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New Characterizations Of Sobolev Spaces On Heisenberg And Carnot Groups And High Order Sobolev Spaces On Euclean Spaces

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NEW CHARACTERIZATIONS OF SOBOLEV SPACES ON HEISENBERG AND CARNOT GROUPS AND HIGH ORDER SOBOLEV SPACES ON EUCLIDEAN SPACES

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DEDICATION

To my family

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CHAPTER 1 INTRODUCTION

1.1 Background and Main Issues

This dissertation focuses on new characterizations of Sobolev spaces . It encompasses an in-depth study of Sobolev spaces on Heisenberg groups, as well as Carnot groups, second order and high order Sobolev spaces on Euclidean spaces. In this introductory chapter, we present the motivation of the study, outline of our approaches, and certain results.

Roughly speaking, a Sobolev space is a space containing functions with adequately many derivatives and endowed with a norm that judges both the size and regularity of a function. Here, the derivatives are in a suitable weak sense to form the space as a Banach space. There are many ways to define a Sobolev space: by the delicate and abstract theory of distributions; by consideration as the natural expansion of monotone, absolutely continuous and BV functions; etc. The most common one is to describe a Sobolev space $W^{k,p}$ as a vector space of functions equipped with a norm that is a combination of L^p-norms of the function itself as well as its weak derivatives up to a given order k.

Since Sobolev spaces appear and play important roles in many branches of modern mathematics such as partial differential equations, calculus of variations, differential geometry, complex analysis, probability theory, optimization and control theory, etc, there has been a substantial effort to characterize Sobolev spaces in many different directions. In his paper [19], Gagliardo used the semi-norm

$$|f|_{W^{s,p}(\Omega)} = (\int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{N+sp}} dxdy)^{1/p},\ p>1.$$

to characterize functions in $W^{s,p}$. However, when $s \to 1^-$, we have that $|f|_{W^{s,p}(\Omega)}$ does not converge to the semi-norm

$$|f|_{W^{1,p}(\Omega)} = (\int_\Omega |\nabla f(x)|^p dx)^{1/p}.$$

In their recent paper [5], Bourgain, Brezis and Mironescu investiaged this situation and found that the scaling $(1-s)^{1/p}$ in front of $|f|_{W^{s,p}(\Omega)}$ reclaims the question.This discovery drove them to a new characterization of the Sobolev space. Indeed, Bourgain, Brezis and Mironescu proved that

Theorem 1.1. Let $\Omega \subset \mathbb{R}^N$ be a smooth, bounded domain, $1 < p < \infty$, ρ_n be a sequence of radial mollifiers in $\mathbb{R}^N$ and $f \in L^p(\Omega)$. Then

$$\lim_{n\to\infty} \int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^p} \rho_n(x-y)\, dxdy = K_{p,N} |f|_{W^{1,p}(\Omega)} \tag{1.1}$$

with the convention that $|f|_{W^{1,p}(\Omega)} = \infty$ if $f \notin W^{1,p}(\Omega)$. Here $K_{p,N}$ depends only on p and N.

We note here that ρ_n is a sequence of radial mollifiers in $\mathbb{R}^N$, i.e.

$$\rho_n(x) = \rho_n(|x|), \ \ \rho_n \geq 0, \ \int_{\mathbb{R}^N} \rho_n(x) dx = 1$$

and

$$\lim_{n\to\infty} \int_\delta^\infty \rho_n(r)\, r^{N-1} dr = 0 \text{ for every } \delta > 0.$$

The case $p = 1$ is more elegant. Indeed, it's interesting to note that though the left hand side of (1.1) is finite, we may still not have $f \in W^{1,1}(\Omega)$. Nevertheless, if $f \in W^{1,1}(\Omega)$, then (1.1) holds. In fact, the following result was showed in [5]:

Theorem 1.2. Let $\Omega \subset \mathbb{R}^N$ be a smooth, bounded domain and ρ_n be a sequence of radial mollifiers in $\mathbb{R}^N$. Then there exists constants $C_1,\ C_2 > 0$ such that for every $f \in L^1(\Omega)$:

$$\begin{aligned} C_1 |Df|(\Omega) &\leq \liminf_{n\to\infty} \int_\Omega \int_\Omega \frac{|f(x)-f(y)|}{|x-y|} \rho_n(x-y)\, dxdy \\ &\leq \limsup_{n\to\infty} \int_\Omega \int_\Omega \frac{|f(x)-f(y)|}{|x-y|} \rho_n(x-y)\, dxdy \leq C_2 |Df|(\Omega) \end{aligned} \tag{1.2}$$

where $|Df|(\Omega)$ is the total variation of the measure Df, the distributional derivative of f, and $|Df|(\Omega) = \infty$ if $f \notin BV(\Omega)$.

Thus the BV semi-norm is really the limiting case of the characterization of $W^{1,p}(\Omega)$ as $p \to 1$. The fact that $C_1 = C_2 = K_{1,N}$ was proved by Bourgain, Brezis and Mironescu in one dimensional case, and by Dávila in [14]. This can be stated as follows:

Theorem 1.3. Let $\Omega \subset \mathbb{R}^N$ be an open, bounded domain with Lipschitz boundary and ρ_n be a sequence of radial mollifiers in $\mathbb{R}^N$. Then for $f \in L^1(\Omega)$:

$$\lim_{n\to\infty}\int_\Omega\int_\Omega \frac{|f(x)-f(y)|}{|x-y|}\rho_n(x-y)\,dxdy = K_{1,N}\,|Df|(\Omega) \tag{1.3}$$

where $|Df|(\Omega)=\infty$ if $f \notin BV(\Omega)$.

Thus the BV semi-norm is really the limiting case of the characterization of $W^{1,p}(\Omega)$ as $p \to 1$.

We can see that the above results contribute to the new characterizations of the Sobolev spaces $W^{1,p}(\Omega)$, $1<p<\infty$, and the space $BV(\Omega)$ when Ω is a smooth domain. We note here that there are statements similar to those in Theorem 1.1 and Theorem 1.3 when the smooth bounded domain Ω is replaced by $\mathbb{R}^N$. However, these characterizations are in general false for arbitrary open and bounded sets, by a construction of Brezis in [7].

In 2006, H-M. Nguyen [34] studied some new characterizations of Sobolev spaces that are closely related to those of Bourgain, Brezis and Mironescu [5]. Indeed, the following two theorems were conjectured by Brezis and confirmed in [34]:

Theorem 1.4. Let $1<p<\infty$.Then

(a) There exists a constant $C_{N,p}$ depending only on N and p such that

$$\iint_{|g(x)-g(y)|>\delta} \frac{\delta^p}{|x-y|^{N+p}}dxdy \le C_{N,p}\int_\Omega |\nabla g(x)|^p\,dx,\ \forall\delta>0,\ g\in W^{1,p}\left(\mathbb{R}^N\right)$$

(b) If $g \in L^p\left(\mathbb{R}^N\right)$ satisfies

$$\sup_{0<\delta<1}\iint_{|g(x)-g(y)|>\delta} \frac{\delta^p}{|x-y|^{N+p}}dxdy < \infty$$

then $g \in W^{1,p}\left(\mathbb{R}^N\right)$.

(c) Moreover, for any $g \in W^{1,p}\left(\mathbb{R}^N\right)$,

$$\lim_{\delta\to 0}\iint_{|g(x)-g(y)|>\delta} \frac{\delta^p}{|x-y|^{N+p}}dxdy = \frac{1}{p}K_{N,p}\int_{\mathbb{R}^N} |\nabla g(x)|^p\,dx$$

where

$$K_{N,p} = \int_{\mathbb{S}^{N-1}} |e\cdot\sigma|^p\,d\sigma \text{ for any } e\in\mathbb{S}^{N-1}.$$

Theorem 1.5. Let $1<p<\infty$.Then

(a) There exists a constant $C_{N,p}$ depending only on N and p such that

$$\sup_{0<\varepsilon<1} \iint_{|g(x)-g(y)|\leq 1} \frac{\varepsilon\left|g(x)-g(y)\right|^{p+\varepsilon}}{\left|x-y\right|^{N+p}} dxdy + \iint_{|g(x)-g(y)|>1} \frac{1}{\left|x-y\right|^{N+p}} dxdy$$

$$\leq C_{N,p} \int_{\mathbb{R}^N} \left|\nabla g\left(x\right)\right|^p dx,\ \forall g \in W^{1,p}\left(\mathbb{R}^N\right).$$

(b) If $g \in L^p\left(\mathbb{R}^N\right)$ satisfies

$$\sup_{0<\varepsilon<1} \iint_{|g(x)-g(y)|\leq 1} \frac{\varepsilon\left|g(x)-g(y)\right|^{p+\varepsilon}}{\left|x-y\right|^{N+p}} dxdy + \iint_{|g(x)-g(y)|>1} \frac{1}{\left|x-y\right|^{N+p}} d < \infty$$

then $g \in W^{1,p}\left(\mathbb{R}^N\right)$.

(c) Moreover, for any $g \in W^{1,p}\left(\mathbb{R}^N\right)$,

$$\lim_{\varepsilon\to 0} \iint_{|g(x)-g(y)|\leq 1} \frac{\varepsilon\left|g(x)-g(y)\right|^{p+\varepsilon}}{\left|x-y\right|^{N+p}} dxdy = K_{N,p} \int_{\mathbb{R}^N} \left|\nabla g\left(x\right)\right|^p dx.$$

Moreover, the delicate case $p=1$, the case of bounded domain with smooth boundary and other variants and generalizations were also demonstrated in [34]. These results were generalized further by Bourgain and H-M. Nguyen [6] and H-M. Nguyen [35].

Attempting to achieve similar characterizations for higher order Sobolev spaces, Borghol studied in [8] some properties of high order Sobolev spaces $W^{k,p}\left(\Omega\right)$ where $1<p<\infty$ and $BV^k\left(\Omega\right)$ when $p=1$ where $k\geq 2$ and Ω is a bounded smooth domain of $\mathbb{R}^N$ in the spirit of Bourgain, Brezis and Mironlescu [5] using $k-th$ differences. More precisely, Borghol verified that

Theorem 1.6. Assume Ω is a convex smooth domain. Let $1<p<\infty$ and let f be a $p-$integrable function in Ω. Then $f \in W^{k,p}\left(\Omega\right)$ iff

$$\liminf_{\varepsilon\to 0} \int_\Omega \int_\Omega \frac{\left|\sum_{j=0}^{k} (-1)^j \binom{k}{j} f\left(\frac{k-j}{k}x + \frac{j}{k}y\right)\right|^p}{\left|x-y\right|^{kp}} \rho_\varepsilon\left(x-y\right) dxdy < \infty.$$

Moreover, when $f \in W^{k,p}\left(\Omega\right)$, we have

$$\lim_{\varepsilon\to 0} \int_\Omega \int_\Omega \frac{\left|\sum_{j=0}^{k} (-1)^j \binom{k}{j} f\left(\frac{k-j}{k}x + \frac{j}{k}y\right)\right|^p}{\left|x-y\right|^{kp}} \rho_\varepsilon\left(x-y\right) dxdy = \int_\Omega \left(\int_{\mathbb{S}^{N-1}} \left|D^k f(x)\left(\sigma,...,\sigma\right)\right|^p d\sigma\right) dx.$$

Theorem 1.7. Assume Ω is a convex smooth domain and let $f \in L^1(\Omega)$. Then $f \in BV^k(\Omega)$ iff

$$\liminf_{\varepsilon\to 0} \int_\Omega \int_\Omega \frac{\left| \sum_{j=0}^{k} (-1)^j \binom{k}{j} f\left(\frac{k-j}{k}x + \frac{j}{k}y\right) \right|}{|x-y|^k} \rho_\varepsilon(x-y)\, dxdy < \infty,$$

and in this case we have

$$\lim_{\varepsilon\to 0} \int_\Omega \int_\Omega \frac{\left| \sum_{j=0}^{k} (-1)^j \binom{k}{j} f\left(\frac{k-j}{k}x + \frac{j}{k}y\right) \right|}{|x-y|^k} \rho_\varepsilon(x-y)\, dxdy = \frac{1}{|\mathbb{S}^{N-1}|} \int_{\mathbb{S}^{N-1}} \left(\int_\Omega \left| D^k f(x)(\sigma, ..., \sigma) \right| dx \right) d\sigma.$$

In 2011, Bojarski, Ihnatsyeva and Kinnunen [4] used a Taylor $(k-1)-$remainder to consider the higher order Sobolev spaces in the sense of Bourgain, Brezis and Mironlescu [5]. Indeed, if we denote

$$T_y^k f(x) = \sum_{|\alpha| \le k} D^\alpha f(y) \frac{(x-y)^\alpha}{\alpha!}$$

$$R^k f(x,y) = f(x) - T_y^k f(x)$$

then Bojarski, Ihnatsyeva and Kinnunen could set up the following results:

Theorem 1.8. Let Ω be an open set in $\mathbb{R}^N$, $1 < p < \infty$, k be a positive integer and let $f \in W^{k-1,p}(\Omega)$. Then $f \in W^{k,p}(\Omega)$ iff

$$\liminf_{\varepsilon\to 0} \int_\Omega \int_\Omega \frac{\left|R^{k-1} f(x,y)\right|^p}{|x-y|^{kp}} \rho_\varepsilon(x-y)\, dxdy < \infty,$$

and in this case

$$\lim_{\varepsilon\to 0} \int_\Omega \int_\Omega \frac{\left|R^{k-1} f(x,y)\right|^p}{|x-y|^{kp}} \rho_\varepsilon(x-y)\, dxdy$$
$$= \int_\Omega \int_{\partial B(0,1)} \left| \sum_{|\alpha|=k} \frac{D^\alpha f(x)}{\alpha!} v^\alpha \right|^p dvdx.$$

We note here that the pointwise characterization of the Sobolev space in the spirit of Hajlasz [20] was made by Bojarski [2].

There have also been definitions and characterizations of Sobolev spaces in metric measure spaces (namely, homogeneous spaces in the sense of Coifman-Weiss [11]). To describe this, we need to introduce some preliminaries on metric spaces.

Let $(\mathcal{S}, \rho, \mu)$ be a metric space with a metric ρ and a doubling measure μ, namely, for all $x, y, z \in \mathcal{S}$, ρ satisfies

$$\rho(x,y) \leq \rho(x,z) + \rho(z,y),$$

and the measure μ satisfies the condition

$$\mu(B(x,2r)) \leq A_\mu \mu(B(x,r)), \qquad x \in \mathcal{S},\ r > 0,$$

for an absolute constant A_μ, where by definition $B(x,r) = \{y \in \mathcal{S} : \rho(x,y) < r\}$, and $\mu(B(x,r))$ denotes the μ-measure of $B(x,r)$. Such a metric space is usually called as a metric space of homogeneous type in the sense of Coifman and Weiss. As usual, we refer to $B(x,r)$ as the ball with center x and radius r, and, if B is a ball, we write x_B for its center, $r(B)$ for its radius and cB for the ball of radius $cr(B)$ having the same center as B. We always assume that $(\mathcal{S}, \rho)$ is locally compact and μ is doubling.

A notion of first order Sobolev space in an metric space using the pointwise estimates was first given by Hajlasz [20]. Another definition of Sobolev spaces of first order on metric spaces using the Poincaré inequalities seemed to have first appeared in the work by Franchi, Lu and Wheeden in [17]. This is in turn based on the equivalence of the representation formulae (pointwise estimates) and the Poincaré inequalities established in [17]. In the article by Lu and Wheeden [30], the authors introduce the notion of polynomials in metric spaces $(\mathcal{S}, \rho, \mu)$ and prove a relationship between high order Poincaré inequalities and representation formulas involving fractional integrals of high order (see also [32]). Motivated by this and the definitions of Sobolev spaces of first order in metric spaces, Liu, Lu and Wheeden define in [25] Sobolev spaces of high order in such metric spaces $(\mathcal{S}, \rho, \mu)$. They prove that several definitions are equivalent if functions of polynomial type exist. In the case of stratified groups, where polynomials do exist and high order Poincaré inequalities do hold (see [27], [28], [32], [10]), we show that our spaces are indeed equivalent to the Sobolev spaces defined by Folland and Stein in [15]. Our results also give some alternate definitions of Sobolev spaces in the classical Euclidean spaces and the relevant notions of Sobolev spaces or gradients in metric spaces in Hajlasz [20], Cheeger [9], Korevaar and Schoen [23], Franchi, Lu and Wheeden [17], Franchi, Hajlasz and Koskela [16], Heinonen and Koskela [21] to high order Sobolev spaces.

To explain in more details what we have described above on Sobolev spaces on metric spaces, we now define what we will mean by polynomial functions on $\mathcal{S}$ following [30] (see [30], [25] for more details of the exposition).

Let $(\mathcal{S}, \rho, \mu)$ be a metric space of homogeneous type, and let Ω be an open set in $\mathcal{S}$. Our results rely on the existence of a linear class of functions $P(x)$ (called polynomial functions) which satisfy both

(P1) For every metric ball $D \subset \Omega$,

$$\text{ess sup}_{x \in D} |P(x)| \leq \frac{C_1(\mu)}{\mu(D)} \int_D |P(y)| d\mu(y),$$

where the essential supremum is taken with respect to μ;

(P2) If D is any metric ball in Ω and E is a subball of D with $\mu(E) > \gamma\mu(D)$, $\gamma > 0$, then

$$\| P \|_{L^\infty_\mu(E)} \geq C_2(\gamma, \mu) \| P \|_{L^\infty_\mu(B)} .$$

In stratified groups, including standard Euclidean space, (P1) and (P2) are known to be true for polynomials in those settings (see [15] and [27]), with constants $C_1(\mu), C_2(\gamma, \mu)$ which depend additionally only on the degree of the polynomial.

The role of degree of a polynomial is replaced by an exponent which measures the order of smoothness of a given locally integrable function f in the sense of assuming that the following Poincaré estimate holds for f and a positive integer k: there exist $q \geq 1$ and a function g such that for every ball $B \subset \Omega$ and some function $P_k(B, f)$,

$$\frac{1}{\mu(B)} \int_B |f - P_k(B, f)| \, d\mu \leq C r(B)^k \left(\frac{1}{\mu(B)} \int_B |g|^q d\mu \right)^{1/q}, \tag{1.4}$$

with C independent of B. $P_k(B, f)$ in (1.4) may be thought of as an approximation to f. In (1.4), g and q are allowed to depend on f but not on B. The function $P_k(B, f)$ may also depend on g, q

and μ, and we sometimes write $P_k(B,f) = P_k(B,f,g,q,\mu)$. The important assumption for us will be that (P1) and (P2) hold for a linear class that contains $P_k(B,f)$ for all $B \subset \Omega$, with constants $C_1(\mu), C_2(\gamma,\mu)$ depending additionally only on k.

When (1.4) holds for a given f, we will say that f satisfies an L^q to L^1 Poincaré inequality of order k for every ball $B \subset \Omega$. For a stratified group, every smooth function f satisfies (1.4) with $q = 1$ and $g = |X^k f|$ for several choices of polynomials of degree $k-1$ by the results in [27], [28].

Thus, using the notion of polynomials, several equivalent definitions of high order Sobolev classes on a domain Ω in a metric space $(\mathcal{S},\rho,\mu)$ with a doubling measure μ are given in [25]. We will use $⨍_E f(x)d\mu(x)$ to denote $\frac{1}{\mu(E)}\int_E f(x)d\mu(x)$, and $||f||_{L^p_\mu(E)}$ to denote the L^p norm of f on E with respect to μ. We now recall two definitions of Sobolev classes given in [25].

Definition 1.9. Given a positive integer m and $1 < p < \infty$, we define the Sobolev class $A^{m,p}(\Omega)$ to be the set of functions $f \in L^p(\Omega)$ so that for each $k = 1, \cdots, m$, there exist q_k with $1 \le q_k < p$, functions $g_k(x)$ with $0 \le g_k \in L^p(\Omega)$, and polynomials $P_k(B,f)$ with

$$⨍_B |f(x) - P_k(B,f)(x)|d\mu(x) \le r(B)^k \left(⨍_B g_k^{q_k}(x)d\mu(x)\right)^{1/q_k} \tag{1.5}$$

for every ball $B \subset \Omega$. The polynomials $P_k(B,f)$ are assumed to belong to a linear class which satisfies (P1) and (P2) with constants depending only on k,γ,μ. If $f \in A^{m,p}(\Omega)$, we define

$$||f||_{A^{m,p}(\Omega)} = ||f||_{L^p(\Omega)} + \inf_{\{g_k\}} \sum_{k=1}^m ||g_k||_{L^p(\Omega)},$$

where the infimum is taken over all sequences such that (1.5) holds for f for $k = 1,\ldots,m$.

It is easy to see that $A^{m,p}(\Omega)$ is a linear space; moreover, $||\cdot||_{A^{m,p}(\Omega)}$ is a norm if all $q_k = 1$.

Definition 1.10. Given a positive integer m and $1 < p < \infty$, we define the Sobolev class $B^{m,p}(\Omega)$ to be the set of functions $f \in L^p(\Omega)$ so that for each $k = 1, \cdots, m$ there exist functions $0 \le g_k \in L^p(\Omega)$ and polynomials $P_k(B,f)$ such that

$$|f(x) - P_k(B,f)(x)| \le r(B)^k g_k(x) \tag{1.6}$$

for $\mu-$a.e. $x \in B$ for every metric ball $B \subset \Omega$. The polynomials $P_k(B,f)$ are assumed to belong to a linear class which satisfies (P1) and (P2) with constants depending only on k, γ, μ. If $f \in B^{m,p}(\Omega)$, let

$$||f||_{B^{m,p}(\Omega)} = ||f||_{L^p(\Omega)} + \inf_{\{g_k\}} \sum_{k=1}^{m} ||g_k||_{L^p(\Omega)}.$$

The class $B^{m,p}(\Omega)$ is a Banach space with norm $|| \cdot ||_{B^{m,p}}$.

Then by a result of [25], we have

Theorem 1.11. Definition 1.9 and Definition 1.10 are equivalent.

In the spirit of H-M. Nguyen's paper [34], I will establish the similar results on Heisenberg groups and Carnot groups in this thesis.

1.2 Preliminary of Heisenberg group and Carnot group

Let $\mathbb{H} = \mathbb{C}^n \times \mathbb{R}$ be the n-dimensional Heisenberg group whose group structure is given by

$$(z, t) \cdot (z', t') = (z + z', t + t' + 2Im(z \cdot \overline{z'})),$$

for any two points (z, t) and (z', t') in $\mathbb{H}$. The Lie algebra of $\mathbb{H}$ is generated by the left invariant vector fileds

$$T = \frac{\partial}{\partial t}, X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}, Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}$$

for $i = 1, \ldots n$. These generators satisfy the non-commutative relationship

$$[X_i, Y_j] = -4\delta_{ij}T.$$

with the Lie bracket given by the commutator

$$([X, Y]f)(x) \doteq (X(Yf))(x) - (Y(Xf))(x),$$

where X, Y are lie algebras onHeisenberg groups. Moreover, all the commutators of length greater than two vanish, and thus this is a nilpotent, graded, and stratified group of step two.

For each real number $r \in \mathbb{R}$, there is a dilation naturally associated with Heisenberg group structure which is usually denoted as

$$\delta_r(z, t) = (rz, r^2 t)$$

However, for simplicity we will write ru to denote $\delta_r u$. The Jacobian determinant of δ_r is r^Q, where

$Q = 2n + 2$ is the homogeneous dimension of $\mathbb{H}$.

We use $\xi = (z, t)$ to denote any point $(z, t) \in \mathbb{H}$ and $\rho(\xi) = (|z|^4 + t^2)^{\frac{1}{4}}$ to denote the homogeneous norm of $\xi \in \mathbb{H}$. With this norm, we can define a Heisenberg ball centered at $\xi = (z, t)$ with radius $R : B(\xi, R) = \{v \in \mathbb{H} : |\xi^{-1}v| < R\}$. The volume of such a ball is $\sigma_Q = C_Q R^Q$ for some constant depending on Q. We also use $\nabla_{\mathbb{H}} f$ to express the horizontal subgradient of the function $f : \mathbb{H} \to \mathbb{R}$:

$$\nabla_{\mathbb{H}} f\,(z, t) = \sum_{j=1}^{n} (X_j f) X_j + Y_j f Y_j)$$

Let Ω be an open set in $\mathbb{H}$. We use $W_0^{1,p}(\Omega)$ to denote the completion of $C_0^\infty(\Omega)$ under the norm $\|f\|_{W^{1,p}(\Omega)} = (\int_\Omega (|\nabla_{\mathbb{H}} f|^p + |f|^p) du)^{1/p}$.

In fact, Heisenberg group is an explicit example of Carnot groups.

Definition 1.12. A **Carnot group** $\mathbb{G}$ is a simply connected Lie group with a stratified Lie algebra $\mathfrak{g}$, i.e. a nilpotent Lie algebra that can be decomposed into a direct sum of linear subspaces called layers,$\{W_j\}_{j=1}^k$, in such a way that the first layer generates the whole algebra:

$$\mathfrak{g} = \bigoplus_{j=1} W_j, W_{j+1} = [W_1, W_j], \forall j < k, W_j = 0, \forall j > k.$$

The postive integer k is called the step of $\mathfrak{g}$.

We now introduce some basic properties of such groups.

Definition 1.13. let $\gamma_{X,x}(t)$ the integral curve of $X \in \mathfrak{g}$ passing through $x \in \mathbb{G}$

$$\gamma'_{X,x}(t) = X(\gamma_{X,x}(t))$$

$$\gamma_{X,x}(0) = x.$$

Then we define the **exponential map** on $\mathfrak{g}$ as

$$\exp(X) \doteq \gamma_{X,\mathfrak{e}}(1) \in \mathbb{G}$$

and it is a local diffeomorphism of a neighborhood of $0 \in \mathfrak{g}$ to a neighborhood of $\mathfrak{e} \in \mathbb{G}$:$d\exp(0) = \mathbb{1}_{\mathfrak{g}}$,which extends to a global diffeomorphism if $\mathbb{G}$ is simply connected.

Definition 1.14. Given a stratified Lie algebra $\mathfrak{g}$, we can define algebra **dilations** as a 1-parameter group $\{\Delta_\lambda\}_{\lambda>0}$ of automorphisms of $\mathfrak{g}$ which act on layers as

$$\Delta_\lambda(X) = \lambda^j X, \forall X \in W_j.$$

We can deduce algebra dilations on $\mathbb{G}$ by the exponential mapping. We call a 1-parameter group of automorphisms of $\mathbb{G}$ group dilations$\{\delta_\lambda\}_{\lambda>0}$:

$$\delta_\lambda(\delta_\mu(x)) = \delta_{\lambda\mu}(x), \delta_\lambda(x \cdot y) = \delta_\lambda(x) \cdot \delta_\lambda(y),\ \forall x, y \in \mathbb{G},\ \forall \lambda, \mu > 0.$$

The quantity $Q = \sum j \dim(W_j)$ is called the homogeneous dimension of $\mathbb{G}$.

IDENTIFICATION WITH $\mathbb{R}^N$: we can identify $\mathbb{G}$ with $\mathbb{R}^N$ endowed with the induced group law since the exponential map is a global diffeomorphism. If we set $m_j = \dim(W_j)$, then $\sum m_j = N$, which differs from Q if $\mathfrak{g}$ has more than one layer. Given a basis $\{X_1, ..., X_{m_1}, X_{m_1+1}, ..., X_{m_1+m_2}, ..., X_N\}$ of $\mathfrak{g}$, the action of group dilations reads

$$\delta_\lambda(x_1, ..., x_N) = (\lambda x_1, ..., \lambda x_{m_1}, \lambda^2 x_{m_1+1}, ..., \lambda^2 x_{m_1+m_2}, ..., \lambda^k x_N).$$

SUB-RIEMANNIAN STRUCTURE: the first layer of the algebra $\mathfrak{g}$ can be identified with a linear subspace of the tangent space of $\mathbb{G}$ at the origin and defines, by left invarients, a canonical subbundle $H\mathbb{G}$ of the tangent bundle $T\mathbb{G}$, whose fibers at point $x \in \mathbb{G}$ will be denoted by $H_x\mathbb{G}$.

Definition 1.15. For a smooth function $f : \mathbb{G} \to \mathbb{R}$, we define its **horizontal gradient** as

$$\nabla_X f = \sum_{j=1}^{m_1} (X_j f) X_j$$

The map $\nabla_X f$ defines a horizontal vector field. The coordinates of $\nabla_X f(x)$ in $H_x\mathbb{G}$ with respect to the chosen orthonomal basis $\{X_1(x), \cdots, X_{m_1}(x)\}$ are given by $\{X_1 f(x), \cdots, X_{m_1} f(x)\}$

Definition 1.16. A **homogeneous norm** is a continuous function $|\cdot|_{\mathbb{G}} : \mathbb{G} \to \mathbb{R}_+$ with the following properties:

$$(1)|x|_{\mathbb{G}} = 0 \Leftrightarrow x = 0;$$

$$(2)|x^{-1}|_{\mathbb{G}} = |x|_{\mathbb{G}};$$

$$(3)|\delta_\lambda(x)|_{\mathbb{G}} = \lambda|x|_{\mathbb{G}}.$$

A homogeneous norm induces a left-invarient homogeneous distance by

$$d(x,y) = |y^{-1} \cdot x|_{\mathbb{G}}$$

with $x, y \in \mathbb{G}$. We define the ball centered at x with radius r in Carnot group $\mathbb{G}$ by $B(x,r) = \{y \in \mathbb{G} \textit{ such that } d(x,y) < r\}$.

The following corollary states three basic properties of the measure under changes of variables.

Corollary 1.17. Let $\mathbb{G}$ be a Carnot group and $f : \mathbb{G} \to \mathbb{R}$ an integrable function on $\mathbb{G}$, then the Harr measure of $\mathbb{G}$

1 . is invariant under left and right translations:

$$\int_{\mathbb{G}} f(y \cdot x)dx = \int_{\mathbb{G}} f(x \cdot y)dx = \int_{\mathbb{G}} f(x)dx \ \forall y \in \mathbb{G}$$

2 . scales under group dilations by the homogeneous dimension of $\mathbb{G}$:

$$\int_{\mathbb{G}} f(x)dx = \lambda^Q \int_{\mathbb{G}} f(x)dx \ \forall \lambda > 0;$$

3 . is affected by $\mathbb{G}$-changes of basis as the Lebesgue measure on $(\mathbb{R}^N, +)$, i.e. putting $\xi = \phi^A(x) = (Ax', x'')$with $x' \in \mathbb{R}^{m_1}$ we have:

$$\int_{\mathbb{G}} f(x)dx = \int_{\mathbb{G}} f(\xi)|detA|d\xi \ \forall A \in GL(m_1, \mathbb{R}).$$

Definition 1.18. A **continuous function** $f : \mathbb{G} \to \mathbb{R}$ is said to be in $\mathfrak{C}^1_{\mathbb{G}}(\mathbb{G}, \mathbb{R})$ if $X_j f : \mathbb{G} \to \mathbb{R}$ exist and are continuous for $j = 1, \cdots, m_1$.

Definition 1.19. (Sobolev space $W^{1,p}_{\mathbb{G}}(\mathbb{G})$) Let $\mathbb{G}$ be a Carnot group and let $p \geq 1$ be fixed. We denote by $W^{1,p}_{\mathbb{G}}(\mathbb{G})$ the set of functions $f \in L^p(\mathbb{G})$ such that $X_j f \in L^p(\mathbb{G})$ for $j = 1, \cdots, m_1$, endowed with the norm

$$\|f\|_{W^{1,p}(\mathbb{G})} = (\int_{\mathbb{G}} |f(x)|^p dx)^{1/p} + (\int_{\mathbb{G}} |\nabla_X f(x)|_X^p dx)^{1/p}.$$

In particular, the classical density result of smooth functions still holds, see [18].

1.3 Outline of the Dissertation

The rest of the dissertation is arranged as follows. In Chapter 2, the new characterizations of Sobolev spaces on Heisenberg groups are studied. We consider the case for $p > 1$ and $p = 1$. It's worthy to note that one of the main techniques in the proof of Theorem 1.4 is to use the uniformity in every directions of the unit sphere in the Euclidean spaces. More precisely, to deal with the general case $\sigma \in \mathbb{S}^{N-1}$, it is often to be assumed that $\sigma = e_N = (0, ..0, 1)$ and hence, one just needs to work on 1-dimensional case. This can be done by using the rotation in the Euclidean spaces. In the case of Heisenberg groups, this type of property is not available because of the structure of the Heisenberg groups, in particular, the dilation. Hence, we need to find a different approach to these characterizations.

In Chapter 3, we study the new characterizations of Sobolev spaces on Carnot groups which is a more general case than the Heisenberg groups. In fact, the Carnot groups have more complicated group structure than the Heisenberg groups. Even the main idea in this project is similar to the characterizations of Sobolev spaces on Heisenberg groups, there are still several lemmas we use in this chapter which are different from Chapter 2.

In Chapter 4, we consider the new characterizations of Sobolev spaces on second order Sobolev spaces on Euclidean spaces. We will present here several types of characterizations: by second order differences and by the Taylor remainder, and by the differences of the first order gradient. Such characterizations are inspired by the works of Bourgain, Brezis and Mironescu [5] and H.M. Nguyen [34, 35] on characterizations of first order Sobolev spaces in the Euclidean space.

In Chapter 5, we extend the results of Chapter 4 to the high order Sobolev spaces $W^{m,p}(\mathbb{R}^N)$ for

$m > 2$ on Euclidean spaces. We also establish several types of characterizations: by mth differences and by the $m-1$th Taylor remainder.

In Chapter 6, we give a new approach for the characterization of Sobolev space on Heisenberg group by considering L^p differentiability of Sobolev functions. In this section, we introduce a variant of the notion of L^p differentiability introduced by Calderón and Zygmund and prove that functions in a Sobolev space on Heisenberg group posses this type of L^p-derivative. We show that our formulation in fact characterizes the Sobolev space on Heisenberg group.

Finally in Chapter 7, concluding remarks are given, and several possible directions are proposed for future study.

1.4 Notation Index

Before proceeding further, we compile the following list of notation index to be used in the entire dissertation.

$\mathbb{C}^n$	n-dimensional complex spaces, where n is a positive integer
$\mathbb{H}$	n-dimensional Heisenberg group
$\mathbb{G}$	Carnot group
$\mathbb{R}_+$	$\{z \in \mathbb{R} : z > 0\}$
$\|x\|$	Euclidean norm of $x \in \mathbb{R}^N$
∇f	gradient of $f(x)$ w.r.t. x
a.e.	almost everywhere
$\square$	end of proof
$GL(m_1, \mathbb{R})$	the group of $m_1 \times m_1$ invertible matrices of real numbers
$\mathfrak{g}$	lie algebra of Carnot group
$\|\cdot\|_{\mathbb{H}}$	homogeneous norm of Heisenberg group

CHAPTER 2 NEW CHARACTERIZATIONS OF SOBOLEV SPACES ON HEISENBERG GROUPS

2.1 Some Useful Lemmas

The following elementary lemma was proved and used in [34]. For the sake of completeness, we include a proof.

Lemma 2.1. Let Ω be a measurable set in $\mathbb{R}^m$, Φ and Ψ be two measurable nonnegative functions on Ω, and $\alpha > -1$. Then

$$\int_0^1 \int_{\Phi(x)>\delta} \delta^\alpha \Psi(x)\, dx d\delta = \int_{\Phi(x)\leq 1} \frac{1}{\alpha+1} \Phi^{\alpha+1}(x)\, \Psi(x)\, dx + \int_{\Phi(x)>1} \frac{1}{\alpha+1} \Psi(x)\, dx.$$

Proof. Using Fubini's theorem, we get

$$\begin{aligned}
&\int_0^1 \int_{\Phi(x)>\delta} \delta^\alpha \Psi(x)\, dx d\delta \\
&= \int_{\Phi(x)>1} \int_0^1 \delta^\alpha \Psi(x)\, d\delta dx + \int_{\Phi(x)\leq 1} \int_0^{\Phi(x)} \delta^\alpha \Psi(x)\, d\delta dx \\
&= \int_{\Phi(x)>1} \frac{1}{\alpha+1} \Psi(x)\, dx + \int_{\Phi(x)\leq 1} \frac{1}{\alpha+1} \Phi^{\alpha+1}(x)\, \Psi(x)\, dx.
\end{aligned}$$

The proof is now completed. □

Next lemma is crucial in establishing our new characterizations of Sobolev spaces on the Heisenberg group $\mathbb{H}$. In the Euclidean spaces, H.M. Nguyen [34] used the property that every two points can be connected by a line-segment and then used the mean-value theorem to control the difference of $|f(x) - f(x + he)|$ (where $h \in \mathbb{R}^N$ and $e \in \mathbb{S}^{N-1}$) by the Hardy-Littlewood maximal function of the partial derivative of f in the direction of e. Such an argument does not work on the Heisenberg group. Therefore, we need to adapt a new argument by using the representation formula on the Heisenberg group $\mathbb{H}$ [26].

Lemma 2.2. Let $g \in W^{1,p}(\mathbb{H})$, $1 < p < \infty$. Then we have

$$\iint\limits_{|g(u)-g(v)|>\delta} \frac{\delta^p}{|u^{-1}v|^{Q+p}} dudv \leq C_{Q,p} \int\limits_{\mathbb{H}} |\nabla_{\mathbb{H}} g(u)|^p \, du, \ \forall \delta > 0$$

where $C_{Q,p}$ is a positive constant depending only on Q and p.

Proof. First, we recall the following pointwise estimate on stratified groups proved in [26], for any metric ball B in $\mathbb{H}$ and every $u \in B$, we have

$$|f(u) - f_B| \leq C \int_{cB} \frac{|\nabla_{\mathbb{H}} f(v)|}{|u^{-1}v|^{Q-1}} dv$$

where f_B is the average of f over B and c is a positive uniform constant bigger than or equal to 1.

Then we can show that

$$|f(u) - f(v)| \leq A_{Q,p}\rho\left(u^{-1} \cdot v\right)\left(M(|\nabla_{\mathbb{H}} f|)(u) + M(|\nabla_{\mathbb{H}} f|)(v)\right) \text{ for a.e. } u, \ v \in \mathbb{H} \tag{2.1}$$

where M denoted the Hardy-Littlewood maximal function

$$M(f)(u) = \sup_{r>0} \frac{1}{|B(u,r)|} \int_{B(u,r)} f(v) dv$$

and $A_{Q,p}$ is the universal constant depending only on Q and p.

Now noting that by (2.1):

$$\begin{aligned}\{|f(u) - f(v)| > \delta\} &\subset \left\{A_{Q,p}\rho\left(u^{-1} \cdot v\right)\left(M(|\nabla_{\mathbb{H}} f|)(u) + M(|\nabla_{\mathbb{H}} f|)(v)\right) > \delta\right\} \\ &\subset \left\{\rho\left(u^{-1} \cdot v\right) M(|\nabla_{\mathbb{H}} f|)(u) > \frac{\delta}{2A_{Q,p}}\right\} \cup \left\{\rho\left(u^{-1} \cdot v\right) M(|\nabla_{\mathbb{H}} f|)(v) > \frac{\delta}{2A_{Q,p}}\right\},\end{aligned}$$

we get

$$\begin{aligned}&\int\limits_{\mathbb{H}}\int\limits_{\mathbb{H}}_{|f(u)-f(v)|>\delta} \frac{\delta^p}{\rho\left(u^{-1} \cdot v\right)^{Q+p}} dudv \\ \leq & \int\limits_{\mathbb{H}}\int\limits_{\mathbb{H}}_{\rho(u^{-1}\cdot v)M(|\nabla_{\mathbb{H}} f|)(u)>\frac{\delta}{2A_{Q,p}}} \frac{\delta^p}{\rho\left(u^{-1} \cdot v\right)^{Q+p}} dudv + \int\limits_{\mathbb{H}}\int\limits_{\mathbb{H}}_{\rho(u^{-1}\cdot v)M(|\nabla_{\mathbb{H}} f|)(v)>\frac{\delta}{2A_{Q,p}}} \frac{\delta^p}{\rho\left(u^{-1} \cdot v\right)^{Q+p}} dudv.\end{aligned}$$

Denote

$$I_1 := \iint\limits_{\substack{\mathbb{H}\ \mathbb{H} \\ \rho(u^{-1}\cdot v)M(|\nabla_{\mathbb{H}}f|)(u) > \frac{\delta}{2A_{Q,p}}}} \frac{\delta^p}{\rho\left(u^{-1}\cdot v\right)^{Q+p}} dudv,$$

$$I_2 := \iint\limits_{\substack{\mathbb{H}\ \mathbb{H} \\ \rho(u^{-1}\cdot v)M(|\nabla_{\mathbb{H}}f|)(v) > \frac{\delta}{2A_{Q,p}}}} \frac{\delta^p}{\rho\left(u^{-1}\cdot v\right)^{Q+p}} dudv.$$

Now, we estimate I_1.

Set

$$v = u \cdot h\sigma$$

where

$$\sigma \in \Sigma = \{u \in \mathbb{H} : |u| = 1\},$$

$$h \in [0, \infty)$$

then

$$\begin{aligned}
I_1 &= \iiint\limits_{\substack{\Sigma\ \mathbb{H}\ 0 \\ hM(|\nabla_{\mathbb{H}}f|)(u) > \frac{\delta}{2A_{Q,p}}}}^{\infty} \frac{\delta^p}{h^{p+1}} dhdud\sigma \\
&\le \int\limits_{\Sigma}\int\limits_{\mathbb{H}} \int\limits_{\frac{\delta}{2A_{Q,p}M|\nabla_{\mathbb{H}}f|(u)}}^{\infty} \frac{\delta^p}{h^{p+1}} dhdud\sigma \\
&= \frac{1}{p}\int\limits_{\Sigma}\int\limits_{\mathbb{H}} \left[2A_{Q,p}M\left|\nabla_{\mathbb{H}}f\right|(u)\right]^p dud\sigma \\
&\le \frac{(2A_{Q,p})^p}{p}\int\limits_{\Sigma}\int\limits_{\mathbb{H}} \left[M\left|\nabla_{\mathbb{H}}f\right|(u)\right]^p dud\sigma \\
&\le C_{Q,p}\int\limits_{\mathbb{H}} \left|\nabla_{\mathbb{H}}f(u)\right|^p du.
\end{aligned}$$

Similarly, to estimate I_2, we put

$$u = v \cdot h\sigma$$

Noting that $\rho\left(u^{-1}\cdot v\right) = \left|v^{-1}u\right|$, we still can get

$$I_2 \le C_{Q,p}\int\limits_{\mathbb{H}} \left|\nabla_{\mathbb{H}}f(u)\right|^p du.$$

The proof now is completed. □

Before we state and prove the next lemma, we like to make the following remark. Let $f \in W^{1,p}(\mathbb{H}),\ 1<p<\infty$. We denote

$$I(\delta)=\int\limits_{\mathbb{H}}\int\limits_{\substack{\mathbb{H}\\ |f(u)-f(v)|>\delta}}\frac{\delta^p}{\rho\left(u^{-1}\cdot v\right)^{Q+p}}dudv.$$

By Lemma 2.2, $\liminf\limits_{\delta\to 0} I(\delta)$ does exist. In the setting of Euclidean spaces [34], this limit is rather easy to evaluate. More precisely, by polar coordinates and the rotations in the Euclidean spaces, it is often assumed in [34] that $\sigma = e_N = (0,..0,1)$. Thus, to deal with the general case $\sigma \in \mathbb{S}^{N-1}$, the author in [34] just needs to consider the one-dimensional case. Then, using real analysis techniques such as the Maximal function, Lebesgue's dominated convergence theorem, Hoai-Minh Nguyen finds successfully the exact value of $\liminf\limits_{\delta\to 0} I(\delta)$. In our setting of Heisenberg group $\mathbb{H}$, this approach is not available because of the underlying geometry. Hence, we need to propose a new method in order to calculate $\liminf\limits_{\delta\to 0} I(\delta)$. Indeed, our main idea is that we will first study the relations of $I(\delta)$ and the following quantity:

$$J(\varepsilon)=\int\limits_{\mathbb{H}}\int\limits_{\substack{\mathbb{H}\\ |f(u)-f(v)|\le 1}}\frac{\varepsilon\left|f(u)-f(v)\right|^{p+\varepsilon}}{\rho\left(u^{-1}\cdot v\right)^{Q+p}}dudv.$$

In fact, we can prove the following results:

Lemma 2.3. Let $f \in W^{1,p}(\mathbb{H}),\ 1<p<\infty$. There hold

$$\liminf_{\varepsilon\to 0} J(\varepsilon) \ge \frac{1}{p}\liminf_{\delta\to 0} I(\delta)$$

$$\limsup_{\varepsilon\to 0} J(\varepsilon) \le \frac{1}{p}\limsup_{\delta\to 0} I(\delta).$$

Proof. By Lemma 2.2, $\liminf\limits_{\delta\to 0} I(\delta)$ and $\limsup\limits_{\delta\to 0} I(\delta)$ exist. Assume that

$$\liminf_{\delta\to 0} I(\delta) = C$$

$$\limsup_{\delta\to 0} I(\delta) = D.$$

We first prove that

$$\liminf_{\varepsilon\to 0}\int_0^1 (p+\varepsilon)\,\varepsilon\delta^{\varepsilon-1} I(\delta)\,d\delta \geq pC. \tag{2.2}$$

Indeed, since

$$\liminf_{\delta\to 0} I(\delta) = C,$$

for every $\tau > 0$, we can find a number $X(\tau) \in (0,1)$ such that

$$I(\delta) > C - \tau \text{ for all } \delta \in (0, X(\tau)).$$

Then

$$\begin{aligned}
&\liminf_{\varepsilon\to 0}\int_0^1 (p+\varepsilon)\,\varepsilon\delta^{\varepsilon-1} I(\delta)\,d\delta \\
&= \liminf_{\varepsilon\to 0}\left[\int_0^{X(\tau)} (p+\varepsilon)\,\varepsilon\delta^{\varepsilon-1} I(\delta)\,d\delta + \int_{X(\tau)}^1 (p+\varepsilon)\,\varepsilon\delta^{\varepsilon-1} I(\delta)\,d\delta\right] \\
&\geq \liminf_{\varepsilon\to 0}\int_0^{X(\tau)} (p+\varepsilon)\,\varepsilon\delta^{\varepsilon-1} I(\delta)\,d\delta \\
&\geq \liminf_{\varepsilon\to 0}\int_0^{X(\tau)} (p+\varepsilon)\,\varepsilon\delta^{\varepsilon-1} (C-\tau)\,d\delta \\
&\geq p(C-\tau).
\end{aligned}$$

Since τ is arbitrary, we now can conclude that

$$\liminf_{\varepsilon\to 0}\int_0^1 \iint_{|f(u)-f(v)|>\delta} \frac{(p+\varepsilon)\,\varepsilon\delta^{p+\varepsilon-1}}{\rho(u^{-1}\cdot v)^{Q+p}}\,dudvd\delta \geq pC.$$

Using Lemma 2.1 with $\alpha = p+\varepsilon-1$, $\Phi(u,v) = |f(u)-f(v)|$, $\Psi(u,v) = \frac{1}{\rho(u^{-1}\cdot v)^{Q+p}}$, we obtain

$$\liminf_{\varepsilon\to 0}\left[\int_{\mathbb{H}}\int_{\mathbb{H}}\limits_{|f(u)-f(v)|\leq 1} \frac{\varepsilon\,|f(u)-f(v)|^{p+\varepsilon}}{\rho(u^{-1}\cdot v)^{Q+p}}\,dudv + \int_{\mathbb{H}}\int_{\mathbb{H}}\limits_{|f(u)-f(v)|>1} \frac{\varepsilon}{\rho(u^{-1}\cdot v)^{Q+p}}\,dudv\right] \geq pC.$$

Noting that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{H}}\int_{\mathbb{H}}\limits_{|f(u)-f(v)|>1} \frac{\varepsilon}{\rho(u^{-1}\cdot v)^{Q+p}}\,dudv = 0,$$

we have

$$\liminf_{\varepsilon\to 0} \iint\limits_{\substack{\mathbb{H}\ \mathbb{H} \\ |f(u)-f(v)|\le 1}} \frac{\varepsilon\left|f(u)-f(v)\right|^{p+\varepsilon}}{\rho\left(u^{-1}\cdot v\right)^{Q+p}}dudv \ge pC.$$

Similarly, since

$$\limsup_{\delta\to 0} I\left(\delta\right) = D,$$

for every $\tau > 0$, we can find a number $X\left(\tau\right) \in (0,1)$ such that

$$I\left(\delta\right) < D+\tau \text{ for all } \delta \in \left(0, X\left(\tau\right)\right).$$

Then

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\int_0^1 \left(p+\varepsilon\right)\varepsilon\delta^{\varepsilon-1} I\left(\delta\right)d\delta \\
&= \limsup_{\varepsilon\to 0}\left[\int_0^{X(\tau)} \left(p+\varepsilon\right)\varepsilon\delta^{\varepsilon-1} I\left(\delta\right)d\delta + \int_{X(\tau)}^1 \left(p+\varepsilon\right)\varepsilon\delta^{\varepsilon-1} I\left(\delta\right)d\delta\right] \\
&\le \limsup_{\varepsilon\to 0}\left[\int_0^{X(\tau)} \left(p+\varepsilon\right)\varepsilon\delta^{\varepsilon-1}\left(D+\tau\right)d\delta + \left(p+\varepsilon\right)\varepsilon X\left(\tau\right)^{\varepsilon-1} C_{Q,p}\int_{\mathbb{H}} \left|\nabla_{\mathbb{H}} f(u)\right|^p du \int_{X(\tau)}^1 1 d\delta\right] \\
&\le \limsup_{\varepsilon\to 0}\int_0^{X(\tau)} \left(p+\varepsilon\right)\varepsilon\delta^{\varepsilon-1}\left(D+\tau\right)d\delta \\
&\le p\left(D+\tau\right).
\end{aligned}$$

Since τ is arbitrary, we now can conclude that

$$\limsup_{\varepsilon\to 0}\int_0^1 \iint\limits_{|f(u)-f(v)|>\delta} \frac{\left(p+\varepsilon\right)\varepsilon\delta^{p+\varepsilon-1}}{\rho\left(u^{-1}\cdot v\right)^{Q+p}}dudvd\delta \le pD.$$

Using Lemma 2.1 with $\alpha = p+\varepsilon-1,\ \Phi\left(u,v\right) = \left|f(u)-f(v)\right|,\ \Psi\left(u,v\right) = \frac{1}{\rho\left(u^{-1}\cdot v\right)^{Q+p}}$, we obtain

$$\limsup_{\varepsilon\to 0}\left[\iint\limits_{\substack{\mathbb{H}\ \mathbb{H} \\ |f(u)-f(v)|\le 1}} \frac{\varepsilon\left|f(u)-f(v)\right|^{p+\varepsilon}}{\rho\left(u^{-1}\cdot v\right)^{Q+p}}dudv + \iint\limits_{\substack{\mathbb{H}\ \mathbb{H} \\ |f(u)-f(v)|>1}} \frac{\varepsilon}{\rho\left(u^{-1}\cdot v\right)^{Q+p}}dudv\right] \le pD.$$

Noting that

$$\lim_{\varepsilon\to 0} \iint\limits_{\substack{\mathbb{H}\ \mathbb{H} \\ |f(u)-f(v)|>1}} \frac{\varepsilon}{\rho\left(u^{-1}\cdot v\right)^{Q+p}}dudv = 0,$$

we have

$$\limsup_{\varepsilon\to 0} \underset{|f(u)-f(v)|\le 1}{\int_{\mathbb{H}}\int_{\mathbb{H}}} \frac{\varepsilon\,|f(u)-f(v)|^{p+\varepsilon}}{\rho\left(u^{-1}\cdot v\right)^{Q+p}} du dv \le pD.$$

□

Lemma 2.4. There holds

$$\frac{1}{p}K_{Q,p}\int_{\mathbb{H}} |\nabla_{\mathbb{H}} f(u)|^p\, du \le \liminf_{\delta\to 0} \underset{|f(u)-f(v)|>\delta}{\int_{\mathbb{H}}\int_{\mathbb{H}}} \frac{\delta^p}{\rho\left(u^{-1}\cdot v\right)^{Q+p}} du dv,$$

for any $f\in W^{1,p}(\mathbb{H})$, $1<p<\infty$.

Proof. First, we notice by the change of variable that

$$\underset{|f(u)-f(v)|>\delta}{\int_{\mathbb{H}}\int_{\mathbb{H}}} \frac{\delta^p}{\rho\left(u^{-1}\cdot v\right)^{Q+p}} du dv = \underset{\left|\frac{f(u\cdot\delta h\sigma)-f(u)}{\delta h}\right|h>1}{\int_{\Sigma}\int_{\mathbb{H}}\int_0^\infty} \frac{1}{h^{p+1}} dh du d\sigma.$$

Now, fix $\sigma=(\sigma',\sigma_{2n+1})\in\Sigma$, with $\sigma'\in\mathbb{C}^n$, we will show that

$$\frac{1}{p}\int_{\mathbb{H}} \left|\nabla_{\mathbb{H}} f(u)\cdot\sigma'\right|^p du \le \liminf_{\delta\to 0} \underset{\left|\frac{f(u\cdot\delta h\sigma)-f(u)}{\delta h}\right|h>1}{\int_{\mathbb{H}}\int_0^\infty} \frac{1}{h^{p+1}} dh du. \tag{2.3}$$

Indeed, since for a.e. $(h,u)\in(0,\infty)\times\mathbb{H}$:

$$\frac{1}{h^{p+1}}\chi_{\left\{\left|\frac{f(u\cdot\delta h\sigma)-f(u)}{\delta h}\right|h>1\right\}}(h,u) \overset{\delta\to 0}{\to} \frac{1}{h^{p+1}}\chi_{\{|\langle\nabla_{\mathbb{H}} f(u),\sigma'\rangle|h>1\}}(h,u),$$

by Fatou's lemma, we have

$$\begin{aligned}\liminf_{\delta\to 0} \underset{\left|\frac{f(u\cdot\delta h\sigma)-f(u)}{\delta h}\right|h>1}{\int_{\mathbb{H}}\int_0^\infty} \frac{1}{h^{p+1}} dh du &\ge \int_{\mathbb{H}}\int_0^\infty \frac{1}{h^{p+1}}\chi_{\{|\langle\nabla_{\mathbb{H}} f(u),\sigma'\rangle|h>1\}}(h,u)\, dh du \\ &= \frac{1}{p}\int_{\mathbb{H}} \left|\langle\nabla_{\mathbb{H}} f(u),\sigma'\rangle\right|^p du.\end{aligned}$$

Now, again by the Fatou's lemma, we obtain

$$\begin{aligned}\liminf_{\delta\to 0}\iint\limits_{|f(u)-f(v)|>\delta}\frac{\delta^p}{\rho\left(u^{-1}\cdot v\right)^{Q+p}}dudv &= \liminf_{\delta\to 0}\int\limits_{\Sigma}\int\limits_{\mathbb{H}}\int\limits_{\substack{0\\ \left|\frac{f(u\cdot\delta h\sigma)-f(u)}{\delta h}\right|h>1}}^{\infty}\frac{1}{h^{p+1}}dhdud\sigma\\ &\geq \int\limits_{\Sigma}\liminf_{\delta\to 0}\int\limits_{\mathbb{H}}\int\limits_{\substack{0\\ \left|\frac{f(u\cdot\delta h\sigma)-f(u)}{\delta h}\right|h>1}}^{\infty}\frac{1}{h^{p+1}}dhdud\sigma\\ &\geq \frac{1}{p}\int\limits_{\Sigma}\int\limits_{\mathbb{H}}\left|\langle\nabla_{\mathbb{H}}f(u),\sigma'\rangle\right|^p dud\sigma\\ &= \frac{1}{p}K_{Q,p}\int\limits_{\mathbb{H}}|\nabla_{\mathbb{H}}f(u)|^p du.\end{aligned}$$

□

Lemma 2.5. Let $f\in C_0^1(\mathbb{H})$. Then

$$\limsup_{\varepsilon\to 0}\int\limits_{\mathbb{H}}\int\limits_{\substack{\mathbb{H}\\ |f(u)-f(v)|\leq 1}}\frac{\varepsilon\,|f(u)-f(v)|^{p+\varepsilon}}{\rho\left(u^{-1}\cdot v\right)^{Q+p}}dudv\leq K_{Q,p}\int\limits_{\mathbb{H}}|\nabla_{\mathbb{H}}f(u)|^p du.$$

Proof. By setting $v=u\cdot h\sigma$, we have

$$\int\limits_{\mathbb{H}}\int\limits_{\substack{\mathbb{H}\\ |f(u)-f(v)|\leq 1}}\frac{\varepsilon\,|f(u)-f(v)|^{p+\varepsilon}}{\rho\left(u^{-1}\cdot v\right)^{Q+p}}dudv=\int\limits_{\Sigma}\int\limits_{\mathbb{H}}\int\limits_{\substack{0\\ |f(u\cdot h\sigma)-f(u)|\leq 1}}^{\infty}\frac{\varepsilon\,|f(u\cdot h\sigma)-f(u)|^{p+\varepsilon}}{h^{p+1}}dudhd\sigma.$$

In the following, C will be a constant independent of $u,\ h,\ \sigma,\ \varepsilon$.

Since $f\in C_0^1(\mathbb{H})$, by triangle inequality and Taylor expansion [31], we have

$$|f(u\cdot h\sigma)-f(u)|\leq\left|\nabla f(u)\,.h\sigma'\right|+Ch^2 \text{ for } (\sigma,u,h)\in\Sigma\times B_A\times(0,R)\,.$$

Also, we can find $M>0$ such that $|\nabla f(u)\,.\sigma'|\leq M$ for all $(\sigma,u)\in\Sigma\times B_A$.

Hence

$$\begin{aligned}|f(u\cdot h\sigma)-f(u)|^{p+\varepsilon} &\leq \left[\left|\nabla f(u)\,.h\sigma'\right|+Ch^2\right]^{p+\varepsilon}\\ &\leq \left|\nabla f(u)\,.h\sigma'\right|^{p+\varepsilon}+Ch^{p+\varepsilon+1}.\end{aligned}$$

Thus

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \underset{|f(u\cdot h\sigma)-f(u)|\le 1}{\int_\Sigma\int_{B_A}\int_0^R} \frac{\varepsilon\,|f(u\cdot h\sigma)-f(u)|^{p+\varepsilon}}{h^{p+1}}dhdud\sigma \\
&\le \limsup_{\varepsilon\to 0}\int_\Sigma\int_{B_A}\int_0^R \frac{\varepsilon\,|\nabla f(u)\,.h\sigma'|^{p+\varepsilon}+C\varepsilon h^{p+\varepsilon+1}}{h^{p+1}}dhdud\sigma \\
&\le \limsup_{\varepsilon\to 0}\int_\Sigma\int_{B_A}\int_0^R \frac{\varepsilon\,|\nabla f(u)\,.h\sigma'|^{p+\varepsilon}}{h^{p+1}}dhdud\sigma \\
&\le \limsup_{\varepsilon\to 0}\int_\Sigma\int_{B_A}\int_0^R \frac{\varepsilon\,|\nabla f(u)\,.\sigma'|^{p+\varepsilon}}{h^{1-\varepsilon}}dhdud\sigma \\
&\le \limsup_{\varepsilon\to 0} R^\varepsilon \int_\Sigma\int_{B_A} \left|\nabla f(u)\,.\sigma'\right|^{p+\varepsilon} dud\sigma \\
&\le \limsup_{\varepsilon\to 0} R^\varepsilon M^\varepsilon \int_\Sigma\int_{B_A} \left|\nabla f(u)\,.\sigma'\right|^{p} dud\sigma \\
&\le K_{Q,p}\int_{\mathbb{H}} |\nabla_{\mathbb{H}} f(u)|^p\,du.
\end{aligned}$$

□

2.2 Main Results

The first aim of this chapter is to prove the following estimates for functions in the Sobolev space $W^{1,p}(\mathbb{H})$:

Theorem 2.6. Let $1<p<\infty$ and $f\in W^{1,p}(\mathbb{H})$. Then

(a) There exists a positive constant $C_{Q,p}$ depending only on Q,p such that

$$\sup_{0<\varepsilon<1} \underset{|f(u)-f(v)|\le 1}{\int_{\mathbb{H}}\int_{\mathbb{H}}} \frac{\varepsilon\,|f(u)-f(v)|^{p+\varepsilon}}{\rho\left(u^{-1}\cdot v\right)^{Q+p}}dudv + \underset{|f(u)-f(v)|>1}{\int_{\mathbb{H}}\int_{\mathbb{H}}} \frac{1}{\rho\left(u^{-1}\cdot v\right)^{Q+p}}dudv \le C_{Q,p}\int_{\mathbb{H}} |\nabla_{\mathbb{H}} f(u)|^p\,du.$$

(b) There holds

$$\lim_{\varepsilon\to 0} \underset{|f(u)-f(v)|\le 1}{\int_{\mathbb{H}}\int_{\mathbb{H}}} \frac{\varepsilon\,|f(u)-f(v)|^{p+\varepsilon}}{\rho\left(u^{-1}\cdot v\right)^{Q+p}}dudv = K_{Q,p}\int_{\mathbb{H}} |\nabla_{\mathbb{H}} f(u)|^p\,du.$$

where $K_{Q,p}$ is a constant defined as follows

$$K_{Q,p} = \int\limits_{\Sigma} \left|\langle e, \sigma' \rangle\right|^{p} d\sigma = \int\limits_{\Sigma} |\langle (e,0), \sigma \rangle|^{p} d\sigma$$

for any $(e, 0) \in \Sigma$.

(c) There exists a positive constant $C_{Q,p}$ such that

$$\underset{|f(u)-f(v)|>\delta}{\int\limits_{\mathbb{H}}\int\limits_{\mathbb{H}}} \frac{\delta^{p}}{\rho\left(u^{-1} \cdot v\right)^{Q+p}} du dv \leq C_{Q,p} \int\limits_{\mathbb{H}} \left|\nabla_{\mathbb{H}} f(u)\right|^{p} du, \ \forall \delta > 0.$$

(d) Moreover

$$\liminf_{\delta \to 0} \underset{|f(u)-f(v)|>\delta}{\int\limits_{\mathbb{H}}\int\limits_{\mathbb{H}}} \frac{\delta^{p}}{\rho\left(u^{-1} \cdot v\right)^{Q+p}} du dv = \frac{1}{p} K_{Q,p} \int\limits_{\mathbb{H}} \left|\nabla_{\mathbb{H}} f(u)\right|^{p} du.$$

Using Theorem 2.6, we set up new characterizations of the Sobolev space $W^{1,p}(\mathbb{H})$ which is the main purpose of this paper. More precisely, we prove that

Theorem 2.7. Let $1 < p < \infty$ and $f \in L^{p}(\mathbb{H})$. Then the following are equivalent:

(1) $f \in W^{1,p}(\mathbb{H})$.

(2)

$$\sup_{0<\varepsilon<1} \underset{|f(u)-f(v)|\leq 1}{\int\limits_{\mathbb{H}}\int\limits_{\mathbb{H}}} \frac{\varepsilon \left|f(u) - f(v)\right|^{p+\varepsilon}}{\rho\left(u^{-1} \cdot v\right)^{Q+p}} du dv + \underset{|f(u)-f(v)|>1}{\int\limits_{\mathbb{H}}\int\limits_{\mathbb{H}}} \frac{1}{\rho\left(u^{-1} \cdot v\right)^{Q+p}} du dv < \infty.$$

(3)

$$\sup_{0<\delta<1} \underset{|f(u)-f(v)|>\delta}{\int\limits_{\mathbb{H}}\int\limits_{\mathbb{H}}} \frac{\delta^{p}}{\rho\left(u^{-1} \cdot v\right)^{Q+p}} du dv < \infty.$$

(4) There exists a nonnegative function $F \in L^{p}(\mathbb{H})$ such that

$$|f(u) - f(v)| \leq \rho\left(u^{-1} \cdot v\right)\left(F(u) + F(v)\right) \text{ for a.e. } u, \ v \in \mathbb{H}.$$

(5) The L^1 to L^p Poincaré inequalities hold for every metric ball B in $\mathbb{H}$. Namely, there exists a function $g \in L^{p}_{loc}(\mathbb{H})$ and an absolute constant $C > 0$ independent of the ball B such that

$$\frac{1}{|B|} \int_{B} |f(u) - f_{B}| du \leq C r(B) \left(\frac{1}{|B|} \int_{B} |g|^{q} du\right)^{\frac{1}{q}}$$

for some $1 \leq q < p$, where $r(B)$ is the radius of the ball B.

The following remarks are in order. First, the proofs of the main theorems (e.g., Theorem B) in [34] rely on the underlying geometry of the Euclidean spaces, such as that any two points can be connected by a line segment. Second, it's worthy to note that one of the main techniques in the proof of Theorem B is to use the uniformity in every directions of the unit sphere in the Euclidean spaces. More precisely, to deal with the general case $\sigma \in \mathbb{S}^{N-1}$, it is often to be assumed that $\sigma = e_N = (0, ..0, 1)$ and hence, one just needs to work on 1-dimensional case. This can be done by using the rotation in the Euclidean spaces. In the case of Heisenberg groups, this type of property is not available because of the structure of the Heisenberg groups, in particular, the dilation. Hence, we need to find a different approach to this characterization. In fact, we will use the representation formula on the Heisenberg group proved in [26] to obtain estimate (4.1). This estimate will allow us to establish a useful lemma (Lemma 2.2 in Section 2.1. Third, as we have shown in [25], (1), (4) and (5) are all equivalent. Therefore, the new ingredient here is that (1), (2) and (3) are equivalent.

2.3 Proof of Main Results

Proof of Theorem 2.6:

(a) First, by Lemma 2.2, we have

$$\int\limits_{\mathbb{H}}\int\limits_{\substack{\mathbb{H} \\ |f(u)-f(v)|>\delta}} \frac{\delta^p}{\rho\left(u^{-1}\cdot v\right)^{Q+p}} dudv \le C_{Q,p} \int\limits_{\mathbb{H}} \left|\nabla_{\mathbb{H}} f(u)\right|^p du, \ \forall \delta > 0. \tag{2.4}$$

As consequences, we get

$$\int\limits_{\mathbb{H}}\int\limits_{\substack{\mathbb{H} \\ |f(u)-f(v)|>1}} \frac{1}{\rho\left(u^{-1}\cdot v\right)^{Q+p}} dudv \le C_{Q,p} \int\limits_{\mathbb{H}} \left|\nabla_{\mathbb{H}} f(u)\right|^p du, \tag{2.5}$$

Now, multiplying (2.4) by $\varepsilon\sigma^{\varepsilon-1}$, $0<\varepsilon<1$ and integrating the expression obtained with respect to σ over $(0,1)$, we can deduce that

$$\int\limits_0^1\int\limits_{\mathbb{H}}\int\limits_{\substack{\mathbb{H} \\ |f(u)-f(v)|>\delta}} \frac{\varepsilon\delta^{p+\varepsilon-1}}{\rho\left(u^{-1}\cdot v\right)^{Q+p}} dudvd\delta \le C_{Q,p} \int\limits_{\mathbb{H}} \left|\nabla_{\mathbb{H}} f(u)\right|^p du.$$

Using Lemma 2.1 with $\alpha = p+\varepsilon-1,\ \Phi(u,v) = |f(u)-f(v)|,\ \Psi(u,v) = \frac{1}{\rho(u^{-1}\cdot v)^{Q+p}}$, we obtain

$$\int\limits_{\mathbb{H}}\int\limits_{\substack{\mathbb{H}\\ |f(u)-f(v)|\le 1}} \frac{\varepsilon\,|f(u)-f(v)|^{p+\varepsilon}}{\rho\left(u^{-1}\cdot v\right)^{Q+p}}dudv + \int\limits_{\mathbb{H}}\int\limits_{\substack{\mathbb{H}\\ |f(u)-f(v)|>1}} \frac{\varepsilon}{\rho\left(u^{-1}\cdot v\right)^{Q+p}}dudv \le C_{Q,p}\int\limits_{\mathbb{H}} |\nabla_{\mathbb{H}} f(u)|^p\,du.$$

Thus,

$$\iint\limits_{|f(u)-f(v)|\le 1} \frac{\varepsilon\,|f(u)-f(v)|^{p+\varepsilon}}{\rho\left(u^{-1}\cdot v\right)^{Q+p}}dudv \le C_{Q,p}\int\limits_{\mathbb{H}} |\nabla_{\mathbb{H}} f(u)|^p\,du. \tag{2.6}$$

By (2.5) and (2.6), we get the assertion (a).

(b) From Lemma 2.3, 2.4, 2.5 and the density argument, we have (b).

(c) This is Lemma 2.2.

(d) This is a consequence of Lemma 2.3, 2.5, 2.4 and the density argument.

□

Proof of Theorem 2.7:

The proof is divided into 6 steps.

Step 1: (1) ⇒ (2). This is a consequence of part (a) of Theorem 2.6 and the fact that $f \in W^{1,p}(\mathbb{H})$.

Step 2: (2) ⇒ (1). First, we will assume further that $f \in L^\infty(\mathbb{H})$. Then from the assumption

$$\sup_{0<\varepsilon<1} \int\limits_{\mathbb{H}}\int\limits_{\substack{\mathbb{H}\\ |f(u)-f(v)|\le 1}} \frac{\varepsilon\,|f(u)-f(v)|^{p+\varepsilon}}{\rho\left(u^{-1}\cdot v\right)^{Q+p}}dudv + \int\limits_{\mathbb{H}}\int\limits_{\substack{\mathbb{H}\\ |f(u)-f(v)|>1}} \frac{1}{\rho\left(u^{-1}\cdot v\right)^{Q+p}}dudv < \infty,$$

it's easy to deduce that

$$L(f) = \sup_{0<\varepsilon<1} \int\limits_{\mathbb{H}}\int\limits_{\mathbb{H}} \frac{\varepsilon\,|f(u)-f(v)|^{p+\varepsilon}}{\rho\left(u^{-1}\cdot v\right)^{Q+p}}dudv < \infty.$$

Now, let (γ_k) be a sequence of smooth mollifiers on $\mathbb{H}$ and set

$$f_k := f * \gamma_k.$$

Since $f_k \in L^p(\mathbb{H}) \cap C^\infty(\mathbb{H})$, so $f_k \in W^{1,p}(\mathbb{H})$. By using Theorem 2.6 (b), we can conclude that

$$K_{Q,p} \int_{\mathbb{H}} |\nabla_{\mathbb{H}} f_k(u)|^p \, du \le \liminf_{\varepsilon \to 0} \underset{|f_k(u)-f_k(v)| \le 1}{\int_{\mathbb{H}} \int_{\mathbb{H}}} \frac{\varepsilon \, |f_k(u) - f_k(v)|^{p+\varepsilon}}{\rho\,(u^{-1} \cdot v)^{Q+p}} dudv$$

$$\le \liminf_{\varepsilon \to 0} \int_{\mathbb{H}} \int_{\mathbb{H}} \frac{\varepsilon \, |f_k(u) - f_k(v)|^{p+\varepsilon}}{\rho\,(u^{-1} \cdot v)^{Q+p}} dudv$$

From Jensen's inequality and the convexity of the function $x^{p+\varepsilon}$, we can obtain

$$\int_{\mathbb{H}} \int_{\mathbb{H}} \frac{\varepsilon \, |f_k(u) - f_k(v)|^{p+\varepsilon}}{\rho\,(u^{-1} \cdot v)^{Q+p}} dudv \le \int_{\mathbb{H}} \int_{\mathbb{H}} \frac{\varepsilon \, |f(u) - f(v)|^{p+\varepsilon}}{\rho\,(u^{-1} \cdot v)^{Q+p}} dudv \le L(f).$$

Hence,

$$K_{Q,p} \int_{\mathbb{H}} |\nabla_{\mathbb{H}} f_k(u)|^p \, du \le \liminf_{\varepsilon \to 0} \int_{\mathbb{H}} \int_{\mathbb{H}} \frac{\varepsilon \, |f(u) - f(v)|^{p+\varepsilon}}{\rho\,(u^{-1} \cdot v)^{Q+p}} dudv \le L(f).$$

Thus, with an extra assumption $f \in L^\infty(\mathbb{H})$, we have that (2) $\Rightarrow$ (1).

For the general case, we make use of the truncated function. For $R > 0$, define

$$f_R(u) = \begin{cases} f(u) \text{ , if } |f(u)| < R \\ \frac{Rf(u)}{|f(u)|}, \text{ otherwise} \end{cases}.$$

It's clear that $f_R \in L^\infty(\mathbb{H})$ and $f_R(u) \overset{R \to \infty}{\to} f(u)$ pointwise for a.e. $u \in \mathbb{H}$. Moreover, it can be checked that

$$|f_R(u) - f_R(v)| \le |f(u) - f(v)| \text{ for all } u, v \in \mathbb{H}.$$

As a consequence, one has

$$\underset{|f_R(u)-f_R(v)| \le 1}{\int_{\mathbb{H}} \int_{\mathbb{H}}} \frac{\varepsilon \, |f_R(u) - f_R(v)|^{p+\varepsilon}}{\rho\,(u^{-1} \cdot v)^{Q+p}} dudv$$

$$= \underset{\substack{|f_R(u)-f_R(v)| \le 1 \\ |f(u)-f(v)| \le 1}}{\int_{\mathbb{H}} \int_{\mathbb{H}}} \frac{\varepsilon \, |f_R(u) - f_R(v)|^{p+\varepsilon}}{\rho\,(u^{-1} \cdot v)^{Q+p}} dudv + \underset{\substack{|f_R(u)-f_R(v)| \le 1 \\ |f(u)-f(v)| > 1}}{\int_{\mathbb{H}} \int_{\mathbb{H}}} \frac{\varepsilon \, |f_R(u) - f_R(v)|^{p+\varepsilon}}{\rho\,(u^{-1} \cdot v)^{Q+p}} dudv$$

$$\le \underset{|f(u)-f(v)| \le 1}{\int_{\mathbb{H}} \int_{\mathbb{H}}} \frac{\varepsilon \, |f(u) - f(v)|^{p+\varepsilon}}{\rho\,(u^{-1} \cdot v)^{Q+p}} dudv + \underset{|f(u)-f(v)| > 1}{\int_{\mathbb{H}} \int_{\mathbb{H}}} \frac{\varepsilon}{\rho\,(u^{-1} \cdot v)^{Q+p}} dudv.$$

Also,

$$\underset{|f_R(u)-f_R(v)| > 1}{\int_{\mathbb{H}} \int_{\mathbb{H}}} \frac{1}{\rho\,(u^{-1} \cdot v)^{Q+p}} dudv \le \underset{|f(u)-f(v)| > 1}{\int_{\mathbb{H}} \int_{\mathbb{H}}} \frac{1}{\rho\,(u^{-1} \cdot v)^{Q+p}} dudv.$$

Thus, we have $f_R \in W^{1,p}(\mathbb{H})$. Moreover, by part (b) of Theorem 2.6, one has

$$K_{Q,p}\int_{\mathbb{H}} |\nabla_{\mathbb{H}} f_R(u)|^p\, du \leq \liminf_{\varepsilon\to 0} \underset{|f_R(u)-f_R(v)|\leq 1}{\int_{\mathbb{H}}\int_{\mathbb{H}}} \frac{\varepsilon\,|f_R(u)-f_R(v)|^{p+\varepsilon}}{\rho\left(u^{-1}\cdot v\right)^{Q+p}} dudv$$

$$\leq \liminf_{\varepsilon\to 0}\left[\underset{|f(u)-f(v)|\leq 1}{\int_{\mathbb{H}}\int_{\mathbb{H}}} \frac{\varepsilon\,|f(u)-f(v)|^{p+\varepsilon}}{\rho\left(u^{-1}\cdot v\right)^{Q+p}} dudv + \underset{|f(u)-f(v)|>1}{\int_{\mathbb{H}}\int_{\mathbb{H}}} \frac{\varepsilon}{\rho\left(u^{-1}\cdot v\right)^{Q+p}} dudv\right]$$

$$= \liminf_{\varepsilon\to 0} \underset{|f(u)-f(v)|\leq 1}{\int_{\mathbb{H}}\int_{\mathbb{H}}} \frac{\varepsilon\,|f(u)-f(v)|^{p+\varepsilon}}{\rho\left(u^{-1}\cdot v\right)^{Q+p}} dudv.$$

Since $R>0$ is arbitrary, we can deduce that $f \in W^{1,p}(\mathbb{H})$.

Step 3: (1) $\Rightarrow$ (3). This is a consequence of part (c) of Theorem 2.6 and the fact that $f \in W^{1,p}(\mathbb{H})$.

Step 4: (3) $\Rightarrow$ (1). Suppose that $f \in L^p(\mathbb{H})$ and there is a constant $C>0$ such that for all $\delta \in (0,1)$:

$$\underset{|f(u)-f(v)|>\delta}{\int_{\mathbb{H}}\int_{\mathbb{H}}} \frac{\delta^p}{\rho\left(u^{-1}\cdot v\right)^{Q+p}} dudv \leq C. \tag{2.7}$$

Multiplying (2.7) by $\varepsilon\delta^{\varepsilon-1}$, $0<\varepsilon<1$, and integrating with respect to δ over $(0,1)$, by Lemma 2.1 with $\alpha = p+\varepsilon-1,\ \Phi(u,v) = |f(u)-f(v)|,\ \Psi(u,v) = \frac{1}{\rho(u^{-1}\cdot v)^{Q+p}}$, one has

$$\underset{|f(u)-f(v)|\leq 1}{\int_{\mathbb{H}}\int_{\mathbb{H}}} \frac{\varepsilon\,|f(u)-f(v)|^{p+\varepsilon}}{\rho\left(u^{-1}\cdot v\right)^{Q+p}} dudv \leq C\,(p+1)\,.$$

Also, by Fatou's lemma, we also get

$$\underset{|f(u)-f(v)|>1}{\int_{\mathbb{H}}\int_{\mathbb{H}}} \frac{1}{\rho\left(u^{-1}\cdot v\right)^{Q+p}} dudv < \infty.$$

As a consequence of Step 2, we have $f \in W^{1,p}(\mathbb{H})$.

The proof is now completed. □

2.4 The Case $p=1$

In this section, we will investigate the special case $p=1$. First, we recall the definition of the space $BV(\Omega)$ of functions with bounded variation in $\Omega \subset \mathbb{H}$.

Definition 2.8 (Horizontal vector fields)**.** The space of smooth sections of $H\Omega$, the horizontal subbundle on Ω, is denoted by $\Gamma(H\Omega)$. The space $\Gamma_c(H\Omega)$ denotes all the elements of $\Gamma(H\Omega)$ with support contained in Ω. Elements of $\Gamma(H\Omega)$ are called horizontal vector fields.

Definition 2.9 ($H-BV$ functions)**.** We say that a function $u \in L^1(\Omega)$ is a function of $H-$bounded variation if

$$|D_H u|(\Omega) = \sup\left\{ \int_\Omega u \operatorname{div} \phi d\xi : \phi \in \Gamma_c(H\Omega), \ |\phi| \le 1 \right\} < \infty,$$

where the symbol div denotes the Riemannian divergence. We denote by $BV(\Omega)$ the space of all functions of $H-$bounded variation.

In this section, we will prove the following property:

Theorem 2.10. Let f be a function in $L^1(\mathbb{H})$ satisfying

$$\sup_{0<\delta<1} \int\limits_{\mathbb{H}} \int\limits_{\substack{\mathbb{H} \\ |f(u)-f(v)|>\delta}} \frac{\delta}{\rho(u^{-1}\cdot v)^{Q+1}} dudv < \infty.$$

Then $f \in BV(\mathbb{H})$.

Proof. Assume that $f \in L^1(\mathbb{H})$ and

$$\int\limits_{\mathbb{H}} \int\limits_{\substack{\mathbb{H} \\ |f(u)-f(v)|>\delta}} \frac{\delta}{\rho(u^{-1}\cdot v)^{Q+1}} dudv < C \tag{2.8}$$

for some positive constant $C > 0$.

Proceeding similarly as in Step 4 of the proof of Theorem 2.7, multiplying (2.8) by $\varepsilon\delta^{\varepsilon-1}, 0 < \varepsilon < 1$, integrating with respect to δ over $(0,1)$, and then using Lemma 2.1, we have

$$\int\limits_{\mathbb{H}} \int\limits_{\substack{\mathbb{H} \\ |f(u)-f(v)|\le 1}} \frac{\varepsilon|f(u)-f(v)|^{1+\varepsilon}}{\rho(u^{-1}\cdot v)^{Q+1}} dudv \le 2C.$$

By Fatou's lemma, from (2.8), we also get

$$\int\limits_{\mathbb{H}} \int\limits_{\substack{\mathbb{H} \\ |f(u)-f(v)|>1}} \frac{1}{\rho(u^{-1}\cdot v)^{Q+1}} dudv < C.$$

Now, we also split the proof into 2 steps:

Step 1: We suppose further that $f \in L^\infty(\mathbb{H})$. Now, we define f_k as in Step 2 of the proof of

Theorem 2.7. Noting that the function $t^{1+\varepsilon}$ is still convex on $\mathbb{R}^+$, we can also have

$$\begin{aligned}
\int_{\mathbb{H}}\int_{\mathbb{H}} \frac{\varepsilon\left|f_k(u)-f_k(v)\right|^{1+\varepsilon}}{\rho\left(u^{-1}\cdot v\right)^{Q+1}}dudv &\leq \int_{\mathbb{H}}\int_{\mathbb{H}} \frac{\varepsilon\left|f(u)-f(v)\right|^{1+\varepsilon}}{\rho\left(u^{-1}\cdot v\right)^{Q+1}}dudv \\
&\leq \underset{|f(u)-f(v)|\leq 1}{\int_{\mathbb{H}}\int_{\mathbb{H}}} \frac{\varepsilon\left|f(u)-f(v)\right|^{1+\varepsilon}}{\rho\left(u^{-1}\cdot v\right)^{Q+1}}dudv \\
&+ \underset{|f(u)-f(v)|> 1}{\int_{\mathbb{H}}\int_{\mathbb{H}}} \frac{\varepsilon\left|f(u)-f(v)\right|^{1+\varepsilon}}{\rho\left(u^{-1}\cdot v\right)^{Q+1}}dudv \\
&\leq 2C+\varepsilon\left[2\left\|f\right\|_{\infty}\right]^{1+\varepsilon}C.
\end{aligned}$$

Now, we can repeat the proofs of (b) in Theorem 2.6 and Step 2 in Theorem 2.7 to conclude that

$f_k \in BV(\mathbb{H})$ and

$$\begin{aligned}
K_{Q,1}\left\|\nabla_{\mathbb{H}} f_k\right\| &\leq \liminf_{\varepsilon\to 0}\int_{\mathbb{H}}\int_{\mathbb{H}} \frac{\varepsilon\left|f_k(u)-f_k(v)\right|^{1+\varepsilon}}{\rho\left(u^{-1}\cdot v\right)^{Q+1}}dudv \\
&\leq \liminf_{\varepsilon\to 0}\{2C+\varepsilon\left[2\left\|f\right\|_{\infty}\right]^{1+\varepsilon}C\} \\
&= 2C.
\end{aligned}$$

Hence $f \in BV(\mathbb{H})$.

Step 2: The general case. Similarly, we also introduce the truncated function

$$f_R(u) = \begin{cases} f(u) \text{ if } |f(u)| < R \\ \frac{Rf(u)}{|f(u)|} \text{ otherwise} \end{cases} \quad \text{for } R > 0.$$

Then one has $f_R \in L^\infty(\mathbb{H})$, $f_R(u) \overset{R\to\infty}{\to} f(u)$ pointwise for a.e. $u \in \mathbb{H}$, and

$$|f_R(u) - f_R(v)| \leq |f(u) - f(v)| \text{ for all } u, v \in \mathbb{H}.$$

As a consequence, one gets

$$\iint\limits_{\substack{\mathbb{H}\ \mathbb{H}\\ |f_R(u)-f_R(v)|\leq 1}} \frac{\varepsilon\left|f_R(u)-f_R(v)\right|^{1+\varepsilon}}{\rho\left(u^{-1}\cdot v\right)^{Q+1}}dudv$$

$$= \iint\limits_{\substack{\mathbb{H}\ \mathbb{H}\\ |f_R(u)-f_R(v)|\leq 1\\ |f(u)-f(v)|\leq 1}} \frac{\varepsilon\left|f_R(u)-f_R(v)\right|^{1+\varepsilon}}{\rho\left(u^{-1}\cdot v\right)^{Q+1}}dudv + \iint\limits_{\substack{\mathbb{H}\ \mathbb{H}\\ |f_R(u)-f_R(v)|\leq 1\\ |f(u)-f(v)|>1}} \frac{\varepsilon\left|f_R(u)-f_R(v)\right|^{1+\varepsilon}}{\rho\left(u^{-1}\cdot v\right)^{Q+1}}dudv$$

$$\leq \iint\limits_{\substack{\mathbb{H}\ \mathbb{H}\\ |f(u)-f(v)|\leq 1}} \frac{\varepsilon\left|f(u)-f(v)\right|^{1+\varepsilon}}{\rho\left(u^{-1}\cdot v\right)^{Q+1}}dudv + \iint\limits_{\substack{\mathbb{H}\ \mathbb{H}\\ |f(u)-f(v)|>1}} \frac{\varepsilon}{\rho\left(u^{-1}\cdot v\right)^{Q+1}}dudv.$$

Also,

$$\iint\limits_{\substack{\mathbb{H}\ \mathbb{H}\\ |f_R(u)-f_R(v)|>1}} \frac{1}{\rho\left(u^{-1}\cdot v\right)^{Q+1}}dudv \leq \iint\limits_{\substack{\mathbb{H}\ \mathbb{H}\\ |f(u)-f(v)|>1}} \frac{1}{\rho\left(u^{-1}\cdot v\right)^{Q+1}}dudv.$$

Thus, we have $f_R \in BV\left(\mathbb{H}\right).$ Moreover,

$$\begin{aligned} K_{Q,1}\left\|\nabla_{\mathbb{H}} f_R\right\| &\leq \liminf_{\varepsilon\to 0} \iint\limits_{\substack{\mathbb{H}\ \mathbb{H}\\ |f_R(u)-f_R(v)|\leq 1}} \frac{\varepsilon\left|f_R(u)-f_R(v)\right|^{1+\varepsilon}}{\rho\left(u^{-1}\cdot v\right)^{Q+1}}dudv \\ &\leq \liminf_{\varepsilon\to 0}\left[\iint\limits_{\substack{\mathbb{H}\ \mathbb{H}\\ |f(u)-f(v)|\leq 1}} \frac{\varepsilon\left|f(u)-f(v)\right|^{1+\varepsilon}}{\rho\left(u^{-1}\cdot v\right)^{Q+1}}dudv + \iint\limits_{\substack{\mathbb{H}\ \mathbb{H}\\ |f(u)-f(v)|>1}} \frac{\varepsilon}{\rho\left(u^{-1}\cdot v\right)^{Q+1}}dudv\right] \\ &= \liminf_{\varepsilon\to 0} \iint\limits_{\substack{\mathbb{H}\ \mathbb{H}\\ |f(u)-f(v)|\leq 1}} \frac{\varepsilon\left|f(u)-f(v)\right|^{1+\varepsilon}}{\rho\left(u^{-1}\cdot v\right)^{Q+1}}dudv. \end{aligned}$$

Since $R > 0$ is arbitrary, we can deduce that $f \in BV\left(\mathbb{H}\right).$

□

Using Theorem2.10, we can also have the following Lipschitz type characterization of BV space:

Theorem 2.11. Let $f \in L^1\left(\mathbb{H}\right)$ be such that there exists a nonnegative function $F \in L^1\left(\mathbb{H}\right)$ satisfying

$$|f(u)-f(v)| \leq \rho\left(u^{-1}\cdot v\right)\left(F(u)+F(v)\right) \text{ for a.e. } u,\ v \in \mathbb{H}. \tag{2.9}$$

Then $f \in BV\left(\mathbb{H}\right).$

Before we begin our proof of this theorem, we like to make some remarks. We note that in the paper [20], the author defined the Sobolev spaces $W^{1,p}(\mathbb{H})$ for $p > 1$ if the above estimate (2.9) holds for $F \in L^p(\mathbb{H})$. But that definition does not hold for $p = 1$. Therefore, our theorem can be viewed as the borderline case of the Sobolev space when $p = 1$ on the Heisenberg group $\mathbb{H}$.

Proof. First, we note here that for all $\delta \in (0,1)$:

$$\begin{aligned}\{|f(u) - f(v)| > \delta\} &\subset \left\{\rho\left(u^{-1} \cdot v\right)(F(u) + F(v)) > \delta\right\} \\ &\subset \left\{\rho\left(u^{-1} \cdot v\right) F(u) > \frac{\delta}{2}\right\} \cup \left\{\rho\left(u^{-1} \cdot v\right) F(v) > \frac{\delta}{2}\right\}.\end{aligned}$$

Hence, one receives

$$\begin{aligned}&\underset{|f(u)-f(v)|>\delta}{\int\limits_{\mathbb{H}}\int\limits_{\mathbb{H}}} \frac{\delta}{\rho\left(u^{-1} \cdot v\right)^{Q+1}} dudv \\ \leq &\underset{\rho(u^{-1}\cdot v)F(u)>\frac{\delta}{2}}{\int\limits_{\mathbb{H}}\int\limits_{\mathbb{H}}} \frac{\delta}{\rho\left(u^{-1} \cdot v\right)^{Q+1}} dudv + \underset{\rho(u^{-1}\cdot v)F(v)>\frac{\delta}{2}}{\int\limits_{\mathbb{H}}\int\limits_{\mathbb{H}}} \frac{\delta}{\rho\left(u^{-1} \cdot v\right)^{Q+1}} dudv.\end{aligned}$$

We denote

$$I_1 := \underset{\rho(u^{-1}\cdot v)F(u)>\frac{\delta}{2}}{\int\limits_{\mathbb{H}}\int\limits_{\mathbb{H}}} \frac{\delta}{\rho\left(u^{-1} \cdot v\right)^{Q+1}} dudv,$$

$$I_2 := \underset{\rho(u^{-1}\cdot v)F(v)>\frac{\delta}{2}}{\int\limits_{\mathbb{H}}\int\limits_{\mathbb{H}}} \frac{\delta}{\rho\left(u^{-1} \cdot v\right)^{Q+1}} dudv.$$

We now estimate I_1.

Setting

$$v = u \cdot h\sigma$$

where

$$\sigma \in \Sigma = \{u \in \mathbb{H} : |u| = 1\},$$

$$h \in [0, \infty)$$

one has

$$I_1 = \int_{\Sigma}\int_{\mathbb{H}}\int_{\substack{0 \\ hF(u)>\frac{\delta}{2}}}^{\infty} \frac{\delta}{h^2} dh du d\sigma$$

$$\leq \int_{\Sigma}\int_{\mathbb{H}}\int_{\frac{\delta}{2F(u)}}^{\infty} \frac{\delta}{h^2} dh du d\sigma$$

$$= 2\int_{\Sigma}\int_{\mathbb{H}} F(u) du d\sigma$$

$$= C_Q \int_{\mathbb{H}} F(u) du.$$

Similarly, by noting that $\rho\left(u^{-1}\cdot v\right) = \left|v^{-1}u\right|$, we also have

$$I_2 \leq C_Q \int_{\mathbb{H}} F(u) du.$$

Hence, we have

$$\sup_{0<\delta<1} \int_{\mathbb{H}}\int_{\substack{\mathbb{H} \\ |f(u)-f(v)|>\delta}} \frac{\delta}{\rho\left(u^{-1}\cdot v\right)^{Q+1}} du dv < \infty.$$

By Theorem 2.10, we obtain $f \in BV(\mathbb{H})$. □

2.5 Some generalizations and variants

In this section, we will study some generalizations of the above results. The next Theorem is a generalized result of Theorem 2.7:

Theorem 2.12. Let $f \in L^p(\mathbb{H}),\ 1 < p < \infty$ and $F : [0,\infty) \to [0,\infty)$ be continuous such that

$$\int_0^{\infty} F(t)\, t^{-p-1} dt = 1.$$

Set

$$F_\delta(t) = \delta^p F\left(\frac{t}{\delta}\right),\ \ \delta > 0.$$

Then we have

(a) If

$$\sup_{0<\delta<1} \int_{\mathbb{H}}\int_{\mathbb{H}} \frac{F_\delta\left(|f(u)-f(v)|\right)}{\rho\left(u^{-1}\cdot v\right)^{Q+p}} du dv < \infty$$

and

$$\int\limits_{\mathbb{H}}\int\limits_{\substack{\mathbb{H}\\ |f(u)-f(v)|>\delta}} \frac{1}{\rho\left(u^{-1}\cdot v\right)^{Q+p}}dudv<\infty,\ \forall\delta>0,$$

then $g\in W^{1,p}\left(\mathbb{H}\right).$

(b) If $g\in W^{1,p}\left(\mathbb{H}\right)$ and

$$\int\limits_{0}^{\infty}F_{\delta}\left(t\right)t^{-p-1}dt<\infty,\ \forall\delta>0,$$

then

$$\int\limits_{\mathbb{H}}\int\limits_{\mathbb{H}}\frac{F_{\delta}\left(|f(u)-f(v)|\right)}{\rho\left(u^{-1}\cdot v\right)^{Q+p}}dudv\leq C_{Q,p}\int\limits_{0}^{\infty}F_{\delta}\left(t\right)t^{-p-1}dt\int\limits_{\mathbb{H}}\left|\nabla_{\mathbb{H}}f(u)\right|^{p}du,\ \forall\delta>0.$$

Proof. (b) Setting

$$D_1(f)=\left\{(u,v)\in\mathbb{H}\times\mathbb{H}:M(|\nabla_{\mathbb{H}}f|)(u)\geq M(|\nabla_{\mathbb{H}}f|)(v)\right\}$$

$$D_2(f)=\left\{(u,v)\in\mathbb{H}\times\mathbb{H}:M(|\nabla_{\mathbb{H}}f|)(u)<M(|\nabla_{\mathbb{H}}f|)(v)\right\},$$

then

$$\int\limits_{\mathbb{H}}\int\limits_{\mathbb{H}}\frac{F_{\delta}\left(|f(u)-f(v)|\right)}{\rho\left(u^{-1}\cdot v\right)^{Q+p}}dudv=\iint\limits_{D_1(f)}\frac{F_{\delta}\left(|f(u)-f(v)|\right)}{\rho\left(u^{-1}\cdot v\right)^{Q+p}}dudv+\iint\limits_{D_2(f)}\frac{F_{\delta}\left(|f(u)-f(v)|\right)}{\rho\left(u^{-1}\cdot v\right)^{Q+p}}dudv.$$

Now, we will first concern

$$I_1=\iint\limits_{D_1(f)}\frac{F_{\delta}\left(|f(u)-f(v)|\right)}{\rho\left(u^{-1}\cdot v\right)^{Q+p}}dudv.$$

Using (4.1), (i) and noting that on the domain $D_1(f)$, one has

$$M(|\nabla_{\mathbb{H}}f|)(u)\geq M(|\nabla_{\mathbb{H}}f|)(v),$$

we get

$$I_1\leq\int\limits_{\mathbb{H}}\int\limits_{\mathbb{H}}\frac{F_{\delta}\left(2A_{Q,p}\rho\left(u^{-1}\cdot v\right)M(|\nabla_{\mathbb{H}}f|)(u)\right)}{\rho\left(u^{-1}\cdot v\right)^{Q+p}}dudv.$$

Now, by the change of variables and Fubini's theorem, we obtain

$$I_1\leq\int\limits_{\Sigma}\int\limits_{\mathbb{H}}\int\limits_{0}^{\infty}\frac{F_{\delta}\left(2A_{Q,p}hM(|\nabla_{\mathbb{H}}f|)(u)\right)}{h^{p+1}}dhdud\sigma.$$

Now, for every $\sigma \in \Sigma$, we can have the following estimate:

$$\int_{\mathbb{H}}\int_{0}^{\infty} \frac{F_\delta\left(2A_{Q,p}hM(|\nabla_{\mathbb{H}} f|)(u)\right)}{h^{p+1}} dhdu = \int_{\mathbb{H}} \left[2A_{Q,p}M(|\nabla_{\mathbb{H}} f|)(u)\right]^p du \int_{0}^{\infty} F_\delta\left(t\right) t^{-p-1} dt$$

$$\leq C_{Q,p} \int_{0}^{\infty} F_\delta\left(t\right) t^{-p-1} dt \int_{\mathbb{H}} |\nabla_{\mathbb{H}} f(u)|^p du.$$

Similarly, by noting that $\rho\left(u^{-1}\cdot v\right) = \left|v^{-1}u\right|$, we can also receive

$$I_2 \leq C_{Q,p} \int_{0}^{\infty} F_\delta\left(t\right) t^{-p-1} dt \int_{\mathbb{H}} |\nabla_{\mathbb{H}} f(u)|^p du.$$

Hence, we can conclude that

$$\int_{\mathbb{H}}\int_{\mathbb{H}} \frac{F_\delta\left(|f(u)-f(v)|\right)}{\rho\left(u^{-1}\cdot v\right)^{Q+p}} dudv \leq C_{Q,p} \int_{0}^{\infty} F_\delta\left(t\right) t^{-p-1} dt \int_{\mathbb{H}} |\nabla_{\mathbb{H}} f(u)|^p du.$$

(a) The assumptions on F, we can find four positive constants m, M, λ and σ with $m < M$ such that

$$|\{t \in [m, M] : F(t) \geq \lambda\}| \geq \sigma.$$

Since F is continuous on $[0,\infty)$, there exists an interval $A \neq \emptyset$ such that

$$A \subset \{t \in [m, M] : F(t) \geq \lambda\}.$$

Sincw,

$$\sup_{0<\delta<1} \int_{\mathbb{H}}\int_{\mathbb{H}} \frac{F_\delta\left(|f(u)-f(v)|\right)}{\rho\left(u^{-1}\cdot v\right)^{Q+p}} dudv < \infty,$$

we get

$$\sup_{0<\delta<1} \int_{\mathbb{H}}\int_{\mathbb{H}} \frac{\delta^p \chi_A\left(\frac{|f(u)-f(v)|}{\delta}\right)}{\rho\left(u^{-1}\cdot v\right)^{Q+p}} dudv < \infty.$$

This implies

$$\sup_{0<\varepsilon<1} \int_{0}^{1}\int_{\mathbb{H}}\int_{\mathbb{H}} \frac{\varepsilon\delta^{\varepsilon+p-1} \chi_A\left(\frac{|f(u)-f(v)|}{\delta}\right)}{\rho\left(u^{-1}\cdot v\right)^{Q+p}} dudvd\delta < \infty.$$

By Fubini's theorem,

$$\sup_{0<\varepsilon<1} \int_{\mathbb{H}}\int_{\substack{\mathbb{H} \\ |f(u)-f(v)|\leq m}} \int_{0}^{1} \frac{\varepsilon\delta^{\varepsilon+p-1} \chi_A\left(\frac{|f(u)-f(v)|}{\delta}\right)}{\rho\left(u^{-1}\cdot v\right)^{Q+p}} d\delta dudv < \infty.$$

Since

$$\frac{|f(u)-f(v)|}{\delta} \leq M,$$

we have

$$\delta^{\varepsilon+p-1} \geq M^{-p-\varepsilon+1} \left| f(u) - f(v) \right|^{\varepsilon+p-1}.$$

Hence

$$\sup_{0<\varepsilon<1} \underset{|f(u)-f(v)|\leq m}{\int_{\mathbb{H}}\int_{\mathbb{H}}} \frac{\varepsilon \left| f(u) - f(v) \right|^{\varepsilon+p-1}}{\rho\left(u^{-1}\cdot v\right)^{Q+p}} \int_0^1 \chi_A \left(\frac{\left| f(u) - f(v) \right|}{\delta} \right) d\delta du dv < \infty. \tag{2.10}$$

Moreover, since $A \subset [m, M]$,

$$\int_0^1 \chi_A \left(\frac{t}{\delta} \right) d\delta = \int_0^\infty \chi_A \left(\frac{t}{\delta} \right) d\delta = t \int_0^\infty \chi_A \left(\frac{1}{\delta} \right) d\delta = C(A)\, t, \ \ t \leq m. \tag{2.11}$$

Here

$$C(A) = \int_0^\infty \chi_A \left(\frac{1}{\delta} \right) d\delta > 0.$$

From (2.10) and (2.11), we get

$$\sup_{0<\varepsilon<1} \underset{|f(u)-f(v)|\leq m}{\int_{\mathbb{H}}\int_{\mathbb{H}}} \frac{\varepsilon \left| f(u) - f(v) \right|^{\varepsilon+p-1}}{\rho\left(u^{-1}\cdot v\right)^{Q+p}} du dv < \infty.$$

Also,

$$\underset{|f(u)-f(v)|>m}{\int_{\mathbb{H}}\int_{\mathbb{H}}} \frac{1}{\rho\left(u^{-1}\cdot v\right)^{Q+p}} du dv < \infty.$$

Setting

$$\widetilde{f} = \frac{f}{m},$$

and using Theorem 2.7, we get $\widetilde{f} \in W^{1,p}(\mathbb{H})$. Hence $f \in W^{1,p}(\mathbb{H})$. $\square$

The second result in this section is to weaken the statement (3) in Theorem 2.7. More precisely, we will prove that

Theorem 2.13. Let $1 < p < \infty$ and $f \in L^p(\mathbb{H})$ be such that

$$\sup_{n\in\mathbb{N}} \underset{|f(u)-f(v)|>\delta_n}{\int_{\mathbb{H}}\int_{\mathbb{H}}} \frac{\delta_n^p}{\rho\left(u^{-1}\cdot v\right)^{Q+p}} du dv < \infty.$$

Here $(\delta_n)_{n\in\mathbb{N}}$ is some arbitrary sequence of positive numbers with

$$\delta_0 = 1$$

$$\delta_n \leq \delta_{n-1} \leq 2\delta_n$$

$$\lim_{n\to\infty} \delta_n = 0.$$

Then $f \in W^{1,p}(\mathbb{H})$.

We notice that one could replace number 2 in the condition of the sequence (δ_n) by an arbitrary number $c > 1$.

Proof. Setting

$$\sup_{n\in\mathbb{N}} \underset{|f(u)-f(v)|>\delta_n}{\int_{\mathbb{H}}\int_{\mathbb{H}}} \frac{\delta_n^p}{\rho(u^{-1}\cdot v)^{Q+p}} dudv < C,$$

then, it's clear that

$$\underset{|f(u)-f(v)|>1}{\int_{\mathbb{H}}\int_{\mathbb{H}}} \frac{1}{\rho(u^{-1}\cdot v)^{Q+p}} dudv < C. \tag{2.12}$$

So we just need to prove for all $\varepsilon \in (0,1)$,

$$\underset{|f(u)-f(v)|\leq 1}{\int_{\mathbb{H}}\int_{\mathbb{H}}} \frac{\varepsilon\,|f(u)-f(v)|^{p+\varepsilon}}{\rho(u^{-1}\cdot v)^{Q+p}} dudv \leq C_{Q,p}$$

since by Theorem 2.7, we get the assertion.

Now, for every $\varepsilon \in (0,1)$, since

$$\underset{|f(u)-f(v)|>\delta_n}{\int_{\mathbb{H}}\int_{\mathbb{H}}} \frac{\delta_n^p}{\rho(u^{-1}\cdot v)^{Q+p}} dudv < C,$$

we get for every $n \geq 0$:

$$\varepsilon(\delta_n - \delta_{n+1})\,\delta_n^{\varepsilon-1} \underset{|f(u)-f(v)|>\delta_n}{\int_{\mathbb{H}}\int_{\mathbb{H}}} \frac{\delta_n^p}{\rho(u^{-1}\cdot v)^{Q+p}} dudv \leq C\varepsilon(\delta_n - \delta_{n+1})\,\delta_n^{\varepsilon-1}.$$

Hence

$$\sum_{n\geq 0} \underset{|f(u)-f(v)|>\delta_n}{\int_{\mathbb{H}}\int_{\mathbb{H}}} \frac{\varepsilon(\delta_n - \delta_{n+1})\,\delta_n^{\varepsilon+p-1}}{\rho(u^{-1}\cdot v)^{Q+p}} dudv \leq C\sum_{n\geq 0}\varepsilon(\delta_n - \delta_{n+1})\,\delta_n^{\varepsilon-1}. \tag{2.13}$$

Now, if we denote $h(\delta) = \varepsilon\delta^{\varepsilon-1}$, then we have by the Lebesgue Dominated Convergence Theorem and noting that h is a decreasing function:

$$
\begin{aligned}
1 &= \int_0^1 h(\delta)\, d\delta \qquad (2.14)\\
&= \sum_{n\geq 0} \int_{\delta_{n+1}}^{\delta_n} h(\delta)\, d\delta \\
&\geq \sum_{n\geq 0} (\delta_n - \delta_{n+1})\, h(\delta_n) \\
&= \sum_{n\geq 0} \varepsilon (\delta_n - \delta_{n+1})\, \delta_n^{\varepsilon-1}.
\end{aligned}
$$

Thus, from (2.13), one has

$$
\sum_{n\geq 0} \underset{|f(u)-f(v)|>\delta_n}{\int_{\mathbb{H}}\int_{\mathbb{H}}} \frac{\varepsilon (\delta_n - \delta_{n+1})\, \delta_n^{\varepsilon+p-1}}{\rho(u^{-1}\cdot v)^{Q+p}} dudv \leq C \qquad (2.15)
$$

Noting that

$$
\begin{aligned}
&\sum_{n\geq 0} \underset{|f(u)-f(v)|>\delta_n}{\int_{\mathbb{H}}\int_{\mathbb{H}}} \frac{\varepsilon (\delta_n - \delta_{n+1})\, \delta_n^{\varepsilon+p-1}}{\rho(u^{-1}\cdot v)^{Q+p}} dudv \qquad (2.16)\\
\geq &\underset{|f(u)-f(v)|\leq 1}{\int_{\mathbb{H}}\int_{\mathbb{H}}} \sum_{n\geq 0} \frac{\varepsilon (\delta_n - \delta_{n+1})\, \delta_n^{\varepsilon+p-1}}{\rho(u^{-1}\cdot v)^{Q+p}} \chi_{\{|f(u)-f(v)|>\delta_n\}}(u,v)\, dudv.
\end{aligned}
$$

Now, fix (u,v) such that

$$
0 < |f(u) - f(v)| \leq 1
$$

and denote $n_{(u,v)}$ the smallest integer number such that

$$
\delta_{n_{(u,v)}} < |f(u) - f(v)|
$$

Then

$$
\begin{aligned}
&\sum_{n\geq 0} \frac{\varepsilon (\delta_n - \delta_{n+1})\, \delta_n^{\varepsilon+p-1}}{\rho(u^{-1}\cdot v)^{Q+p}} \chi_{\{|f(u)-f(v)|>\delta_n\}}(u,v) \qquad (2.17)\\
= &\sum_{n\geq n_{(u,v)}} \frac{\varepsilon (\delta_n - \delta_{n+1})\, \delta_n^{\varepsilon+p-1}}{\rho(u^{-1}\cdot v)^{Q+p}} \chi_{\{|f(u)-f(v)|>\delta_n\}}(u,v). \\
= &\sum_{n\geq n_{(u,v)}} \frac{\varepsilon (\delta_n - \delta_{n+1})\, \delta_n^{\varepsilon+p-1}}{\rho(u^{-1}\cdot v)^{Q+p}}.
\end{aligned}
$$

We claim that

$$
\frac{|f(u) - f(v)|}{2} \leq \delta_{n_{(u,v)}} < |f(u) - f(v)|. \qquad (2.18)
$$

Indeed, we could suppose by contradiction that

$$\delta_{n_{(u,v)}} < \frac{|f(u)-f(v)|}{2} < |f(u)-f(v)| \leq \delta_{n_{(u,v)}-1},$$

then

$$\begin{aligned}
\delta_{n_{(u,v)}-1} - \delta_{n_{(u,v)}} &> |f(u)-f(v)| - \frac{|f(u)-f(v)|}{2} \\
&= \frac{|f(u)-f(v)|}{2} \\
&> \delta_{n_{(u,v)}}
\end{aligned}$$

which is impossible by the assumption of the sequence (δ_n).

Set

$$k(\delta) = \frac{\varepsilon \delta^{\varepsilon+p-1}}{\rho(u^{-1}\cdot v)^{Q+p}} \text{ on the interval } 0 \leq \delta < |f(u)-f(v)|.$$

Noting that this function is increasing, arguing as (2.14), we obtain by (2.18):

$$\begin{aligned}
\frac{1}{(p+1)\,2^{p+1}} \frac{\varepsilon\,|f(u)-f(v)|^{p+\varepsilon}}{\rho(u^{-1}\cdot v)^{Q+p}} &\leq \frac{1}{(p+\varepsilon)2^{p+\varepsilon}} \frac{\varepsilon\,|f(u)-f(v)|^{p+\varepsilon}}{\rho(u^{-1}\cdot v)^{Q+p}} \\
&= \int_0^{\frac{|f(u)-f(v)|}{2}} k(\delta)\,d\delta \\
&\leq \int_0^{\delta_{n_{(u,v)}}} k(\delta)\,d\delta \\
&= \sum_{n \geq n_{(u,v)}} \int_{\delta_{n+1}}^{\delta_n} k(\delta)\,d\delta \\
&\leq \sum_{n \geq n_{(u,v)}} (\delta_n - \delta_{n+1})\,k(\delta_n) \\
&= \sum_{n \geq n_{(u,v)}} \frac{\varepsilon(\delta_n - \delta_{n+1})\,\delta_n^{\varepsilon+p-1}}{\rho(u^{-1}\cdot v)^{Q+p}}. \qquad (2.19)
\end{aligned}$$

Hence, by (2.15), (2.16), (2.17) and (2.19), we get

$$\begin{aligned} C &\geq \sum_{n\geq 0} \underset{|f(u)-f(v)|>\delta_n}{\int_{\mathbb{H}}\int_{\mathbb{H}}} \frac{\varepsilon\left(\delta_n-\delta_{n+1}\right)\delta_n^{\varepsilon+p-1}}{\rho\left(u^{-1}\cdot v\right)^{Q+p}} dudv \\ &\geq \underset{|f(u)-f(v)|\leq 1}{\int_{\mathbb{H}}\int_{\mathbb{H}}} \sum_{n\geq 0} \frac{\varepsilon\left(\delta_n-\delta_{n+1}\right)\delta_n^{\varepsilon+p-1}}{\rho\left(u^{-1}\cdot v\right)^{Q+p}} \chi_{\{|f(u)-f(v)|>\delta_n\}}\left(u,v\right) dudv \\ &= \underset{|f(u)-f(v)|\leq 1}{\int_{\mathbb{H}}\int_{\mathbb{H}}} \sum_{n\geq n_{(u,v)}} \frac{\varepsilon\left(\delta_n-\delta_{n+1}\right)\delta_n^{\varepsilon+p-1}}{\rho\left(u^{-1}\cdot v\right)^{Q+p}} dudv \\ &\geq \underset{|f(u)-f(v)|\leq 1}{\int_{\mathbb{H}}\int_{\mathbb{H}}} \frac{1}{(p+1)\,2^{p+1}} \frac{\varepsilon\left|f(u)-f(v)\right|^{p+\varepsilon}}{\rho\left(u^{-1}\cdot v\right)^{Q+p}} dudv. \end{aligned}$$

Thus we can conclude the assertion

$$\sup_{0<\varepsilon<1} \underset{|f(u)-f(v)|\leq 1}{\int_{\mathbb{H}}\int_{\mathbb{H}}} \frac{\varepsilon\left|f(u)-f(v)\right|^{p+\varepsilon}}{\rho\left(u^{-1}\cdot v\right)^{Q+p}} dudv < \infty.$$

By Theorem 2.7 (statement 2), we have $f \in W^{1,p}\left(\mathbb{H}\right)$. □

CHAPTER 3 NEW CHARACTERIZATIONS OF SOBOLEV SPACES ON CARNOT GROUPS

3.1 Some Useful Lemmas

In this section, we will introduce several useful lemmas which are used to prove the main results.

Lemma 3.1. Let $f \in W^{1,p}(\mathbb{G})\,,\ 1 < p < \infty$. Then we have

$$\iint\limits_{|f(u)-f(v)|>\delta} \frac{\delta^p}{|u^{-1}v|_{\mathbb{G}}^{Q+p}} dudv \le C_{Q,p} \int\limits_{\mathbb{G}} |\nabla_{\mathbb{H}} f(u)|^p \, du,\ \forall \delta > 0$$

where $C_{Q,p}$ is a positive constant depending only on Q and p.

The proof is similar to the proof of Lemma 2.2.

Lemma 3.2. Let $f \in W^{1,p}(\mathbb{G})\,,\ 1 < p < \infty$. We denote

$$I(\delta) = \underset{|f(u)-f(v)|>\delta}{\int\limits_{\mathbb{G}} \int\limits_{\mathbb{G}}} \frac{\delta^p}{|u^{-1} \cdot v|_{\mathbb{G}}^{Q+p}} dudv$$

$$J(\varepsilon) = \underset{|f(u)-f(v)|\le 1}{\int\limits_{\mathbb{G}} \int\limits_{\mathbb{G}}} \frac{\varepsilon\,|f(u) - f(v)|^{p+\varepsilon}}{|u^{-1} \cdot v|_{\mathbb{G}}^{Q+p}} dudv.$$

Then

$$\liminf_{\varepsilon \to 0} J(\varepsilon) \ge p \liminf_{\delta \to 0} I(\delta)$$

$$\limsup_{\varepsilon \to 0} J(\varepsilon) \le p \limsup_{\delta \to 0} I(\delta)\,.$$

The proof of this lemma is similar to Lemma 4.3.

Remark: In the setting of Euclidean spaces, $\liminf\limits_{\delta \to 0}$ is rather easy to evaluate. However, in our setting of Carnot group, the similar approach is not available because of the underlying geometry. That is why we try to find the relations of $I(\delta)$ and $J(\varepsilon)$.

The following is from [1]

Lemma 3.3. If $f \in C^1_{\mathbb{G}}(\mathbb{G}, \mathbb{R})$ then

$$\omega_x(h) \doteq |f(x \cdot h) - f(x) - \langle \nabla_X f(x), h' \rangle_X|$$

is such that

i) $\frac{\omega_x(h)}{|h|_{\mathbb{G}}} \to 0$ as $h \to 0$, uniformly for x on compact sets;

ii) $\frac{\omega_x(h)}{|h|_{\mathbb{G}}} \leq C_K \ \forall x \in K compact, \forall |h|_{\mathbb{G}} < 1$.

Lemma 3.4. There holds

$$\tfrac{1}{p} K_{Q,p} \int\limits_{\mathbb{G}} |\nabla_X f(u)|^p du \leq \liminf_{\delta \to 0} I(\delta),$$

for any $f \in C^1_{\mathbb{G}}(\mathbb{G}, \mathbb{R})$, $1 < p < \infty$ and $K_{Q,p} = \int\limits_{\Sigma} |\langle v', y' \rangle|^p dy$ with $y', v' \in \mathbb{R}^{m_1}$ and $|v'|^{m_1}_{\mathbb{R}} = 1$.

Proof. First, we notice by the change of variable that

$$\int_{\mathbb{G}} \int_{\mathbb{G}}\limits_{|f(u)-f(v)|>\delta} \frac{\delta^p}{|u^{-1} \cdot v|_{\mathbb{G}}^{Q+p}} du dv = \int_{\sum} \int_{\mathbb{G}} \int_0^\infty\limits_{|\frac{f(u \cdot \delta h \sigma) - f(u)}{\delta h}|h>1} \frac{1}{h^{p+1}} dh du d\sigma.$$

Now, fix $\sigma = (\sigma', \sigma'') \in \sum$, with $\sigma' \in \mathbb{R}^{m_1}$, we will show that

$$\tfrac{1}{p} \int\limits_{\mathbb{G}} |\langle \nabla_X f(u), \sigma' \rangle_X|^p du \leq \liminf_{\delta \to 0} \int\limits_{\mathbb{G}} \int\limits_0^\infty\limits_{|\frac{f(u \cdot \delta h \sigma) - f(u)}{\delta h}|h>1} \tfrac{1}{h^{p+1}} dh d\delta.$$

Here, $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^{m_1}$. Indeed, since for a.e. $(h, u) \in (0, \infty) \times \mathbb{G}$, by lemma 3.3 :

$$\frac{1}{h^{p+1}} \chi_{\left\{\left|\frac{f(u \cdot \delta h \sigma) - f(u)}{\delta h}\right| h > 1\right\}} (h, u) \overset{\delta \to 0}{\to} \frac{1}{h^{p+1}} \chi_{\{|\langle \nabla_X f(u), \sigma' \rangle_X| h > 1\}} (h, u),$$

by Fatou's lemma, we have

$$\liminf_{\delta \to 0} \int\limits_{\mathbb{G}} \int\limits_0^\infty\limits_{\left|\frac{f(u \cdot \delta h \sigma) - f(u)}{\delta h}\right| h > 1} \frac{1}{h^{p+1}} dh du \geq \int\limits_{\mathbb{G}} \int\limits_0^\infty \frac{1}{h^{p+1}} \chi_{\{|\langle \nabla_X f(u), \sigma' \rangle_X| h > 1\}} (h, u)\, dh du$$

$$= \frac{1}{p} \int\limits_{\mathbb{G}} \left|\langle \nabla_X f(u), \sigma' \rangle_X\right|^p du.$$

Now, again by the Fatou's lemma, we obtain

$$\liminf_{\delta\to 0} \iint\limits_{|f(u)-f(v)|>\delta} \frac{\delta^p}{\rho\left(u^{-1}\cdot v\right)^{Q+p}} dudv = \liminf_{\delta\to 0} \int\limits_{\Sigma}\int\limits_{\mathbb{G}}\int\limits_{0}^{\infty}\limits_{\left|\frac{f(u\cdot\delta h\sigma)-f(u)}{\delta h}\right|h>1} \frac{1}{h^{p+1}} dhdud\sigma$$

$$\geq \int\limits_{\Sigma} \liminf_{\delta\to 0} \int\limits_{\mathbb{G}}\int\limits_{0}^{\infty}\limits_{\left|\frac{f(u\cdot\delta h\sigma)-f(u)}{\delta h}\right|h>1} \frac{1}{h^{p+1}} dhdud\sigma$$

$$\geq \frac{1}{p}\int\limits_{\Sigma}\int\limits_{\mathbb{G}} \left|\langle\nabla_X f(u),\sigma'\rangle_X\right|^p dud\sigma$$

$$= \frac{1}{p}K_{Q,p}\int\limits_{\mathbb{G}} |\nabla_X f(u)|^p\, du.$$

Here, $K_{Q,p} = \int\limits_{\Sigma} |\langle v', y'\rangle|^p dy$ with $y', v' \in \mathbb{R}^{m_1}$ and $|v'|_{\mathbb{R}}^{m_1} = 1$.

This expression does not depend on v', since by an orthogonal $\mathbb{G}$-change of basis which does not alter the measure nor the homogeneous norm(so that also the domain of integration does not change), it is possible to choose any other unitrary vector of $\mathbb{R}^{m_1}$ by rotation: set $A \in O(m_1, \mathbb{R})$ and $u' = A^T v'$,then $\int\limits_{\Sigma} |\langle v', y'\rangle|^p dy = \int\limits_{\Sigma} |\langle v', Ay'\rangle|^p dy = \int\limits_{\Sigma} |\langle u', y'\rangle|^p dy.$ □

Lemma 3.5. Let $f \in C^1_{\mathbb{G}}$ with compact support in $\mathbb{G}$. Then

$$\limsup_{\varepsilon\to 0} J(\varepsilon) \leq K_{Q,p}\int\limits_{\mathbb{G}} |\nabla_X f(u)|^p\, du.$$

The proof of this lemma is similar to Lemma 2.5.

3.2 Main Results

Theorem 3.6. Let $1 < p < \infty$ and $f \in W^{1,p}(\mathbb{G})$. Then

(a) There exists a positive constant $C_{Q,p}$ depending only on Q, p such that

$$\sup_{0<\varepsilon<1} \int\limits_{\mathbb{G}}\int\limits_{\mathbb{G}}\limits_{|f(u)-f(v)|\leq 1} \frac{\varepsilon\,|f(u)-f(v)|^{p+\varepsilon}}{|u^{-1}\cdot v|_{\mathbb{G}}^{Q+p}} dudv + \int\limits_{\mathbb{G}}\int\limits_{\mathbb{G}}\limits_{|f(u)-f(v)|>1} \frac{1}{|u^{-1}\cdot v|_{\mathbb{G}}^{Q+p}} dudv \leq C_{Q,p}\int\limits_{\mathbb{G}} |\nabla_X f(u)|^p\, du.$$

(b) There holds

$$K_{Q,p}\int\limits_{\mathbb{G}} |\nabla_{\mathbb{G}} f(u)|^p\, du = \lim_{\varepsilon\to 0} \int\limits_{\mathbb{G}}\int\limits_{\mathbb{G}}\limits_{|f(u)-f(v)|\leq 1} \frac{\varepsilon\,|f(u)-f(v)|^{p+\varepsilon}}{\rho\left(u^{-1}\cdot v\right)^{Q+p}} dudv.$$

where $K_{Q,p}$ is a constant defined as follows

$$K_{Q,p} = \int\limits_{\Sigma} |\langle v', y' \rangle|^p dy$$

with $y', v' \in \mathbb{R}^{m_1}$ and $|v'|_{\mathbb{R}}^{m_1} = 1$.

(c) There exists a positive constant $C_{Q,p}$ such that

$$\underset{|f(u)-f(v)|>\delta}{\int\limits_{\mathbb{G}}\int\limits_{\mathbb{G}}} \frac{\delta^p}{|u^{-1}\cdot v|_{\mathbb{G}}^{Q+p}} dudv \leq C_{Q,p} \int\limits_{\mathbb{G}} |\nabla_X f(u)|^p \, du, \ \forall \delta > 0.$$

(d) Moreover

$$\frac{1}{p} K_{Q,p} \int\limits_{\mathbb{G}} |\nabla_X f(u)|^p \, du = \liminf_{\delta \to 0} \underset{|f(u)-f(v)|>\delta}{\int\limits_{\mathbb{G}}\int\limits_{\mathbb{G}}} \frac{\delta^p}{|u^{-1}\cdot v|_{\mathbb{G}}^{Q+p}} dudv.$$

Theorem 3.7. Let $1 < p < \infty$ and $f \in L^p(\mathbb{G})$. Then the following are equivalent:

(1) $f \in W^{1,p}(\mathbb{G})$.

(2)

$$\sup_{0<\varepsilon<1} \underset{|f(u)-f(v)|\leq 1}{\int\limits_{\mathbb{G}}\int\limits_{\mathbb{G}}} \frac{\varepsilon\, |f(u)-f(v)|^{p+\varepsilon}}{|u^{-1}\cdot v|_{\mathbb{G}}^{Q+p}} dudv + \underset{|f(u)-f(v)|>1}{\int\limits_{\mathbb{G}}\int\limits_{\mathbb{G}}} \frac{1}{|u^{-1}\cdot v|_{\mathbb{G}}^{Q+p}} dudv < \infty.$$

(3)

$$\sup_{0<\delta<1} \underset{|f(u)-f(v)|>\delta}{\int\limits_{\mathbb{G}}\int\limits_{\mathbb{G}}} \frac{\delta^p}{|u^{-1}\cdot v|_{\mathbb{G}}^{Q+p}} dudv < \infty.$$

(4) There exists a nonnegative function $F \in L^p(\mathbb{G})$ such that

$$|f(u) - f(v)| \leq |u^{-1} \cdot v|_{\mathbb{G}} \left(F(u) + F(v)\right) \text{ for a.e. } u, \ v \in \mathbb{G}.$$

The proofs of Theorem 3.6 and 3.7 are similar to Theorem 2.6 and 2.7.

3.3 The Case $p = 1$

In this section, we will investigate the special case $p = 1$. At the beginning, we recall the definition of the space $BV(\Omega)$ of functions with bounded variation in $\Omega \subset \mathbb{G}$.

Definition Let $\Omega \subseteq \mathbb{G}$ be open and $f \in L^1(\Omega)$. Then, f has bounded $\mathbb{G}$-variation in Ω if

$$|\nabla_{\mathbb{G}} f|(\Omega) \doteq \sup\{\textstyle\int_{\Omega} f div_{\mathbb{G}}(\psi) dL^n : \psi \in C_0^1(\Omega, H\mathbb{G}), |\psi| \leq 1\} < \infty$$

where $|\nabla_{\mathbb{G}f}|(\Omega)$ is called $\mathbb{G}$-variation of f in Ω. We denote by $BV_{\mathbb{G}}(\Omega)$ the vector space of functions of bounded $\mathbb{G}$-variation in Ω.

Theorem 3.8. Let f be a function in $L^1(\mathbb{G})$ satisfying

$$\sup_{0<\delta<1} \underset{|f(u)-f(v)|>\delta}{\int_{\mathbb{G}}\int_{\mathbb{G}}} \frac{\delta}{|u^{-1}\cdot v|_{\mathbb{G}}^{Q+1}} dudv < \infty.$$

Then $f \in BV(\mathbb{G})$.

The proof of this theorem is similar to Theorem 2.10.

Using Theorem 3.8, we can also have the following Lipschitz type characterization of BV space:

Proposition 3.9. Let $f \in L^1(\mathbb{G})$ be such that there exists a nonnegative function $F \in L^1(\mathbb{G})$ satisfying

$$|f(u)-f(v)| \leq |u^{-1}\cdot v|_{\mathbb{G}} \left(F(u)+F(v)\right) \text{ for a.e. } u,\ v \in \mathbb{G}.$$

Then $f \in BV(\mathbb{G})$.

The proof of this proposition is similar to Theorem 2.11.

3.4 Some generalizations and variants

In this section, we will study some generalizations of the above results. The next Theorem is a generalized result of Theorem 3.7:

Theorem 3.10. Let $f \in L^p(\mathbb{G})$, $1 < p < \infty$ and $F : [0,\infty) \to [0,\infty)$ be continuous such that

$$\int_0^\infty F(t)\, t^{-p-1} dt = 1.$$

Set

$$F_\delta(t) = \delta^p F\left(\frac{t}{\delta}\right),\ \delta > 0.$$

Then we have

(a) If

$$\sup_{0<\delta<1} \int_{\mathbb{G}}\int_{\mathbb{G}} \frac{F_\delta(|f(u)-f(v)|)}{|u^{-1}\cdot v|_{\mathbb{G}}^{Q+p}} dudv < \infty$$

and

$$\int\limits_{\mathbb{G}}\int\limits_{\mathbb{G}}_{|f(u)-f(v)|>\delta} \frac{1}{|u^{-1}\cdot v|_{\mathbb{G}}^{Q+p}} dudv < \infty,\ \forall \delta > 0,$$

then $g \in W^{1,p}(\mathbb{G})$.

(b) If $g \in W^{1,p}(\mathbb{G})$ and

$$\int\limits_0^\infty F_\delta(t)\, t^{-p-1} dt < \infty,\ \forall \delta > 0,$$

then

$$\int\limits_{\mathbb{G}}\int\limits_{\mathbb{G}} \frac{F_\delta(|f(u)-f(v)|)}{|u^{-1}\cdot v|_{\mathbb{G}}^{Q+p}} dudv \le C_{Q,p} \int\limits_0^\infty F_\delta(t)\, t^{-p-1} dt \int\limits_{\mathbb{G}} |\nabla_{\mathbb{H}} f(u)|^p \, du,\ \forall \delta > 0.$$

The proof of this Theorem is similar to Theorem 2.12.

CHAPTER 4 NEW CHARACTERIZATIONS OF SECOND ORDER SOBOLEV SPACES ON EUCLIDEAN SPACES

4.1 Main Results

The first purpose of this chapter is to prove the following estimates for functions in the Sobolev spaces $W^{2,p}(\mathbb{R}^N)$.

Theorem 4.1. Let $g \in W^{2,p}\left(\mathbb{R}^N\right),\ 1 < p < \infty$. Then there exists a constant $C_{N,p}$ such that

(1)

$$\underset{\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|>\delta}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{\delta^p}{|x-y|^{N+2p}} dxdy \le C_{N,p} \int_{\mathbb{R}^N} |\Delta g|^p \, dx, \ \forall \delta > 0.$$

(2)

$$\lim_{\delta \to 0} \underset{\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|>\delta}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{\delta^p}{|x-y|^{N+2p}} dxdy = \frac{1}{2^{2p+1}p} \int_{\mathbb{S}^{N-1}} \int_{\mathbb{R}^N} \left|D^2 g\left(x\right)\left(\sigma,\sigma\right)\right|^p dxd\sigma.$$

(3)

$$\begin{aligned} &\sup_{0<\varepsilon<1} \underset{\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|\le 1}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{\varepsilon \left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|^{p+\varepsilon}}{|x-y|^{N+2p}} dxdy \\ &+ \underset{\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|>1}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{1}{|x-y|^{N+2p}} dxdy \\ &\le C_{N,p} \int_{\mathbb{R}^N} |\Delta g|^p \, dx. \end{aligned}$$

(4)

$$\begin{aligned} &\lim_{\varepsilon \to 0} \underset{\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|\le 1}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{\varepsilon \left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|^{p+\varepsilon}}{|x-y|^{N+2p}} dxdy \\ &= \frac{1}{2^{2p+1}} \int_{\mathbb{S}^{N-1}} \int_{\mathbb{R}^N} \left|D^2 g\left(x\right)\left(\sigma,\sigma\right)\right|^p dxd\sigma. \end{aligned}$$

Here we have used the notation

$$\left|D^2 g\left(x\right)\left(\sigma,\sigma\right)\right| = \sum_{1 \le i_1, i_2 \le N} \sigma_{i_1}\sigma_{i_2} \frac{\partial^2 g}{\partial x_{i_1} \partial x_{i_2}}\left(x\right).$$

We will use this notation frequently throughout this chapter..

Theorem 4.2. Let $g \in W^{2,p}(\mathbb{R}^N)$, $1 < p < \infty$. Then there exists a constant $C_{N,p}$ such that

1.
$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}_{|g(x)-g(y)-\nabla g(y)(x-y)|>\delta} \frac{\delta^p}{|x-y|^{N+2p}} dxdy \leq C_{N,p} \int_{\mathbb{R}^N} |\triangle g|^p dx, \forall \delta > 0.$$

2.
$$\lim_{\delta\to 0} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}_{|g(x)-g(y)-\nabla g(y)(x-y)|>\delta} \frac{\delta^p}{|x-y|^{N+2p}} dxdy = \frac{1}{2^{p+1}p} \int_{\mathbb{S}^{N-1}} \int_{\mathbb{R}^N} |D^2 g(x)(\sigma,\sigma)|^p dx d\sigma.$$

3.
$$\sup_{0<\varepsilon<1} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}_{|g(x)-g(y)-\nabla g(y)(x-y)|\leq 1} \frac{\varepsilon |g(x)-g(y)-\nabla g(y)(x-y)|^{p+\varepsilon}}{|x-y|^{N+2p}} dxdy$$
$$+ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}_{|g(x)-g(y)-\nabla g(y)(x-y)|>1} \frac{1}{|x-y|^{N+2p}} dxdy \leq C_{N,p} \int_{\mathbb{R}^N} |\triangle g|^p dx.$$

4.
$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}_{|g(x)-g(y)-\nabla g(y)(x-y)|\leq 1} \frac{\varepsilon |g(x)-g(y)-\nabla g(y)(x-y)|^{p+\varepsilon}}{|x-y|^{N+2p}} dxdy$$
$$= \frac{1}{2^{p+1}} \int_{\mathbb{S}^{N-1}} \int_{\mathbb{R}^N} |D^2 g(x)(\sigma,\sigma)|^p dx d\sigma.$$

Using Theorems 4.1 and 4.2, We can set up the new characterizations of the Sobolev space $W^{2,p}(\mathbb{R}^N)$ using the method of second order differences and the Taylor remainder of first order which are our main aims of this chapter. Indeed, we prove the following two theorems:

Theorem 4.3. Let $g \in A^p(\mathbb{R}^N)$, $1 < p < \infty$ where $A^p(\mathbb{R}^N)$ is the set of all $g \in L^p\left(\mathbb{R}^N\right)$ such that $\exists \{g_n\}$ and $A(g) > 0$:

1) $\| g_n\|_p \leq A(g)$;

2) $\left| g_n(x) + g_n(y) - 2g_n\left(\frac{x+y}{2}\right)\right| \leq A(g)$;

3) $\left| g_n(x) + g_n(y) - 2g_n\left(\frac{x+y}{2}\right)\right| \leq A(g) \left| g(x) + g(y) - 2g\left(\frac{x+y}{2}\right)\right|$ a.e. $x, y \in \mathbb{R}^N$;

4)$g_n \to g$ a.e.

Then the following are equivalent:

(1) $g \in W^{2,p}\left(\mathbb{R}^N\right)$.

(2)

$$\sup_{0<\delta<1} \underset{\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|>\delta}{\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}} \frac{\delta^p}{|x-y|^{N+2p}}dxdy < \infty.$$

(3)

$$\sup_{0<\varepsilon<1} \underset{\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|\leq 1}{\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}} \frac{\varepsilon\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|^{p+\varepsilon}}{|x-y|^{N+2p}}dxdy$$

$$+ \underset{\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|>1}{\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}} \frac{1}{|x-y|^{N+2p}}dxdy$$

$$< \infty.$$

Theorem 4.4. Let $g \in B^p(\mathbb{R}^N)$, $1 < p < \infty$ where $B^p(\mathbb{R}^N)$ is the set of all $g \in L^p\left(\mathbb{R}^N\right)$ such that $\exists \{g_n\}$ and $B(g) > 0$:

1) $\| g_n\|_p \leq B(g)$;

2) $|g_n(x) - g_n(y) - \nabla g_n(y)(x-y)| \leq B(g)$;

3) $|g_n(x) - g_n(y) - \nabla g_n(y)(x-y)| \leq B(g)\,|g(x) - g(y) - \nabla g(y)(x-y)|$ a.e. $x, y \in \mathbb{R}^N$;

4) $g_n \to g$ a.e.

Then the following are equivalent:

(1) $g \in W^{2,p}(\mathbb{R}^N)$

(2)

$$\sup_{0<\delta<1} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}{}_{|g(x)-g(y)-\nabla g(y)(x-y)|>\delta} \frac{\delta^p}{|x-y|^{N+2p}}dxdy < \infty$$

(3)

$$\sup_{0<\varepsilon<1} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}{}_{|g(x)-g(y)-\nabla g(y)(x-y)|>1} \frac{\varepsilon|g(x)-g(y)-\nabla g(y)(x-y)|^{p+\varepsilon}}{|x-y|^{N+2p}}dxdy$$

$$+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\limits_{|g(x)-g(y)-\nabla g(y)(x-y)|>1}\frac{1}{|x-y|^{N+2p}}dxdy<\infty$$

Next, we will also study the characterizations of $W^{2,p}\left(\mathbb{R}^N\right)$ by the differences of the first order gradient in the spirit of Bourgain, Brezis and Mironescu [5] and H.M. Nguyen [34,35]. More precisely, we will prove that

Theorem 4.5. *Let* $g\in W^{1,p}\left(\mathbb{R}^N\right),\ 1<p<\infty$. *Then* $g\in W^{2,p}\left(\mathbb{R}^N\right)$ *iff*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|(\nabla g(x)-\nabla g(y))\cdot(x-y)|^p}{|x-y|^{2p}}\rho_n\left(|x-y|\right)dxdy\le C,\ \forall n\ge 1,$$

for some constant $C>0$. *Moreover,*

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|(\nabla g(x)-\nabla g(y))\cdot(x-y)|^p}{|x-y|^{2p}}\rho_n\left(|x-y|\right)dxdy=\int_{\mathbb{R}^N}\int_{\mathbf{S}^{N-1}}\left|D^2g\left(x\right)\left(\sigma,\sigma\right)\right|^p d\sigma dx.$$

Here $\left(\rho_n\right)_{n\in\mathbb{N}}$ *is a sequence of nonnegative radial mollifiers satisfying*

$$\lim_{n\to\infty}\int_{\tau}^{\infty}\rho_n\left(r\right)r^{N-1}dr=0,\ \forall\tau>0,$$

$$\lim_{n\to\infty}\int_{0}^{\infty}\rho_n\left(r\right)r^{N-1}dr=1.$$

Theorem 4.6. Let $g\in C^p(\mathbb{R}^N)$, $1<p<\infty$ where $C^p(\mathbb{R}^N)$ is the set of all $g\in L^p\left(\mathbb{R}^N\right)$ such that $\exists\left\{g_n\right\}$ and $C\left(g\right)>0$:

1) $\|\ g_n\|_p\le C\left(g\right);$

2) $|(\nabla g_n\left(x\right)-\nabla g_n\left(y\right))\cdot(x-y)|\le C\left(g\right);$

3) $|(\nabla g_n\left(x\right)-\nabla g_n\left(y\right))\cdot(x-y)|\le C\left(g\right)|(\nabla g\left(x\right)-\nabla g\left(y\right))\cdot(x-y)|$ a.e.$x,y\in\mathbb{R}^N$;

4) $g_n\to g$ a.e.

Then the following are equivalent:

(1) $g\in W^{2,p}(\mathbb{R}^N)$

(2)

$$\sup_{0<\delta<1}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\limits_{|(\nabla g(x)-\nabla g(y))\cdot(x-y)|>\delta}\frac{\delta^p}{|x-y|^{N+2p}}dxdy<\infty$$

(3)

$$\sup_{0<\varepsilon<1} \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}_{|(\nabla g(x)-\nabla g(y))\cdot(x-y)|>1} \frac{\varepsilon|\left(\nabla g\left(x\right)-\nabla g\left(y\right)\right)\cdot\left(x-y\right)|^{p+\varepsilon}}{|x-y|^{N+2p}}dxdy + \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}_{|(\nabla g(x)-\nabla g(y))\cdot(x-y)|>1} \frac{1}{|x-y|^{N+2p}}dxdy < \infty$$

It is worthy noting that if we use the term $|\nabla g\left(x\right)-\nabla g\left(y\right)|$ instead of $|\left(\nabla g\left(x\right)-\nabla g\left(y\right)\right)\cdot(x-y)|$, then Theorem 4.5 is just a easy consequence of Theorem 1.1. Indeed, if

$$\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|(\nabla g(x)-\nabla g(y))|^p}{|x-y|^p}\rho_n\left(|x-y|\right)dxdy \le C,\ \forall n\ge 1,$$

then

$$\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{\left|\left(\frac{\partial g}{\partial x_i}(x)-\frac{\partial g}{\partial x_i}(y)\right)\right|^p}{|x-y|^p}\rho_n\left(|x-y|\right)dxdy \le C,\ \forall n\ge 1,\ \forall i=1,...,N.$$

Hence, by Theorem 1.1, $\frac{\partial g}{\partial x_i}\in W^{1,p}\left(\mathbb{R}^N\right)\ \forall i=1,...,N$ which means that $g\in W^{2,p}\left(\mathbb{R}^N\right)$. However, in our case, we have

$$\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|(\nabla g(x)-\nabla g(y))\cdot(x-y)|^p}{|x-y|^{2p}}\rho_n\left(|x-y|\right)dxdy \le \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|(\nabla g(x)-\nabla g(y))|^p}{|x-y|^p}\rho_n\left(|x-y|\right)dxdy.$$

4.2 Characterizations Using the Second Order Differences

In this section, we will investigate the characterizations of second order Sobolev spaces $W^{2,p}\left(\mathbb{R}^N\right)$ in terms of the second order differences, namely Theorems 4.1 and 4.3.

In order to prove the above two theorems, we will study the following useful lemmas. First of all, we will need to use the following basic lemma from Fourier analysis.

Lemma 4.7. Let $1<p<\infty$. Then there exists a constant $C_{N,p}>0$ such that for every $1\le i\le N$ we have for every $g\in L^p(\mathbb{R}^N)$

$$||\frac{\partial^2}{\partial^2 x_i}g||_{L^p(\mathbb{R}^N)} \le C_{N,p}||\triangle g||_{L^p(\mathbb{R}^N)}.$$

Proof. It suffices to prove that the operator $T=\frac{\partial^2}{\partial^2 x_i}\cdot\triangle^{-1}$ is bounded on $L^p(\mathbb{R}^N)$. It is easy to see that the operator T is a multiplier operator with the symbol $\frac{\xi_i^2}{|\xi|^2}$ which is a Marcinkiewicz multiplier which is known to be bounded on $L^p(\mathbb{R}^N)$. The operator T can also be viewed as a composition of two Riesz transforms and is known to be bounded on $L^p(\mathbb{R}^N)$. We refer to Stein's book [38]. $\square$

Lemma 4.8. There exists a constant $C_{N,p} > 0$ such that for all $\delta > 0$, all $g \in W^{2,p}\left(\mathbb{R}^N\right)$:

$$\int\limits_{\mathbb{R}^N}\int\limits_{\substack{\mathbb{R}^N \\ \left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|>\delta}} \frac{\delta^p}{|x-y|^{N+2p}} dxdy \leq C_{N,p} \int\limits_{\mathbb{R}^N} |\Delta g|^p \, dx. \tag{4.1}$$

Proof. First, using the polar coordinates, we get

$$\int\limits_{\mathbb{R}^N}\int\limits_{\substack{\mathbb{R}^N \\ \left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|>\delta}} \frac{\delta^p}{|x-y|^{N+2p}} dxdy = \int\limits_{\mathbb{S}^{N-1}}\int\limits_{\mathbb{R}^N}\int\limits_{\substack{0 \\ \left|g(x)+g(x+h\sigma)-2g\left(x+\frac{h}{2}\sigma\right)\right|>\delta}}^{\infty} \frac{\delta^p}{h^{2p+1}} dhdxd\sigma.$$

Hence, to prove (4.1), it's enough to prove that for every $\sigma \in \mathbb{S}^{N-1}$, we can obtain

$$\int\limits_{\mathbb{R}^N}\int\limits_{\substack{0 \\ \left|g(x)+g(x+h\sigma)-2g\left(x+\frac{h}{2}\sigma\right)\right|>\delta}}^{\infty} \frac{\delta^p}{h^{2p+1}} dhdx \leq C_{N,p} \int\limits_{\mathbb{R}^N} |\Delta g|^p \, dx. \tag{4.2}$$

Because of the rotation, we now can assume without loss of generality that $\sigma = e_N = (0, ..., 0, 1)$.

By the mean value theorem, one has

$$\begin{aligned}
&\left|g(x)+g(x+he_N)-2g\left(x+\frac{h}{2}e_N\right)\right| \\
&=\left|\left[g(x+he_N)-g\left(x+\frac{h}{2}e_N\right)\right]-\left[g\left(x+\frac{h}{2}e_N\right)-g(x)\right]\right| \\
&=\left|\int\limits_{x_N+\frac{h}{2}}^{x_N+h}\frac{\partial g}{\partial x_N}(x',s)\,ds-\int\limits_{x_N}^{x_N+\frac{h}{2}}\frac{\partial g}{\partial x_N}(x',s)\,ds\right| \\
&=\left|\int\limits_{x_N}^{x_N+\frac{h}{2}}\left[\frac{\partial g}{\partial x_N}\left(x',s+\frac{h}{2}\right)-\frac{\partial g}{\partial x_N}(x',s)\right]ds\right| \\
&=\left|\int\limits_{x_N}^{x_N+\frac{h}{2}}\left(\int\limits_{s}^{s+\frac{h}{2}}\frac{\partial^2 g}{\partial x_N^2}(x',t)\,dt\right)ds\right| \\
&\le\int\limits_{x_N}^{x_N+h}\int\limits_{x_N}^{x_N+h}\left|\frac{\partial^2 g}{\partial x_N^2}(x',t)\right|dtds \\
&\le\int\limits_{x_N}^{x_N+h}hM_N\left(\frac{\partial^2 g}{\partial x_N^2}\right)(x)\,ds \\
&\le h^2M_N\left(\frac{\partial^2 g}{\partial x_N^2}\right)(x)\,.
\end{aligned}$$

Thus,

$$\begin{aligned}
\int\limits_{\mathbb{R}^N}\int\limits_{0}^{\infty}\limits_{\left|g(x)+g(x+h\sigma)-2g\left(x+\frac{h}{2}\sigma\right)\right|>\delta}\frac{\delta^p}{h^{2p+1}}dhdx &\le \int\limits_{\mathbb{R}^N}\int\limits_{0}^{\infty}\limits_{h^2M_N\left(\frac{\partial^2 g}{\partial x_N^2}\right)(x)>\delta}\frac{\delta^p}{h^{2p+1}}dhdx \\
&=\int\limits_{\mathbb{R}^N}\int\limits_{\sqrt{\frac{\delta}{M_N\left(\frac{\partial^2 g}{\partial x_N^2}\right)(x)}}}^{\infty}\frac{\delta^p}{h^{2p+1}}dhdx \\
&=\frac{1}{2p}\int\limits_{\mathbb{R}^N}\left|M_N\left(\frac{\partial^2 g}{\partial x_N^2}\right)(x)\right|^p dx \\
&\le C_{N,p}\int\limits_{\mathbb{R}^N}|\Delta g|^p\,dx.
\end{aligned}$$

The proof of Lemma 4.8 is now completed. ☐

Lemma 4.9. There holds

$$\lim_{\delta\to 0}\underset{\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|>\delta}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}}\frac{\delta^p}{|x-y|^{N+2p}}dxdy=\frac{1}{2^{2p+1}p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}\left|D^2g(x)(\sigma,\sigma)\right|^p dxd\sigma$$

for all $g\in W^{2,p}\left(\mathbb{R}^N\right),\ 1<p<\infty.$

Proof. Again, by changing of variables, we obtain

$$\underset{\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|>\delta}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}}\frac{\delta^p}{|x-y|^{N+2p}}dxdy=\underset{\left|\frac{g(x)+g(x+\sqrt{\delta}h\sigma)-2g\left(x+\frac{h}{2}\sqrt{\delta}\sigma\right)}{h^2\delta}\right|h^2>1}{\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}\int_0^\infty}\frac{1}{h^{2p+1}}dhdxd\sigma.$$

Define $g_\delta:\mathbb{S}^{N-1}\to\mathbb{R}$ by

$$g_\delta(\sigma)=\underset{\left|\frac{g(x)+g(x+\sqrt{\delta}h\sigma)-2g\left(x+\frac{h}{2}\sqrt{\delta}\sigma\right)}{h^2\delta}\right|h^2>1}{\int_{\mathbb{R}^N}\int_0^\infty}\frac{1}{h^{2p+1}}dhdx.$$

We first prove that for all $\sigma\in\mathbb{S}^{N-1},\ \forall\delta>0:$

$$g_\delta(\sigma)\le C_{N,p}\int_{\mathbb{R}^N}|\Delta g|^p\,dx. \tag{4.3}$$

Indeed, again, without loss of generality, we assume that $\sigma=e_N=(0,...,0,1)$. Hence, we need to verify that

$$\underset{\left|\frac{g(x',x_N)+g(x',x_N+\sqrt{\delta}h\sigma)-2g\left(x',x_N+\frac{h}{2}\sqrt{\delta}\sigma\right)}{h^2\delta}\right|h^2>1}{\int_{\mathbb{R}^N}\int_0^\infty}\frac{1}{h^{2p+1}}dhdx\le C_{N,p}\int_{\mathbb{R}^N}|\Delta g|^p\,dx. \tag{4.4}$$

Similar to what is done in Lemma 4.8, we have

$$\left|\frac{g(x',x_N)+g(x',x_N+\sqrt{\delta}h\sigma)-2g\left(x',x_N+\frac{h}{2}\sqrt{\delta}\sigma\right)}{h^2\delta}\right|\le M_N\left(\frac{\partial^2 g}{\partial x_N^2}\right)(x).$$

Thus,

$$\underset{\left|\frac{g(x',x_N)+g(x',x_N+\sqrt{\delta}h\sigma)-2g\left(x',x_N+\frac{h}{2}\sqrt{\delta}\sigma\right)}{h^2\delta}\right|h^2>1}{\int_{\mathbb{R}^N}\int_0^\infty}\frac{1}{h^{2p+1}}dhdx\le\underset{h^2M_N\left(\frac{\partial^2 g}{\partial x_N^2}\right)(x)>1}{\int_{\mathbb{R}^N}\int_0^\infty}\frac{1}{h^{2p+1}}dhdx$$

$$\le C_{N,p}\int_{\mathbb{R}^N}|\Delta g|^p\,dx.$$

Next, we will show that

$$g_\delta(\sigma) \to \frac{1}{2^{2p+1}p} \int\limits_{\mathbb{R}^N} \left|D^2 g(x)(\sigma,\sigma)\right|^p dx \text{ as } \delta \to 0 \text{ for every } \sigma \in \mathbb{S}^{N-1} \tag{4.5}$$

where

$$\left|D^2 g(x)(\sigma,\sigma)\right| = \sum_{1 \le i_1, i_2 \le N} \sigma_{i_1}\sigma_{i_2} \frac{\partial^2 g}{\partial x_{i_1}\partial x_{i_2}}(x).$$

Again, without loss of generality, we suppose that $\sigma = e_N = (0,...,0,1)$. We write

$$g_\delta(\sigma) = \int\limits_{\mathbb{R}^N} \int\limits_0^\infty G_\delta(x,h)\, dh dx$$

where

$$G_\delta(x,h) = \frac{1}{h^{2p+1}} \chi_{\left\{\left|\frac{g(x)+g(x+\sqrt{\delta}h\sigma)-2g\left(x+\frac{h}{2}\sqrt{\delta}\sigma\right)}{h^2\delta}\right| h^2 > 1\right\}}(x,h).$$

Noting that for all $\sigma \in \mathbb{S}^{N-1}$:

$$G_\delta(x,h) \to \frac{1}{h^{2p+1}} \chi_{\{|D^2 g(x)(\sigma,\sigma)| h^2 > 4\}}(x,h) \text{ as } \delta \to 0 \text{ for a.e. } (x,h) \in \mathbb{R}^N \times [0,\infty) ,$$

and

$$G_\delta(x,h) \le \frac{1}{h^{2p+1}} \chi_{\left\{h^2 M_N\left(\frac{\partial^2 g}{\partial x_N^2}\right)(x) > 1\right\}}(x,h) \in L^1\left(\mathbb{R}^N \times [0,\infty)\right).$$

Hence, by Lebesgue's dominated convergence theorem, we get (4.5).

Using (4.3) and (4.5) and the Lebesgue dominated convergence theorem again, we can conclude that

$$\lim_{\delta \to 0} \int\limits_{\mathbb{R}^N} \int\limits_{\substack{\mathbb{R}^N \\ \left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right| > \delta}} \frac{\delta^p}{|x-y|^{N+2p}} dx dy = \frac{1}{2^{2p+1}p} \int\limits_{\mathbb{S}^{N-1}} \int\limits_{\mathbb{R}^N} \left|D^2 g(x)(\sigma,\sigma)\right|^p dx d\sigma.$$

□

Proof of Theorem 4.1:

(1) and (2) are consequences of Lemma 4.8 and Lemma 4.9.

Now we will prove (3). By (1), we get

$$\int\limits_{\mathbb{R}^N} \int\limits_{\substack{\mathbb{R}^N \\ \left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right| > \delta}} \frac{\delta^p}{|x-y|^{N+2p}} dx dy \le C_{N,p} \int\limits_{\mathbb{R}^N} |\Delta g|^p dx, \ \forall \delta > 0. \tag{4.6}$$

In particular,

$$\int\limits_{\substack{\mathbb{R}^N\mathbb{R}^N \\ \left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|>1}}\int \frac{1}{|x-y|^{N+2p}}dxdy \le C_{N,p}\int\limits_{\mathbb{R}^N}|\Delta g|^p\,dx. \tag{4.7}$$

Now, from (4.6), one has

$$\int\limits_0^1\int\limits_{\substack{\mathbb{R}^N\mathbb{R}^N \\ \left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|>\delta}}\int \frac{\varepsilon\delta^{p+\varepsilon-1}}{|x-y|^{N+2p}}dxdyd\delta \le C_{N,p}\int\limits_{\mathbb{R}^N}|\Delta g|^p\,dx.$$

Using Lemma 2.1, we deduce

$$\int\limits_{\substack{\mathbb{R}^N\mathbb{R}^N \\ \left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|\le 1}}\int \frac{\varepsilon\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|^{p+\varepsilon}}{|x-y|^{N+2p}}dxdy + \int\limits_{\substack{\mathbb{R}^N\mathbb{R}^N \\ \left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|>1}}\int \frac{\varepsilon}{|x-y|^{N+2p}}dxdy \tag{4.8}$$

$$\le C_{N,p}\int\limits_{\mathbb{R}^N}|\Delta g|^p\,dx.$$

From (4.7) and (4.8), we get the assertion (3).

Now, set

$$G(\delta) = \int\limits_{\substack{\mathbb{R}^N\mathbb{R}^N \\ \left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|>\delta}}\int \frac{\delta^p}{|x-y|^{N+2p}}dxdy.$$

So by the previous results, we have

$$G(\delta) \le C_{N,p}\int\limits_{\mathbb{R}^N}|\Delta g|^p\,dx,\ \forall\delta>0$$

and

$$\lim_{\delta\to 0}G(\delta) = \frac{1}{2^{2p+1}p}\int\limits_{\mathbb{S}^{N-1}}\int\limits_{\mathbb{R}^N}\left|D^2g(x)(\sigma,\sigma)\right|^p dxd\sigma.$$

Now, we claim that

$$\lim_{\varepsilon\to 0}\int\limits_0^1 (p+\varepsilon)\,\varepsilon\delta^{\varepsilon-1}G(\delta)\,d\delta = \frac{1}{2^{2p+1}}\int\limits_{\mathbb{S}^{N-1}}\int\limits_{\mathbb{R}^N}\left|D^2g(x)(\sigma,\sigma)\right|^p dxd\sigma. \tag{4.9}$$

Indeed, for every $\epsilon>0$, we can find a number $X(\epsilon)\in(0,1)$ such that

$$\left|G(\delta) - \frac{1}{2^{2p+1}p}\int\limits_{\mathbb{S}^{N-1}}\int\limits_{\mathbb{R}^N}\left|D^2g(x)(\sigma,\sigma)\right|^p dxd\sigma\right| < \epsilon \text{ for all } \delta\in(0,X(\epsilon)).$$

Now, we have:

$$\lim_{\varepsilon\to 0}\left|\int_{X(\epsilon)}^{1}(p+\varepsilon)\,\varepsilon\delta^{\varepsilon-1}\left[G(\delta)-\frac{1}{2^{2p+1}p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}\left|D^2g(x)(\sigma,\sigma)\right|^p dxd\sigma\right]d\delta\right|$$

$$\leq\int_{X(\epsilon)}^{1}\lim_{\varepsilon\to 0}(p+\varepsilon)\,\varepsilon\delta^{\varepsilon-1}\left|G(\delta)-\frac{1}{2^{2p+1}p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}\left|D^2g(x)(\sigma,\sigma)\right|^p dxd\sigma\right|d\delta$$

$$\leq\int_{X(\epsilon)}^{1}\lim_{\varepsilon\to 0}(p+\varepsilon)\,\varepsilon X(\epsilon)^{\varepsilon-1}\left[C_{N,p}\int_{\mathbb{R}^N}|\Delta g|^p\,dx\right]d\delta$$

$$=0.$$

Moreover,

$$\lim_{\varepsilon\to 0}\left|\int_{0}^{X(\epsilon)}(p+\varepsilon)\,\varepsilon\delta^{\varepsilon-1}\left[G(\delta)-\frac{1}{2^{2p+1}p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}\left|D^2g(x)(\sigma,\sigma)\right|^p dxd\sigma\right]d\delta\right|$$

$$\leq\lim_{\varepsilon\to 0}\int_{0}^{X(\epsilon)}(p+\varepsilon)\,\varepsilon\delta^{\varepsilon-1}\left|G(\delta)-\frac{1}{2^{2p+1}p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}\left|D^2g(x)(\sigma,\sigma)\right|^p dxd\sigma\right|d\delta$$

$$\leq\lim_{\varepsilon\to 0}\int_{0}^{X(\epsilon)}(p+\varepsilon)\,\varepsilon\delta^{\varepsilon-1}\epsilon d\delta$$

$$\leq p\epsilon.$$

Thus,

$$\lim_{\varepsilon\to 0}\left|\int_{0}^{1}(p+\varepsilon)\,\varepsilon\delta^{\varepsilon-1}\left[G(\delta)-\frac{1}{2^{2p+1}p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}\left|D^2g(x)(\sigma,\sigma)\right|^p dxd\sigma\right]d\delta\right|\leq p\epsilon,\ \forall\epsilon>0.$$

Hence we can get

$$\lim_{\varepsilon\to 0}\int_{0}^{1}(p+\varepsilon)\,\varepsilon\delta^{\varepsilon-1}G(\delta)\,d\delta=\lim_{\varepsilon\to 0}\int_{0}^{1}(p+\varepsilon)\,\varepsilon\delta^{\varepsilon-1}\left[\frac{1}{2^{2p+1}p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}\left|D^2g(x)(\sigma,\sigma)\right|^p dxd\sigma\right]d\delta$$

$$=\frac{1}{2^{2p+1}}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}\left|D^2g(x)(\sigma,\sigma)\right|^p dxd\sigma.$$

Consequently, we have

$$\lim_{\varepsilon\to 0}\int_{0}^{1}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\limits_{\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|>\delta}\frac{(p+\varepsilon)\,\varepsilon\delta^{p+\varepsilon-1}}{|x-y|^{N+2p}}dxdyd\delta=\frac{1}{2^{2p+1}}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}\left|D^2g(x)(\sigma,\sigma)\right|^p dxd\sigma.$$

Now, using Lemma 2.1 with $\alpha=p+\varepsilon-1,\ \Phi(x,y)=\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|,\ \Psi(x,y)=$

$\frac{1}{|x-y|^{N+2p}}$, we obtain

$$\lim_{\varepsilon\to 0}\left[\iint\limits_{\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|\le 1}\frac{\varepsilon\left|g(x)+g(y)-2g(\frac{x+y}{2})\right|^{p+\varepsilon}}{|x-y|^{N+2p}}dxdy + \iint\limits_{\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|>1}\frac{\varepsilon}{|x-y|^{N+2p}}dxdy\right]$$

$$=\frac{1}{2^{2p+1}}\int\limits_{\mathbb{S}^{N-1}}\int\limits_{\mathbb{R}^N}\left|D^2g\left(x\right)\left(\sigma,\sigma\right)\right|^p dxd\sigma.$$

Noting that

$$\lim_{\varepsilon\to 0}\iint\limits_{\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|>1}\frac{\varepsilon}{|x-y|^{N+2p}}dxdy=0,$$

we have

$$\lim_{\varepsilon\to 0}\iint\limits_{\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|\le 1}\frac{\varepsilon\left|g(x)+g(y)-2g(\frac{x+y}{2})\right|^{p+\varepsilon}}{|x-y|^{N+2p}}dxdy=\frac{1}{2^{2p+1}}\int\limits_{\mathbb{S}^{N-1}}\int\limits_{\mathbb{R}^N}\left|D^2g\left(x\right)\left(\sigma,\sigma\right)\right|^p dxd\sigma.$$

We have the statement (4). $\square$

Proof of Theorem 4.3:

First, it is clear that statements (1) $\Longrightarrow$ (2) and (1) $\Longrightarrow$ (3) are consequences of Theorem 4.1.

Now, we will prove (3) $\Longrightarrow$ (1) :

First, we assume further that $\left|g(x)+g(y)-2g(\frac{x+y}{2})\right|$ is bounded by $M\left(g\right)$ on $\mathbb{R}^N\times\mathbb{R}^N$. Then

since

$$\sup_{0<\varepsilon<1}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\limits_{\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|\le 1}\frac{\varepsilon\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|^{p+\varepsilon}}{|x-y|^{N+2p}}dxdy$$

$$+\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\limits_{\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|>1}\frac{1}{|x-y|^{N+2p}}dxdy$$

$$<\infty$$

we get

$$\sup_{0<\gamma<1}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\gamma|g(x)+g(y)-2g(\frac{x+y}{2})|^{p+\gamma}}{|x-y|^{N+2p}}dxdy=C\left(g,M\left(g\right)\right)<\infty.$$

Let η_ε be any sequence of smooth mollifiers and set $g^\varepsilon=g*\eta_\varepsilon$. Then we can get $g^\varepsilon\in L^p(\mathbb{R}^N)\cap$

$C^\infty(\mathbb{R}^N) \subseteq W^{2,p}(\mathbb{R}^N)$.

Using (4) of Theorem 4.1, we can have

$$\begin{aligned}
&C_{N,p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}|D^2g^\varepsilon(x)(\sigma,\sigma)|^p dxd\sigma\\
&\leq \sup_{0<\gamma<1}\underset{|g^\varepsilon(x)+g^\varepsilon(y)-2g^\varepsilon(\frac{x+y}{2})|\leq 1}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}}\frac{\gamma|g^\varepsilon(x)+g^\varepsilon(y)-2g^\varepsilon(\frac{x+y}{2})|^{p+\gamma}}{|x-y|^{N+2p}}dxdy\\
&\leq \sup_{0<\gamma<1}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\gamma|g^\varepsilon(x)+g^\varepsilon(y)-2g^\varepsilon(\frac{x+y}{2})|^{p+\gamma}}{|x-y|^{N+2p}}dxdy\\
&= \sup_{0<\gamma<1}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\gamma|\int_{\mathbb{R}^N}\{g(x-z)+g(y-z)-2g(\frac{x+y}{2}-z)\}\eta^\varepsilon(z)dz|^{p+\gamma}}{|x-y|^{N+2p}}dxdy.
\end{aligned}$$

Since the function $x^{p+\varepsilon}$ is convex on $[0,\infty)$, by Jensen's inequality, we can deduce

$$\begin{aligned}
&\sup_{0<\gamma<1}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\gamma|\int_{\mathbb{R}^N}\{g(x-z)+g(y-z)-2g(\frac{x+y}{2}-z)\}\eta^\varepsilon(z)dz|^{p+\gamma}}{|x-y|^{N+2p}}dxdy\\
&\leq \sup_{0<\gamma<1}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\gamma|g(x-z)+g(y-z)-2g(\frac{x+y}{2}-z)|^{p+\gamma}\int_{\mathbb{R}^N}\eta^\varepsilon(z)dz}{|x-y|^{N+2p}}dxdy\\
&= \sup_{0<\gamma<1}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\gamma|g(x)+g(y)-2g(\frac{x+y}{2})|^{p+\gamma}}{|x-y|^{N+2p}}dxdy \leq C\left(g,M\left(g\right)\right).
\end{aligned}$$

So $\|g^\varepsilon\|_{W^{2,p}(\mathbb{R}^N)}$ is bounded. Since $g^\varepsilon\to g$ a.e, we get $g\in W^{2,p}(\mathbb{R}^N)$.

In the general case, since $g\in A^p\left(\mathbb{R}^N\right)$, we can find a sequence $\{g_n\}$ and $A\left(g\right)>0$ such that

$\left|g_n(x)+g_n(y)-2g_n\left(\frac{x+y}{2}\right)\right|$ is bounded by $A\left(g\right)$ and $\left|g_n(x)+g_n(y)-2g_n\left(\frac{x+y}{2}\right)\right|\leq A\left(g\right)\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|$

a.e. $x, y \in \mathbb{R}^N$. Then it is clear that $g_n \in W^{2,p}(\mathbb{R}^N)$ since

$$\sup_{0<\varepsilon<1} \iint\limits_{\substack{\mathbb{R}^N \mathbb{R}^N \\ \left|\frac{g_n}{A(g)}(x)+\frac{g_n}{A(g)}(y)-2\frac{g_n}{A(g)}\left(\frac{x+y}{2}\right)\right|\leq 1}} \frac{\varepsilon\left|\frac{g_n}{A(g)}(x)+\frac{g_n}{A(g)}(y)-2\frac{g_n}{A(g)}\left(\frac{x+y}{2}\right)\right|^{p+\varepsilon}}{|x-y|^{N+2p}} dxdy$$

$$+ \iint\limits_{\substack{\mathbb{R}^N \mathbb{R}^N \\ \left|\frac{g_n}{A(g)}(x)+\frac{g_n}{A(g)}(y)-2\frac{g_n}{A(g)}\left(\frac{x+y}{2}\right)\right|> 1}} \frac{1}{|x-y|^{N+2p}} dxdy$$

$$= \sup_{0<\varepsilon<1} \iint\limits_{\substack{\mathbb{R}^N \mathbb{R}^N \\ \left|\frac{g_n}{A(g)}(x)+\frac{g_n}{A(g)}(y)-2\frac{g_n}{A(g)}\left(\frac{x+y}{2}\right)\right|\leq 1 \\ \left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|\leq 1}}$$

$$+ \iint\limits_{\substack{\mathbb{R}^N \mathbb{R}^N \\ \left|\frac{g_n}{A(g)}(x)+\frac{g_n}{A(g)}(y)-2\frac{g_n}{A(g)}\left(\frac{x+y}{2}\right)\right|\leq 1 \\ \left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|> 1}} \frac{\varepsilon\left|\frac{g_n}{A(g)}(x)+\frac{g_n}{A(g)}(y)-2\frac{g_n}{A(g)}\left(\frac{x+y}{2}\right)\right|^{p+\varepsilon}}{|x-y|^{N+2p}} dxdy$$

$$\leq \sup_{0<\varepsilon<1} \iint\limits_{\substack{\mathbb{R}^N \mathbb{R}^N \\ \left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|\leq 1}} \frac{\varepsilon\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|^{p+\varepsilon}}{|x-y|^{N+2p}} dxdy$$

$$+ \iint\limits_{\substack{\mathbb{R}^N \mathbb{R}^N \\ \left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|> 1}} \frac{1}{|x-y|^{N+2p}} dxdy$$

$$< \infty.$$

Moreover, by (4) in Theorem 4.1, we have

$$\frac{1}{2^{2p+1}} \int_{\mathbb{S}^{N-1}} \int_{\mathbb{R}^N} \left|D^2 g_n(x)(\sigma,\sigma)\right|^p dx d\sigma$$

$$= \lim_{\varepsilon\to 0} \iint_{\substack{\mathbb{R}^N\mathbb{R}^N \\ \left|g_n(x)+g_n(y)-2g_n\left(\frac{x+y}{2}\right)\right|\le 1}} \frac{\varepsilon\left|g_n(x)+g_n(y)-2g_n\left(\frac{x+y}{2}\right)\right|^{p+\varepsilon}}{|x-y|^{N+2p}} dxdy$$

$$\le \lim_{\varepsilon\to 0} C(A(g)) \iint_{\substack{\mathbb{R}^N\mathbb{R}^N \\ \left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|\le 1}} \frac{\varepsilon\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|^{p+\varepsilon}}{|x-y|^{N+2p}} dxdy$$

$$\le C(g, A(g)).$$

Hence, $\|g_n\|_{W^{2,p}(\mathbb{R}^N)}$ is bounded. Since $g_n \to g$ a.e. $\mathbb{R}^N$, we get $g \in W^{2,p}(\mathbb{R}^N)$. □

4.3 Characterizations Using the Taylor Remainder

The man purpose of this section is to establish Theorems 4.2 and 4.4, namely, characterizing the second order Sobolev spaces $W^{2,p}(\mathbb{R}^N)$ using the method of the Taylor remainder of first order.

In order to prove these two theorems, we will need to adapt the following useful lemmas:

Lemma 4.10. There exists a constant $C_{N,p} > 0$ such that for all $\delta > 0$, all $g \in W^{2,p}\left(\mathbb{R}^N\right)$:

$$\iint_{\substack{\mathbb{R}^N\mathbb{R}^N \\ |g(x)-g(y)-\nabla g(y)(x-y)|>\delta}} \frac{\delta^p}{|x-y|^{N+2p}} dxdy \le C_{N,p} \int_{\mathbb{R}^N} |\Delta g|^p dx. \tag{4.10}$$

Proof. Again, using the polar coordinates, we get

$$\iint_{\substack{\mathbb{R}^N\mathbb{R}^N \\ |g(x)-g(y)-\nabla g(y)(x-y)|>\delta}} \frac{\delta^p}{|x-y|^{N+2p}} dxdy = \int_{\mathbb{S}^{N-1}} \int_{\mathbb{R}^N} \int_{\substack{0 \\ |g(x+h\sigma)-g(x)-h\nabla g(x)\sigma|>\delta}}^{\infty} \frac{\delta^p}{h^{2p+1}} dhdxd\sigma.$$

Thus, again, to prove (4.10), it's enough to prove that for every $\sigma \in \mathbb{S}^{N-1}$, we get

$$\int_{\mathbb{R}^N} \int_{\substack{0 \\ |g(x+h\sigma)-g(x)-h\nabla g(x)\sigma|>\delta}}^{\infty} \frac{\delta^p}{h^{2p+1}} dhdx \le C_{N,p} \int_{\mathbb{R}^N} |\Delta g|^p dx. \tag{4.11}$$

Because of the rotation, we assume without loss of generality that $\sigma = e_N = (0, ..., 0, 1)$.

Now, by the mean value theorem, one has

$$\begin{aligned}
|g(x+he_N)-g(x)-h\nabla g(x)e_N| &= \left|\int_0^1 \frac{\partial g}{\partial x_N}\left(x',x_N+sh\right)hds - h\frac{\partial g}{\partial x_N}(x)\right| \\
&= \left|h\int_0^1 \left[\frac{\partial g}{\partial x_N}\left(x',x_N+sh\right)hds - \frac{\partial g}{\partial x_N}(x',x_N)\right]ds\right| \\
&= \left|h\int_0^1\int_{x_N}^{x_N+sh} \frac{\partial^2 g}{\partial x_N^2}\left(x',t\right)dtds\right| \\
&\leq h\int_0^1 shM_N\left(\frac{\partial^2 g}{\partial x_N^2}\right)(x)\,ds \\
&\leq \frac{1}{2}h^2 M_N\left(\frac{\partial^2 g}{\partial x_N^2}\right)(x)\,.
\end{aligned}$$

Thus,

$$\begin{aligned}
\int\limits_{\substack{\mathbb{R}^N \\ |g(x+h\sigma)-g(x)-h\nabla g(x)\sigma|>\delta}}\int\limits_0^\infty \frac{\delta^p}{h^{2p+1}}dhdx &\leq \int\limits_{\substack{\mathbb{R}^N \\ h^2 M_N\left(\frac{\partial^2 g}{\partial x_N^2}\right)(x)>\delta}}\int\limits_0^\infty \frac{\delta^p}{h^{2p+1}}dhdx \\
&= \int\limits_{\mathbb{R}^N}\int\limits_{\sqrt{\frac{\delta}{M_N\left(\frac{\partial^2 g}{\partial x_N^2}\right)(x)}}}^\infty \frac{\delta^p}{h^{2p+1}}dhdx \\
&= \frac{1}{2p}\int\limits_{\mathbb{R}^N}\left|M_N\left(\frac{\partial^2 g}{\partial x_N^2}\right)(x)\right|^p dx \\
&\leq C_{N,p}\int\limits_{\mathbb{R}^N}\left|\frac{\partial^2 g}{\partial x_N^2}\right|^p dx \\
&\leq C_{N,p}\int\limits_{\mathbb{R}^N}|\Delta g|^p\,dx.
\end{aligned}$$

□

Lemma 4.11. There holds

$$\lim_{\delta\to 0}\int_{\mathbb{R}^N}\int_{\substack{\mathbb{R}^N \\ |g(x)-g(y)-\nabla g(y)(x-y)|>\delta}} \frac{\delta^p}{|x-y|^{N+2p}}dxdy = \frac{1}{2^{p+1}p}\int_{\mathbb{S}^N}\int_{\mathbb{R}^N}|D^2 g(x)(\sigma,\sigma)|^p dxd\sigma$$

for all $g \in W^{2,p}(\mathbb{R}^N)$,$1<p<\infty$.

Proof. Again, by changing of variables, we obtain

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}_{|g(x)-g(y)-\nabla g(y)(x-y)|>\delta} \frac{\delta^p}{|x-y|^{N+2p}}dxdy = \int_{\mathbb{S}^N}\int_{\mathbb{R}^N}\int_0^\infty{}_{|\frac{g(x+\sqrt{\delta}h\sigma)-g(x)-\sqrt{\delta}h\nabla g(x)\sigma}{h^2\delta}|h^2>1} \frac{1}{h^{2p+1}}dhdxd\sigma$$

Define $g_\delta : \mathbb{S}^{N-1} \to \mathbb{R}$ by

$$g_\delta(\sigma) = \int_{\mathbb{R}^N}\int_0^\infty{}_{|\frac{g(x+\sqrt{\delta}h\sigma)-g(x)-\sqrt{\delta}h\nabla g(x)\sigma}{h^2\delta}|h^2>1} \frac{1}{h^{2p+1}}dhdx$$

We first prove that for all $\sigma \in \mathbb{S}^{N-1}$, $\forall \delta > 0$:

$$g_\delta(\sigma) \le C_{N,p}\int_{\mathbb{R}^N}|\triangle g|^p dx \tag{4.12}$$

Indeed, again, without loss of generality, we assume that $\sigma = e_N = (0, ..., 0, 1)$. Hence, we need to verify that

$$\int_{\mathbb{R}^N}\int_0^\infty{}_{|\frac{g(x',x_N+\sqrt{\delta}h)-g(x',x_N)-\nabla g(x',x_N)\sqrt{\delta}he_N}{h^2\delta}|h^2>1} \frac{1}{h^{2p+1}}dhdx \le C_{N,p}\int_{\mathbb{R}^N}|\triangle g|^p dx \tag{4.13}$$

Similar to the proof of Lemma 4.8, we have

$$|\frac{g(x', x_N+\sqrt{\delta}h) - g(x', x_N) - \nabla g(x', x_N)\sqrt{\delta}he_N}{h^2\delta}| \le \frac{1}{2}M_N(\frac{\partial^2 g}{\partial x_N^2})(x)$$

Thus,

$$\int_{\mathbb{R}^N}\int_0^\infty{}_{|\frac{g(x',x_N+\sqrt{\delta}h)-g(x',x_N)-\nabla g(x',x_N)\sqrt{\delta}he_N}{h^2\delta}|h^2>1} \frac{1}{h^{2p+1}}dhdx \le \int_{\mathbb{R}^N}\int_0^\infty{}_{h^2M_N(\frac{\partial^2 g}{\partial x_N^2})(x)>2} \frac{1}{h^{2p+1}}dhdx$$

So we can get

$$\int_{\mathbb{R}^N}\int_0^\infty{}_{|\frac{g(x',x_N+\sqrt{\delta}h)-g(x',x_N)-\nabla g(x',x_N)\sqrt{\delta}he_N}{h^2\delta}|h^2>1} \frac{1}{h^{2p+1}}dhdx \le C_{N,p}\int_{\mathbb{R}^N}|\triangle g|^p dx$$

Next we will show that

$$g_\delta(\sigma) \to \frac{1}{2^{p+1}p}\int_{\mathbb{R}^N}|D^2g(x)(\sigma,\sigma)|^p dx \text{ as } \delta \to 0 \text{ for every } \sigma \in \mathbb{S}^{N-1} \tag{4.14}$$

where

$$|D^2g(x)(\sigma,\sigma)| = \sum_{1\le i_1,i_2\le N}\sigma_{i_1}\sigma_{i_2}\frac{\partial^2 g}{\partial x_{i_1}\partial x_{i_2}(x)}.$$

Again, with loss of generality, we suppose that $\sigma = e_N = (0, ..., 0, 1)$. We write

$$g_\delta(\sigma) = \int_{\mathbb{R}^N}\int_0^\infty G_\delta(x,h)dhdx$$

where

$$G_\delta(x,h) = \frac{1}{h^{2p+1}}\chi_{\{|\frac{g(x+\sqrt{\delta}h\sigma)-g(x)-\nabla g(x)\sqrt{\delta}h\sigma}{h^2\delta}|h^2>1\}}(x,h).$$

Noting that for all $\sigma \in \mathbb{S}^{N-1}$:

$$G_\delta(x,h) \to \frac{1}{h^{2p+1}}\chi_{\{|D^2g(x)(\sigma,\sigma)|h^2>2\}} \text{ as } \delta \to 0 \text{ for } a.e.(x,h) \in \mathbb{R}^N \times [0,\infty),$$

and

$$G_\delta(x,h) \leq \frac{1}{h^{2p+1}}\chi_{\{h^2 M_N(\frac{\partial^2 g}{\partial x_N^2})(x)>2\}}(x,h) \in L^1(\mathbb{R}^N \times [0,\infty)).$$

Hence, by the Lebesgue dominated convergence theorem, we get (3.5) Using (3.3), (3.5) and the Lebesgue dominated convergence theorem again, we can conclude that

$$\lim_{\delta\to 0} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\limits_{|g(x)-g(y)-\nabla g(y)(x-y)|>\delta} \frac{\delta^p}{|x-y|^{N+2p}}dxdy = \frac{1}{2^{p+1}p}\int_{\mathbb{S}^N}\int_{\mathbb{R}^N}|D^2g(x)(\sigma,\sigma)|^p dxd\sigma.$$

□

Proof of Theorem 4.2: (1) and (2) are consequences of Lemma 4.10 and Lemma 4.11. Now we will prove (3). By (1), we get

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\limits_{|g(x)-g(y)-\nabla g(y)(x-y)|>\delta} \frac{\delta^p}{|x-y|^{N+2p}}dxdy \leq C_{N,p}\int_{\mathbb{R}^N}|\triangle g|^p dx, \forall\delta>0. \tag{4.15}$$

In particular,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\limits_{|g(x)-g(y)-\nabla g(y)(x-y)|>1} \frac{1}{|x-y|^{N+2p}}dxdy \leq C_{N,p}\int_{\mathbb{R}^N}|\triangle g|^p dx, \forall\delta>0. \tag{4.16}$$

Now, from (3.6), one has

$$\int_0^1\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\limits_{|g(x)-g(y)-\nabla g(y)(x-y)|>\delta} \frac{\varepsilon\delta^{p+\varepsilon-1}}{|x-y|^{N+2p}}dxdyd\delta \leq C_{N,p}\int_{\mathbb{R}^N}|\triangle g|^p dx \tag{4.17}$$

Using Lemma 2.1, we deduce

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\limits_{|g(x)-g(y)-\nabla g(y)(x-y)|\leq 1} \frac{\varepsilon|g(x)-g(y)-\nabla g(y)(x-y)|^{p+\varepsilon}}{|x-y|^{N+2p}}dxdy \\
&+ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\limits_{|g(x)-g(y)-\nabla g(y)(x-y)|>1} \frac{1}{|x-y|^{N+2p}}dxdy \\
&\leq C_{N,p}\int_{\mathbb{R}^N}|\triangle g|^p dx.
\end{aligned}$$

Hence, we get the assertion (3). Now set

$$H(\delta) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\limits_{|g(x)-g(y)-\nabla g(y)(x-y)|>\delta} \frac{\delta^p}{|x-y|^{N+2p}}.$$

So from what we have proved, we have

$$H(\delta) \leq C_{N,p}\int_{\mathbb{R}^N} |\triangle g|^p dx, \forall \delta > 0.$$

and

$$\lim_{\delta\to 0} H(\delta) = \frac{1}{2^{p+1}p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N} |D^2 g(x)(\sigma,\sigma)|^p dx d\sigma. \tag{4.18}$$

Indeed, for every $\theta > 0$, we can find a number $X(\theta) \in (0,1)$ such that

$$|H(\delta) - \frac{1}{2^{p+1}p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N} |D^2 g(x)(\sigma,\sigma)|^p dx d\sigma| < \theta \text{ for all } \delta \in (0, X(\theta)).$$

Now, we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}|\int_{X(\theta)}^{1}(p+\varepsilon)\varepsilon\delta^{\varepsilon-1}\left[H(\delta) - \frac{1}{2^{p+1}p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N} |D^2 g(x)(\sigma,\sigma)|^p dx d\sigma\right] d\delta| \\
&\leq \lim_{\varepsilon\to 0}\int_{X(\theta)}^{1}(p+\varepsilon)\varepsilon\delta^{\varepsilon-1}|H(\delta) - \frac{1}{2^{p+1}p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N} |D^2 g(x)(\sigma,\sigma)|^p dx d\sigma| d\delta \\
&\leq \lim_{\varepsilon\to 0}\int_{X(\theta)}^{1}(p+\varepsilon)\varepsilon\delta^{\varepsilon-1}\left[C_{N,p}\int_{\mathbb{R}^N}|\triangle g|^p dx\right] d\delta \\
&= \lim_{\varepsilon\to 0}(p+\varepsilon)(1 - X(\theta)^{\varepsilon})\left[C_{N,p}\int_{\mathbb{R}^N}|\triangle g|^p dx\right] \\
&= 0.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\lim_{\varepsilon\to 0}|\int_{0}^{X(\theta)}(p+\varepsilon)\varepsilon\delta^{\varepsilon-1}\left[H(\delta) - \frac{1}{2^{p+1}p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N} |D^2 g(x)(\sigma,\sigma)|^p dx d\sigma\right] d\delta| \\
&\leq \lim_{\varepsilon\to 0}\int_{0}^{X(\theta)}(p+\varepsilon)\varepsilon\delta^{\varepsilon-1}|H(\delta) - \frac{1}{2^{p+1}p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N} |D^2 g(x)(\sigma,\sigma)|^p dx d\sigma| d\delta \\
&\leq \lim_{\varepsilon\to 0}\int_{0}^{X(\theta)}(p+\varepsilon)\varepsilon\delta^{\varepsilon-1}\theta d\delta \\
&\leq p\theta.
\end{aligned}$$

Thus, $\forall$ sufficiently small $\theta > 0$:

$$\lim_{\varepsilon\to 0}|\int_{0}^{1}(p+\varepsilon)\varepsilon\delta^{\varepsilon-1}\left[H(\delta) - \frac{1}{2^{p+1}p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N} |D^2 g(x)(\sigma,\sigma)|^p dx d\sigma\right] d\delta| \leq p\theta.$$

Hence we can get

$$\lim_{\varepsilon\to 0}\int_0^1 (p+\varepsilon)\varepsilon\delta^{\varepsilon-1}H(\delta)d\delta = \lim_{\varepsilon\to 0}\int_0^1 (p+\varepsilon)\varepsilon\delta^{\varepsilon-1}\left[\frac{1}{2^{p+1}p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}|D^2g(x)(\sigma,\sigma)|^p dxd\sigma\right]d\delta$$
$$= \frac{1}{2^{p+1}}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}|D^2g(x)(\sigma,\sigma)|^p dxd\sigma.$$

Consequently, we have

$$\lim_{\varepsilon\to 0}\underset{|g(x)-g(y)-\nabla g(y)(x-y)|>\delta}{\int_0^1\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}}\frac{\varepsilon(p+\varepsilon)\delta^{p+\varepsilon-1}}{|x-y|^{N+2p}}dxdyd\delta = \frac{1}{2^{p+1}}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}|D^2g(x)(\sigma,\sigma)|^p dxd\sigma.$$

Now, using Lemma 2.1 with

$$\alpha = p+\varepsilon-1,\ \Phi(x,y) = |g(x)-g(y)-\nabla g(y)(x-y)|,\ \Psi(x,y) = \frac{1}{|x-y|^{N+2p}},$$

we obtain

$$\lim_{\varepsilon\to 0}\left[\underset{|g(x)-g(y)-\nabla g(y)(x-y)|\le 1}{\iint}\frac{\varepsilon|g(x)-g(y)-\nabla g(y)(x-y)|^{p+\varepsilon}}{|x-y|^{N+2p}}dxdy + \underset{|g(x)-g(y)-\nabla g(y)(x-y)|>1}{\iint}\frac{\varepsilon}{|x-y|^{N+2p}}dxdy\right]$$
$$= \frac{1}{2^{p+1}}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}\left|D^2g\left(x\right)\left(\sigma,\sigma\right)\right|^p dxd\sigma.$$

Noting that

$$\lim_{\varepsilon\to 0}\underset{|g(x)-g(y)-\nabla g(y)(x-y)|>1}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}}\frac{\varepsilon}{|x-y|^{N+2p}}dxdy = 0,$$

we have

$$\lim_{\varepsilon\to 0}\underset{|g(x)-g(y)-\nabla g(y)(x-y)|\le 1}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}}\frac{\varepsilon|g(x)-g(y)-\nabla g(y)(x-y)|^{p+\varepsilon}}{|x-y|^{N+2p}}dxdy$$
$$= \frac{1}{2^{p+1}}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}|D^2g(x)(\sigma,\sigma)|^p dxd\sigma.$$

□

Proof of Theorem 4.4 is similar to the proof of Theorem 4.3 and will be omitted.

4.4 More Characterizations of Second Order Spaces: Combinations of First Order Difference and Taylor Remainder

In this section, we will study some other characterizations of the second order Sobolev spaces. Namely, we will give characterizations motivated by the observation that $g \in W^{2,p}(\mathbb{R}^N)$ is essentially equivalent to $\nabla g \in W^{1,p}(\mathbb{R}^N)$.

Characterization of Bourgain-Brezis-Mironescu type

Lemma 4.12. Let $g \in W^{2,p}\left(\mathbb{R}^N\right)$. Then

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|(\nabla g(x) - \nabla g(y)) \cdot (x-y)|^p}{|x-y|^{2p}} \rho_n\left(|x-y|\right) dxdy \le C, \ \forall n \ge 1, \tag{4.19}$$

for some constant $C > 0$. Moreover,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|(\nabla g(x) - \nabla g(y)) \cdot (x-y)|^p}{|x-y|^{2p}} \rho_n\left(|x-y|\right) dxdy = \int_{\mathbb{R}^N}\int_{\mathbf{S}^{N-1}} \left|D^2 g\left(x\right)\left(\sigma,\sigma\right)\right|^p d\sigma dx. \tag{4.20}$$

Proof. Since $g \in W^{2,p}\left(\mathbb{R}^N\right)$, $\frac{\partial g}{\partial x_i} \in W^{1,p}\left(\mathbb{R}^N\right)$. Using Theorem 1.1 (noting Theorem 1.1 still holds when $\Omega = \mathbb{R}^N$), we get

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|\frac{\partial g}{\partial x_i}(x) - \frac{\partial g}{\partial x_i}(y)\right|^p}{|x-y|^p} \rho_n\left(|x-y|\right) dxdy \le C.$$

Hence,

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|(\nabla g(x) - \nabla g(y)) \cdot (x-y)|^p}{|x-y|^{2p}} \rho_n\left(|x-y|\right) dxdy \\ &\le \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|(\nabla g(x) - \nabla g(y))|^p}{|x-y|^p} \rho_n\left(|x-y|\right) dxdy \le C \end{aligned}$$

for some constant $C > 0$.

Now, suppose that $g \in C_0^\infty\left(\mathbb{R}^N\right)$. Then by Taylor's formula, we have

$$\begin{aligned} |(\nabla g(x+h) - \nabla g(x)) \cdot h| &= \left|\sum \left(\frac{\partial g}{\partial x_i}(x+h) - \frac{\partial g}{\partial x_i}(x)\right) h_i\right| \\ &\le \left|D^2 g\left(x\right)\left(h,h\right)\right| + c\left|h\right|^3. \end{aligned}$$

Hence, for every $\theta > 0$:

$$|(\nabla g(x+h) - \nabla g(x)) \cdot h|^p \le (1+\theta)\left|D^2 g\left(x\right)\left(h,h\right)\right|^p + c_\theta \left|h\right|^{3p}.$$

Thus,

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|(\nabla g(x)-\nabla g(y))\cdot(x-y)|^p}{|x-y|^{2p}}\rho_n(|x-y|)\,dxdy \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|(\nabla g(x+h)-\nabla g(x))\cdot h|^p}{|h|^{2p}}\rho_n(|h|)\,dhdx \\
&\leq (1+\theta)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\rho_n(|h|)}{|h|^{2p}}\left|D^2 g(x)(h,h)\right|^p dhdx + c_\theta\,|\mathrm{supp}(g)|\int_{\mathbb{R}^N}\rho_n(|h|)\,|h|^p\,dh.
\end{aligned}$$

Also,

$$\int_{\mathbb{R}^N} \frac{\rho_n(|h|)}{|h|^{2p}}\left|D^2 g(x)(h,h)\right|^p dh = \int_0^\infty \rho_n(r)\,r^{N-1}dr \int_{\mathbf{S}^{N-1}} \left|D^2 g(x)(\sigma,\sigma)\right|^p d\sigma.$$

Now, let $n\to\infty$ and then $\theta\to 0$, we get

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|(\nabla g(x)-\nabla g(y))\cdot(x-y)|^p}{|x-y|^{2p}}\rho_n(|x-y|)\,dxdy \leq \int_{\mathbb{R}^N}\int_{\mathbf{S}^{N-1}} \left|D^2 g(x)(\sigma,\sigma)\right|^p d\sigma dx. \tag{4.21}$$

Now, for any $A>0$, then again by Taylor's formula:

$$\left|(\nabla g(x+h)-\nabla g(x))\cdot h - D^2 g(x)(h,h)\right| \leq C_A\,|h|^3 \text{ a.e. } x\in B(0,A);\; h\in B(0,1).$$

Then

$$\left|D^2 g(x)(h,h)\right|^p \leq (1+\theta)\,|(\nabla g(x+h)-\nabla g(x))\cdot h|^p + C_{A,\theta}\,|h|^{3p} \text{ a.e. } x\in B(0,A);\; h\in B(0,1).$$

Hence

$$\begin{aligned}
&\int_{B(0,A)}\int_{B(0,1)} \frac{\rho_n(|h|)}{|h|^{2p}}\left|D^2 g(x)(h,h)\right|^p dhdx \\
&\leq (1+\theta)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|(\nabla g(x+h)-\nabla g(x))\cdot h|^p}{|h|^{2p}}\rho_n(|h|)\,dhdx + C_{A,\theta}\,|B(0,A)|\int_{B(0,1)}\rho_n(|h|)\,|h|^p\,dh.
\end{aligned}$$

Let $n\to\infty$ and $\theta\to 0:$

$$\begin{aligned}
\int_{B(0,A)}\int_{\mathbf{S}^{N-1}} \left|D^2 g(x)(\sigma,\sigma)\right|^p d\sigma dx &\leq \liminf_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|(\nabla g(x+h)-\nabla g(x))\cdot h|^p}{|h|^{2p}}\rho_n(|h|)\,dhdx \\
&= \liminf_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|(\nabla g(x)-\nabla g(y))\cdot(x-y)|^p}{|x-y|^{2p}}\rho_n(|x-y|)\,dxdy.
\end{aligned}$$

Since $A>0$ is arbitrary, we get (4.20) for $C_0^\infty\left(\mathbb{R}^N\right)$ −functions.

By density argument, we also have (4.20) for $W^{2,p}\left(\mathbb{R}^N\right)$ −functions. □

Proof of Theorem 4.5: Assume that

$$\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|(\nabla g(x)-\nabla g(y))\cdot(x-y)|^p}{|x-y|^{2p}}\rho_n(|x-y|)\,dxdy \le C(g)\,,\ \forall n\ge 1.$$

Let $g_k = g * \eta_k$ where η_k is a sequence of smooth mollifiers. Noting that $D^\alpha(g_k) = D^\alpha(g) * \eta_k$, we have from the convexity of the function t^p on $[0,\infty)$ that

$$\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|(\nabla g_k(x)-\nabla g_k(y))\cdot(x-y)|^p}{|x-y|^{2p}}\rho_n(|x-y|)\,dxdy$$
$$\le \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|(\nabla g(x)-\nabla g(y))\cdot(x-y)|^p}{|x-y|^{2p}}\rho_n(|x-y|)\,dxdy \le C(g)\,.$$

Moreover, since g_k is smooth, by Lemma 4.12, we get

$$\int\limits_{\mathbb{R}^N}\int\limits_{\mathbf{S}^{N-1}} \left|D^2 g_k(x)(\sigma,\sigma)\right|^p d\sigma dx \le C(g)$$

which means that

$$\int\limits_{\mathbb{R}^N} |\Delta g_k(x)|^p\,dx \le C_1(g)\,.$$

Since $g_k \to g$ a.e. $\mathbb{R}^N$, we can conclude that $g \in W^{2,p}\left(\mathbb{R}^N\right)$. □

Characterization of H.-M. Nguyen type

Lemma 4.13. There exists a constant $C_{N,p} > 0$ such that for all $\delta > 0$, all $g \in W^{2,p}\left(\mathbb{R}^N\right)$:

$$\underset{|(\nabla g(x)-\nabla g(y))\cdot(x-y)|>\delta}{\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}} \frac{\delta^p}{|x-y|^{N+2p}}dxdy \le C_{N,p}\int\limits_{\mathbb{R}^N}|\Delta g|^p\,dx.$$

Proof. By Lemma 4.10, we have

$$\underset{|(\nabla g(x)-\nabla g(y))\cdot(x-y)|>\delta}{\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}} \frac{\delta^p}{|x-y|^{N+2p}}dxdy$$
$$\le \underset{|g(x)-g(y)-\nabla g(y)(x-y)|>\frac{\delta}{2}}{\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}} + \underset{|g(y)-g(x)-\nabla g(x)\cdot(y-x)|>\frac{\delta}{2}}{\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}} \frac{\delta^p}{|x-y|^{N+2p}}dxdy$$
$$\le C_{N,p}\int\limits_{\mathbb{R}^N}|\Delta g|^p\,dx.$$

□

Lemma 4.14. There holds

$$\lim_{\delta\to 0} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}_{|(\nabla g(x)-\nabla g(y))(x-y)|>\delta} \frac{\delta^p}{|x-y|^{N+2p}}dxdy = \frac{1}{2p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N} |D^2g(x)(\sigma,\sigma)|^p dxd\sigma$$

for all $g \in W^{2,p}(\mathbb{R}^N)$, $1 < p < \infty$.

Proof. By changing of variables, we obtain

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}_{|(\nabla g(x)-\nabla g(y))\cdot(x-y)|>\delta} \frac{\delta^p}{|x-y|^{N+2p}}dxdy = \int_{\mathbb{S}^N}\int_{\mathbb{R}^N}\int_0^\infty{}_{|\frac{(\nabla g(x+\sqrt{\delta}h\sigma)-\nabla g(x))\cdot\sqrt{\delta}h\sigma}{h^2\delta}|h^2>1} \frac{1}{h^{2p+1}}dhdxd\sigma$$

Define $g_\delta : \mathbb{S}^{N-1} \to \mathbb{R}$ by

$$g_\delta(\sigma) = \int_{\mathbb{R}^N}\int_0^\infty{}_{|\frac{(\nabla g(x+\sqrt{\delta}h\sigma)-\nabla g(x))\cdot\sqrt{\delta}h\sigma}{h^2\delta}|h^2>1} \frac{1}{h^{2p+1}}dhdx$$

Then by the same argument as in Lemma 4.13 and by Lemma 4.11, we can prove that for all $\sigma \in \mathbb{S}^{N-1}$, $\forall\delta > 0$:

$$g_\delta(\sigma) \leq C_{N,p}\int_{\mathbb{R}^N} |\triangle g|^p dx.$$

Now we will show that

$$g_\delta(\sigma) \to \frac{1}{2p}\int_{\mathbb{R}^N} |D^2g(x)(\sigma,\sigma)|^p dx \text{ as } \delta \to 0 \text{ for every } \sigma \in \mathbb{S}^{N-1} \tag{4.22}$$

Indeed, again, with loss of generality, we suppose that $\sigma = e_N = (0, ..., 0, 1)$. We write

$$g_\delta(\sigma) = \int_{\mathbb{R}^N}\int_0^\infty G_\delta(x,h)dhdx$$

where

$$G_\delta(x,h) = \frac{1}{h^{2p+1}}\chi_{\{|\frac{(\nabla g(x+\sqrt{\delta}h\sigma)-\nabla g(x))\cdot\sqrt{\delta}h\sigma}{h^2\delta}|h^2>1\}}(x,h).$$

Noting that for all $\sigma \in \mathbb{S}^{N-1}$:

$$G_\delta(x,h) \to \frac{1}{h^{2p+1}}\chi_{\{|D^2g(x)(\sigma,\sigma)|h^2>1\}} \text{ as } \delta \to 0 \text{ for } a.e.(x,h) \in \mathbb{R}^N \times [0,\infty),$$

and

$$G_\delta(x,h) \leq \frac{1}{h^{2p+1}}\chi_{\{h^2 M_N(\frac{\partial^2 g}{\partial x_N^2})(x)>1\}}(x,h) \in L^1(\mathbb{R}^N \times [0,\infty)).$$

Hence, by the Lebesgue dominated convergence theorem, we get (4.22). Using the Lebesgue dominated convergence theorem again, we can conclude that

$$\lim_{\delta\to 0} \underset{|(\nabla g(x)-\nabla g(y))\cdot(x-y)|>\delta}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{\delta^p}{|x-y|^{N+2p}}dxdy = \frac{1}{2p}\int_{\mathbb{S}^N}\int_{\mathbb{R}^N} |D^2 g(x)(\sigma,\sigma)|^p dxd\sigma.$$

□

Theorem 4.15. Let $g \in W^{2,p}(\mathbb{R}^N)$, $1 < p < \infty$. Then there exists a constant $C_{N,p}$ such that

1.
$$\underset{|(\nabla g(x)-\nabla g(y))\cdot(x-y)|>\delta}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{\delta^p}{|x-y|^{N+2p}}dxdy \le C_{N,p}\int_{\mathbb{R}^N} |\triangle g|^p dx, \forall \delta > 0.$$

2.
$$\lim_{\delta\to 0} \underset{|(\nabla g(x)-\nabla g(y))\cdot(x-y)|>\delta}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{\delta^p}{|x-y|^{N+2p}}dxdy = \frac{1}{2p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N} |D^2 g(x)(\sigma,\sigma)|^p dxd\sigma.$$

3.
$$\sup_{0<\varepsilon<1} \underset{|(\nabla g(x)-\nabla g(y))\cdot(x-y)|\le 1}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{\varepsilon|(\nabla g(x)-\nabla g(y))\cdot(x-y)|^{p+\varepsilon}}{|x-y|^{N+2p}}dxdy$$
$$+ \underset{|(\nabla g(x)-\nabla g(y))\cdot(x-y)|>1}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{1}{|x-y|^{N+2p}}dxdy \le C_{N,p}\int_{\mathbb{R}^N} |\triangle g|^p dx.$$

4.
$$\lim_{\varepsilon\to 0} \underset{|(\nabla g(x)-\nabla g(y))\cdot(x-y)|\le 1}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{\varepsilon|(\nabla g(x)-\nabla g(y))\cdot(x-y)|^{p+\varepsilon}}{|x-y|^{N+2p}}dxdy$$
$$= \frac{1}{2}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N} |D^2 g(x)(\sigma,\sigma)|^p dxd\sigma.$$

Proof. (3) is just a consequence of (1), the Fatou lemma and Lemma 2.1. So we just have to prove (4). Set

$$H(\delta) = \underset{|(\nabla g(x)-\nabla g(y))\cdot(x-y)|>\delta}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{\delta^p}{|x-y|^{N+2p}}.$$

So from what we have proved, we get

$$H(\delta) \le C_{N,p}\int_{\mathbb{R}^N} |\triangle g|^p dx, \forall \delta > 0$$

and

$$\lim_{\delta\to 0} H(\delta) = \frac{1}{2p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N} |D^2 g(x)(\sigma,\sigma)|^p dx d\sigma.$$

Thus, for every $\theta > 0$, we can find a number $X(\theta) \in (0,1)$ such that

$$|H(\delta) - \frac{1}{2p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N} |D^2 g(x)(\sigma,\sigma)|^p dx d\sigma| < \theta \text{ for all } \delta \in (0, X(\theta)).$$

Now, we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0} |\int_{X(\theta)}^{1} (p+\varepsilon)\varepsilon\delta^{\varepsilon-1}\left[H(\delta) - \frac{1}{2p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N} |D^2 g(x)(\sigma,\sigma)|^p dx d\sigma\right] d\delta| \\
&\leq \lim_{\varepsilon\to 0} \int_{X(\theta)}^{1} (p+\varepsilon)\varepsilon\delta^{\varepsilon-1} |H(\delta) - \frac{1}{2p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N} |D^2 g(x)(\sigma,\sigma)|^p dx d\sigma| d\delta \\
&\leq \lim_{\varepsilon\to 0} \int_{X(\theta)}^{1} (p+\varepsilon)\varepsilon\delta^{\varepsilon-1}\left[C_{N,p}\int_{\mathbb{R}^N} |\triangle g|^p dx\right] d\delta \\
&= \lim_{\varepsilon\to 0} (p+\varepsilon)(1 - X(\theta)^{\varepsilon})\left[C_{N,p}\int_{\mathbb{R}^N} |\triangle g|^p dx\right] \\
&= 0.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\lim_{\varepsilon\to 0} |\int_{0}^{X(\theta)} (p+\varepsilon)\varepsilon\delta^{\varepsilon-1}\left[H(\delta) - \frac{1}{2p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N} |D^2 g(x)(\sigma,\sigma)|^p dx d\sigma\right] d\delta| \\
&\leq \lim_{\varepsilon\to 0} \int_{0}^{X(\theta)} (p+\varepsilon)\varepsilon\delta^{\varepsilon-1} |H(\delta) - \frac{1}{2p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N} |D^2 g(x)(\sigma,\sigma)|^p dx d\sigma| d\delta \\
&\leq \lim_{\varepsilon\to 0} \int_{0}^{X(\theta)} (p+\varepsilon)\varepsilon\delta^{\varepsilon-1}\theta d\delta \\
&\leq p\theta.
\end{aligned}$$

Thus,

$$\lim_{\varepsilon\to 0} |\int_{0}^{1} (p+\varepsilon)\varepsilon\delta^{\varepsilon-1}\left[H(\delta) - \frac{1}{2p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N} |D^2 g(x)(\sigma,\sigma)|^p dx d\sigma\right] d\delta| \leq p\theta, \forall \text{ sufficiently small } \theta > 0.$$

Hence we can get

$$\begin{aligned}
\lim_{\varepsilon\to 0} \int_0^1 (p+\varepsilon)\varepsilon\delta^{\varepsilon-1} H(\delta) d\delta &= \lim_{\varepsilon\to 0} \int_0^1 (p+\varepsilon)\varepsilon\delta^{\varepsilon-1}\left[\frac{1}{2p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N} |D^2 g(x)(\sigma,\sigma)|^p dx d\sigma\right] d\delta \\
&= \frac{1}{2}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N} |D^2 g(x)(\sigma,\sigma)|^p dx d\sigma.
\end{aligned}$$

Consequently, we have

$$\lim_{\varepsilon\to 0}\underset{|(\nabla g(x)-\nabla g(y))\cdot(x-y)|>\delta}{\int_0^1\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}}\frac{\varepsilon(p+\varepsilon)\delta^{p+\varepsilon-1}}{|x-y|^{N+2p}}dxdyd\delta=\frac{1}{2}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}|D^2g(x)(\sigma,\sigma)|^p dxd\sigma.$$

Now, using Lemma 2.1 with

$$\alpha=p+\varepsilon-1,\ \Phi(x,y)=|(\nabla g(x)-\nabla g(y))\cdot(x-y)|,\ \Psi(x,y)=\frac{1}{|x-y|^{N+2p}},$$

we obtain

$$\lim_{\varepsilon\to 0}\left[\begin{array}{c}\iint\limits_{|(\nabla g(x)-\nabla g(y))\cdot(x-y)|\leq 1}\frac{\varepsilon|(\nabla g(x)-\nabla g(y))\cdot(x-y)|^{p+\varepsilon}}{|x-y|^{N+2p}}dxdy\\ +\iint\limits_{|(\nabla g(x)-\nabla g(y))\cdot(x-y)|>1}\frac{\varepsilon}{|x-y|^{N+2p}}dxdy\end{array}\right]$$

$$=\frac{1}{2}\int\limits_{\mathbb{S}^{N-1}}\int\limits_{\mathbb{R}^N}\left|D^2g\left(x\right)\left(\sigma,\sigma\right)\right|^p dxd\sigma.$$

We have the statement (4) by noting that

$$\lim_{\varepsilon\to 0}\underset{|(\nabla g(x)-\nabla g(y))\cdot(x-y))|>1}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}}\frac{\varepsilon}{|x-y|^{N+2p}}dxdy=0.$$

□

Proof of Theorem 4.6 is similar to the proof of Theorem 4.3; Theorem 4.4 and will be omitted.

CHAPTER 5 NEW CHARACTERIZATIONS OF HIGH ORDER SOBOLEV SPACES ON EUCLIDEAN SPACE

5.1 Main Results

Motivated by our work of the new characterizations of the second order Sobolev space, a natural next step is to study the characterizations of the high order Sobolev spaces $W^{m,p}$ with $m \geq 3$ which is the main purpose of this paper. Nevertheless, those characterizations are more complicated than the second order case and we use different methods to handle the high order case. We write

$$T_y^m f(x) = \sum_{|\alpha| \leq m} D^\alpha f(y) \tfrac{(x-y)^\alpha}{\alpha!}$$

and

$$R^m f(x,y) = f(x) - T_y^m f(x)$$

for the Taylor polynomial of order m and the Taylor remainder of order m, respectively.

If we write $\triangle_h f(x) = f(x+h) - f(x)$,for$x, h \in \mathbb{R}^N$ then the second difference can be written as

$$\triangle_h^{(2)} f(x) = \triangle_h(\triangle_h f)(x) = f(x+2h) - 2f(x+h) + f(x)$$

Similarly, we can repeat this progress to get m-th difference which is

$$\triangle_h^{(m)} f(x) = f(x+mh) - \tbinom{m}{1} f(x+(m-1)h) + \cdots + (-1)^{m-1}\tbinom{m}{m-1} f(x+h) + (-1)^m f(x)$$

If we set $y = x + mh$, then we can rewrite the m-th difference to be

$$\begin{aligned}\triangle^{(m)} f(x,y) &= f(y) - \binom{m}{1} f(\frac{x+(m-1)y}{m}) + \cdots + (-1)^{m-1}\binom{m}{m-1} f(\frac{(m-1)x+y}{m}) + (-1)^m f(x) \\ &= \sum_{k=0}^{m} (-1)^k \binom{m}{k} f(\frac{kx+(m-k)y}{m})\end{aligned}$$

where $\binom{m}{k} = \frac{m!}{(m-k)!k!}$.

In this paper, we first prove the following two theorems

Theorem 5.1. Let $f \in W^{m,p}\left(\mathbb{R}^N\right),\ 1 < p < \infty$. Then there exists a constant $C_{N,p}$ such that

(1)

$$\iint\limits_{\substack{\mathbb{R}^N\mathbb{R}^N \\ \left|\triangle^{(m)} f(x,y)\right|>\delta}} \frac{\delta^p}{|x-y|^{N+mp}} dxdy \le C_{N,p} \int\limits_{\mathbb{R}^N} |\nabla^m f(x)|^p\, dx,\ \forall\delta > 0.$$

(2)

$$\lim_{\delta\to 0} \iint\limits_{\substack{\mathbb{R}^N\mathbb{R}^N \\ \left|\triangle^{(m)} f(x,y)\right|>\delta}} \frac{\delta^p}{|x-y|^{N+mp}} dxdy = \frac{1}{m^{mp+1}p} \int\limits_{\mathbb{S}^{N-1}} \int\limits_{\mathbb{R}^N} |D^m f\,(x)\,(\sigma,\cdots,\sigma)|^p\, dxd\sigma.$$

(3)

$$\sup_{0<\varepsilon<1} \iint\limits_{\substack{\mathbb{R}^N\mathbb{R}^N \\ \left|\triangle^{(m)} f(x,y)\right|\le 1}} \frac{\varepsilon\, |\triangle^m f(x,y)|^{p+\varepsilon}}{|x-y|^{N+mp}} dxdy + \iint\limits_{\substack{\mathbb{R}^N\mathbb{R}^N \\ \left|\triangle^{(m)} f(x,y)\right|>1}} \frac{1}{|x-y|^{N+mp}} dxdy$$

$$\le C_{N,p} \int\limits_{\mathbb{R}^N} |\nabla^m f|^p\, dx.$$

(4)

$$\lim_{\varepsilon\to 0} \iint\limits_{\substack{\mathbb{R}^N\mathbb{R}^N \\ \left|\triangle^{(m)} f(x,y)\right|\le 1}} \frac{\varepsilon\left|\triangle^{(m)} f(x,y)\right|^{p+\varepsilon}}{|x-y|^{N+mp}} dxdy = \frac{1}{m^{mp+1}} \int\limits_{\mathbb{S}^{N-1}} \int\limits_{\mathbb{R}^N} |D^m f\,(x)\,(\sigma,\cdots,\sigma)|^p\, dxd\sigma.$$

Here

$$|D^m f\,(x)\,(\sigma,\cdots,\sigma)| = \sum_{1\le i_1,\cdots,i_m\le N} \sigma_{i_1}\cdots\sigma_{i_m} \frac{\partial^m f}{\partial x_{i_1}\cdots\partial x_{i_m}}\,(x)\,.$$

Theorem 5.2. Let $f \in W^{m,p}\left(\mathbb{R}^N\right),\ 1 < p < \infty$. Then there exists a constant $C_{N,m,p}$ such that

(1)

$$\iint\limits_{\substack{\mathbb{R}^N\mathbb{R}^N \\ |R^{m-1} f(x,y)|>\delta}} \frac{\delta^p}{|x-y|^{N+mp}} dxdy \le C_{N,m,p} \int\limits_{\mathbb{R}^N} |\nabla^m f(x)|^p\, dx,\ \forall\delta > 0.$$

(2)

$$\lim_{\delta\to 0} \iint\limits_{\substack{\mathbb{R}^N\mathbb{R}^N \\ |R^{m-1} f(x,y)|>\delta}} \frac{\delta^p}{|x-y|^{N+mp}} dxdy = \frac{1}{(m!)^p mp} \int\limits_{\mathbb{S}^{N-1}} \int\limits_{\mathbb{R}^N} |D^m f\,(x)\,(\sigma,\cdots,\sigma)|^p\, dxd\sigma.$$

(3)

$$\sup_{0<\varepsilon<1} \iint\limits_{\substack{\mathbb{R}^N\mathbb{R}^N \\ |R^{m-1}f(x,y)|\le 1}} \frac{\varepsilon \left|R^{m-1}f(x,y)\right|^{p+\varepsilon}}{|x-y|^{N+mp}} dxdy + \iint\limits_{\substack{\mathbb{R}^N\mathbb{R}^N \\ |R^{m-1}f(x,y)|>1}} \frac{1}{|x-y|^{N+mp}} dxdy$$

$$\le C_{N,m,p} \int\limits_{\mathbb{R}^N} |\nabla^m f|^p \, dx.$$

(4)

$$\lim_{\varepsilon\to 0} \iint\limits_{\substack{\mathbb{R}^N\mathbb{R}^N \\ |R^{m-1}f(x,y)|\le 1}} \frac{\varepsilon \left|R^{m-1}f(x,y)\right|^{p+\varepsilon}}{|x-y|^{N+mp}} dxdy = \frac{1}{(m!)^p m} \int\limits_{\mathbb{S}^{N-1}} \int\limits_{\mathbb{R}^N} |D^m f(x)(\sigma,\cdots,\sigma)|^p \, dxd\sigma.$$

Here

$$|D^m f(x)(\sigma,\cdots,\sigma)| = \sum_{1\le i_1,\cdots,i_m\le N} \sigma_{i_1}\cdots\sigma_{i_m} \frac{\partial^m f}{\partial x_{i_1}\cdots\partial x_{i_m}}(x).$$

Using Theorems 5.1 and 5.2, we can get the main aims of our paper: the new characterizations of high order Sobolev spaces $W^{m,p}$ with $m \ge 3$:

Theorem 5.3. Let $f \in L^p(\mathbb{R}^N) \cap L^1(\mathbb{R}^N),\ 1<p<\infty$. Then the following are equivalent:

(1) $f \in W^{m,p}(\mathbb{R}^N)$.

(2)

$$\sup_{0<\delta<1} \iint\limits_{\substack{\mathbb{R}^N\mathbb{R}^N \\ \left|\triangle^{(m)}f(x,y)\right|>\delta}} \frac{\delta^p}{|x-y|^{N+mp}} dxdy < \infty.$$

(3)

$$\sup_{0<\varepsilon<1} \iint\limits_{\substack{\mathbb{R}^N\mathbb{R}^N \\ \left|\triangle^{(m)}f(x,y)\right|\le 1}} \frac{\varepsilon \left|\triangle^{(m)}f(x,y)\right|^{p+\varepsilon}}{|x-y|^{N+mp}} dxdy + \iint\limits_{\substack{\mathbb{R}^N\mathbb{R}^N \\ \left|\triangle^{(m)}f(x,y)\right|>1}} \frac{1}{|x-y|^{N+mp}} dxdy < \infty.$$

Theorem 5.4. Let $f \in W^{m-1,p}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N),\ 1<p<\infty$. Then the following are equivalent:

(1) $f \in W^{m,p}(\mathbb{R}^N)$.

(2)

$$\sup_{0<\delta<1} \iint\limits_{\substack{\mathbb{R}^N\mathbb{R}^N \\ |R^{m-1}f(x,y)|>\delta}} \frac{\delta^p}{|x-y|^{N+mp}} dxdy < \infty.$$

(3)

$$\sup_{0<\varepsilon<1} \underset{|R^{m-1}f(x,y)|\leq 1}{\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}} \frac{\varepsilon \left|R^{m-1}f(x,y)\right|^{p+\varepsilon}}{|x-y|^{N+mp}} dxdy + \underset{|R^{m-1}f(x,y)|>1}{\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}} \frac{1}{|x-y|^{N+mp}} dxdy < \infty.$$

5.2 Characterization Using mth-Differences

In this section, we will investigate the characterizations of high order Sobolev spaces $W^{m,p}\left(\mathbb{R}^N\right)$ in terms of the m-th differences. We will study some following useful lemmas to prove Theorems 5.1 and 5.3.

Lemma 5.5. There exists a constant $C_{N,m,p} > 0$ such that for all $\delta > 0$, all $f \in W^{m,p}\left(\mathbb{R}^N\right), m \geq 3$:

$$\underset{\left|\triangle^{(m)}(x,y)\right|>\delta}{\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}} \frac{\delta^p}{|x-y|^{N+mp}} dxdy \leq C_{N,m,p} \int\limits_{\mathbb{R}^N} |\nabla^m f|^p \, dx. \tag{5.1}$$

Proof. First, using the polar coordinates, set $y = x + mh\sigma$ with $h > 0$ and $\sigma \in \mathbb{S}^{N-1}$,we get

$$\underset{\left|\triangle^{(m)}(x,y)\right|>\delta}{\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}} \frac{\delta^p}{|x-y|^{N+mp}} dxdy = \frac{1}{m^{mp}} \underset{\left|\triangle^{(m)}(x,x+mh\sigma)\right|>\delta}{\int\limits_{\mathbb{S}^{N-1}}\int\limits_{\mathbb{R}^N}\int\limits_0^\infty} \frac{\delta^p}{h^{mp+1}} dhdxd\sigma.$$

Hence, to prove (5.1), it's enough to prove that for every $\sigma \in \mathbb{S}^{N-1}$, we can obtain

$$\underset{\left|\triangle^{(m)}(x,x+mh\sigma)\right|>\delta}{\int\limits_{\mathbb{R}^N}\int\limits_0^\infty} \frac{\delta^p}{h^{mp+1}} dhdx \leq C_{N,p} \int\limits_{\mathbb{R}^N} |\nabla^m f|^p \, dx. \tag{5.2}$$

Because of the rotation, we now can assume that $\sigma = e_N = (0, ..., 0, 1)$.We want to prove it by induction, first we claim for $k \geq 3$

$$\triangle^{(k)} f(x, x + khe_N) = \int_{x_N}^{x_N+h} \int_{s_1}^{s_1+h} \cdots \int_{s_{k-1}}^{s_{k-1}+h} \frac{\partial^k f}{\partial x_N^k}(x', s_k) ds_k ds_{k-1} \cdots ds_1$$

For k=3, by the mean value theorem, one has

$$\begin{aligned}
\triangle^{(3)} f(x, x+3he_N) &= f(x+3he_N) - 3f(x+2he_N) + 3f(x+he_N) - f(x) \\
&= [(f(x+3he_N) - f(x+2he_N)) - (f(x+2he_N) - f(x+he_N))] \\
&- [(f(x+2he_N) - f(x+he_N)) - (f(x+he_N) - f(x))] \\
&= (\int_{x_N+2h}^{x_N+3h} \frac{\partial f}{\partial x_N}(x', s)\, ds - \int_{x_N+h}^{x_N+2h} \frac{\partial f}{\partial x_N}(x', s) ds) \\
&- (\int_{x_N+h}^{x_N+2h} \frac{\partial f}{\partial x_N}(x', s) ds - \int_{x_N}^{x_N+h} \frac{\partial f}{\partial x_N}(x', s)\, ds) \\
&= \int_{x_N}^{x_N+h} \left[(\frac{\partial f}{\partial x_N}(x', s+2h) - \frac{\partial f}{\partial x_N}(x', s+h)) - (\frac{\partial f}{\partial x_N}(x', s+h) - \frac{\partial f}{\partial x_N}(x', s)) \right] ds \\
&= \int_{x_N}^{x_N+h} [\int_{s+h}^{s+2h} \frac{\partial^2 f}{\partial x_N^2}(x', t) dt - \int_{s}^{s+h} \frac{\partial^2 f}{\partial x_N^2}(x', t) dt] ds \\
&= \int_{x_N}^{x_N+h} [\int_{s}^{s+h} (\frac{\partial^2 f}{\partial x_N^2}(x', t+h) - \frac{\partial^2 f}{\partial x_N^2}(x', t)) dt] ds \\
&= \int_{x_N}^{x_N+h} \int_{s}^{s+h} \int_{t}^{t+h} \frac{\partial^3 f}{\partial x_N^3}(x', \tau) d\tau dt ds.
\end{aligned}$$

Now we assume the claim holds for $k = m-1$, we only need to prove the claim also holds for

$k = m$. Since

$$\begin{aligned}
\triangle^{(m)} f(x, x+mhe_N) &= f(x+mhe_N) - C_m^1 f\left(x+(m-1)he_N\right) + C_m^2 f\left(x+(m-2)he_N\right) \\
&+ \cdots + (-1)^{m-1} C_m^{m-1} f(x+he_N) + (-1)^m f(x) \\
&= (f(x+mhe_N) - f(x+(m-1)he_N)) - C_{m-1}^1 (f(x+(m-1)he_N) \\
&- f(x+(m-2)he_N)) + \cdots + (-1)^{m-2} C_{m-1}^{m-2} (f(x+2he_N) \\
&- f(x+he_N)) + (-1)^{m-1}(f(x+he_N) - f(x)) \\
&= \int_{x_N+(m-1)h}^{x_N+mh} \frac{\partial f}{\partial x_N}\left(x', s\right) ds - C_{m-1}^1 \int_{x_N+(m-2)h}^{x_N+(m-1)h} \frac{\partial f}{\partial x_N}(x', s) ds \\
&+ \cdots + (-1)^{m-2} C_{m-1}^{m-2} \int_{x_N+h}^{x_N+2h} \frac{\partial f}{\partial x_N}(x', s) ds + (-1)^{m-1} \int_{x_N}^{x_N+h} \frac{\partial f}{\partial x_N}\left(x', s\right) ds \\
&= \int\limits_{x_N}^{x_N+h} [\frac{\partial f}{\partial x_N}\left(x', s+(m-1)h\right) - C_{m-1}^1 \frac{\partial f}{\partial x_N}\left(x', s+(m-2)h\right) \\
&+ \cdots + (-1)^{m-2} C_{m-1}^{m-2} \frac{\partial f}{\partial x_N}\left(x', s+h\right) + (-1)^{m-1} \frac{\partial f}{\partial x_N}\left(x', s\right)] ds \\
&= \int_{x_N}^{x_N+h} [\triangle^{(m-1)}(\frac{\partial f}{\partial x_N})(\tilde{x}, \tilde{x}+(m-1)he_N)] ds, \text{ with } \tilde{x} = (x', s) \\
&= \int_{x_N}^{x_N+h} \int_s^{s+h} \int_{s_1}^{s_1+h} \cdots \int_{s_{m-2}}^{s_{m-2}+h} \frac{\partial^{m-1} f}{\partial x_N^{m-1}}(\frac{\partial f}{\partial x_N})(x', s_{m-1}) ds_{m-1} ds_{m-2} \cdots ds_1 ds \\
&= \int_{x_N}^{x_N+h} \int_{s_1}^{s_1+h} \cdots \int_{s_{m-1}}^{s_{m-1}+h} \frac{\partial^m f}{\partial x_N^m}(x', s_m) ds_m ds_{m-1} \cdots ds_1.
\end{aligned}$$

So

$$\begin{aligned}
| \triangle^{(m)} f(x, x+mhe_N)| &\leq \int_{x_N}^{x_N+h} \int_{s_1}^{s_1+h} \cdots \int_{s_{m-2}}^{s_{m-2}+h} hM_N(\frac{\partial^m f}{\partial x_N^m})(x) ds_{m-1} \cdots ds_1 \\
&\leq h^m M_N(\frac{\partial^m f}{\partial x_N^m})(x)
\end{aligned}$$

Thus,

$$\int\limits_{\mathbb{R}^N}\int\limits_{\substack{0 \\ \left|\triangle^{(m)}f(x,x+mhe_N)\right|>\delta}}^{\infty}\frac{\delta^p}{h^{mp+1}}dhdx \leq \int\limits_{\mathbb{R}^N}\int\limits_{\substack{0 \\ h^m M_N\left(\frac{\partial^m f}{\partial x_N^m}\right)(x)>\delta}}^{\infty}\frac{\delta^p}{h^{mp+1}}dhdx$$

$$= \int\limits_{\mathbb{R}^N}\int\limits_{\left(\frac{\delta}{M_N\left(\frac{\partial^m f}{\partial x_N^m}\right)(x)}\right)^{\frac{1}{m}}}^{\infty}\frac{\delta^p}{h^{2p+1}}dhdx$$

$$= \frac{1}{mp}\int\limits_{\mathbb{R}^N}\left|M_N\left(\frac{\partial^m f}{\partial x_N^m}\right)(x)\right|^p dx$$

$$\leq C_{N,p}\int\limits_{\mathbb{R}^N}\left|\frac{\partial^m f}{\partial x_N^m}\right|^p dx$$

$$\leq C_{N,p}\int\limits_{\mathbb{R}^N}\left|\nabla^m f\right|^p dx.$$

The proof of Lemma 5.5 is now completed. □

Lemma 5.6. There holds

$$\lim_{\delta\to 0}\int\limits_{\mathbb{R}^N}\int\limits_{\substack{\mathbb{R}^N \\ \left|\triangle^{(m)}f(x,y)\right|>\delta}}\frac{\delta^p}{|x-y|^{N+mp}}dxdy = \frac{1}{m^{mp+1}p}\int\limits_{\mathbb{S}^{N-1}}\int\limits_{\mathbb{R}^N}\left|D^m f(x)(\sigma,\cdots,\sigma)\right|^p dxd\sigma$$

for all $f\in W^{m,p}\left(\mathbb{R}^N\right), m\geq 3,\ 1<p<\infty.$

Proof. Again, by changing of variables and set $y = x + \sqrt[m]{\delta}h\sigma$ we obtain

$$\int\limits_{\mathbb{R}^N}\int\limits_{\substack{\mathbb{R}^N \\ \left|\triangle^{(m)}f(x,y)\right|>\delta}}\frac{\delta^p}{|x-y|^{N+mp}}dxdy = \int\limits_{\mathbb{S}^{N-1}}\int\limits_{\mathbb{R}^N}\int\limits_{\substack{0 \\ \left|\frac{\triangle^{(m)}f(x,x+\sqrt[m]{\delta}h\sigma)}{h^m\delta}\right|h^m>1}}^{\infty}\frac{1}{h^{mp+1}}dhdxd\sigma.$$

Define $F_\delta:\mathbb{S}^{N-1}\to\mathbb{R}$ by

$$F_\delta(\sigma) = \int\limits_{\mathbb{R}^N}\int\limits_{\substack{0 \\ \left|\frac{\triangle^{(m)}f(x,x+\sqrt[m]{\delta}h\sigma)}{h^m\delta}\right|h^m>1}}^{\infty}\frac{1}{h^{mp+1}}dhdx.$$

We first prove that for all $\sigma\in\mathbb{S}^{N-1},\ \forall\delta>0:$

$$F_\delta(\sigma)\leq C_{N,p}\int\limits_{\mathbb{R}^N}\left|\nabla^m f\right|^p dx. \tag{5.3}$$

Indeed, again, without loss of generality, we assume that $\sigma = e_N = (0,...,0,1)$. Hence, we need to verify that

$$\int\limits_{\mathbb{R}^N}\int\limits_{\substack{0 \\ \left|\frac{\triangle^{(m)} f(x,x+\sqrt[m]{\delta}he_N)}{h^m\delta}\right|h^m>1}}^{\infty} \frac{1}{h^{2p+1}} dhdx \leq C_{N,p}\int\limits_{\mathbb{R}^N} \left|\nabla^m f\right|^p dx. \tag{5.4}$$

Similarly as in the Lemma 5.5, we have

$$\left|\frac{\triangle^{(m)} f(x, x+\sqrt[m]{\delta}he_N)}{h^m\delta}\right| \leq M_N\left(\frac{\partial^m f}{\partial x_N^m}\right)(x).$$

Thus,

$$\int\limits_{\mathbb{R}^N}\int\limits_{\substack{0 \\ \left|\frac{\triangle^{(m)} f(x,x+\sqrt[m]{\delta}he_N)}{h^m\delta}\right|h^m>1}}^{\infty} \frac{1}{h^{mp+1}} dhdx \leq \int\limits_{\mathbb{R}^N}\int\limits_{\substack{0 \\ h^m M_N\left(\frac{\partial^m f}{\partial x_N^m}\right)(x)>1}}^{\infty} \frac{1}{h^{mp+1}} dhdx$$

$$\leq C_{N,p}\int\limits_{\mathbb{R}^N} \left|\nabla^m f\right|^p dx.$$

Next, we will show that

$$F_\delta(\sigma) \to \frac{1}{m^{mp+1}p}\int\limits_{\mathbb{R}^N} \left|D^m f(x)(\sigma, \cdots, \sigma)\right|^p dx \text{ as } \delta \to 0 \text{ for every } \sigma \in \mathbb{S}^{N-1} \tag{5.5}$$

where

$$\left|D^m f(x)(\sigma, \cdots, \sigma)\right| = \sum_{1\leq i_1,\cdots,i_m\leq N} \sigma_{i_1}\cdots\sigma_{i_m}\frac{\partial^m f}{\partial x_{i_1}\cdots\partial x_{i_m}}(x).$$

Again, without loss of generality, we suppose that $\sigma = e_N = (0, ..., 0, 1)$. We write

$$F_\delta(\sigma) = \int\limits_{\mathbb{R}^N}\int\limits_0^\infty G_\delta(x, h)\, dhdx$$

where

$$G_\delta(x, h) = \frac{1}{h^{mp+1}}\chi_{\left\{\left|\frac{\triangle^{(m)} f(x,x+\sqrt[m]{\delta}he_N)}{h^m\delta}\right|h^m>1\right\}}(x, h).$$

Noting that for all $\sigma \in \mathbb{S}^{N-1}$:

$$G_\delta(x, h) \to \frac{1}{h^{mp+1}}\chi_{\{|D^m f(x)(\sigma,\cdots,\sigma)|h^m>m^m\}}(x, h) \text{ as } \delta \to 0 \text{ for a.e. } (x, h) \in \mathbb{R}^N \times [0, \infty)\ ,$$

and

$$G_\delta(x, h) \leq \frac{1}{h^{mp+1}}\chi_{\left\{h^m M_N\left(\frac{\partial^m f}{\partial x_N^m}\right)(x)>1\right\}}(x, h) \in L^1\left(\mathbb{R}^N \times [0, \infty)\right).$$

Hence, by the Lebesgue's dominated convergence theorem, we get (5.5).

Using (5.3) and (5.5), again, using the Lebesgue's dominated convergence theorem, we can conclude that

$$\lim_{\delta\to 0}\underset{\left|\triangle^{(m)}(x,y)\right|>\delta}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}}\frac{\delta^p}{|x-y|^{N+mp}}dxdy=\frac{1}{m^{mp+1}p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}|D^m f(x)(\sigma,\cdots,\sigma)|^p\,dxd\sigma.$$

□

Lemma 5.7. Assume that $f\in L^p\left(\mathbb{R}^N\right)\cap C^\infty\left(\mathbb{R}^N\right)$ satisfying

$$C(f):=\sup_{0<\varepsilon<1}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\varepsilon\left|\triangle^{(m)}(x,y)\right|^{p+\varepsilon}}{|x-y|^{N+mp}}dxdy<\infty.$$

Then $f\in W^{m,p}\left(\mathbb{R}^N\right)$ and

$$\frac{1}{m^{mp+1}}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}|D^m f(x)(\sigma,\cdots,\sigma)|^p\,dxd\sigma\le\liminf_{\varepsilon\to 0}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\varepsilon\left|\triangle^{(m)}(x,y)\right|^{p+\varepsilon}}{|x-y|^{N+mp}}dxdy.$$

Proof. From the assumptions, and set $y=x+t\sigma$ we have

$$\sup_{0<\varepsilon<1}\int_{\mathbb{S}^{N-1}}\int_{B_R}\int_0^1\frac{\varepsilon\left|\triangle^{(m)}(x,x+t\sigma)\right|^{p+\varepsilon}}{t^{mp+1}}dtdxd\sigma\le C(f).$$

By Taylor expansion, since $f\in L^p\left(\mathbb{R}^N\right)\cap C^\infty\left(\mathbb{R}^N\right)$, we can have that

$$\frac{1}{m^m}\left|D^m f(x)(t\sigma,\cdots,t\sigma)\right|\le\left|\triangle^{(m)}(x,x+t\sigma)\right|+Ct^{m+1}$$

for all $(\sigma,x,t)\in\mathbb{S}^{N-1}\times B_R\times(0,1)$. Also, since

$$\left|\triangle^{(m)}(x,x+t\sigma)\right|\le Ct^m\text{ for all }(\sigma,x,t)\in\mathbb{S}^{N-1}\times B_R\times(0,1),$$

we have

$$\begin{aligned}\frac{1}{m^{m(p+\varepsilon)}}\left|D^m f(x)(t\sigma,\cdots,t\sigma)\right|^{p+\varepsilon}&\le\left[\left|\triangle^{(m)}(x,x+t\sigma)\right|+Ct^{m+1}\right]^{p+\varepsilon}\\&\le\left|\triangle^{(m)}(x,x+t\sigma)\right|^{p+\varepsilon}+Ct^{mp+\varepsilon+1},\end{aligned}$$

for all $(\sigma,x,t)\in\mathbb{S}^{N-1}\times B_R\times(0,1)$.

Now, noting that

$$\liminf_{\varepsilon\to 0}\int_{\mathbb{S}^{N-1}}\int_{B_R}\int_0^1\frac{\varepsilon t^{mp+\varepsilon+1}}{t^{mp+1}}dtdxd\sigma=0,$$

we can deduce

$$\liminf_{\varepsilon\to 0}\int\limits_{\mathbb{S}^{N-1}}\int\limits_{B_R}\int\limits_0^1 \frac{1}{m^{m(p+\varepsilon)}}\frac{\varepsilon\left|D^m f(x)(t\sigma,\cdots,t\sigma)\right|^{p+\varepsilon}}{t^{mp+1}}dtdxd\sigma$$

$$\leq \liminf_{\varepsilon\to 0}\int\limits_{\mathbb{S}^{N-1}}\int\limits_{B_R}\int\limits_0^1 \frac{\varepsilon\left|\triangle^{(m)}(x,x+t\sigma)\right|^{p+\varepsilon}}{t^{mp+1}}dtdxd\sigma.$$

As a consequence, we get

$$\frac{1}{m^{mp+1}}\int\limits_{\mathbb{S}^{N-1}}\int\limits_{B_R}\left|D^m f(x)(\sigma,\cdots,\sigma)\right|^p dxd\sigma \leq \liminf_{\varepsilon\to 0}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{\varepsilon\left|\triangle^{(m)}(x,y)\right|^{p+\varepsilon}}{|x-y|^{N+mp}}dxdy.$$

Hence, we can conclude that $f\in W^{m,p}\left(\mathbb{R}^N\right)$ and

$$\frac{1}{m^{mp+1}}\int\limits_{\mathbb{S}^{N-1}}\int\limits_{\mathbb{R}^N}\left|D^m f(x)(\sigma,\cdots,\sigma)\right|^p dxd\sigma \leq \liminf_{\varepsilon\to 0}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{\varepsilon\left|\triangle^{(m)}(x,y)\right|^{p+\varepsilon}}{|x-y|^{N+mp}}dxdy.$$

□

Proof of Theorem 5.1:

(1) and (2) are consequences of Lemma 5.5 and Lemma 5.6.

Now we will prove (3). By (1), we get

$$\int\limits_{\mathbb{R}^N}\int\limits_{\substack{\mathbb{R}^N\\ |\triangle^m f(x,y)|>\delta}}\frac{\delta^p}{|x-y|^{N+mp}}dxdy \leq C_{N,p}\int\limits_{\mathbb{R}^N}\left|\nabla^m f\right|^p dx,\ \forall\delta>0. \tag{5.6}$$

As a consequence, by Fatou's lemma,

$$\int\limits_{\mathbb{R}^N}\int\limits_{\substack{\mathbb{R}^N\\ |\triangle^m f(x,y)|>1}}\frac{1}{|x-y|^{N+mp}}dxdy \leq C_{N,p}\int\limits_{\mathbb{R}^N}\left|\nabla^m f\right|^p dx. \tag{5.7}$$

Now, from (5.6), one has

$$\int\limits_0^1\int\limits_{\mathbb{R}^N}\int\limits_{\substack{\mathbb{R}^N\\ |\triangle^m f(x,y)|>\delta}}\frac{\varepsilon\delta^{p+\varepsilon-1}}{|x-y|^{N+mp}}dxdyd\delta \leq C_{N,p}\int\limits_{\mathbb{R}^N}\left|\nabla^m f\right|^p dx.$$

Using Lemma 2.1, we deduce

$$\int\limits_{\mathbb{R}^N}\int\limits_{\substack{\mathbb{R}^N\\ |\triangle^m f(x,y)|\leq 1}}\frac{\varepsilon\left|\triangle^m f(x,y)\right|^{p+\varepsilon}}{|x-y|^{N+mp}}dxdy+\int\limits_{\mathbb{R}^N}\int\limits_{\substack{\mathbb{R}^N\\ |\triangle^m f(x,y)|>1}}\frac{\varepsilon}{|x-y|^{N+mp}}dxdy \leq C_{N,p}\int\limits_{\mathbb{R}^N}\left|\nabla^m f\right|^p dx. \tag{5.8}$$

From (5.7) and (5.8), we get the assertion (3).

Now, set

$$H(\delta) = \iint\limits_{\substack{\mathbb{R}^N\mathbb{R}^N \\ |\triangle^m f(x,y)|>\delta}} \frac{\delta^p}{|x-y|^{N+mp}} dxdy.$$

So be previous results, we have

$$H(\delta) \le C_{N,p} \int\limits_{\mathbb{R}^N} |\nabla^m f|^p \, dx, \ \forall \delta > 0$$

and

$$\lim_{\delta\to 0} H(\delta) = \frac{1}{m^{mp+1}p} \int\limits_{\mathbb{S}^{N-1}} \int\limits_{\mathbb{R}^N} |D^m f(x)(\sigma, \cdots, \sigma)|^p \, dxd\sigma.$$

Now, we claim that

$$\lim_{\varepsilon\to 0} \int\limits_0^1 (p+\varepsilon)\varepsilon\delta^{\varepsilon-1} H(\delta)\, d\delta = \frac{1}{m^{mp+1}} \int\limits_{\mathbb{S}^{N-1}} \int\limits_{\mathbb{R}^N} |D^m f(x)(\sigma, \cdots, \sigma)|^p \, dxd\sigma. \tag{5.9}$$

Indeed, for every $\epsilon > 0$, we can find a number $X(\epsilon) \in (0,1)$ such that

$$\left| H(\delta) - \frac{1}{m^{mp+1}p} \int\limits_{\mathbb{S}^{N-1}} \int\limits_{\mathbb{R}^N} |D^m f(x)(\sigma, \cdots, \sigma)|^p \, dxd\sigma \right| < \epsilon \text{ for all } \delta \in (0, X(\epsilon)).$$

Now, we have:

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \left| \int\limits_{X(\epsilon)}^1 (p+\varepsilon)\varepsilon\delta^{\varepsilon-1} \left[H(\delta) - \frac{1}{m^{mp+1}p} \int\limits_{\mathbb{S}^{N-1}} \int\limits_{\mathbb{R}^N} |D^m f(x)(\sigma, \cdots, \sigma)|^p \, dxd\sigma \right] d\delta \right| \\
&\le \int\limits_{X(\epsilon)}^1 \lim_{\varepsilon\to 0} (p+\varepsilon)\varepsilon\delta^{\varepsilon-1} \left| H(\delta) - \frac{1}{m^{mp+1}p} \int\limits_{\mathbb{S}^{N-1}} \int\limits_{\mathbb{R}^N} |D^m f(x)(\sigma, \cdots, \sigma)|^p \, dxd\sigma \right| d\delta \\
&\le \int\limits_{X(\epsilon)}^1 \lim_{\varepsilon\to 0} (p+\varepsilon)\varepsilon X(\epsilon)^{\varepsilon-1} \left[C_{N,p} \int\limits_{\mathbb{R}^N} |\nabla^m f|^p \, dx \right] d\delta \\
&= 0.
\end{aligned}$$

Moreover,

$$\lim_{\varepsilon\to 0}\left|\int_0^{X(\epsilon)}(p+\varepsilon)\,\varepsilon\delta^{\varepsilon-1}\left[H(\delta)-\frac{1}{m^{mp+1}p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}|D^m f(x)(\sigma,\cdots,\sigma)|^p\,dxd\sigma\right]d\delta\right|$$

$$\leq \lim_{\varepsilon\to 0}\int_0^{X(\epsilon)}(p+\varepsilon)\,\varepsilon\delta^{\varepsilon-1}\left|H(\delta)-\frac{1}{m^{mp+1}p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}|D^m f(x)(\sigma,\cdots,\sigma)|^p\,dxd\sigma\right|d\delta$$

$$\leq \lim_{\varepsilon\to 0}\int_0^{X(\epsilon)}(p+\varepsilon)\,\varepsilon\delta^{\varepsilon-1}\epsilon d\delta$$

$$\leq \lim_{\varepsilon\to 0}\int_0^{1}(p+\varepsilon)\,\varepsilon\delta^{\varepsilon-1}\epsilon d\delta$$

$$= p\epsilon.$$

Thus,

$$\lim_{\varepsilon\to 0}\left|\int_0^{1}(p+\varepsilon)\,\varepsilon\delta^{\varepsilon-1}\left[H(\delta)-\frac{1}{m^{mp+1}p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}|D^m f(x)(\sigma,\cdots,\sigma)|^p\,dxd\sigma\right]d\delta\right|\leq p\epsilon,\ \forall\epsilon>0.$$

Hence we can get

$$\lim_{\varepsilon\to 0}\int_0^{1}(p+\varepsilon)\,\varepsilon\delta^{\varepsilon-1}H(\delta)\,d\delta=\lim_{\varepsilon\to 0}\int_0^{1}(p+\varepsilon)\,\varepsilon\delta^{\varepsilon-1}\left[\frac{1}{m^{mp+1}p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}|D^m f(x)(\sigma,\cdots,\sigma)|^p\,dxd\sigma\right]d\delta$$

$$=\frac{1}{m^{mp+1}}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}|D^m f(x)(\sigma,\cdots,\sigma)|^p\,dxd\sigma.$$

Consequently, we have

$$\lim_{\varepsilon\to 0}\int_0^{1}\underset{|\triangle^m f(x,y)|>\delta}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}}\frac{(p+\varepsilon)\,\varepsilon\delta^{p+\varepsilon-1}}{|x-y|^{N+2p}}dxdyd\delta=\frac{1}{m^{mp+1}}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}|D^m f(x)(\sigma,\cdots,\sigma)|^p\,dxd\sigma.$$

Now, using Lemma 2.1 with $\alpha=p+\varepsilon-1,\ \Phi(x,y)=|\triangle^m f(x,y)|,\ \Psi(x,y)=\frac{1}{|x-y|^{N+mp}}$, we obtain

$$\lim_{\varepsilon\to 0}\left[\iint_{|\triangle^m f(x,y)|\leq 1}\frac{\varepsilon\,|\triangle^m f(x,y)|^{p+\varepsilon}}{|x-y|^{N+mp}}dxdy+\iint_{|\triangle^m f(x,y)|>1}\frac{\varepsilon}{|x-y|^{N+mp}}dxdy\right]$$

$$=\frac{1}{m^{mp+1}}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}|D^m f(x)(\sigma,\cdots,\sigma)|^p\,dxd\sigma.$$

Noting that

$$\lim_{\varepsilon\to 0}\iint_{|\triangle^m f(x,y)|>1}\frac{\varepsilon}{|x-y|^{N+mp}}dxdy=0,$$

we have

$$\lim_{\varepsilon\to 0}\iint\limits_{|\triangle^m f(x,y)|\le 1}\frac{\varepsilon\,|\triangle^m f(x,y)|^{p+\varepsilon}}{|x-y|^{N+mp}}dxdy=\frac{1}{m^{mp+1}}\int\limits_{\mathbb{S}^{N-1}}\int\limits_{\mathbb{R}^N}|D^m f\,(x)\,(\sigma,\cdot\cdot\cdot,\sigma)|^p\,dxd\sigma.$$

We have the statement (4). □

Proof of Theorem 5.3:

First, it is clear that statements (1) $\Longrightarrow$ (2) and (1) $\Longrightarrow$ (3) are consequences of Theorem 5.1.

Now, we will prove (3) $\Longrightarrow$ (1) : Let η_ε be any sequence of smooth mollifiers and set

$$f^\varepsilon = f * \eta_\varepsilon$$

Then we can get $f^\varepsilon \in L^p(\mathbb{R}^N)\cap C^\infty(\mathbb{R}^N)\subseteq W^{m,p}(\mathbb{R}^N)$.

Using the (4) of the Theorem 5.1, we can have

$$\begin{aligned}
&C_{N,p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}|D^m f^\varepsilon(x)(\sigma,\cdot\cdot\cdot,\sigma)|^p dxd\sigma\\
&\le \sup_{0<\gamma<1}\underset{|\triangle^{(m)}f^\varepsilon(x,y)|\le 1}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}}\frac{\gamma|\triangle^{(m)}\ f^\varepsilon(x,y)|^{p+\gamma}}{|x-y|^{N+mp}}dxdy\\
&= \sup_{0<\gamma<1}\underset{|\triangle^{(m)}f^\varepsilon(x,y)|\le 1}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}}\frac{\gamma|\int_{\mathbb{R}^N}\{\triangle^{(m)}f(x-z,y-z)\}\eta^\varepsilon(z)dz|^{p+\gamma}}{|x-y|^{N+mp}}dxdy
\end{aligned}$$

Since the function $x^{p+\varepsilon}$ is convex on $[0,\infty)$, by Jensen's inequality, we can deduce

$$\begin{aligned}
&\sup_{0<\gamma<1}\underset{|\triangle^{(m)}f^\varepsilon(x,y)|\le 1}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}}\frac{\gamma|\int_{\mathbb{R}^N}\{\triangle^{(m)}f(x-z,y-z)\}\eta^\varepsilon(z)dz|^{p+\gamma}}{|x-y|^{N+mp}}dxdy\\
&\le \sup_{0<\gamma<1}\underset{|\triangle^{(m)}f^\varepsilon(x,y)|\le 1}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}}\frac{\gamma|\ \triangle^{(m)}\ f(x-z,y-z)|^{p+\gamma}\int_{\mathbb{R}^N}\eta^\varepsilon(z)dz}{|x-y|^{N+mp}}dxdy\\
&= \sup_{0<\gamma<1}\underset{|\int_{\mathbb{R}^N}\{\triangle^{(m)}f(x,y)\}\eta^\varepsilon(z)dz|\le 1}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}}\frac{\gamma|\ \triangle^{(m)}\ f(x,y)|^{p+\gamma}}{|x-y|^{N+mp}}dxdy
\end{aligned}$$

The last step comes from changing variable and $\int_{\mathbb{R}^N}\eta^\varepsilon(z)dz=1$. Since

$$|\int_{\mathbb{R}^N}\{\triangle^{(m)}f(x,y)\}\eta^\varepsilon(z)dz| = |\ \triangle^{(m)}\ f(x,y)|$$

Thus,

$$
\begin{aligned}
& C_{N,p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}|D^m f^\varepsilon(x)(\sigma,\cdots,\sigma)|^p dxd\sigma \\
& \leq \sup_{0<\gamma<1} \underset{|\int_{\mathbb{R}^N}\{\triangle^m f(x,y)\eta^\varepsilon(z)dz|\leq 1}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{\gamma|\triangle^m f(x,y)|^{p+\gamma}}{|x-y|^{N+mp}}dxdy \\
& = \sup_{0<\gamma<1} \underset{|\triangle^m f(x,y)|\leq 1}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{\gamma|\triangle^m f(x,y)|^{p+\gamma}}{|x-y|^{N+mp}}dxdy \\
& \leq \infty from (3)
\end{aligned}
$$

So $\|f^\varepsilon\|_{W^{m,p}(\mathbb{R}^N)}$ is bounded. Then there exists a subsequence of f^ε, denoted as f^ε, such that

$$f^\varepsilon \rightharpoonup g \text{ in } W^{2,p}(\mathbb{R}^N).$$

On the other hand, from the property of mollifier, we know

$$f^\varepsilon \to f \text{ a.e.}$$

From the uniqueness of the limit, we get $f = g$ a.e. in $W^{m,p}(\mathbb{R}^N)$. That is , $f \in W^{m,p}(\mathbb{R}^N)$. □

5.3 Characterization Using (m-1)th-Taylor Remainder

In this section, we will investigate the characterizations of high order Sobolev spaces $W^{m,p}\left(\mathbb{R}^N\right)$ in terms of the Taylor-remainder of order $m-1$.

Lemma 5.8. There exists a constant $C_{N,m,p} > 0$ such that for all $\delta > 0$, all $f \in W^{m,p}\left(\mathbb{R}^N\right)$:

$$\underset{|R^{m-1}f(x,y)|>\delta}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{\delta^p}{|x-y|^{N+mp}}dxdy \leq C_{N,m,p}\int_{\mathbb{R}^N}|\nabla^m f|^p\, dx. \tag{5.10}$$

Proof. Again, using the polar coordinates, we get

$$\underset{|R^{m-1}f(x,y)|>\delta}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{\delta^p}{|x-y|^{N+mp}}dxdy = \underset{|R^{m-1}f(x+h\sigma,x)|>\delta}{\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}\int_0^\infty} \frac{\delta^p}{h^{mp+1}}dhdxd\sigma.$$

Thus, again, to prove (5.10), it's enough to prove that for every $\sigma \in \mathbb{S}^{N-1}$, we get

$$\underset{|R^{m-1}f(x+h\sigma,x)|>\delta}{\int_{\mathbb{R}^N}\int_0^\infty} \frac{\delta^p}{h^{mp+1}}dhdx \leq C_{N,p}\int_{\mathbb{R}^N}|\nabla^m f|^p\, dx. \tag{5.11}$$

Because of the rotation, we assume without loss of generality that $\sigma = e_N = (0,...,0,1)$. We claim for any $k \geq 3$,

$$R^{k-1}f(x+he_N,x) =$$

$$h^k \int_0^1 \cdots \int_0^1 \frac{\partial^k f}{\partial x_N^k}(x', x_N + s_k s_{k-1} s_{k-2} \cdots s_1 h) s_1^{k-1} s_2^{k-2} \cdots s_{k-2}^2 s_{k-1} ds_k ds_{k-1} \cdots ds_2 ds_1$$

We first check the claim holds for $k = 3$. Now, by the mean value theorem, one has

$$\begin{aligned}
R^2 f(x+he_N,x) &= f(x+he_N) - f(x) - h\nabla f(x) e_N - \frac{h^2}{2!}\frac{\partial^2 f}{\partial x_N^2}(x) \\
&= \int_0^1 \frac{\partial f}{\partial x_N}\left(x', x_N + sh\right) h ds - h\frac{\partial f}{\partial x_N}(x) - \frac{h^2}{2!}\frac{\partial^2 f}{\partial x_N^2}(x) \\
&= h\int_0^1 \left[\frac{\partial f}{\partial x_N}\left(x', x_N + sh\right) h ds - \frac{\partial f}{\partial x_N}(x', x_N)\right] ds - \frac{h^2}{2!}\frac{\partial^2 f}{\partial x_N^2}(x) \\
&= h\int_0^1 \int_{x_N}^{x_N+sh} \frac{\partial^2 f}{\partial x_N^2}\left(x', t\right) dt ds - \frac{h^2}{2!}\frac{\partial^2 f}{\partial x_N^2}(x) \\
&= h^2 \int_0^1 \int_0^1 \frac{\partial^2 f}{\partial x_N^2}\left(x', x_N + tsh\right) s dt ds - h^2 \int_0^1 \int_0^1 \frac{\partial^2 f}{\partial x_N^2}(x', x_N) s dt ds \\
&= h^2 \int_0^1 \int_0^1 \int_{x_N}^{x_N+tsh} \frac{\partial^3 f}{\partial x_N^3}(x', \tau) d\tau s dt ds \\
&= h^3 \int_0^1 \int_0^1 \int_0^1 \frac{\partial^3 f}{\partial x_N^3}(x', x_N + \tau s t h) s t d\tau s dt ds.
\end{aligned}$$

By induction, assume it holds for $k = m-1$.We want to prove it holds for $k = m$.Indeed,

$$R^{m-1}f(x+he_N,x) = R^{m-2}f(x+he_N,x) - \frac{h^{m-1}}{(m-1)!}\frac{\partial^{m-1}f}{\partial x_N^{m-1}}(x)$$

$$= h^{m-1}\int_0^1\cdots\int_0^1 \frac{\partial^{m-1}f}{\partial x_N^{m-1}}(x', x_N + s_{m-1}s_{m-2}\cdots s_1 h)$$

$$s_1^{m-2}s_2^{m-3}\cdots s_{m-3}^2 s_{m-2} ds_{m-1}\cdots ds_1 - \frac{h^{m-1}}{(m-1)!}\frac{\partial^{m-1}f}{\partial x_N^{m-1}}(x)$$

$$= h^{m-1}\int_0^1\cdots\int_0^1 \frac{\partial^{m-1}f}{\partial x_N^{m-1}}(x', x_N + s_{m-1}s_{m-2}\cdots s_1 h)$$

$$s_1^{m-2}s_2^{m-3}\cdots s_{m-3}^2 s_{m-2} ds_{m-1}\cdots ds_1$$

$$- h^{m-1}\int_0^1\cdots\int_0^1 \frac{\partial^{m-1}f}{\partial x_N^{m-1}}(x', x_N) s_1^{m-2}s_2^{m-3}\cdots s_{m-3}^2 s_{m-2} ds_{m-1}\cdots ds_1$$

$$= h^{m-1}\int_0^1\cdots\int_0^1 \{\frac{\partial^{m-1}f}{\partial x_N^{m-1}}(x', x_N + s_{m-1}s_{m-2}\cdots s_1 h) - \frac{\partial^{m-1}f}{\partial x_N^{m-1}}(x', x_N)\}$$

$$s_1^{m-2}s_2^{m-3}\cdots s_{m-3}^2 s_{m-2} ds_{m-1}\cdots ds_1$$

$$= h^{m}\int_0^1\cdots\int_0^1 \int_{x_N}^{x_N+s_m s_{m-1}\cdots s_1 h} \frac{\partial^{m}f}{\partial x_N^{m}}(x', s_m) ds_m$$

$$s_1^{m-2}s_2^{m-3}\cdots s_{m-3}^2 s_{m-2} ds_{m-1}\cdots ds_1$$

$$= h^{m}\int_0^1\cdots\int_0^1 \frac{\partial^{m}f}{\partial x_N^{m}}(x', x_N + s_m s_{m-1}\cdots s_1)$$

$$s_1^{m-1}s_2^{m-2}\cdots s_{m-2}^2 s_{m-1} ds_m\cdots ds_1$$

Thus

$$|R^{m-1}f(x+he_N,x)| \le h^m\int_0^1\cdots\int_0^1 M_N(\frac{\partial^m f}{\partial x_N^m})(x) s_1^{m-1}s_2^{m-2}\cdots s_{m-2}^2 s_{m-1} ds_m\cdots ds_1$$

$$= \frac{1}{m!}h^m M_N(\frac{\partial^m f}{\partial x_N^m})(x).$$

Now we can prove 5.11

$$\iint\limits_{\substack{\mathbb{R}^N \; 0 \\ |R^{m-1}f(x+he_N,x)|>\delta}}^{\infty} \frac{\delta^p}{h^{mp+1}} dhdx \leq \iint\limits_{\substack{\mathbb{R}^N \; 0 \\ \frac{1}{m!}h^m M_N(\frac{\partial^m f}{\partial x_N^m})(x)>\delta}}^{\infty} \frac{\delta^p}{h^{mp+1}} dhdx$$

$$= \int_{\mathbb{R}^N} \int_{\left(\frac{\delta m!}{M_N(\frac{\partial^m f}{\partial x_N^m})(x)}\right)^{\frac{1}{m}}}^{\infty} \frac{\delta^p}{h^{mp+1}} dhdx$$

$$= \frac{1}{(m!)^p mp} \int_{\mathbb{R}^N} \left| M_N(\frac{\partial^m f}{\partial x_N^m})(x) \right|^p dx$$

$$\leq C_{N,p} \int_{\mathbb{R}^N} \left| \frac{\partial^m f}{\partial x_N^m}(x) \right|^p dx$$

$$\leq C_{N,p} \int\limits_{\mathbb{R}^N} |\nabla^m f|^p \, dx.$$

□

Lemma 5.9. There holds

$$\lim_{\delta \to 0} \iint\limits_{\substack{\mathbb{R}^N \mathbb{R}^N \\ |R^{m-1}f(x,y)|>\delta}} \frac{\delta^p}{|x-y|^{N+mp}} dxdy = \frac{1}{(m!)^p mp} \int\limits_{\mathbb{S}^{N-1}} \int\limits_{\mathbb{R}^N} |D^m f(x)(\sigma, \cdots, \sigma)|^p \, dxd\sigma$$

for all $f \in W^{m,p}(\mathbb{R}^N)$, $m \geq 3$, $1 < p < \infty$.

Proof. Again, by changing of variables and set $y = x + \sqrt[m]{\delta} h\sigma$ we obtain

$$\iint\limits_{\substack{\mathbb{R}^N \mathbb{R}^N \\ \left|R^{(m-1)}f(x,y)\right|>\delta}} \frac{\delta^p}{|x-y|^{N+mp}} dxdy = \int\limits_{\mathbb{S}^{N-1}} \iint\limits_{\substack{\mathbb{R}^N \; 0 \\ \left|\frac{R^{(m-1)}f(x,x+\sqrt[m]{\delta}h\sigma)}{h^m \delta}\right| h^m > 1}}^{\infty} \frac{1}{h^{mp+1}} dhdxd\sigma.$$

Define $F_\delta : \mathbb{S}^{N-1} \to \mathbb{R}$ by

$$F_\delta(\sigma) = \iint\limits_{\substack{\mathbb{R}^N \; 0 \\ \left|\frac{R^{m-1}f(x,x+\sqrt[m]{\delta}h\sigma)}{h^m \delta}\right| h^m > 1}}^{\infty} \frac{1}{h^{mp+1}} dhdx.$$

We first prove that for all $\sigma \in \mathbb{S}^{N-1}$, $\forall \delta > 0$:

$$F_\delta(\sigma) \leq C_{N,p} \int\limits_{\mathbb{R}^N} |\nabla^m f|^p \, dx. \tag{5.12}$$

Indeed, again, without loss of generality, we assume that $\sigma = e_N = (0, ..., 0, 1)$. Hence, we need to verify that

$$\int\limits_{\mathbb{R}^N}\int\limits_{\substack{0 \\ \left|\frac{R^{m-1}f(x,x+\sqrt[m]{\delta}he_N)}{h^m\delta}\right|h^m>1}}^{\infty}\frac{1}{h^{mp+1}}dhdx \leq C_{N,p}\int\limits_{\mathbb{R}^N}|\nabla^m f|^p\,dx. \tag{5.13}$$

Similarly as in the Lemma 5.6, we have

$$\left|\frac{R^{m-1}f(x,x+\sqrt[m]{\delta}he_N)}{h^m\delta}\right| \leq \frac{1}{m!}M_N\left(\frac{\partial^m f}{\partial x_N^m}\right)(x)\,.$$

Thus,

$$\int\limits_{\mathbb{R}^N}\int\limits_{\substack{0 \\ \left|\frac{R^{m-1}f(x,x+\sqrt[m]{\delta}he_N)}{h^m\delta}\right|h^m>1}}^{\infty}\frac{1}{h^{mp+1}}dhdx \leq \int\limits_{\mathbb{R}^N}\int\limits_{\substack{0 \\ h^mM_N\left(\frac{\partial^m f}{\partial x_N^m}\right)(x)>m!}}^{\infty}\frac{1}{h^{mp+1}}dhdx$$

$$\leq C_{N,p}\int\limits_{\mathbb{R}^N}|\nabla^m f|^p\,dx.$$

Next, we will show that

$$F_\delta(\sigma) \to \frac{1}{(m!)^p mp}\int\limits_{\mathbb{R}^N}|D^m f(x)(\sigma,\cdots,\sigma)|^p\,dx \text{ as } \delta\to 0 \text{ for every } \sigma\in\mathbb{S}^{N-1} \tag{5.14}$$

where

$$|D^m f(x)(\sigma,\cdots,\sigma)| = \sum_{1\leq i_1,\cdots,i_m\leq N}\sigma_{i_1}\cdots\sigma_{i_m}\frac{\partial^m f}{\partial x_{i_1}\cdots\partial x_{i_m}}(x)\,.$$

Again, without loss of generality, we suppose that $\sigma = e_N = (0,...,0,1)$. We write

$$F_\delta(\sigma) = \int\limits_{\mathbb{R}^N}\int\limits_0^\infty G_\delta(x,h)\,dhdx$$

where

$$G_\delta(x,h) = \frac{1}{h^{mp+1}}\chi_{\left\{\left|\frac{R^{m-1}f(x,x+\sqrt[m]{\delta}he_N)}{h^m\delta}\right|h^m>1\right\}}(x,h)\,.$$

Noting that for all $\sigma\in\mathbb{S}^{N-1}$:

$$G_\delta(x,h) \to \frac{1}{h^{mp+1}}\chi_{\{|D^m f(x)(\sigma,\cdots,\sigma)|h^m>m!\}}(x,h) \text{ as } \delta\to 0 \text{ for a.e. } (x,h)\in\mathbb{R}^N\times[0,\infty)\ ,$$

and

$$G_\delta(x,h) \leq \frac{1}{h^{mp+1}}\chi_{\left\{h^mM_N\left(\frac{\partial^m f}{\partial x_N^m}\right)(x)>m!\right\}}(x,h)\in L^1\left(\mathbb{R}^N\times[0,\infty)\right).$$

Hence, by the Lebesgue's dominated convergence theorem, we get (5.14).

Using (5.12) and (5.14), again, using the Lebesgue's dominated convergence theorem, we can conclude that

$$\lim_{\delta\to 0}\int\int_{\substack{\mathbb{R}^N\mathbb{R}^N \\ |R^{m-1}f(x,y)|>\delta}} \frac{\delta^p}{|x-y|^{N+mp}}dxdy = \frac{1}{(m!)^p mp}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N} |D^m f(x)(\sigma,\cdots,\sigma)|^p\, dxd\sigma.$$

□

Lemma 5.10. Assume that $f \in W^{m-1,p}(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$ satisfying

$$C(f) := \sup_{0<\varepsilon<1}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\varepsilon\left|R^{m-1}f(x,y)\right|^{p+\varepsilon}}{|x-y|^{N+mp}}dxdy < \infty.$$

Then $f \in W^{m,p}(\mathbb{R}^N)$ and

$$\frac{1}{(m!)^p m}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N} |D^m f(x)(\sigma,\cdots,\sigma)|^p\, dxd\sigma \le \liminf_{\varepsilon\to 0}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\varepsilon\left|R^{m-1}f(x,y)\right|^{p+\varepsilon}}{|x-y|^{N+mp}}dxdy.$$

Proof. From the assumptions, and set $y = x + t\sigma$ we have

$$\sup_{0<\varepsilon<1}\int_{\mathbb{S}^{N-1}}\int_{B_R}\int_0^1 \frac{\varepsilon\left|R^{m-1}f(x,x+t\sigma)\right|^{p+\varepsilon}}{t^{mp+1}}dtdxd\sigma \le C(f).$$

By Taylor expansion, since $f \in L^p(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$, we can have that

$$\frac{1}{m!}|D^m f(x)(t\sigma,\cdots,t\sigma)| \le \left|R^{m-1}f(x,x+t\sigma)\right| + Ct^{m+1}$$

for all $(\sigma,x,t) \in \mathbb{S}^{N-1}\times B_R\times(0,1)$. Also, since

$$\left|R^{m-1}f(x,x+t\sigma)\right| \le Ct^m \text{ for all } (\sigma,x,t) \in \mathbb{S}^{N-1}\times B_R\times(0,1),$$

we have

$$\begin{aligned}\frac{1}{(m!)^{(p+\varepsilon)}}|D^m f(x)(t\sigma,\cdots,t\sigma)|^{p+\varepsilon} &\le \left[\left|R^{m-1}f(x,x+t\sigma)\right| + Ct^{m+1}\right]^{p+\varepsilon}\\ &\le \left|R^{m-1}f(x,x+t\sigma)\right|^{p+\varepsilon} + Ct^{mp+\varepsilon+1},\end{aligned}$$

for all $(\sigma,x,t) \in \mathbb{S}^{N-1}\times B_R\times(0,1)$.

Now, noting that

$$\liminf_{\varepsilon\to 0}\int_{\mathbb{S}^{N-1}}\int_{B_R}\int_0^1 \frac{\varepsilon t^{mp+\varepsilon+1}}{t^{mp+1}}dtdxd\sigma = 0,$$

we can deduce

$$\liminf_{\varepsilon\to 0}\int_{\mathbb{S}^{N-1}}\int_{B_R}\int_0^1 \frac{1}{(m!)^{p+\varepsilon}}\frac{\varepsilon\left|D^m f(x)(t\sigma,\cdots,t\sigma)\right|^{p+\varepsilon}}{t^{mp+1}}dtdxd\sigma$$

$$\leq \liminf_{\varepsilon\to 0}\int_{\mathbb{S}^{N-1}}\int_{B_R}\int_0^1 \frac{\varepsilon\left|R^{m-1}f(x,x+t\sigma)\right|^{p+\varepsilon}}{t^{mp+1}}dtdxd\sigma.$$

As a consequence, we get

$$\frac{1}{(m!)^p m}\int_{\mathbb{S}^{N-1}}\int_{B_R}\left|D^m f(x)(\sigma,\cdots,\sigma)\right|^p dxd\sigma \leq \liminf_{\varepsilon\to 0}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\varepsilon\left|R^{m-1}f(x,y)\right|^{p+\varepsilon}}{|x-y|^{N+mp}}dxdy.$$

Hence, we can conclude that $f\in W^{m,p}\left(\mathbb{R}^N\right)$ and

$$\frac{1}{(m!)^p m}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}\left|D^m f(x)(\sigma,\cdots,\sigma)\right|^p dxd\sigma \leq \liminf_{\varepsilon\to 0}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\varepsilon\left|R^{m-1}f(x,y)\right|^{p+\varepsilon}}{|x-y|^{N+mp}}dxdy.$$

□

Proof of Theorem 5.2:

(1) and (2) are consequences of Lemma 5.8 and Lemma 5.9.

Now we will prove (3). By (1), we get

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\limits_{|R^{m-1}f(x,y)|>\delta}\frac{\delta^p}{|x-y|^{N+mp}}dxdy \leq C_{N,m,p}\int_{\mathbb{R}^N}\left|\nabla^m f\right|^p dx,\ \forall\delta>0. \tag{5.15}$$

As a consequence, by Fatou's lemma,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\limits_{|R^{m-1}f(x,y)|>1}\frac{1}{|x-y|^{N+mp}}dxdy \leq C_{N,m,p}\int_{\mathbb{R}^N}\left|\nabla^m f\right|^p dx. \tag{5.16}$$

Now, from (5.15), one has

$$\int_0^1\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\limits_{|R^{m-1}f(x,y)|>\delta}\frac{\varepsilon\delta^{p+\varepsilon-1}}{|x-y|^{N+mp}}dxdyd\delta \leq C_{N,m,p}\int_{\mathbb{R}^N}\left|\nabla^m f\right|^p dx.$$

Using Lemma 2.1, we deduce

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\limits_{|R^{m-1}f(x,y)|\leq 1}\frac{\varepsilon\left|R^{m-1}f(x,y)\right|^{p+\varepsilon}}{|x-y|^{N+mp}}dxdy + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\limits_{|R^{m-1}f(x,y)|>1}\frac{\varepsilon}{|x-y|^{N+mp}}dxdy \leq C_{N,m,p}\int_{\mathbb{R}^N}\left|\nabla^m f\right|^p dx. \tag{5.17}$$

From (5.16) and (5.17), we get the assertion (3).

Now, set

$$H(\delta) = \int\limits_{\mathbb{R}^N}\int\limits_{\substack{\mathbb{R}^N \\ |R^{m-1}f(x,y)|>\delta}} \frac{\delta^p}{|x-y|^{N+mp}} dxdy.$$

So be previous results, we have

$$H(\delta) \leq C_{N,m,p} \int\limits_{\mathbb{R}^N} |\nabla^m f|^p \, dx, \ \forall \delta > 0$$

and

$$\lim_{\delta \to 0} H(\delta) = \frac{1}{(m!)^p mp} \int\limits_{\mathbb{S}^{N-1}} \int\limits_{\mathbb{R}^N} |D^m f(x)(\sigma, \cdots, \sigma)|^p \, dxd\sigma.$$

Now, we claim that

$$\lim_{\varepsilon \to 0} \int_0^1 (p+\varepsilon)\,\varepsilon\delta^{\varepsilon-1} H(\delta)\, d\delta = \frac{1}{(m!)^p m} \int\limits_{\mathbb{S}^{N-1}} \int\limits_{\mathbb{R}^N} |D^m f(x)(\sigma, \cdots, \sigma)|^p \, dxd\sigma. \tag{5.18}$$

Indeed, for every $\epsilon > 0$, we can find a number $X(\epsilon) \in (0,1)$ such that

$$\left| H(\delta) - \frac{1}{(m!)^p mp} \int\limits_{\mathbb{S}^{N-1}} \int\limits_{\mathbb{R}^N} |D^m f(x)(\sigma, \cdots, \sigma)|^p \, dxd\sigma \right| < \epsilon \text{ for all } \delta \in (0, X(\epsilon)).$$

Now, we have:

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \left| \int_{X(\epsilon)}^1 (p+\varepsilon)\,\varepsilon\delta^{\varepsilon-1} \left[H(\delta) - \frac{1}{(m!)^p mp} \int\limits_{\mathbb{S}^{N-1}} \int\limits_{\mathbb{R}^N} |D^m f(x)(\sigma, \cdots, \sigma)|^p \, dxd\sigma \right] d\delta \right| \\
&\leq \int_{X(\epsilon)}^1 \lim_{\varepsilon \to 0} (p+\varepsilon)\,\varepsilon\delta^{\varepsilon-1} \left| H(\delta) - \frac{1}{(m!)^p mp} \int\limits_{\mathbb{S}^{N-1}} \int\limits_{\mathbb{R}^N} |D^m f(x)(\sigma, \cdots, \sigma)|^p \, dxd\sigma \right| d\delta \\
&\leq \int_{X(\epsilon)}^1 \lim_{\varepsilon \to 0} (p+\varepsilon)\,\varepsilon X(\epsilon)^{\varepsilon-1} \left[C_{N,p} \int\limits_{\mathbb{R}^N} |\nabla^m f|^p \, dx \right] d\delta \\
&= 0.
\end{aligned}$$

Moreover,

$$\lim_{\varepsilon\to 0}\left|\int_0^{X(\epsilon)}(p+\varepsilon)\,\varepsilon\delta^{\varepsilon-1}\left[H(\delta)-\frac{1}{(m!)^p mp}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}|D^m f(x)(\sigma,\cdots,\sigma)|^p\,dxd\sigma\right]d\delta\right|$$

$$\le \lim_{\varepsilon\to 0}\int_0^{X(\epsilon)}(p+\varepsilon)\,\varepsilon\delta^{\varepsilon-1}\left|H(\delta)-\frac{1}{(m!)^p mp}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}|D^m f(x)(\sigma,\cdots,\sigma)|^p\,dxd\sigma\right|d\delta$$

$$\le \lim_{\varepsilon\to 0}\int_0^{X(\epsilon)}(p+\varepsilon)\,\varepsilon\delta^{\varepsilon-1}\epsilon d\delta$$

$$\le \lim_{\varepsilon\to 0}\int_0^{1}(p+\varepsilon)\,\varepsilon\delta^{\varepsilon-1}\epsilon d\delta$$

$$= p\epsilon.$$

Thus,

$$\lim_{\varepsilon\to 0}\left|\int_0^{1}(p+\varepsilon)\,\varepsilon\delta^{\varepsilon-1}\left[H(\delta)-\frac{1}{(m!)^p mp}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}|D^m f(x)(\sigma,\cdots,\sigma)|^p\,dxd\sigma\right]d\delta\right|\le p\epsilon,\ \forall\epsilon>0.$$

Hence we can get

$$\lim_{\varepsilon\to 0}\int_0^{1}(p+\varepsilon)\,\varepsilon\delta^{\varepsilon-1}H(\delta)\,d\delta=\lim_{\varepsilon\to 0}\int_0^{1}(p+\varepsilon)\,\varepsilon\delta^{\varepsilon-1}\left[\frac{1}{m^{mp+1}p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}|D^m f(x)(\sigma,\cdots,\sigma)|^p\,dxd\sigma\right]d\delta$$

$$=\frac{1}{(m!)^p m}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}|D^m f(x)(\sigma,\cdots,\sigma)|^p\,dxd\sigma.$$

Consequently, we have

$$\lim_{\varepsilon\to 0}\int_0^{1}\underset{|R^{m-1}f(x,y)|>\delta}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}}\frac{(p+\varepsilon)\,\varepsilon\delta^{p+\varepsilon-1}}{|x-y|^{N+2p}}dxdyd\delta=\frac{1}{(m!)^p m}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}|D^m f(x)(\sigma,\cdots,\sigma)|^p\,dxd\sigma.$$

Now, using Lemma 2.1 with $\alpha=p+\varepsilon-1,\ \Phi(x,y)=\left|R^{m-1}f(x,y)\right|,\ \Psi(x,y)=\frac{1}{|x-y|^{N+mp}}$, we obtain

$$\lim_{\varepsilon\to 0}\left[\iint_{|R^{m-1}f(x,y)|\le 1}\frac{\varepsilon\left|R^{m-1}f(x,y)\right|^{p+\varepsilon}}{|x-y|^{N+mp}}dxdy+\iint_{|R^{m-1}f(x,y)|>1}\frac{\varepsilon}{|x-y|^{N+mp}}dxdy\right]$$

$$=\frac{1}{(m!)^p m}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}|D^m f(x)(\sigma,\cdots,\sigma)|^p\,dxd\sigma.$$

Noting that

$$\lim_{\varepsilon\to 0}\iint_{|R^{m-1}f(x,y)|>1}\frac{\varepsilon}{|x-y|^{N+mp}}dxdy=0,$$

we have

$$\lim_{\varepsilon\to 0} \iint\limits_{|R^{m-1}f(x,y)|\le 1} \frac{\varepsilon\left|R^{m-1}f(x,y)\right|^{p+\varepsilon}}{|x-y|^{N+mp}} dxdy = \frac{1}{(m!)^p m} \int\limits_{\mathbb{S}^{N-1}} \int\limits_{\mathbb{R}^N} |D^m f(x)(\sigma,\cdots,\sigma)|^p \, dxd\sigma.$$

We have the statement (4). □

Proof of Theorem 5.4:

First, it is clear that statements (1) $\Longrightarrow$ (2) and (1) $\Longrightarrow$ (3) are consequences of Theorem 5.2.

Now, we will prove (3) $\Longrightarrow$ (1) : Let η_ε be any sequence of smooth mollifiers and set

$$f^\varepsilon = f * \eta_\varepsilon$$

Then we can get $f^\varepsilon \in W^{m-1,p}(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N) \subseteq W^{m,p}(\mathbb{R}^N)$.

Using the (4) of the Theorem 3.1, we can have

$$\begin{aligned}
&C_{N,m,p} \int_{\mathbb{S}^{N-1}} \int_{\mathbb{R}^N} |D^m f^\varepsilon(x)(\sigma,\cdots,\sigma)|^p dxd\sigma \\
&\le \sup_{0<\gamma<1} \underset{|R^{m-1}f^\varepsilon(x,y)|\le 1}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{\gamma|R^{m-1}f^\varepsilon(x,y)|^{p+\gamma}}{|x-y|^{N+mp}} dxdy \\
&= \sup_{0<\gamma<1} \underset{|R^{m-1}f^\varepsilon(x,y)|\le 1}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{\gamma|\int_{\mathbb{R}^N}\{R^{m-1}f(x-z,y-z)\}\eta^\varepsilon(z)dz|^{p+\gamma}}{|x-y|^{N+mp}} dxdy
\end{aligned}$$

Since the function $x^{p+\varepsilon}$ is convex on $[0,\infty)$, by Jensen's inequality, we can deduce

$$\begin{aligned}
&\sup_{0<\gamma<1} \underset{|R^{m-1}f^\varepsilon(x,y)|\le 1}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{\gamma|\int_{\mathbb{R}^N}\{R^{m-1}f(x-z,y-z)\}\eta^\varepsilon(z)dz|^{p+\gamma}}{|x-y|^{N+mp}} dxdy \\
&\le \sup_{0<\gamma<1} \underset{|R^{m-1}f^\varepsilon(x,y)|\le 1}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{\gamma|R^{m-1}f(x-z,y-z)|^{p+\gamma}\int_{\mathbb{R}^N}\eta^\varepsilon(z)dz}{|x-y|^{N+mp}} dxdy \\
&= \sup_{0<\gamma<1} \underset{|\int_{\mathbb{R}^N}\{R^{m-1}f(x,y)\}\eta^\varepsilon(z)dz|\le 1}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{\gamma|R^{m-1}f(x,y)|^{p+\gamma}}{|x-y|^{N+mp}} dxdy
\end{aligned}$$

The last step comes from changing variable and $\int_{\mathbb{R}^N}\eta^\varepsilon(z)dz = 1$. Since

$$|\int_{\mathbb{R}^N}\{R^{m-1}f(x,y)\}\eta^\varepsilon(z)dz| = |R^{m-1}f(x,y)|$$

Thus,

$$
\begin{aligned}
& C_{N,p} \int_{\mathbb{S}^{N-1}} \int_{\mathbb{R}^N} |D^m f^\varepsilon(x)(\sigma, \cdots, \sigma)|^p dx d\sigma \\
& \leq \sup_{0<\gamma<1} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\gamma |R^{m-1} f(x,y)|^{p+\gamma}}{|x-y|^{N+mp}} dx dy \\
& \qquad\qquad \scriptstyle{|\int_{\mathbb{R}^N} \{R^{m-1} f(x,y) \eta^\varepsilon(z) dz| \leq 1} \\
& = \sup_{0<\gamma<1} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\gamma |R^{m-1} f(x,y)|^{p+\gamma}}{|x-y|^{N+mp}} dx dy \\
& \qquad\qquad \scriptstyle{|R^{m-1} f(x,y)| \leq 1} \\
& < \infty \text{ from (3)}
\end{aligned}
$$

So $\|f^\varepsilon\|_{W^{m,p}(\mathbb{R}^N)}$ is bounded. Then there exists a subsequence of f^ε, denoted as f^ε, such that

$$f^\varepsilon \rightharpoonup g \text{ in } W^{m,p}(\mathbb{R}^N).$$

On the other hand, from the property of mollifiers, we know

$$f^\varepsilon \to f \text{ a.e.}$$

From the uniqueness of the limit, we get $f = g$ a.e. in $W^{m,p}(\mathbb{R}^N)$. That is , $f \in W^{m,p}(\mathbb{R}^N)$. □

CHAPTER 6 L^P-DIFFERENTIABILITY OF THE FUNCTIONS IN SOBOLEV SPACE ON HEISENBERG GROUPS

6.1 main results

Throughout this chapter, Ω will denote an open set of $\mathbb{H} = (\mathbb{C}^N \times \mathbb{R}, \circ, \delta_\lambda)$ whose points we denote by (z,t) with $z \in \mathbb{R}$ and $z = (z_1, ..., z_N) \in \mathbb{C}^N$.

The main theorems we want to study in this chapter are

Theorem 6.1. Let$1 \leq p < \infty$ and $f \in W^{1,p}(\mathbb{H})$. Then

$$\lim_{\varepsilon \to 0} \int_{\mathbb{H}} (\fint_{B(\mathbf{e},\varepsilon)} \frac{|f(x\circ h) - f(x) - \langle \nabla_{\mathbb{H}} f(x), h' \rangle|^{pq}}{|h|^{pq}} dh)^{\frac{1}{q}} = 0$$

where $1 \leq q \leq \frac{Q}{Q-p}$ if $1 \leq p < Q$, $1 \leq q < \infty$ if $p = Q$, and $1 \leq q < \infty$ if $p > Q$.Here, $e = (0, ..., 0) \in \mathbb{H}$ and $h = (h', h_{2N+1}) \in \mathbb{H}$,$\langle \cdot, \cdot \rangle$is the inner product on $\mathbb{R}^{2N}$.

Theorem 6.2. Let$1 < p < \infty$ and suppose $f \in L^p(\mathbb{H})$. Then the following are equivalent:

(1) $f \in W^{1,p}(\mathbb{H})$.

(2)there exists a $v \in L^p(\mathbb{H}; \mathbb{R}^{2N})$ such that

$$\lim_{\varepsilon \to 0} \int_{\mathbb{H}} (\fint_{B(\mathbf{e},\varepsilon)} \frac{|f(x\circ h) - f(x) - \langle v(x), h' \rangle|^{pq}}{|h|^{pq}} dh)^{\frac{1}{q}} = 0$$

where $1 \leq q \leq \frac{Q}{Q-p}$ if $1 \leq p < Q$, $1 \leq q < \infty$ if $p = Q$, and $1 \leq q < \infty$ if $p > Q$.Here, $e = (0, ..., 0) \in \mathbb{H}$ and $h = (h', h_{2N+1}) \in \mathbb{H}$.

6.2 Some Useful Lemmas

In order to prove the above two theorems, we will study the following useful lemma:

Lemma 6.3. Suppose $f \in W_0^{1,p}(\mathbb{H})$ for some $1 \leq p < \infty$, and that $1 \leq q \leq \frac{Q}{Q-p}$ if $1 \leq p < Q$, $1 \leq q < \infty$ if $p = Q$ and $1 \leq q \leq \infty$ if $p > Q$. Then there exsits a positive $C = C(p, q, Q)$ such that for all $0 < r < 1$

$$\frac{1}{r^{Q+pq}}\int_{B(\mathbf{e},r)}|f(x\circ h)-f(x)-\langle\nabla_{\mathbb{H}}f(x),h'\rangle|^{pq}dh$$
$$\leq C(\fint_{B(\mathbf{e},r)}|\nabla_{\mathbb{H}}f(x\circ h)-\nabla_{\mathbb{H}}f(x)|^{p})^{q}$$
$$+C(\int_0^1\fint_{B(\mathbf{e},sr)}|\nabla_{\mathbb{H}}f(x\circ z)-\nabla_{\mathbb{H}}f(x)|^{p}dzds)^{q}$$

Proof. Since $f\in W_0^{1,p}(\mathbb{H})$, then we can first assume that $f\in C_0^1(\mathbb{H})$. Then we have for a.e. $x\in\mathbb{H}$,

$$1)\lim_{r\to 0}\fint_{B(x,r)}|f(y)-f(x)|^{p}dy=0$$
$$2)\lim_{r\to 0}\fint_{B(x,r)}|\nabla_{\mathbb{H}}f(y)-\nabla_{\mathbb{H}}f(x)|^{p}dy=0$$

Fix such a point x. Select $\varphi\in C_0^1(B(\mathbf{e},r))$ with $\|\varphi\|_{L^{p'}(B(\mathbf{e},r))}$with $1/p+1/p'=1$,$1\leq p<\infty$.We calculate

$$\fint_{B(\mathbf{e},r)}\varphi(h)(f(x\circ h)-f(x)-\langle\nabla_{\mathbb{H}}f(x),h'\rangle)dh$$
$$=\fint_{B(\mathbf{e},r)}\varphi(h)(\int_0^1\langle\nabla_{\mathbb{H}}f(x\circ\delta_s(h)),h'\rangle ds-\langle\nabla_{\mathbb{H}}f(x),h'\rangle)dh$$
$$=\fint_{B(\mathbf{e},r)}\varphi(h)(\int_0^1\langle\nabla_{\mathbb{H}}f(x\circ\delta_s(h))-\nabla_{\mathbb{H}}f(x),h'\rangle ds)dh$$
$$=\int_0^1\fint_{B(\mathbf{e},r)}\varphi(h)\langle\nabla_{\mathbb{H}}f(x\circ\delta_s(h))-\nabla_{\mathbb{H}}f(x),h'\rangle dhds$$
$$=\int_0^1\frac{1}{s}\fint_{B(\mathbf{e},sr)}\varphi(\delta_{s^{-1}}(z))\langle\nabla_{\mathbb{H}}f(x\circ z)-\nabla_{\mathbb{H}}f(x),z'\rangle dzds$$

The last step above comes from changing variable:$z=\delta_s(h)$,and we can get $z\in B(\mathbf{e},sr)$, $sh'=z'$,

$dz = s^Q dh$ and $|B(\mathbf{e}, sr)| = s^Q |B(\mathbf{e}, r)|$. Using hölder's inequality, we have

$$\begin{aligned}
&\fint_{B(\mathbf{e},r)} \varphi(h)(f(x\circ h) - f(x) - \langle \nabla_{\mathbb{H}} f(x), h'\rangle)dh \\
&= \int_0^1 \frac{1}{s} \fint_{B(\mathbf{e},sr)} \varphi(\delta_{s^{-1}}(z))\langle \nabla_{\mathbb{H}} f(x\circ z) - \nabla_{\mathbb{H}} f(x), z'\rangle dz ds \\
&\le \int_0^1 \frac{1}{s} \fint_{B(\mathbf{e},sr)} |\varphi(\delta_{s^{-1}}(z))||\nabla_{\mathbb{H}} f(x\circ z) - \nabla_{\mathbb{H}} f(x)||z| dz ds \\
&\le r \int_0^1 (\fint_{B(\mathbf{e},sr)} |\varphi(\delta_{s^{-1}}(z))|^{p'} dz)^{\frac{1}{p'}} (\fint_{B(\mathbf{e},sr)} |\nabla_{\mathbb{H}} f(x\circ z) - \nabla_{\mathbb{H}} f(x)|^p dz)^{\frac{1}{p}} ds
\end{aligned}$$

Since

$$\fint_{B(\mathbf{e},sr)} |\varphi(\delta_{s^{-1}}(z))|^{p'} dz = \fint_{B(\mathbf{e},r)} |\varphi(y)|^{p'} dy$$

we obtain

$$\begin{gathered}
\fint_{B(\mathbf{e},r)} \varphi(h)(f(x\circ h) - f(x) - \langle \nabla_{\mathbb{H}} f(x), h'\rangle)dh \le \\
r(\fint_{B(\mathbf{e},r)} |\varphi(y)|^{p'} dy)^{\frac{1}{p'}} \int_0^1 \fint_{B(\mathbf{e},sr)} |\nabla_{\mathbb{H}} f(x\circ z) - \nabla_{\mathbb{H}} f(x)|^p dz)^{\frac{1}{p}} ds
\end{gathered}$$

Taking the supremum over all φ as above gives there exists a φ such that

$$\begin{aligned}
&\fint_{B(\mathbf{e},r)} \varphi(h)(f(x\circ h) - f(x) - \langle \nabla_{\mathbb{H}} f(x), h'\rangle)dh \\
&= (\fint_{B(\mathbf{e},r)} |f(x\circ h) - f(x) - \langle \nabla_{\mathbb{H}} f(x), h'\rangle|^p dh)^{\frac{1}{p}} (\fint_{B(\mathbf{e},r)} |\varphi(y)|^{p'} dy)^{\frac{1}{p'}}.
\end{aligned}$$

Then we have

$$\begin{gathered}
(\fint_{B(\mathbf{e},r)} |f(x\circ h) - f(x) - \langle \nabla_{\mathbb{H}} f(x), h'\rangle|^p dh)^{\frac{1}{p}} (\fint_{B(\mathbf{e},r)} |\varphi(y)|^{p'} dy)^{\frac{1}{p'}} \le \\
r(\fint_{B(\mathbf{e},r)} |\varphi(y)|^{p'} dy)^{\frac{1}{p'}} \int_0^1 \fint_{B(\mathbf{e},sr)} |\nabla_{\mathbb{H}} f(x\circ z) - \nabla_{\mathbb{H}} f(x)|^p dz)^{\frac{1}{p}} ds.
\end{gathered}$$

That is, for any $1 \le p < \infty$, we have

$$(\fint_{B(\mathbf{e},r)} |f(x\circ h) - f(x) - \langle \nabla_{\mathbb{H}} f(x), h'\rangle|^p dh)^{\frac{1}{p}} \le r \int_0^1 \fint_{B(\mathbf{e},sr)} |\nabla_{\mathbb{H}} f(x\circ z) - \nabla_{\mathbb{H}} f(x)|^p dz)^{\frac{1}{p}} ds.$$

Firstly, we consider the case $1 \le q \le \frac{Q}{Q-p}$ and $1 \le p < Q$.

Now we have the claim: There exists a constant $C = C(Q, p)$ such that

$$(\fint_{B(x,r)} |g|^{\frac{Qp}{Q-p}} dy)^{\frac{Q-p}{Qp}} \leq Cr(\fint_{B(\mathbf{e},r)} |\nabla_{\mathbb{H}} g|^p dh)^{\frac{1}{p}} + C(\fint_{B(\mathbf{e},r)} |g|^p dh)^{\frac{1}{p}}$$

for all $g \in W^{1,p}(\mathbb{H})$.

Apply the claim to $g(h) = f(x \circ h) - f(x) - \langle \nabla_{\mathbb{H}} f(x), h' \rangle$,we can get

$$\begin{aligned}
&(\fint_{B(\mathbf{e},r)} |f(x \circ h) - f(x) - \langle \nabla_{\mathbb{H}} f(x), h' \rangle|^{\frac{Qp}{Q-p}} dh)^{\frac{Q-p}{Qp}} \\
&\leq Cr(\fint_{B(\mathbf{e},r)} |\nabla_{\mathbb{H}} f(x \circ z) - \nabla_{\mathbb{H}} f(x)|^p dz)^{1/p} \\
&+ C(\fint_{B(\mathbf{e},r)} |f(x \circ h) - f(x) - \langle \nabla_{\mathbb{H}} f(x), h' \rangle|^p dh)^{\frac{1}{p}}
\end{aligned}$$

Now we prove the claim by using Poincare's inequality and embedding theorem: Since we have known

$$\|g\|_{L^{pq}(B(\mathbf{e},r))} \leq C\|g\|_{L^{p^*}(B(\mathbf{e},r))} \text{ with } 1 \leq p \leq \tfrac{Q}{Q-p} \text{ and } p^* = \tfrac{Qp}{Q-p}$$

and

$$(\fint_{B(\mathbf{e},r)} |g - g_{B(\mathbf{e},r)dh}|^{p^*})^{\frac{1}{p^*}} \leq Cr(\fint_{B(\mathbf{e},r)} |\nabla_{\mathbb{H}} g|^p dh)^{\frac{1}{p}} \text{ with } g_{B(\mathbf{e},r)} = \fint_{B(\mathbf{e},r)} g dh$$

Then we have

$$\begin{aligned}
&(\fint_{B(\mathbf{e},r)} |g|^{p^*} dh)^{\frac{1}{p^*}} \\
&\leq (\fint_{B(\mathbf{e},r)} (|g - g_{B(\mathbf{e},r)dh}| + |g_{B(\mathbf{e},r)}|)^{p^*} dh)^{\frac{1}{p^*}} \\
&\leq C(\fint_{B(\mathbf{e},r)} |g - g_{B(\mathbf{e},r)}|^{p^*} + |g_{B(\mathbf{e},r)}|^{p^*} dh)^{\frac{1}{p^*}} \\
&= C(\fint_{B(\mathbf{e},r)} |g - g_{B(\mathbf{e},r)dh}|^{p^*})^{\frac{1}{p^*}} + |\fint_{B(\mathbf{e},r)} g dh| \\
&\leq Cr(\fint_{B(\mathbf{e},r)} |\nabla_{\mathbb{H}} g|^p dh)^{\frac{1}{p}} + C(\fint_{B(\mathbf{e},r)} |g|^p dh)^{\frac{1}{p}}
\end{aligned}$$

which completes the claim.

Finally, we have

$$
\begin{aligned}
&(\fint_{B(\mathbf{e},r)} |f(x\circ h)-f(x)-\langle \nabla_{\mathbb{H}}f(x),h'\rangle|^{\frac{Qp}{Q-p}}dh)^{\frac{Q-p}{Qp}}\\
&\le Cr(\fint_{B(\mathbf{e},r)} |\nabla_{\mathbb{H}}f(x\circ z)-\nabla_{\mathbb{H}}f(x)|^p dz)^{1/p}\\
&+C(\fint_{B(\mathbf{e},r)} |f(x\circ h)-f(x)-\langle \nabla_{\mathbb{H}}f(x),h'\rangle|^{p}dh)^{\frac{1}{p}}\\
&\le Cr(\fint_{B(\mathbf{e},r)} |\nabla_{\mathbb{H}}f(x\circ z)-\nabla_{\mathbb{H}}f(x)|^p dz)^{1/p}\\
&+Cr\int_0^1 \fint_{B(\mathbf{e},sr)} |\nabla_{\mathbb{H}}f(x\circ z)-\nabla_{\mathbb{H}}f(x)|^p dz)^{\frac{1}{p}}ds\\
&\le Cr(\fint_{B(\mathbf{e},r)} |\nabla_{\mathbb{H}}f(x\circ z)-\nabla_{\mathbb{H}}f(x)|^p dz)^{1/p}\\
&+Cr(\int_0^1 \fint_{B(\mathbf{e},sr)} |\nabla_{\mathbb{H}}f(x\circ z)-\nabla_{\mathbb{H}}f(x)|^p dzds)^{\frac{1}{p}}
\end{aligned}
$$

Since we have

$$\|g\|_{L^{pq}(B(\mathbf{e},r))}\le C\|g\|_{L^{p^*}(B(\mathbf{e},r))} \text{ with } 1\le p\le \tfrac{Q}{Q-p} \text{ and } p^*=\tfrac{Qp}{Q-p}$$

So

$$
\begin{aligned}
&(\fint_{B(\mathbf{e},r)} |f(x\circ h)-f(x)-\langle \nabla_{\mathbb{H}}f(x),h'\rangle|^{pq}dh)^{\frac{1}{pq}}\\
&\le C(\fint_{B(\mathbf{e},r)} |f(x\circ h)-f(x)-\langle \nabla_{\mathbb{H}}f(x),h'\rangle|^{\frac{Qp}{Q-p}}dh)^{\frac{Q-p}{Qp}}\\
&\le Cr(\fint_{B(\mathbf{e},r)} |\nabla_{\mathbb{H}}f(x\circ z)-\nabla_{\mathbb{H}}f(x)|^p dz)^{1/p}\\
&+Cr(\int_0^1 \fint_{B(\mathbf{e},sr)} |\nabla_{\mathbb{H}}f(x\circ z)-\nabla_{\mathbb{H}}f(x)|^p dzds)^{\frac{1}{p}}
\end{aligned}
$$

Take the powerpq to both sides of the inequality above

$$
\begin{aligned}
&\fint_{B(\mathbf{e},r)} |f(x\circ h)-f(x)-\langle \nabla_{\mathbb{H}}f(x),h'\rangle|^{pq}dh\\
&\le Cr^{pq}(\fint_{B(\mathbf{e},r)} |\nabla_{\mathbb{H}}f(x\circ z)-\nabla_{\mathbb{H}}f(x)|^p dz)^{q}\\
&+Cr^{pq}(\int_0^1 \fint_{B(\mathbf{e},sr)} |\nabla_{\mathbb{H}}f(x\circ z)-\nabla_{\mathbb{H}}f(x)|^p dzds)^{q}
\end{aligned}
$$

Divide r^{pq} to both sides of the above inequality and rewrite the left side, than we are done for the first case.

Secondly, we consider the case$1 \le q < \infty$ and $p = Q$.

For any $Q \le pq < \infty$, we can find a $\tilde{p} = Q - \varepsilon$with a small ε such that $\tilde{p}^* = pq$, then use the similar method for the first case, we can get the same conclusion.

Finally, we consider the case$1 \le q \le \infty$ and $p > Q$ by using the Morrey's estimate [29]

Since for a.e $x \in \mathbb{H}$, we have

$$\lim_{r \to 0} \fint_{B(x,r)} |\nabla_{\mathbb{H}} f(y) - \nabla_{\mathbb{H}} f(x)|^p dy = 0$$

Choose such a point x, and write

$$g(h) = f(x \circ h) - f(x) - \langle \nabla_{\mathbb{H}} f(x), h' \rangle$$

for $h \in B(\mathbf{e}, r)$.

Employing Morrey's estimate, we deduce

$$|g(h) - g(\mathbf{e})| \le Cr(\fint_{B(\mathbf{e},r)} |\nabla_{\mathbb{H}} g(h)|^p dh)^{1/p}$$

Since $g(\mathbf{e}) = 0$, and $\nabla_{\mathbb{H}} g(h) = \nabla_{\mathbb{H}} f(x \circ h) - \nabla_{\mathbb{H}} f(x)$, we have

$$\begin{aligned}
&\fint_{B(\mathbf{e},r)} |f(x \circ h) - f(x) - \langle \nabla_{\mathbb{H}} f(x), h' \rangle|^{pq} dh \\
&= \fint_{B(\mathbf{e},r)} |g(h)|^{pq} dh \\
&\le C \fint_{B(\mathbf{e},r)} r^{pq} (\fint_{B(\mathbf{e},r)} |\nabla_{\mathbb{H}} f(x \circ z) - \nabla_{\mathbb{H}} f(x)|^p dz)^q dh \\
&= Cr^{pq} \fint_{B(\mathbf{e},r)} |\nabla_{\mathbb{H}} f(x \circ z) - \nabla_{\mathbb{H}} f(x)|^p dz)^q
\end{aligned}$$

Thus

$$\begin{aligned}
&\frac{1}{r^{Q+pq}} \int_{B(\mathbf{e},r)} |f(x \circ h) - f(x) - \langle \nabla_{\mathbb{H}} f(x), h' \rangle|^{pq} dh \\
&\le C(\fint_{B(\mathbf{e},r)} |\nabla_{\mathbb{H}} f(x \circ h) - \nabla_{\mathbb{H}} f(x)|^p)^q
\end{aligned}$$

Then we are done with all three cases. □

Lemma 6.4. Suppose $\rho_z \in L^1(\mathbb{H})$ and $1 \le p < \infty$. Then the operator $\mathfrak{G}_\rho : W^{1,p}(\mathbb{H}) \to L^p(\mathbb{H}; \mathbb{R}^{2N})$

is a bounded operator with the estimate

$$||\mathfrak{G}_\rho f||_{L^p} \le C||\rho||_{L^1}||\nabla_{\mathbb{H}} f||_{L^p}$$

for all $f \in W^{1,p}(\mathbb{H})$. Similarly $\mathfrak{G}_\rho : BV(\mathbb{H}) \to L^1(\mathbb{H};\mathbb{R}^{2N})$ with the estimate

$$||\mathfrak{G}_\rho f||_{L^p} \le C||\rho||_{L^1}||D_H f||_{L^p}$$

for all $f \in BV(\mathbb{H})$. Here, $C = C(p,Q) > 0$, $\mathfrak{G}_\rho f(x) = Q\int_{\mathbb{H}} \frac{f(x)-f(y)}{|x^{-1}\circ y|}\frac{(x^{-1}\circ y)'}{|x^{-1}\circ y|}\rho_i(x^{-1}\circ y)dy$.

Proof. Applying Hölder's inequality we have

$$\begin{aligned}
&\int_{\mathbb{H}}(\int_{\mathbb{H}} \frac{|f(x)-f(y)|}{|x^{-1}\circ y|}\rho(x^{-1}\circ y)dy)^p dx \\
&\le \int_{\mathbb{H}}[\int_{\mathbb{H}} \frac{|f(x)-f(y)|^p}{|x^{-1}\circ y|^p}\rho(x^{-1}\circ y)dy][\int_{\mathbb{H}}\rho(x^{-1}\circ y)dy]^{p-1}dx \\
&||\rho||_{L^1}^{p-1}\int_{\mathbb{H}}\int_{\mathbb{H}} \frac{|f(x)-f(y)|^p}{|x^{-1}\circ y|^p}\rho(x^{-1}\circ y)dydx
\end{aligned}$$

Since we have

$$\int_{\mathbb{H}}\int_{\mathbb{H}} \frac{|f(x)-f(y)|^p}{|x^{-1}\circ y|^p}\rho(x^{-1}\circ y)dydx \le C(p,Q)||\rho||_{L^1}||\nabla_H f||_{L^p}^p$$

The we obtain

$$\int_{\mathbb{H}}|\mathfrak{G}_\rho f(x)|^p dx \le Q^p\int_{\mathbb{H}}(\int_{\mathbb{H}} \frac{|f(x)-f(y)|}{|x^{-1}\circ y|}\rho(x^{-1}\circ y)dy)^p dx \le Q^p C(p,Q)||\rho||_{L^1}||\nabla_H f||_{L^p}^p.$$

The subsequent statement for BV functions follows from the density of $C^\infty(\mathbb{H})\cap W^{1,1}(\mathbb{H})$ in BV with respect to the strict convergence. □

6.3 Proof of main results

Proof of Theorem 6.1:

First, we expand the integrand on concentric rings with $0 < \varepsilon < 1$

$$⨍_{B(\mathbf{e},\varepsilon)} \frac{|f(x\circ h)-f(x)-\langle\nabla_{\mathbb{H}} f(x), h'\rangle|^{pq}}{|h|^{pq}}dh = \sum_{k=0}^{\infty}\frac{1}{\varepsilon^Q|B(\mathbf{e},1)|}\int_{B(\mathbf{e},\frac{\varepsilon)}{2^k}\setminus B(\mathbf{e},\frac{\varepsilon}{2^{k+1}})}\frac{|f(x\circ h)-f(x)-\langle\nabla_{\mathbb{H}} f(x), h'\rangle|^{pq}}{|h|^{pq}}d$$

Fix $k \in \mathbb{N}$, we make estimates

$$\frac{1}{\varepsilon^Q|B(\mathbf{e},1)|}\int_{B(\mathbf{e},\frac{\varepsilon)}{2^k})\setminus B(\mathbf{e},\frac{\varepsilon}{2^{k+1}})}\frac{|f(x\circ h)-f(x)-\langle\nabla_{\mathbb{H}}f(x),h'\rangle|^{pq}}{|h|^{pq}}dh$$
$$\leq \frac{1}{\varepsilon^Q|B(\mathbf{e},1)|}(\frac{\varepsilon}{2^{k+1}})^{-pq}\int_{B(\mathbf{e},\frac{\varepsilon)}{2^k})\setminus B(\mathbf{e},\frac{\varepsilon}{2^{k+1}})}|f(x\circ h)-f(x)-\langle\nabla_{\mathbb{H}}f(x),h'\rangle|^{pq}dh$$
$$\leq \frac{2^{pq}}{|B(\mathbf{e},1)|}\frac{1}{2^{kQ}}(\frac{\varepsilon}{2^k})^{-Q-pq}\int_{B(\mathbf{e},\frac{\varepsilon)}{2^k})}|f(x\circ h)-f(x)-\langle\nabla_{\mathbb{H}}f(x),h'\rangle|^{pq}dh$$

Applying the lemma we already proved, we have

$$(\frac{\varepsilon}{2^k})^{-Q-pq}\int_{B(\mathbf{e},\frac{\varepsilon)}{2^k})}|f(x\circ h)-f(x)-\langle\nabla_{\mathbb{H}}f(x),h'\rangle|^{pq}dh$$
$$\leq C(\fint_{B(\mathbf{e},\frac{\varepsilon)}{2^k})}|\nabla_{\mathbb{H}}f(x\circ h)-\nabla_{\mathbb{H}}f(x)|^p dh)^q$$
$$+C(\int_0^1\fint_{B(\mathbf{e},\frac{s\varepsilon)}{2^k})}|\nabla_{\mathbb{H}}f(x\circ z)-\nabla_{\mathbb{H}}f(x)|^p dzds)^q$$

Therefore, summing in k and applying the basic inequality $(\sum_k |a_k|)^{\frac{1}{q}} \leq \sum_k |a_k|^{\frac{1}{q}}$.

we have

$$(\fint_{B(\mathbf{e},\varepsilon)}\frac{|f(x\circ h)-f(x)-\langle\nabla_{\mathbb{H}}f(x),h'\rangle|^{pq}}{|h|^{pq}}dh)^{\frac{1}{q}}$$
$$\leq C\sum_{k=0}^{\infty}(\frac{1}{2^k})^{\frac{Q}{q}}\fint_{B(\mathbf{e},\frac{\varepsilon)}{2^k})}|\nabla_{\mathbb{H}}f(x\circ h)-\nabla_{\mathbb{H}}f(x)|^p dh$$
$$+C\sum_{k=0}^{\infty}(\frac{1}{2^k})^{\frac{Q}{q}}\int_0^1\fint_{B(\mathbf{e},\frac{s\varepsilon)}{2^k})}|\nabla_{\mathbb{H}}f(x\circ z)-\nabla_{\mathbb{H}}f(x)|^p dzds$$

Intergrating the preceding inequality over $x \in \mathbb{H}$ and making use of Tonelli's theorem we obtain

$$\int_{\mathbb{H}}(\fint_{B(\mathbf{e},\varepsilon)}\frac{|f(x\circ h)-f(x)-\langle\nabla_{\mathbb{H}}f(x),h'\rangle|^{pq}}{|h|^{pq}}dh)^{\frac{1}{q}}dx$$
$$\leq C\sum_{k=0}^{\infty}(\frac{1}{2^k})^{\frac{Q}{q}}\fint_{B(\mathbf{e},\frac{\varepsilon)}{2^k})}\int_{\mathbb{H}}|\nabla_{\mathbb{H}}f(x\circ h)-\nabla_{\mathbb{H}}f(x)|^p dxdh$$
$$+C\sum_{k=0}^{\infty}(\frac{1}{2^k})^{\frac{Q}{q}}\int_0^1\fint_{B(\mathbf{e},\frac{s\varepsilon)}{2^k})}\int_{\mathbb{H}}|\nabla_{\mathbb{H}}f(x\circ z)-\nabla_{\mathbb{H}}f(x)|^p dxdzds$$

However, since $h \in B(\mathbf{e},\frac{\varepsilon)}{2^k})$ and $z \in B(\mathbf{e},\frac{s\varepsilon)}{2^k})$ with $0 < s < 1$, we have

$$max\{\int_{\mathbb{H}}|\nabla_{\mathbb{H}}f(x\circ h)-\nabla_{\mathbb{H}}f(x)|^p dx,\int_{\mathbb{H}}|\nabla_{\mathbb{H}}f(x\circ z)-\nabla_{\mathbb{H}}f(x)|^p dx\} \leq$$

$$\sup_{\eta\in B(\mathbf{e},\varepsilon)} \int_{\mathbb{H}} |\nabla_{\mathbb{H}} f(x\circ\eta) - \nabla_{\mathbb{H}} f(x)|^p dx$$

and this bound above is independent of k. Thus,

$$\int_{\mathbb{H}} (\fint_{B(\mathbf{e},\varepsilon)} \frac{|f(x\circ h) - f(x) - \langle \nabla_{\mathbb{H}} f(x), h'\rangle|^{pq}}{|h|^{pq}} dh)^{\frac{1}{q}} dx$$
$$\leq \sup_{\eta\in B(\mathbf{e},\varepsilon)} \int_{\mathbb{H}} |\nabla_{\mathbb{H}} f(x\circ\eta) - \nabla_{\mathbb{H}} f(x)|^p dx (C\sum_{k=0}^{\infty} (\frac{1}{2^k})^{\frac{Q}{q}}).$$

As the infinite series is summable, the result follows from sending $\varepsilon \to 0$ and using the continuity of invariant in $L^p(\mathbb{H})$. □

Proof of Theorem 6.2:

As we have shown that $f \in W^{1,p}(\mathbb{H})$ implies the L^p-convergence in Theorem 6.1, it remains to show the converse. We first treat the case $1 < p < \infty$. Let us therefore suppose that there exists $v \in L^p(\mathbb{H};\mathbb{R}^{2n})$ such that

$$\lim_{\varepsilon\to 0} \int_{\mathbb{H}} (\fint_{B(\mathbf{e},\varepsilon)} \frac{|f(x\circ h) - f(x) - \langle v(x), h'\rangle|^{pq}}{|h|^{pq}} dh)^{\frac{1}{q}} = 0.$$

We then estimate

$$\int_{\mathbb{H}} \fint_{B(\mathbf{e},\varepsilon)} \frac{|f(x\circ h) - f(x)|^p}{|h|^p} dhdx$$
$$\leq 2^{p-1} \int_{\mathbb{H}} \fint_{B(\mathbf{e},\varepsilon)} \frac{|f(x\circ h) - f(x) - \langle v(x), h'\rangle|^p}{|h|^p} dhdx$$
$$+ 2^{p-1} \int_{\mathbb{H}} \fint_{B(\mathbf{e},\varepsilon)} |\frac{\langle v(x), h'\rangle}{|h|}|^p dhdx$$
$$\leq 2^{p-1} \int_{\mathbb{H}} \fint_{B(\mathbf{e},\varepsilon)} \frac{|f(x\circ h) - f(x) - \langle v(x), h'\rangle|^p}{|h|^p} dhdx$$
$$+ 2^{p-1} \int_{\mathbb{H}} \fint_{B(\mathbf{e},\varepsilon)} |v(x)|^p dhdx$$

Now our assumption(we can take $q = 1$) is that the first term on the right hand side tends to zero as $\varepsilon \to 0$, while the second is bounded by a constant times the L^p norm of v. Take a sequence ε_n such that $\varepsilon_n \to 0$ as $n \to \infty$ and we have

$$\underset{n\to 0}{limsup} \int_{\mathbb{H}} \fint_{B(\mathbf{e},\varepsilon_n)} \frac{|f(x\circ h) - f(x)|^p}{|h|^p} dhdx < \infty$$

This left hand side can be rewrite to be

$$\int_{\mathbb{H}} \fint_{B(\mathbf{e},\varepsilon_n)} \frac{|f(x\circ h)-f(x)|^p}{|h|^p} dhdx = \int_{\mathbb{H}} \int_{\mathbb{H}} \frac{|f(x\circ h)-f(x)|^p}{|h|^p} \rho_n(h) dhdx$$

with $\rho_n(h) = \frac{\chi_{B(\mathbf{e},\varepsilon_n)}(h)}{|B(\mathbf{e},\varepsilon_n)|}$. We can see $\rho_n(h) : \mathbb{H} \to \mathbb{R}^+ \cup \{0\}$ is a family of radial mollifiers with respect to the norm $|\cdot|_{\mathbb{H}}$ satisfies the following properties:

$$i) \int_{\mathbb{H}} \rho_n(h) dh = 1, \forall n > 0;$$

$$ii) \int_{\mathbb{H}\setminus B(\mathbf{e},\delta)} \rho_n(h) dh \to 0 \text{ as } n \to \infty, \forall \delta > 0.$$

Now we prove that if $f \in L^p(\mathbb{H})$ and satisfies the inequality

$$\underset{n\to 0}{limsup} \int_{\mathbb{H}} \int_{\mathbb{H}} \frac{|f(x\circ h)-f(x)|^p}{|h|^p} \rho_n(h) dhdx < \infty,$$

then $f \in W^{1,p}(\mathbb{H})$.

We first assume $f \in C^1_{\mathbb{H}}(\mathbb{H})$ then extends to any $f \in L^p(\mathbb{H})$ by density [18]. In order to prove this, we first prove a claim:

let K be a compact set, B is the unit ball with center $\mathbf{e} = (0, ...0) \in \mathbb{H}$, and $K_n = \int_B \frac{|\langle x', h'\rangle|^p}{|h|^p} \rho_n(h) dh$ with x' a unit vector of $\mathbb{R}^{2N}$, then

$$\int_K \int_B \frac{|\langle \nabla_{\mathbb{H}} f(x), h'\rangle|^p}{|h|^p} \rho_n(h) dhdx = K_n \int_K |\nabla_{\mathbb{H}} f(x)|^p dx.$$

and K_n does not depend on x'.

To see this we use a lemma in [[15], Proposition 1.13]:

$$K_n = \int_{\mathbb{H}} (\frac{|\langle x', h'\rangle|^p}{|h|^{p+Q}}) \rho_n(h) |h|^Q \chi_{[0,1]}(|h|) dh$$

$$= M(\frac{|\langle x', h'\rangle|^p}{|h|^{p+Q}}) \int_0^1 \rho_n(r) r^{Q-1} dr.$$

On the other hand,

$$\int_B |\langle x', h'\rangle|^p dh = \int_{\mathbb{H}} (\frac{|\langle x', h'\rangle|^p}{|h|^{p+Q}})|h|^{p+Q}\chi_{[0,1]}(|h|)dh$$
$$= M(\frac{|\langle x', h'\rangle|^p}{|h|^{p+Q}})\int_0^1 r^{p+Q-1} = \frac{1}{p+Q}M(\frac{|\langle x', h'\rangle|^p}{|h|^{p+Q}})$$

So that

$$M(\frac{|\langle x', h'\rangle|^p}{|h|^{p+Q}}) = (p+Q)\int_B |\langle x', h'\rangle|^p dh$$

This expression does not depend on x', since by an orthogonal $\mathbb{H}$-change basis which does not alter the measure nor the homogenous norm, it is possible to choose any other unitary vector of $\mathbb{R}^{2N}$ by rotation: set $A \in O(2N, \mathbb{R})$ and call $y' = A^T x' \in \mathbb{R}^{2N}$, then

$$\int_B |\langle x', h'\rangle|^p dh = \int_B |\langle x', Ah'\rangle|^p dh = \int_B |\langle y', h'\rangle|^p dh$$

Finally, we can see

$$K_n = (p+Q)\int_B |\langle x', h'\rangle|^p dh \int_0^1 \rho_n(r) r^{Q-1} dr.$$

does not depend on x'.

To prove the claim, since

$$\int_K \int_B \frac{|\langle \nabla_{\mathbb{H}} f(x), h'\rangle|^p}{|h|^p}\rho_n(h)dhdx = \int_K |\nabla_{\mathbb{H}} f(x)|^p \int_B \frac{|< \nu_f(x), h' >|^p}{|h|^p}\rho_n(h)dhdx$$

where $\nu_f(x) = \frac{\nabla_{\mathbb{H}} f(x)}{|\nabla_{\mathbb{H}} f(x)|}$ is a unit norm horizontal vector in $\mathbb{R}^{2N}$, so that $\int_B \frac{|\langle \nu_f(x), h'\rangle|^p}{|h|^p}\rho_n(h)dh = K_n$.

Claim is then proved.

Now, we want to prove $f \in W^{1,p}(\mathbb{H})$. Using the triangular inequality, we get

$$|\langle \nabla_{\mathbb{H}} f(x), h'\rangle| \leq |f(x \circ h) - f(x)| + |f(x \circ h) - f(x) - \langle \nabla_{\mathbb{H}} f(x), h'\rangle|$$

So that for any $p > 1$, $\theta > 0$ there exists a $C_{p,\theta} > 0$ such that

$$|\langle \nabla_{\mathbb{H}} f(x), h'\rangle|^p \leq (1+\theta)|f(x \circ h) - f(x)|^p + C_{p,\theta}|f(x \circ h) - f(x) - \langle \nabla_{\mathbb{H}} f(x), h'\rangle|^p.$$

Combining this relation with the claim we proved, with θ arbitrarily fixed, we get

$$K_n \int_K |\nabla_{\mathbb{H}} f(x)|^p dx \leq (1+\theta) \int_K \int_B \frac{|f(x\circ h) - f(x)|^p}{|h|^p} \rho_n(h) dh dx$$
$$+ C_{p,\theta} \int_K \int_B \frac{|f(x\circ h) - f(x) - \langle \nabla_{\mathbb{H}} f(x), h' \rangle|^p}{|h|^p} dh dx$$

Set

$$J_n = \int_K \int_B \frac{|f(x\circ h) - f(x) - \langle \nabla_{\mathbb{H}} f(x), h' \rangle|^p}{|h|^p} dh dx$$

For any $\delta \in (0,1)$, we can split the integral into two parts:

$$\begin{aligned} J_n &= \int_K \int_{B(\mathbf{e},\delta)} \frac{|f(x\circ h) - f(x) - \langle \nabla_{\mathbb{H}} f(x), h' \rangle|^p}{|h|^p} dh dx \\ &+ \int_K \int_{B\setminus B(\mathbf{e},\delta)} \frac{|f(x\circ h) - f(x) - \langle \nabla_{\mathbb{H}} f(x), h' \rangle|^p}{|h|^p} dh dx \\ &= J_{1,n} + J_{2,n} \end{aligned}$$

By the mean value theorem, there exists a $s^* \in (0,1)$ such that

$$f(x\circ h) - f(x) = \langle \nabla_{\mathbb{H}} f(x \circ \delta_{s^*}(h)), h' \rangle$$

so

$$\begin{aligned} |f(x\circ h) - f(x) - \langle \nabla_{\mathbb{H}} f(x), h' \rangle| &= |\langle \nabla_{\mathbb{H}} f(x \circ \delta_{s^*}(h)) - \nabla_{\mathbb{H}} f(x), h' \rangle| \\ &\leq |h'|_{\mathbb{R}^{2N}} |\nabla_{\mathbb{H}} f(x \circ \delta_{s^*}(h)) - \nabla_{\mathbb{H}} f(x)| \\ &\leq |h| |\nabla_{\mathbb{H}} f(x \circ \delta_{s^*}(h)) - \nabla_{\mathbb{H}} f(x)| \end{aligned}$$

Finally, since $f \in C^1_{\mathbb{H}}(\mathbb{H})$, we have

1) $\dfrac{|f(x\circ h) - f(x) - \langle \nabla_{\mathbb{H}} f(x), h' \rangle|}{|h|} \to 0$, as $h \to 0$, uniformly for x on compact sets;

2) $\dfrac{|f(x\circ h) - f(x) - \langle \nabla_{\mathbb{H}} f(x), h' \rangle|}{|h|} \leq C_K, \forall x \in K$ compact $, \forall |h| < 1$.

Then $J_{1,n}$ is arbitrarily small for any δ sufficiently small by the uniform convergence of $|f(x\circ h) - f(x) - \langle \nabla_{\mathbb{H}} f(x), h' \rangle|$. To estimate $J_{2,n}$, we can control $|f(x\circ h) - f(x) - \langle \nabla_{\mathbb{H}} f(x), h' \rangle|$ and use the tail property of $\rho_n(h)$. More precisely,

(1) for any $\lambda > 0$ there exists $0 < C(\lambda, K) < 1$ such that $|f(x\circ h) - f(x) - \langle \nabla_{\mathbb{H}} f(x), h' \rangle| < (\frac{\lambda}{2|K|})^{1/p} |h|$ for all $|h| < C(\lambda, K)$ and all $x \in K$. We then have

$$J_{1,n} < \frac{\lambda}{2|K|}|K| \int_{B(\mathbf{e},\delta)} \rho_n(h)dh < \frac{\lambda}{2}, \forall \delta < C(\lambda, K), \forall n;$$

(2) Since $|f(x \circ h) - f(x) - \langle \nabla_{\mathbb{H}} f(x), h' \rangle| \leq C_K^{1/p}|h|$ for all $x \in K$, we have

$$J_{2,n} \leq C_K |K| \int_{B \setminus B(\mathbf{e},\delta)} \rho_n(h)dh, \ \forall \delta \in (0,1)$$

and the integral over the annulus can be made arbitrarily small for n large due to the tail property of $\rho_n(h)$, so that

$$J_{2,n} \leq C_K |K| \frac{\lambda}{2C_K|K|} = \frac{\lambda}{2}, \ \forall n > N(\delta, \lambda, K)$$

Thus, for any fixed $\lambda > 0$, there exists an $N(\lambda, K)$ such that $J_n < \lambda$ provided $n > N(\lambda)$. We then end up with

$$K_n \int_K |\nabla_{\mathbb{H}} f(x)|^p dx \leq (1+\theta) \int_K \int_B \frac{|f(x \circ h) - f(x)|^p}{|h|^p} \rho_n(h) dh dx + C_{p,\theta}\lambda$$

Which is true for any n sufficiently large. This provides

$$\begin{aligned}
\int_K |\nabla_{\mathbb{H}} f(x)|^p dx &\leq \frac{1+\theta}{K_n} \underset{n\to 0}{liminf} \int_{\mathbb{H}} \int_{\mathbb{H}} \frac{|f(x \circ h) - f(x)|^p}{|h|^p} \rho_n(h) dh dx \\
&\leq \frac{1+\theta}{K_n} \underset{n\to 0}{limsup} \int_{\mathbb{H}} \int_{\mathbb{H}} \frac{|f(x \circ h) - f(x)|^p}{|h|^p} \rho_n(h) dh dx \\
&< \infty
\end{aligned}$$

Then we get $f \in W^{1,p}(\mathbb{H})$ by the density argument. $\square$

CHAPTER 7 CONCLUDING REMARKS AND FUTURE DIRECTIONS

In this dissertation, we have discussed new characterizations of Sobolev spaces on both stratified lie groups and Euclidean spaces. In chapter 2 and 3 , we established several new characterizations of Sobolev spaces on Heisenberg groups and Carnot groups. One difficulties considered here is to estimate $|f(x) - f(y)|$. Our results include the cases $1 < p < \infty$ and $p = 1$. In the future study, it will be interesting to study the new characterizations of the high order Sobolev space $W^{m,p}$ for $m \geq 2$ on Heisenberg goups and Carnot groups. In addition, the study of the high order BV space on Heisenberg groups and Carnot groups will also be a worthwhile undertaking.

Chapter 4 and Chapter 5 have been devoted to the new characterizations of high order Sobolev spaces in Euclidean spaces. We used two approaches: by the m-th order differences($m \geq 2$) and by the $m-1$-th Taylor remainder. The question is: can we design a more general function ω in Chapter 5 and 6 such that ω is convex. With some more additional assumptions of ω, we can have a new characterizations of high order Sobolev spaces. In fact, If we let $\omega(t) = t^p$, we indeed get the cases of Chapter 5 and 6.

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ABSTRACT

NEW CHARACTERIZATIONS OF SOBOLEV SPACES ON HEISENBERG GROUPS AND CARNOT GROUPS AND HIGH ORDER SOBOLEV SPACES ON EUCLIDEAN SPACES

by

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December 2015

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This dissertation concerns the new characterizations of Sobolev spaces on Heisenberg groups, Carnot groups and high order Sobolev spaces on Euclidean space. It contains two parts. The first part focus on the characterizations of the Sobolev space on Heisenberg groups and Carnot groups. Throughout this dissertation, the representation formula of Sobolev functions on Heisenberg goups and Carnot groups are used to estimate the difference of function values on two different points in Heisenberg groups and Carnot groups.

In Chapter 1, we introduce the motivation of the dissertation and give a brief review of some known characterizations of Sobolev spaces in Euclidean spaces. We also give the preliminary of the Heisenberg groups and Carnot groups which have different groups structures from the Euclidean spaces.

In Chapter 2 and 3, we study the first order Sobolev spaces on Heisenberg groups and Carnot groups. It originates from studying the asymptotic behavior of the fractional Sobolev space$W^{s,p}(0 < s < 1)$ as $s \to 1$. In Euclidean space, one of the main techniques to characterize Sobolev space is to

use the uniformity in every directions of the unit sphere in the Euclidean spaces. More precisely, to deal with the general $\delta \in \mathbb{S}^{N-1}$, it is often to be assumed that $\delta = e_N = (0, \cdots, 0, 1)$ and hence, one just needs to work on 1-dimensional case. This can be done by using the rotation in the Euclidean spaces. In the types of Heisenberg groups and Carnot groups, this type of property is not available because of the group structure of these two types of groups. Therefore we find a different approach to this characterization. We also study the cases for $p = 1$ which are the characterizations of BV space.

In Chapter 4 and Chapter 5, we consider the characterizations of high order Sobolev spaces $W^{m,p}(m \geq 2)$ in Euclidean spaces. Chapter 4 focus on the second order Sobolev space's characterizations. We present several types of characterizations: by second order differences, by the first order Taylor remainder and by the differences of the first order gradient. In Chapter 5, we study the characterizations of high order Sobolev spaces $W^{m,p}(\mathbb{R}^N)$ in Euclidean spaces by the m th order differences and the $m-1$th order Taylor remainder which we define both in Chapter 5.

In Chapter 6, we present that the functions in a Sobolev space possess the type of L^p-derivative which is introduced by Calder$\acute{o}$n and Zygumnd. In fact, our construction of the condition characterize the Sobolev functions on Heisenberg groups.

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