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# On the LQG Theory with Bounded Control

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# On the LQG theory with bounded control

D.V. Iourtchenko, J.L. Menaldi and A.S. Bratus

**Abstract.** We consider a stochastic optimal control problem in the whole space, where the corresponding HJB equation is degenerate, with a quadratic running cost and coefficients with linear growth. In this paper we provide a full mathematical details on the key estimate relating the asymptotic behavior of the solution as the space variable goes to infinite.

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In the previous papers [2] a problem of stochastic optimal control of a *dynamic* system was considered. Such problems arise in different fields of engineering, namely mechanical, electrical, thermal and others. An equation of motion of this type of systems usually is governed by a stochastic differential equation (SDE) of second or higher order. Written in a state-space form this SDE is transformed into a set of first order SDEs, with noise entering only some of them. As a result the covariance matrix will be degenerate, which may serve as a characteristic feature of this type of systems.

The dynamic programming approach (DPA) may be used to study the problem of optimal control [4, 5]. It converts the problem of finding an optimal control policy to a problem of finding a solution to degenerate, multidimensional parabolic PDE - Hamilton-Jacobi-Bellman (HJB) equation. The major difficulty here is that the stated Cauchy problem for the HJB equation should be solved in the entire state-space, whereas it cannot be solved numerically, since the asymptotic behavior of the Bellman function is unknown. There is also no general approach to find an . analytical solution to the nonlinear, multidimensional, degenerate parabolic equation. To overcome this complication the hybrid solution method has been proposed. This method suggests finding an analytical *function*, which satisfies the HJB equation and the initial conditions within a certain "outer" domain. This function can be used as a boundary condition for numerical simulation of the corresponding HJB equation, thereby solving the HJB equation within the entire state-space domain. On the other hand, because no boundary conditions are used on the "outer" domain, this degenerated parabolic PDE cannot have a unique solution. However, if we prove that the constructed function provides an asymptotic behavior of the Bellman function, then it can be used as a boundary condition for numerical simulation of the HJB equation. This approach with tedious proof has been given in [2]. In this paper we propose a much shorter proof, which can easily be extended to a dynamic system with multiple-degrees-of-freedom, to dynamic systems under Poisson noise, nonautonomous systems and even deterministic system.

## 1. Problem statement

Consider a dynamical system governed by the following set of SDE:

$$\begin{cases} \dot{x_1}(s) = x_2(s) \\ \dot{x_2}(s) = -2\alpha x_2(s) - \beta^2 x_1(s) + v(s) + \sigma \dot{B}(s), \quad t < s \leq T, \\ x(t) = x_0, \ \dot{x}(t) = \dot{x}_0 \end{cases} \tag{1.1}$$

Here $\alpha, \beta$, $\sigma$ are positive constants, $B = B(s)$ is a Wiener process, derivative of which should be understood formally and the control $v = v(s)$ is an adapted random process satisfying $|v(s)| \leq R$, for a fixed constant $R > 0$. The control goal is to minimize the quadratic cost function:

$$J_{x_1,x_2,t}(v) = E\left\{ \frac{a}{2}[\beta^2 x_1^2(T) + x_2^2(T)] + \int_t^T \frac{b}{2}[\beta^2 x_1^2(s) + x_2^2(s)]ds \right\} \tag{1.2}$$

where $T$ is given constant. A special case of $a = 1/2$ or $b = 1/2$ corresponds to the minimization of the mean total system response energy. Following the dynamic programming approach to solve the problem (1.1),(1.2) we introduce the Bellman function

$$u(x_1, x_2, t) = \inf \{ J_{x_1,x_2,t}(v) : \ |v(\cdot)| \leq R \}, \tag{1.3}$$

which satisfies the following Hamilton-Jacobi-Bellman equation

$$\begin{aligned} &\frac{\partial u}{\partial t} + Lu + \inf_{|v|\leq R} \left\{ v \frac{\partial u}{\partial x_2} \right\} + F = 0, \\ &F(x_1, x_2) = b/2(\beta^2 x_1^2 + x_2^2), \\ &Lu = x_2 \frac{\partial u}{\partial x_1} - (2\alpha x_2 + \beta^2 x_1) \frac{\partial u}{\partial x_2} + \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x_2^2}. \end{aligned} \tag{1.4}$$

Equation (1.4) is degenerate parabolic equation, to be solved in the entire state-space with the terminal condition

$$u(x_1, x_2, T) = \frac{a}{2}(\beta^2 x_1^2 + x_2^2). \tag{1.5}$$

Note that $\inf_{|v|\leq R} \left\{ v \frac{\partial u}{\partial x_2} \right\} = -R \left| \frac{\partial u}{\partial x_2} \right|$.

## 2. Construction of a "solution" and the first order estimate

It is possible to construct a function, which would satisfy HJB equation (1.4) and initial condition (1.5) within a certain "outer" domain. Since this function does not satisfies the HJB equation in the entire state-space, we cannot call it the solution. To express the following arguments as clear as possible consider, for the time being, the case of $\alpha = 0$ (more general case can be treated with minor modifications).

To construct the solution within the "outer" domain assume the existence of a domain $\Gamma(x_1, x_2, t)$, which does not contain a switching line ($\partial u/\partial x_2 = 0$) for any $x_1, x_2$ and $t > 0$, see [2] for details. Thus, within this domain $z = \text{sign}(\partial u/\partial x_2)$ is a constant. Then looking for a solution in the form $f_{ij}(t)x_i x_j$ ($x_0 = 1$) results in a set of ODEs for $f_{ij}(t)$ with appropriate initial conditions. Solving analytically these ODEs one derives the "solution" to HJB equation (1.4) (with $b = 0$ for simplicity) as

$$\begin{aligned}\tilde{u}(x_1, x_2, \tau) =& \frac{a}{2}\left(\left[x_2 - \frac{Rz}{\beta}\sin(\beta[T-t])\right]^2 + \right.\\ &\left. + \left[\beta x_1 + \frac{Rz}{\beta}(1 - \cos(\beta[T-t]))\right]^2\right) + \frac{\sigma^2(T-t)}{2}a.\end{aligned} \tag{2.1}$$

Note that the last term in the equation comes from integrating the noise intensity with respect to time, so if $\sigma = \sigma(\tau)$ then the last term would be $1/2\int_0^\tau \sigma^2(s)ds$. This function $\tilde{u}$ satisfies the HJB equation (1.4) and initial conditions (1.5) within the following "outer" domain

$$|x_2| \geq \frac{R}{\beta}|\sin\beta(T-t)|. \tag{2.2}$$

A substitution shows that $\tilde{u}$ defined by (2.1) satisfies the HJB equation (1.4) within the "outer" domain given by the condition (2.2). Once the analytical "solution" is known, it can be used as a boundary condition to solve the HJB equation numerically within the remaining "inner" domain, thereby finding a solution to the corresponding HJB equation in the entire state-space. It is worth mentioning that, since the system (1.1) does not depend on time explicitly, a "solution" for the case of $b \neq 0$, can be obtained by integrating (2.1) with respect to explicit time $\tau$.

In order to derive the first approximation, let's consider the quadratic part of the function (2.1) and its integral only ($a \geq 0, b \geq 0$ are constants)

$$\Psi(x_1, x_2, t) = \frac{1}{2}(\beta^2 x_1^2 + x_2^2)[a + b(T-t)], \tag{2.3}$$

which satisfies $L\Psi = \frac{\sigma^2}{2}[a+b(T-t)]$. Define $w(x_1, x_2, t) = u(x_1, x_2, t) - \Psi(x_1, x_2, t)$, where $u(x_1, x_2, t)$ is the solution of HJB equation (1.4) with $\alpha = 0$. Then in the paper [2] the authors have obtained the following estimate

$$\lim_{|x_1|+|x_2|\to\infty} \frac{|u(x_1, x_2, t) - \Psi(x_1, x_2, t)|}{|x_1|^2 + |x_2|^2} = 0, \tag{2.4}$$

Note, that the inequality (2.4) holds in the case of poisson noise and the system with linear friction ($\alpha \neq 0$).

## 3. Complete estimate

A priori, the expression (2.1) defining the function $\tilde{u}$ is only valid on the "outer" domain, so that we need a suitable extension to the "inner" domain. to accomplish this task, we keep almost the same expression, but we replace the function $z = \text{sign}$ with a suitable smooth and bounded function $\tilde{z} = \tilde{z}(x,t)$ such that:

$$\tilde{z}(x_1, x_2, t) = \text{sign}(x_2) \quad \text{if} \quad |x_2| \geq R_0, \tag{3.1}$$

with $R_0$ satisfying $R_0 \geq R/\beta$, and some other conditions to be determined below. Recall that $R$ and $\beta$ are the initial constants of the model, and $R/\beta$ represents the size of the "outer" domain.

For the sake of simplicity, in this section we use the local notation $x = (x_1, x_2)$ and $a = 1$, $b = 0$. If $u$ is the optimal cost and $w = u - \psi$, where $\psi$ is given by (2.1), then we obtain the following nonlinear equation ($\tau = T - t$)

$$\partial_\tau w = Lw + \inf_{|v| \leq R} \Big\{ v \big[ \varphi_2 + \partial_2 w \big] \Big\} + \varphi, \tag{3.2}$$

where $\varphi_2 = \big(x_2 - \frac{R}{\beta} \tilde{z} \sin \beta\tau\big)\big(1 - \frac{R}{\beta} \partial_2 \tilde{z} \sin \beta\tau\big)$, $\varphi = \big(x_2 R \tilde{z} + \frac{R^2}{\beta} \tilde{z}^2 \sin \beta\tau\big) + \varphi_1$, and

$$\begin{aligned}
\varphi_1 = & \Big[ \Big( x_1 x_2 R + 2 x_2 \frac{R^2}{\beta^2} \tilde{z} \Big) (1 - \cos \beta\tau) - x_2^2 \frac{R}{\beta} \sin \beta\tau \Big] \partial_1 \tilde{z} + \\
& + \Big[ \big( x_1 x_2 \beta - \frac{\sigma^2}{\beta} \big) R \sin \beta\tau + R x_1 (x_1 \beta^2 + 2 \tilde{z} R)(\cos \beta\tau - 1) \Big] \partial_2 \tilde{z} + \\
& + \big( \sigma^2 \frac{R^2}{\beta^2} \big) (1 - \cos \beta\tau) \big[ (\partial_2 \tilde{z})^2 + \tilde{z} \partial_2^2 \tilde{z} \big] + \\
& + \big( \sigma^2 \frac{R}{2\beta} \big) \Big[ x_1 \beta (1 - \cos \beta\tau) - x_2 \sin \beta\tau \Big] \big( \partial_2^2 \tilde{z} \big).
\end{aligned}$$

Recall that we could have $\tilde{z} = \tilde{z}(x,t)$, but only the dependency in $x_2$ is relevant. In view of (3.1), on the region where $|x_2| \geq R_0$ we have $\tilde{z}(x,t) = \text{sign}(x_2)$ and therefore $\partial_1 \tilde{z} = \partial_2 \tilde{z} = 0$ and $\varphi_1 = 0$. Moreover, if there are constants $C_1, k > 0$ such that

$$|\partial_1 \tilde{z}| + |\partial_2 \tilde{z}| + \sigma^2 |\partial_2^2 \tilde{z}| \leq C_1 (1 + |x_1|)^{-k-1}, \quad \forall x_1, x_2, t, \tag{3.3}$$

then for another constant $C > 0$ we have $|\varphi_1| \leq C(1 + |x_1| + |x_2|)^{-k}$, for every $x_1$, $x_2$, and $t$. Note that if $\sigma = 0$ then the second derivative $\partial_2^2 \tilde{z}$ does not intervene, and calculations are simpler.

The expression (3.2) is the Hamilton-Jacobi-Bellman equation corresponding to the following optimal control problem:

$$\tilde{J}_{x,t}(v) = \mathbb{E} \Big\{ \int_t^T \big[ v(s) \varphi_2(x(s), s) + \varphi(x(s), s) \big] \mathrm{d}s \Big\}$$

and

$$w(x,t) = \inf_{|v(\cdot)|\le R} \big\{\tilde{J}_{x,t}(v)\big\}. \tag{3.4}$$

This optimal cost $w$, which is equals to $u - \psi$, has to be estimated in the "outer domain" to deduce the desired result.

The running cost $\ell(x,t,v) = v\varphi_2(x,t) + \varphi(x,t)$ can be written as $\ell(x,t,v) = \ell_1(x,t,v) + \ell_0(x,t,v)) + \varphi_1(x,t)$ with

$$\begin{aligned}\ell_1(x,t,v) &= \Big[v\Big(1 - \frac{R}{\beta}\partial_2\tilde{z}(x,t)\sin\beta\tau\Big) + R\tilde{z}(x,t)\Big]x_2 + \\ &\qquad\qquad + v\frac{R^2}{\beta^2}\tilde{z}(x,t)\partial_2\tilde{z}(x,t)\sin^2\beta\tau, \\ \ell_0(x,t,v) &= \Big(\frac{R}{\beta}\tilde{z}(x,t)\sin\beta\tau\Big)\big(-v - R\tilde{z}(x,t)\big).\end{aligned}$$

Notice that if $\tilde{z} = \mathrm{sign}(x_2)$ then $\ell_1 = (v+R)x_2$ and $\ell_0 = -\frac{R}{\beta}\sin\beta\tau(v+R)$, so $x_2 - \frac{R}{\beta}\sin\beta\tau$ determines the sign of $\ell = \ell_1 + \ell_0$, since $\varphi_1 = 0$. Thus, if $\tilde{z}$ and $\partial_2\tilde{z}$ are bounded then $\ell_0$ is a bounded function, and moreover, within the region $|x_2| \le R_0$ we can bound the function $\ell_1$, i.e., $|\ell_1| \le C(1+|x_1|)^{-k}$ for any $x, t$ and some constant $C$. Similarly, for $|x_2| \ge R_0$, we have two possibilities: (1) $x_2 > 0$ which implies $\tilde{z} = 1$ and $\partial_2\tilde{z} = 0$, and we deduce $\ell_1 = (v+R)x_2 \ge 0$, for every $|v| \le R$; and (2) $x_2 < 0$ which implies $\tilde{z} = -1$ and $\partial_2\tilde{z} = 0$, and we deduce $\ell_1 = (v-R)x_2 \ge 0$, for every $|v| \le R$. This proves that under the conditions (3.1) and (3.3) there exists constants $C_1, C_0 > 0$ such that

$$\begin{aligned}&\ell_1(x,t,v) \ge 0, \quad \forall x,t,v, \ \text{ with } \ |x_2| \ge R_0, \\ &\ell_1(x,t,v) \ge -C_1\big(1+|x_1|+|x_2|\big)^{-k}, \quad \forall x,t,v, \\ &\ell_0(x,t,v) \ge -C_0, \quad \forall x,t,v,\end{aligned}$$

Thus, we deduce the estimate

$$\tilde{J}_{x,t}(v) \ge -C\big(1+|x_2|\big)^{-k}(T-t), \quad \forall x,t, \tag{3.5}$$

for some suitable constant $C > 0$ depending only on the data (e.g., $R$). Actually, since for every $m > 0$ there exists a constant $C_m > 0$ such that $\ell_1(x,t,v) \ge -C_m\big(1+|x_1|+|x_2|^m\big)^{-k}$, for every $x,t,v$, we deduce that (3.5) holds with $|x_2|$ replaced with $|x_2|^m$. Moreover, if we assume that for $v^*(x) = -R\mathrm{sign}(x_2)$ we have

$$\ell(x,t,v^*(x)) \ge -C(1+|x_1|)^{-k}, \quad \forall x,t,v, \tag{3.6}$$

then for every $m > 0$ there exists a constant $C_m > 0$ such that

$$\tilde{J}_{x,t}(v^*) \ge -C_m\big(1+|x_1|+|x_2|^m\big)^{-k}(T-t), \quad \forall x,t, \tag{3.7}$$

where $v^*(t) = v^*(x(t))$ is assumed to be an optimal feedback for the initial control problem. Note that if $|x_2| \ge R_0$ then $\tilde{z}(x,t) = \mathrm{sign}(x_2)$ and therefore $\ell_0(x,t,v^*(x)) = 0$, which means that condition (3.6) is only required when $|x_2| < R_0$. For instance, if we assume that $|\tilde{z}| \le 1$, $\mathrm{sign}(\tilde{z}) = \mathrm{sign}(x_2)$ and $\sin\beta\tau \ge 0$ then we deduce $\ell_0(x,t,v^*(x)) \ge 0$.

On the other hand, to get a bound from above, we use the smooth function $\tilde{z}$ to define the feedback $v_0(t) = -R\tilde{z}(x(t), t)$, which produce a running cost

$$\ell(x,t,v_0) = x_2 \frac{R^2}{\beta} \tilde{z}(x,t)\partial_2 \tilde{z}(x,t) \sin \beta\tau - \\ - \frac{R^3}{\beta^2} \tilde{z}^2(x,t)\partial_2 \tilde{z}(x,t) \sin^2 \beta\tau + \varphi_1(x,t),$$

which has the property that $|\ell_0(x, t, v_0(t))| \leq C(1 + |x_1|)^{-k}$, for every $x, t$ and some suitable constant $C > 0$, and also, $\ell(x, t, v_0) = 0$ if $|x_2| \geq R_0$. This shows that for every $m > 0$ there exists a constant $C_m > 0$, which depend on the data and $m$, such that

$$\tilde{J}_{x,t}(v_0) \leq C_m \big(1 + |x_1| + |x_2|^m\big)^{-k} (T - t), \quad \forall x, t. \tag{3.8}$$

We have proven the following

**Theorem 3.1.** *Let $u$ be the optimal cost of our initial control problem. Then for every $m > 0$ there exists a constant $C_m > 0$ depending only on the data such that*

$$u(x,t) - \psi(x,t) \leq C_m \big(1 + |x_1| + |x_2|^m\big)^{-k} (T - t), \quad \forall x, t,$$

*where $\psi$ is defined by* (2.1) *with some smooth and bounded function $\tilde{z}$ satisfying* (3.1) *and* (3.3). *Moreover, if we accept that $v^*(x) = -Rsign(x_2)$ is an optimal feedback for the initial control problem then*

$$u(x,t) - \psi(x,t) \geq -C_m \big(1 + |x_1| + |x_2|^m\big)^{-k} (T - t), \quad \forall x, t,$$

*provided* (3.6) *is satisfied. Otherwise, we only have*

$$u(x,t) - \psi(x,t) \geq -C_m \big(1 + |x_2|\big)^{-m} (T - t), \quad \forall x, t,$$

*for some suitable constant $C_m$.* □

Notice that we use that fact that a solution (with the regularities of $u$) of the HJB equation (3.2) is indeed the optimal cost (3.4). For instance, we know that $u$ is the maximum sub-solution (or the unique continuous viscosity solution) [1, 3] of the corresponding HJB equation, and because $\psi$ is smooth, $w = u - \psi$ is the maximum sub-solution of the HJB equation (3.2), and therefore, the optimal cost.

To actually see that $v^* = -R\,\mathrm{sign}(x_2)$ is an optimal feedback for the control problem we need to establish the existence of week solution of non-degenerate stochastic differential equations with measurable coefficients, in our case the measurable coefficient is due to the discontinuous feedback. This can be accomplished by modifying Krylov's arguments [6] or with an explicit probabilistic construction similar to the solution of the typical one dimensional example $\mathrm{d}x = \mathrm{sign}(x)\mathrm{d}w(t)$. Even, an argument with $\varepsilon$-optimal controls could be sufficient.

Finally, to construct a function $\tilde{z}$ satisfying the above condition, we may take the function $\mathrm{sign}(x_2)$ for $|x_2| \geq R_0$ and make an odd $C^2$ extension for $|x_2| \leq R_0$, satisfying $|\tilde{z}| \leq 1$ and $\mathrm{sign}(\tilde{z}) = \mathrm{sign}(x_2)$.

## 4. Conclusions

Given mathematical proof resulted in the Theorem 3.1 shows that a specially constructed solution provides asymptotic behavior of the Bellman function as the space variable goes to infinite. Consequently, this solution may be used as boundary conditions to solve the HJB equation numerically. That means that the proposed earlier *hybrid solution method* of finding the solution to the stochastic LQ problem of optimal control is a completely formulated and valid new technique. It is important to stress that Theorem 1 applies also for for deterministic optimal control LQ problem with bounded control.

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