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1. Introduction

We are interested in the asymptotic behavior of optimal stopping time problems for general continuous time Markov-Feller processes.

This kind of problems has been investigated by Bensoussan and Lions [2] for reflected diffusions with smooth coefficients. The case of reflected diffusion with jumps has been studied by [6] for bounded measures and in Garroni and Menaldi [4] under fairly general conditions. On the other hand, in Robin [10] and Stettner [11] the case of general ergodic semigroup has been considered, even for more general control problems. We refer also to Lions and Perthame [5] and Perthame [8] form impulse control problems and to Menaldi, Perthame and Robin [7] for switching control problems.

In this paper, we consider a fairly general class of semigroup with some ergodic property. The typical example on hand is the reflected diffusion processes with jumps.

The first section, §2, present the problem to be studied. In §3, we need to establish some a priori bounds of Lewy-Stampacchia type to be used later. Next, in §4, we study a case where the invariant distribution is not necessary. Finally, under ergodicity assumptions we treat a general case.

2. Statement of the Problem

Let $(X_t, t \geq 0)$ be a Markov-Feller process with respect to the filtration $(\mathcal{F}_t, t \geq 0)$ satisfying the usual conditions, with values in some state space E, a compact metric space. Denote by $(\Phi(t), t \geq 0)$ its semigroup defined on $C(E)$, the space of continuous functions from E into $\mathcal{R}$; and by L its infinitesimal generator defined on $D(L)$, subspace of $C(E)$.

Let T be the set of all stopping time adapted to $(\mathcal{F}_t, t \geq 0)$. Given two functions and a constant,

$$(1) \qquad f, \psi \in C(E), \quad \alpha > 0,$$

we are interested in the behavior of the optimal cost function

$$(2) \qquad u_\alpha(x) = \inf\{J_x^\alpha(\tau) : \tau \in T\},$$

$$J_x^\alpha(\tau) = E\{\int_0^\tau e^{-\alpha t} f(X_t)dt + e^{\alpha\tau}\psi(X_\tau)\},$$

as the positive number α vanishes.

It is clear that this involves ergodic properties of the Markov-Feller process $(X_t, t \geq 0)$. Actually, we are concerned with particular processes for which ergodic properties are recently known, e.g. reflected diffusions processes with jumps.

Classic results (cfr. Bensoussan [1], Robin [9]) provided a characterization of u_α as the maximum element of the set of function v satisfying

$$(3) \qquad v \in C(E), v \leq \psi,$$

$$v \leq e^{-\alpha t}\Phi(t)v + \int_0^t e^{-\alpha s}\Phi(s) f ds, \;\; \forall t \geq 0.$$

If u_α is a function in $D(L)$ then

$$(4) \qquad (Lu_\alpha - \alpha u_\alpha + f) \wedge (\psi - u_\alpha) = 0, \quad \wedge = \text{ minimum},$$

Unfortunate, u_α does not belong to $D(L)$ generally, even for smooth data f, ψ. However, if we complete the space $D(L)$ allowing discontinuities then (4) becomes true. This is referred to as the strong formulation of the variational inequality (cfr. Bensoussan and Lions [2]) for diffusion processes with jumps.

Our plan is to establish (4) for general Feller-Markov processes and then the case of reflected diffusions processes with jumps is studied. First for Poisson jumps and finally for general jumps.

A priori bounds

Let us assume that for some Radon measure ν on E the semigroup $(\Phi(t), t \geq 0)$ leaves invariant the sets of zero μ-measure, i.e.

(5) $$\forall t, \varepsilon > 0 \ \exists \delta > 0 \text{ such that } \forall v \in C(E) \text{ satisfying}$$

$$\mu(\{x : v(x) > 0\}) < \delta \text{ we have } \nu(\{x : \Phi(t)v(x) > 0\} < \varepsilon.$$

Then we can extend $(\Phi(t), t \geq 0)$ into a weakly-star continuous semigroup on $L^\infty(E)$. Its weakly-star infinitesimal generator, still denoted by L, has domain $D^\infty(L)$, a subspace of $L^\infty(E)$ characterized by

(6) $$v \in D^\infty(L) \text{ iff } t^{-1}(\Phi(t)v - v), t > 0, \text{ is bounded in } L^\infty(E).$$

Moreover,

(7) $$\text{if } v \in D^\infty(L) \text{ then } t^{-1}(\Phi(t)v - v) \rightharpoonup Lv \text{ weakly-star as } t \to 0.$$

Also the equation

$$Lu - \alpha u = v, \quad u \text{ in } D^\infty(L)$$

has a unique solution for any $\alpha > 0$, v in $L^\infty(E)$.

Recall the maximum principle satisfied by L in $D(L)$:

(8) If $v \in D(L) \subset C(E)$ attains its global maximum at a point

(9) $$x_0 \in E \ \text{ then } Lv(x_0) \leq 0.$$

Theorem 1

Under the assumptions (1), (5) and

(10) there exists a sequence of functions in $D(L)$

(11) $\{\psi_n\}_{n=1}^{\infty}$, such that $\wedge_{n=1}^{k} \psi_n \to \psi$ as $k \to \infty$,

(12) and $L\psi_n$ is uniformly in n bounded from above in $L^\infty(E)$,

the problem

(13) $u_\alpha \in C(E) \cap D^\infty(L), (Lu_\alpha - \alpha u_\alpha + f) \wedge (\psi - u_\alpha) = 0$

has a unique solution, explicitly given by (2). Moreover, u_α satisfies the Lewy-Stampacchia inequality

(14) $0 \leq Lu_\alpha - \alpha u_\alpha + f \leq [\max_n (L\psi_n - \alpha\psi_n) + f]^+,$

where $[\cdot]^+$ denotes the positive part.

Proof

We use the technique of penalization and we give only the main steps.

Define the mapping $\tau_\varepsilon v = u$ as the unique solution of the linear equation

$$Lu - (\alpha + \frac{1}{\varepsilon})u + \frac{1}{\varepsilon}(v \wedge \psi) + f = 0.$$

Since τ_ε maps $C(E)$ into $D(L)$, the maximum principle (8) applied to the function

$$w = \pm(u - \tilde{u}) - (1 + \varepsilon\alpha)^{-1} \|v - \tilde{v}\|_{C(E)},$$

where $\|\cdot\|_{C(E)}$ denotes the supremum norm, gives $w \leq 0$, i.e.

$$\|\tau_\varepsilon v - \tau_\varepsilon \tilde{v}\|_{C(E)} \leq (1 + \varepsilon\alpha)^{-1} \|v - \tilde{v}\|_{C(E)}.$$

Hence τ_ε is a contraction on $C(E)$, which implies that the penalized problem

$$Lu_\varepsilon - \alpha u_\varepsilon - \frac{1}{\varepsilon}(u_\varepsilon - \psi)^+ + f = 0 \tag{15}$$

has a unique solution in $D(L)$.

By using the maximum principle (8) with the function

$$w = u_{\varepsilon^1} - u_\varepsilon, 0 < \varepsilon^1 < \varepsilon,$$

$$Lw = \alpha w + (\frac{1}{\varepsilon^1} - \frac{1}{\varepsilon})(u_{\varepsilon^1} - \psi)^+ + \frac{1}{\varepsilon}[(u_{\varepsilon^1} - \psi)^+ - (u_\varepsilon - \psi)^+],$$

we get $w \leq 0$, i.e.

$$u_{\varepsilon^1} \leq u_\varepsilon, \;\; 0 < \varepsilon^1 < \varepsilon. \tag{16}$$

Let z_ε^k be the unique solution of the linear equation

$$Lz_\varepsilon^k - (\alpha + \frac{1}{\varepsilon})z_\varepsilon^k + \frac{1}{\varepsilon}[\max_{n\leq k}(L\psi_n - \alpha\psi_n) + f]^+ = 0, \tag{17}$$

and u_ε^k be the solution of the penalized problem with ψ replaced by $\wedge_{n=1}^k \psi_n, \psi_n$ given by (19). Now, from the maximum principle (8) applied to the function

$$w_k = u_\varepsilon^k - \psi_n - \varepsilon z_\varepsilon^k, \;\; n \leq k,$$

$$\begin{aligned} Lw_k = (\alpha - \frac{1}{\varepsilon})w_k + [\max_{i\leq k}(L\psi_i - \alpha\psi_i) + f]^- + \\ +[\max_{i\leq k}(L\psi_i - \alpha\psi_i) - (L\psi_n - \alpha\psi_n) + \\ +\frac{1}{\varepsilon}(u_\varepsilon^k - \wedge_{i=1}^k \psi_i)^- + \frac{1}{\varepsilon}(\psi_n - \wedge_{i=1}^k \psi_i), \end{aligned}$$

where $[\cdot]^-$ is the negative part, we deduce $w_k \leq 0$, i.e.

$$\frac{1}{\varepsilon}(u_\varepsilon^k - \wedge_{n=1}^k \psi_n)^+ \leq z_\varepsilon^k, \varepsilon > 0.$$

Hence, by letting $k \to \infty$ we establish

$$\frac{1}{\varepsilon}(u_\varepsilon - \psi)^+ \leq z_\varepsilon, \ \ \varepsilon > 0, \tag{18}$$

where z_ε is the solution in $D^\infty(L)$ of the linear equation (3.10 for $k = \infty$. Notice that

$$\|z_\varepsilon\|_{C(E)} \leq (1 + \varepsilon\alpha)^{-1} \|[\max_{n\geq 1}(L\psi_n - \alpha\psi_n) + f]^+\|, \varepsilon > 0. \tag{19}$$

Going back to (13), we may use the maximum principle (8) with the function

$$w = u_\varepsilon - u_{\varepsilon'} - \frac{1}{\varepsilon'}\|(u_{\varepsilon'} - \psi)^+\|_{C(E)}(\varepsilon - \varepsilon'), \ \ 0 < \varepsilon' < \varepsilon,$$

$$Lw = \alpha w + \frac{1}{\varepsilon}(y_\varepsilon - \psi)^+ - \frac{1}{\varepsilon'}(u_{\varepsilon'})^+,$$

to get $w \leq 0$, i.e.

$$0 \leq u_\varepsilon - u_{\varepsilon'} \leq \|\frac{1}{\varepsilon'}(u_{\varepsilon'} - \psi)^+\|_{C(E)}(\varepsilon - \varepsilon'), \ \ 0 < \varepsilon' < \varepsilon. \tag{20}$$

Notice that we have use the fact that $w > 0$ implies

$$0 < \frac{1}{\varepsilon^1}(u_{\varepsilon^1} - \psi)^+ \leq \frac{1}{\varepsilon}(u_\varepsilon - \psi)^+.$$

Similarly, the maximum principle (8) applied to the function

$$w = \pm(u_\varepsilon - \tilde{u}_\varepsilon) - \max\{\frac{1}{\alpha}\|f - \tilde{f}\|_{C(E)}, \|\psi - \tilde{\psi}\|_{C(E)},$$

$$Lw = \alpha w \pm \frac{1}{\varepsilon}[(u_\varepsilon - \psi)^+ - (\tilde{u}_\varepsilon - \tilde{\psi})^+] \pm (\tilde{f} - f),$$

gives $w \leq 0$, i.e.

$$\|u_\varepsilon - \tilde{u}_\varepsilon\|_{C(E)} \leq \max\{\frac{1}{\alpha}\|f - \tilde{f}\|_{C(E)}, \|\psi - \tilde{\psi}\|_{C(E)}\}, \tag{21}$$

where u_ε and $\tilde{u}_\varepsilon$ denote the solutions of the penalized problems (12) with data f, ψ and $\tilde{f}, \tilde{\psi}$.

Now we are ready to pass to the limit as ε vanishes. In view of (15),...,(17) we get a limiting function u_α in $C(E) \cap D^\infty(L)$ satisfying (10). The Lewy-Stampacchia inequality (11) follows from (15) and the fact that

$$z_\varepsilon \to [\max_{n \geq 1}(L\psi_n - \alpha\psi_n) + f]^+ \text{ as } \varepsilon \to 0,$$

weakly star in $L^\infty(E)$. The estimate (18) gives continuity of the solution u_α w.r.t. data.

One way to show the uniqueness of solution to the problems (10) is to identify any solution with the value function (2). That can be achieved by using a weak version of Dynkin formula for function in $C(E) \cap D^\infty(L)$.

An alternative way is to establish the fact that u_α is indeed the maximum subsolution, i.e. any v in $C(E) \cap D^\infty(L)$ satisfying

$$Lv - \alpha v + f \geq 0, \quad v \leq \psi \tag{22}$$

should be $v \leq u_\alpha$. To that effect, we consider the problem

$$u \in C(E) \cap D^\infty(L), (Lu - \alpha u + f) \wedge (u \wedge u_\alpha - u) = 0.$$

We claim $v \leq u$ which implies $v \leq u_\alpha$. Indeed

$$L(v - u) - \alpha(v - u) = g$$

where $g \geq 0$. Because $v - u$ belongs to $D^\infty(L)$ we deduce $v \leq u$ ν- a.e., and continuity gives $v \leq u$ in $C(E)$. $\square$

Poisson Jumps

We assume here that the infinitesimal generator L has a Poisson jumps part, i.e.

$$(23) \qquad L = L_0 + I,$$

$$I\ v(x) = \lambda(x) \int_E [v(y) - v(x)]m(dy),$$

where L_0 is the infinitesimal generator of a semigroup $(\Phi_0(t), t \geq 0)$ satisfying the same assumptions as $(\Phi(t), t \geq 0)$, and $m(\cdot)$ is a probability measure on E and

$$(24) \qquad \lambda \in C(E), \quad \lambda(x) \geq \lambda_0 > 0, \quad \forall x \in E.$$

Let us study the behavior as α vanishes in the equation satisfied by the optimal cost (2), namely

$$(25) \qquad u_\alpha \in C(E) \cap D^\infty(L), (Lu_\alpha - \alpha u_\alpha + f) \wedge (\psi - u_\alpha) = 0.$$

.

Theorem 2

Assume (1), (5), (9), (20) and (21). Then two possibilities may occur as α vanishes:

(i) either $m(u_\alpha) = \int_E u_\alpha(y)m(dy)$ is bounded

(ii) or $m(u_\alpha)$ diverges to $-\infty$ (it is always bounded from above).

In the first case (i), the function u_α converges weakly star to u_0 in $D^\infty(E)$, where u_0 is the maximum element of the set of functions u satisfying

$$(26) \qquad u \in D^\infty(L), \quad (Lu + f) \wedge (\psi - u) = 0,$$

provided $\psi \leq 0$. For the second case (ii), the function $v_\alpha = u_\alpha - M(u_\alpha)$ converges to weakly star to v_0 in $D^\infty(E)$, where v_0 is the unique solution of the equation,

$$v_0 \in D^\infty(L), \;\; m(v_0) = 0, \tag{27}$$

$$Lv_0 + f = r, \text{ for some real number } r,$$

provided L satisfies the strong maximum principle, namely: the only solutions of the equation $Lv = c$, c constant are constants function v, with $c = 0$.

Proof

From the Lewy-Stampacchia, inequality (11) we have

$$Lu_\alpha - \alpha u_\alpha + f_\alpha = 0,$$

with f_α bounded in $L^\infty(E)$ as α vanishes. Because

$$L_0 v_\alpha - (\lambda + \alpha) v_\alpha = Lu_\alpha - \alpha u_\alpha + \alpha m(u_\alpha)$$

we deduce

$$\|v_\alpha\|_{C(E)} \leq \left(\frac{1}{\lambda_0 + \alpha}\right)\|f_\alpha + \alpha m(u_\alpha)\|_{L^\infty(E)}, \; \forall \alpha > 0. \tag{28}$$

Also, since

$$\|u_\alpha\|_{C(E)} \leq \frac{1}{\alpha}\|f_\alpha\|_{L^\infty(E)}$$

we have

$$|\alpha m(u_\alpha)| \leq \|f_\alpha\|_{L^\infty(E)}, \;\; \forall \alpha > 0. \tag{29}$$

Then in either cases (i), (ii), we can find a function v_0 in $D^\infty(L)$ such that

$$v_\alpha \rightharpoonup v_0 \text{ and } Lv_\alpha \rightharpoonup Lv_0 \text{ weakly star in } L^\infty(E)$$

as $\alpha \to 0$ for some subsequence.

Now, if (i) holds than we have

$$u_\alpha \rightharpoonup u_0,$$

which is clearly a solution of (23). To show that u_0 is the maximum subsolution (solutions) we denote by u a solution of (23) and by $\tilde{u}_\alpha$ the solution of problem (10) with data $f + \alpha u$, ψ. By Theorem 1, we have

$$u \leq \tilde{u}_\alpha,$$

and because $u \leq \psi \leq 0$, the monotonicity in the data implies

$$\tilde{u}_\alpha \rightharpoonup \tilde{u}_0 \leq u_0, \ as\ \alpha \to 0.$$

Hence $u \leq u_0$.

for the second case (ii) we notice that

$$v_\alpha \leq \psi - m(u_\alpha) = \psi_\alpha$$

Since v_α bounded, $m(u_\alpha)$ should be bounded from above. In this case (ii), ψ_α diverges to $+\infty$ and limiting equation is (24), for

$$\alpha m(u_\alpha) \to r.$$

The uniqueness for the problem (24) is part of the assumption on the strong maximum principle satisfied by L. $\square$

Remark 1

Suppose that the resolvent operator corresponding to L is compact in $C(E)$ i.e., if $f_n \rightharpoonup f$ weakly star in $L^\infty(E)$ then the solution u_n of

$$Lu_n - \alpha u_n + f_n = 0, \quad \alpha > 0 \text{ fixed},$$

converges in $C(E)$ to the solution u of the limiting equation. We deduce that the limiting functions either u_0 or v_0 are in $C(E)$. $\square$

Remark 2

Notice that the measure $m(\cdot)$ is not in general an invariant measure for the semigroup, $(\Phi(t), t \geq 0)$.

Remark 3

Most of the results can be extended to the case where E is locally compact metric space. Also other kind of control problem can be studied with this technique. $\square$

5. General Jumps

When we allow the probability measure $m(\cdot)$ in (20) to depend on x, the method of §4 does not work anymore. However, the technique based on the invariant measure can be carried out. We have in mind the case of reflected diffusion with jumps studied in [6]. On the other hand, if we want to include cases with accumulation of jumps, e.g.

$$Iv(x) = \int_F [v(x+\gamma,\xi)) - v(x)]\beta(x,\xi)\pi(d\xi), \tag{30}$$

with π a σ-finite measure on F,

$$0 \leq \beta(x,\xi) \leq 1, \;\; 0 < \gamma(x,\xi) \leq \gamma_0(\xi), \tag{31}$$

$$\int_F \gamma_0(\xi) T(dz) < \infty, \tag{32}$$

$$x + \gamma(x, \xi) \in E, \ \forall x \in E, \ \beta(x, \xi) \neq 0, \tag{33}$$

then we need to go through precise estimates on the corresponding transition density function to show the existence of an invariant density measure, cfr. Garroni and Menaldi [4].

Herein, even if we are thinking of the reflected diffusion with jumps, we state all results for general semigroup with nice ergodic properties.

Assume that there exists an invariant distribution $m(\cdot)$ for the semigroup $(\Phi(t), t \geq 0)$ which is exponentially stable, i.e.

$$\|\Phi(t)v - m(v)\|_{C(E)} \leq Ce^{\nu t}\|v\|_{C(E)}, \forall v \in C(E), \tag{34}$$

for some constant $C, \nu > 0$ and where $m(\cdot)$ is a probability measure on E and

$$m(v) = \int_E v(y) m(dy), \forall v \in C(E). \tag{35}$$

Theorem 3

Let us assume (1), (5), (9) and (29). Then the limit of the solution u_α of problem (22) as α vanishes is characterized as follows:

(i) if $m(f) \geq 0$ then u_α converges weakly star to u_0 in $D^\infty(L)$, where u_0 is the maximum element of the set of functions u satisfying

$$u \in D^\infty(L), (Lu + f) \wedge (\psi - u) = 0, \tag{36}$$

provided $\psi \leq 0$;

(ii) if $m(f) < 0$ then $v_\alpha = u_\alpha - m(u_\alpha)$ is bounded in $D^\infty(L)$ and any weakly star limit v satisfies

$$v \in D^\infty(L), m(v) = 0, Lv + f = r, \text{ for some constant } r. \tag{37}$$

Moreover, if the operator L satisfies the strong maximum principle mentioned in Theorem 2, then we have three alternatives

(i) if $m(f) > 0$ then u_0 is the unique solution of (31),

(ii) if $m(f) = 0$ then u_0 is the unique solution of the problem

$$u_0 \in D^\infty(L), Lu_0 + f = 0, \min\{\psi - u_0\} = 0, \tag{38}$$

(iii) if $m(f) < 0$ then v_0 is the unique solution of (32) and $r = m(f)$.

Proof

Again by Lewy-Stampacchia inequality (11) we have

$$Lu_\alpha - \alpha u_\alpha + f_\alpha = 0,$$

where f_α remains bounded in $L^\infty(E)$ as α vanishes. In view of (29) we get

$$\|v_\alpha\|_{C(E)} \leq \frac{1}{\alpha + \nu}\}\|f_\alpha - \alpha u_\alpha\|_{L^\infty(E)}, \ \forall \alpha > 0, \tag{39}$$

after noticing that $v_\alpha = u_\alpha - m(u_\alpha), m(v_\alpha) = 0, m(f_\alpha - \alpha u_\alpha) = 0$.

We can then assume that

$$v_\alpha \rightharpoonup v_0, Lv_\alpha \rightharpoonup Lv_0, \alpha m(v_\alpha) \to r, \tag{40}$$

at least for some sequence in α and the two first convergences are weakly star in $L^\infty(E)$.

Since $m(u_\alpha)$ is always bounded from above and

$$\alpha m(u_\alpha) = m(f_\alpha) \leq m(f),$$

we show that $m(u_\alpha)$ bounded implies $m(f) \geq 0$. To see the opposite condition, we look at the stopping set

$$S_\alpha = \{x \in E : u_\alpha(x) = \psi(x)\}.$$

Because ψ and v_α are bounded, there exist $\alpha_0 > 0$ such that S_α is empty for $0 < \alpha < \alpha_0$, if we have assumed $m(u_\alpha)$ unbounded. In this case $f_\alpha = f$ for $0 < \alpha < \alpha_0$, which implies $m(f) \leq 0$. If actually $m(f) = 0$ then we can construct a subsolution as follows: w_α solution of

$$w_\alpha \in D(L), \;\; Lw_\alpha - \alpha w_\alpha + f = 0$$

and

$$\bar{w}_\alpha = w_\alpha - \|\psi - w_\alpha\|_{C(E)}.$$

The maximum principle yields $u_\alpha \geq \bar{w}_\alpha$. Since w_α is bounded, because $m(f) = 0$, we should have u_α bounded from below, which contradicts the fact that $m(u_\alpha)$ is unbounded. Summing up, we have established the following:

(41) (i) $m(u_\alpha)$ is bounded if and only if $m(f) \geq 0$

(42) (ii) if $m(u_\alpha)$ is unbounded there exists $\alpha_0 > 0$

(43) such that $f_\alpha = f$ for $0 < \alpha < \alpha_0$.

Hence, (34), (35) and (36) allows us to pass to the limit as in Theorem 4.1 to complete the proof, after showing (33). To that effect, notice that $f_\alpha \leq f$

and f_α converges weakly to f_0 as goes to 0. If $m(f) = 0$ then we have

$$u_0 \in D^\infty(L) \;\; Lu_0 + f_0 = 0, \;\; f_0 \leq f,$$

with u_0 being the a weak limit of u_α. But $m(f_0) = m(f) = 0$, which implies $f_0 = f$. $\square$

Remark

When $m(f) > 0$, still we have (31) for any continuous function ψ, not necessarily negative. $\square$

Remark

Comments similar to those of §4 can be stated. $\square$

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