2-1-1999

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July 27, 2000

Abstract

Our purpose is to study an ergodic linear equation associated to diffusion processes with jumps in the whole space. This integro-differential equation plays a fundamental role in ergodic control problems of second order Markov processes. The key result is to prove the existence and uniqueness of an invariant density function for a jump diffusion, whose lower order coefficients are only Borel measurable. Based on this invariant probability, existence and uniqueness (up to an additive constant) of solutions to the ergodic linear equation are established.

Key words and phrases: Jump diffusion, interior Dirichlet problem, exterior Dirichlet problem, ergodic optimal control, Green function, Girsanov transformation, Doeblin condition.

1991 AMS Subject Classification. Primary: 35J25, 60J60, 60J75. Secondary: 45K05, 46N20, 49A60, 93D05, 93E20.

Introduction

Ergodic properties of diffusion processes and its relation with partial differential equations are well know in the classic literature. However, similar questions for diffusion processes with jumps are not so popular, only recently was some attention given, cf. [18], Garroni and Menaldi [8] and reference therein.

Due to applications in stochastic control (in particular the action of a feedback function), we have to be able to treat diffusions with jumps with only Borel measurable lower order coefficients (where the control is applied). This gives particular complications, even in the purely diffusion case, cf. Bensoussan [1]. Moreover, since we are interested in the whole space, an assumption relative to the existence of a Liapunov function is needed. This produce a drift of linear growth at infinity and the existence and regularity of the Green function or transition density function (even in the purely partial differential equations case) as proved by Garroni and Menaldi [8] does not apply.

Most of the arguments are based on the so-called *Doeblin condition*, which in turn is based in the strict positivity of the Green functions deduced from the strong maximum principle. We refer to the books of Borkar [4] and Ethier and Kurtz [6] for a related discussion.

Now, we describe, without all the technical assumptions, the ergodic problem we want to be able to consider. Let $k(x)$ be a Borel measurable function from $\mathbb{R}^d$ into V (i.e., a measurable feedback). The dynamic of the system (for a given feedback) follows a diffusion with jumps in $\mathbb{R}^d$, i.e. a (strong) Markov process $(\Omega, P, X_t, t \geq 0)$ with semigroup $(\Phi_k(t), t \geq 0)$ and infinitesimal generator A_k, as discussed in the next section. A long run average cost is associated to the controlled system by

$$J(k) = \int_{\mathbb{R}^d} f(x, k(x)) \mu_k(dx), \tag{0.1}$$

where f is the running cost and μ_k is the invariant probability measure associated with the system. Usually the purpose is to give a characterization of the optimal cost

$$\lambda = \inf\{J(k) : k(\cdot)\} \tag{0.2}$$

and to construct an optimal feedback control $\hat{k}$.

A formal application of the dynamic principle (e.g. Fleming and Soner [7]) yields the following Hamilton-Jacobi-Bellman equation

$$\inf_k \{A_k u(x)\} = \lambda \ \text{ in } \ \mathbb{R}^d, \tag{0.3}$$

where the infimum is calculated for each fixed x, and $k = v$ in V. An optimal feedback control is obtained as the minimizer $\hat{k}(x)$ in (0.3).

In order to study the Hamilton-Jacobi-Bellman equation (0.3) we need some previous discussion. This research is dedicated to the linear problem. A subsequent paper [22] will deal with the about stated problem. In Section 1, we give some details on the construction of the diffusion with jumps in the whole space $\mathbb{R}^d$, under convenient assumptions. Next, most of the effort is dedicated to the construction of the invariant probability measure μ_k, for any measurable feedback k. This will extends classic results, e.g. Bensoussan [1], Khasminskii [12]. Thus, in Section 2, we study some preliminary properties on the integro-differential operator needed later. In Section 3, we give a detailed summary of the (linear) interior Dirichlet problem for the integro-differential operator A_k, which is mainly based Bensoussan and Lions [3], Garroni and Menaldi [8], Gimbert and Lions [11]. In Section 4, we consider the (linear) exterior Dirichlet problem. This will give some conditions under which the diffusion with jumps is (positive) recurrent. Finally, in Section 5, we construct the invariant probability measure.

1. Diffusions with Jumps

Consider an integro-differential operator of the form

$$I_0\varphi(x) = \int_{\mathbb{R}^d_*} [\varphi(x+z) - \varphi(x) - z \cdot \nabla\varphi(x)] M_0(x, dz), \tag{1.1}$$

where the Levy kernel $M_0(x, dz)$ is a Radon measure on $\mathbb{R}^d_* = \mathbb{R}^d \setminus \{0\}$ for any fixed x, and satisfies

$$\int_{|z|<1} |z|^2 M_0(x, dz) + \int_{|z|\geq 1} |z| M_0(x, dz) < \infty, \quad \forall x \in \mathbb{R}^d. \tag{1.2}$$

It is clear that this operator is associated with a jump process.

Similarly, let L_0 be a second order uniformly elliptic operator associated with a diffusion process in the whole space, i.e.

$$L_0 = \sum_{i,j=1}^{d} a_{ij}(x)\partial_{ij} + \sum_{i=1}^{d} b_i(x)\partial_i, \tag{1.3}$$

where the coefficients (a_{ij}) are bounded and Lipschitz continuous, i.e. for some $c_0, M > 0$ and $0 < \alpha < 1$,

$$\begin{cases} c_0|\xi|^2 \leq \sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \leq c_0^{-1}|\xi|^2, \quad \forall x, \xi \in \mathbb{R}^d, \\ |a_{ij}(x) - a_{ij}(x')| \leq M|x - x'|, \quad \forall x, x' \in \mathbb{R}^d, \end{cases} \tag{1.4}$$

$a_{ij} = a_{ji}$, and the first order coefficients (b_i) are Lipschitz continuous, i.e. for some $M > 0$,

$$\begin{cases} |b_i(x) - b_i(x')| \leq M|x - x'|, \quad \forall x, x' \in \mathbb{R}^d \\ b_i(0) = 0, \quad i = 1, \ldots, d. \end{cases} \tag{1.5}$$

The fact that $b = (b_i)$ vanishes on the origin and on assumption of the type

$$-\sum_{i=1}^{d} b_i(x)x_i \geq c_1|x|^2, \quad \forall x \in \mathbb{R}^d, |x| \geq r_1, \tag{1.6}$$

for some constants $c_1, r_1 > 0$, will allow us to show some "stability" on the system (cf. Section 4)

The Levy kernel $M_0(x, dz)$ is assumed to have a particular structure, namely

$$M_0(x, A) = \int_{\{\zeta : j(x,\zeta) \in A\}} m_0(x, \zeta)\pi(d\zeta), \tag{1.7}$$

where $\pi(\cdot)$ is a σ-finite measure on the measurable space $(F, \mathcal{F})$, the functions $j(x, \zeta)$ and $m_0(x, \zeta)$ are measurable for (x, ζ) in $\mathbb{R}^d \times F$, and there exist a measurable and positive function $j_0(\zeta)$ and constants $C_0 > 0$, $1 \le \gamma \le 2$ [γ is the order of I] such that for every x, ζ we have

$$\begin{cases} |j(x,\zeta)| \le j_0(\zeta), \qquad 0 \le m_0(x,\zeta) \le 1, \\ \int_F |j_0(\zeta)|^p (1 + j_0(\zeta))^{-1} \pi(d\zeta) \le C_0, \quad \forall p \in [\gamma, 2], \end{cases} \tag{1.8}$$

the function $j(x, \zeta)$ is continuously differentiable in x for any fixed ζ and there exists a constant $c_0 > 0$ such that for any (x, ζ) we have

$$c_0 \le \det(\mathbf{1} + \theta \nabla j(x,\zeta)) \le c_0^{-1}, \quad \forall \theta \in [0, 1], \tag{1.9}$$

where $\mathbf{1}$ denotes the identity matrix in $\mathbb{R}^d, \nabla$ is the gradient operator in x, and $\det(\cdot)$ denotes the determinant of a matrix.

Depending on the assumptions on the coefficients of the operators L_0, I_0 and on the domain $\mathcal{O}$ of $\mathbb{R}^d$, we can construct the corresponding Markov-Feller process. The reader is referred to the books by Bensoussan and Lions [3], Gikhman and Skorokhod [9] (among others) and references therein. Usually, more regularity on the coefficients $j(x, \zeta)$ and $m_0(x, \zeta)$ is needed, e.g.

$$\begin{cases} |m_0(x,\zeta) - m_0(x',\zeta)| \le M|x - x'|, \quad \forall x, x' \in \mathbb{R}^d, \\ |j(x,\zeta) - j(x',\zeta)| \le j_0(\zeta)|x - x'|, \quad \forall x, x' \in \mathbb{R}^d, \end{cases} \tag{1.10}$$

for some constant $M > 0$ and the same function $j_0(\zeta)$ as in assumption (1.8). Thus the integro-differential operator I_0 has the form

$$I_0\varphi(x) = \int_F [\varphi(x + j(x,\zeta)) - \varphi(x) - j(x,\zeta) \cdot \nabla\varphi(x)] m_0(x,\zeta)\pi(d\zeta). \tag{1.11}$$

It is possible to show that the Markov-Feller process associated with the infinitesimal generator $L_0 + I_0$ (which is referred to as the "diffusion with jumps") has a transition probability density function $G_0(x, t, y)$, which is smooth in some sense (cf. Garroni and Menaldi [8]).

Since our purpose is to treat control problems, we remark that (in general) the optimal feedback is not smooth. This forces us to consider some coefficients (e.g. of first order) which are only measurable. To that effect, we will use the so-called Girsanov's transformation.

Let $\Omega = D([0, +\infty), \mathbb{R}^d)$ be the canonical space of right continuous functions with left-hand limits ω from $[0, +\infty)$ into $\mathbb{R}^d$ endowed with the Skorokhod topology. Denote by either X_t or $X(t)$ the canonical process and by F_t the filtration generate by $\{X_s : s \le t\}$ (universally completed and right-continuous). Now let $(\Omega, P^0, F_t, X_t, t \ge 0)$ be the (homogeneous) Markov-Feller process with transition density function $G_0(x, t, y)$ associated with the integro-differential operator $L_0 + I_0$, i.e. the density w.r.t. the Lebesgue measure of $P^0\{X(t) \in dy \mid X(s) = x\}$ is equal to $G_0(x, t-s, x)$. For the sake of simplicity, we refer to $(P_x^0, X(t), t \geq 0)$ as the above Markov-Feller process, where P_x^0 denote the conditional probability w.r.t. $\{X(0) = x\}$.

Hence, for any smooth function $\varphi(x)$ the process

$$Y_\varphi(t) = \varphi(X(t)) - \int_0^t (L_0 + I_0)\varphi(X(s))ds \tag{1.12}$$

is a P_x-martingale. This follow immediately from the representation

$$\begin{cases} E_x\{\varphi(X(t))\} &= \int_{\mathbb{R}^d} G_0(x,t,y)\varphi(y)dy + \\ & \quad + \int_0^t ds \int_{\mathbb{R}^d} G_0(x, t-s, y)(L_0 + I_0)\varphi(y)dy, \end{cases} \tag{1.13}$$

and the Markov property. Moreover, it is also possible to express the process X_t as follows

$$dX(t) = a^{1/2}(X(t))dw(t) + \int_{\mathbb{R}_\star^d} z\mu_X(dt, dz) + b(X(t))dt, \tag{1.14}$$

where $(w(t), t \geq 0)$ is a standard Wiener process in $\mathbb{R}^d$, $a^{1/2}(x)$ is the positive square root of the matrix $(a_{ij}(x))$ and $b(x)$ is the vector $(b_i(x))$. The process μ_X is the martingale measure associated with the process $(X(t), t \geq 0)$, i.e. if $\eta_X(t, A)$ denotes the integer random measure defined as the number of jumps of the process $X(\cdot)$ on $(0, t]$ with values in $A \subset \mathbb{R}_\star^d$ then

$$\mu_X(dt, A) + \pi_X(dt, A) = \eta_X(dt, A), \tag{1.15}$$

where $\mu_X(t, A)$ is a square integral (local) martingale quasi-left continuous and $\pi_X(t, A)$ is a predictable increasing process obtained via the Doob-Meyer decomposition, and

$$\pi_X(dt, dz) = M_0(X(t-), dz)dt, \tag{1.16}$$

where $M_0(x, dz)$ is the Levy kernel used to define the integro-differential operator I_0 given by (1.1).

Let $g(x) = (g_1(x), \ldots, g_d(x))$ and $c(x, z)$ be functions defined for x in $\mathbb{R}^d$, $z \in \mathbb{R}_\star^d$ such that

$$\begin{cases} g_i, c \text{ are bounded, measurable and,} \\ 0 \leq c(x,z) \leq C_0(1 \wedge |z|), \quad \forall x, z, \end{cases} \tag{1.17}$$

where C_0 is a constant.

Consider the exponential martingale $(e(t), t \geq 0)$ as the solution of the stochastic differential equation

$$\begin{cases} de(t) &= e(t)[r_X(t)dw(t) + \int_{\mathbb{R}_\star^d} \gamma_X(t,z)\mu_X(dt,dz)], \\ e(0) &= 1, \end{cases} \tag{1.18}$$

where

$$\begin{cases} r_X(t) = a^{-1/2}(X(t))g(X(t)), \\ \gamma_X(t,z) = zc(X(t), z), \end{cases} \tag{1.19}$$

i.e.,

$$\begin{cases} e(t) &= \exp\{\int_0^t r_X(s)dw(s) + \int_0^t \int_{\mathbb{R}^d_\star} \gamma_X(s,z)\mu_X(ds,dz) - \\ & - \int_0^t |r_X(s)|^2 ds - \int_0^t \int_{\mathbb{R}^d_\star} [\gamma_X(s,z) - \ln(1+\gamma_X(s,z))]\pi_X(ds,dz)\}. \end{cases} \tag{1.20}$$

If we denote by

$$L = L_0 + \sum_{i=1}^d g_i(x)\partial_i \tag{1.21}$$

and

$$I\varphi(x) = I_0\varphi(x) + \int_{\mathbb{R}^d_\star} [\varphi(x+z) - \varphi(x)]c(x,z)M_0(x,dz), \tag{1.22}$$

then, by means of Itô's formula one can prove that for any smooth function φ, the process

$$Z_\varphi(t) = \varphi(X(t)) - \int_0^t (L+I)\varphi(X(s))ds \tag{1.23}$$

is a P_x-martingale, where the new probability measure is defined as

$$dP_x = e(t)dP_x^0 \ \text{ on } \ F_t. \tag{1.24}$$

Notice that the probability measures P_x^0 and P_x are absolutely continuous, one with respect to the other. Also, a representation of the form (1.14) is valid under the new probability measure P_x, i.e.

$$dX(t) \quad = \quad a^{1/2}(X(t))dw(t) + [b(X(t)) + g(X(t), v(t))]dt + \int_{\mathbb{R}^d_\star} z\mu_v(dt,dz), \tag{1.25}$$

where $(w(t), t \geq 0)$ is again a standard Wiener process and μ_v is the martingale measure associated with the (canonical) process $X(t)$ under the new measure P_x.

Remark 1.1 *Due to the linear growth of the coefficients $b_i(x), i = 1, \ldots, d$, we can not use directly the construction in Garroni and Menaldi [8] of the Green function (or transition density).* □

2. Preliminary Properties

Before considering the interior and exterior Dirichlet problem for the linear operator $L+I$, we need to point-out some essential properties of the integro-differential operator used in our discussion later on. As mentioned in the previous section, we assume

$$\begin{cases} g_i, c \text{ are bounded, measurable and,} \\ 0 \leq c(x,z) \leq C_0(1 \wedge |z|), \quad \forall x, z, \end{cases} \tag{2.1}$$

and define the first order operators

$$L_1 = \sum_{i=1}^{d} g_i(x))\partial_i, \tag{2.2}$$

$$I_1\varphi(x) = \int_{\mathbb{R}^d_\star} [\varphi(x+z) - \varphi(x)]c(x,z)M_0(x,dz). \tag{2.3}$$

Thus, the infinitesimal generator A associated with the diffusion with jumps has the form

$$(L_0 + L_1) + (I_0 + I_1), \tag{2.4}$$

where L_0 and I_0 are the principal part given by (1.3) and (1.1), respectively.

The main assumptions for L_0 are (1.4) and (1.5), i.e., uniformly elliptic second order differential operator with Lipschitz coefficients, bounded second order coefficients and without a zero order coefficient. Condition (1.6) is used to construct a Liapunov function, which will be discussed later. For the integro-differential operator I_0 we assume (1.2), which briefly states that I_0 is the sum of an almost local second order term and a bounded (zero-order) non-local operator. Conditions (1.7), ..., (1.10) specify the x-dependency of the kernel (measure, singular at zero but smooth at infinity) $M_0(x,dz)$ in (1.1), so that a representation (1.11) is valid. On the other hand, L_1 is a first order differential operator with (Borel) measurable and bounded coefficients and without a zero order coefficient. The Levy kernel

$$M_1(x,dz) = c(x,z)M_0(x,dz) \tag{2.5}$$

associated with the integro-differential operator I_1 is of first order [cf. assumption (1.12) on $c(x,z)$], but the density $m_1(x,z)$ is only (Borel) measurable and bounded instead of Lipschitz continuous and bounded as in (1.10).

Denote by $C^\alpha = C^\alpha(\overline{\mathcal{O}})$, $\overline{\mathcal{O}}$ closure of an open subset $\mathcal{O}$ of $\mathbb{R}^d$, the space of Hölder continuous (with exponent α) and bounded function on $\overline{\mathcal{O}}$, $0 \leq \alpha \leq 1$, for $\alpha = 0$ the space of continuous and bounded functions and for $\alpha = 1$ the space of Lipschitz continuous and bounded functions. On the other hand, $L^p = L^p(\mathcal{O})$, $1 \leq p \leq \infty$ denotes the Lebesgue space of p-integrable (essentially bounded for $p = \infty$) functions. If $\mathcal{O}$ is an open subset of $\mathbb{R}^d$ and $\varepsilon > 0$, then $\mathcal{O}_\varepsilon = \mathcal{O} + \varepsilon B_1$ where B_1 is the open ball centered at the origin with radius 1.

Proposition 2.1 (ε-estimates) *Let $\mathcal{O}$ be an open subset of $\mathbb{R}^d$ and let the assumptions $(1.7), \ldots, (1.9)$ hold. Then for any given $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that for any smooth function φ we have*

$$\|I_0\varphi\|_{L^p(\mathcal{O})} \leq \varepsilon\|\nabla^2\varphi\|_{L^p(\mathcal{O}_\varepsilon)} + C_\varepsilon[\|\nabla\varphi\|_{L^p(\mathcal{O})} + \|\varphi\|_{L^p(\mathbb{R}^d)}], \tag{2.6}$$

where $\nabla^2\varphi$ is the Hessian of φ (i.e., the matrix of all second order partial derivatives) and $\nabla\varphi$ the gradient of φ. Similarly, if assumption (1.17) relative to $c(x,z)$ holds then

$$\|I_1\varphi\|_{L^p(\mathcal{O})} \leq \varepsilon\|\nabla\varphi\|_{L^p(\mathcal{O}_\varepsilon)} + C_\varepsilon\|\varphi\|_{L^p(\mathbb{R}^d)}. \tag{2.7}$$

Moreover, if assumption (1.10) holds then

$$\|I_0\varphi\|_{C^\alpha(\overline{\mathcal{O}})} \leq \varepsilon\|\nabla^2\varphi\|_{C^\alpha(\overline{\mathcal{O}}_\varepsilon)} + C_\varepsilon[\|\nabla\varphi\|_{C^\alpha(\overline{\mathcal{O}})} + \|\varphi\|_{C^\alpha(\mathbb{R}^d)}]. \tag{2.8}$$

Proof. We refer to Garroni and Menaldi [8, pp. 52–57] for the main argument. For instance, let us take

$$I_1\varphi(x) = \int_F [\varphi(x + j(x,\zeta)) - \varphi(x)] m_1(x,\zeta)\pi(d\zeta),$$

with

$$m_1(x,\zeta) = c(x, j(x,\zeta)) m_0(x,\zeta).$$

By means of (1.8) and (1.12) we obtain

$$0 \le m_1(x,\zeta) \le C_0(1 \wedge j_0(\zeta)).$$

Since

$$\varphi(x + j(x,\zeta)) - \varphi(x) = \int_0^1 \nabla\varphi(x + \theta j(x,\zeta)) \cdot j(x,\zeta) d\theta$$

we may define

$$I_{1,\eta}\varphi(x) = \int_0^1 d\theta \int_{F_\eta} \nabla\varphi(x + \theta j(x,\zeta)) \cdot j(x,\zeta) m_1(x,\zeta)\pi(d\zeta),$$
$$\text{with} \quad F_\eta = \{\zeta \in F \ : \ 0 < j_0(\zeta) < \eta\}, \quad \eta > 0$$

to get

$$\|I_{1,\eta}\varphi\|_{L^\infty(\mathcal{O})} \le C(\eta)\|\nabla\varphi\|_{L^\infty(\mathcal{O}_\eta)} \tag{2.9}$$

and

$$\|I_{1,\eta}\varphi\|_{L^1(\mathcal{O})} \le c_0^{-1} C(\eta)\|\nabla\varphi\|_{L^1(\mathcal{O}_\eta)}, \tag{2.10}$$

where

$$C(\eta) = C_0 \int_{F_\eta} |j_0(\zeta)|^2 (1 + j_0(\zeta))^{-1}\pi(d\zeta),$$

and c_0 the constant in assumption (1.9). Notice that $x \longmapsto x + j(x,\zeta)$ is a continuously differentiable 1–1 map, so that is preserves zero-measure sets [justifying (2.9)] and it allows a change of variables to establish (2.10). Because $\mathcal{O}_\eta$ is monotone in η and $C(\eta) \to 0$ as $\eta \to 0$, we deduce from (2.9) and (2.10) the first term of (2.7) for $p = \infty$ and $p = 1$.

On the other hand,

$$I_1\varphi(x) - I_{1,\eta}\varphi(x) = \int_{F'_\eta} [\varphi(x + j(x,\zeta)) - \varphi(x)] m_1(x,\zeta)\pi(d\zeta)$$

where

$$F'_\zeta = \{\zeta \in F \ : \ j_0(\zeta) \ge \eta\}.$$

Because

$$\eta(1+\eta)^{-1} \int_{F^1_\eta} m_1(x,\zeta)\pi(d\zeta) \le C,$$

where C depends only on the constants c_0 in assumptions (1.8) and (1.12), we obtain the estimate (2.7) with $C_\varepsilon = C\eta^{-1}(1+\eta)$ for $p = \infty$ and $C_\varepsilon = c_0^{-1} C\eta^{-1}(1+\eta)$ for $p = 1$. It is clear that $\eta > 0$ is selected so small that $\eta < \varepsilon$ and $C(\eta) < \varepsilon$.

Similar arguments are used for $1 < p < \infty$ and the other estimates (2.6) and (2.8). □

Remark 2.2 *The estimates in Proposition 2.1 can be used with* $\mathcal{O} = \mathbb{R}^d$*, so that in this case* $\mathcal{O}_\varepsilon = \mathbb{R}^d$ *too. For instance, in the sense of estimate (2.7) we say that the integro-differential operator* I_1 *is the sum of an "almost local first-order" term (i.e.* $I_{1,\eta}$*) and a bounded operator.* □

Remark 2.3 *It is possible to normalize the second constant* (C_ε) *instead of the first constant* (ε) *in the estimates of Proposition 2.1. For instance, we have*

$$\|I_0\varphi\|_{L^p(\mathcal{O})} \leq \alpha(\varepsilon)\|\nabla^2\varphi\|_{L^p(\mathcal{O}_\varepsilon)} + C_0[\varepsilon^{-1}\|\nabla\varphi\|_{L^p(\mathcal{O})} + \varepsilon^{-2}\|\varphi\|_{L^p(\mathbb{R}^d)}], \tag{2.11}$$

and

$$\|I_1\varphi\|_{L^p(\mathcal{O})} \leq \alpha(\varepsilon)\|\nabla\varphi\|_{L^p(\mathcal{O}_\varepsilon)} + C_0\varepsilon^{-1}\|\varphi\|_{L^p(\mathbb{R}^d)}, \tag{2.12}$$

where $\alpha(\varepsilon) \to 0$ *as* $\varepsilon \to 0$ *and* $C_0 > 0$ *is a constant independent of* ε *and* φ*. Moreover, if* $\mathcal{O}$ *is bounded then we can replace* $\|\varphi\|_{L^p(\mathbb{R}^d)}$ *in estimates (2.6), (2.7), (2.11), (212) for* $[diam(\mathcal{O})]^{1/p}\|\varphi\|_{L^\infty(\mathbb{R}^d)}$*. Furthermore, if* I_0 *is at most of order* γ *[cf. condition (1.8) on* γ*] then we can estimate* $\alpha(\varepsilon)$ *as* $C\varepsilon^{2-\gamma}$ *and improve the exponent of* ε^{-1} *and* ε^{-2} *as* $\varepsilon^{1-\gamma}$ *and* $\varepsilon^{-\gamma}$*.* □

A direct application of Proposition 2.1 is the "almost local estimates" for the integro-differential operator. For instance we have

Proposition 2.4 (almost local estimates) *Let* $\mathcal{O}' \subset \mathcal{O}$ *be bounded open subsets of* $\mathbb{R}^d$ *with* $dist(\partial\mathcal{O}', \partial\mathcal{O}) \geq \delta > 0$*. Suppose* u *in* $W^{2,p}_{loc}(\mathcal{O}) \cap L^\infty(\mathbb{R}^d), 1 < p < \infty$*, is a solution of the equation*

$$L_0 u + I_0 u = f \ \text{ in } \ \mathcal{O}, \tag{2.13}$$

where the coefficients satisfy (1.4), (1.5), (1.7), (1.8) and (1.9). Then there exists a constant c*, depending only on* d, p, δ*,* $diam(\mathcal{O})$ *and the bounds imposed through the assumptions, such that*

$$\|u\|_{W^{2,p}(\mathcal{O}')} \leq C\left[\|f\|_{L^p(\mathcal{O})} + \|u\|_{L^\infty(\mathbb{R}^d)}\right]. \tag{2.14}$$

Proof. We proceed as in Gilbarg and Trudinger [10, Theorem 9.11, p. 236]. For σ in (0, 1), we denote by η a cutoff function in $C^2_0(B_R)$ satisfying $0 \leq \eta \leq 1$, $\eta = 1$ in $B_{\sigma R}$, $\eta = 0$ for $|x| \geq \sigma' R$, $\sigma' = (1+\sigma)/2$, $|\nabla\eta| \leq 4/(1-\sigma)R$, $|\nabla^2\eta| \leq 16/(1-\sigma)^2R^2$. Then, for $v = \eta u$ we have

$$\begin{aligned} \|\sum_{i,j} a_{ij}\sigma_{ij}v\|_{L^P(B_R)} &\leq \|\eta\sum_{i,j} a_{ij}\partial_{ij}u + 2\sum_{i,j} a_{ij}\partial_i\eta\partial_j u + u\sum_{i,j} a_{ij}\partial_{ij}\eta\|_{L^p(B_R)} \leq \\ &\leq C\Big[\|\eta f\|_{L^p(B_R)} + \|\eta I_0 u\|_{L^p(B_R)} + \\ &\quad + \frac{1}{(1-\sigma)R}\|\nabla u\|_{L^p(B_{\sigma' R})} + \frac{1}{(1-\sigma)^2R^2}\|u\|_{L^p(B_R)}\Big]. \end{aligned}$$

Now

$$\eta(x)I_0u(x) = I_0v(x) - u(x)I_0\eta(x) - \varphi(x),$$

where

$$\varphi(x) = \int_{\mathbb{R}^d_\star} [u(x+z) - u(x)][\eta(x+z) - \eta(x)] M_0(x,dz).$$

To estimate φ we start with

$$\begin{aligned}\varphi(x) &= \int_{|z|<\varepsilon} M_0(x,dz) \int_0^1 z \cdot \nabla u(x+\theta z) \infty \int_0^1 z \cdot \nabla \eta(x+\theta' z) d\theta' + \\ &+ \int_{|z|\geq\varepsilon} [u(x+z) - u(x)][\eta(x+z) - \eta(x)] M_0(x,dz),\end{aligned}$$

for $\varepsilon = (\sigma'' - \sigma')R$, $\sigma'' = (1+\sigma')/2$, $\sigma'' - \sigma' = (1-\sigma')/2$, we get

$$\begin{aligned}\|\varphi\|_{L^p(B_{\sigma^1 R})} &\leq C\Big[\frac{1}{(1-\sigma)R} \|\nabla u\|_{L^P(B_{\sigma'' R})} + \\ &+ \frac{1}{(1-\sigma')^2 R^2} \|u\|_{L^\infty(\mathbb{R}^d)}\Big].\end{aligned}$$

Since the matrix a_{ij} is positive and R small, a variation of (2.6) shows that

$$\begin{aligned}\|I_0 v\|_{L^p(B_{\sigma' R})} &\leq \varepsilon \|\sum_{i,j} a_{ij} \partial_{ij} v\|_{L^p(B_{\sigma' R} + B_\varepsilon)} + \\ &+ C_\varepsilon \Big[\|\nabla v\|_{L^p(B_{\sigma' R})} + \|v\|_{L^p(\mathbb{R}^d)}\Big].\end{aligned}$$

Because v has support in $B_{\sigma' R}$ we may replace $B_{\sigma^1 R} + B_\varepsilon$ by $B_{\sigma^1 R}$ to obtain

$$\begin{aligned}\|I_0 v\|_{L^p(B_{\sigma' R})} &\leq \varepsilon \|\sum_{i,j} a_{ij} \sigma_{ij} v\|_{L^p(B_{\sigma' R})} + \\ &+ C_\varepsilon \Big[\|\nabla v\|_{L^p(B_{\sigma' R})} + \frac{1}{(1-\sigma)R} \|u\|_{L^p(B_{\sigma' R})}\Big].\end{aligned}$$

It is clear that

$$\|I_0 \eta\|_{L^p(B_{\sigma^1 R})} \leq C \frac{1}{(1-\sigma)^2 R^2}.$$

Collecting all pieces we deduce

$$\begin{aligned}\|\sum_{i,j} a_{ij} \sigma_{ij} v\|_{L^p(B_{\sigma' R})} &\leq C\Big[\|f\|_{L^p(B_{\sigma' R})} + (C_\varepsilon + \frac{1}{(1-\sigma)R}) \|\nabla u\|_{L^p(B_{\sigma'' R})} + \\ &+ (C_\varepsilon + \frac{1}{1-\sigma)R})(\frac{1}{(1-\sigma)R}) \|u\|_{L^p(B_R)} + \\ &+ \frac{1}{(1-\sigma')^2 R^2} \|u\|_{L^\infty(\mathbb{R}^d)} + \varepsilon \|\sum_{i,j} a_{ij} \partial_{ij} v\|_{L^p(B_{\sigma' R})}\Big],\end{aligned}$$

for some constant $C > 0$. Using the fact that $(1-\sigma)/2 = (1-\sigma')$ and taking ε so small that $C\varepsilon \leq 1/2$, we have

$$\begin{aligned}(1-\sigma)^2 R^2 \|\sum_{i,j} a_{ij} \partial_{ij} v\|_{L^p(B_{\sigma' R})} &\leq C\Big[R^2 \|f\|_{L^p(B_R)} + \\ &+ (1+R)(1-\sigma')R \|\nabla u\|_{L^p(B_{\sigma' R})} + \\ &+ (1+R)(1-\sigma')^2 R^2 \|u\|_{L^p(B_{\sigma' R})} + \|u\|_{L^\infty(\mathbb{R}^d)}\Big],\end{aligned}$$

for some constant $C > 0$. By means of the weighted seminorms

$$\Phi_k = \sup_{0<\sigma<1} \{(1-\sigma)^k R^k \|D^k u\|_{L^p(B_\sigma R)}\}, \quad k = 0,1,2$$

we obtain

$$\begin{cases} \Phi_2 & \leq & C\Big[R^2 \|f\|_{L^p(B_R)} + \|u\|_{L^\infty(\mathbb{R}^d)} + \\ & & +(1+R)\Phi_1 + (1+R)\Phi_0\Big]. \end{cases} \tag{2.15}$$

Hence, the interpolation inequality

$$\Phi_1 \leq \varepsilon \Phi_2 + \frac{C(\eta)}{\varepsilon} \Phi_0 \tag{2.16}$$

provides the desired estimate (2.14), after taking $\sigma = \frac{1}{2}$ and covering $\mathcal{O}'$ with a finite number of ball of radius $R/2$. □

Remark 2.5 *In the estimate (2.14) we may replace the term* $\|u\|_{L^\infty(\mathbb{R}^d)}$ *by the following norm* $\|u\|_{L^p(\mathcal{O})}$ *plus a term of the form*

$$\sup_z \{ (\int_{\mathcal{O}} |u(x + j(x,z))|^p dx)^{1/p} \}, \tag{2.17}$$

where $j(x,z)$ *is the function defining* I_0. *This is to use the norm in* $L^p_{unif}(\mathbb{R}^d)^1$ *instead of in* $L^\infty(\mathbb{R}^d)$. □

Remark 2.6 *The continuity of the first order coefficients* $b_i(x)$ *is not used in the proof of Proposition 2.4, only the fact that* $(b_i(x))$ *are bounded on each bounded subset* $\mathcal{O}$ *of* $\mathbb{R}^d$ *is needed. Thus, the estimate (2.14) remains true when* $L_0 + L_1$ *and* $I_0 + I_1$ *replace* L_0 *and* I_0 *in (2.13).* □

Let us turn our attention to the operators L_0, L_1, I_0, I_1 acting on the Sobolev spaces $W^{1,p}_0(\mathcal{O})$, $1 \leq p \leq \infty$, for a bounded open subset $\mathcal{O}$ of $\mathbb{R}^d$.

It is clear that because the second order coefficients [cf. (1.4)] are Lipschitz and the first order coefficient bounded in $\mathcal{O}$ [cf. (1.5)] we have

$$L_0 : W^{1,p}_0(\mathcal{O}) \to W^{-1,p}(\mathcal{O}) \tag{2.18}$$

where $W^{-1,p}(\mathcal{O})$ denotes the dual of $W^{1,q}_0(\mathcal{O})$, $1/p + 1/q = 1$. We also have

$$L_1 : W^{1,p}_0(\mathcal{O}) \to L^p(\mathcal{O}). \tag{2.19}$$

On the other hand, for the non-local operators I_0, I_1 we need to have a function defined in the whole space $\mathbb{R}^d$. For the space $W^{1,p}_0(\mathcal{O})$ the natural extension to $\mathbb{R}^d$ is by zero. So, unless explicitly stated the contrary, we implicitly assume that any functions in $W^{1,p}_0(\mathcal{O})$ has been extended by zero outside of $\mathcal{O}$, prior the application of the operator I_0 or I_1.

[1] i.e., the norm $\big(\sup_x \{\int_{\{y\,:\,|x+y|<1\}} |u(y)|^p dy\}\big)^{1/p}$.

Thus, we regard $W_0^{1,p}(\mathcal{O})$ as $W_{\overline{\mathcal{O}}}^{1,p}(\mathbb{R}^d)$, functions in $W^{1,p}(\mathbb{R}^d)$ with support in $\overline{\mathcal{O}}$. Since I_1 is a first order operator, we have

$$I_1 : W_0^{1,p}(\mathcal{O}) \to L^p(\mathcal{O}). \tag{2.20}$$

However, to prove that

$$I_0 : W_0^{1,p}(\mathcal{O}) \to W^{-1,p}(\mathcal{O}), \tag{2.21}$$

we need some work. Here we make use of assumption (1.10). Indeed, the critical part is the "almost local" operator

$$I_{0,\eta}\varphi(x) = \int_{|z|<\eta} [\varphi(x+z) - \varphi(x) - z \cdot \nabla\varphi(x)] M_0(x, dz), \tag{2.22}$$

for $\eta > 0$. First, we re-write (2.22) as

$$I_{0,\eta}\varphi(x) = \int_0^1 (1-\theta) d\theta \int_{|z|<\eta} z\nabla^2\varphi(x+\theta z) \cdot z M_0(x, dz),$$

and we consider

$$\langle I_{0,\eta}\varphi, \psi\rangle = \int_{\mathcal{O}} I_{0,\eta}\varphi(x)\psi(x) dx$$

for smooth (test) functions. Using the explicitly x-dependency of the Levy kernel $M_0(x, dz)$ we have to consider an expression of the form

$$\sum_{i,k} \int_{\mathcal{O}} j_i(x,\zeta)\partial_{ik}\varphi(x+\theta j(x,\zeta)) j_k(x,\zeta) m_0(x,\zeta)\psi(x) dx. \tag{2.23}$$

Denote by $T(x)$ the inverse diffeomorphism $x \longmapsto x + \theta j(x,\zeta)$ for a fixed (θ,ζ). Therefore $\partial_i(\partial_k\varphi(x+\theta j(x,\zeta))) = \partial_\ell[\partial_k\varphi(x+\theta j(x,\zeta))]\partial_i T_\ell(x)$. Setting

$$\sigma_{k\ell}(x,\zeta,\theta) = \sum_i j_i(x,\zeta) j_k(x,\zeta) m_0(x,\zeta)\partial_i T_\ell(x) \tag{2.24}$$

we can integrate by parts (2.23) to get

$$-\sum_{k,\ell} \int_{\mathcal{O}} \partial_k\varphi(x+\theta j(x,\zeta))\partial_\ell[\sigma_{k\ell}(x,\zeta,\theta)\psi(x)] dx.$$

Therefore, in order to establish (2.21) we need to assume that $j(x,\zeta)$ has a bounded second derivative in x, i.e. there exist $\delta > 0$ such that

$$\|\nabla_x^2 j(\cdot,\zeta)\|_{L^\infty(\mathbb{R}^d)} \le C, \quad \forall \zeta \in F_\delta \tag{2.25}$$

for some constant $C > 0$, and $F_\delta = \{\zeta \in F \;:\; j_0(\zeta) < \delta\}$.

We will state the property (2.21) for further reference.

Proposition 2.7 *Let the assumptions* (1.7), ..., (1.10) *and (2.25) hold. Then for any given* $\varepsilon > 0$ *there exists* $C_\varepsilon > 0$ *such that for any* φ *in* $W^{1,p}(\mathbb{R}^d)$, $1 \le p \le \infty$, ψ *in* $W_0^{1,q}(\mathcal{O})$, $1/p + 1/q = 1$, *with* $\|\psi\|_{W_0^{1,q}(\mathcal{O})} \le 1$ *we have*

$$|\int_{\mathcal{O}} \psi(x) I_0\varphi(x) dx| \le \varepsilon \|\nabla\varphi\|_{L^p(\mathcal{O}_\varepsilon)} + C_\varepsilon \|\varphi\|_{L^p(\mathbb{R}^d)}, \tag{2.26}$$

where $\mathcal{O}_\varepsilon = \mathcal{O} + \varepsilon B_1$, $\mathcal{O}$ *is an smooth domain*[2]. *In particular (2.21) is satisfied. Moreover, we can replace the* $L^p(\mathbb{R}^d)$*-norm by the* $L^\infty(\mathbb{R}^d)$*-norm in (2.26) if* $\mathcal{O}$ *is bounded.* □

With the same principle of integration by parts and in view of the equality

$$2\varphi(x) I_0\varphi(x) = (I_0\varphi^2)(x) - \int_{\mathbb{R}^d_\star} [\varphi(x+z) - \varphi(x)]^2 M_0(x, dz) \tag{2.27}$$

we can prove the following estimate

Proposition 2.8 *Under the assumptions* (1.7), ..., (1.9) *and (2.25), for any given* $\varepsilon > 0$ *there exists a constant* $C_\varepsilon > 0$ *such that*

$$\langle I_0\varphi, \varphi\rangle \le \varepsilon \|\varphi\| |\varphi| + C_\varepsilon |\varphi|^2, \quad \forall \varphi \in H_0^1(\mathcal{O}), \tag{2.28}$$

where $\langle \cdot, \cdot \rangle$ *is the duality pairing in* $H_0^1(\mathcal{O})$ *and* $H^{-1}(\mathcal{O})$, $\mathcal{O}$ *smooth domain in* $\mathbb{R}^d$, *and* $\|\cdot\|$ *and* $|\cdot|$ *denotes the norms in* $H_0^1(\mathcal{O})$ *and* $L^2(\mathcal{O})$, *respectively.* □

Remark 2.9 *Another key-property used in Bensoussan and Lions [2] is the following*

$$\langle I_0\varphi, \varphi^+\rangle \le C \|\varphi^+\| |\varphi^+|, \quad \forall \varphi \in H_0^1(\mathcal{O}), \tag{2.29}$$

for some constant $C > 0$. *This can be proved similarly to (2.28).* □

Since the first order coefficients of the differential operator L_0 have a linear growth, we are forced to use spaces with some weight at infinity. Denote by $L^2_r = L^2_r(\mathbb{R}^d)$ the Lebesgue space with the norm

$$\|\varphi\|_{L^2_r} = \Big(\int_{\mathbb{R}^d} |\varphi(x)|^2 (1 + |x|^2)^{-r} dx \Big)^{1/2}, \tag{2.30}$$

and by $H^1_r = H^1_r(\mathbb{R}^d)$ the (first order) Sobolev space with the norm

$$\|\varphi\|_{H^1_r} = (\|\varphi\|^2_{L^2_r} + \|\nabla\varphi\|^2_{L^2_r})^{1/2}, \tag{2.31}$$

for any $r \ge 0$. It is clear that L^2_r and H^1_r are Hilbert spaces, and if $s \le r$ then

$$L^2_s \subset L^2_r, \qquad \|\varphi\|_{L^2_r} \le \|\varphi\|_{L^2_s}. \tag{2.32}$$

The same technique used in Proposition 2.7 yields

[2] $\mathcal{O}$ is sufficiently smooth so that $W_0^{1,p}(\mathcal{O}) = W^{1,p}_{\overline{\mathcal{O}}}(\mathbb{R}^d)$, i.e. the extension by zero of functions in $W_0^{1,p}(\mathcal{O})$ belongs to $W^{1,p}(\mathbb{R}^d)$.

Proposition 2.10 *Under the assumptions* (1.7), ..., (1.10) *and (2.25) for any* $\varepsilon > 0$ *there exists* $C_\varepsilon > 0$ *such that for any* φ, ψ *in* H^1_r*, with* $\|\psi\|_{H^1_r} \leq 1$ *we have*

$$|\int_{\mathbb{R}^d} \psi(x) I_0 \varphi(x)(1 + |x|^2)^{-r} dx| \leq \varepsilon \|\nabla\varphi\|_{L^2_r} + C_\varepsilon \|\varphi\|_{L^2_r} \quad \square \tag{2.33}$$

Thus, the non-local operator I_0 maps H^1_r into its dual, denoted by H^{-1}_{-r}. However this is not true for the differential operator L_0, since $b_i(x)$ may (and should) growth linearly in x. For smooth functions φ, ψ we can bound the expression

$$\int_{\mathbb{R}^d} \psi(x) b(x) \cdot \nabla\varphi(x)(1 + |x|^2)^{-r} dx \leq \left(\sup_x \frac{2|b(x)|}{1 + |x|} \right) \|\nabla\varphi\|_{L^2_{r-1}} \|\psi\|_{L^2_r}. \tag{2.34}$$

Therefore, L_0 maps only H^1_{r-1} into the dual space H^{-1}_{-r}. All this gives some complications when looking at the bilinear form

$$a_0(\varphi, \psi) \;=\; -\int_{\mathbb{R}^d} \psi(x)[L_0 + I_0]\varphi(x)(1 + |x|^2)^{-r} dx. \tag{2.35}$$

Any way, we can prove the following result.

Proposition 2.11 *Let the assumptions (1.4), (1.5),* (1.7), ..., (1.10) *and (2.5) hold. Then the bilinear form (2.35) is not continuous in* H^1_r*, but we have*

$$|a_0(\varphi, \psi)| \leq C_0(\|\nabla\varphi\|_{L^2_{r-1}} + \|\varphi\|_{L^2_r})\|\psi\|_{H^1_r}, \tag{2.36}$$

for any φ, ψ *and some constant* C_0. *Moreover* $a_0(\cdot, \cdot)$ *is coercive in* H^1_r*, i.e., there exist* $c_0, \lambda_0 > 0$ *such that*

$$a_0(\varphi, \varphi) + \lambda_0 |\varphi|^2_{L^2_r} \geq c_0 \|\varphi\|^2_{H^1_r} \tag{2.37}$$

for any φ*. The constants* C_0, c_0 *and* λ_0 *depends only on the dimension* d *and the bounds imposed through the hypotheses.* $\square$

Therefore, the Lax-Milgram theory did not apply directly and some "regularization" is needed.

3. Interior Dirichlet Problem

Let L and I be the second order differential operator (1.21) and the integro-differential operator (1.22) as before. For a given bounded and smooth domain $\mathcal{O}$, we consider first the interior Dirichlet problem

$$\begin{cases} -(L + I)u + a_0 u &= f \text{ in } \mathcal{O}, \\ u &= h \text{ in } \mathbb{R}^d \setminus \mathcal{O}, \end{cases} \tag{3.1}$$

and next exterior Dirichlet problem

$$\begin{cases} -(L + I)u + a_0 u &= f \text{ in } \mathbb{R}^d \setminus \overline{\mathcal{O}}, \\ u &= h \text{ in } \overline{\mathcal{O}}, \end{cases} \tag{3.2}$$

where a_0, f, h are given measurable and bounded functions, $a_0(x) \geq 0$.

Notice the non-local character of the integro-differential operator I. So that for the interior problem (3.1) [exterior problem (3.2)] we need the solution u to be defined in a neighborhood of the closure $\overline{\mathcal{O}}[\mathbb{R}^d \setminus \mathcal{O}$, respectively]. Thus, we seek the solution as defined in the whole space $\mathbb{R}^d$.

A natural way to handle the non-homogeneous boundary conditions is the following two-steps problems. First we do suitable extension of the boundary (or exterior) data h to the whole space, for instance if h is defined in $\mathbb{R}^d \setminus \mathcal{O}$ then we extend h to the whole $\mathbb{R}^d$ preserving its regularity properties. Next, we solve an homogeneous problem (like (3.1) with $h = 0$) for $u - h$, where we use the zero-extension to define the non-local operator I. With this in mind, we can re-consider the interior Dirichlet problem (3.1) [or the exterior Dirichlet problem (3.2)] as

$$\begin{cases} -(L+I)u + a_0 u &= f \quad \text{in } \mathcal{O}, \\ u &= h \quad \text{in } \partial\mathcal{O}. \end{cases} \tag{3.3}$$

Actually, we means $u = v + w$ where v solves a non-homogeneous Dirichlet boundary conditions second-order differential equation

$$\begin{cases} -Lv + a_0 v &= 0 \quad \text{in } \mathcal{O}, \\ u &= h \quad \text{in } \partial\mathcal{O}, \end{cases} \tag{3.4}$$

and w solves an homogeneous (interior) Dirichlet problem

$$\begin{cases} -(L+I)w + a_0 w &= f + Iv \quad \text{in } \mathcal{O}, \\ w &= 0 \quad \text{in } \mathbb{R}^d \setminus \mathcal{O}, \end{cases} \tag{3.5}$$

for the whole integro-differential operator $L + I$. Sufficient conditions to solve the PDE (3.4) are well known (cf. Gilbarg and Trudinger [10], Ladyzhenskaya and Uraltseva [14]) so we will state results concerning the existence, uniqueness and regularity for the solutions of the homogeneous interior Dirichlet problem (3.5) with an integro-differential operator of the form (1.1) and (1.3).

Therefore, the primary purpose of this section is to state several results relative to the homogeneous Dirichlet problems (3.3) [with $h = 0$]. This is re-statement of results from Bensoussan and Lions [3], Gimbert and Lions [11] with some natural extensions based on Garroni and Menaldi [8]. For the sake of the reader convenience, we will give some details on key points of the proofs. Thus recall that $I = I_0 + I_1$, I_0 given by (1.11) and

$$I_1\varphi(x) = \int_F [\varphi(x + j(x,\zeta)) - \varphi(x)] m_1(x,\zeta)\pi(d\zeta), \tag{3.6}$$

where $m_1(x,\zeta)$ is a measurable density satisfying

$$0 \leq m_1(x,\zeta) \leq C_0 j_0(\zeta), \quad \forall x, \zeta \tag{3.7}$$

for some constant $C_0 > 0$ and the same $j_0(\zeta)$ as in (1.8). The differential operator L takes the form

$$L = \sum_{i,j=1}^{d} a_{ij}(x)\partial_{ij} + \sum_{i=1}^{d} (a_i(x) + b_i(x))\partial_i \tag{3.8}$$

where (a_{ij}) and (b_i) satisfy (1.4) and (1.5), and (a_i) are measurable and bounded functions

$$|a_i(x)| \leq C_1, \quad \forall x. \tag{3.9}$$

The (homogeneous) interior Dirichlet problem is

$$\begin{cases} -(L+I)u &= f \quad \text{in } \mathcal{O}, \\ u &= 0 \quad \text{in } \mathbb{R}^d \setminus \mathcal{O}, \end{cases} \tag{3.10}$$

for a given function f. The assumptions on the coefficients are (1.4), (1.5), (3.9), (1.8), (1.9), (1.10), (2.25) and (3.7). Before starting the discussion let us mention that because (1) the higher order coefficients possess bounded derivatives [there are Lipschitz continuous, cf. (1.4), (1.10)] instead of being only Hölder continuous and (2) the jump-modulation function is smooth [cf. (2.25)], the whole integro-differential operator $L+I$ can be put in "divergence form". This was not possible under the assumption in Garroni and Menaldi [8].

As it was pointed-out in Bensoussan and Lions [3] and discussed with great detail in Gimbert and Lions [11], a key difficulty is the fact we do not have (in general) the property of mapping $W^{2,p}(\mathcal{O}) \cap W_0^{1,p}(\mathcal{O})$ into $L^p(\mathcal{O})$ for the whole operator $L+I$. The problem is due to the non-local operator I_0 which requires a zero-extension. The non-variational formulation of (3.10) would need a solution u in $W^{2,p}_{loc}(\mathcal{O})$ plus a meaning for the boundary condition, e.g. u in $W_0^{1,p}(\mathcal{O})$ or in $C(\overline{\mathcal{O}})$. To have such a strong solution some restrictions on I_0 are needed, for instance

$$\begin{cases} |j(x,\zeta)| m_0(x,\zeta) \leq j_1(\zeta) \quad \forall x, \zeta \text{ such that} \\ \qquad x \in \mathcal{O}_\varepsilon, \ x + j(x,\zeta) \notin \mathcal{O}, \text{ with} \\ \int_F [j_1(\zeta)]^{1+p} \pi(d\zeta) \leq C_1, \quad \forall p \in [\gamma_1, 1] \end{cases} \tag{3.11}$$

where $C_1, \gamma_1 > 0$. The constant γ_1 (actually $1+\gamma_1$) may be referred to as the "order" of I_0 on the boundary $\partial\mathcal{O}$.

Theorem 3.1 (Strong Solution) *Let us assume*[3] *(1.4), (1.5), (1.8), (1.9), (1.10), (3.7), (3.9) and (3.11) with $0 \leq \gamma_1 < 1/d$. Then for any f in $L^p(\mathcal{O})$, $d < p < 1/\gamma_1$, there exists unique solution of (3.10) in $W^{2,p}(\mathcal{O})$. Moreover, if u denotes the solution of the non-homogeneous interior Dirichlet problem (3.1) [with h sufficiently smooth to be able to solve the PDE (3.4)] then we have the following stochastic representation*

$$\begin{cases} u(x) &= E_x\{\int_0^\tau f(X(t)) \exp(-\int_0^t a_0(X(s))ds)dt + \\ & \quad + h(X(\tau)) \exp(-\int_0^\tau a_0(X(t))dt)\}, \end{cases} \tag{3.12}$$

where τ is the first exit time of the process $X(t)$ from the closed set $\overline{\mathcal{O}}$, i.e.

$$\tau = \inf\{t \geq 0 \ : \ X(t) \notin \overline{\mathcal{O}}\}, \tag{3.13}$$

$E_x\{\cdot\}$ is the mathematical expectation w.r.t. the measure P_x, $(P_x, X(t), t \geq 0)$ is the diffusion with jumps corresponding to $L+I$. □

[3]In (1.4) and (1.10) we may replace Lipschitz by Hölder continuity for the higher order coefficients.

Only some indications of the proof is given, since this is a variation [extension in some sense] of results established in Bensoussan and Lions [3], Gimbert and Lions [11].

Remark 3.2 *By taking $a_0 = 0$ in the above theorem, we have established the existence and the uniqueness of the interior Dirichlet problem*

$$\begin{cases} -(L+I)u = f & \text{in } \mathcal{O}, \\ u = h & \text{in } \mathbb{R}^d \setminus \mathcal{O}, \end{cases} \tag{3.14}$$

in $W^{2,p}(\mathcal{O})$. Notice that u belongs to $W^{1,\infty}(\mathbb{R}^d)$, but u does not necessarily belongs to $W^{2,p}(\mathbb{R}^d)$. The gradient ∇u may have a jump across the boundary $\partial\mathcal{O}$. □

Another important point is the Maximum principle in Sobolev spaces, e.g. Krylov [13], Lions [15]. There are several formulations of this principle. A practical one is the following, as proved in Gimbert and Lions [11].

Proposition 3.3 (Maximum Principle) *Assume (1.4), (1.5), (1.8), (1.9), (1.10), (3.7) and (3.9). Suppose that a bounded and continuous function u in $\mathbb{R}^d$ attains its global maximum at point $\bar{x}$ in $\mathcal{O}$, and that u belongs to $W^{2,d}_{loc}(\mathcal{O})$. Then*

$$\begin{cases} \lim\limits_{\varepsilon\to 0} ess\ inf_{|x-\bar{x}|<\varepsilon}\{Lu(x)\} \leq 0, \\ \lim\limits_{\varepsilon\to 0} ess\ inf_{|x-\bar{x}|<\varepsilon}\{Iu(x)\} \leq 0. \ \square \end{cases} \tag{3.15}$$

Notice that a local version of the ϵ-estimates, namely (here $\gamma = 1$)

$$\|I\varphi\|_{L^p(D)} \leq \varepsilon\|\nabla\varphi\|_{L^p(D_\varepsilon)} + C_\varepsilon\|\varphi\|_{L^\infty(\mathbb{R}^d)} \tag{3.16}$$

where D is bounded and $D_\varepsilon = \{x \in D \ : \ \text{dist}(x, D) \leq \delta_\varepsilon\}$ allows the definition of $I\varphi$ for φ in $W^{2,d}_{loc}(\mathcal{O}) \cap L^\infty(\mathbb{R}^d)$, cf. Proposition 2.1 for details.

Now, returning to the proof of Theorem 3.1, let u be a solution of (3.1) in $W^{2,p}(\mathcal{O})$. By means of the (weak) maximum principle (Proposition 3.3) applied to the function $\lambda\underline{u} - (u - k)$, with $\lambda\beta > \|(f + Lh + Ih)^+\|_{L^\infty(\mathcal{O})}$, $\lambda\beta > \|h^-\|_{L^\infty(\mathcal{O})}$ and $\underline{u}, \beta$ as in (2.12), (2.16), we deduce (by contradiction) that $\lambda\underline{u} - (u - h) \geq 0$. This provides the estimate

$$\begin{cases} u \leq h + \lambda\underline{u}, & \text{in } \mathcal{O}, \\ \lambda\beta = \max\{\|h^-\|_{L^\infty(\mathcal{O})}, \|(f + Lh + Ih)^+\|_{L^\infty(\mathcal{O})}\}, \end{cases} \tag{3.17}$$

which implies

$$\|u\|_{L^\infty(\mathbb{R}^d)} \leq C(\|h\|_{W^{2,\infty}(\mathbb{R}^d)} + \|f\|_{L^\infty(\mathcal{O})}), \tag{3.18}$$

for some constant C depending (essentially) on $\|\underline{u}\|_{L^\infty(\mathcal{O})}$. Hence, the global L^p-estimate for the differential elliptic equation and (3.18) yield the a priori estimate

$$\|u\|_{W^{2,p}(\mathcal{O})} + \|u\|_{W^{1,\infty}(\mathbb{R}^d)} \leq C_p(\|h\|_{W^{2,\infty}(\mathbb{R}^d)} + \|f\|_{L^\infty(\mathcal{O})}). \tag{3.19}$$

By means of the estimate (3.17), the existence of a solution is proved under the extra assumption $a_0(x) \geq \alpha$, with α sufficiently large. Finally, by a fixed-point argument the extra condition on $a_0(\cdot)$ is removed. □

Another interesting strong version of the maximum principle is the following:

Proposition 3.4 (Strong Maximum Principle) *Let (1.4), (1.5), (1.8), (1.9), (1.10), (3.7) and (3.9) hold. Suppose that a bounded and continuous function u in $\mathbb{R}^d$ attains its global maximum at a point $\overline{x}$ in $\mathcal{O}$. If $\mathcal{O}$ is connected, u belongs to $W^{2,d}_{loc}(\mathcal{O})$ and*

$$(L+I)u \geq 0 \ \ \textit{in} \ \ \mathcal{O}, \tag{3.20}$$

then u is constant in $\mathcal{O}$. □

The proof is a direct consequence of the weak maximum principle (Proposition 3.3) and a barrier function. Indeed, as in the classic case (e.g. Protter and Weinberger[19]), if we assume that u is not a constant then there is a point x_0 at which u attains its (global) maximum value M, and two balls (inside $\mathcal{O}$) B_1 and B_2 such that x_0 is the center of B_2 and belongs to the boundary ∂B_1, and for some $\delta > 0$

$$u \leq M - \delta \quad \text{on} \ \ \overline{B}_1 \setminus B_2. \tag{3.21}$$

Hence, the contradiction follows after applying Proposition 3.3 to the function $w \doteq u + \varepsilon \bar{u}$, with $0 < \varepsilon < \delta / \max\{\bar{u}(x) \ : \ x \in \overline{B}_1 \setminus B_2\}$. The function $\overline{u}$ is a barrier function satisfying.

$$\begin{cases} \bar{u} > 0 \ \text{ in } \ B_1, \qquad \bar{u} < 0 \ \text{ outside } \ B_1 \\ (L+I)\bar{u} > 0 \ \text{ in } \ B_2 \end{cases} \tag{3.22}$$

To construct such a barrier function, we call $\bar{x}$ and $\bar{r}$ the center and the radius of the ball B_1. Define

$$\bar{u}(x) = \exp(-\lambda |x - \bar{x}|^2) - \exp(-\lambda \bar{r}^2), \tag{3.23}$$

for a constant $\lambda > 0$ to be determined below. It is clear that the first two conditions of (3.22) are satisfied. Computations show that

$$\begin{aligned} L\bar{u}(x) \ &= \ \Big[4\lambda^2 \sum_{i,j} a_{ij}(x)(x_i - \bar{x}_i)(x_j - \bar{x}_j) - 2\lambda \sum_i a_{ii}(x) - \\ &\quad -2\lambda \sum_i (a_i(x) + b_i(x))(x_i - \bar{x}_i)\Big]\Big[\exp(-\lambda|x - \bar{x}|^2)\Big] \end{aligned}$$

and

$$\begin{aligned} I\bar{u}(x) \ &= \ -2\lambda \Big[\int_{\mathbb{R}^d_\star} (x - \bar{x}) \cdot z M(x, dz)\Big]\Big[\exp(-\lambda|x - \bar{x}|^2)\Big] + \\ &\quad + \int_0^1 (1-\theta) d\theta \int_{\mathbb{R}^d_\star} z \cdot \nabla^2 \bar{u}(x + \theta z) z M(x, dz). \end{aligned}$$

Since $\bar{u}$ is concave, we have

$$\begin{cases} (L+I)\bar{u}(x) \ \geq \ 2\lambda \Big[\exp(-\lambda|x - \bar{x}|^2)\Big] \times \\ \qquad\qquad \times \Big[\lambda c_0 |x - \bar{x}|^2 - d c_0^{-1} - (c_1 + M|x|)|x - \bar{x}|\Big], \end{cases} \tag{3.24}$$

where c_0 is the constant in (1.4), M is the constant in (1.5) and

$$c_1 = \sup_x \Big\{ (\sum_i |a_i(x)|^2)^{1/2} + \int_{\mathbb{R}^d_\star} |z| M(x, dz) \Big\}. \tag{3.25}$$

Thus, for any $0 < r < R$ we can choose $\lambda \geq \lambda(c_0, c_1, r, R)$ such that

$$(L + I)\bar{u}(x) \geq \beta > 0 \quad \text{if} \;\; r \leq |x - \bar{x}| \leq R. \tag{3.26}$$

In particular (3.22) holds. □

When the integro-differential operator I is almost of order $\gamma = 1$, one can prove the following local L^p-estimates.

Proposition 3.5 (Local L^p-estimates) *Let the assumptions (1.4), (1.5), (1.8) with $\gamma = 1$, (1.10), (3.7) and (3.9) hold. Suppose u is a function in $W^{1,p}(\mathcal{O}) \cap L^\infty(\mathbb{R}^d)$ satisfying*

$$-(Lu + Iu) = f \;\; \text{in} \;\; \mathcal{O}. \tag{3.27}$$

Then for any bounded domain B with closure in $\mathcal{O}$ we can find a constant C such that

$$\|u\|_{W^{2,p}(B)} \leq C[\|f\|_{L^p(\mathcal{O})} + \|u\|_{L^\infty(\mathbb{R}^d)}]. \tag{3.28}$$

Proof. By means of a smooth cutoff function β, with compact support in $\mathcal{O}$ and $\beta = 1$ on B, as in Gilbarg and Trudinger [10, Theorem 9.11, p. 236].

$$\begin{cases} -(L + I)(u\beta) &= f\beta + g \quad \text{in} \quad \mathcal{O}, \\ u\beta &= 0 \quad \text{in} \quad \mathbb{R}^d \setminus \mathcal{O}, \end{cases} \tag{3.29}$$

where $g = g_1 + g_2$,

$$g_1 = \beta L u - L(u\beta), \qquad g_2 = \beta I u - I(u\beta).$$

The contribution of g_1 is a first order differential operator in u, which can be handled in the usual way. The nonlocal expression takes the form

$$g_2(x) = -\int_{\mathbb{R}^d} [\beta(x + z) - \beta(x)] u(x + z) M(x, dz)$$

which yields

$$\|g_2\|_{L^2(\mathcal{O})} \leq C_\beta \|u\|_{L^\infty(\mathbb{R}^d)},$$

for a constant C_β depending on β and the constant C_0 in (1.8). Hence, by means of (3.29)

$$\|u\|_{W^{2,p}(B)} \leq C_p \|f\beta + g\|_{L^p(\mathcal{O})}.$$

Therefore, introducing weighted seminorms and using an interpolation inequality we deduce (3.28). □

Remark 3.6 *Estimate (3.28) in Proposition 3.5 holds for integro-differential operators of order $\gamma = 2$, but the proof is a little more complicated, cf. Proposition 2.4.* □

Let us turn our attention to the variational formulation of the (homogeneous) interior Dirichlet problem (3.10), i.e. a solution in $W_0^{1,p}(\mathcal{O})$. The key point here is to establish that $L + I$ maps $W_0^{1,p}(\mathcal{O})$ into $W^{1,p}(\mathcal{O})$ as discussed in Section 2. Assumptions (1.10), (2.25) are used in Proposition 2.7 to make sense of $I_0\varphi$ for φ in $W_0^{1,p}(\mathcal{O})$. Next, estimate (2.28) is necessary to show that the bilinear form

$$a(\varphi, \psi) = -\langle L\varphi, \psi \rangle - \langle I\varphi, \psi \rangle \tag{3.30}$$

is continuous and coercive in $H_0^1(\mathcal{O})$. We state the main results in this direction.

Theorem 3.7 (Weak Solution) *Let the assumptions (1.4), (1.5), (1.8), (1.9), (1.10), (2.25), (3.7) and (3.9) hold. Then for any f in $W^{-1,p}(\mathcal{O})$, $1 < p < \infty$, there exists a unique solution u of the (homogeneous) interior Dirichlet problem*

$$-(L+I)u = f \quad in \ W^{1,p}(\mathcal{O}), \tag{3.31}$$

in $W^{1,p}_0(\mathcal{O})$. Moreover, if f belongs to $L^p_{loc}(\mathcal{O})$ then u belongs to $W^{2,p}_{loc}(\mathcal{O})$. Furthermore, for f in $L^p(\mathcal{O})$, $p > d$, stochastic representation (3.12) is valid. □

The above results for $j(x,\zeta) = j(\zeta)$, independent of x, have been proved in Bensoussan and Lions [3] with $p \geq 2$ and the general case $(1 < p < \infty)$ in Gimbert and Lions [11]. We give only some detail of the proof.

First, the fact that the bilinear form (3.30) is continuous and coercive yields the existence and uniqueness of the solution in $H^1_0(\mathcal{O})$. Next the almost local estimates (cf. Proposition 2.4) proves the regularity result. Finally, approximating I by a zero-order integro-differential operator I_ε (cf. arguments in Proposition 2.1) we are able to approximate "weak solutions" by "strong solution", which establish the stochastic representation (3.12). A crucial point is a weak version of the maximum principle, which follows from estimate (2.29).

Proposition 3.8 (Weak Maximum Principle) *Let us assume (1.4), (1.5), (1.8), (1.9), (1.10), (2.25), (3.7) and (3.9). Suppose a function u in $W^{1,p}(\mathcal{O})$, $d \leq p < \infty$, satisfies*

$$\begin{cases} (L+I)u \geq 0 & in \ W^{1,p}(\mathcal{O}), \\ u \geq 0 & on \ \partial\mathcal{O}, \end{cases} \tag{3.32}$$

then $u \geq 0$ a.e. in $\mathcal{O}$. □

4. Exterior Dirichlet Problem

We are going to study two cases of the exterior Dirichlet problem. First the case related to the recurrence of the jump-diffusion process, namely

$$\begin{cases} (L+I)u = 0 & \text{in } \mathbb{R}^d \setminus \overline{\mathcal{O}} \\ u = h & \text{in } \overline{\mathcal{O}}. \end{cases} \tag{4.1}$$

Next, we will consider the case associated with the positive recurrence of the jump-diffusion process, i.e.

$$\begin{cases} -(L+I)u = f & \text{in } \mathbb{R}^d \setminus \overline{\mathcal{O}} \\ u = 0 & \text{in } \overline{\mathcal{O}}. \end{cases} \tag{4.2}$$

To prove the existence and uniqueness of solutions for the above exterior Dirichlet problems (4.1) and (4.2), we will make use of Liapunov functions. Assume that there exists a function ψ such that

$$\begin{cases} \psi > 0, \qquad \psi \in W^{2,p}_{loc}(\mathbb{R}^d), \\ \psi(x) \to +\infty \ \text{ as } |x| \to \infty, \\ (L+I)\psi \leq 0 \ \text{ in } \mathbb{R}^d \setminus \overline{\mathcal{O}}, \end{cases} \tag{4.3}$$

for some $p \geq d$. The above condition (4.3) hides a growth assumption for ψ so that $I\psi$ makes sense. As we will see later, a typical example for a Liapunov function ψ has logarithm growth. Since I accepts functions with a linear growth, $I\psi$ is well defined. In general, we need to assume that the jumps are bounded (in this case any growth is acceptable) or to suppose that ψ is uniformly integrable w.r.t. to the Levy measure of I.

Our first interest is to look for bounded solutions of (4.1) and (4.2). Then, we turn to the probabilistic interpretation of (4.1) and (4.2). Denote by $(\Omega, P, F_t, X_t, t \geq 0)$ a canonical realization of the diffusion with jumps (Markov-Feller) process whose infinitesimal generator coincides with $L + I$ on smooth functions. Let τ be the first exit time of the process $X(t) \doteq X_t$ from the closed set $\mathbb{R}^d \setminus \mathcal{O}$, i.e.

$$\tau = \inf\{t \geq 0 : X(t) \in \mathcal{O}\}, \tag{4.4}$$

where $\tau = \infty$ if $X(t) \in \mathbb{R}^d \setminus \mathcal{O}$, $\forall t \geq 0$. A probabilistic solution of (4.1) is a function $u(x)$ satisfying:

$$\begin{cases} u(X(t))\mathbf{1}_{(t<\tau)} + h(X(\tau))\mathbf{1}_{(t\geq\tau)} \\ \qquad \text{is a } F_t - \text{(local) martingale.} \end{cases} \tag{4.5}$$

Similarly, a probabilistic solution of (4.2) is a function $(u(x)$ satisfying:

$$\begin{cases} u(X(t))\mathbf{1}_{(t<\tau)} + \int_0^{\tau\wedge t} f(X(s))ds \\ \qquad \text{is a } F_t - \text{(local) martingale.} \end{cases} \tag{4.6}$$

Remarking that the process $X(t)$ is continuous from the right, we see that $P(\tau = 0 \mid X(0) = x) = 0$ for any x in the open set $\mathbb{R}^d \setminus \overline{\mathcal{O}}$. Thus, if we assume "recurrence" for the process $X(t)$, i.e.

$$P(\tau < \infty \mid X(0) = x) = 1 \quad \forall x \in \mathbb{R}^d \setminus \overline{\mathcal{O}}, \tag{4.7}$$

then any bounded solution u of (4.5) must satisfy

$$u(x) = E\{h(X(\tau)) \mid X(0) = x\} \quad \forall x \in \mathbb{R}^d \setminus \overline{\mathcal{O}}, \tag{4.8}$$

so that the bounded probabilistic solutions of (4.1) are unique. Similarly, if we suppose "positive recurrence" for the process $X(t)$, i.e.

$$E\{\tau \mid X(0) = x\} < \infty \quad \forall x \in \mathbb{R}^d \setminus \overline{\mathcal{O}} \tag{4.9}$$

then any bounded solution u of (4.6) must satisfy

$$u(x) = E\{\int_0^\tau f(X(t))dt \mid X(0) = x\} \quad \forall x \in \mathbb{R}^d \setminus \overline{\mathcal{O}}, \tag{4.10}$$

again, bounded probabilistic solutions of (4.2) are unique.

The probabilistic formulations included both conditions in (4.1) [or (4.2)] simultaneously. In particular, the boundary condition in (4.1) is satisfied in the following sense

$$\lim_{t\to\infty} u(X(\tau \wedge t)) = h(X(\tau)) \text{ a.s.}, \tag{4.11}$$

i.e. a pathway limits holds. Notice that for (4.5) and (4.6) we are implicitly assuming that the data h and f are Borel measurable. Actually, because the semigroup associated to the diffusion process preserves negligible sets, we may work with Lebesgue measurable functions instead of Borel measurable functions.

Theorem 4.1 (Recurrence) *Suppose h is a Borel measurable and bounded function $\overline{\mathcal{O}}$. Under the assumptions (1.4), (1.5), (1.8), (1.9), (1.10), (3.7), (3.9) and (4.3), the non-homogeneous exterior Dirichlet problem (4.1) has a unique probability solution [i.e., (4.5) holds] u belonging to $L^\infty(\mathbb{R}^d) \cap W^{2,p}_{loc}(\mathbb{R}^d \setminus \overline{\mathcal{O}})$ for any $p < \infty$. Moreover*

$$(L + I)u = 0 \quad a.e. \ in \ \mathbb{R}^d \setminus \overline{\mathcal{O}} \tag{4.12}$$

and the stochastic representation (4.8) is valid.

Proof First, replacing h by $h + \|h\|_{L^\infty}$ we can assume $h \geq 0$ without any loss of generality. Existence is shown as in Bensoussan [1], i.e., for $n > 0$ sufficiently large so that the ball B_n (centered at the origin) with radius n contains $\overline{\mathcal{O}}$, we consider the solution of the Dirichlet problem (in a bounded region)

$$\begin{cases} (L + I)u_n &= 0 \ \text{ in } \ B_n \setminus \overline{\mathcal{O}}, \\ u_n &= h \ \text{ in } \ \overline{O}, \\ u_n &= 0 \ \text{ in } \ \mathbb{R}^d \setminus B_n. \end{cases} \tag{4.13}$$

Notice that a priori, we need h to be defined in $\overline{\mathcal{O}}$. However, the most relevant part is its definition on $\partial\mathcal{O}$ as expected.

Now, the weak maximum principle (cf. Proposition 3.9) implies that $u_n \geq 0$. Thus, again the weak maximum principle applied to the difference $u_n - u_m$, with $m < n$, shows that $u_n \geq u_m$. Therefore, we have an increasing sequence satisfying

$$0 \leq u_m \leq u_n \leq \|h\|_{L^\infty}. \tag{4.14}$$

Hence, the almost local estimate (cf. Proposition 2.4) proves that $\{u_n\}$ is uniformly bounded in $W^{2,p}_{loc}(\mathbb{R}^d \setminus \overline{\mathcal{O}})$, for any $p < \infty$, and that (4.12) holds for the limiting function u.

To prove (4.5) and (4.7), we proceed us in Khasminskii [12]. Denote by τ^n the first exit time from B_n. By means of Itô is formula and (4.3) we have

$$E\{\psi(X(\tau \wedge \tau^n)) \mid X(0) = x\} \leq \psi(x)$$

which yields

$$E\{\psi(X(\tau^n))\mathbf{1}_{(\tau > \tau^n)}\} \leq \psi(x).$$

Thus, if

$$\alpha_n = \inf\{\psi(x) \ : \ |x| \geq n\}$$

we deduce

$$P(\tau > \tau^n) \geq \frac{\psi(x)}{\alpha_n}$$

which implies (4.7), so the recurrence properly holds.

On the bounded domain $B_n \setminus \overline{\mathcal{O}}$ we have

$$\begin{cases} u_n(X(t))\mathbf{1}_{(t<\tau^n \wedge \tau)} + h(X(\tau))\mathbf{1}_{(t \geq \tau = \tau^n)} \\ \qquad \text{is a } \ F_t - \text{martingale.} \end{cases} \tag{4.15}$$

Since $u_n(X(t))\mathbf{1}_{(t<\tau^n\wedge\tau)}$ increases to $u(X(t))\mathbf{1}_{(t<\tau)}$ and $\mathbf{1}_{(\tau<\tau^n)}$ decreases to zero, we obtain (4.5). □

Remark 4.2 *Without the assumption (2.25) we need to suppose that the boundary function data h is smooth enough and that condition (3.11) holds in order to look for strong solution on the bounded domain $B_n \setminus \overline{\mathcal{O}}$, cf. Theorem 3.1. Actually, u_n is the probability solution of (4.13). This argument is well known for degenerate diffusion processes, e.g. Stroock and Varadhan [21], Menaldi [16], Robin [20]. An alternative approach is to use the so-called viscosity solutions (cf. Crandall et al. [5]), which is better adapted to nonlinear problem.* □

Remark 4.3 *As mentioned before, if we want to prescribe the data function h only on the boundary $\partial\mathcal{O}$, then a canonical "zero-extension" is assumed, i.e. the two-steps (3.4), (3.5) is used to solve (3.3). In our case, a simple Borel measurable and bounded data function h on $\partial\mathcal{O}$ is not good enough to give a $W^{1,p}(B_n \setminus \mathcal{O})$ meaning to (4.13), the boundary condition can be regarded in a sense similar to (4.11).* □

In order to study the homogeneous exterior Dirichlet problem (4.2) we need to add the condition

$$(L + I)\psi \leq -1 \quad \text{in } \mathbb{R}^d \setminus \mathcal{O} \tag{4.16}$$

to the function ψ satisfying (4.3).

Theorem 4.4 (Positive Recurrence) *Let the assumptions (1.4), (1.5), (1.8), (1.9), (1.10), (3.7), (3.9), (4.3), (4.16) and*

$$f \in L^\infty(\mathbb{R}^d \setminus \overline{\mathcal{O}}) \tag{4.17}$$

hold. Then the homogeneous exterior Dirichlet problem (4.2) has a unique probability solution [i.e. (4.6) is satisfied] u such that u/ψ is bounded. Moreover, u belongs to $W^{2,p}_{loc}(\mathbb{R}^d \setminus \overline{\mathcal{O}})$, for any $p < \infty$,

$$-(L + I)u = f \quad \text{a.e. in } \mathbb{R}^d \setminus \overline{\mathcal{O}} \tag{4.18}$$

and the stochastic representation (4.10) is valid.

Proof First, by linearity, we may consider the problem for f^+ and f^- independently. This allows us to assume $f \geq 0$, without any loss of generality.

Again we proceed as in Bensoussan [1] to prove the existence and as in Khasminskii [12] to obtain the uniqueness, similar to Theorem 4.1.

On the bounded domain $B_n \setminus \overline{\mathcal{O}}$ we consider the (homogeneous) Dirichlet problem

$$\begin{cases} -(L + I)u_n &= f \quad \text{in } B_n \setminus \overline{\mathcal{O}}, \\ \quad u_n &= 0 \quad \text{in } \overline{\mathcal{O}} \cup (\mathbb{R}^d \setminus B_n). \end{cases} \tag{4.19}$$

The weak maximum principle (cf. Proposition 3.9) implies $u_n \geq 0$ and $u_n \geq u_m$, for $n > m$. Similarly, the weak maximum principle applied to the function $u_n - c\psi$, with $c \geq \|f\|_{L^\infty}$, yields an uniform bound for the increasing sequence u_n, i.e.

$$0 \leq u_m \leq u_n \leq \psi \|f\|_{L^\infty}, \quad \forall m < n. \tag{4.20}$$

Thus, the almost local estimate (cf. Proposition 2.4) proves that the sequence $\{u_n\}$ is uniformly bounded in $W^{2,p}_{loc}(\mathbb{R}^d \setminus \overline{\mathcal{O}})$, for any $p < \infty$ and that (4.18) holds for the limiting function u.

To show the validity of the positive recurrence property, we start with

$$E\{\psi(X(t \wedge \tau \wedge \tau^n)) \mid X(0) = 0\} = \psi(x) + \\ + E\{\int_0^{t\wedge\tau\wedge\tau^n} (L + I)\psi(X(s))ds \mid X(0) = x\},$$

and in view of (4.16), as $t \to \infty$ we get

$$E\{\tau \wedge \tau^n \mid X(0) = x\} \leq \psi(x), \quad \forall x \in \mathbb{R}^d \setminus \overline{\mathcal{O}},$$

where τ^n is the first exit time from B_n. This proves (4.9).

Now, as in (4.15), on the bounded domain $B_n \setminus \overline{\mathcal{O}}$ we have

$$\begin{cases} u_n(X(t))\mathbf{1}_{(t<\tau^n\wedge\tau)} + \int_0^{\tau^n\wedge\tau\wedge t} f(X(s))ds \\ \qquad \text{is a } \ F_t - \text{martingale.} \end{cases}$$

Since $u_n(X(t))\mathbf{1}_{(t<\tau^n\wedge\tau)}$ increases to $u(X(t))\mathbf{1}_{(t<\tau)}$ we obtain (4.6), even if u_n is unbounded. □

Remark 4.5 *If we add the assumption (3.11) with $0 \leq \gamma_1 < 1/d$, $d < p < 1/\gamma_1$, then the arguments of the strong solution (cf. Theorem 3.1) apply for the exterior Dirichlet problem (4.1) [if h belongs to $W^{2,p}(\mathcal{O})$] and (4.2), i.e. the solutions of (4.13) and (4.19) u_n are in $W^{2,p}(B_n \setminus \overline{\mathcal{O}})$ and the limiting function u belongs to $W^{2,p}(B \setminus \overline{\mathcal{O}})$, for any ball $B \subset \overline{\mathcal{O}}$. For instance, we may use an almost local estimate of the type (2.14) up the boundary $\partial\mathcal{O}$. However, it is not developed here.* □

Usually we seek a Liapunov function ψ as the logarithm of a positive definite quadratic form, e.g.

$$\psi(x) = \ln(|x|^2 + 1). \tag{4.21}$$

Calculations show that

$$(L + I)\psi(x) \geq -c, \quad \forall x, \ |x| \geq r_1 \tag{4.22}$$

which provides a Liapunov function for any domain outside of the ball of center 0 and radius r_1.

Other types of Liapunov functions are the one considered in [17], namely

$$\psi_q(x) = (2 + |x|^2)^{q/2}, \quad q > 0. \tag{4.23}$$

We have proved that if the constants r_1 or c_1 in (1.6) are sufficiently large, then the function ψ_q given by (4.23) satisfies

$$L\psi_q(x) + I\psi_q(x) \leq -\alpha_q\psi_q(x), \quad \forall x, \ |x| \geq r_1 \tag{4.24}$$

for a positive constant α_q depending only on the various bounds imposed by the assumptions (1.4), (1.6) and the extra condition

$$\sup_x \int_{|z|\geq 1} |z|^q M(x, dz) < \infty. \tag{4.25}$$

It is clear that also we have

$$|L\psi_q(x)| + |I\psi_q(x)| \leq c_q \psi_q(x), \quad \forall x,\ |x| \geq r_1 \tag{4.26}$$

for some constant c_q. At this point, most of the results valid for the operator $-(L+I)+\lambda$ can be extended to the case $\lambda = 0$. In particular a variational formulation of (4.2) is studied and the estimate

$$\|u\psi_{-q}\|_{L^\infty} \leq \frac{1}{\alpha_q} \|f\psi_{-q}\|_{L^\infty} \tag{4.27}$$

holds.

5. Invariant Measure

First, we recall a classic result on ergodicity of Doob (cf. Bensoussan [1]).

Let $(X, \mathcal{F})$ be compact metric space endowed with the Borel σ-algebra. Suppose that P is a linear operator from $B(X)$ into itself (the Banach space of bounded and Borel measurable functions from X into $\mathbb{R}$) such that

$$\begin{cases} \|P\varphi\| \leq \|\varphi\|, \quad \forall \varphi \in B(X), \\ P\varphi = \varphi \ \text{ if } \ \varphi = 1, \end{cases} \tag{5.1}$$

where $\|\cdot\|$ denotes the supremum norm in X. Define

$$\lambda(x, y, F) = P\mathbf{1}_F(x) - P\mathbf{1}_F(y), \tag{5.2}$$

for any x, y in X and any Borel subset F of X, where $\mathbf{1}_F$ is the characteristic function of the set B.

Theorem 5.1 (Doob's Ergodicity) *Under the assumptions (5.1) and*

$$\exists \delta > 0 \ / \ \lambda(x, y, F) \leq 1 - \delta, \quad \forall x, y \in X,\ \forall F \in \mathcal{F}, \tag{5.3}$$

there exists a unique probability measure on $(X, \mathcal{F})$ denoted by μ such that

$$|P^n \varphi(x) - \int_X \varphi d\mu| \leq K e^{-\rho n} \|\varphi\|, \tag{5.4}$$

where $\rho = -\ell n(1-\delta)$, $K = 2/(1-\delta)$. The measure μ is the unique invariant probability on $(X, \mathcal{F})$, i.e. the unique probability on X such that

$$\int_X \varphi d\mu = \int_X P\varphi d\mu, \quad \forall \varphi \in B(X).\square \tag{5.5}$$

Usually, this result is applied after verifying the Doeblin condition (5.3), which is based on the strict positivity of the transition density function of the underlying Markov process. This strict positivity of the Green function is a natural consequence of the parabolic strong maximum principle.

Let $\mathcal{O}$ be a sufficiently large smooth and bounded domain (e.g. a ball) so that the non-homogeneous exterior Dirichlet problem

$$\begin{cases} (L+I)u = 0 & \text{in } \mathbb{R}^d \setminus \overline{\mathcal{O}}, \\ u = \varphi & \text{in } \overline{\mathcal{O}}, \end{cases} \tag{5.6}$$

can be solved in $W^{2,p}_{loc}(\mathbb{R}^d \setminus \overline{\mathcal{O}}) \cap W^{1,p}_{loc}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ for non-negative φ in $W^{1,p}(\mathcal{O}) \cap L^\infty(\mathbb{R}^d)$. Now, consider the non-homogeneous interior Dirichlet problem in a larger domain (ball) $B \supset \overline{\mathcal{O}}$,

$$\begin{cases} (L+I)v = 0 & \text{in } B, \\ v = u & \text{in } \mathbb{R}^d \setminus B, \end{cases} \tag{5.7}$$

which can be solved in $W^{2,p}_{loc}(B) \cap W^{1,p}(B) \cap L^\infty(\mathbb{R}^d)$, for any v in $W^{1,p}_{loc}(\mathbb{R}^d \setminus \overline{\mathcal{O}}) \cap L^\infty(\mathbb{R}^d)$. Therefore we can define the linear operator

$$\begin{cases} P : W^{1,p}(\mathcal{O}) \cap L^\infty(\mathbb{R}^d) \to W^{1,p}(\mathcal{O}) \cap L^\infty(\mathbb{R}^d), \\ P\varphi = v, \end{cases} \tag{5.8}$$

where the solution u of (5.7) has been restricted to the domain $\overline{\mathcal{O}}$. The point is to prove that P is an ergodic operator, i.e. defining λ by (5.2) we have (5.3) for $X = \overline{\mathcal{O}}$.

By means of the weak maximum principle, we can prove that

$$\varphi \geq 0 \quad \text{implies} \quad P\varphi \geq 0. \tag{5.9}$$

Since $P\varphi = 1$ for $\varphi = 1$, the operator P can be identified with a probability measure on $(X, \mathcal{F})$, so that

$$\begin{cases} P : B(\overline{\mathcal{O}}) \to B(\overline{\mathcal{O}}), \\ P\varphi(x) = \displaystyle\int_X \varphi(y) P(x, dy). \end{cases} \tag{5.10}$$

Proposition 5.2 *Under the assumptions (1.4), (1.5), (1.6), (1.8), (1.9) (1.10), (2.25), (3.7) and (3.9) we have (5.3) for $X = \overline{\mathcal{O}}$.*

Proof Similarly to Bensoussan [1], an argument by contradiction based on the strong maximum principle yields the result as follows.

Assuming that (5.3) is not true, we can find sequences $\{x_k, y_k, F_k\}$ such that

$$u_k(x_k) \to 1, \quad u_k(y_k) \to 0, \tag{5.11}$$

where x_k, y_k belong to $\overline{\mathcal{O}}$ and $v_k = P\mathbf{1}_{F_k}$. Actually, we may replace $\mathbf{1}_{F_k}$ by a smooth function φ_k, $0 \leq \varphi_k \leq 1$, without any loss of generality. Thus, P is defined by (5.6) and (5.7). By means of the almost local L^p-estimates (cf. Proposition 2.4) we see that u_k is bounded in $W^{2,p}_{loc}(B \setminus \overline{\mathcal{O}}) \cap L^\infty(\mathbb{R}^d)$. Therefore, u_k is also bounded in $W^{2,p}_{loc}(B) \cap L^\infty(\mathbb{R}^d)$. Hence, a subsequence of $\{u_k\}$ converges to u_0, uniformly in $\overline{B}$ and weakly in $W^{2,p}_{loc}(B)$, where u_0 is a solution of

$$(L + I)u_0 = 0 \quad \text{in } \; B. \tag{5.12}$$

Since $\{x_k\}$ and $\{y_k\}$ are in $\overline{\mathcal{O}}$, we can find two limit point x_0 and y_0 such that

$$u_0(x_0) = 1, \quad u_0(y_0) = 0, \quad x_0, y_0 \in \overline{\mathcal{O}}, \tag{5.13}$$

after using (4.8). Thus x_0 is an interior point in B where u_0 attains its global maximum value. Applying the strong maximum principle (cf. Proposition 3.4) on B for (5.12) we deduce that u_0 must be a constant. Thus gives a contradiction with (5.10). □

We can associate to the operator P a Markov's chain $\{Y_n\}$ with states in $\overline{\mathcal{O}}$ as follows:

$$Y_n = X(\tau_n), \tag{5.14}$$

where τ_n is the exit time from $\overline{B}$ after attending the set $\mathbb{R}^d \setminus \overline{\otimes}$, i.e. by induction with $\tau_0 = 0$ we have

$$\begin{cases} \tau_n' &= \inf\{t \geq \tau_{n-1} \; : \; X(t) \in \overline{\mathcal{O}}\}, \\ \tau_n &= \inf\{t \geq \tau_n' \; : \; X(t) \in \mathbb{R}^d \setminus B\}, \end{cases} \tag{5.15}$$

for $n = 1, 2, \ldots$. It is clear that the representation formula in the previous sections shows that

$$P\varphi(x) = E_x\varphi(Y_n) = E_x\varphi(Y_1) \tag{5.16}$$

for any x in $\overline{\mathcal{O}}$.

By means of Theorem 5.1 and Proposition 5.2 we can find a unique invariant probability measure for the operator P (i.e. the Markov's chain $\{Y_n\}$), denoted by $\tilde{\mu}$. Then we define a measure $\tilde{\nu}$ on $\mathbb{R}^d$ (unnormalized) by

$$\int_{\mathbb{R}^d} f(x)\tilde{\nu}(dx) = \int_{\mathcal{O}} \tilde{\mu}(dx)E_x\{\int_0^{\tau_1} f(X(t))dt\}. \tag{5.17}$$

Notice that if

$$u(x) = \int_0^{\tau_1} f(X(t))dt \tag{5.18}$$

then $u = u''$, where

$$\begin{cases} u'(x) &= \displaystyle\int^{\tau_1'} f(X(t))dt, \\ u''(x) &= \displaystyle\int_{\tau_1'}^{\tau_1} f(X(t))dt + u'(X(\tau_1')), \end{cases} \tag{5.19}$$

and

$$\begin{cases} -(L + I)u' &= f \quad \text{in } \; \mathbb{R}^d \setminus \mathcal{O}, \\ u' &= 0 \quad \text{in } \; \overline{\mathcal{O}}, \end{cases} \tag{5.20}$$

$$\begin{cases} -(L+I)u'' = f & \text{in } B, \\ u'' = u' & \text{in } \mathbb{R}^d \setminus B. \end{cases} \tag{5.21}$$

Therefore, going back to the definition of the operator P and the convergence (5.4) we have

$$\begin{cases} (L+I)v^n = 0 & \text{in } \mathbb{R}^d \setminus \overline{\mathcal{O}}, \\ v^n = u^n & \text{in } \overline{\mathcal{O}}. \end{cases} \tag{5.22}$$

$$\begin{cases} (L+I)u^n = 0 & \text{in } B, \\ u^n = v^{n-1} & \text{in } \mathbb{R}^d \setminus B, \end{cases} \tag{5.23}$$

with $v_0 = u$. Hence

$$\int_{\mathbb{R}^d} f(x)\tilde{\nu}(dx) = \lim_n u_n = \int_{\otimes} P^n v_0(x)\mu(dx), \tag{5.24}$$

which is a constant in x. If we take $f = 1$, the maximum principle applied to (5.20) and (5.21) implies

$$\begin{cases} \inf\{E_x(\tau_1') \,:\, x \in \overline{\mathcal{O}}\} \geq c_0 > 0, \\ \sup\{E_x(\tau_1) \,:\, x \in \overline{\mathcal{O}}\} \leq C_0 < \infty. \end{cases} \tag{5.25}$$

In particular, $\tilde{\nu}(\mathbb{R}^d) < \infty$.

Define the probability measure ν by

$$\nu(F) = \frac{\tilde{\nu}(F)}{\tilde{\nu}(\mathbb{R}^d)}, \quad \forall F \in \mathcal{B}(\mathbb{R}^d), \tag{5.26}$$

with $\tilde{\nu}$ given by (5.17). We have

Theorem 5.3 (Invariant Measure) *Let the assumptions (1.4), (1.5), (1.6), (1.8), (1.9), (1.10), (2.25), (3.7) and (3.9) hold. Then ν, given by (5.26), is an invariant probability measure for the diffusion with jumps in $\mathbb{R}^d$, i.e.*

$$\int_{\mathbb{R}^d} E_x\{f(X(t))\}\nu(dx) = \int_{\mathbb{R}^d} f(x)\nu(dx), \tag{5.27}$$

for any bounded and Borel measurable function f.

Proof Clearly, it suffices to prove (5.27) with $\tilde{\nu}$ instead of ν and for smooth functions f, say continuous with compact support.

From the definition of $\tilde{\nu}$ we get

$$\int_{\mathbb{R}^d} E_x\{f(X(t))\}\tilde{\nu}(dx) = \int_{\mathcal{O}} \tilde{\mu}(dx) E_x\{\int_0^{\tau_1} g(X(s))ds\},$$

where

$$g(x) = E_x\{f(X(t))\}.$$

By the Markov's property we have

$$E_x\{\int_0^{\tau_1} g(X(s))ds\} = E_x\{\int_0^{\tau_1} f(X(t+s))ds\},$$

and therefore

$$\int_{\mathbb{R}^d} E_x\{f(X(t))\}\tilde{\nu}(dx) = \int_{\mathcal{O}} \tilde{\mu}(dx)E_x\{\int_t^{t+\tau_1} f(X(s))ds\}. \tag{5.28}$$

If we write the integral in the variable s on the region $[t, t+\tau_1]$ into three pieces, on $[0, \tau_1]$, on $[\tau_1, \tau_1 + t]$ and on $[0, t]$, we obtain

$$E_x\{\int_{\tau_1}^{\tau_1+t} f(X(s))ds\} = E_x\{E_{X_1}[\int_0^t f(X(s))ds]\} = E_x g(Y_1),$$

where $\{Y_n\}$ is the Markov's chain associated with the operator P, by (5.14). Since $\tilde{\mu}$ is an invariant probability measure for the Markov's chain, we have

$$\int_{\mathcal{O}} \tilde{\mu}(dx)E_x\{\int_{\tau_1}^{\tau_1+t} f(X(s))ds\} = \int_{\mathcal{O}} \tilde{\mu}(dx)E_x\{\int_0^t f(X(s))ds\}.$$

Thus, the integral in s over $[\tau_1, \tau_1 + t]$ cancels with the integral over $[0, t]$ and we deduce from (5.28)

$$\int_{\mathbb{R}^d} E_x\{f(X(t))\}\tilde{\nu}(dx) = \int_{\mathcal{O}} \tilde{\mu}(dx)E_x\{\int_0^{\tau_1} f(X(s))ds\},$$

which is indeed the required invariant condition. □

As in Khasminskii [12, pp. 121–124] (Theorem 5.1 and its Corollaries), we can prove the following results.

Corollary 5.4 *Under the assumptions of Theorem 5.3 the invariant probability measure ν is unique and we have*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T E_x\{f(X(t))\}dt = \int_{\mathbb{R}^d} f(x)\nu(dx), \tag{5.29}$$

for any bounded and Borel measurable function f. □

Remark 5.5 *In view of the definition (5.17)..., (5.24) of the invariant measure and the results in previous sections, we see that $f = 0$ a.e. implies $\nu(f) = 0$. Thus the measure ν is absolutely continuous w.r.t the Lebesgue measure. We can write*

$$\int_{\mathbb{R}^d} f(x)\nu(dx) = \int_{\mathbb{R}^d} f(x)m(x)dx, \tag{5.30}$$

where the invariant density $m(x)$ satisfies

$$m \geq 0, \qquad \int_{\mathbb{R}^d} m(x)dx = 1. \tag{5.31}$$

Moreover, if $\Phi^\star(t)$ denotes the dual semigroup then $\Phi^\star(t)m = m$, $\forall t \geq 0$. □

Now, we want to consider the linear integro-differential equation in the whole space, i.e.

$$-(L+I)u + a_0 u = f \quad \text{in } \mathbb{R}^d \tag{5.32}$$

and

$$-(L+I)u = f \quad \text{in } \mathbb{R}^d \tag{5.33}$$

where a_0, f are given (bounded) functions, $a_0(x) \geq \alpha_0 > 0$.

Except for the fact that the coefficients $b_i(x)$ have linear growth, the treatment of (5.32) is rather standard. We state the results with only some indication of the arguments used to prove them.

Consider the function

$$\psi_{q,\lambda}(x) = (\lambda + |x|^2)^{q/2}, \quad q > 0, \ \lambda \geq 1. \tag{5.34}$$

As in Section 4, we get

$$\begin{aligned} L\psi_{q,\lambda}(x) &= q\Big[(q-2)(\lambda+|x|^2)^{-2}\sum_{i,j} a_{ij}(x)x_i x_j + \\ &\quad +(\lambda+|x|^2)^{-1}\sum_i (a_{ii}(x) + a_i(x)x_i) + \\ &\quad +(\lambda+|x|^2)^{-1}\sum_i b_i(x)x_i\Big]\psi_{q,\lambda}(x). \end{aligned}$$

Define

$$\left\{ \begin{aligned} \alpha_1(q,\lambda) &= \sup_{x\in\mathbb{R}^d} \Big\{(q-2)(\lambda+|x|^2)^{-2}\textstyle\sum_{i,j} a_{ij}(x)x_i x_j + \\ &\quad +(\lambda+|x|^2)^{-1}\textstyle\sum_i (a_{ii}(x) + a_i(x)x_i)\Big\}, \end{aligned} \right. \tag{5.35}$$

$$\alpha_2(q,\lambda) = \sup_{x\in\mathbb{R}^d} \Big\{(\lambda+|x|^2)^{-1}\sum_i b_i(x)x_i\Big\}. \tag{5.36}$$

By means of the assumptions (1.4) and (3.9) we have

$$\begin{aligned} \alpha_1(q,\lambda) &\leq \sup_x \{q c_0^{-1}|x|^2(\lambda+|x|^2)^{-2} + \\ &\quad +(d c_0^{-1} + c_1|x|)(\lambda+|x|^2)^{-1}\} \leq \frac{C}{\sqrt{\lambda}}, \end{aligned}$$

for a constant C independent of $\lambda \geq 1$. Similarly, the assumption (1.6) [even with $c_1 = 0$] implies

$$\alpha_2(q,\lambda) \leq \sup_{|x|\leq r_1} \{(\lambda+|x|^2)^{-1}\sum_i b_i(x)x_i\} \leq \frac{C}{\lambda}$$

for some constant C depending only on r_1 and the bound of $b_i(x)$ for $|x| \leq r_1$. On the other hand, since

$$\begin{aligned} |\nabla\psi_{q,\lambda}(x+z)| &\leq q(2+|z|^2)^{\frac{q-1}{2}}(\lambda+|x|^2)^{\frac{q-1}{2}}, \\ |\nabla^2\psi_{q,\lambda}(x+z)| &\leq q(q-1)(2+|z|^2)^{q/2-1}(\lambda+|x|^2)^{q/2-1}, \end{aligned}$$

we obtain

$$|I\psi_{q,\lambda}(x)| \le q\alpha_3(q,\lambda)\psi_{q,\lambda}(x),$$

where

$$\begin{aligned}\alpha_3(q,\lambda) &= \sup_x \Big\{\lambda^{-\frac{1}{2}} \int_{|z|\ge 1} |z|(2+|z|^2)^{\frac{q-1}{2}} M(x,dz) + \\ &\quad + \lambda^{-1} 3^{q/2-1} \int_{|z|<1} |z|^2 M(x,dz)\Big\}.\end{aligned} \tag{5.37}$$

Collecting all, we deduce

$$(L+I)\psi_{q,\lambda} \le \alpha(q,\lambda)\psi_{q,\lambda} \quad \text{in } \mathbb{R}^d \tag{5.38}$$

and

$$|L\psi_{q,\lambda}| + |I\psi_{q,\lambda}| \le C_{q,\lambda}\psi_{q,\lambda} \quad \text{in } \mathbb{R}^d, \tag{5.39}$$

for some constant $C_{q,\lambda}$ and $\alpha(q,\lambda) = \sum_i \alpha_i(q,\lambda)$,

$$\alpha(q,\lambda) \to 0 \text{ as } \lambda \to \infty, \tag{5.40}$$

for any fixed $q > 0$.

Proposition 5.6 (Positive Zero-Order Coefficient) *Let the assumptions (1.4), (1.5), (1.6) [even with $c_1 = 0$], (1.8), (1.9), (1.10), (2.25), (3.7), (3.9), (4.48) and*

$$f\psi_{-q},\ a_0 \in L^\infty(\mathbb{R}^d), \qquad a_0(x) \ge \alpha_0 > 0 \quad \forall x, \tag{5.41}$$

hold. Then the integro-differential equation (5.32) possesses one and only one solution u in $W^{2,p}_{loc}(\mathbb{R}^d)$ such that $u\psi_{-q}$ belongs to $L^\infty(\mathbb{R}^d)$. Moreover we have estimate

$$\|u\psi_{-q,\lambda}\|_{L^\infty(\mathbb{R}^d)} \le \frac{1}{\alpha_0 - \alpha(q,\lambda)} \|f\psi_{-q,\lambda}\|_{L^\infty(\mathbb{R}^d)}, \tag{5.42}$$

where $\alpha(q,\lambda)$ is given by (5.38), and the following stochastic representation is valid

$$u(x) = E\{\int_0^\infty f(X(t)) \exp(-\int_0^t a_0(X(s))ds)dt\}. \tag{5.43}$$

Proof The arguments are very similar to those of Theorem 4.4. A key point is the property (5.38) on the constant $\alpha(q,\lambda)$.

The weak maximum principle yields the a priori estimate (5.42). Next the regularization technique applied to the variational form of (5.32) provides the desired result. □

To study the linear equation without a zero-order coefficient (5.33) the arguments are very different from the above.

We consider the space

$$L^\infty_q(\mathbb{R}^d) = \{\varphi \ :\ \varphi\psi_{-q} \in L^\infty(\mathbb{R}^d)\}, \tag{5.44}$$

for $q > 0$ and $\psi_{-q}(x) = (2+|x|^2)^{-q/2}$. The linear equation is then

$$\begin{cases} u \in W^{2,p}_{loc}(\mathbb{R}^d) \cap L^\infty_q(\mathbb{R}^d), \quad p \ge d,\ q > 0, \\ -(L+I)u = f \ \text{ a.e. in } \mathbb{R}^d. \end{cases} \tag{5.45}$$

Theorem 5.7 (Linear Equation) *Let the assumptions (1.4) , . . . , (1.10), (2.25), (3.7), (3.9), (4.48) and*

$$f \in L_q^\infty(\mathbb{R}^d), \quad q > 0$$

hold. The linear integro-differential equation (5.45) has a solution u (unique up to an additive constant) if and only f has a zero-mean, i.e.

$$\nu(f) \doteq \int_{\mathbb{R}^d} f(x)\nu(dx) = 0, \tag{5.46}$$

where $\nu(dx)$ is the unique invariant probability measure defined by (5.26).

Proof First we remark that the a priori estimate of the type (4.27) applied to the exterior Dirichlet problem (5.21) lets us conclude that the property (5.29) on the invariant probability measure ν remains valid for any Borel measurable function f such that $f\psi_{-q}$ is bounded.

To prove that the solution is unique up to an additive constant, we denote by u_0 the solution of the equation (5.45) for $f = 0$. We have

$$E_x\{u_0(X(t))\} = u_0(x),$$

i.e.

$$\frac{1}{T}\int_0^T E_x\{u_0(X(t))\}dt = u_0(x),$$

By means of Corollary 5.4, as $T \to \infty$ we deduce

$$\nu(u_0) = u_0(x), \quad \forall x \in \mathbb{R}^d$$

so that u_0 is constant. Notice that it is possible to use an argument based on the strong maximum principle to obtain the same result.

In order to construct a solution of (5.45) we proceed as in (5.20), . . . , (5.21). For given f satisfying $\nu(f) = 0$ we define u_0 as the solution of an interior Dirichlet problem

$$\begin{cases} -(L+I)u_0 = f & \text{in } B, \\ u_0 = 0 & \text{in } \mathbb{R}^d \setminus B, \end{cases}$$

and v_0 as the solution of an exterior Dirichlet problem

$$\begin{cases} -(L+I)v_0 = f & \text{in } \mathbb{R}^d \setminus \overline{\mathcal{O}}, \\ v_0 = u_0 & \text{in } \overline{\mathcal{O}}. \end{cases}$$

Since

$$\|u_0\|_{L^\infty(B)} \le C_0\|f\|_{L^\infty(B)}$$

we get

$$\|v_0\psi_{-q}\|_{L^\infty(\mathbb{R}^d\setminus\overline{\mathcal{O}})} \le C_q\|f\psi_{-q}\|_{L^\infty(\mathbb{R}^d)}.$$

Now, define the sequences $\{v_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ by

$$\begin{cases} (L+I)u_n &= 0 \quad \text{in } B, \\ u_n &= v_{n-1} \quad \text{in } \mathbb{R}^d \setminus B, \end{cases}$$

and

$$\begin{cases} (L+I)v_n &= 0 \quad \text{in } \mathbb{R}^d \setminus \overline{\mathcal{O}}, \\ v_n &= u_n \quad \text{in } \overline{\mathcal{O}}. \end{cases}$$

Hence, if we set $\tilde{u}_n = u_0 + u_1 + \cdots + u_n$ and $\tilde{v}_n = v_0 + v_1 + \cdots + v_n$ we obtain

$$\begin{cases} -(L+I)\tilde{u}_n &= f \quad \text{in } B, \\ \tilde{u}_n &= \tilde{v}_{n-1} \quad \text{in } \mathbb{R}^d \setminus B, \end{cases} \tag{5.47}$$

and

$$\begin{cases} -(L+I)\tilde{v}_n &= f \quad \text{in } \mathbb{R}^d \setminus \overline{\mathcal{O}}, \\ \tilde{v}_n &= \tilde{u}_n \quad \text{in } \overline{\mathcal{O}}. \end{cases} \tag{5.48}$$

We have the estimates

$$\|\tilde{u}_n \psi_{-q}\|_{L^\infty(B)} \leq C_q(\|f\psi_{-q}\|_{L^\infty(B)} + \|\tilde{v}_{n-1}\psi_{-q}\|_{L^\infty(\mathbb{R}^d \setminus B)}) \tag{5.49}$$

and

$$\|\tilde{v}_n \psi_{-q}\|_{L^\infty(\mathbb{R}^d \setminus \overline{\mathcal{O}})} \leq C_q(\|f\psi_{-q}\|_{L^\infty(\mathbb{R}^d \setminus \overline{\mathcal{O}})} + \|\tilde{u}_n \psi_{-q}\|_{L^\infty(\overline{\mathcal{O}})}). \tag{5.50}$$

Since

$$0 = \tilde{\nu}(f) = \lim_n u_n = \mu(v_0),$$

the ergodic estimates (5.4) of Theorem 5.1 proves that

$$\|u_n\|_{L^\infty(\overline{\mathcal{O}})} \leq K_q e^{-\rho n} \|f\psi_{-q}\|_{L^\infty(\mathbb{R}^d)},$$

which implies that $\tilde{u}_n$ converges in $L^\infty(\overline{\mathcal{O}})$ and that

$$\|\tilde{u}_n\|_{L^\infty(\overline{\mathcal{O}})} \leq \frac{K_q}{1 - e^{-\rho}} \|f\psi_{-q}\|_{L^\infty(\mathbb{R}^d)}. \tag{5.51}$$

Therefore, $\tilde{v}_n$ and $\tilde{u}_n$ converges in $L_q^\infty(\mathbb{R}^d)$ to $\tilde{v}$ and $\tilde{u}$, solutions of

$$\begin{cases} -(L+I)\tilde{u} &= f \quad \text{in } B, \\ \tilde{u} &= \tilde{v} \quad \text{in } \mathbb{R}^d \setminus B. \end{cases} \tag{5.52}$$

and

$$\begin{cases} -(L+I)\tilde{v} &= f \quad \text{in } \mathbb{R}^d \setminus \overline{\mathcal{O}}, \\ \tilde{v} &= \tilde{u} \quad \text{in } \overline{\mathcal{O}}. \end{cases} \tag{5.53}$$

Hence $\tilde{u} = \tilde{v}$ in $\overline{B} \setminus \mathcal{O}$, and the function

$$u(x) = \begin{cases} \tilde{u}(x) & \text{if } x \in B, \\ \tilde{v}(x) & \text{if } x \in \mathbb{R}^d \setminus \overline{\mathcal{O}}, \end{cases} \tag{5.54}$$

satisfies (5.45), and we have the a priori estimate

$$\|u\psi_{-q}\|_{L^\infty(\mathbb{R}^d)} \leq C_q \|f\psi_{-q}\|_{L^\infty(\mathbb{R}^d)}, \tag{5.55}$$

for some constant $C_q > 0$. □

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