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SOME ESTIMATES FOR FINITE DIFFERENCE APPROXIMATIONS*

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Abstract. Some estimates for the approximation of optimal stochastic control problems by discrete time problems are obtained. In particular an estimate for the solutions of the continuous time versus the discrete time Hamilton-Jacobi-Bellman equations is given. The technique used is more analytic than probabilistic.

Key words. diffusion process, Markov chain, dynamic programming, finite difference, Hamilton-Jacobi-Bellman equations

AMS(MOS) subject classifications. 65K10, 65G99, 49D25, 93E25, 93E20

Introduction. We are interested in the approximation of optimal control problems for diffusion processes by means of finite difference methods. It is well known (e.g., Kushner [16], [17]) that a basic probabilistic counterpart is the approximation of a diffusion process by a Markov chain. A typical problem in stochastic control theory is the following.

In a complete filtered probability space $(\Omega, P, \mathcal{F}, \mathcal{F}(t), t \geqq 0)$ suppose we have two progressively measurable processes $(y(t), \lambda(t), t \geqq 0)$ satisfying the following stochastic differential equation in the Itô sense:

$$(0.1) \qquad \begin{aligned} &dy(t) = g(y(t), \lambda(t))\, dt + \sigma(y(t), \lambda(t))\, dw(t), \qquad t \geqq 0, \\ &y(0) = x, \end{aligned}$$

for given x, g, σ, and some n-dimensional Wiener process $(w(t), t \geqq 0)$. The processes $(y(t), t \geqq 0)$ and $(\lambda(t), t \geqq 0)$ represent the state in $\mathcal{R}^d$ and the control in Λ (a compact metric space) of the dynamic system, respectively.

The cost functional is given by

$$(0.2) \qquad J(x, \lambda) = E\left\{\int_0^\tau f(y(t), \lambda(t))\, e^{-\alpha t}\, dt\right\},$$

where f is a given function, $\alpha > 0$, and τ is the first exit time of a domain D in $\mathcal{R}^d$ for the process $(y(t), t \geqq 0)$.

The associated Hamilton-Jacobi-Bellman (HJB) equation (e.g., Bensoussan and Lions [2], Fleming and Rishel [9], Krylov [14]) to be satisfied by the optimal cost

$$(0.3) \qquad u(x) = \inf\{J(x, \lambda)\colon \text{any control } \lambda(\cdot)\}$$

is indeed

$$(0.4) \qquad \begin{aligned} &\alpha u = \inf\{L(\lambda)u + f(\cdot, \lambda)\colon \lambda \in \Lambda\} \quad \text{in } D, \\ &u = 0 \quad \text{on } \in \partial D, \end{aligned}$$

with the differential operator

$$(0.5) \qquad L(\lambda) = \frac{1}{2}\sum_{i,j=1}^{d}\left(\sum_{k=1}^{n}\sigma_{ik}(\cdot, \lambda)\sigma_{jk}(\cdot, \lambda)\right)\partial_{ij} + \sum_{i=1}^{d} g_i(\cdot, \lambda)\,\partial_i,$$

where $\partial_{ij}, \partial_i$ denote the partial derivatives and $g = (g_i, i = 1, \cdots, d)$, $\sigma = (\sigma_{ik},\ i = 1, \cdots, d,\ k = 1, \cdots, n)$.

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Let $\mathscr{R}_h^d$ denote a h-finite difference grid in $\mathscr{R}^d$. Consider the finite difference operator

$$
\begin{aligned}
L_h(\lambda)\varphi(x) = h^{-1} \sum_{k=1}^{n} \{&\beta_k^+(x,\lambda,h)[\varphi(x+\gamma_k^+(x,\lambda,h))-\varphi(x)] \\
&+\beta_k^-(x,\lambda,h)[\varphi(x+\gamma_k^-(x,\lambda,h))-\varphi(x)]\},
\end{aligned} \tag{0.6}
$$

where the coefficients satisfy

$$
\begin{aligned}
&\beta_k^\pm(x,\lambda,h) \geqq 0 \quad \forall x, \lambda, h, \\
&x+\gamma_k^\pm(x,\lambda,h) \in \mathscr{R}_h^d \quad \forall x \in \mathscr{R}_h^d, \quad \lambda \in \Lambda.
\end{aligned} \tag{0.7}
$$

The finite difference approximation of the HJB equation (0.4) using the operator (0.6) is

$$
\begin{aligned}
&\alpha u_h = \inf\{L_h(\lambda)u_n + f(\cdot,\lambda) : \lambda \in \Lambda\} \quad \text{in } D_h, \\
&u_h = 0 \quad \text{in } \mathscr{R}_h^d \backslash D_h,
\end{aligned} \tag{0.8}
$$

where D_h is the set of points in $\mathscr{R}_h^d$ belonging to D.

Our purpose is to estimate the difference

$$
\sup\{|u(x)-u_h(x)| : x \in D_h\} \tag{0.9}
$$

in terms of the parameter h. We expect to dominate (0.9) by

$$
\begin{aligned}
&\sup\{\inf\{|l(x,\lambda)-l(x',\lambda)| : x' \in \mathscr{R}_h^d\} : x \in \mathscr{R}^d, \lambda \in \Lambda\}, \\
&\qquad\qquad \text{for } l = f, g_i, \sigma_{ik}, i = 1, \cdots, d, k = 1, \cdots, n.
\end{aligned} \tag{0.10}
$$

For instance, if f, g, σ are Lipschitz-continuous in x, then

$$
|u(x)-u_h(x)| \leqq Ch^{1/2} \quad \forall x \in D_h, \quad h \in (0,1], \tag{0.11}
$$

for some constant C independent of x and h.

Let us mention that finite difference operators of the form (0.6) satisfy automatically the so-called discrete maximum principle. Problem (0.8) is indeed the discrete HJB equation associated with some suitable optimal control problem of a Markov chain. We remark that several computational methods are available for the discrete HJB equation (0.8) (e.g., Kushner and Kleinman [18], Puterman [29], Puterman and Brumelle [30], Quadrat [31], and Theosys [33]).

Actually, the objective of the paper is to show how the underlying technique can be used with a typical problem (0.1)-(0.5). The probabilistic interpretation of the finite difference operator (0.6) is part of the key idea. From a purely stochastic control viewpoint, an estimate on an approximation to the optimal cost is certainly of great value. However, we may question how optimal the discrete optimal feedback is when it is applied to the actual continuous time problem. Toward an answer to the preceding questions, we can argue in the following way. First of all, what really matters for the optimizers is to know how far they are from the minimum cost in the real model. The stochastic equation (0.1) is only an approximation of the real evolution, as well as being the Markov chain associated with the operator (0.6). Our claim is that by preserving the structure of the problem, i.e., to have a probabilistic interpretation of the approximating HJB equation (0.8), and by getting some estimates of the convergence of the corresponding optimal costs, we cannot be far away from the real model.

Even if the Markov chain associated with the operator (0.6) always has finite state, we may want to discretize the set Λ, just to improve the implementation of the infimum in equation (0.8). In this case, we can replace Λ in (0.7) and (0.8) by a discretization $\Lambda(h)$, and similar results hold true (cf. [24]).

Deterministic versions along with the same kind of ideas can be found in Capuzzo-Dolcetta [4], Capuzzo-Dolcetta and Ishii [5], Crandall and Lions [6], Falcone [8], Gonzalez and Rofman [12], Menaldi and Rofman [27], and Souganidis [36].

The cases where the discount factor α is actually a function, the coefficients g, σ are time-dependent, the horizon is finite, the HJB equation is indeed a set of inequalities, and the domain D is unbounded can also be studied.

In § 1 we consider the one-dimensional case. Even if this case is very restrictive, we obtain enough information from it to deal with the multidimensional case. Moreover, this section can stand by itself, but we believe it is a natural step in the technique to be developed. General problems are treated in § 2.

1. One-dimensional case. It is clear that for one-dimensional problems we dispose of many classic tricks, probably more efficient in practice than the one we will describe. However, we claim that by studying this simple case we may obtain some nonstandard ways of looking at a multidimensional finite difference scheme.

Let g, σ be real continuous functions on $\mathcal{R}\times\Lambda$ such that

$$\text{(1.1)}\qquad \begin{aligned} &|g(x,\lambda)|+|\sigma(x,\lambda)|\leqq C \quad \forall x\in\mathcal{R},\quad \lambda\in\Lambda,\\ &|g(x,\lambda)-g(x',\lambda)|+|\sigma(x,\lambda)-\sigma(x',\lambda)|\leqq K|x-x'| \quad \forall x,x'\in\mathcal{R},\quad \lambda\in\Lambda, \end{aligned}$$

for some constants $C=C(g,\sigma)$ and $K=K(g,\sigma)$. The set Λ is a compact metric space, generally a compact subset of $\mathcal{R}^m$.

On a complete Wiener space $(\Omega, P, \mathcal{F}, \mathcal{F}(t), w(t), t\geqq 0)$, i.e., $(\Omega, P, \mathcal{F})$ is a complete probability space, $(\mathcal{F}(t), t\geqq 0)$ is a right-continuous family of complete sub-σ-algebras of $\mathcal{F}$, $(w(t), t\geqq 0)$ is a one-dimensional standard Wiener process adapted to $(\mathcal{F}(t), t\geqq 0)$; we consider the controlled diffusion process

$$\text{(1.2)}\qquad dy(t)=g(y(t),\lambda(t))\,dt+\sigma(y(t),\lambda(t))\,dw(t),\qquad t>0,$$

where the control $(\lambda(t), t\geqq 0)$ is a progressively measurable process taking values in Λ. Its associated infinitesimal generator $L(\lambda)$ is the second-order differential operator

$$\text{(1.3)}\qquad L(\lambda)\varphi=\tfrac{1}{2}\sigma^2(\cdot,\lambda)\varphi''+g(\cdot,\lambda)\varphi',\qquad \sigma(\cdot,\cdot)\geqq 0,$$

where φ' and φ'' are the first and second derivatives of φ.

For the moment, let us forget about the h-finite grid $\mathcal{R}_h$, i.e., the last condition of (0.7) is disregarded. Consider the finite difference operator

$$\text{(1.4)}\qquad \begin{aligned} &L_h(\lambda)\varphi=\frac{1}{h}\left[\frac{1}{2}\varphi(\cdot+gh+\sigma\gamma\sqrt{h})+\frac{1}{2}\varphi(\cdot+gh-\sigma\gamma\sqrt{h})-\varphi\right],\\ &g=g(x,\lambda),\quad \sigma=\sigma(x,\lambda),\quad \gamma=\gamma(x,\lambda,h); \end{aligned}$$

the function $\gamma\geqq 0$ is to be chosen later (cf. (1.8)).

In § 1.1 we will construct a controlled Markov chain associated with the finite difference operator, from which a piecewise constant (on stochastic time intervals) process $(y_h(t); t\geqq 0)$ is defined in such a way that

$$\text{(1.5)}\qquad E\sup\{|y(t)-y_h(t)|^p e^{-\alpha t}: t\geqq 0\}\leqq Ch^{p/2}\quad \forall h\in(0,1],$$

for some constants C, $\alpha>0$ depending only on g, σ, and $p>0$ uniformly with respect to a class of controls to be specified.

Next, we use this estimate to obtain (0.11) for a linear equation, i.e., without control λ.

In § 1.3 we realize the above technique gives only a one-sided estimate of the type (0.11) for nonlinear problems. The difficulty is the lack of information on the optimal control $\lambda(\cdot)$. At this point, we need to use analytic techniques to obtain (0.11).

1.1. A Markov chain. Define

$$\begin{aligned} &\tau(x, \lambda, h, w) = \inf\{t \geqq 0\colon g(x,\lambda)(t-h) + \sigma(x,\lambda)w(t) \\ &\qquad\qquad\qquad \text{equals either } \delta(x,\lambda,h) \text{ or } -\delta(x,\lambda,h)\}, \\ &\delta(x,\lambda,h) = \sigma(x,\lambda)\gamma(x,\lambda,h)\sqrt{h}, \qquad w(0)=0, \\ &\xi(x,\lambda,h,w) = g(x,\lambda)\tau(x,\lambda,h,w) + \sigma(x,\lambda)w(\tau(x,\lambda,h,w)). \end{aligned} \tag{1.6}$$

Note that $w(\cdot)$ is a standard Wiener process and $\tau = h$ and $\xi = gh$ whenever σ vanishes.

Let $\lambda(\cdot)$ be a feedback control, i.e., a Borel-measurable function from $\mathscr{R}$ into Λ. We construct by induction the sequences of random variables $(X_n, \theta_n, n = 0, 1, \cdots)$ as follows. For a given initial data x,

$$\begin{aligned} &X_0 = x, \quad \theta_0 = 0, \quad w_0(t) = w(t), \\ &X_{n+1} = X_n + \xi(X_n, \lambda(X_n), h, w_n), \\ &\theta_{n+1} = \theta_n + \tau(X_n, \lambda(X_n), h, w_n), \\ &w_{n+1}(t) = w(t+\theta_n) - w(\theta_n), \qquad n = 0, 1, \cdots. \end{aligned} \tag{1.7}$$

If instead of a feedback control $\lambda(\cdot)$ we have a nonanticipating control $(\lambda_n, n = 0, 1, \cdots)$, where λ_n is a random variable valued in Λ and adapted to $(X_0, \cdots, X_{n-1})$, then the procedure (1.7) still works.

Let us define the function $\gamma(x, \lambda, h)$ by

$$\begin{aligned} &\gamma = 0 \quad \text{if } 0 \leqq \sigma < |g|\sqrt{h}, \\ &\gamma = 1 \quad \text{if } g = 0, \\ &\gamma = \gamma_0(g\sigma^{-1}\sqrt{h}) \quad \text{if } 0 < |g|\sqrt{h} \leqq \sigma, \end{aligned} \tag{1.8}$$

where

$$\gamma_0(r) = (2r)^{-1} \ln [e^{2r^2} + \operatorname{sign}(r)(e^{4r^2} - 1)^{1/2}], \tag{1.9}$$

for $r \neq 0$, $-1 \leqq r \leqq 1$. Note that $\gamma_0(r) > 0$; moreover,

$$0 < \gamma_0(r) - 1 \leqq |r| \quad \forall r \in [-1, 0) \cup (0, 1]. \tag{1.10}$$

This implies the inequality

$$|\sigma(x,\lambda)\gamma(x,\lambda,h) - \sigma(x,\lambda)| \leqq 2|g(x,\lambda)|\sqrt{h}, \tag{1.11}$$

for every x, λ, h.

THEOREM 1.1. *If we choose $\gamma(x, \lambda, h)$ by* (1.8), *then for any feedback $\lambda(\cdot)$ the procedure* (1.7) *defines a Markov chain $(X_n, n = 0, 1, \cdots)$ with transition probability determined by*

$$\begin{aligned} &E(\varphi(X_{n+1}) \mid X_n = x) = \Pi_h(\lambda(x))\varphi(x), \\ &\Pi(\lambda)\varphi(x) = \tfrac{1}{2}\varphi(x + g(x,\lambda)h + \sigma(x,\lambda)\gamma(x,\lambda,h)\sqrt{h}) \\ &\qquad + \tfrac{1}{2}\varphi(x + g(x,\lambda)h - \sigma(x,\lambda)\gamma(x,\lambda,h)\sqrt{h}), \end{aligned} \tag{1.12}$$

and a sequence $(\theta_n, n = 0, 1, \cdots)$ of stopping times relative to $(\mathscr{F}(t), t \geqq 0)$, with independent increments $\tau_n = \theta_n - \theta_{n-1}$,

$$E\tau_n = h \quad \forall n = 1, 2, \cdots. \tag{1.13}$$

Proof. Without loss of generality, we may assume g and σ constants. Consider the two functions $u(x)$ and $v(x)$ defined by the equations

$$\tfrac{1}{2}\sigma^2 u'' + gu' = -1 \quad \text{in } (\delta, \delta), \quad u(-\delta) = u(\delta) = 0,$$

$$\tfrac{1}{2}\sigma^2 v'' + gv' = 0 \quad \text{in } (-\delta, \delta), \quad v(-\delta) = 0, \quad v(\delta) = 1,$$

where $\delta = \sigma\gamma\sqrt{h}$. If $-\delta \leqq -gh \leqq \delta$, then

$$E\tau(h, \cdot) = u(-gh), \qquad P(\xi(h, \cdot) = gh + \delta) = v(-gh), \tag{1.14}$$

with the notation (1.6). Since we can compute explicitly,

$$u(x) = -\frac{x}{g} + \frac{\delta}{g}\left(\frac{e^{\alpha\delta} + e^{-\alpha\delta} - 2e^{\alpha x}}{e^{\alpha\delta} - e^{-\alpha\delta}}\right), \quad \alpha = 2g\sigma^{-2}, \quad v(x) = \frac{e^{\alpha\delta} - e^{-\alpha x}}{e^{\alpha\delta} - e^{-\alpha\delta}},$$

yielding

$$u(-gh) = h + \frac{\gamma h}{r}\left(\frac{e^{2r\gamma} + e^{-2r\gamma} - 2e^{2r^2}}{e^{2r\gamma} - e^{-2r\gamma}}\right), \quad v(-gh) = \frac{e^{2r\gamma} - e^{2r^2}}{e^{2r\gamma} - e^{-2r\gamma}}, \quad r = g\sigma^{-1}\sqrt{h}.$$

Suppose we have chosen γ such that $u(-gh) = h$, i.e.,

$$e^{2r\gamma} + e^{-2r\gamma} - 2e^{2r^2} = 0, \qquad \gamma > 1. \tag{1.15}$$

Since

$$1 - v(-gh) = \frac{e^{2r^2} - e^{2r\gamma}}{e^{2r\gamma} - e^{-2r\gamma}},$$

the relation (1.15) implies $v(-gh) = \frac{1}{2}$, i.e.,

$$E\tau(h, \cdot) = h, \qquad P(\xi(h, \cdot) = gh \pm \sigma\gamma\sqrt{h}) = \tfrac{1}{2}, \tag{1.16}$$

whenever $0 < |g|\sqrt{h} \leqq \sigma\gamma$. Note that (1.16) still holds if we take $\gamma = 1$ for $g = 0$ and that (1.15) gives $\gamma = \gamma_0(r)$ as in (1.9), for $0 < |g|\sqrt{h} \leqq \sigma$, because $\gamma > 1$.

If $0 \leqq \sigma < |g|\sqrt{h}$, then the equalities (1.14) hold true for functions u and v satisfying

$$\tfrac{1}{2}\sigma^2 u'' + gu' = -1 \quad \text{in } (-\infty, -\delta), \quad u(-\delta) = 0, \quad u \text{ with polynomial growth},$$

$$\tfrac{1}{2}\sigma^2 v'' + gv' = 0 \quad \text{in } (-\infty, -\delta), \quad v(-\delta) = 1, \quad v \text{ with polynomial growth},$$

for $g > 0$ and replacing the interval $(-\infty, -\delta)$ by $(\delta, +\infty)$ if $g < 0$. It is clear that $v = 1$ and

$$u(x) = -\frac{x}{g} + \frac{\delta}{|g|}.$$

So

$$u(-gh) = h + |g|^{-1}\sigma\gamma\sqrt{h},$$

and $\gamma = 0$ is the right choice.

All of the above proves (1.12) and (1.13) after using standard facts on Brownian motions. □

Remark 1.1. If $g = 0$ and $\sigma = 1$ then the construction (1.7) coincides with the classic Skorokhod's representation (cf. [35]).

For a given feedback $\lambda(\cdot)$ let us denote by $(y_x^h(t, \lambda(\cdot)), t \geqq 0)$ and $(\lambda^h(t), t \geqq 0)$ the processes

$$\text{(1.17)} \qquad \begin{aligned} &y_x^h(t, \lambda(\cdot)) = X_n \quad \text{if } \theta_n \leqq t < \theta_{n+1}, \quad n = 0, 1, \cdots, \\ &\lambda^h(t) = \lambda(X_n) \quad \text{if } \theta_n \leqq t < \theta_{n+1}, \quad n = 0, 1, \cdots, \end{aligned}$$

where $(X_n, \theta_n, n = 0, 1, \cdots)$ are defined by (1.7) with the choice (1.8) of function γ.

Note that these processes are adapted to $(\mathscr{F}(t), t \geqq 0)$ and piecewise constants on stochastic intervals. This approach is different from the one used by Pardoux and Talay [28]. Our partition is on the range, i.e., X_n takes values in a variable grid of $\mathscr{R}$, and the time partition is chosen accordingly; our time intervals are random.

Now consider the controlled diffusion process $(y(t) = y_x(t, \lambda^h), t \geqq 0)$ given by the stochastic equation (1.2) with initial data $y(0) = x$ and control process $\lambda(t) = \lambda^h(t)$.

THEOREM 1.2. *Let the assumption* (1.1) *and the choice* (1.8) *hold. Then for any positive number p there exist two positive constants C, α depending only on p and the constants $C(g, \sigma)$, $K(g, \sigma)$ of* (1.1) *such that*

$$\text{(1.18)} \qquad E \sup \{|y_x(t, \lambda^h) - y_x^h(t, \lambda(\cdot))|^p e^{-\alpha t}: t \geqq 0\} \leqq C h^{p/2},$$

uniformly for any feedback $\lambda(\cdot)$ and x in $\mathscr{R}$.

Proof. Based on the procedure (1.6), (1.7) we have

$$X_{n+1} = X_n + g(X_n, \lambda(X_n))(\theta_{n+1} - \theta_n) + \sigma(X_n, \lambda)(w(\theta_{n+1}) - w(\theta_n)), \qquad n = 0, 1, \cdots,$$

which gives

$$X_n = x + \int_0^{\theta_n} g(y^h(t), \lambda^h(t))\, dt + \int_0^{\theta_n} \sigma(y^h(t), \lambda^h(t))\, dw(t),$$

where $y^h(t) = y_x^h(t, \lambda(\cdot))$ and $\lambda^h(t)$ are the processes defined by (1.17). If we set

$$q^h(t) = x + \int_0^t g(y^h(s), \lambda^h(s))\, ds + \int_0^t \sigma(y^h(s))\, dw(s), \qquad t \geqq 0,$$

then

$$q^h(t) - y^h(t) = g(X_n, \lambda(X_n))(t - \theta_n) + \sigma(X_n, \lambda(X_n))(w(t) - w(\theta_n)) \quad \text{if } \theta_n \leqq t < \theta_{n+1}.$$

Again, in view of the definition (1.6) we deduce

$$\text{(1.19)} \qquad |q^h(t) - y^h(t)| \leqq C(g, \sigma)\sqrt{h} \quad \forall t \geqq 0, \quad 0 < h \leqq 1,$$

where $C(g, \sigma)$ is the constant of the assumption (1.1).

Now, consider the process $z(t) = y(t) - q^h(t)$, with $y(t) = y_x(t, \lambda^h)$,

$$\begin{aligned} dz(t) = {} & [g(y(t), \lambda^h(t)) - g(y^h(t), \lambda^x(t))]\, dt \\ & + [\sigma(y(t), \lambda^h(t)) - \sigma(y^h(t), \lambda^h(t))]\, dw(t), \qquad t \geqq 0, \quad z(0) = 0, \end{aligned}$$

and apply Itô's formula to the function

$$\varphi(z, t) = (\beta^2 + z^2)^{p/2} e^{-\alpha t}, \qquad \alpha, \beta, p > 0,$$

to get

$$\begin{aligned} d\varphi(z(t), t) = {} & \{p z(t)[g(y(t), \lambda^h(t)) - g(y^h(t), \lambda^h(t))](\beta^2 + z^2(t)) \\ & + \tfrac{1}{2}(p\beta^2 + p(p-1)z^2(t))][\sigma(y(t), \lambda^h(t)) - \sigma(y^h(t), \lambda^h(t))]^2 \\ & - \alpha(\beta^2 + z^2(t))^2\}(\beta^2 + z^2(t))^{p/2-2} e^{-\alpha t}\, dt \\ & + p z(t)[\sigma(y(t), \lambda^h(t)) - \sigma(y^h(t), \lambda^h(t))](\beta^2 + z^2(t))^{p/2-1} e^{-\alpha t}\, dw(t). \end{aligned}$$

If we take $\alpha > \alpha_p$,

$$\alpha_p = p \sup \{[g(x,\lambda) - g(x',\lambda)](x-x')^{-1} + (p \vee 1)[\sigma(x,\lambda) - \sigma(x',\lambda)]^2 (x-x')^{-2} \colon x \neq x' \text{ in } \mathcal{R},\ \lambda \text{ in } \Lambda\}, \tag{1.20}$$

where $\vee$ denotes the maximum, and

$$\beta = \sup \{|q^h(t) - y^h(t)| \colon t \geqq 0,\ \omega \text{ in } \Omega\};$$

then we obtain

$$d\varphi(z(t),t) \leqq (\alpha_p - \alpha)\varphi(z(t),t)\,dt + dM_t, \qquad t \geqq 0,$$
$$|y(t) - y^h(t)|^p e^{-\alpha t} \leqq \varphi(z(t),t), \qquad t \geqq 0,$$

with M_t being the martingale term. In virtue of (1.19) we deduce

$$E\left\{|y(t)-y^h(t)|^p e^{-\alpha t} + (\alpha - \alpha_p)\int_0^t |y(s)-y^h(s)|^p e^{-\alpha s}\,ds\right\} \leqq [C(g,\sigma)\sqrt{h}]^p \quad \forall t \geqq 0, \quad 0 < h \leqq 1. \tag{1.21}$$

Next, by means of the stochastic inequality

$$E \sup \left\{\left|\int_0^t \varphi(s)\,dw(s)\right| \colon t \geqq 0\right\} \leqq 3E\left\{\left(\int_0^\infty \varphi^2(t)\,dt\right)^{1/2}\right\},$$

we bound the martingale term

$$E \sup \left\{\left|\int_0^t dM_s\right| \colon t \geqq 0\right\} \leqq 3pK(g,\sigma)\left(E\left\{\int_0^\infty \varphi^2(z(t),t)\,dt\right\}\right)^{1/2},$$
$$\{\varphi^2(z(t),t)\} \leqq [C(g,\sigma)\sqrt{h}]^{2p} e^{-(2\alpha - \alpha_{2p})t}, \quad t \geqq 0.$$

Hence we obtain (1.18) for $2\alpha > \alpha_{2p}$ as in (1.20) and

$$C = C(g,\sigma)[1 + 3pK(g,\sigma)(2\alpha - \alpha_{2p})^{-1}],$$

where $C(g,\sigma)$ and $K(g,\sigma)$ are the constants of (1.1). □

Remark 1.2. Notice that the constant α_p defined by (1.20) is bounded by $p(p \vee 1)K(g,\sigma)$, the constant of (1.1). It is clear then that α_p vanishes as p goes to zero.

Remark 1.3. Similarly, we can show for any $t \geqq 0$ the estimate

$$E\{|y_x^h(t,\lambda(\cdot)) - y_{x'}^h(t,\lambda(\cdot))|^p e^{-\alpha_p t}\} \leqq (C^2(g,\sigma)h + |x-x'|^2)^{p/2}, \tag{1.22}$$

where $C(g,\sigma)$, α_p are the constants of (1.1), (1.20) and x, x' belong to $\mathcal{R}$, h in $(0,1]$, and the feedback $\lambda(\cdot)$ is arbitrary.

1.2. The linear equation. In this section we consider the case without controls, i.e., the set Λ reduces to one element, and we drop it.

Recall the stochastic differential equation

$$dy(t) = g(y(t)) + \sigma(y(t))\,dw(t), \qquad t \geqq 0, \quad y(0) = x, \tag{1.23}$$

where g, σ are bounded and Lipschitz-continuous. For a bounded and uniformly continuous function f we set

$$u(x,t) = E\left\{\int_0^t f(y_x(s))\,ds\right\}, \quad \forall x \in \mathcal{R}, \quad t \geqq 0. \tag{1.24}$$

This function is the unique solution of the following partial differential equation that is bounded and uniformly continuous:

$$\partial_t u(x,t) = Lu(x,t) + f(x) \quad \forall x \in \mathcal{R}, \quad t > 0, \quad u(x,0) = 0 \quad \forall x \in \mathcal{R} \tag{1.25}$$

where ∂_t denotes the partial derivative in t and L is the differential operator

$$L\varphi(x,t) = \tfrac{1}{2}\sigma^2(x)\partial_x^2\varphi(x,t) + g(x)\partial_x\varphi(x,t). \tag{1.26}$$

The partial differential equation (PDE) (1.25) is understood in the Schwartz' distributions sense. On the other hand, we set

$$u_h(x, nh) = h \sum_{i=0}^{n} E\{f(X_i)\}, \qquad n = 0, 1, \cdots,$$

where $(X_n, \theta_n, n = 0, 1, \cdots)$ is the sequence of Theorem 1.1. It is clear that

$$u_h(x, nh) = E\left\{\int_0^{\theta_n} f(y_x^h(t))\, dt\right\}, \qquad n = 0, 1, \cdots, \tag{1.27}$$

with the notation of Theorem 1.2. For convenience we set

$$u_h(x,t) = u_h(x, nh) \quad \text{if } nh \leqq t < (n+1)h.$$

By means of Markov's property, we can deduce

$$\begin{aligned} &\nabla_h u_h(x,t) = L_h u_h(x,t) + f(x) \quad \forall x \in \mathcal{R}, \quad t > 0, \\ &u_h(x,0) = 0 \quad \forall x \in \mathcal{R}, \end{aligned} \tag{1.28}$$

where L_h is now the finite difference operator

$$\begin{aligned} L_h\varphi(x) = \frac{1}{2h}[&\varphi(x + g(x)h + \sigma(x)\gamma(x,h)\sqrt{h}) \\ &+ \varphi(x + g(x)h - \sigma(x)\gamma(x,h)\sqrt{h}) - 2\varphi(x)] \quad \forall h \text{ in } (0,1], \end{aligned} \tag{1.29}$$

and

$$\nabla_h\varphi(t) = \frac{1}{h}[\varphi(t+h) - \varphi(t)] \quad \forall h \text{ in } (0,1]. \tag{1.30}$$

Note that x belongs to $\mathcal{R}$, so our Markov chain has states in $\mathcal{R}$. Actually, we discretize first the time variable and then the state variable.

Denote by $\rho(r)$ the modulus of continuity of f, i.e.,

$$\rho(r) = \sup\{|f(x) - f(x')|: x, x' \in \mathcal{R}, |x - x'| \leqq r\}. \tag{1.31}$$

THEOREM 1.3. *Under the assumptions of Theorem* 1.2, *for any p, $T > 0$ there exists a constant C depending only on p, T, the Lipschitz constants of g, σ, and the bound of f, such that*

$$|u(x,t) - u_h(x,t)| \leqq C[\sqrt{h} + \rho(r) + (r^{-1}\sqrt{h})^p] \quad \forall r > 0, \tag{1.32}$$

valid for any x in $\mathcal{R}$, t in $[0, T]$, h in $(0, 1]$.

Proof. In view of the representations (1.24) and (1.27) we have

$$|u(x,t) - u_h(x,t)| \leqq E\left\{\int_0^t |f(y(s)) - f(y^h(s))|\, ds\right\} + C(f)E\{|\theta_n - t|\} = I + II,$$

where

$$|f(x)-f(x')| \leqq C(f) \quad \forall x, x' \in \mathcal{R}.$$

Thus,

$$I \leqq \alpha^{-1}(e^{\alpha t}-1)C(f)r^{-p}E\left\{\int_0^t |y(s)-y^h(s)|^p e^{-\alpha s}\, ds\right\}$$
$$+t\rho(r) \leqq \alpha^{-1}(e^{\alpha T}-1)C(f)C_1(r^{-1}\sqrt{h})^p + T\rho(r),$$

with α, $C=C_1$ being the constants of (1.18) in Theorem 1.2. Also,

$$E\{|\theta_n - t|\} \leqq h + E\{|\theta_n - nh|\} \leqq h + (E\{|(\tau_1-h)+\cdots+(\tau_n-h)|^2\})^{1/2}$$
$$= h + (nE\{(\tau_1-h)^2\})^{1/2} \leqq h + C_2\sqrt{th},$$

where $\tau_i = \theta_i - \theta_{i-1}$, and C_2 is a constant such that

$$E\{(\tau_1-h)^2\} \leqq (C_2 h)^2 \quad \forall h \in (0,1]. \tag{1.33}$$

It is clear that the above proves (1.32) provided we have established (1.33).

To show (1.33), we see that if $-\delta \leqq -gh \leqq \delta$, $\delta = \sigma\gamma\sqrt{h}$ then the characteristic function of τ_1,

$$u(x,s) = E\{e^{s\tau_1}\}, \qquad s>0 \text{ fixed},$$

is the solution of the differential equation

$$\tfrac{1}{2}\sigma^2 u'' + gu' - su = 0 \quad \text{in } (-\delta,\delta), \quad u(-\delta,s) = u(-\delta,s) = 1,$$

and

$$E\{(\tau_1)^2\} = \frac{\partial^2 u}{\partial s}(-gh, 0).$$

Hence, after some calculations we obtain (1.33). □

Remark 1.4. Analogously to the above theorem, and by means of Remark 1.3, we can prove that

$$|u_h(x,t) - u_h(x',t)| \leqq C\{\sqrt{h} + \rho(r) + r^{-p}[h + |x-x'|^2]^{p/2}\} \quad \forall r > 0 \tag{1.34}$$

for any h in $(0,1]$, x in $\mathcal{R}$, t in $[0,T]$ and some constant C depending only on the bound of f, the Lipschitz constants of g, σ and the constants T, $p>0$. Actually, we can do better, i.e., in the estimates (1.22) and (1.34) we may have the right-hand side with $h=0$, but this requires the use of another explicit Markov chain, the one used in § 1.3.

1.3. Fully nonlinear equation. Let us return to the control problem (0.1)-(0.5) for one dimension, i.e., D is the whole real line $\mathcal{R}$, Λ is some compact subset of $\mathcal{R}$, $n=d=1$ in (0.5). Recall that for any adapted control process $(\lambda(t), t \geqq 0)$ we obtain the state process $(y(t) = y_x(t,\lambda), t \geqq 0)$ as the solution of the stochastic differential equation (1.2) with initial condition $y(0)=x$. Next, the cost functional is defined by

$$J(x,\lambda) = E\left\{\int_0^\infty f(y(t), \lambda(t))\, e^{-\alpha t}\, dt\right\}, \tag{1.35}$$

and the optimal cost is

$$u(x) = \inf\{J(x,\lambda)\colon \lambda \text{ any control process}\}. \tag{1.36}$$

The associated HJB equation is

$$\alpha u = \inf\{L(\lambda)u + f(\cdot, \lambda): \lambda \in \Lambda\} \quad \text{in } \mathcal{R}, \tag{1.37}$$

which is indeed an ordinary differential equation in the real line, since $L(\lambda)$ is given by (1.3). If the data are smooth and the operator is uniformly elliptic, then the HJB equation (1.37) has one and only one solution with Lipschitz second derivative (cf. Krylov [13]). In general we use either the viscosity solution (cf. Lions [20]) or the maximum subsolution in the Schwartz' distribution sense (cf. Lions and Menaldi [21]).

The approximate control is then

$$\begin{aligned} &J_h(x, \lambda(\cdot)) = E\left\{\int_0^\infty f(y^h(t), \lambda(y^h(t)))\chi_h^\alpha(t)\, dt\right\}, \\ &\chi_h^\alpha(t) = (1+\alpha h)^{-n} \quad \text{if } \theta_n \leqq t < \theta_{n+1}, \quad n = 0, 1, \cdots, \end{aligned} \tag{1.38}$$

where $(y^h(t) = y_x^h(t, \lambda(\cdot)), t \geqq 0)$ and $(\theta_n, n = 0, 1, \cdots)$ are defined by (1.17) and (1.7). Note that

$$J_h(x, \lambda(\cdot)) = E\left\{h \sum_{n=0}^\infty f(X_n, \lambda(X_n))(1+\alpha+\alpha h)^{-n}\right\}. \tag{1.39}$$

The optimal cost is

$$u_h(x) = \inf\{J_h(x, \lambda(\cdot)): \lambda(\cdot) \text{ feedback control}\}, \tag{1.40}$$

$$\alpha u_h = \inf\{L_h(\lambda)u_h + f(\cdot, \lambda): \lambda \in \Lambda\} \quad \text{in } \mathcal{R}. \tag{1.41}$$

It is clear that an estimate of the type (1.18) will provide only a one-side bound for the rate of convergence of u_h toward u. Then we will modify the continuous time control problem as follows.

To simplify the exposition we assume g, σ Lipschitz-continuous in the control variable, i.e.,

$$|g(x, \lambda) - g(x, \lambda')| + |\sigma(x, \lambda) - \sigma(x, \lambda')| \leqq K|\lambda - \lambda'| \quad \forall x \in \mathcal{R}, \lambda, \quad \lambda' \in \Lambda, \tag{1.42}$$

for some constant $K = K(g, \sigma)$, and we call $\lambda(\cdot)$ an M-feedback control if $\lambda(\cdot)$ is Lipschitz-continuous, i.e.,

$$|\lambda(x) - \lambda(x')| \leqq M|x - x'| \quad \forall x, x' \in \mathcal{R}. \tag{1.43}$$

Consider the M-optimal cost

$$u(x, M) = \inf\{J_h(x, \lambda(\cdot)): \lambda(\cdot) M\text{-feedback control}\}, \tag{1.44}$$

for any $M > 0$, M destined to become infinite.

It is clear that $u(x, M) \geqq u(x)$ and, under reasonable assumptions we will have

$$u(x, M) \to u(x) \quad \forall x \in \mathcal{R} \quad \text{as } M \to \infty.$$

Moreover, sometimes the M-optimal cost is meaningful by itself.

THEOREM 1.4. *Let the assumptions of Theorem* 1.2 *and* (1.42) *hold. Then for any* $M, p > 0$ *there exist two constants* $C(M)$, $C > 0$ *depending only on* p, α, *the bound of* f, *and the constants of hypothesis* (1.1); $C(M)$ *depends also on* M *and the* $K(g, \sigma)$ *of* (1.42), *such that*

$$\begin{aligned} &u(x) - u_h(x) \leqq C[\sqrt{h} + \rho(r) + (r^{-1}\sqrt{h})^p] \quad \forall r > 0, \\ &u_h(x) - u(x, M) \leqq C(M)[\sqrt{h} + \rho(r) + (r^{-1}\sqrt{h})^p] \quad \forall r > 0, \end{aligned} \tag{1.45}$$

for any x *in* $\mathcal{R}$, h *in* $(0, 1]$ *and* $\rho(r)$ *given by* (1.31), *uniformly for* λ *in* Λ.

Proof. Starting from

$$u(x)-u_h(x)\leqq \sup\{J(x,\lambda_h)-J_h(x,\lambda(\cdot)) : \lambda(\cdot)\},$$

$$e^{-\alpha t}\leqq \chi_h^\alpha(t)\leqq e^{-\alpha(t-h)}\quad \forall t\geqq 0$$

and in view of the estimate (1.18), we deduce the first part of (1.45) as in Theorem 1.3.

For the second part of (1.45) we use

$$u_h(x)-u(x,M)\leqq \sup\{J_h(x,\lambda(\cdot))-J(x,\lambda(\cdot)) : \lambda(\cdot) \text{ any } M\text{-feedback control}\}$$

and we prove

$$E\sup\{|y_x(t,\lambda(\cdot))-y_x^h(t,\lambda(\cdot))|^p e^{-\alpha t} : t\geqq 0\}\leqq C(M)h^{p/2}, \tag{1.46}$$

as in Theorem 1.2, but now, $C(M)$ depends also on the Lipschitz constant M of the feedback control $\lambda(\cdot)$, as well as on the constant $K(g,\sigma)$ of (1.42). Thus, we complete the proof of the estimate (1.45). □

Until now, we have used only estimates on the stochastic state equation to obtain some bounds for the rate of convergence of the discrete HJB toward the continuous-time HJB.

Now we will look at the approximation problem in a more analytic way.

Suppose $\varphi(x)$ is a smooth function; then we can write

$$L_h(\lambda)\varphi=\frac{1}{2}\sigma^2\int_{-1}^{1}\varphi''(\cdot+gh+t\sigma\sqrt{h})(1-|t|)\,dt+g\int_0^1\varphi'(\cdot+tgh)\,dt \tag{1.47}$$

where the primes denote derivatives and we must take $\gamma=1$ in (1.4), i.e., for $g=g(x,\lambda)$, $\sigma=\sigma(x,\lambda)$,

$$L_h(\lambda)\varphi=\frac{1}{h}\left[\frac{1}{2}\varphi(\cdot+gh+\sigma\sqrt{h})+\frac{1}{2}\varphi\,\bigg|\,(\cdot+gh-\sigma\sqrt{h})-\varphi(x)\right]. \tag{1.48}$$

First,

$$|L(\lambda)\varphi(x)-L_h\lambda\varphi(x)|\leqq C_\varphi h^{p/2}\quad \forall x\in\mathcal{R},\quad h\in(0,1], \tag{1.49}$$

and λ in Λ, and some constant C_φ depending on the bounds of g,σ,φ'', and the p-Hölder constant of φ'', i.e., the constant $K=K(\varphi'')$ satisfying

$$|\varphi''(x)-\varphi''(x')|\leqq K|x-x'|^p\quad \forall x,x'\in\mathcal{R}, \tag{1.50}$$

for some exponent $0<p\leqq 1$.

Let us define $[\varphi]_p$ as the infimum of the set of all constant C satisfying

$$\begin{aligned}\inf\{L(\lambda)\varphi(y) : |y-x|\leqq C\sqrt{h}\}-Ch^{p/2}&\leqq L_h(\lambda)\varphi(x)\\ &\leqq \sup\{L(\lambda)\varphi(y) : |y-x|\leqq C\sqrt{h}\}+Ch^{p/2}\quad \forall h\in(0,1],\end{aligned} \tag{1.51}$$

for any x in Λ. It is clear that $|\varphi|_p$ can be bounded by the constant C_φ of (1.49). However, here we can do better:

(i) $[\varphi]_1$ is dominated by the bounds of the second derivative φ'' and the constants $C(g,\sigma)$, $K(g,\sigma)$ of hypothesis (1.1).

(ii) If $\sigma=\sigma(\lambda)$, i.e., constant in x, then $[\varphi]_p$ is dominated by the p-Hölder constant and the bound of the first derivative φ', and $C(g,\sigma)$, $K(g,\sigma)$.

(iii) If $g=0$ and $\sigma=\sigma(\lambda)$, then $[\varphi]_1$ is dominated by the bound of σ and does not depend on φ.

Suppose that f is bounded continuous and for some constant C, $K>0$, $0<p\leqq 1$,

$$
\begin{aligned}
&|f(x,\lambda)|\leqq C \quad \forall x\in\mathcal{R}, \quad \lambda\in\Lambda,\\
&|f(x,\lambda)-f(x',\lambda)|\leqq K|x-x'|^p \quad \forall x,x'\in\mathcal{R}, \quad \lambda\in\Lambda,
\end{aligned}
\tag{1.52}
$$

and

$$\alpha>\alpha_p, \text{ the constant given by (1.20).} \tag{1.53}$$

THEOREM 1.5. *Under the assumptions* (1.1), (1.52), *and* (1.53) *there exists a constant C depending only on the constants $C(g,\sigma)$, $K(g,\sigma)$, $C(f)$, $K(f)$ of* (1.1), (1.52), *the constant α of* (1.53), *and the value $[u]_p$ with u being the maximum solution of the* HJB *equations* (1.37), *such that*

$$|u(x)-u_h(x)|\leqq Ch^{p/2} \quad \forall x\in\mathcal{R}, \quad h\in(0,1], \tag{1.54}$$

where $u_h(x)$ is solution of the discrete HJB *equation* (1.41) *with the finite difference operator* (1.48).

Proof. We remark that the fact that $[u]_p$ is finite is implicit. To check that the discrete HJB equation (1.41) has a solution, we rewrite it as follows:

$$
\begin{aligned}
&u_h=\inf\{\Pi_h^\alpha(\lambda)u_h+hf(\cdot,\lambda):\lambda\in\Lambda\} \quad \text{in } \mathcal{R},\\
&\Pi_h^\alpha(\lambda)\varphi=(1+\alpha h)^{-1}[hL_h(\lambda)\varphi-\varphi],
\end{aligned}
\tag{1.55}
$$

and we note that the operator involved is a contraction map in the space of bounded continuous functions on $\mathcal{R}$.

First we will show that for some constants C, $K>0$ depending only on the constants in the assumptions (1.1), (1.52), and (1.53) such that

$$
\begin{aligned}
&|u_h(x)|+|u(x)|\leqq C \quad \forall x\in\mathcal{R},\\
&|u_h(x)-u_h(x')|+|u(x)-u(x')|\leqq K|x-x'|^p \quad \forall x,x'\in\mathcal{R},
\end{aligned}
\tag{1.56}
$$

for any h in $(0,1]$, $0<p\leqq 1$, the same p as in (1.52). It is relatively easy to obtain (1.50) for u from the stochastic representation (1.36); however, we prefer to use analytic arguments to present the technique used.

Consider the function

$$m(x,q,\varepsilon)=(\varepsilon^2+x^2)^{q/2} \quad \forall x\in\mathcal{R}, \tag{1.57}$$

for $q,\varepsilon>0$ fixed, and the solution $u(x)$ of the HJB equation (1.37). To prove the second part of (1.56) we look at the point (x_0,y_0) of $\mathcal{R}\times\mathcal{R}$ where the function

$$w(x,y)=u(x)-u(y)-Km(x-y,p,\varepsilon)m(x+y,q,1)$$

attains its maximum value, for a fixed K to be selected later. We want to show that $w(x_0,y_0)\leqq 0$ for an appropriate choice of K.

The extended operator

$$
\begin{aligned}
\tilde{L}(\lambda)\varphi(x,y)=&\tfrac12\sigma^2(x,\lambda)\varphi''_{xx}+\sigma(x,\lambda)\sigma(y,\lambda)\varphi''_{xy}\\
&+\tfrac12\sigma^2(y,\lambda)\varphi''_{yy}+g(x,\lambda)\varphi'_x+g(y,\lambda)\varphi'_y,
\end{aligned}
\tag{1.58}
$$

is elliptic and satisfies

$$\tilde{L}(\lambda)[u(x)-u(y)]=L(\lambda)u(x)-L(\lambda)u(y).$$

After some calculations, we have

$$\begin{aligned}\tilde{L}(\lambda)[m(x-y,p,\varepsilon)m(x+y,q,1)] = &\frac{p}{2}[\sigma(x,\lambda)-\sigma(y,\lambda)]^2\\ &\cdot m(x-y,p-2,\varepsilon)m(x+y,q,1)\\ &\cdot[(p-1)(x-y)^2 m(x-y,-2,\varepsilon)+1]\\ &+\frac{q}{2}[\sigma(x,\lambda)+\sigma(y,\lambda)]^2 m(x-y,p,\varepsilon)\\ &\cdot m(x+y,1,q-2)\\ &\cdot[(q-1)(x+y)^2 m(x+y,-2,1)]\\ &+pq[\sigma(x,\lambda)+\sigma(y,\lambda)][\sigma(x,\lambda)-\sigma(y,\lambda)]\\ &\cdot(x-y)(x+y)m(x-y,p-2,\varepsilon)\\ &\cdot m(x+y,q-2,1)+p[g(x,\lambda)-g(y,\lambda)]\\ &\cdot(x-y)m(x-y,p-2,\varepsilon)m(x+y,q,1)\\ &+q[g(x,\lambda)+g(y,\lambda)]\\ &\cdot(x+y)m(x-y,p,\varepsilon)m(x+y,q-2,1),\end{aligned}$$

which shows that

$$\tilde{L}(\lambda)[m(x-y,p,\varepsilon)m(x+y,q,1)] \leqq (\alpha_p - qC)m(x-y,p,\varepsilon)m(x+y,q,1), \tag{1.59}$$

where α_p is the constant defined by (1.20) and C is a constant independent of λ, x, y, ε, p, and $0<q<1$. We choose $q>0$ such that $\alpha - \alpha_p + qC \geqq \alpha_0 > 0$.

Now, by means of the maximum principle, we have $L(\lambda)w(x_0, y_0) \leqq 0$, i.e.,

$$L(\lambda)u(x_0) - L(\lambda)u(y_0) \leqq (\alpha-\alpha_0)Km(x_0-y_0,p,\varepsilon)m(x_0+y_0,1,q), \tag{1.60}$$

assuming that u is smooth and after using (1.59). But, from HJB equation (1.37) we deduce

$$\alpha[u(x_0)-u(y_0)] \leqq [K(f)+(\alpha-\alpha_0)K]m(x_0-y_0,p,\varepsilon)m(x_0+y_0,1,q),$$

where $K(f)$ is the p-Hölder Lipschitz of f in (1.52). Hence, if we choose $K = \alpha_0^{-1}K(f)$, then we conclude that $w(x_0, y_0) \leqq 0$. Therefore, we should have

$$u(x)-u(y) \leqq Km(x-y,p,\varepsilon)m(x+y,q,1).$$

Because the constant K does not depend on ε, q, we send ε, q to zero to obtain the second part of (1.56) for u, assuming that u is smooth.

Similarly, we show the Hölder-continuous estimate for u_h. In that case we use the extended operator

$$\begin{aligned}&\tilde{L}_h(\lambda)\varphi(x,y) = \frac{1}{h}\left[\frac{1}{2}\varphi(z^+(x,\lambda), z^+(y,\lambda)+\frac{1}{2}\varphi(z^-(x,\lambda), z^-(y,\lambda))-\varphi(x,y)\right],\\ &z^{\pm}(\cdot,\lambda) = \cdot + g(\cdot,\lambda)h \pm \sigma(\cdot,\lambda)\sqrt{h}.\end{aligned} \tag{1.61}$$

Note that if u is not smooth then we have to approximate u by a smooth function u_e, either by regularization, i.e., $\sigma+\varepsilon$ replaces σ, or by the so-called infimum convolution. The proof of the first part of (1.56) uses a technique analogous to the above.

Let us prove the estimate (1.54). Consider the function

$$w(x,y) = u_h(x) - u(y) - C_1 m(x-y,p,\varepsilon)m(x+y,q,1) - C_2 h^{p/2}$$

for some constants C_1, C_2, q, $\varepsilon>0$ to be selected later, and let (x_0, y_0) be a point where $w(x, y)$ attains its maximum value. A calculation similar to the one to obtain (1.59) shows that

$$\tilde{L}_h(\lambda)[m(x-y, p, \varepsilon)m(x+y, q, 1)] \leqq (\alpha_p - rq)m(x-y, p, \varepsilon)m(x+y, q, 1), \tag{1.62}$$

for any x, y in $\mathscr{R}$, λ in Λ, h, q in $(0, 1]$, some constant $r>0$ and the same α_p of (1.20). We take $q>0$ such that $\alpha_p - rq \leqq \alpha - \alpha_0$, $\alpha_0>0$.

Because $\tilde{L}_h(\lambda)w(x_0, y_0)\leqq 0$ we deduce

$$L_h(\lambda)u_h(x_0) - L_h(\lambda)u(y_0) \leqq (\alpha-\alpha_0)C_1 m(x_0-y_0, p, \varepsilon)m(x_0+y_0, q, 1),$$

and in view of (1.51),

$$L_h(\lambda)u(y_0) \leqq L(\lambda)u(y_1) + [u]_p h^{p/2}, \qquad |y_0 - y_1| \leqq [u]_p\sqrt{h}. \tag{1.63}$$

From the HJB equations satisfied by u_h and u we obtain

$$\alpha[u_h(x_0)-u(y_1)] \leqq \sup\{|f(x_0,\lambda)-f(y_1,\lambda)|: \lambda\in\Lambda\} + (\alpha-\alpha_0)Cm(x_0-y_0, p, \varepsilon)m(x_0+y_0, q, 1) + [u]_p h^{p/2},$$

and by means of (1.52), (1.56), (1.63) we get

$$|f(x_0,\lambda)-f(y_1,\lambda)| + |u(y_0)-u(y_1)| \leqq [K(f)+K(u)]m(x_0-y_0, p, \varepsilon),$$

provided $\varepsilon = [u]_p\sqrt{h}$.

Collecting all, we have

$$\alpha[u_h(x_0)-u(y_0)] \leqq [K(f)+K(u)+(\alpha-\alpha_0)C_1]m(x_0-y_0, p, \varepsilon) \cdot m(x_0+y_0, q, 1) + [u]_p h^{p/2}.$$

Hence, if we choose

$$C_1 = \alpha_0^{-1}[K(f)+K(u)], \quad C_2 = [u]_p, \quad \varepsilon = [u]_p\sqrt{h},$$

then $w(x_0, y_0)\leqq 0$, i.e.,

$$u_h(x) - u(y) \leqq C_1 m(x-y, p, \varepsilon)m(x+y, q, 1) + C_2 h^{p/2}, \tag{1.64}$$

for any x, y in $\mathscr{R}$, h, q in $(0, 1]$. Letting q vanish and taking $x=y$, we establish one side of (1.54).

Reversing the role of u_h and u we complete the proof. □

Remark 1.5. Note that in the proof of Theorem 1.5 we assume implicitly that the function u is smooth. However, once the estimates have been established, we can remove that assumption on u, only $[u]_p$ needs to be finite.

1.4. Extension and comments. The fact that the functions g, σ are bounded is not really important, we need only to assume linear growth, i.e.,

$$|g(x)| + |\sigma(x)| \leqq C(1+|x|) \quad \forall x\in\mathscr{R}, \tag{1.65}$$

for some constant $C = C(g, \sigma)$. In this case the estimate (1.18) of Theorem 1.2 becomes

$$E\sup\{|y_x(t,\lambda^h) - y_x^h(t,\lambda(\cdot))|^p e^{-\alpha t}: t\geqq 0\} \leqq C(1+|x|^2)^{p/2}h^{p/2} \quad \forall x\in\mathscr{R}, \tag{1.66}$$

for some constants C, $\alpha>0$.

To adapt Theorem 1.1 to the time-dependent case, we modify the construction (1.6), (1.7), for instance,

$$\tau(x, t, \lambda, h, w) = \inf\{s\geqq 0: g(x, s+t, \lambda)(s-h) + \sigma(x, s+t, \lambda)w(s) \text{ equals } \pm\delta(x, t, \lambda, h)\}. \tag{1.67}$$

A generalization to dimension $d \geqq 2$ is possible but more delicate. Let us describe the procedure. We write σ as the matrix formed by the column vectors $\sigma_1, \sigma_2, \cdots, \sigma_n$; the drift vector g is expressed as $g = g^1 e_1 + \cdots + g^n e_n$, where g^i are scalars and e_i are vectors in the direction σ_i, i.e., $\sigma_i = \sigma^i e_i$, σ^i is scalar. We want to define τ_i as the first time for which

$$g^i e_i(\tau_i - h) + \sigma^i e_i w_i(\tau_i) = \pm \sigma^i \gamma^i \sqrt{h}\, e_i.$$

This is the same as cancelling the vector e_i and defining τ_i as in (1.6) with g, σ, γ, w, replaced by $g^i, \sigma^i, \gamma^i, w_i$. Then we are interested in the stopping time $\tau = \min\{\tau_i: i = 1, \cdots, n\}$, which is the first exit time of the box in $\mathscr{R}^n$ bounded by the hyperplane $z_i = \pm\sigma^i\gamma^i\sqrt{h}$, $z = (z_1, \cdots, z_n)$ in $\mathscr{R}^n$, for a Wiener process in $\mathscr{R}^n$ with drift $(g^1, \cdots, g^n)$ and diffusion term the diagonal matrix $(\sigma^1, \cdots, \sigma^n)$, starting at the point $(-g^1\sqrt{h}, \cdots, -g^n\sqrt{h})$. Details of this construction will be presented in a future work.

In Theorems 1.3, 1.4, and 1.5 we can allow the functions g, σ to satisfy (1.65) and the function f to have polynomial growth, i.e.,

$$(1.68)\qquad \begin{aligned} &|f(x, \lambda)| \leqq C(1+x^2)^{q/2} \quad \forall x \in \mathscr{R}, \\ &|f(x, \lambda) - f(y, \lambda)| \leqq K|x-y|^p(1+x^2+y^2)^{r/2} \quad \forall x, y \in \mathscr{R}, \end{aligned}$$

for $q > 0$, $0 < q \leqq 1$, $r = \max\{q-p, 0\}$. The estimate (1.54) is modified accordingly. For the estimates (1.32) and (1.45) we use

$$(1.69)\qquad \rho(r) = \inf\{|f(x, \lambda) - f(y, \lambda)|(1+x^2+y^2)^{q/2}: x, y \text{ in } \mathscr{R}, \lambda \text{ in } \Lambda\}.$$

A discretization in Λ can also be incorporated. In that case, a term of the form

$$(1.70)\qquad r(h) = \sup\{\inf\{|l(x, \lambda) - l(x, \lambda')|: \lambda' \in \Lambda(h)\}: x \in \mathscr{R}^d, \lambda \in \Lambda, l = f, g, \sigma\}$$

will appear in the estimates (1.32), (1.45), and (1.54) of Theorems 1.3, 1.4, and 1.5. Here $\Lambda(h)$ is a discretization of Λ.

The constant $\alpha > 0$ can be replaced by a function $\alpha(x, \lambda)$.

The fact that we made only the discretization in the time variable is just the first step. To discretize the space variable, we can add the second part of condition (0.7), as in the next section. An alternative is to use finite elements to solve the discrete HJB of the type (1.41). This issue is reserved for a future work.

2. General problems. In this section we will consider the typical control problem (0.1)-(0.5) in a bounded open subset D of $\mathscr{R}^d$.

Let g and σ be bounded continuous functions from $\mathscr{R}^d \times \Lambda$ into $\mathscr{R}^d$ and $\mathscr{R}^d \times \mathscr{R}^n$, respectively, such that $g = (g_i, i = 1, \cdots, d)$, $\sigma = (\sigma_{ik}, i = 1, \cdots, d, k = 1, \cdots, n)$,

$$(2.1)\qquad \begin{aligned} &|g(x, \lambda)|\sigma(x, \lambda)| \leqq C \quad \forall x \in \mathscr{R}^d \quad \forall \lambda \in \Lambda, \\ &|g(x, \lambda) - g(x', \lambda)| + |\sigma(x, \lambda) - \sigma(x', \lambda)| \leqq K|x-x'| \quad \forall x, x' \in \mathscr{R}^d, \quad \lambda \in \Lambda, \end{aligned}$$

for some constants $C = C(g, \sigma)$, $K = K(g, \sigma)$, some locally compact metric space Λ and where $|\cdot|$ denotes the Euclidean norm in the corresponding space.

On a complete Wiener space $(\Omega, P, \mathscr{F}, \mathscr{F}(t), w(t), t \geqq 0)$ in $\mathscr{R}^n$ we consider the state equation

$$(2.2)\qquad dy(t) = g(y(t), \lambda(t))\, dt + \sigma(y(t), \lambda(t))\, dw(t), \quad t > 0, \quad y(0) = x,$$

where the control $(\lambda(t), t \geqq 0)$ is a progressively measurable process taking values in Λ. Denote by τ the first exit time of $\bar{D}$, closure of D, for the process $(y(t), t \geqq 0)$, i.e.,

$$(2.3)\qquad \tau = \inf\{t \geqq 0: y(t) \notin \bar{D}\}.$$

For a given real bounded-continuous function f on $\mathcal{R}^d \times \Lambda$ such that

$$|f(x, \lambda)| \leqq C \quad \forall x \in \mathcal{R}^d, \quad \lambda \in \Lambda, \tag{2.4}$$
$$|f(x, \lambda) - f(x', \lambda)| \leqq K|x - x'|^p \quad \forall x, x' \in \mathcal{R}^d, \quad \lambda \in \Lambda,$$

for some constants C, $K > 0$, $0 < p \leqq 1$, we define

$$J(x, \lambda) = E\left\{\int_0^\tau f(y(t), \lambda(t))\, e^{-\alpha t}\, dt\right\}, \qquad \alpha > 0, \tag{2.5}$$

and the optimal cost function

$$u(x) = \inf\{J(x, \lambda)\text{: any control process } \lambda\}. \tag{2.6}$$

The HJB equation is

$$\alpha u = \inf\{L(\lambda)u + f(\cdot, \lambda): \lambda \in \Lambda\} \quad \text{in } D, \quad u = 0 \quad \text{on } \partial D, \tag{2.7}$$

where the differential operator

$$L(\lambda) = \frac{1}{2}\sum_{i,j=1}^{d}\left(\sum_{k=1}^{n} \sigma_{ik}(\cdot, \lambda)\sigma_{jk}(\cdot, \lambda)\right)\partial_{ij} + \sum_{i=1}^{d} g_i(\cdot, \lambda)\partial_i, \tag{2.8}$$

and the bounded domain D has a uniform exterior sphere, i.e.,

(2.9) there exists $r > 0$ such that for any x in ∂D there is y in $\mathcal{R}^d \backslash D$ such that $\{z: |y - z| \leqq r\} \cap \bar{D} = \{x\}$,

and $L(\lambda)$ is not degenerate on the boundary, i.e.,

$$\sum_{k=1}^{n}\sum_{i=1}^{d} |\sigma_{ik}(x, \lambda)\eta_i(x)| \geqq \nu_0 > 0 \quad \forall x \in \partial D, \quad \lambda \in \Lambda, \tag{2.10}$$

with $\eta = (\eta_1, \cdots, \eta_d)$ being a normal direction to ∂D.

In § 2.1 we will give some properties of the finite difference operator (0.6). Next, we study the discrete HJB equations and its associated Markov chain. We present the main estimate in § 2.3 and then we give some comments and extensions.

2.1. The finite difference operator. Recall the operator (0.6),

$$\begin{aligned} L_h(\lambda)\varphi(x) = h^{-1}\sum_{k=1}^{n} \{&\beta_k^+(x, \lambda, h)[\varphi(x + \gamma_k^+(x, \lambda, h)) - \varphi(x)] \\ &+ \beta_k^-(x, \lambda, h)[\varphi(x + \gamma_k^-(x, \lambda, h)) - \varphi(x)]\}, \end{aligned} \tag{2.11}$$

where $\beta_k^\pm$, $\gamma_k^\pm$ are bounded Borel-measurable functions in x, λ for h fixed,

$$\beta_k^\pm(x, \lambda, h) \geqq 0, \quad x + \gamma_k^\pm(x, \lambda, h) \in \mathcal{R}_h^d \quad \forall x \in \mathcal{R}_h^d, \quad \lambda \in \Lambda. \tag{2.12}$$

The h-finite difference grid $\mathcal{R}_h^d$ is given, $0 < h \leqq 1$.

We denote

$$L_h(k, \lambda)\varphi(x) = h^{-1}\{\beta_k^+[\varphi(x + \gamma_k^+) - \varphi(x)] + \beta_k^-[\varphi(x + \gamma_k^-) - \varphi(x)]\}, \tag{2.13}$$

and

$$\gamma_k^0 = (\beta_k^+ + \beta_k^-)^{-1}(\beta_k^+\gamma_k^+ + \beta_k^-\gamma_k^-), \tag{2.14}$$

where the variables x, λ, h have been omitted.

We can rewrite

$$L_h(k,\lambda)\varphi(x)=h^{-1}\{\beta_k^+[\varphi(x+\gamma_k^+)-\varphi(x+\gamma_k^0)] \\ +\beta_k^-[\varphi(x+\gamma_k^-)-\varphi(x+\gamma_k^0)]+(\beta_k^++\beta_k^-)[\varphi(x+\gamma_k^0)-\varphi(x)]\},$$

and when φ is smooth,

$$\varphi(x+\gamma_k^\pm)-\varphi(x+\gamma_k^0)=\sum_{i=1}^d(\gamma_{ik}^\pm-\gamma_{ik}^0)\int_0^1\partial_i\varphi(x+\gamma_k^0+t(\gamma_k^\pm-\gamma_k^0))\,dt,$$

$$\varphi(x+\gamma_k^0)-\varphi(x)=\sum_{i=1}^d\gamma_{ik}^0\int_0^1\partial_i\varphi(x+t\gamma_k^0)\,dt,$$

with $\gamma_{ik}^\pm$, γ_{ik}^0 being the components of $\gamma_k^\pm$, γ_k^0. Using the fact that

$$\gamma_k^\pm-\gamma_k^0=\pm\beta_k^\pm(\beta_k^++\beta_k^-)^{-1}(\gamma_k^+-\gamma_k^-),$$

we have

$$\beta_k^+[\varphi(x+\gamma_k^+)-\varphi(x+\gamma_k^0)]+\beta_k^-[\varphi(x+\gamma_k^-)-\varphi(x+\gamma_k^0)] \\ =\beta_k^+\beta_k^-(\beta_k^++\beta_k^-)^{-2}\sum_{i,j=1}^d(\gamma_{ik}^+-\gamma_{ik}^-)(\gamma_{jk}^+-\gamma_{jk}^-) \\ \cdot\int_0^1dt\int_{t\beta_k^+}^{t\beta_k^-}\partial_{ij}\varphi(x+\delta_k(s))\,ds,$$

where

$$\delta_k(s)=(\beta_k^++\beta_k^-)^{-1}[(\beta_k^+-s)\gamma_k^++(\beta_k^--s)]\gamma_k^-.$$

If

$$\chi_k(s)=\begin{cases}2(\beta_k^++\beta_k^-)^{-1}[1-s(\beta_k^-)^{-1}] & \text{if } s\geqq 0,\\ 2(\beta_k^++\beta_k^-)^{-1}[1+s(\beta_k^+)^{-1}] & \text{if } s\leqq 0,\end{cases}$$

then

$$hL_h(k,\lambda)\varphi(x)=\frac{1}{2}\beta_k^+\beta_k^-(\beta_k^++\beta_k^-)^{-1}\sum_{i,j=1}^d(\gamma_{ik}^+-\gamma_{ik}^-)(\gamma_{jk}^+-\gamma_{jk}^-) \\ \cdot\int_{-\beta_k^+}^{\beta_k^-}\partial_{ij}\varphi(x+\delta_k(s))\chi_k(s)\,ds \\ +\sum_{i=1}^d(\beta_k^+\gamma_{ik}^++\beta_k^-\gamma_{ik}^-)\int_0^1\partial_i\varphi(x+t\gamma_k^0)\,dt. \tag{2.15}$$

Note that

$$\int_{-\beta_k^+}^{\beta_k^-}\chi_k(s)\,ds=1$$

and that $\delta_k(s)$, γ_k^0 are convex combinations of γ_k^+ and γ_k^-.

Therefore, let us assume that for some constant $C>0$ and any $i,j=1,\cdots,d$, we have

$$\left|\sum_{k=1}^n[\sigma_{ik}\sigma_{jk}h-(\beta_k^++\beta_k^-)^{-1}\beta_k^+\beta_k^-(\gamma_{ik}^+-\gamma_{ik}^-)(\gamma_{jk}^+-\gamma_{jk}^-)]\right|\leqq C^2h^{3/2}, \\ \left|g_ih-\sum_{k=1}^n(\beta_k^+\gamma_{ik}^++\beta_k^-\gamma_{ik}^-)\right|\leqq Ch^{3/2}, \\ |\gamma_{ik}^+|+|\gamma_{ik}^-|\leqq Ch^{1/2} \tag{2.16}$$

uniformly for the variables x in $\mathcal{R}_h^d$, λ in Λ, h in $(0, 1]$. Then we have the following estimate:

$$|L_h(\lambda)\varphi(x) - L(\lambda)\varphi(x)| \leqq C_\varphi h^{p/2} \quad \forall x, \lambda, h, \tag{2.17}$$

where C_φ is a constant depending only on the bounds of functions g, σ, ∂_{ij}, φ and the Hölder-continuous constants of the second derivatives $\partial_{ij}\varphi$ with exponent $0 < p \leqq 1$, i.e., the constant $K(\partial_{ij}\varphi)$ of (2.4) for $\partial_{ij}\varphi$ in lieu of f.

Typical examples where the assumption (2.16) holds are the following cases: any $\gamma_{ik}^\pm$, $\beta_k^\pm$, g_{ik}, σ_{ik} satisfying

$$\begin{aligned} &\gamma_{ik}^\pm = g_{ik}(x, \lambda, h)\beta_k^{-2}(x, \lambda, h)h \pm \sigma_{ik}(x, \lambda, h)\beta_k^{-1}(x, \lambda, h)\sqrt{h}, \\ &\beta_k^\pm = \frac{1}{2}\beta_k(x, \lambda, h), \quad \beta_k(x, \lambda, h) > 0, \quad k = 1, \cdots, n, \\ &\left| g_i(x, \lambda) - \sum_{k=1}^n g_{ik}(x, h, \lambda) \right| \leqq Ch^{1/2}, \\ &|\sigma_{ik}(x, \lambda)\sigma_{jk}(x, \lambda) - \sigma_{ik}(x, \lambda, h)\sigma_{jk}(x, \lambda, h)| \leqq Ch^{1/2}, \end{aligned} \tag{2.18}$$

uniformly in x, λ, h and for some constant $C > 0$. A more classic possibility is to choose $n = d$,

$$\gamma_{ik}^\pm(x, \lambda, h) = \begin{cases} \pm\sqrt{h} & \text{if } i = k, \\ 0 & \text{otherwise,} \end{cases} \tag{2.19}$$

and accordingly the coefficients $\beta_k^\pm(x, \lambda, h)$ to insure (2.16). Also, we may take $n = d + 1$,

$$\begin{aligned} &\gamma_{ik}^\pm = \pm\sigma_{ik}(x, \lambda, h)\beta_k^{-1}(x, \lambda, h)\sqrt{h} \quad \text{if } k = 1, \cdots, n-1, \\ &\beta_k^\pm = \tfrac{1}{2}\beta_k(x, \lambda, h) > 0 \quad \text{if } k = 1, \cdots, n-1, \\ &\gamma_{in}^\pm = g_i(x, \lambda, h)\beta_n^{-2}(x, \lambda, h)h, \qquad \beta_n > 0, \\ &\beta_n^+ = \beta_n(x, \lambda, h), \qquad \beta_n^- = 0, \quad \sigma_{in}(x, \lambda) = 0 \quad \forall \\ &|g_i(x, \lambda) - g_i(x, \lambda, h)| \leqq Ch^{1/2} \quad \forall i, \\ &|\sigma_{ik}(x, \lambda)\sigma_{jk}(x, \lambda) - \sigma_{ik}(x, \lambda, h)\sigma_{jk}(x, \lambda, h)| \leqq Ch^{1/2} \quad \forall i, jk, \quad k \neq n, \end{aligned} \tag{2.20}$$

uniformly in x, λ, h for some constant $C > 0$.

When the differential operator (2.8) is degenerate with constant order of degeneration, i.e.,

$$L(\lambda) = \frac{1}{2} \sum_{i,j=1}^m \left(\sum_{k=1}^n \sigma_{ik}(\cdot, \lambda)\sigma_{jk}(\cdot, \lambda) \right) \partial_{ij} + \sum_{i=1}^d g_i(\cdot, \lambda)\partial_i, \tag{2.21}$$

where $0 \leqq m \leqq d$, $n \geqq 0$ and clearly $d - m$ is the order of degeneration, it is convenient to choose (2.18) or (2.20) instead of (2.19). In this case the constant C_φ of (2.17) will depend only on the constants $K(\partial_{ij}, \varphi)$ and bounds of g, σ, $\partial_{ij}\varphi$, for $i, j = 1, \cdots, m$.

Denote by $\mathcal{R}_h^d$ an h-finite difference grid in $\mathcal{R}^d$, i.e.,

$$\begin{aligned} &\mathcal{R}_h^d = \{x = (x_1, \cdots, x_d) \colon \forall i = 1, \cdots, d, \exists k = 0, \pm 1, \cdots, \text{such that } x_i = hr_i(k)\}, \\ &1 \leqq r_i(k) - r_i(k-1) < 2 \quad \forall k = 0, \pm 1, \cdots. \end{aligned} \tag{2.22}$$

For the open bounded subset D of $\mathcal{R}^d$ we denote

$$D_h = \{x \in D \cap \mathcal{R}_h^d \colon x + \gamma_k^\pm(x, \lambda, h) \in \bar{D}, \forall \lambda, k\}, \tag{2.23}$$

and its boundary

$$(2.24)\qquad \begin{aligned}&\Gamma_h = \bigcup\{\Gamma_k^+(\lambda)\cup\Gamma_k^-(\lambda): \lambda, k\},\\ &\Gamma_k^\pm(\lambda) = \{x\in D\cap R_h^d: x+\gamma_k^\pm(x,\lambda,h)\in \bar D\cap \mathcal{R}_h^d \text{ and } x+\gamma_k^\mp(x,\lambda,h)\notin \bar D\cap\mathcal{R}_h^d\},\end{aligned}$$

for a fixed h in $(0, 1]$.

Under the assumptions (2.11), (2.12) we can easily prove the discrete maximum principle for the finite difference operator $L_h(\lambda)$. It is as follows. If a function $u_h(x)$ defined on $\bar D_h = D_h\cup\Gamma_h$ attains its maximum value at some point x_0, then

$$(2.25)\qquad \begin{aligned}&\text{(i)}\quad \text{If } x_0\in D_h, \text{ then } L_h(\lambda)u_h(x_0)\leqq 0\quad \forall\lambda\in\Lambda;\\ &\text{(ii)}\quad \text{If } x_0\in\Gamma_k^\pm(\lambda), \text{ then } \nabla_k^\pm(\lambda)u_h(x_0)\leqq 0,\end{aligned}$$

where $\nabla_k^\pm(\lambda)$ is the operator given by

$$(2.26)\qquad \nabla_k^\pm(\lambda)\varphi(x) = h^{-1}\beta_k^\pm(x,\lambda,h)[\varphi(x+\gamma_k^\pm(x,\lambda,h))-\varphi(x)],$$

for any φ.

2.2. Study of the discrete equation. Here, we are interested in the discrete HJB equation (0.8), i.e., in finding a function $u_h(x)$, x in $\bar D_h$ such that

$$(2.27)\qquad \alpha u_h = \inf\{L_h(\lambda)u_h + f(\cdot,\lambda): \lambda\in\Lambda\}\quad \text{in } D_h,\quad u_h=0\quad \text{on }\Gamma_h,$$

where $D_h, \Gamma_h, \bar D_h$ are defined by (2.23), (2.24), and the finite difference operator $L_h(\lambda)$ is given by (2.11), (2.12).

First, we will associate an optimal control problem of a Markov chain to the HJB equation (2.27).

Let

$$(2.28)\qquad G(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\xi}\exp\left(-\frac{t^2}{2}\right)dt,\qquad -\infty\leqq\xi\leqq+\infty,$$

and

$$(2.29)\qquad \begin{aligned}&G(\tilde\beta_1^+) = \beta^{-1}\beta_1^+,\qquad G(\tilde\beta_1^-) = \beta^{-1}(\beta_1^+ + \beta_1^-),\\ &G(\tilde\beta_k^+) = \beta^{-1}(\beta_1^+ + \beta_1^- + \cdots + \beta_k^+),\qquad k=2,\cdots,n,\\ &G(\tilde\beta_k^-) = \beta^{-1}(\beta_1^+ + \beta_1^- + \cdots + \beta_k^+ + \beta_k^-),\qquad k=2,\cdots,n-1,\\ &\beta = \beta_1^+ + \beta_1^- + \cdots + \beta_n^+ + \beta_n^-,\quad \beta_0^- = -\infty,\quad \tilde\beta_n^- = +\infty\end{aligned}$$

where the variables x, λ, h have been omitted. For a random variable η with Gaussian density (2.28), we define the random fields $\xi_k^\pm(w) = \xi_k^\pm(x,\lambda,h;\eta)$ by

$$(2.30)\qquad \xi_k^\pm(w) = \begin{cases}1 & \text{if } w\in A_k^\pm,\\ 0 & \text{otherwise}\end{cases}$$

where

$$(2.31)\qquad \begin{aligned}&A_k^+ = \{w: \tilde\beta_{k-1}^- < \eta(w) < \tilde\beta_k^+\},\qquad \tilde\beta_0^- = -\infty,\\ &A_k^- = \{w: \tilde\beta_k^+ < \eta(w) < \tilde\beta_k^-\},\qquad k=1,2,\cdots,n.\end{aligned}$$

Suppose we are given a sequence (η_i: $i=1, 2, \cdots$) of independent random variables with the same Gaussian density (2.28) in a complete probability space (Ω, P, $\mathcal{F}$). Then we consider the following controlled Markov chain:

$$
\begin{aligned}
X^{i+1} = X^i + \sum_{k=1}^{n} [&\gamma_k^+(X^i, \lambda(X^i), h)\xi_k^+(X^i, \lambda(X^i), h; \eta_{i+1}) \\
&+ \gamma_k^-(X^i, \lambda(X^i), h)\xi_k^-(X^i, \lambda(X^i), h; \eta_{i+1})], \qquad i=0, 1, \cdots, \\
&X^0 \text{ given in } \mathcal{R}_h^d,
\end{aligned} \tag{2.32}
$$

where $\lambda = \lambda(\cdot)$ is a feedback control, i.e., a Borel-measurable function from $\mathcal{R}^d$ into Λ; actually it suffices that λ be defined only on $\mathcal{R}_h^d$.

A simple calculation shows that the transition probability operators of the Markov chain (2.32) is given by

$$
\begin{aligned}
&E\{\varphi(X^{i+1}) \,|\, X^i = x\} = \pi_h(\lambda(x))\varphi(x), \\
&\pi_h(\lambda)\varphi(x) = \beta^{-1}(x, \lambda, h) \sum_{k=1}^{n} [\beta_k^+(x, \lambda, h)\varphi(x + \gamma_k^+(x, \lambda, h)) \\
&\qquad\qquad + \beta_k^-(x, \lambda, h)\varphi(x + \gamma_k^-(x, \lambda, h))] \quad \forall \varphi, \\
&\beta(x, \lambda, h) = \sum_{k=1}^{n} [\beta_k^+(x, \lambda, h) + \beta_k^-(x, \lambda, h)] \quad \text{for any } x, \lambda, h.
\end{aligned} \tag{2.33}
$$

Standard arguments of the discrete optimal control theory (e.g., Bertsekas and Shreve [3], Gihkman and Skorokhod [10], Ross [32]) show that the optimal cost function

$$
\begin{aligned}
&u_h(x) = \inf\{J_h(x, \lambda): \lambda(\cdot) \text{ any feedback control}\}, \\
&J_h(x, \lambda) = E\left\{\sum_{i=0}^{\nu-1} hf(X^i, \lambda(X^i))[q_h(X^i, \lambda(X^i))]^i \,|\, X^0 = x\right\}, \\
&q_h(x, \lambda) = [h\alpha + \beta(x, \lambda, h)]^{-1}\beta(x, \lambda, h), \\
&\nu = \inf\{i = 0, 1, \cdots : X^i \in \Gamma_h\}
\end{aligned} \tag{2.34}
$$

satisfies the discrete HJB equation (2.27). Note that

$$
\begin{aligned}
&E\{X^{i+1} \,|\, X^i = x\} = x + \sum_{k=1}^{n} \gamma_k^0, \\
&\operatorname{Var}\{X^{i+1} \,|\, X^i = x\} = \sum_{k,i,j} [(\gamma_k^+ - \gamma_i^0)(\gamma_k^+ - \gamma_i^0)^* + (\gamma_k^- - \gamma_i^0)(\gamma_k^- - \gamma_i^0)^*], \\
&\gamma_k^0 = \beta^{-1}(\beta_k^+\gamma_k^+ + \beta_k^-\gamma_k^-),
\end{aligned} \tag{2.35}
$$

which are related to the condition (2.16).

We remark that the random fields (2.30) possess the following property:

$$
\begin{aligned}
P\{w \in \Omega: \xi_k^\pm(x, \lambda, h; \eta(w)) \neq \xi_k^\pm(x', \lambda', h; \eta(w))\} \leqq 2\beta^{-1}(x, \lambda, h) \\
\cdot \sum_{k=1}^{n} [|\beta_k^+(x, \lambda, h) - \beta_k^+(x', \lambda', h)| + |\beta_k^-(x, \lambda, h) - \beta_k^-(x', \lambda', h)|],
\end{aligned} \tag{2.36}
$$

for any x, λ, h. All the above properties are useful for directly studying the dependence on the data of the optimal cost (2.34) (cf. [24]).

On the other hand, we can use the technique of barrier functions used in continuous time control problems (e.g., Lions [20], Lions and Menaldi [21]) to construct subsolutions of the discrete HJB equation; this method uses the assumptions (2.9) and (2.10). So, we suppose that:

(2.37) There exist functions $\bar{u}_h(x)$ defined on $\mathscr{R}^d$ that are bounded and Hölder-continuous uniformly in h with exponent $0<p\leqq 1$ such that for some constants $\beta_p\geqq 0$, $K>0$ we have $L_h(\lambda)\bar{u}_h\leqq -1+\beta_p\bar{u}_h$, in D_h, $\forall\lambda\in\Lambda$, $\bar{u}_h(x)=0$, $\forall x\in\Gamma_h$, $h\in(0,1]$, $|\bar{u}_h(x)-\bar{u}_h(x')|\leqq K|x-x'|^p$, $\forall x, x'\in\mathscr{R}^d$.

THEOREM 2.1. *Let us assume* (2.11), (2.12), (2.37). *Then for any bounded Borel-measurable function $f(x,\lambda)$ and any constant $\alpha>0$ there exist a unique solution of the discrete* HJB *equation* (2.27). *Moreover, for two data $f, \tilde{f}$* we have

$$\|u_h-\tilde{u}_h\|\leqq \alpha^{-1}\|f-\tilde{f}\| \quad \forall h\in(0,1], \tag{2.38}$$

where $\tilde{u}_h$ denotes the solution corresponding to $\tilde{f}$ and $\|\cdot\|$ is the supremum norm. Furthermore, if $\alpha\geqq\beta_p$ in (2.37) *then*

$$|u_h(x)|\leqq\|f\|\bar{u}_h(x)\quad \forall x\in\bar{D}_h,\quad h\in(0,1]. \tag{2.39}$$

Proof. It is possible to use the Markov chain (2.32) to establish the results as in [24], but we prefer to illustrate its analytic counterpart.

First of all, we rewrite the discrete HJB equation (2.27) as

$$u_h=\inf\{g_h(\cdot,\lambda)\pi_h(\lambda)u_h+f_h(\cdot,\lambda):\lambda\in\Lambda\}\quad \text{in } D_h,\quad u_h=0\quad \text{on } \Gamma_h, \tag{2.40}$$

where

$$f_h(x,\lambda)=h[h\alpha+\beta(x,\lambda,h)]^{-1}f(x,\lambda), \tag{2.41}$$

and $\pi_h(\lambda)$, $\beta(x,\lambda,h)$, $q_h(x,\lambda)$ are defined by (2.33), (2.34). If we denote by $T_h(u_h)$ the right-hand side of (2.40), then T_h is a contraction mapping on the space of bounded Borel-measurable functions from $\bar{D}_h$ onto $\mathscr{R}$ (actually just functions, since $\bar{D}_h$ is a finite set) with the norm

$$\|v\|=\sup\{|v(x)|: x\in\bar{D}_h\}. \tag{2.42}$$

Hence there exists a unique solution to (2.40).

Since we can express for any u_h^0 given the following:

$$u_h=\lim_{i\to\infty}u_h^i,\quad u_h^{i+1}=T_h(u_h^i),\quad i=0,1,\cdots,$$

we easily deduce (2.38), where $\|\cdot\|$ denotes the supremum norm in the corresponding space, i.e., for $\|f\|$ we take the supremum over $\bar{D}_h\times\Lambda$ or $\mathscr{R}_h^d\times\Lambda$.

To check (2.39), we use the discrete maximum principle on the function $w=\pm u_h-\|f\|\bar{u}_h$. □

Consider the extended finite difference operator $\tilde{L}_h(\lambda)$ given by

$$\begin{aligned}\tilde{L}_h(\lambda)(x,y)&=h^{-1}\sum_{k=1}^{n}[\tilde{L}_h^+(k,\lambda)\varphi(x,y)+\tilde{L}_h^-(k,\lambda)\varphi(x,y)],\\ h\tilde{L}_h^{\pm}(k,\cdot)\varphi(x,y)&=p_k^{\pm}(x,y,h)[\varphi(x+\gamma_k^{\pm}(x,h),y+\gamma_k^{\pm}(y,h))-\varphi(x,y)]\\ &\quad+q_k^{\pm}(x,y,h)[\varphi(x+\gamma_k^{\pm}(x,h),y)-\varphi(x,y)]\\ &\quad+q_k^{\pm}(y,x,h)[\varphi(x,y+\gamma_k^{\pm}(y,h))-\varphi(x,y)],\end{aligned} \tag{2.43}$$

where

$$
\begin{aligned}
&p_k^{\pm}(x, y, \lambda, h) = \beta_k^{\pm}(y, \lambda, h), \quad \wedge = \text{minimum},\\
&q_k^{\pm}(x, y, \lambda, h) = \beta_k^{\pm}(x, \lambda, h) - \beta_k^{\pm}(x, \lambda, h) \wedge \beta_k^{\pm}(y, \lambda, h).
\end{aligned} \tag{2.44}
$$

Note that our choice implies that

$$
\tilde{L}_h(\lambda)[\varphi(x) + \psi(y)] = L_h(\lambda)\varphi(x) + L_h(\lambda)\psi(y), \tag{2.45}
$$

for any functions $\varphi(x)$, $\psi(y)$, and $L_h(\lambda)$ as in (2.11), (2.12). It is clear that

$$
\alpha_p(h) = \sup\{m(x-y, -p, h)\tilde{L}_h(\lambda)m(x-y, h): x, y \in \mathscr{R}_h^d, \lambda \in \Lambda\}, \tag{2.46}
$$

is finite, for h in $(0, 1]$, $0 < p \leqq 1$, and

$$
m(x, p, \varepsilon) = (\varepsilon + |x|^2)^{p/2}, \qquad x \in \mathscr{R}^d, \quad \varepsilon > 0. \tag{2.47}
$$

Suppose that (2.1), (2.16), and

$$
\begin{aligned}
&|\beta_k^{\pm}(x, \lambda, h) - \beta_k^{\pm}(y, \lambda, h)| \leqq K|x-y|,\\
&|\gamma_k^{\pm}(x, \lambda, h) - \gamma_k^{\pm}(y, \lambda, h)| \leqq Kh^{1/2}|x-y|,\\
&|q_k^{+}(x, y, h)\gamma_k^{+}(x, \lambda, h) + q_k^{-}(x, y, \lambda, h)\gamma_k^{-}(x, \lambda, h)| \leqq Kh|x-y|,
\end{aligned} \tag{2.48}
$$

for some constant $K > 0$, uniformly for the variables x, y in $\mathscr{R}_h^d$, λ in Λ, h in $(0, 1]$, $k = 1, 2, \cdots, n$, hold true. Then, for some constant C depending on the various constants of the hypotheses (2.1), (2.16), and (2.48), we have

$$
\alpha_p(h) \leqq \alpha_p + Ch^{p/2}, \qquad p > 0, \tag{2.49}
$$

where

$$
\begin{aligned}
\alpha_p = k \sup\Bigg\{ &|x-y|^{-2} \sum_{i=1}^{d} [g_i(x, \lambda) - g_i(y, \lambda)](x_i - y_i)\\
&+ (p \vee 1)|x-y|^4 \sum_{k=1}^{n} \sum_{i,j=1}^{d} (x_i - y_i)(x_j - y_j)[\sigma_{ik}(x, \lambda) - \sigma_{jk}(y, \lambda)]\\
&\cdot [\sigma_{jk}(x, \lambda) - \sigma_{jk}(y, \lambda)] + |x-y|^{-2} \sum_{k=1}^{n} \sum_{i=1}^{d} [\sigma_{ik}(x, \lambda) - \sigma_{ik}(x, \lambda)]^2:\\
&x \neq y' \text{ in } \mathscr{R}^d, \lambda \text{ in } \Lambda \Bigg\}.
\end{aligned} \tag{2.50}
$$

Note that $\alpha_p(h)$ and α_p vanish as p goes to zero. The condition (2.48) is almost equivalent to the Lipschitz condition of (2.1), i.e., that (2.16) and (2.48) imply (2.1) and (2.48) is expected to hold if we wish to insure (2.1).

THEOREM 2.2. *Under the assumptions* (2.4), (2.11), (2.12), (2.37), *and*

$$
\alpha > \max\{\alpha_p(h), \beta_p\} \quad \forall h \in (0, 1], \tag{2.51}
$$

the unique solution u_h to the discrete HJB *equation* (2.27) *satisfies*

$$
|u_h(x) - u_h(y)| \leqq K|x-y|^p \quad \forall x, y \in \bar{D}_h, \quad h \in (0, 1], \tag{2.52}
$$

for some constant K depending only on the various constants appearing in the hypotheses (2.4), (2.37), *and* (2.51).

Proof. Consider the function

$$
w(x, y) = u_h(x) - u_h(y) - Km(x-y, h, p),
$$

where $m(\cdot,\cdot,\cdot)$ is given by (2.47) and $K>0$ is a constant to be selected later. We want to show that $w(x,y)\leqq 0$ at any point x,y of $\bar{D}_h$, which implies (2.52), since $|x-y|\geqq h$, if $x\neq y$. Let (x_0, y_0) be a point in $\bar{D}_h\times\bar{D}_h$ where the function $w(x,y)$ attains its maximum value; such a point exists always. If either $x_0\in\Gamma_h$ or $y_0\in\Gamma_h$ then $w(x_0,y_0)\leqq 0$ provided $K\geqq\|f\|K(\bar{u}_h)$, the constant of (2.37). Herein we have used the estimate (2.39). Let us look at the case when x_0, y_0 belong to D_h.

By means of the discrete maximum principle for the extended operator $\tilde{L}_h(\lambda)$ we have $\tilde{L}_h(\lambda)w(x_0,y_0)\leqq 0$, which implies

$$L_h(\lambda)u_h(x_0)-L_h(\lambda)u_h(y_0)\leqq K\tilde{L}_h(\lambda)m(x_0-y_0,h,p)$$

after using (2.45). If we choose $0<\alpha_0\leqq\alpha-\alpha_p(h)$, for any h in $(0,1]$, then in view of (2.46) we get

$$\tilde{L}_h(\lambda)m(x_0-y_0,p,h)\leqq(\alpha-\alpha_0)m(x_0-y_0,p,h)$$

for any l in Λ. Since u_h satisfies the discrete HJB equation (2.27) at x_0 and y_0, we deduce

$$\begin{aligned}\alpha[u_h(x_0)-u_h(y_0)]&\leqq\sup\{|f(x_0,\lambda)-f(y_0,\lambda)|:\lambda\in\Lambda\}\\&\quad+K\sup\{\tilde{L}_h(\lambda)m(x_0-y_0,p,h):\lambda\in\Lambda\}\\&\leqq[K(f)+(\alpha-\alpha_0)K]m(x_0-y_0,p,h)\end{aligned}$$

where $K(f)$ is the constant of hypothesis (2.4). Hence if we take $K=\alpha_0^{-1}K(f)$, then $w(x_0,y_0)\leqq 0$, i.e.,

$$u_h(x)-u_h(y)\leqq Km(x-y,p,h)\quad\forall x,y\in\bar{D}_h,\quad h\in(0,1].$$

Thus, the estimate (2.52) follows. □

Remark 2.1. Note that in the assumption (2.51) we suppose implicitly that (2.16) and (2.48) hold true.

2.3. Main estimate. Let us look at the continuous time HJB equation (2.7), i.e.,

(2.53) $$\alpha u=\inf\{L(\lambda)u+f(\cdot,\lambda):\lambda\in\Lambda\}\quad\text{in } D,\quad u=0\quad\text{on }\partial D,$$

where the differential operator is

(2.54) $$L(\lambda)=\sum_{i,j=1}^{d}a_{ij}(\cdot,\lambda)\partial_{ij}+\sum_{i=1}^{d}a_i(\cdot,\lambda)\partial_i,$$

and we have identified the coefficients

(2.55) $$\begin{aligned}&\sum_{k=1}^{n}\sigma_{ik}(x,\lambda)\sigma_{jk}(x,\lambda)=2a_{ij}(x,\lambda)\quad\forall x\in\bar{D},\quad\lambda\in\Lambda,\\&g_i(x,\lambda)=a_i(x,\lambda)\quad\forall x\in\bar{D},\quad\lambda\in\Lambda.\end{aligned}$$

Suppose that

(2.56) D is a bounded domain in $\mathcal{R}^d$ with smooth boundary ∂D, say $C^{2,p}$ for some $0<p\leqq 1$,

and

(2.57) $\alpha\geqq 0$, and for some $\nu_0>0$ we have $\nu_0|x|^2\leqq\sum_{i,j=1}^{d}a_{ij}(x,\lambda)\xi_i\xi_j\leqq\nu_0^{-1}|\xi|^2$, $\forall x\in\bar{D}$, $\lambda\in\bar{D}$, $\lambda\in\Lambda$, $\xi\in\mathcal{R}^d$.

It has been proven independently by Evans [7] and Krylov [14] (cf. Gilbarg and Trudinger [11]) that under the assumptions (2.56), (2.57), and

(2.58) a_{ij}, a_i, f are smooth, say C^3 in x uniformly in λ, the HJB equation (2.53) has a unique classic solution, which is continuous on the closure $\bar{D}$ and its first- and second-order derivatives are Hölder-continuous for some exponent $0<p_0<1$ on the open domain D.

This result has been improved by Krylov [15] to show that under the same assumptions, the first- and second-order derivatives of the solution u are indeed Hölder-continuous on the closure $\bar{D}$.

Then an almost optimal result due to Safonov [34] provides an equivalent of Schauder estimates for HJB equations. Precisely, under the assumptions (2.56), (2.57), and

$$\|\varphi\|_{(p)} \leqq K \quad \forall \varphi = a_{ij}(\cdot, \lambda), a_i(\cdot, \lambda) \quad \forall \lambda \in \Lambda, \tag{2.59}$$

where $\|\cdot\|_{(p)}$ denotes the Hölder norm in $C^p(\bar{D})$, $0<p\leqq 1$, there exists a constant $p_0(\nu_0, d)$ in $(0, 1]$ such that the estimate

$$\|u\|_{(2+p)} \leqq C \sup\{\|f(\cdot, \lambda)\|_{(p)}: \lambda \in \Lambda\}, \tag{2.60}$$

holds for some constant C depending only in K, ν_0, p and the domain D, provided $0<p<p_0(\nu_0, d)$. Note that $\|\cdot\|_{(2+p)}$ denotes the Hölder norm in the space $C^{2,p}(\bar{D})$.

Another case in the quasilinear equation is

$$a_{ij}(x, \lambda) = a_{ij}(x) \quad \forall x \in \bar{D} \quad \forall \lambda \in \Lambda. \tag{2.61}$$

Thus we do not control the diffusion term. Under the conditions (2.56), (2.57), (2.59), and (2.61) the estimate (2.60) holds for every $0<p<1$ (cf. Ladyzhenskaya and Uraltseva [19]).

It is known (cf. Lions [20], Lions and Menaldi [21], Krylov [13]) that under the assumptions (2.1), (2.4), (2.9), and (2.10) the HJB equation (2.53) has a unique solution in some weak sense, e.g., either as the maximum subsolution with $L(\lambda)$ acting in the Schwarz distributions sense or as the unique viscosity solution. Moreover, if we denote by u_ε the solution of the HJB equation (2.53) with $L(\lambda)$ replaced by $L(\lambda)+\varepsilon\Delta$, Δ the Laplacian operator, then we can assert that

$$u_\varepsilon \in C^2(\bar{D}), \quad u_\varepsilon \to u \quad \text{in } C(\bar{D}), \tag{2.62}$$

where $C(\bar{D})$ is the space of continuous functions on $\bar{D}$.

For a smooth function φ, say $C^{2,p}(\bar{D})$, $0<p\leqq 1$, let us define $[\varphi]_p$ as the infimum of the set of all constant C satisfying

$$\begin{aligned} &\inf\{L(\lambda)\varphi(y): |y-x|\leqq C\sqrt{h}\} - Ch^{p/2} \leqq L_h(\lambda)\varphi(x) \\ &\qquad \leqq \sup\{L(\lambda)\varphi(y): |y-x|\leqq C\sqrt{h}\} + Ch^{p/2} \quad \forall h \in (0, 1], \end{aligned} \tag{2.63}$$

for any x in D_h, λ in Λ.

It is clear that $[\varphi]_p$ can be bounded by the constant C_φ of the estimate (2.17). However, occasionally we can do better:

(i) $[\varphi]_1$ is dominated by the bounds of the second-order derivatives of φ and the constants $C(g, \sigma)$, $K(g, \sigma)$, C of hypotheses (2.1), (2.16), provided $n=1$. This means that only one-dimensional Brownian motion is allowed, e.g., the system associated with an equation of order d perturbed by a white noise.

(ii) If $\sigma=\sigma(\lambda)$ and $n=1$, i.e., constant in x and only one Brownian motion, and

$$\sigma_{i1}\sigma_{j1}h = (\beta_1^+ + \beta_1^-)^{-1}\beta_1^+\beta_1^-(\gamma_{i1}^+ - \gamma_{i1}^-)(\gamma_{j1}^+ - \gamma_{j1}^-) \quad \forall i, j, \lambda,$$

then $[\varphi]_p$ is dominated by the p-Hölder norm of the first-order derivatives of φ and the constants $C(g, \sigma)$, $K(g, \sigma)$,

(iii) If $g=0$ and (ii) holds, then $[\varphi]_1$ is dominated by the bound of σ and does not depend on φ.

THEOREM 2.3. *Let the assumptions* (2.1), (2.4), (2.9)-(2.12), (2.16), (2.37), (2.48), *and*

$$\alpha > \max\{\alpha_p, \beta_p\}, \qquad 0 < p \leqq 1 \tag{2.64}$$

hold true. Then there exists a constant $C>0$ *depending only on the various constants of the hypotheses* (2.4), (2.37), (2.48), (2.64), *and* $[u]_p$ *as in* (2.63), *such that*

$$|u_h(x) - u(x)| \leqq Ch^{p/2} \quad \forall x \in \bar{D}_h, \quad h \in (0, 1], \tag{2.65}$$

where u_h *is the solution of the discrete* HJB *equation* (2.27) *and* u *is the viscosity solution of the* HJB *equation* (2.53).

Proof. We remark that we are using (2.62) to suppose $[u]_p$ finite and defined as the limit of $[u_\varepsilon]_p$.

First, we will give a proof where the constant C of (2.65) depends on the C_φ, $\varphi = u$ of the convergence property (2.17), i.e., the p-Hölder norm of the second-order derivatives of u. This argument uses implicitly the discrete maximum principle in a way similar to Lions and Mercier [22].

Indeed, let us define the nonlinear resolvent operators

$$R(\varphi) = v \text{ iff } v = 0 \text{ on } \partial D \text{ and } \alpha v = \varphi + \inf\{L(\lambda)v + f(\cdot, \lambda) \colon \lambda \in \Lambda\} \text{ in } D, \tag{2.66}$$

and

$$\begin{aligned} &R_h(\varphi_h) = v_h \text{ iff } v_h = 0 \text{ on } \Gamma_h \text{ and} \\ &\alpha v_h = \varphi_h + \inf\{L_h(\lambda)v_h + f(\cdot, \lambda) \colon \lambda \in \Lambda\} \text{ in } D_h. \end{aligned} \tag{2.67}$$

It is clear that if u_h and u denote the solutions to the HJB equations (2.27) and (2.53), then

$$u_h - u = R_h(0) - R(0) = R_h[R^{-1}(R(0))] - R_h[R_h^{-1}(R(0))],$$

where R^{-1} and R_h^{-1} are the inverse operators. By means of Theorem 2.1, the inequality (2.38) gives

$$\|R_h(\varphi) - R_h(\psi)\| \leqq \alpha^{-1}\|\varphi - \psi\| \quad \forall h \in (0, 1],$$

for any functions φ, ψ and with $\|\cdot\|$ denoting the supremum norm on $\bar{D}_h$. Hence

$$\|u_h - u\| \leqq \alpha^{-1}\|R^{-1}(u) - R_h^{-1}(u)\|.$$

Since we can bound

$$\begin{aligned} |R^{-1}(u) - R_h^{-1}(u)| &= |\inf\{L(\lambda)u + f(\cdot, \lambda) \colon \lambda \in \Lambda\} - \inf\{L_h(\lambda)u + f(\cdot, \lambda) \colon \lambda \in \Lambda\}| \\ &\leqq \sup\{|L(\lambda)u - L_h(\lambda)u| \colon \lambda \in \Lambda\} \leqq C_u h^{p/2}, \end{aligned}$$

where C_u is the constant in (2.17), we deduce the estimate (2.65) with $C = \alpha^{-1}C_u$.

Next, to fully prove (2.65) we will show first that

$$|u(x)| \leqq C[\operatorname{dist}(x, \partial D)]^p \quad \forall x \in \bar{D}, \tag{2.68}$$

$$|u(x) - u(y)| \leqq K|x-y|^p \quad \forall x, y \in \bar{D}, \tag{2.69}$$

for some constants C, K depending only on the various constants of (2.4), (2.9), (2.10), and (2.64). To that effect, we construct a p-Hölder-continuous subsolution $\bar{u}$, i.e., a function $\bar{u}$ satisfying in a weak sense,

$$\begin{aligned} &L(\lambda)\bar{u} \leqq -1 + \beta_p \bar{u} \quad \text{in } D, \quad \bar{u} = 0 \quad \text{on } \partial D, \\ &|\bar{u}(x) - \bar{u}(y)| \leqq K|x-y|^p \quad \forall x, y \in \bar{D}, \end{aligned} \tag{2.70}$$

for some constants $K = K(\bar{u})$, $\beta_p \geqq 0$, as in (2.37). The maximum principle applied to the function

$$w(x) = \pm u(x) - \|f\|\bar{u}(x)$$

yields $w(x) \leqq 0$, i.e., (2.68).

To obtain (2.69) we proceed as in Theorem 2.2. We consider the extended differential operator

$$\begin{aligned} \tilde{L}(\lambda) = \frac{1}{2} \sum_{i,j=1}^{d} \sum_{k=1}^{n} [&\sigma_{ik}(x, \lambda)\sigma_{jk}(x, \lambda)\partial_{ij}^{x} + \sigma_{ik}(x, \lambda)\sigma_{jk}(y, \lambda)\partial_{ij}^{xy} \\ &+ \sigma_{ik}(y, \lambda)\sigma_{jk}(x, \lambda)\partial_{ij}^{yx} + \sigma_{ik}(y, \lambda)\sigma_{jk}(y, \lambda)\partial_{ij}^{y} \\ &+ \sum_{i=1}^{d} [g_i(x, \lambda)\partial_i^x + g_i(y, \lambda)\partial_i^y]], \end{aligned} \tag{2.71}$$

where ∂_{ij}^{x} and ∂_{ij}^{xy} denote the derivatives with respect to x_i, x_j and x_i, y_j, respectively. A simple calculation shows that

$$\tilde{L}(\lambda)m(x-y, p, \varepsilon) \leqq \alpha_p m(x-y, p, \varepsilon) \quad \forall x, y \in \mathcal{R}^d, \quad \lambda \in \Lambda, \tag{2.72}$$

any $\varepsilon > 0$ and with α_p being the constant (2.50). The function $m(\cdot, \cdot, \cdot)$ is given by (2.47).

Now, define the function

$$w(x, y) = u(x) - u(y) - Km(x-y, p, \varepsilon),$$

for some constant $K > 0$ to be selected. Let (x_0, y_0) be a point in $\bar{D} \times \bar{D}$ where $w(x, y)$ attains its maximum values. We want to prove that $w(x_0, y_0) \leqq 0$, which implies (2.69) as ε vanishes. In fact, if either $x_0 \in \partial D$ or $y_0 \in \partial D$, then $w(x_0, y_0) \leqq 0$ provided $K \geqq C(u)$, the constant in (2.68). So, if x_0, y_0 belong to D, then the maximum principle yields $\tilde{L}(\lambda)w(x_0, y_0) \leqq 0$, i.e.,

$$L(\lambda)u(x_0) - L(\lambda)u(y_0) \leqq K\tilde{L}(\lambda)m(x_0 - y_0, p, \varepsilon).$$

In view of the HJB equation (2.53) and the inequality (2.72) we get

$$\begin{aligned} \alpha[u(x_0) - u(y_0)] &\leqq \sup\{|f(x_0, \lambda) - f(y_0, \lambda)| : \lambda \in \Lambda\} \\ &\quad + K \sup\{\tilde{L}(\lambda)m(x_0 - y_0, p, \varepsilon) : \lambda \in \Lambda\} \\ &\leqq [K(f) + \alpha_p K]m(x_0 - y_0, p, \varepsilon), \end{aligned}$$

where $K(f)$ is the constant of hypotheses (2.4). Hence, take $K = (\alpha - \alpha_p)^{-1}K(f)$ to obtain $w(x_0, y_0) \leqq 0$.

Let us prove the estimate (2.65). To that effect, we consider the function

$$w(x, y) = u_h(x) - u(y) - C_1 m(x-y, p, h) - C_2 h^{p/2},$$

for any x, y in $\bar{D}_h$, and some constants $C_1, C_2 > 0$ to be selected. We want to show that at the point (x_0, y_0) in $\bar{D}_h \times \bar{D}_h$ where $w(x, y)$ attains its maximum value, we have $w(x_0, y_0) \leqq 0$, from which (2.65) follows immediately. Indeed if either $x_0 \in \Gamma_h$ or $y_0 \in \Gamma_h$ then $w(x_0, y_0) \leqq 0$, provided $C_1 = \max\{K(u_h), K(u)\}$, the p-Hölder constants given by (2.52) and (2.69). Actually, the constants in (2.39) and (2.68) suffice, i.e., p-Hölder constants near the boundary. When $x_0, y_0 \in D_h$, we can use the discrete maximum principle for the extended operator $\tilde{L}_h(\lambda)$, defined by (2.43), to get $\tilde{L}_h(\lambda)w(x_0, y_0) \leqq 0$, i.e.,

$$L_h(\lambda)u_h(x_0) - L_h(\lambda)u(y_0) \leqq C_1\tilde{L}_h(\lambda)m(x_0 - y_0, p, h).$$

In view of the discrete HJB equation (2.27) we get

$$L_h(\lambda)u_h(x_0) - L_h(\lambda)u(y_0) \geqq \alpha u_h(x_0) - [L(\lambda)u(y) + f(y, \lambda)] + [L(\lambda)u(y) - L_h(\lambda)u(y_0)] + [f(y, \lambda) - f(x_0, \lambda)].$$

Thus, by taking the supremum over λ in Λ and y such that

$$|y - y_0| \leqq [u]_p \sqrt{h},$$

we deduce

$$\alpha[u_h(x_0) - u(y_0)] - [u]_p h^{p/2} - [\alpha K(u) + K(f)](1 + [u]_p^p) m(x_0 - y_0, p, h) \leqq \alpha_p(h) C_1 m(x_0 - y_0, p, h),$$

after using the definition (2.46) and (2.63), and the p-Hölder constants $K(f)$, $K(u)$ of (2.4), (2.69). Since we need only to show (2.65) for $h > 0$ sufficiently small, the hypothesis (2.64) and the inequality (2.49) permit us to choose

$$C_1 \geqq [\alpha - \alpha_p(h)]^{-1}(1 + [u]_p^p)[\alpha K(u) + K(f)],$$

$$C_2 = \alpha^{-1}[u]_p,$$

in order to have $w(x_0, y_0) \leqq 0$, i.e.,

$$u_h(x) - u(y) \leqq Cm(x - y, p, h) \quad \forall x, y \in \mathscr{R}_h^d, \quad h \in (0, 1],$$

where $C = a + C_2$; this implies one side of (2.65).

By symmetry we obtain (2.65) after using the estimate (2.52) of Theorem 2.2. □

2.4. Final comments. Sometimes we need to discretize the set Λ, where the control takes values. In this case, a new term of the form

$$\sup\{\inf\{|l(x, \lambda) - l(x, \lambda')| : \lambda' \in \Lambda(h)\} : x \in \mathscr{R}^d, \lambda \in \Lambda\}, \tag{2.73}$$

for $l = f, g_i, \sigma_{ik}, i = 1, \cdots, d, k = 1, \cdots, n$, will appear in the right-hand side of the estimate (2.65). Here $\Lambda(h)$ is a discretization of Λ. Also, the constant α could be a function $\alpha(x, \lambda)$, for which the preceding results extend. If the domain D is unbounded and the data f has polynomial growth, then the solutions $u(x)$, $u_h(x)$ will have also polynomial growth, and some weight function is needed to obtain an estimate similar to (2.65) (cf. [24]).

We may be interested in the performance of the optimal control of the discrete problem, when suitably extended and applied to the initial problem. That issue is not considered here. However, the optimizer will face the problem of actually computing $u_h(x)$. In general, only an approximation $\tilde{u}_h(x)$ is computed and from that, a control policy $\tilde{\lambda}_h(\cdot)$ is derived. This $\tilde{\lambda}_h(\cdot)$ allows us to simulate a trajectory $\tilde{y}_h(\cdot)$. To this policy $(\tilde{\lambda}_h(\cdot), \tilde{y}_h(\cdot))$ a new cost $\hat{u}_h(x)$ is associated. Then, starting from (2.65) we need really to estimate $|\tilde{u}_h(x) - \hat{u}_h(x)|$. Again, this issue is not addressed here.

As mentioned in the theorems, the assumption on uniform ellipticity (2.57) is not required, at least explicitly. For instance, the case of a one-dimensional Brownian motion can be considered. This includes the control of a one-dimensional ordinary equation of order n, perturbed by a white noise.

We have assumed (2.10) for simplicity and to have the Dirichlet condition on the whole boundary ∂D. However, we need only to correctly identify the part Γ of the boundary where the diffusion process exists, and then we can use the technique described in this paper. This requires supposing that the operator $L(\lambda)$ is degenerate with constant order of degeneration, i.e., (2.21).

From the practical point of view, the estimate $h^{1/2}$ is not relevant, since better results are usually expected. However, this gives a precise relation between the grid for the space variable and the control variable, when $\Lambda(h)$ is used. The constant $\alpha_p(h)$ defined by (2.46) plays an important role in the stability of the numerical schemes. This is not obtained in classical schemes.

Perhaps the most interesting part is the fact that the finite difference operator (0.6) does not require any condition of "stability." It is stable in nature, and most estimates valid for the differential operator (0.5) have an equivalent in the discrete case.

In Bancora-Imbert, Chow, and Menaldi [1], the numerical solution of an optimal correction problem for a damped random linear oscillator is studied. The HJB equation takes the form of a variational inequality, namely,

$$\begin{aligned} &\partial_t u + Lu \geqq 0 \quad \text{in } \mathcal{R}_2 \times [0, T), \qquad -c \leqq \partial_2 u \leqq c \quad \text{in } \mathcal{R}_2 \times [0, T), \\ &(\partial_t u + Lu)(\partial_2 u + c)(\partial_2 u - c) = 0 \quad \text{in } \mathcal{R}_2 \times [0, T), \\ &u(\cdot, T) = f \quad \text{in } \mathcal{R}_2, \end{aligned} \tag{2.74}$$

where the differential operator is given by

$$Lu(x_1, x_2, t) = \tfrac{1}{2} r^2 \partial_2^2 u(x_1, x_2, t) - (p x_2 + q^2 x_1) \partial_2 u(x_1, x_2, t) + x_2 u(x_1, x_2, t),$$

and r, p, q, c are constants; $r, q, c > 0$; and f is a given function. A precise algorithm is described and used there. The solution of the discrete problem is found as the common limit of two sequences, one decreasing and the other increasing. This allows us to bound the error and to give an almost optimal policy. We refer also to Sun and Menaldi [37], [25]. Note that in the case of (2.74), the solution u is Lipschitz-continuous together with its second derivative in the x_2 variable.

In a subsequent paper, the (quasi-) variational inequalities will be studied. It is well known that for those problems the solution is not smooth, i.e., the second derivative must have a jump. For that reason, only the second approach of Theorem 2.3, i.e., using $[u]_p$, seems to be appropriate. Perhaps a combination with finite elements of the type used by Menaldi and Rofman [26], [23] could be of some help.

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