $(\mathcal{F}_t : t \ge 0)$ and a standard d-dimensional $\mathcal{F}_t$-Brownian motion process $(B_t : t \ge 0)$. We are given V a compact metric space, $X = [(\mathbb{R}^d) \mod (1)] \simeq]0, 1]^d$

$$b : X \times V \to \mathbb{R}^d, \qquad \varphi : X \times V \to \mathbb{R}, \tag{3.1}$$

where $b(x,v)$ and $\varphi(x,v)$ are periodic in x with period 1 in each coordinate (as functions defined on $\mathbb{R}^d$), b is continuous in $X \times V$ and Lipschitz continuous in x, namely,

$$|b(x,v) - b(x',v)| \leq M|x - x'|, \quad \forall x,\, x' \in X, \tag{3.2}$$

φ is continuous and nonnegative.

The state equation is given by

$$\begin{cases} \mathrm{d}x_t = b(x_t, v_t)\mathrm{d}t + \sqrt{2}\mathrm{d}B_t, \quad t > 0, \\ x_0 = x \in X, \end{cases} \tag{3.3}$$

where $(v_t : t \geq 0)$ is any progressively measurable process with values in V.

As above, the cost is given by

$$I_\alpha(\theta, x, v) = \frac{1}{\theta} \ln \mathbb{E}_x \Big\{ \exp\Big(\theta \int_0^\infty \mathrm{e}^{-\alpha t} \varphi(x_t, v_t)\mathrm{d}t\Big)\Big\}, \tag{3.4}$$

where $\alpha > 0$ is the discount factor and θ is a real parameter. For the sake of simplicity, we will consider only the case $\theta > 0$. The optimal cost function is

$$u_\alpha(\theta, x) = \inf_v I_\alpha(\theta, x, v). \tag{3.5}$$

Remark 3.1. One could avoid the assumption (3.2) that b is Lipschitz continuous and then define the state equation using the Girsanov transformation (e.g., see Bensoussan [1, Chapter 6]). □

As seen in Section 2, the HJB-equation for (3.5) is

$$A_\theta u_\alpha + \alpha u_\alpha = H(\theta, x, Du_\alpha), \tag{3.6}$$

with u_α periodic in x,

$$A_\theta u := \alpha\theta\partial_\theta u - \Delta u - \theta|Du|^2,$$
$$H(\theta, x, p) := \inf_v \big\{\varphi(x,v) + b(x,v)\cdot p\big\},$$

and

$$u_\alpha(0, x) = u_\alpha^0(x), \tag{3.7}$$

with

$$A_0 u_\alpha^0 = H(0, x, Du_\alpha^0), \tag{3.8}$$

and u_α^0 periodic. Note that Du, Δu and $\partial_\theta u$ denote the gradient in x, the Laplacian in x, and the partial derivative in θ, respectively.

It is well known (e.g., see Bensoussan and Lions [2, 3]) that (3.8) has a unique solution in $W^{2,p}(X)$, $2 \leq p < \infty$. Without any lost of generality, we consider (3.6) with θ in $]0,1[$.

First we study an auxiliary equation in w, namely,

$$\alpha\theta\partial_\theta w - \Delta w = \inf_v \big\{\theta\varphi w + b\cdot Dw\big\}, \tag{3.9}$$

with w periodic in x and $w(0,x) = 1$.

Proposition 3.2. *Assuming* (3.1) *and* (3.3)*, there is a unique solution w of* (3.9) *in* $H^1(]0,1[\times X)$ *such that w and $\partial_\theta w$ belong to* $L^\infty(]0,1[\times X)$.

Proof. We begin with the following equation for ε in $]0,1[$,

$$\begin{aligned} &\alpha\theta\partial_\theta w^\varepsilon - \Delta w^\varepsilon = \inf_v \big\{\theta\varphi w^\varepsilon + b\cdot Dw^\varepsilon\big\}, \quad \theta \in]\varepsilon, 1[, \\ &w^\varepsilon(\varepsilon, x) = h_\varepsilon(x), \quad x \in X, \end{aligned} \tag{3.10}$$

with w^ε periodic in x and

$$h^\varepsilon(x) = \mathrm{e}^{\frac{\varepsilon}{\alpha}\|\varphi\|}, \tag{3.11}$$

where

$$\|\varphi\| := \sup_{x,v} |\varphi(x,v)|,$$

and clearly $h_\varepsilon \to 1$ as $\varepsilon \to 0$.

Since θ belongs to $]\varepsilon,1[$, equation (3.10) can be seen as a standard Cauchy problem and there is a unique solution w^ε in $W^{1,2}_p(]\varepsilon,1[\times X)$, $2 \le p < \infty$. Therefore, we can interpret $w^\varepsilon(\theta,x)$ as the following optimal cost

$$w^\varepsilon(\theta,x) = \inf_v \mathbb{E}_x\Big\{h_\varepsilon \exp\Big(\theta\int_0^{T_\varepsilon} \mathrm{e}^{-\alpha t}\varphi(x_t,v_t)\mathrm{d}t\Big)\Big\}, \tag{3.12}$$

by applying Itô formula to $\psi_T w(\theta_T, x_T)$ with

$$\theta_t := \theta \mathrm{e}^{-\alpha t}, \quad \psi_T = \exp\Big(\int_0^T \theta_t\varphi(x_t,v_t)\mathrm{d}t\Big),$$

and where we have taken

$$T_\varepsilon = \inf\big\{t \ge 0 : \theta_t = \varepsilon\big\}, \quad \text{i.e. } T_\varepsilon = \frac{\ln(\frac{\theta}{\varepsilon})}{\alpha}.$$

Then we deduce

$$0 \le w^\varepsilon(\theta,x) \le \mathrm{e}^{\frac{\theta}{\alpha}\|\varphi\|}, \tag{3.13}$$

for every $\varepsilon > 0$.

To show that $\partial_\theta w^\varepsilon$ is uniformly (in $\varepsilon > 0$) bounded in $L^\infty(]\varepsilon,1[\times X)$ for a fixed $\alpha > 0$, we consider the expression

$$\begin{aligned} \Big|\mathbb{E}_x\Big\{h_\varepsilon \exp\Big((\theta+\delta)\int_0^{T_\varepsilon^\delta} \mathrm{e}^{-\alpha t}\varphi(x_t,v_t)\mathrm{d}t\Big)\Big\} - \\ - \mathbb{E}_x\Big\{h_\varepsilon \exp\Big(\theta\int_0^{T_\varepsilon} \mathrm{e}^{-\alpha t}\varphi(x_t,v_t)\mathrm{d}t\Big)\Big\}\Big| \le I_1 + I_2, \end{aligned}$$

with

$$(\theta+\delta)\mathrm{e}^{-\alpha T_\varepsilon^\delta}=\varepsilon,\quad \text{i.e.}\ \ T_\varepsilon^\delta=\frac{\ln(\frac{\theta+\delta}{\varepsilon})}{\alpha}$$

and

$$I_1=\Big|\mathbb{E}_x\Big\{h_\varepsilon\exp\Big((\theta+\delta)\int_0^{T_\varepsilon^\delta}\mathrm{e}^{-\alpha t}\varphi(x_t,v_t)\mathrm{d}t\Big)\Big\}- \\ -\mathbb{E}_x\Big\{h_\varepsilon\exp\Big(\theta\int_0^{T_\varepsilon^\delta}\mathrm{e}^{-\alpha t}\varphi(x_t,v_t)\mathrm{d}t\Big)\Big\}\Big|,$$

$$I_2=\Big|\mathbb{E}_x\Big\{h_\varepsilon\exp\Big(\theta\int_0^{T_\varepsilon^\delta}\mathrm{e}^{-\alpha t}\varphi(x_t,v_t)\mathrm{d}t\Big)\Big\}- \\ -\mathbb{E}_x\Big\{h_\varepsilon\exp\Big(\theta\int_0^{T_\varepsilon}\mathrm{e}^{-\alpha t}\varphi(x_t,v_t)\mathrm{d}t\Big)\Big\}\Big|,$$

for $\delta>0$ and any arbitrary control. Now

$$I_1\le\big|h_\varepsilon\big|\mathbb{E}_x\Big\{\exp\Big(\theta\int_0^{T_\varepsilon^\delta}\mathrm{e}^{-\alpha t}\varphi(x_t,v_t)\mathrm{d}t\Big)\Big|\exp\Big(\delta\int_0^{T_\varepsilon^\delta}\mathrm{e}^{-\alpha t}\varphi(x_t,v_t)\mathrm{d}t\Big)-1\Big|\Big\}\le \\ \le\big|h_\varepsilon\big|\delta\frac{\|\varphi\|}{\alpha}\exp\Big(\frac{(\theta+\delta)\|\varphi\|}{\alpha}\Big),$$

while

$$I_2\le\big|h_\varepsilon\big|\mathbb{E}_x\Big\{\exp\Big(\theta\int_0^{T_\varepsilon}\mathrm{e}^{-\alpha t}\varphi(x_t,v_t)\mathrm{d}t\Big)\Big|\exp\Big(\theta\int_{T_\varepsilon}^{T_\varepsilon^\delta}\mathrm{e}^{-\alpha t}\varphi(x_t,v_t)\mathrm{d}t\Big)-1\Big|\Big\}\le \\ \le\big|h_\varepsilon\big|\exp\Big(\frac{\theta\|\varphi\|}{\alpha}\Big)\Big[\exp\Big(\frac{\theta\|\varphi\|}{\alpha}\big(\mathrm{e}^{-\alpha T_\varepsilon}-\mathrm{e}^{-\alpha T_\varepsilon^\delta}\big)\Big)-1\Big],$$

but $\theta\mathrm{e}^{-\alpha T_\varepsilon}=\varepsilon$ so that

$$\theta\mathrm{e}^{-\alpha T_\varepsilon}-\theta\mathrm{e}^{-\alpha T_\varepsilon^\delta}=\delta\mathrm{e}^{-\alpha T_\varepsilon^\delta}=\frac{\varepsilon\delta}{\theta+\delta}$$

and

$$I_2\le\big|h_\varepsilon\big|\exp\Big(\frac{\theta\|\varphi\|}{\alpha}\Big)\Big[\exp\Big(\frac{\varepsilon\delta\|\varphi\|}{\alpha(\theta+\delta)}\Big)-1\Big].$$

Similarly for $\delta<0$, and we deduce a bound of the type

$$|w^\varepsilon(\theta+\delta,x)-w^\varepsilon(\theta,x)|\le C|h_\varepsilon|\mathrm{e}^{\frac{\theta}{\alpha}\|\varphi\|}\frac{\|\varphi\|}{\alpha}|\delta|,$$

and so $\partial_\theta w^\varepsilon$ is uniformly (in $\varepsilon>0$) bounded for a fixed $\alpha>0$.

Now we show that for any θ in $]\varepsilon,1[$ the function $x\mapsto w^\varepsilon(\theta,x)$ is bounded in $W^{2,p}(X)$, uniformly with respect to ε and θ. Indeed, for $\lambda>0$ sufficiently large, we write the equation in w^ε as

$$-\Delta w^\varepsilon+\lambda w^\varepsilon=\inf_v\big\{\psi^\varepsilon(\cdot,v)+b(\cdot,v)\cdot Dw^\varepsilon\big\},$$

with $\psi^\varepsilon = \theta\varphi w^\varepsilon + \lambda w^\varepsilon - \alpha\theta\partial_\theta w^\varepsilon$. Since w^ε and $\partial_\theta w^\varepsilon$ are bounded uniformly in ε and θ, classic results show that

$$\|w^\varepsilon(\theta,\cdot)\|_{W^{2,p}(X)} \leq C,$$

where the constant C depends only on the bounds of ψ^ε, b and the constant λ.

Define $\tilde{w}^\varepsilon$ on $]0,1[\times X$ as

$$\tilde{w}^\varepsilon(\theta,x) = \begin{cases} w^\varepsilon(\theta,x), & \theta > \varepsilon, \\ h_\varepsilon(x), & \theta \leq \varepsilon, \end{cases},$$

which satisfies the same estimates (uniformly in ε) as w^ε, i.e., $\tilde{w}^\varepsilon \geq 0$, bounded and continuous in $]0,1[\times X$, with $\partial_\theta \tilde{w}^\varepsilon$ bounded in $L^\infty(]0,1[\times X)$ and $\tilde{w}^\varepsilon(\theta,\cdot)$ bounded in $W^{2,p}(X)$, uniformly in θ. Thus, by extracting a subsequence, we have in particular,

$$\tilde{w}^\varepsilon \to w \quad \text{in} \quad L^2(0,1;H^2(X)) \text{ weakly},$$

and

$$\partial_\theta\tilde{w}^\varepsilon \to \partial_\theta w \quad \text{in} \quad L^2(]0,1[\times X) \text{ weakly}.$$

These estimates allow to pass to the limit as $\varepsilon \to 0$ in

$$\int_0^1 \alpha\theta\langle\partial_\theta\tilde{w}^\varepsilon, z\rangle \mathrm{d}\theta + \int_0^1 \langle D\tilde{w}^\varepsilon, Dz\rangle \mathrm{d}\theta - \\ - \int_0^1 \big\langle \inf_v \big\{\theta\varphi\tilde{w}^\varepsilon + b(\cdot,v)\cdot D\tilde{w}^\varepsilon\big\}, z\big\rangle \mathrm{d}\theta = \int_0^\varepsilon \big\langle \inf_v \big\{\theta\varphi h_\varepsilon\big\}, z\big\rangle \mathrm{d}\theta$$

to obtain (3.9). □

We are ready to state

Theorem 3.3. *Assume* (3.1)–(3.3)*, then there exits a unique solution u to the equation* (3.6)*,* (3.7) *such that u and $\partial_\theta u$ belong to $L^\infty(]0,1[\times X)$, the functions $x \mapsto u(\theta,x)$ belong to $W^{2,p}(X)$ and $u = u_\alpha(\theta,x)$ given by* (3.5).

Proof. By means of the Itô formula, first with an arbitrary control and next with $\hat{v}$ defined as the minimizer

$$\hat{v} = \operatorname{argmin}\big\{\theta\varphi(\cdot,v)w + b(\cdot,v)\cdot Dw\big\},$$

we obtain

$$w_\alpha(\theta,x) = \inf_v \mathbb{E}_x\Big\{\exp\Big(\theta\int_0^\infty \mathrm{e}^{-\alpha t}\varphi(x_t,v_t)\mathrm{d}t\Big)\Big\}.$$

Now define u as

$$\mathrm{e}^{\theta u} = w_\alpha, \quad \theta > 0,$$

to get

$$\alpha(u + \theta\partial_\theta u) - \Delta u - \theta|Du|^2 = \inf_v\{\varphi(\cdot, v) + b(\cdot, v) \cdot Du\}.$$

For $\theta = 0$, we define $u(0, x) = \bar{u}$ as the solution of

$$\alpha\bar{u} - \Delta\bar{u} = \inf_v\{\varphi(\cdot, v) + b(\cdot, v) \cdot D\bar{u}\}, \quad \bar{u} \in W^{2,p},$$

which is known to exist (see Bensoussan and Lions [2]).

From the definition of u we obtain

$$u(\theta, x) = \inf_v I_\alpha(\theta, x, v),$$

which conclude the proof, in view of the regularity of w_α. □

4 Asymptotics

The first step is to obtain estimates on u_α independent of α.

•**Estimate of** αu_α:
As seen before, for $\theta > 0$ and $\varphi \geq 0$, we have

$$1 \leq w_\alpha \leq \mathrm{e}^{\frac{\theta\|\varphi\|}{\alpha}},$$

and therefore

$$0 \leq u_\alpha \leq \frac{\|\varphi\|}{\alpha},$$

so

$$0 \leq \alpha u_\alpha(x) \leq \|\varphi\|, \quad \forall\alpha > 0. \tag{4.1}$$

•**Estimate of** $\alpha(u_\alpha + \theta\partial_\theta u_\alpha) = \alpha\partial_\theta(\theta u_\alpha)$:
Define

$$\Phi_\alpha := \int_0^\infty \mathrm{e}^{-\alpha t}\varphi(x_t, v_t)\mathrm{d}t,$$
$$\Psi_\alpha := \ln \mathbb{E}_x\{\mathrm{e}^{\theta\Phi_\alpha}\} = \ln \int_\Omega \mathrm{e}^{\theta\Phi_\alpha(\omega)} P_x(\mathrm{d}\omega).$$

Clearly

$$\Psi_\alpha(x, v, \theta + \delta) = \Psi_\alpha(x, v, \theta) + \delta\partial_\theta\Psi_\alpha(x, v, \theta + \eta\delta),$$

for some η in $(0, 1)$. Since

$$\partial_\theta\Psi_\alpha = \frac{\mathbb{E}_x\{\Phi_\alpha \mathrm{e}^{\theta\Phi_\alpha}\}}{\mathbb{E}_x\{\mathrm{e}^{\theta\Phi_\alpha}\}},$$

if $K = \|\varphi\|$ then we have

$$0 \leq \partial_\theta \Psi_\alpha \leq \frac{K\mathbb{E}_x\{e^{\theta\Phi}\}}{\alpha\mathbb{E}_x\{e^{\theta\Phi}\}} = \frac{K}{\alpha},$$

and

$$|\Psi_\alpha(x,v,\theta+\delta) - \Psi_\alpha(x,v,\theta)| \leq |\delta|\frac{K}{\alpha}.$$

Therefore

$$|(\theta+\delta)u_\alpha(\theta+\delta,x) - \theta u_\alpha(\theta,x)| \leq |\delta|\frac{\theta K}{\alpha}$$

so

$$\left|\partial_\theta\big(\theta u_\alpha(\theta,x)\big)\right| \leq \frac{\theta K}{\alpha}, \tag{4.2}$$

i.e., $\alpha\partial_\theta(\theta u_\alpha)$ is bounded uniformly in α.

•**Estimate of** $|Du_\alpha|_{L^2}$:

The equation in u_α can be written as

$$-\Delta u_\alpha - b_\alpha \cdot Du_\alpha = \theta|Du_\alpha|^2 + \psi_\alpha - \alpha u_\alpha, \tag{4.3}$$

with

$$b_\alpha = b(x,v_\alpha), \qquad \psi_\alpha = \varphi(x,v_\alpha) - \alpha\theta\partial_\theta u_\alpha,$$
$$v_\alpha(x) = \text{argmin}\big\{\varphi(\cdot,v) + b(\cdot,v)\cdot Du_\alpha(x)\big\}.$$

Let m_α be the density invariant probability measure corresponding to the operator $-\Delta - b_\alpha \cdot D$ (e.g., see Bensoussan [1]), which satisfies

$$0 < \delta_0 \leq m_\alpha \leq \delta_1.$$

Multiplying (4.3) by m_α and using the equation for m_α, we deduce

$$0 = \theta\int_X |Du_\alpha|^2 m_\alpha \mathrm{d}x + \int_X \big(\psi_\alpha - \alpha u_\alpha\big)m_\alpha \mathrm{d}x. \tag{4.4}$$

Since δ_0 and δ_1 depend only on the L^∞ norm of b, they are independent of α and θ. Therefore (4.4) gives

$$\theta|Du_\alpha|^2_{L^2(X)} \leq C, \quad \forall \alpha,\, \theta, \tag{4.5}$$

i.e., a bound on $|Du_\alpha|_{L^2(X)}$ uniformly in $\alpha > 0$ and θ in $[\varepsilon, 1]$, for every $\varepsilon > 0$.

•**Estimate of** $u_\alpha - \bar{u}_\alpha$:

Let us define

$$\bar{u}_\alpha(\theta) := \int_X u_\alpha(\theta,x)\mathrm{d}x \quad \text{and} \quad \Lambda_\alpha(\theta,x) := u_\alpha(\theta,x) - \bar{u}_\alpha(\theta).$$

The equation for Λ_α is

$$-\Delta\Lambda_\alpha = -\alpha\partial_\theta(\theta u_\alpha) + \theta|D\Lambda_\alpha|^2 + \inf_v \big\{\varphi(\cdot, v) + b(\cdot, v)\cdot D\Lambda_\alpha\big\}. \tag{4.6}$$

and by Poincaré inequality we have

$$|\Lambda_\alpha|_{L^2(X)} \leq C|Du_\alpha|_{L^2(X)}.$$

Considering θ as a parameter in (4.6) and since $\alpha\partial_\theta(\alpha u_\alpha)$ is bounded, we have

$$\sqrt{\theta}|\Lambda_\alpha|_{L^2(X)} \leq C,$$

moreover, we can mimic the arguments in Lemmas 4.7 and 4.8 of Bensoussan and Frehse [5] to obtain

$$\sqrt{\theta}|\Lambda_\alpha|_{L^\infty(X)} \leq C, \tag{4.7}$$

for some constant $C > 0$, uniformly in α and θ. Furthermore, considering $z_\alpha(\theta, x) = \theta\Lambda_\alpha(\theta, a)$, which satisfies

$$-\Delta z_\alpha = -\alpha\theta\partial_\theta(\theta u_\alpha) + |Dz_\alpha|^2 + \inf_v \big\{\theta\varphi(\cdot, v) + b(\cdot, v)\cdot Dz_\alpha(\theta, \cdot)\big\},$$

so that one can apply Theorem 3.7 of Bensoussan and Frehse [5] to deduce

$$\|z_\alpha\|_{C^\delta(X)} \leq C,$$

i.e.,

$$\theta|\Lambda_\alpha|_{C^\delta(X)} \leq C, \tag{4.8}$$

for some constant $C > 0$, uniformly in α and θ.

•**Passage to the limit a** $\alpha \to 0$:

(a) First we look at $\alpha u_\alpha(\theta, x)$. In view of (4.1), (4.2) and (4.8), taking a sub-sequence we have

$$\alpha u_\alpha \to \xi, \tag{4.9}$$

uniformly on every compact subset of $Q =]0, 1[\times X$. Let us show that ξ does not depend on x. Indeed, since

$$\sqrt{\theta}\Lambda_\alpha = \sqrt{\theta}[u_\alpha(\theta, x) - \bar{u}_\alpha(\theta)]$$

is bounded, we have $\alpha\sqrt{\theta}\Lambda_\alpha \to 0$ and therefore

$$\lim_{\alpha\to 0} \alpha[u_\alpha(\theta, x) - \bar{u}_\alpha(\theta)] = 0, \quad \forall x \in X,\ \theta > 0.$$

On the other hand, since $u_\alpha(0, x) = u^0_\alpha(x)$ we know that $\alpha u^0_\alpha(x)$ must converge to a constant too.

Now, since $\theta\partial_\theta(\alpha u_\alpha)$ is bounded, we deduce that

$$\theta\partial_\theta(\alpha u_\alpha) \to \theta\frac{d\xi}{d\theta}$$

weakly-star in L^∞,
(b) Then we pass to the limit in the equation of Λ_α, for each $\theta > 0$ fixed. By means of the equation (4.6) and the previous bounds on u_α, in particular (4.2), (4.5) and (4.8), we can find a subsequence such that

$$\Lambda_\alpha \to u \quad \text{in } H^1(X) \text{ weakly and } L^\infty(X) \text{ strongly}$$

as $\alpha \to 0$. Therefore

$$\int_X \Delta\Lambda_\alpha\,(\Lambda_\alpha - u)\mathrm{d}x \to 0,$$

since $\Delta\Lambda_\alpha$ is bounded in $L^1(X)$. This is,

$$\int_X D\Lambda_\alpha \cdot D\Lambda_\alpha \mathrm{d}x \to \int_X D\Lambda_\alpha \cdot Du\mathrm{d}x.$$

However, due to the weak convergence in $H^1(X)$ we have

$$\int_X D\Lambda_\alpha \cdot Du\mathrm{d}x \to \int_X Du \cdot Du\mathrm{d}x,$$

which yields

$$\int_X |D\Lambda_\alpha - Du|^2\mathrm{d}x \to 0,$$

i.e., $\Lambda_\alpha \to u(\theta, \cdot)$ strongly in $H^1(X)$.

Hence, if we call $\chi(\theta)$ the limit of $\alpha\partial_\theta(\theta u_\alpha)$ we see that the couple (χ, u) satisfies

$$\begin{cases} \chi - \Delta u = \theta|Du|^2 + \inf\limits_v \big\{\varphi(\cdot, v) + b(\cdot, v)\cdot Du(\cdot)\big\}, \quad u \in H^1(X), \\ \displaystyle\int_X u(\theta, x)\mathrm{d}x = 0, \quad \forall\theta > 0. \end{cases} \tag{4.10}$$

But form Nagai [10] (who treats a more difficult case in $\mathbb{R}^d$ and unbounded φ, and therefore the result applies a fortiori to our simple case) there exists a unique pair (χ, u) satisfying (4.10) and

$$\chi(\theta) = \lim_{T\to\infty} \frac{u(T, x)}{T},$$

with $u(T, x)$ given by (1.5). Therefore we conclude that

$$\frac{\mathrm{d}\big(\theta\xi(\theta)\big)}{\mathrm{d}\theta} = \chi(\theta),$$

which gives

$$\xi(\theta) = \frac{1}{\theta} \int_0^\theta \chi(r)\mathrm{d}r,$$

i.e.,

$$\lim_{\alpha \to 0} \alpha u_\alpha(\theta, x) = \frac{1}{\theta} \int_0^\theta \chi(r)\mathrm{d}r. \tag{4.11}$$

We have shown the desired result summarized as

Theorem 4.1. *Under the assumptions of Section 3 we have*

$$\lim_{\alpha \to 0} \alpha \partial_\theta \big(\alpha u_\alpha(\theta, x)\big) = \chi(\theta),$$

$$\lim_{\alpha \to 0} \left[u_\alpha(\theta, x) - \int_X u_\alpha(\cdot, x)\mathrm{d}x\right] = u(\theta, x),$$

where (χ, u) *is the unique solution of* (4.10)*,*

$$\chi(\theta) = \liminf_{T \to \infty} \frac{1}{T} \inf_v \left[\frac{1}{\theta} \ln \mathbb{E}_x \Big\{ \exp \Big(\theta \int_0^T \varphi(x_t, v_t)\mathrm{d}t\Big)\Big\}\right],$$

and (4.11) *holds.* □

To conclude, let us mention that certainly, the above result remain true for reflected diffusion processes in a bounded region of $\mathbb{R}^d$. The case in the whole space $\mathbb{R}^d$ or diffusion with jumps requires a more elaborated technique, and it may be the subject of future research.

References

[1] A. Bensoussan, *Perturbation methods in optimal control*, Wiley, New York, 1988.

[2] A. Bensoussan and J.L. Lions, *Applications des inéquations variationnelles en contrôle stochastique*, Dunod, Paris 1978.

[3] A. Bensoussan and J.L. Lions, *Contrôle impulsionnel et inéquations quasi variationnelles*, Dunod, Paris 1982.

[4] A. Bensoussan and J.H. Van Schuppen, Optimal control of partially observable stochastic systems with an exponential of integral performance index, *SIAM J. Control Optim.*, **23** (1985), 599–613.

[5] A. Bensoussan and J. Frehse, *Regularity results for nonlinear elliptic systems and applications*, Springer-Verlag, New-York, 2002.

[6] W.H. Fleming and W.M. McEneaney, Risk-sensitive control on an infinite time horizon, *SIAM J. Control Optim.*, **33** (1995), 1881–1915.

[7] D.H. Jacobson, Optimal stochastic linear systems with exponential performance criteria and relation to detewninistic differential games, *IEEE Trans. Automat. Control*, **AC-18** (1973), 124-131.

[8] W.M. McEneaney, *Connections between risk-sensitive stochastic control, differential games and H^∞-control: the non linear case*, Brown University, PhD Thesis, 1993.

[9] H. Nagai, Ergodic control problems on the whole Euclidean space and convergence of symmetric diffusions, *Forum Math.*, **4** (1992), 159-173.

[10] H. Nagai, Bellman equations of risk-sensitive control, *SIAM J. Control Optim.* **34** (1996), 74–101.

[11] T. Runolfsson, Stationary risk-sensitive LQG control and its relation to LQG and H-infinity control, *Proc. 29th CDC Conference*, Honolulu, HI, 1990, 1018-1023.

[12] P. Whittle, *Risk-sensitive Optimal Control*, Wiley, New-York, 1990.