1-2-2012

Nodal geometry of eigenfunctions on smooth manifolds and hardy-littlewood-sobolev inequalities on the heisenberg group

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NODAL GEOMETRY OF EIGENFUNCTIONS ON SMOOTH MANIFOLDS AND HARDY-LITTLEWOOD-SOBOLEV INEQUALITIES ON THE HEISENBERG GROUP

by

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DISSERTATION

Submitted to the Graduate School

of Wayne State University,

Detroit, Michigan

in partial fulfillment of the requirements

for the degree of

DOCTOR OF PHILOSOPHY

2012

MAJOR: MATHEMATICS

Approved by:

Advisor Date

DEDICATION

To my family

Yunxia Fu, Suiqing Han, Xiaoli Han, and Dan Wang

ACKNOWLEDGEMENTS

First, I express my deepest gratitude to my advisor, Professor Guozhen Lu, who has brought me into this fantastic mathematical world. Without his exceptional guidance, constant inspiration, consistent support, and endless care, I could never have finished this dissertation.

I am taking this opportunity to thank Professors William Cohn, Jose-Luis Menaldi, Claude Pruneau, Peiyong Wang, and Chengzhong Xu for serving in my committee.

During my graduate study at Wayne State University, the faculty members in Department of Mathematics have helped me so much to grow as both a researcher and a teacher, including Professors John Breckenridge, Robert Bruner, Pau-Liu Chow, Daniel Isakson, Rafail Khasminskii, Tao Mei, Choon-Jai Rhee, Tze-Chien Sun, George Yin, Zhimin Zhang, Mss. Patricia Bonesteel, Mary Klamo, and Mr. James Veneri (who should be peacefully resting in heaven now). I thank them and all the unnamed ones.

I am indebted to my parents and my sister for their unconditional and unlimited love, support, and inspiration since I was born. This dissertation is dedicated to my dearest mother Yunxia Fu, father Suiqing Han, and sister Xiaoli Han.

Special thanks go to my wife, Dan Wang, for her love, support, encouragement, and confidence in me. I also want to thank my colleagues Dingzhou Cao, Qi He, Shaobai Kan, Nguyen Lam, Weiyuan Li, Lijing Sun, Yu Sun, Yayuan Xiao, Huiqing Zhu, and Jiuyi Zhu, with whom my Ph.D. journey became much more colorful and memorable.

Finally, but not least, I would like to express my appreciation to the entire Department of Mathematics for their hospitality and services. I enjoyed the warm and friendly atmosphere in the department, and I appreciate the support I have received during my study at Wayne State University.

PREFACE

This dissertation is divided into two separate parts:

- **Part I** We study eigenfunctions of Laplacian on smooth manifolds, by analyzing properties of high-energy states, we describe the nodal geometry of these functions in terms of the estimates on the size of nodal sets and the geometry of nodal domains.

- **Part II** We study Hardy-Littlewood-Sobolev inequalities on the Heisenberg group, by analyzing the sharp versions, we prove the existence of maximizers and give upper bounds of sharp constants in all admissible cases.

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Part I

Nodal geometry of eigenfunctions on smooth manifolds

1 Introduction

1.1 Background and major questions

Let $(\mathbb{M}, g)$ be a n dimensional smooth, compact, and connected Riemannian manifold without boundary, write the Laplace-Beltrami operator on $\mathbb{M}$ as

$$\Delta = \Delta_g = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(g^{ij} \sqrt{g} \frac{\partial}{\partial x_j} \right).$$

Consider the partial differential equation (PDE) on $\mathbb{M}$:

$$-\Delta u = \lambda u, \tag{1.1}$$

that is, u is an eigenfunction with eigenvalue λ. There are generally three directions of studying (1.1):

(1) $\lambda = 0$, i.e., analysis of harmonic functions;

(2) λ is the principal eigenvalue, i.e., analysis of principal eigenfunction (ground state);

(3) $\lambda \to \infty$, i.e., analysis of asymptotic behavior of eigenfunctions (limit of high-energy states).

We concentrate on Case (3) from above, and there is a vast literature on analytical results about the properties of eigenfunctions, which can be roughly categorized by their scopes: local or global. In Chapter 2, we mention the classical ones which we further use in the following chapters (i.e. geometric estimates of nodal domains in Chapter 3 and Hausdorff measure estimates of nodal sets in Chapters 4 and 5). We do not include the proofs for most of the theorems therein and provide the reference instead.

In Chapter 3, we continue to investigate the properties of eigenfunctions: BMO (bounded mean oscillation) estimates. It was developed by Donnelly and Fefferman [DF3], and further improved by Chanillo and Muckenhoupt [CM], Lu [Lu1, Lu2]. We here provide the best known results following this line. On the pass of improving these results, a δ-Besicovitch cover lemma is used, (and in fact it is a natural acquisition cooperated with the mechanism,) which is of independent interest. Our covering lemma improves the ones in [CM, Lu1, Lu2], and is indeed very "near" its optimal version as we offer a conjecture in Appendix A.

Using there tools aforementioned, we describe the nodal geometry of eigenfunctions: Let $\mathcal{N} = \{p \in \mathbb{M} : u(p) = 0\}$ denotes the nodal set of u (where the function vanishes), and the nodal domain be a connected component of $\mathbb{M} \setminus \mathcal{N}$. Therefore, all nodal domains can be naturally grouped into positive and negative families. Our primary interest is to understand the asymptotic geometric behavior of the nodal sets and nodal domains as $\lambda \to \infty$.

We first establish a local version of Courant's nodal domain theorem in Chapter 3 using the BMO estimates proved therein. Then, we switch our attention to one of the natural questions concerning the size of $\mathcal{N}$, whose first taste is provided in §1.3 as proving that $\mathcal{N} \neq \emptyset$. Yau [Y] conjectured that

Conjecture 1.1 (Yau)**.**

$$c\sqrt{\lambda} \leq \mathcal{H}^{n-1}(\mathcal{N}) \leq C\sqrt{\lambda},$$

in which the constants c and C depend only on $\mathbb{M}$, and $\mathcal{H}^{n-1}$ is the $n-1$ dimensional Hausdorff measure.

The equality is achieved by the eigenfunction

$$u(x) = \sin(k_1 x_1)\sin(k_2 x_2)\cdots\sin(k_n x_n)$$

on the torus $\mathbb{T}^n = \mathbb{S}^1 \times \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$.

Conjecture 1.1 was verified by Donnelly and Fefferman [DF1] on analytic manifolds. While on smooth manifolds, the known results are still far away from these optimal bounds. The discussion on lower and upper bounds of $\mathcal{H}^{n-1}(\mathcal{N})$ is expanded in Chapters 4 and 5, respectively, and we shall begin with an approach which uses the BMO estimates in Chapter 3 to deduce polynomially decreasing lower bounds as a quick consequence of the isoperimetric inequality.

1.2 Notational index

We list the frequently used notations in this note as follows.

- $(\mathbb{M}, g)$: a n dimensional smooth, compact, and connected Riemannian manifold (without boundary) equipped with metric g.
- $\mathbb{S}^{n-1}$: unit sphere in $\mathbb{R}^n$.
- $\Delta = \Delta_g$: the Laplace-Beltrami operator on $\mathbb{M}$.
- $\mathcal{H}^k$: k dimensional Hausdorff measure for $k \geq 0$. Particularly, $\mathcal{H}^n$ and $\mathcal{H}^{n-1}$ denote the volume and sphere measures, we also use $|\cdot|$ to represent the volume (or sphere) measure.

- $B(p,r)$: the geodesic ball centered at p and of radius r. $2B$ denotes the concentric ball with twice the radius of B's.

- $⨏_{B(p,r)} f$ and $⨏_{\partial B(p,r)} f$: average integral of f in $B(p,r)$ and on $\partial B(p,r)$.

- Ω: an open and bounded domain in $\mathbb{R}^n$ or in $\mathbb{M}$.

- $\bigtriangledown_i f = \partial f/\partial x_i$: i-th derivative of f.

- $\bigtriangledown f$: gradient of f, i.e., $\bigtriangledown f = (\bigtriangledown_1 f, \cdots, \bigtriangledown_n f)$.

- $\mathcal{S} = \{p \in \mathbb{M} : u(p) = |\bigtriangledown u(p)| = 0\}$: the singular set of eigenfunction u, where both u and its gradient vanish.

- q': the conjugate index of the index $1 \le q \le \infty$, that is,

$$\frac{1}{q} + \frac{1}{q'} = 1.$$

- C and c: generic positive constants depending only on $\mathbb{M}$, and we shall not pursue the explicit dependence upon the geometry of $\mathbb{M}$, we also use $c_1 = c_1(c_2)$ to describe the particular dependence of c_1 upon c_2.

1.3 Nodal set is nonempty

In this section, we show $\mathcal{N} \neq \emptyset$, some classical results about eigenfunctions on $\mathbb{R}^n$ shall also be investigated.

A simple observation on $\mathbb{S}^1$

$$\lambda \oint_{\mathbb{S}^1} u = -\oint_{\mathbb{S}^1} u'' = 0$$

shows that u must not be positive or negative, thus zero exists.

Let us switch directly to the rigorous proof, and first recall the boundary value problem

$$\begin{cases} -\Delta w = \lambda w, & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

It can be showed that the set Σ of eigenvalues is at most countable. Next we state two theorems about eigenvalues and eigenfunctions of (1.2) (cf. Theorems 1 and 2 in §6.5.1 of [E]).

Theorem 1.2 (Eigenvalues and eigenfunctions)**.** *Each eigenvalue is real, and if we repeat each eigenvalue according to its (finite) multiplicity, we have*

$$\Sigma = \{\lambda_k\}_{k=1}^{\infty},$$

in which

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots,$$

and $\lambda_k \to \infty$ *as* $k \to \infty$. *Furthermore, there exists an orthogonal basis* $\{w_k\}_{k=1}^{\infty}$ *of* $L^2(\Omega)$, *where* $w_k \in H_0^1(\Omega)$ *is an eigenfunction corresponding to* λ_k.

Remark.

(a) The main tool to prove Theorem 1.2 is Fredholm alternative, and it does not depend on the structure of $\mathbb{R}^n$, thus we can apply it to other settings with no difficulty.

(b) Here we have $w_k \in C^\infty(\Omega)$, if we furthermore assume that $\partial\Omega$ is smooth, then $w_k \in C^\infty(\overline{\Omega})$, for $k = 1, 2, \cdots$.

(c) We call λ_1 and w_1 the principal eigenvalue and eigenfunction[i].

Theorem 1.3 (Principal eigenvalue and eigenfunction)**.** *We have*

$$\lambda_1 = \min\{B[u,u] | u \in H_0^1(\Omega), \|u\|_2 = 1\},$$

in which $B[u,v] = -\int_\Omega v\Delta u = \int_\Omega \nabla u \cdot \nabla v$ *for* $u, v \in H_0^1(\Omega)$*. While* w_1 *is positive in* Ω*, and any weak solution of (1.2) when* $\lambda = \lambda_1$ *is a multiple of* w_1*.*

Next we introduce a lemma about principal eigenvalues and eigenfunctions in balls.

Lemma 1.4 (Principal eigenvalue and eigenfunction on a ball)**.** *Let* λ_1 *and* w_1 *be principal eigenvalue and eigenfunction on* $B(x,1) \subset \mathbb{R}^n$*, then* w_1 *is radially symmetric and strictly decreasing with respect to* x*.*

We write $w_1 = w_1(r)$ *for* $0 \leq r \leq 1$*, then* $R^{-2}\lambda_1$ *and* $w_1(r/R)$ *are principal eigenvalue and eigenfunction of on* $B(x,R)$*.*

Proof of Lemma 1.4. To verify w_1 is radially symmetric and strictly decreasing with respect to x, we use moving plane method developed by Serrin [S] and Gidas, Ni and Nirenberg [GNN]. (See also §9.5.2 in [E].) We are able to use it since it is assured that principal eigenfunction is in $C^\infty(B(x,R))$ and positive on $B(x,R)$. We omit the proof here since it is a standard and direct application of moving plane method.

Thus, $w_1 = w_1(r)$, and by computing the radial derivatives, we have

$$w_1'' + \frac{n-1}{r} w_1' + \lambda_1 w_1 = 0$$

[i] In some literature (e.g. [CoMi]), λ_1 is referred as the fist Dirichlet eigenvalue, and w_1 as the first Dirichlet eigenfunction.

for $0 \le r \le 1$ and $w_1(1) = 0$ (the boundary condition in (1.2)). Given $\widetilde{w}_1(r) = w_1(r/R)$ for $0 \le r \le R$, then, it is easy to check that

$$\widetilde{w}_1'' + \frac{n-1}{r}\widetilde{w}_1' + \frac{\lambda_1}{R^2}\widetilde{w}_1 = 0,$$

and $\widetilde{w}_1(R) = w_1(1) = 0$. By uniqueness of principal eigenvalue and eigenfunction, we conclude that $R^{-2}\lambda_1$ and $w_1(r/R)$ are principal eigenvalue and eigenfunction on $B(x, R)$. $\square$

Now we can prove the existence of zeros (cf. Lemma 6.2.1 in [HL] and Lemma 1 in [CoMi]).

Theorem 1.5 (Wavelength)**.** *u vanishes at some point in each ball of radius at most $c\lambda^{-1/2}$.*

Proof of Theorem 1.5. Given $B(x, c\lambda^{-1/2}) \subset \mathbb{R}^n$, we denote λ_1 and w_1 as principal eigenvalue and eigenfunction on $B(x, c\lambda^{-1/2})$, then $\lambda_1 \simeq c^{-2}\lambda$ by Lemma 1.4.

Without loss of generality, assume u is positive on $B(p, c\lambda^{-1/2})$, then $v = w_1/u$ assumes an interior maximum since $v = 0$ on $\partial B(x, c\lambda^{-1/2})$. At this interior maximum point,

$$\bigtriangledown_i v = \frac{\bigtriangledown_i w_1 - w_1 \bigtriangledown_i u}{u^2} = 0,$$

and

$$\Delta v = \frac{u\Delta w_1 - w_1 \Delta u}{u^2} = \frac{(\lambda - \lambda_1)w_1}{u} \le 0.$$

Thus, contradiction occurs when c is sufficiently large and the principal eigenvalue $\lambda_1 \simeq c^{-2}\lambda < \lambda/2$. $\square$

Remark. We fix the quantity $r_\lambda = O(\lambda^{-1/2})$, which is also called the "wavelength" of eigenfunction, that is, u vanishes at some point in every ball with radius r_λ. For technical reasons, we may assume u vanishes at some point in the middle half or middle one-third of the ball by choose $r_\lambda = 2c\lambda^{-1/2}$ or $r_\lambda = 3c\lambda^{-1/2}$.

One can obtain the corresponding version of the above theorem on $\mathbb{M}$ by modifying the proof: On a geodesic ball $B \subset \mathbb{M}$, $\lambda_1 = \min \int_B |\bigtriangledown \psi|^2 / \int_B |\psi^2|$, Bishop-Gromov volume comparison theorem together with the positive/negative assumption guarantee that u vanishes at some point in a local scale $r_\lambda = O(\lambda^{-1/2})$ (cf. [CoMi]).

To further estimate the measure of nodal set of u, Hausdorff measure is the appropriate and powerful tool to use. Here, we use [F] as a classical reference on geometric measure theory.

2 Local and global analysis

In this chapter, we discuss local and global properties of eigenfunctions, which provide most of the tools used in further study of their nodal geometry in Chapters 3 and 4. We roughly categorize them as follows.

(1) Local results, which often hold in a local scale[i] (dependent or independent on λ), including

- doubling condition,
- growth estimate,
- vanishing order estimate,
- Bernstein's estimates,
- local maximum principle,
- frequency functions and monotonicity formula, etc.

These methods apply to large classes of functions: harmonic functions, eigenfunctions, polynomials, solutions to elliptic and parabolic PDEs, etc. Among the rich literature in this area, there exists a well written survey [HL].

(2) Global results, including

- Dong-Sogge-Zelditch's integral formula,
- Sogge's L^p estimate,

[i]We point out that local properties of eigenfunctions on a scale of $r_\lambda = O(\lambda^{-1/2})$ are of extreme importance, particularly, eigenfunctions resemble harmonic functions in this scale (as we see in Chapter 1).

- global frequency functions, etc.

A good reference is [Z].

Next we present the ones from above which are used in our investigation.

2.1 Local properties

In this section, we review some local properties of eigenfunctions.

Theorem 2.1 (Doubling condition)**.** *There exists r_1 depending only on $\mathbb{M}$ such that*

$$\fint_{B(p,2r)} u^2 \leq c^{\sqrt{\lambda}} \fint_{B(p,r)} u^2$$

for any $p \in \mathbb{M}$ and $r \in (0, r_1]$.

A beautiful growth estimate was proved in Theorem 1(A) of [DF3], thus Theorem 2.1 follows as an evident consequence by iteration.

Theorem 2.2 (Growth estimate)**.** *With the same r_1 as in Theorem 2.1,*

$$\int_{B(p,(1+\lambda^{-\frac{1}{2}})r)} u^2 \leq c \int_{B(p,r)} u^2$$

for any $p \in \mathbb{M}$ and $r \in (0, r_1]$.

A related vanishing order estimate is

Theorem 2.3 (Vanishing order estimate)**.** *With the same r_1 as in Theorem 2.1,*

$$\max_{B(p,R)} |u| \leq \left(\frac{cR}{r}\right)^{c\sqrt{\lambda}} \max_{B(p,r)} |u|$$

for any $p \in \mathbb{M}$ and $0 < r \leq R \leq r_1$.

Remark. There are two approaches to get the above theorems: Donnelly and Fefferman [DF1, DF2, DF3] used Carleman's inequalities to give quantitative unique continuation results, and Lin [L] used the frequency function and monotonicity formula. See also [HL, Z] for more discussion.

Theorem 2.4 (Bernstein's estimates)**.** *With the same* r_1 *as in Theorem 2.1,*

(a) L^2 *Bernstein's estimate*

$$\left(\int_{B(p,r)} |\bigtriangledown u|^2\right)^{1/2} \leq \frac{c\sqrt{\lambda}}{r}\left(\int_{B(p,r)} |u|^2\right)^{1/2},$$

(b) L^∞ *Bernstein's estimate, with* $K = \frac{n+2}{4}$*,*

$$\max_{B(p,r)} |\bigtriangledown u| \leq \frac{c\lambda^K}{r} \max_{B(p,r)} |u|$$

for any $p \in \mathbb{M}$ *and* $r \in (0, r_1]$.

Remark. It was conjectured by Donnelly and Fefferman [DF3] that in the L^∞ Bernstein's estimate, $K = 1/2$, but it is still unknown, while Dong [D3] verified it in dimension two for $r \leq c\lambda^{-1/4}$.

Next we mention the local maximum principle, which is a standard a priori estimate for solutions to elliptic PDEs, see [LS].

Theorem 2.5 (Local maximum principle)**.** *There exists* r_2 *depending only on* $\mathbb{M}$ *such that*

$$\sup_{B(p,r)} u^2 \leq c \fint_{B(p,2r)} u^2$$

for any $p \in \mathbb{M}$ *and* $r \in (0, r_2]$.

A simple corollary states

Corollary 2.6. *Assume the same conditions as in Theorem 2.5, and $\delta > 0$ is a small number, we have*

$$\sup_{B(p,r)} u^2 \leq \frac{c}{\delta^n} \fint_{B(p,(1+\delta)r)} u^2$$

for any $p \in \mathbb{M}$ and $r \in (0, r_2]$.

2.2 Global properties

The following two theorems involving the global estimates are of great importance.

Theorem 2.7 (Dong-Sogge-Zelditch's integral formula)**.**

$$\lambda \int_{\mathbb{M}} |u| d\mathcal{H}^n = 2 \int_{\mathcal{N}} | \bigtriangledown u| d\mathcal{H}^{n-1}. \tag{2.1}$$

More generally, for $f \in C^2(\mathbb{M})$, we have

$$\int_{\mathbb{M}} \big[(\Delta + \lambda) f\big] |u| d\mathcal{H}^n = 2 \int_{\mathcal{N}} f | \bigtriangledown u| d\mathcal{H}^{n-1}. \tag{2.2}$$

Remark.

(i) (2.1) follows if we take $f = 1$ in (2.2).

(ii) (2.2) was proved by Dong [D1] for $f = q^{-1/2}$, where

$$q = | \bigtriangledown u|^2 + \frac{\lambda u^2}{n}. \tag{2.3}$$

See also Alt, Caffarelli, and Friedman [ACF] for its inspiration.

(iii) (2.2) was verified by Sogge and Zelditch [SZ] with two different proofs.

Theorem 2.8 (Sogge's L^p estimate)**.**

$$\frac{\|u\|_{L^p(\mathbb{M})}}{\|u\|_{L^2(\mathbb{M})}} \leq \begin{cases} c\lambda^{\frac{n(p-2)-p}{4p}}, \ \textit{if } \frac{2(n+1)}{n-1} \leq p \leq \infty, \\ \\ c\lambda^{\frac{(n-1)(p-2)}{8p}}, \ \textit{if } 2 \leq p \leq \frac{2(n+1)}{n-1}. \end{cases}$$

It was proved by Sogge [Sog]. A simple consequence on the lower bound of $\|u\|_{L^1(\mathbb{M})}$ is

Corollary 2.9. *For normalized u, i.e., $\|u\|_{L^2(\mathbb{M})} = 1$, we have*

$$\|u\|_{L^1(\mathbb{M})} \geq c\lambda^{\frac{1-n}{8}}.$$

Proof of Corollary 2.9. Let $0 < \theta < 1$, wit from Hölder's inequality,

$$\begin{aligned} 1 &= \int_{\mathbb{M}} |u|^2 \\ &\leq \left(\int_{\mathbb{M}} |u|\right)^{\theta} \left(\int_{\mathbb{M}} |u|^{\frac{2-\theta}{1-\theta}}\right)^{1-\theta} \\ &\leq \|u\|^{\theta}_{L^1(\mathbb{M})} \|u\|^{2-\theta}_{L^{\frac{2-\theta}{1-\theta}}(\mathbb{M})} \\ &\leq \frac{c\theta(n-1)}{8} \|u\|^{\theta}_{L^1(\mathbb{M})}, \end{aligned}$$

if we choose $\frac{2-\theta}{1-\theta} \leq \frac{2(n+1)}{n-1}$, and the corollary is done by canceling terms. $\square$

3 BMO estimates and geometry of nodal domains

In this chapter, we investigate the BMO estimates of eigenfunctions[i], from which we derive some geometric estimates on nodal domains, i.e. the connected region of $\{u \neq 0\}$. The material in this chapter has been published in [HaLu1]. For technical reasons, we separate our discussion into higher dimensional and two dimensional cases.

3.1 A covering lemma

In this section, we discuss a δ-Besicovitch covering lemma on $\mathbb{R}^n$ which plays a crucial role in the BMO estimates. Itself has independent interest as well.

Theorem 3.1 (δ-Besicovitch covering lemma). *Given any finite collection of balls $\{B_\alpha\}_{\alpha \in I}$ in $\mathbb{R}^n$, let $\delta > 0$ be small enough, then one can select a subcollection $\{B_1, \cdots, B_N\}$ such that*

$$\bigcup_{\alpha \in I} B_\alpha \subset \bigcup_{i=1}^{N} (1+\delta) B_i, \tag{3.1}$$

and

$$\sum_{i=1}^{N} \chi_{B_i}(x) \leq c(n) \delta^{-\frac{n}{2}} \log \frac{1}{\delta} \tag{3.2}$$

for all $x \in \mathbb{R}^n$, where $c(n)$ depends on the dimension n, but is independent of δ and the given collection of balls.

This covering lemma was introduced by Chanillo and Muckenhoupt [CM] with the right-hand side of (3.2) replaced by $c\delta^{-n}$, and further sharpened by Lu [Lu1, Lu2] to $c\delta^{-n+\frac{1}{2}} \log \frac{1}{\delta}$. Therefore, one sees that Theorem 3.1 significantly improves these previous results.

[i] See [G] for reference on BMO spaces.

Though we do not know yet if the above theorem is the best possible result, we believe that it is fairly sharp and close to the optimal one. Indeed, we conjecture the following covering lemma.

Conjecture 3.2. *The upper bound in (3.2) can be improved to* $c(n)\delta^{-\frac{n-1}{2}}$*, and it is sharp.*

The sharpness of the above conjecture, if it is true, is demonstrated by an example in Appendix A. However, we still do not have a proof that it is indeed true.

Let us give some definitions that are used to prove Theorem 3.1. Note that all the collections of balls here are finite.

Definition 3.3 (δ-proper cover)**.** *Given* $\delta \geq 0$*, a subcollection of balls* $\{B_1, \cdots, B_N\} \subset \{B_\alpha\}_{\alpha\in I}$ *is called a* δ*-proper cover of* $\{B_\alpha\}_{\alpha\in I}$ *if*

$$\bigcup_{\alpha\in I} B_\alpha \subset \bigcup_{i=1}^{N}(1+\delta)B_i \tag{3.3}$$

and

$$B_j \not\subset \bigcup_{i=1, i\neq j}^{N} (1+\delta)B_i \tag{3.4}$$

for every $j = 1, \cdots, N$.

If $C = \{B_1, \cdots, B_N\} \subset \{B_\alpha\}_{\alpha\in I}$ *satisfies (3.3), then it is called a* δ*-cover of* $\{B_\alpha\}_{\alpha\in I}$. *If* $S = \{B_1, \cdots, B_N\} \subset \{B_\alpha\}_{\alpha\in I}$ *satisfies (3.4), then it is called a* δ*-proper subcollection of* $\{B_\alpha\}_{\alpha\in I}$.

Lemma 3.4. *Given a collection of balls* $\{B_\alpha\}_{\alpha\in I}$*, there exists* $\delta_0 > 0$ *such that for all* $\delta \in [0, \delta_0]$*, there exists a* δ*-proper cover of* $\{B_\alpha\}_{\alpha\in I}$.

Proof of Lemma 3.4. We prove by induction on the cardinality of the collection of balls, namely $|I|$.

It is obvious that a collection of a single ball has a δ-proper cover for any $\delta \geq 0$. If this lemma is true for every collection $\{B_\alpha\}_{\alpha\in I}$ with $|I| \leq k$, we prove that it is also true for $\{B_\alpha\}_{\alpha\in I}$ with $|I| = k+1$.

Case 1: If $\exists\ C_0 = \{B_1, \cdots, B_N\} \not\subset \{B_\alpha\}_{\alpha\in I}$ is a 0-cover of $\{B_\alpha\}_{\alpha\in I}$ (then $1 \leq N \leq k$), i.e.,

$$\bigcup_{\alpha\in I} B_\alpha \subset \bigcup_{i=1}^{N} B_i,$$

then by induction, $\exists \delta_0 > 0$, such that for $\forall \delta \in [0, \delta_0]$, there exists a δ-proper cover of C_0, which is then also a δ-proper cover of $\{B_\alpha\}_{\alpha\in I}$.

Case 2: Assume that there is no 0-cover of $\{B_\alpha\}_{\alpha\in I}$ with $|I| \leq k$. Let $\{B_\alpha\}_{\alpha\in I} = \{B_{\alpha_1}, \cdots, B_{\alpha_{k+1}}\}$. Then

$$B_{\alpha_j} \not\subset \bigcup_{i=1, i\neq j}^{k+1} B_{\alpha_i}$$

for every $j = 1, \cdots, k+1$. Therefore, there exists a sufficiently small $\delta_j > 0$ such that

$$B_{\alpha_j} \not\subset \bigcup_{i=1, i\neq j}^{k+1} (1+\delta_j) B_{\alpha_i}.$$

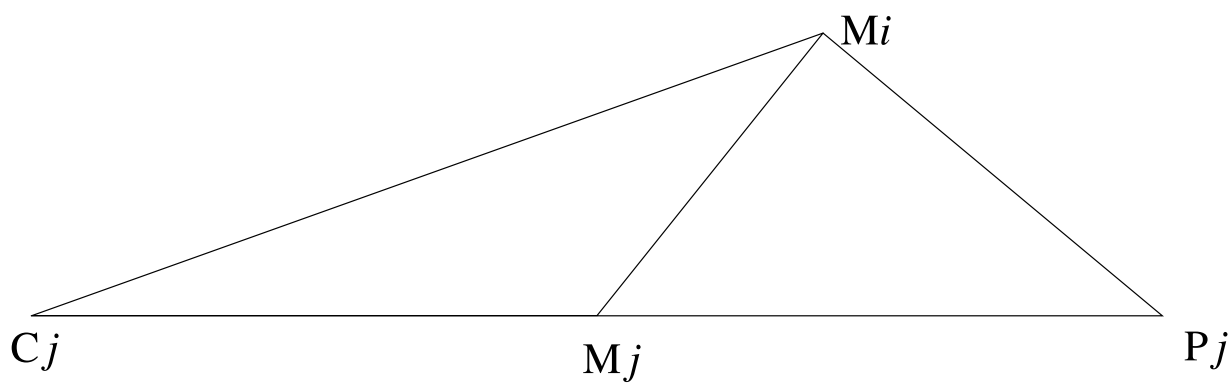
Let $\delta_0 = \min(\delta_1, \cdots, \delta_{k+1}) > 0$, we see that

$$B_{\alpha_j} \not\subset \bigcup_{i=1, i\neq j}^{k+1} (1+\delta_0) B_{\alpha_i} \subset \bigcup_{i=1, i\neq j}^{k+1} (1+\delta_j) B_i$$

for every $j = 1, \cdots, k+1$. Therefore, $\{B_{\alpha_1}, \cdots, B_{\alpha_{k+1}}\}$ is the δ-proper subcollection of itself for $\delta \in [0, \delta_0]$, and thus the δ-proper cover of itself for $\delta \in [0, \delta_0]$. $\square$

Lemma 3.5. *Let $\delta > 0$ be small enough, then given any collection of balls $\{B_\alpha\}_{\alpha\in I}$ and $S = \{B_1, \cdots, B_N\} \subset \{B_\alpha\}_{\alpha\in I}$ is a δ-proper subcollection of $\{B_\alpha\}_{\alpha\in I}$ with $r \leq r_i \leq 2r$ for $i = 1, \cdots, N$, where $r > 0$. Then,*

$$\sum_{i=1}^{N} \chi_{B_i}(x) \leq c\delta^{-\frac{n}{2}} \tag{3.5}$$



for all $x \in \mathbb{R}^n$*, where* c *depends only on* n*.*

Proof of Lemma 3.5. Let $x_0 \in \bigcap_{i=1}^M B_i$, $M = M(x_0)$. By a translation, we may suppose $x_0 = 0$. For $x \in \mathbb{R}^n$, define $T_r(x) = x/r$, then $\{T_r(B_1), \cdots, T_r(B_M)\}$ is a δ-proper subcollection with the radius between 1 and 2. Without loss of generality, we may assume $1 \leq r_i \leq 2$ for $i = 1, \cdots, M$.

Now $0 \in \bigcap_{i=1}^M B_i$ and

$$B_j \not\subset \bigcup_{i=1, i\neq j}^{M} (1+\delta)B_i,$$

which means $\forall j = 1, \cdots, M$, $\exists P_j \in B_j$ and $\overline{P_j C_i} \geq (1+\delta) r_i$, for all $i = 1, \cdots, M$ and $i \neq j$, in which C_i is the center of B_i. Thus, we have

$$|C_i| < 2, \ |P_i| < 4, \ \text{for } i = 1, \cdots, M.$$

Denote M_j as the midpoint of C_j and P_j:

$$|M_j| = |\frac{C_j + P_j}{2}| \leq 4.$$

Compute that

$$\overline{C_jM_i} + \overline{M_iP_i} \geq \overline{C_jP_i} \geq (1+\delta)r_j,$$

$$\overline{P_jM_i} + \overline{M_iC_i} \geq \overline{P_jC_i} \geq (1+\delta)r_i,$$

$$\overline{M_iP_i} = \overline{M_iC_i} \leq \frac{1}{2}r_i.$$

Thus,

$$\overline{P_jM_i} \geq (1+\delta)r_i - \frac{1}{2}r_i > 0,$$

and

$$\overline{C_jM_i} \geq (1+\delta)r_j - \frac{1}{2}r_i > 0.$$

Observe that

$$\overline{C_jM_j}^2 + \overline{M_jM_i}^2 - 2\cos(\pi-\theta)\cdot\overline{C_jM_j}\cdot\overline{M_jM_i} = \overline{C_jM_i}^2, \tag{3.6}$$

and

$$\overline{P_jM_j}^2 + \overline{M_jM_i}^2 - 2\cos\theta\cdot\overline{P_jM_j}\cdot\overline{M_jM_i} = \overline{P_jM_i}^2. \tag{3.7}$$

Combining (3.6) and (3.7), and noticing that $\overline{M_jP_j} = \overline{M_jC_j} \leq \frac{1}{2}r_j$,

$$\begin{aligned}
2\overline{M_jM_i}^2 &= \overline{C_jM_i}^2 + \overline{P_jM_i}^2 - \overline{C_jM_j}^2 - \overline{P_jM_j}^2 \\
&\geq \left[(1+\delta)r_i - \frac{1}{2}r_i\right]^2 + \left[(1+\delta)r_j - \frac{1}{2}r_i\right]^2 - 2\left[\frac{1}{2}r_j\right]^2 \\
&\geq (\frac{1}{2}+\delta)^2 r_i^2 + (1+\delta)r_j^2 + \frac{1}{4}r_i^2 - (1+\delta)r_ir_j \\
&\geq (\frac{\delta}{2}+\delta^2)r_i^2 + \frac{\delta}{2}r_j^2 \\
&\geq \delta + \delta^2,
\end{aligned}$$

where in the above we have used the inequality that $r_ir_j \leq \frac{r_i^2+r_j^2}{2}$, $r_i \geq 1$, and $r_j \geq 1$.

Then, we get

$$\overline{M_j M_i} \geq \frac{\sqrt{\delta}}{\sqrt{2}}.$$

Hence, $M(\sqrt{\delta})^n \leq (4\sqrt{2})^n$, i.e., $M \leq 4^{n+\frac{1}{4}}\delta^{-\frac{n}{2}}$, and (3.5) follows. □

Now we will prove the main covering lemma: Theorem 3.1.

Proof of Theorem 3.1. By Lemma 3.4, there exists $\delta_0 > 0$ such that for all $\delta \in [0, \delta_0]$, there exists $\{B_1, \cdots, B_N\} \subset \{B_\alpha\}_{\alpha \in I}$ as its δ-proper cover. Then, it clearly satisfies (3.1) of Theorem 3.1, we now prove (3.2).

Let $x_0 \in \bigcap_{i=1}^M B_i$, $M = M(x_0)$. By a translation, we may suppose $x_0 = 0$. Now $0 \in \bigcap_{i=1}^M B_i$, and

$$B_j \not\subset \bigcup_{i=1, i \neq j}^{M} (1+\delta)B_i.$$

Without loss of generality, we may assume $r_1 \leq \cdots \leq r_M$, since $0 \in B_1 \cap B_M$ and $B_1 \not\subset (1+\delta)B_M$, we have $2r_1 \geq \delta r_M$, then let

$$K = K(0) = 2 + \lfloor \log_2 \frac{1}{\delta} \rfloor,$$

where $\lfloor \cdot \rfloor$ denotes the largest integer part, and let

$$S_j = \{B_i | 2^{j-1} r_1 \leq r_i < 2^j r_1\}$$

for $j = 1, \cdots, K$. We see that S_j is a δ-proper subcollection of $\{B_\alpha\}_{\alpha \in I}$ for $j = 1, \cdots, K$, and since $2^K r_1 \geq r_M$, we have

$$\{B_1, \cdots, B_M\} \subset \bigcup_{j=1}^{K} S_j.$$

Denote $K_j = |S_j|$, thus $K_j \leq 4^n \delta^{-\frac{n}{2}}$ by Lemma 3.5. Then,

$$M = \sum_{j=1}^{K} K_j \leq K 4^n \delta^{-\frac{n}{2}} \leq c\delta^{-\frac{n}{2}} \log \frac{1}{\delta},$$

and (3.2) in Theorem 3.1 follows. $\square$

3.2 In higher dimensions

3.2.1 BMO estimate of $\log|u|$

Theorem 3.6 (BMO estimate of $\log|u|$)**.** *For $n \geq 3$,*

$$\|\log|u|\|_{BMO} \leq c\lambda^{\frac{3n}{4}}(\log\lambda)^2.$$

An equivalent recording of Theorem 3.6 is

Theorem 3.7. *For $n \geq 3$, let $E \subset B \subset \mathbb{M}$, then*

$$\sup_B |u| \leq \left(\frac{c|B|}{|E|}\right)^{c\lambda^{\frac{3n}{4}}(\log\lambda)^2} \sup_E |u|.$$

Remark. The BMO estimate of $\log|u|$ was first investigated by Donnelly and Fefferman [DF3], and further developed by Chanillo and Muckenhoupt [CM], Lu [Lu1, Lu2], the author and Lu [HaLu1], we list the results in this line as follows.

- $\lambda^{\frac{n(n+2)}{4}}$ (Donnelly and Fefferman [DF3])
- $\lambda^n \log\lambda$ (Chanillo and Muckenhoupt [CM])
- $\lambda^{n-\frac{1}{8}}(\log\lambda)^2$ (Lu [Lu2])

Our results further improve these ones. However, we still do not know what the optimal bounds are.

Lemma 3.8 (Reverse Hölder's inequality)**.**

$$\left(\frac{1}{|B|}\int_B |u|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{2n}} \leq c\sqrt{\lambda}\left(\frac{1}{|B|}\int_B |u|^2\right)^{1/2}.$$

Proof of Lemma 3.8. By the Poincaré's inequality, for any ball B, we have

$$\left(\frac{1}{|B|}\int_B |u-u_B|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{2n}} \le c|B|^{\frac{1}{n}}\left(\frac{1}{|B|}\int_B |\bigtriangledown u|^2\right)^{\frac{1}{2}},$$

where $u_B = \frac{1}{|B|}\int_B u$. Applying Hölder's inequality and L^2 Bernstein's estimate in Lemma 2.4, we obtain

$$\left(\frac{1}{|B|}\int_B |u-u_B|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{2n}} \le c\sqrt{\lambda}\left(\frac{1}{|B|}\int_B |u|^2\right)^{1/2}.$$

By Minkowski's inequality, Lemma 3.8 follows. □

The following lemma is really the key to derive the BMO estimate in Theorem 3.6.

Lemma 3.9. *Suppose $n \ge 3$ and $w > 0$, and assume that for any ball B,*

$$\int_{(1+\lambda^{-\frac{1}{2}})B} w \le c_1 \int_B w, \tag{3.8}$$

and

$$\left(\frac{1}{|B|}\int_B w^{\frac{n}{n-2}}\right)^{\frac{n-2}{n}} \le \frac{c_2\lambda}{|B|}\int_B w. \tag{3.9}$$

Then,

$$\|\log w\|_{BMO} \le c\lambda^{\frac{3n}{4}}(\log\lambda)^2,$$

in which c depends on the constants c_1 and c_2.

Remark. Substituting w by $|u|^2$ in the above lemma, the BMO bound of $\log|u|$ in Theorem 3.6 follows since condition (3.8) is assured by the growth condition in Theorem 2.2, and condition (3.9) is the reverse Hölder's inequality in Lemma 3.8.

To show Lemma 3.9, we first need to show the following

Lemma 3.10. *Let w satisfy the hypothesis of Lemma 3.9, and k is an integer, let B be a fixed ball, and $E \subset B$, then there exist c_3 and c_4 such that if*

$$|E| \geq \left[1 - c_3 \lambda^{-\frac{3n}{4}} (\log \lambda)^{-1}\right]^k |B|,$$

then

$$\int_E w \geq \left[c_4 \lambda^{-\frac{n}{4}} (\log \lambda)^{-1}\right]^k \int_B w,$$

in which $c_3 = c_3(c_2)$ and $c_4 = c_4(c_1)$.

Proof of Lemma 3.10. The proof rests on an induction on k. We first want to show if $|E| \geq (1 - \bar{c}\lambda^{-\frac{n}{2}})|B|$ for some appropriate $\bar{c} = \bar{c}(c_2)$, then $\int_E w \geq \frac{1}{2} \int_B w$. To show this, we first note that $|B \setminus E| \leq \bar{c}\lambda^{-\frac{n}{2}}|B|$. Thus, by Hölder's inequality and (3.9),

$$\int_{B \setminus E} w \leq \left(\int_B w^{\frac{n}{n-2}}\right)^{\frac{n-2}{n}} |B \setminus E|^{\frac{2}{n}} \leq c_2 \bar{c}^{\frac{2}{n}} \int_B w.$$

If we choose $\bar{c}$ such that $c_2 \bar{c}^{\frac{2}{n}} < \frac{1}{2}$, then

$$\int_{B \setminus E} w \leq \frac{1}{2} \int_B w,$$

this implies

$$\int_E w > \frac{1}{2} \int_B w.$$

Thus, if $c_3 \leq \bar{c}$, and $|E| \geq (1 - c_3 \lambda^{-\frac{3n}{4}} (\log \lambda)^{-1})|B|$, then $|E| \geq (1 - \bar{c}\lambda^{-\frac{n}{2}})|B|$, and therefore,

$$\int_E w \geq \frac{1}{2} \int_B w \geq c_4 \left[\lambda^{-\frac{n}{4}} (\log \lambda)^{-1}\right] \int_B w,$$

and we are done for the case $k = 1$.

Now we assume the statement is true for $k - 1$. We may assume $|E| \leq (1 - \bar{c}\lambda^{-\frac{n}{2}})|B|$, otherwise, there is nothing to prove. Thus, for each density point x of E, we can select a ball

$B_x \subset B$ such that $x \in B_x$, and

$$\frac{|B_x \cap E|}{|B_x|} = 1 - \bar{c}\lambda^{-\frac{n}{2}}.$$

Applying the cover lemma Theorem 3.1 when $n \geq 3$ to the balls B_x with the choice $\delta = \lambda^{-1/2}$, and without loss of generality, assume $\{B_x\}$ is finite, thus there exists a finite number of balls $\{B_i\}_{i=1}^N$ such that

$$\bigcup_x B_x \subset \bigcup_{i=1}^N (1 + \lambda^{-\frac{1}{2}})B_i,$$

and

$$\sum_{i=1}^N \chi_{B_i}(x) \leq c\lambda^{\frac{n}{4}} \log \lambda,$$

in which c is a constant independent of λ.

We then define

$$E_1 = \left[\bigcup_{i=1}^N (1 + \lambda^{-\frac{1}{2}})B_i\right] \cap B.$$

Then, $E \subset E_1 \subset B$, and we shall show

$$|E| \leq \left[1 - c_3\lambda^{-\frac{3n}{4}}(\log \lambda)^{-1}\right]|E_1|. \tag{3.10}$$

Wit that

$$\begin{aligned}
|E_1| &= |E| + |\left[\bigcup_{i=1}^N (1 + \lambda^{-\frac{1}{2}})B_i\right] \cap B \setminus E| \\
&\geq |E| + |\left(\bigcup_{i=1}^N B_i\right) \cap B \setminus E| \\
&= |E| + |\bigcup_{i=1}^N B_i \setminus E| \\
&\geq |E| + c(n)^{-1}\lambda^{-\frac{n}{4}}(\log \lambda)^{-1} \sum_i^N |B_i \setminus E|,
\end{aligned}$$

where we used the overlapping condition in the covering lemma above. By our selection,

$$|B_i \setminus E| = \overline{c}\lambda^{-\frac{n}{2}}|B_i|,$$

and therefore,

$$\begin{aligned}
|E_1| &\geq |E| + c(n)^{-1}\lambda^{-\frac{n}{4}}(\log\lambda)^{-1}\sum_i^N |B_i \setminus E| \\
&= |E| + c(n)^{-1}\overline{c}\lambda^{-\frac{3n}{4}}(\log\lambda)^{-1}(1+\lambda^{-\frac{1}{2}})^{-n}\sum_i^N |(1+\lambda^{-\frac{1}{2}})B_i| \\
&\geq |E| + c_3\lambda^{-\frac{3n}{4}}(\log\lambda)^{-1}|E_1|
\end{aligned}$$

by setting $c^{-1}\overline{c}(1+\lambda^{-1/2})^{-n} = c_3$, note that $c_3 \leq \overline{c}$ and (3.10) follows. Thus,

$$|E_1| \geq \left[1 - c_3\lambda^{-\frac{3n}{4}}(\log\lambda)^{-1}\right]^{k-1}|B|,$$

and the proof is finished by induction if we can show

$$\int_E w \geq c_4\lambda^{-\frac{n}{4}}(\log\lambda)^{-1}\int_{E_1} w. \tag{3.11}$$

Using the growth property (3.8) as assumed in Lemma 3.9 and combining it with the covering lemma in Theorem 3.1, we get

$$\int_{E_1} w \leq \sum_{i=1}^N \int_{(1+\lambda^{-\frac{1}{2}})B_i} w \leq c_1\sum_{i=1}^N \int_{B_i} w.$$

By the choice of each B_i: $|B_x \bigcap E| = (1-\overline{c}\lambda^{-\frac{n}{2}})|B_x|$ and the induction assumption on $k=1$, one can show that

$$\int_{B_i\cap E} w \geq \frac{1}{2}\int_{B_i} w,$$

which is

$$\int_{B_i} w \leq 2\int_{B_i\cap E} w,$$

and therefore,

$$\begin{aligned}\int_{E_1} w &\leq 2c_1 \sum_{i=1}^{N} \int_{B_i} w \\ &= 2c_1 \sum_{i=1}^{N} \int_{E} \chi_{B_i} w \\ &= 2c_1 \int_E (\sum_{i=1}^{N} \chi_{B_i}) w \\ &\leq c(n) c_1 \lambda^{\frac{n}{4}} \log \lambda \int_E w.\end{aligned}$$

Therefore, (3.11) is proved for some $c_3 = c_3(c_2)$, $c_4 = c_4(c_1)$. This completes the proof of Lemma 3.10. $\square$

Now with the help of Lemma 3.10, we are ready to show Lemma 3.9, which deduces Theorem 3.6.

Proof of Lemma 3.9. Without loss of generality, we assume

$$\frac{1}{|B|} \int_B w = 1,$$

and it suffices to show that for $t > 0$,

$$|B \cap \{w > t\}| \leq t^{-c\lambda^{-\frac{3n}{4}} (\log \lambda)^{-2}} |B|.$$

Write $E = B \cap \{w < t\}$, select k such that

$$|E| \sim \left[1 - c_3 \lambda^{-\frac{3n}{4}} (\log \lambda)^{-1}\right]^k |B|,$$

then,

$$k \sim c\lambda^{\frac{3n}{4}} \log \lambda \log(|B|/|E|).$$

Therefore, by Lemma 4.10 and normalization of w,

$$|B| = \int_B w \le \left(c_4 \lambda^{\frac{n}{4}} \log \lambda\right)^k \int_E w \le \left(c_4 \lambda^{\frac{n}{4}} \log \lambda\right)^k t|E|.$$

Hence,

$$\frac{|B|}{|E|} \le \left(c_4 \lambda^{\frac{n}{4}} \log \lambda\right)^k t \le \left(\frac{|B|}{|E|}\right)^{c\lambda^{\frac{3n}{4}} (\log \lambda)^2},$$

which implies

$$|B \cap \{w > t\}| \le t^{-c\lambda^{-\frac{3n}{4}} (\log \lambda)^{-2}} |B|.$$

$\square$

Remark. The proof of Theorem 3.7 is similar to the one of Theorem 3.6, instead one use L^∞ norms, see [DF3] and [CM] for details.

3.2.2 Geometric estimate of nodal domains

In this subsection, we apply Theorems 3.6 and 3.7 in the previous subsection on the BMO estimates of $\log |u|$ to obtain a geometric estimate of nodal domains locally.

Theorem 3.11 (Geometric estimate of nodal domains)**.** *For $n \ge 3$, let $B \subset \mathbb{M}$ be any ball, and $\Omega \subset B$ be any of the connected components of $\{p \in B : u(p) \neq 0\}$. If $\Omega \cap \frac{1}{2}B \neq \emptyset$, then*

$$|\Omega| \ge c\lambda^{-\frac{3n^2}{4} - \frac{n}{2}} (\log \lambda)^{-2n} |B|.$$

Remark. We point out the results (replacing the coefficient on the right-hand side of the inequality in Theorem 3.11):

- $\lambda^{-\frac{n^2(n+2)+n}{2}}$ (Donnelly and Fefferman [DF3])
- $\lambda^{-\frac{n(4n+1)}{2}} (\log \lambda)^{-2n}$ (Chanillo and Muckenhoupt [CM])

- $\lambda^{-\frac{n(8n+1)}{4}}(\log\lambda)^{-4n}$ (Lu [Lu2])

First, we impose a lemma from [DF3]:

Lemma 3.12. *Suppose* $0 < R < \lambda^{-1/2}$, $\Omega \subset B(p,R) \cap \{u > 0\}$, *and* $p \in \Omega$ *with*

$$\frac{|\Omega|}{|B(p,R)|} < \eta^n < \frac{1}{2}\eta_0^n < \frac{1}{2}.$$

Then, there is a positive number r *such that*

(1)

$$0 < r < \frac{\eta}{\eta_0}R,$$

(2)

$$\frac{|\Omega \cap B(p,r)|}{B(p,r)} \geq \eta_0^n,$$

(3)

$$\sup_{\Omega\cap B(p,r)} |u| \leq \left(\frac{r}{R}\right)^{c_5/\eta} \sup_{B(p,R)} |u|.$$

Now we show a consequence of the above lemma with the help Theorem 3.7, by which the geometric estimate of nodal domains in Theorem 4.11 follows easily.

Corollary 3.13. *Assume the same conditions as in Lemma 3.12, then*

$$\frac{|\Omega|}{|B(p,R)|} \geq c\lambda^{-\frac{3n^2}{4}}(\log\lambda)^{-2n}. \tag{3.12}$$

Proof of Corollary 3.13. Assume the condition

$$\frac{|\Omega|}{|B(p,R)|} < \eta^n < \frac{1}{2}\eta_0^n < \frac{1}{2}$$

holds. Then, we prove (3.12) contradiction from a suitable choice of η.

By Theorem 3.7 and (2) in Lemma 3.12,

$$\begin{aligned}
\sup_{B(p,r)} |u| &\leq \left[\frac{c|B(p,r)|}{|\Omega \cap B(p,r)|}\right]^{c\lambda^{\frac{3n}{4}}(\log\lambda)^2} \sup_{\Omega\cap B(p,r)} |u| \\
&\leq (c\eta_0^{-n})^{c\lambda^{\frac{3n}{4}}(\log\lambda)^2} \sup_{\Omega\cap B(p,r)} |u| \\
&\leq (c\eta_0^{-n})^{c\lambda^{\frac{3n}{4}}(\log\lambda)^2} \left(\frac{r}{R}\right)^{c_5/\eta} \sup_{B(p,R)} |u|.
\end{aligned}$$

However, from the vanishing order estimate in Theorem 2.3:

$$\max_{B(p,R)} |u| \leq \left(\frac{cR}{r}\right)^{c\sqrt{\lambda}} \max_{B(p,r)} |u|,$$

which implies

$$(c\eta_0^{-n})^{c\lambda^{\frac{3n}{4}}(\log\lambda)^2} \left(\frac{cr}{R}\right)^{c_5/\eta - c\sqrt{\lambda}} \geq 1,$$

if we assume further $c_5/\eta - c\sqrt{\lambda} \geq 0$, then by (1) in Lemma 3.12,

$$(c\eta_0^{-n})^{c\lambda^{\frac{3n}{4}}(\log\lambda)^2} \left(\frac{c\eta}{\eta_0}\right)^{\frac{c}{\eta} - \sqrt{\lambda}} \geq 1.$$

Now we choose $\eta = \eta_0^2$ and $\eta_0 = c_6\lambda^{-\frac{3n}{8}}(\log\lambda)^{-1}$. This choice forces $c_5/\eta - c\sqrt{\lambda} \geq 0$ and also yields

$$(c\eta_0)^{c_5/\eta - c\sqrt{\lambda} - cn\lambda^{\frac{3n}{4}}(\log\lambda)^2} \geq 1.$$

This is a contradiction as $c\eta_0 < 1$ for small c_6. Therefore,

$$\frac{|\Omega|}{|B(p,R)|} \geq \eta^n = c\lambda^{-\frac{3n^2}{4}}(\log\lambda)^{-2n}.$$

$\square$

Remark. The upper bound in (3.12) is $\lambda^{-\frac{n^2(n+2)}{2}}$ in [DF3], $\lambda^{-2n^2}(\log\lambda)^{-2n}$ in [CM], and $\lambda^{-\frac{n(8n-1)}{4}}(\log\lambda)^{-4n}$ in [Lu2].

Now we prove Theorem 3.11.

Proof of Theorem 3.11. Given any $B \subset \mathbb{M}$ and $\Omega = B \cap \{u > 0\}$, let $p \in \Omega \cap \frac{1}{2}B$, write R_0 as the radius of B and

$$R = \min\left\{\frac{1}{2}\lambda^{-\frac{1}{2}}, \frac{1}{2}R_0\right\},$$

Applying (3.12) in Corollary 3.13 for $B(p, R)$, wit the fact that

$$\Omega \cap B(p, R) \subset \Omega \cap B,$$

therefore,

$$\begin{aligned}
& \frac{|\Omega \cap B|}{|B|} \\
\geq\ & \frac{|\Omega \cap B(p,R)|}{|B|} \\
\geq\ & c\lambda^{-\frac{3n^2}{4}}(\log\lambda)^{-2n}\frac{|B(p,R)|}{|B|} \\
\geq\ & c\lambda^{-\frac{3n^2}{4}}(\log\lambda)^{-2n}\frac{R^n}{R_0^n} \\
\geq\ & c\lambda^{-\frac{3n^2}{4}-\frac{n}{2}}(\log\lambda)^{-2n},
\end{aligned}$$

which completes the proof of Theorem 3.11. □

3.3 In dimension two

In this section, we carry out the computation in dimension two of the following two theorems, they are the analogues of Theorems 3.6 and 3.11, we present the proofs in order to get precise estimates.

Theorem 3.14 (BMO estimate of $\log|u|$)**.** *For $n = 2$,*

$$\|\log|u|\|_{BMO} \leq c\lambda^{\frac{3}{2}+\epsilon},$$

in which $c = c(\epsilon, \mathbb{M})$ depends only on ϵ and $\mathbb{M}$.

Theorem 3.15 (Geometric estimate of nodal domains)**.** *For $n = 2$, let $B \subset \mathbb{M}$ be any ball, and $\Omega \subset B$ be any of the connected components of $\{x \in B : u(x) \neq 0\}$. If $\Omega \cap \frac{1}{2}B \neq \emptyset$, then for any given $\epsilon > 0$,*

$$|\Omega| \geq c\lambda^{-4-\epsilon}|B|,$$

in which $c = c(\epsilon, \mathbb{M})$ depends only on ϵ and $\mathbb{M}$.

Lemma 3.16 (Reverse Hölder's inequality)**.** *Let $1 \leq q < \infty$, then*

$$\left(\frac{1}{|B|}\int_B |u|^q\right)^{\frac{1}{q}} \leq c\sqrt{\lambda}\left(\frac{1}{|B|}\int_B |u|^2\right)^{\frac{1}{2}},$$

in which c depends only on q.

Proof of Lemma 3.15. By the Poincaré's inequality, for any ball B, we have

$$\left(\frac{1}{|B|}\int_B |u - u_B|^q\right)^{\frac{1}{q}} \leq c|B|^{\frac{1}{2}}\left(\frac{1}{|B|}\int_B |\bigtriangledown u|^p\right)^{\frac{1}{p}},$$

where $u_B = \frac{1}{|B|}\int_B u$, $1 < p < 2$, $1/q = 1/p - 1/2$, and $c = c(p)$. Applying Hölder's inequality and L^2 Bernstein's estimate in Lemma 2.4, we obtain

$$\left(\frac{1}{|B|}\int_B |u - u_B|^q\right)^{\frac{1}{q}} \leq c\sqrt{\lambda}\left(\frac{1}{|B|}\int_B |u|^2\right)^{\frac{1}{2}}.$$

By Minkowski's inequality, Lemma 3.16 follows for $2 < q < \infty$, for the case $1 < q \leq 2$, we can apply Hölder's inequality again. $\square$

Theorem 3.14 follows from the following

Lemma 3.17. *Suppose $w > 0$, $q > 2$, $\epsilon > 0$, and $(1 + \epsilon)/q' \geq 1$. Assume also that for any ball B,*

$$\int_{(1+\lambda^{-\frac{1}{2}})B} w \leq c_1 \int_B w, \tag{3.13}$$

and

$$\left(\frac{1}{|B|}\int_B w^q\right)^{\frac{1}{q}} \leq \frac{c_2\lambda}{|B|}\int_B w. \tag{3.14}$$

Then,

$$\|\log w\|_{BMO} \leq c\lambda^{\frac{3}{2}+\epsilon},$$

in which $c = c(c_1, c_2, \epsilon)$ *depends on* c_1, c_2, *and* ϵ.

In order to prove Lemma 3.17, we need the following

Lemma 3.18. *Let* w, q, *and* $0 < \epsilon < 1$ *satisfy the hypothesis of Lemma 3.17,* k *is an integer, let* B *be a fixed ball,* $E \subset B$, *then there exist* c_3 *and* c_4 *such that if*

$$|E| \geq \left[1 - c_3\lambda^{-\frac{3}{2}-\epsilon}(\log\lambda)^{-1}\right]^k |B|,$$

then

$$\int_E w \geq \left[c_4\lambda^{-\frac{1}{2}}(\log\lambda)^{-1}\right]^k \int_B w,$$

in which $c_3 = c_3(c_2)$ *and* $c_4 = c_4(c_1)$.

Proof of Lemma 3.18. We will use induction on k as done in §3.2 in higher dimensions. (See also [Lu1] for the two dimension case). We first verify the lemma for $k = 1$. To do so, we claim that if $\epsilon > 0$, and $|E| \geq (1 - \overline{c}\lambda^{-1-\epsilon})|B|$ for some appropriate $\overline{c} = \overline{c}(c_2)$, then $\int_E w \geq \frac{1}{2}\int_B w$. To show this, we first note that $|B \setminus E| \leq \overline{c}\lambda^{-1-\epsilon}|B|$. If we choose $q > 2$ such that $\frac{1+\epsilon}{q'} \geq 1$. Thus, by Hölder's inequality and (3.14),

$$\begin{aligned}
\int_{B\setminus E} w &\leq \left(\int_B w^q\right)^{1/q} |B \setminus E|^{1/q'} \\
&\leq c_2\overline{c}^{\frac{1}{q'}}\lambda^{-\frac{1+\epsilon}{q'}+1}\int_B w \\
&\leq c_2\overline{c}^{\frac{1}{q'}}\int_B w.
\end{aligned}$$

If we choose $\overline{c}$ such that $c_2\overline{c}^{\frac{1}{q'}} < \frac{1}{2}$, then

$$\int_{B\setminus E} w \leq \frac{1}{2}\int_B w,$$

this implies

$$\int_E w > \frac{1}{2}\int_B w.$$

Note that the choice of $\overline{c}$ is dependent on ϵ since $c_2 = c_2(q)$, and q is dependent on ϵ. Thus, if $c_3 \leq \overline{c}$, and $|E| \geq (1 - c_3\lambda^{-\frac{3}{2}-\epsilon}(\log\lambda)^{-1})|B|$, then

$$\int_E w \geq \frac{1}{2}\int_B w \geq c_4\left[\lambda^{-\frac{1}{2}}(\log\lambda)^{-1}\right]\int_B w,$$

and we are done for the case $k = 1$.

Now we assume the statement is true for $k-1$. We may assume $|E| \leq (1 - \overline{c}\lambda^{-1-\epsilon})|B|$, otherwise, there is nothing to prove. Thus, for each density point x of E, we can select a ball $B_x \subset B$ such that $x \in B_x$, and

$$\frac{|B_x \cap E|}{|B_x|} = 1 - \overline{c}\lambda^{-1-\epsilon}.$$

Applying the cover lemma Theorem 3.1 when $n = 2$ to the balls B_x with the choice $\delta = \lambda^{-1/2}$, and without loss of generality, assume $\{B_x\}$ is finite, thus there exist a finite number of balls $\{B_i\}_{i=1}^N$ such that

$$\bigcup_x B_x \subset \bigcup_{i=1}^N (1 + \lambda^{-\frac{1}{2}})B_i,$$

and

$$\sum_{i=1}^N \chi_{B_i}(x) \leq c\lambda^{\frac{1}{4}}\log\lambda,$$

in which c is a constant independent of λ.

We then define

$$E_1 = \left[\bigcup_{i=1}^{N} (1 + \lambda^{-\frac{1}{2}}) B_i \right] \cap B.$$

Then, $E \subset E_1 \subset B$, and similarly as we did in §3.2, we can show

$$|E| \leq \left[1 - c_3 \lambda^{-\frac{3}{2}-\epsilon} (\log \lambda)^{-1} \right] |E_1|,$$

and

$$\int_E w \geq c_4 \lambda^{-\frac{1}{2}} (\log \lambda)^{-1} \int_{E_1} w$$

for some $c_3 = c_3(c_2)$ and $c_4 = c_4(c_1)$. This suffices to complete the proof. □

Now we prove Theorem 3.14.

Proof of Theorem 3.14. Without loss of generality, we assume

$$\frac{1}{|B|} \int_B w = 1,$$

and it suffices to show that for $t > 0$,

$$|B \cap \{w < t\}| \leq t^{c\lambda^{-\frac{3}{2}-\epsilon} (\log \lambda)^{-2}} |B|.$$

Write $E = B \cap \{w < t\}$. Select k such that

$$|E| \sim \left[1 - c_3 \lambda^{-\frac{3}{2}-\epsilon} (\log \lambda)^{-1} \right]^k |B|,$$

then,

$$k \sim c \left(\lambda^{\frac{3}{2}+\epsilon} \log \lambda \right) \log(|B|/|E|).$$

Therefore, by Lemma 3.18 and the normalization of w, we have

$$|B| = \int_B w \leq \left(c_4^{-1} \lambda^{\frac{1}{2}} \log \lambda \right)^k \int_E w \leq \left(c_4^{-1} \lambda^{\frac{1}{2}} \log \lambda \right)^k t|E|.$$

Thus,

$$\begin{aligned}
\frac{|B|}{|E|} &\leq te^{k\log\left(c_4^{-1}\lambda^{\frac{1}{2}}\log\lambda\right)} \\
&\leq t\left(\frac{|B|}{|E|}\right)^{\left(c\lambda^{\frac{3}{2}+\epsilon}\log\lambda\right)\log\left(c_4^{-1}\lambda^{\frac{1}{2}}\log\lambda\right)} \\
&\leq t\left(\frac{|B|}{|E|}\right)^{c\lambda^{\frac{3}{2}+\epsilon}(\log\lambda)^2}.
\end{aligned}$$

That is,

$$|E| \leq t^{c\lambda^{-\frac{3}{2}-\epsilon}(\log\lambda)^{-2}}|B|.$$

Since ϵ is arbitrary, we can then have

$$|E| \leq t^{c\lambda^{-\frac{3}{2}-\epsilon}}|B|.$$

$\square$

We omit the proof of Theorem 3.15 since it is similar to the one of Theorem 3.11.

4 Hausdorff measure estimates of nodal sets: lower bounds

In this chapter, we discuss the lower bounds of $\mathcal{H}^{n-1}$ measure estimates of nodal sets of eigenfunctions, i.e., the left-hand side of Conjecture 1.1.

The conjecture was verified by Donnelly and Fefferman [DF1] on analytic manifolds. The real analytic assumption is used in a crucial way: the eigenfunctions are analytically continued to holomorphic functions with bounded growth, and then the problem is reduced to a problem about polynomials.

The problem on smooth manifolds seems much more difficult, several sequences of findings lie in this area, and we present these results as follows (omitting the constants):

- Two dimensional: $\sqrt{\lambda}$ (Brüning [B], Yau [Y]) *Optimal!*
- $\lambda^{\frac{(1-n)(n+1)^2}{2}}$ (Donnelly and Fefferman [DF3])
- $\lambda^{\frac{(1-n)(4n+1)}{2}}(\log\lambda)^{2(1-n)}$ (Chanillo and Muckenhoupt [CM])
- $\lambda^{\frac{(1-n)(8n+1)}{4}}(\log\lambda)^{4(1-n)}$ (Lu [Lu2])
- $c^{-\sqrt{\lambda}}$ (Han and Lin [HL])
- $\lambda^{\frac{(1-n)(3n+2)}{4}}(\log\lambda)^{2(1-n)}$ (H. and Lu [HaLu1])
- $\lambda^{\frac{7-3n}{8}}$ (Sogge and Zelditch [SZ])
- $\lambda^{\frac{3-n}{4}}$ (Colding and Minicozzi [CoMi], Hezari and Sogge [HeSo]) *Best!*
- $\lambda^{\frac{3-n}{2}-\frac{1}{2n}}$ (Mangoubi [M])
- $\lambda^{\frac{1-n}{2}}$ (H. and Lu [HaLu2])

- $\lambda^{\frac{17-5n}{16}}$ (Hezari and Wang [HW])

In which the lower bound $c\lambda^{\frac{3-n}{4}}$ by Colding and Minicozzi [CoMi] and Hezari and Sogge [HeSo] is the best known so far. However, it is still far from the optimal bound.

4.1 Isoperimetric inequalities and polynomially decreasing lower bounds

Let us begin by a remark [HaLu1] of a polynomially decreasing lower bound from the geometric estimate of nodal domains in Theorem 3.11: Choosing $B = \mathbb{M}$ such that Ω is a connected component of $\{u > 0\}$, then

$$|\Omega| \geq c\lambda^{-\frac{3n^2}{4}-\frac{n}{2}}(\log\lambda)^{-2n}.$$

Then, from isoperimetric inequality on $\mathbb{M}$ (See e.g. [O].)

$$|\partial\Omega| \geq c|\Omega|^{\frac{n-1}{n}},$$

we immediately derive that

$$\mathcal{H}^{n-1}(\mathcal{N}) \geq |\partial\Omega| \geq c\lambda^{\frac{(1-n)(3n+2)}{4}}(\log\lambda)^{2(1-n)}.$$

One can similarly derive the lower bounds in [DF3, CM, Lu2] as showed in the beginning of this chapter. Next we improve these results by introducing more delicate strategies.

Generally on smooth manifolds, there are two major directions of proving lower bounds in Conjecture 1.1:

(1) local to global estimate,

(2) direct global estimate.

In (1), one begins by examining the size of nodal set locally(with a local scale we shall discuss in the following sections), then uses a covering to recover the size of $\mathcal{N}$ globally. This idea was originated by Donnelly and Fefferman [DF1], and the major tool to get the local $\mathcal{H}^{n-1}$ lower bounds of $\mathcal{N}$ is the isoperimetric inequality. (See [F].)

Theorem 4.1 (Isoperimetric inequality)**.** *Given a ball $B \subset \mathbb{R}^n$ and a continuous function f on B, denote $B^+ = B \cap \{f > 0\}$, and $B^- = B \setminus B^+$, then*

$$\mathcal{H}^{n-1}(B \cap \mathcal{N}) \geq c(n) \left[\min\{\mathcal{H}^n(B^+), \mathcal{H}^n(B^-)\}\right]^{\frac{n-1}{n}},$$

in which $\mathcal{N} = \{f = 0\}$.

Then, in every small ball, one of the following three cases occurs.

(a) $\mathcal{H}^n(B^+) \sim \mathcal{H}^n(B^-)$,

(b) $\mathcal{H}^n(B^+) \gg \mathcal{H}^n(B^-)$,

(c) $\mathcal{H}^n(B^+) \ll \mathcal{H}^n(B^-)$.

Remark. Following this line, excluding cases (b) and (c), one has the lower bound of $\min\{\mathcal{H}^n(B^+), \mathcal{H}^n(B^-)\}/\mathcal{H}^n(B)$, and therefore, the lower bound of $\mathcal{H}^{n-1}(B \cap \mathcal{N})$ follows from the isoperimetric inequality in Theorem 4.1.

To demonstrate the idea in (1), we now first combine the isoperimetric inequality in Theorem 3.1 together with geometric estimates of nodal domain in Corollary 3.13 to prove a polynomially decreasing lower bound, which improves the one in [HaLu1].

Theorem 4.2.

$$\mathcal{H}^{n-1}(\mathcal{N}) \geq c\lambda^{-\frac{3n^2-3n-2}{4}}(\log\lambda)^{2(1-n)}.$$

Proof of Theorem 4.2. Choose a maximum family of disjoint balls $B \subset \mathbb{M}$ with radius $O(\lambda^{-1/2}) > 3\lambda^{-1/2}$ such that u vanishes at some point in the middle one-third of every ball, i.e. there exist $p_1, p_2 \in \frac{1}{3}B$ such that $u(p_1) > 0$ and $u(p_2) < 0$. By taking $R = \lambda^{-1/2}$ in (3.12) of Corollary 3.13, we have

$$|B^+(p_1, R)| \geq c\lambda^{-\frac{3n^2}{4}}(\log\lambda)^{-2n}|B(p_1, R)|,$$

and

$$|B^-(p_2, R)| \geq c\lambda^{-\frac{3n^2}{4}}(\log\lambda)^{-2n}|B(p_2, R)|.$$

Observe that $B^+(p_1, R) \subset B^+$ and $B^-(p_2, R) \subset B^-$, therefore,

$$|B^+| \geq |B^+(p_1, R)| \geq c\lambda^{-\frac{n(3n+2)}{4}}(\log\lambda)^{-2n},$$

and

$$|B^-| \geq |B^-(p_1, R)| \geq c\lambda^{-\frac{n(3n+2)}{4}}(\log\lambda)^{-2n},$$

applying the isoperimetric inequality in Theorem 4.1, we arrive at

$$\mathcal{H}^{n-1}(B \cap \mathcal{N}) \geq c\lambda^{-\frac{(n-1)(3n+2)}{4}}(\log\lambda)^{2(1-n)}.$$

The theorem is proved by noting the number of such balls is $O(\lambda^{n/2})$. $\square$

Remark. Using the estimates in (3.12) of Corollary 3.13 in [DF3, CM, Lu2], one improves the corresponding polynomially decreasing lower bounds to

- $\lambda^{-\frac{n^3+n^2-3n}{2}}$ (Donnelly and Fefferman [DF3])

- $\lambda^{-\frac{n(4n-5)}{2}}(\log\lambda)^{2(1-n)}$ (Chanillo and Muckenhoupt [CM])

- $\lambda^{-\frac{8n^2-11n+1}{4}}(\log\lambda)^{4(1-n)}$ (Lu [Lu2])

In (2), Dong-Sogge-Zelditch's integral formula and Sogge's L^p estimate play important roles, and we shall investigate in §4.3.

4.2 Local to global approach

We begin this section by showing a exponentially decreasing lower bound, which is not better that the one in Theorem 3.2. However, this detour is beneficial since an approach is developed in the pass, and substantial improvement is to be made.

4.2.1 $c^{-\sqrt{\lambda}}$–$\mathcal{H}^{n-1}$ lower bound

First, we need a crucial lemma which shows the comparison between the integrals of $|u|$ over positive and negative domains. (See Lemma 5 in [CoMi], cf. also Lemma 6.2.2 in [HL] for a simpler but more restrictive version.)

Lemma 4.3. *There exists $r_3 > 0$ depending only on $\mathbb{M}$ such that if $r \in (0, r_3]$, and $u(p) = 0$, then*

$$\left|\int_{B(p,r)} u\right| \leq \frac{1}{3}\int_{B(p,r)} |u|, \tag{4.1}$$

in which $u^+ = \max\{u, 0\}$ and $u^- = u^+ - u$.

Remark. Note that

$$\left|\int_{B(p,r)} u\right| = \left|\int_{B(p,r)} u^+ - \int_{B(p,r)} u^-\right|,$$

and

$$\int_{B(p,r)} |u| = \int_{B(p,r)} u^+ + \int_{B(p,r)} u^-,$$

thus,

$$\begin{aligned}\int_{B(p,r)} u^+ &= \frac{1}{2}\left(\int_{B(p,r)} |u| - \int_{B(p,r)} u^- + \int_{B(p,r)} u^+\right)\\ &\geq \left(\int_{B(p,r)} |u| - \left|\int_{B(p,r)} u\right|\right)\\ &\geq \frac{1}{3}\int_{B(p,r)} |u|,\end{aligned}$$

similarly,

$$\int_{B(p,r)} u^- \geq \frac{1}{3}\int_{B(p,r)} |u|,$$

and these imply

$$\min\left\{\int_{B(p,r)} u^+, \int_{B(p,r)} u^-\right\} \geq \frac{1}{3}\int_{B(p,r)} |u|.$$

Therefore, we see that Lemma 4.3 establishes the comparability of $\int_{B(p,r)} u^+$ and $\int_{B(p,r)} u^-$, in order to pass this property to $\mathcal{H}^n(B^+(p,r))$ and $\mathcal{H}^n(B^-(p,r))$, we need the following lemma (cf. Lemma 6.2.4 in [HL]).

Lemma 4.4. *If there exist c_1 and c_2 positive such that*

$$\min\left\{\int_{B(p,r)} u^+, \int_{B(p,r)} u^-\right\} \geq c_1 \int_{B(p,r)} |u| \tag{4.2}$$

and

$$\int_{B(p,r)} u^2 \leq c_2 \left(\int_{B(p,r)} |u|\right)^2, \tag{4.3}$$

then

$$\mathcal{H}^{n-1}(B(p,r) \cap \mathcal{N}) \geq c \cdot c_2^{\frac{1-n}{n}},$$

in which $c = c(c_1)$ depends on c_1.

Proof of Lemma 4.4. By Hölder's inequality and (4.3), we have for any measurable set $G \subset B(p,r)$,

$$\left(\int_G |u|\right)^2 \leq \mathcal{H}^n(G) \int_G u^2 \leq \mathcal{H}^n(G) \int_{B(p,r)} u^2 \leq c_2 \mathcal{H}^n(G) \left(\int_{B(p,r)} |u|\right)^2,$$

therefore,

$$\mathcal{H}^n(G) \geq c_2^{-1} \frac{\left(\int_G |u|\right)^2}{\left(\int_{B(p,r)} |u|\right)^2}.$$

By (4.2) and taking $G = B^{\pm}(p,r)$, we get

$$\min\{\mathcal{H}^n(B^+(p,r)), \mathcal{H}^n(B^-(p,r))\} \geq c_1^2 c_2^{-1},$$

together with the isoperimetric inequality in Theorem 4.1,

$$\mathcal{H}^{n-1}(B(p,r) \cap \mathcal{N}) \geq c \left[\min\{\mathcal{H}^n(B^+(p,r)), \mathcal{H}^n(B^-(p,r))\}\right]^{\frac{n-1}{n}} \geq c \cdot c_2^{\frac{1-n}{n}}.$$

□

Next we fix a local scale $r_\lambda = O(\lambda^{-1/2})$ (i.e., the "wavelength" in Theorem 1.5) such that u vanishes at some point in the middle half of every ball with radius r_λ, and $c^{-\sqrt{\lambda}}$–$\mathcal{H}^{n-1}$ lower bound of $\mathcal{N}$ follows easily from the doubling condition in Theorem 2.1 if we cover $\mathbb{M}$ by these balls.

Theorem 4.5 ($c^{-\sqrt{\lambda}}$–$\mathcal{H}^{n-1}$ lower bound)**.**

$$\mathcal{H}^{n-1}(\mathcal{N}) \geq c^{-\sqrt{\lambda}}.$$

Proof of Theorem 4.5. We choose a maximum disjoint family of balls B with radius r_λ in $\mathbb{M}$, then the number of these balls $\sim \lambda^{n/2}$, and there exists $p \in \frac{1}{2}B$ such that $u(p) = 0$. That is, $B(p, \frac{1}{2}r_\lambda)$'s are also mutually disjoint.

Applying a priori estimate in Theorem 2.5,

$$\sup_{B(p,\frac{1}{2}r_\lambda)} u^2 \le c \fint_{B(p,r_\lambda)} u^2 \le c^{\sqrt{\lambda}} \fint_{B(p,\frac{1}{2}r_\lambda)} u^2,$$

thus,

$$\left(\int_{B(p,\frac{1}{2}r_\lambda)} u^2\right)^2 \le \sup_{B(p,\frac{1}{2}r_\lambda)} u^2 \left(\int_{B(p,\frac{1}{2}r_\lambda)} |u|\right)^2 \le c^{\sqrt{\lambda}} \fint_{B(p,\frac{1}{2}r_\lambda)} u^2 \left(\int_{B(p,\frac{1}{2}r_\lambda)} |u|\right)^2,$$

and

$$\int_{B(p,\frac{1}{2}r_\lambda)} u^2 \le r_\lambda^{-n} c^{\sqrt{\lambda}} \left(\int_{B(p,\frac{1}{2}r_\lambda)} |u|\right)^2.$$

By Lemma 4.4 with $c_1 = \frac{1}{3}$ from Lemma 4.3 and $c_2 = r_\lambda^{-n} c^{\sqrt{\lambda}} = \lambda^{\frac{n}{2}} c^{\sqrt{\lambda}}$ from the above inequality, we have

$$\mathcal{H}^{n-1}(B(p, \frac{1}{2}r_\lambda) \cap \mathcal{N}) \ge c \cdot c_2^{\frac{1-n}{n}} = \lambda^{\frac{1-n}{2}} c^{-\sqrt{\lambda}}.$$

We complete the proof here by noticing the number of such balls $\sim \lambda^{n/2}$. □

4.2.2 $\lambda^{\frac{1-n}{2}}$–$\mathcal{H}^{n-1}$ lower bound

In this subsection, we choose a different local scale independent of λ cooperated with local maximum principle in Corollary 2.6 to achieve local lower bound of nodal set. It coincides with [HaLu2] in this subsection.

Theorem 4.6 ($\lambda^{\frac{1-n}{2}}$–$\mathcal{H}^{n-1}$ lower bound)**.**

$$\mathcal{H}^{n-1}(\mathcal{N}) \ge c\lambda^{\frac{1-n}{2}}.$$

Proof of Theorem 4.6. We fix a nodal point p (i.e., $u(p) = 0$) and $0 < R \le \min\{r_1, r_2, r_3\}$ such that the growth estimate in Theorem 2.2, the local maximum principle in Corollary

2.6, and (4.1) in Lemma 4.3 all hold. Then, in Lemma 4.4, we take $c_1 = \frac{1}{3}$. For c_2, we take $\delta = \lambda^{-1/2}$,

$$
\begin{aligned}
&\left(\int_{B(p,R)} u^2\right)^2 \\
\leq &\sup_{B(p,R)} u^2 \left(\int_{B(p,R)} |u|\right)^2 \\
\leq &c(\lambda^{-1/2}R)^{-n} \left(\int_{B(p,(1+\lambda^{-\frac{1}{2}})R)} u^2\right) \left(\int_{B(p,R)} |u|\right)^2 \\
\leq &c(\lambda^{-1/2}R)^{-n} \left(\int_{B(p,R)} u^2\right) \left(\int_{B(p,R)} |u|\right)^2,
\end{aligned}
$$

where we applied Theorem 2.2 and Corollary 2.6. Thus,

$$
\int_{B(p,R)} u^2 \leq c(\lambda^{-1/2}R)^{-n} \left(\int_{B(p,R)} |u|\right)^2.
$$

Choosing $c_2 = c(\lambda^{-1/2}R)^{-n} = c\lambda^{n/2}$ in Lemma 4.4, we deduce

$$
\mathcal{H}^{n-1}(B(p,R) \cap \mathcal{N}) \geq c \cdot c_2^{\frac{1-n}{n}} = c\lambda^{\frac{1-n}{2}},
$$

which implies the global lower bound in Theorem 4.6. □

Remark. The local scale R depends only on $\mathbb{M}$. However, there is an intimate connection between the results involving local scales R and r_λ, it will be revealed in §4.2.3 cooperated with an innovative tool invented by Colding and Minicozzi [CoMi], by which the best known lower bound is deduced from Sogge's L^p estimate.

4.2.3 $\lambda^{\frac{3-n}{4}}$–$\mathcal{H}^{n-1}$ lower bound

First, we normalize the eigenfunction, that is, $\|u\|_{L^2(\mathbb{M})} = 1$. Then, we set the same local scale $r_\lambda = O(\lambda^{-1/2})$ for which every ball with radius r_λ vanishes at some point in the middle half, one knows from Theorem 2.1 that doubling condition holds with an exponentially increasing (of λ) constant, (and therefore, the exponentially decreasing local lower bound follows as in Theorem 4.5.) With a stronger doubling condition however, one can easily improve this local result.

Corollary 4.7. *Assume there exists $d \geq 1$ such that the doubling condition*

$$\int_{B(p,2r_\lambda)} u^2 \leq d \int_{B(p,r_\lambda)} u^2 \tag{4.4}$$

holds for $B(p, r_\lambda)$, then

$$\mathcal{H}^{n-1}(B(p, r_\lambda) \cap \mathcal{N}) \geq c(d)\lambda^{\frac{1-n}{2}}, \tag{4.5}$$

in which $c(d)$ depends on d.

Remark. In analytic case, there are sufficiently many good balls (proportional to the volume of $\mathbb{M}$, and therefore, the number of such balls $\sim \lambda^{n/2}$), the optimal lower bound follows by a covering.

A simple observation shows that this local lower bound (4.5) coincides with the one in Theorem 4.6, and the connection is proved by the following theorem.

Theorem 4.8. *There is at least one d-good ball in $B(p, R)$ if d is sufficiently large, where a d-good ball satisfies (4.4).*

Proof of Theorem 4.8. We cover $B(p, R)$ by finite balls $\{B_j | B_j \subset B(p, (1 + \lambda^{-1/2})R)\}$ with radius r_λ, and the overlapping is bounded by a constant C depending only on $\mathbb{M}$ since $r_\lambda \sim \lambda^{-1/2}$. Then, there exists a constant c such that

$$2B_j \subset B(p, (1 + c\lambda^{-\frac{1}{2}})R).$$

Thus, if all balls are d-bad, that is,

$$\int_B u^2 \leq d^{-1} \int_{2B} u^2, \tag{4.6}$$

then, omitting the integrating function $|u|^2$,

$$\int_{B(p,R)} \leq \int_{\bigcup B_j} \leq \sum_j \int_{B_j} \leq d^{-1} \sum \int_{2B_j} \leq Cd^{-1} \int_{\bigcup_j 2B_j} \leq Cd^{-1} \int_{B(p,(1+c\lambda^{-\frac{1}{2}})R)}.$$

The contradiction comes from Theorem 2.2 that

$$\int_{B(p,R)} u^2 \sim \int_{B(p,(1+c\lambda^{-\frac{1}{2}})R)} u^2$$

by choosing d large, which means there is at leat one d-good ball in $B(p,R)$. $\square$

Remark. From proof of Theorem 4.8, we see that the L^2 weight of u on the bad region (union of bad balls) is relatively small comparing with the L^2 weight on the good region (union of good balls). However, existence of good balls in $B(p,R)$ is independent of $\int_{B(p,R)} u^2$.

We end our analysis in this subsection by outlining Colding and Minicozzi's proof [CoMi] on estimating the number of disjoint good balls. Recall Sogge's L^p estimate in Theorem 2.8,

Corollary 4.9. *For $\Omega \subset \mathbb{M}$,*

$$\|u\|_{L^2(\Omega)} \leq |\Omega|^{\frac{p-2}{2p}} \|u\|_{L^p(\Omega)} \leq |\Omega|^{\frac{p-2}{2p}} \|u\|_{L^p(\mathbb{M})}.$$

Particularly, if $|\Omega| = \lambda^{-s}$ for $s \geq 0$, choosing $p = \frac{2(n+1)}{n-1}$ in Theorem 2.8, we have

$$\|u\|_{L^2(\Omega)} \leq \lambda^{\frac{1}{2(n+1)}(\frac{n-1}{2}-2s)},$$

which implies $\|u\|_{L^2(\Omega)} \to 0$ if $s > \frac{n-1}{4}$ as $\lambda \to \infty$.

Remark. The local Weyl law yields $\|u\|_{L^\infty(\mathbb{M})} \leq c\lambda^{\frac{n-1}{4}}$ (i.e. $p = \infty$ in Sogge's L^p estiamtes), thus,

$$\|u\|_{L^2(\Omega)} \leq \lambda^{\frac{n-1}{4}-\frac{s}{2}},$$

this implies $\|u\|_{L^2(\Omega)} \to 0$ if $s > \frac{n-1}{2}$, which is weaker than the result of Corollary 4.9. In fact, the estimate when $p = \frac{2(n+1)}{n-1}$ is sharpest among all $p \geq 2$ and the one from Sobolev's inequality, see Remarks 1, 2, and 3 in [CoMi].

Since the L^2 weight concentrates on the good region (union of good balls with controllable overlap), that is, $\|u\|_{L^2(G)} \geq c$ where G is the good region. Thus, $|G|$ is at least $\lambda^{\frac{1-n}{4}}$ by Corollary 4.9, and the number of d-good balls is at least $\lambda^{\frac{n+1}{4}}$, and consequently the $\lambda^{\frac{3-n}{4}}$ lower bound follows. We refer the reader to [CoMi] for more details.

4.3 Direct global approach

In the second approach, Dong-Sogge-Zelditch's integral formula in Theorem 2.7 and Sogge's L^p estimates in Theorem 2.8 play important roles.

One arrives at the lower bound of $\mathcal{H}^{n-1}(\mathcal{N})$ from (2.1)

$$\lambda\|u\|_{L^1(\mathbb{M})} \leq \lambda \int_{\mathbb{M}} |u| \; d\mathcal{H}^n = 2\int_{\mathcal{N}} |\bigtriangledown u| d\mathcal{H}^{n-1} \leq 2\mathcal{H}^{n-1}(\mathcal{N})\|\bigtriangledown u\|_{L^\infty(\mathcal{N})} \tag{4.7}$$

if one has lower bound of $\|u\|_{L^1(\mathbb{M})}$ and upper bound of $\|\bigtriangledown u\|_{L^\infty(\mathcal{N})}$ for normalized u. Accordingly,

$$\|\bigtriangledown u\|_{L^\infty(\mathbb{M})} \leq c\lambda^{\frac{n+1}{4}}$$

by local Weyl's law, and from Corollary 2.9,

$$\|u\|_{L^1(\mathbb{M})} \geq c\lambda^{\frac{1-n}{8}}.$$

$\lambda^{\frac{7-3n}{8}}$ lower bound follows by combing exponents (Sogge and Zeltitch [SZ]). The above estimates of $\|u\|_{L^1(\mathbb{M})}$ and of $\| \bigtriangledown u\|_{L^\infty(\mathcal{N})}$ are sharp, which are achieved by zonal spherical harmonics and highest weight spherical harmonic on $\mathbb{S}^2$. However, as pointed out by Sogge and Zelditch [SZ] and proved by Herazi and Wang [HW], one replaces the right-hand side of (4.7) with a bound involving L^p bound of $\bigtriangledown u$:

$$2\left[\mathcal{H}^{n-1}(\mathcal{N})\right]^{\frac{2}{3}} \| \bigtriangledown u\|_{L^3(\mathcal{N})},$$

which can be estimated by plugging $f = | \bigtriangledown u|^2$ into (2.2) of Theorem 2.7,

$$\int_{\mathbb{M}} \left[(\Delta + \lambda)| \bigtriangledown u|^2\right]|u|d\mathcal{H}^n = 2\int_{\mathcal{N}} | \bigtriangledown u|^3 d\mathcal{H}^{n-1} = 2\| \bigtriangledown u\|^3_{L^3(\mathcal{N})}.$$

A better lower bound of $\lambda^{\frac{17-5n}{16}}$ follows by further utilizing Bochner's indentity for $\Delta|\bigtriangledown u|^2$ on the left-hand side of above equation (Herazi and Wang [HW]). Recently, using this direct global approach, the lower bound was pushed to $\lambda^{\frac{3-n}{4}}$ (the best know as first shown by Colding and Minicozzi [CoMi], see §3.2.3) by Herazi and Sogge [HeSo] choosing a different test function $f = (1 + \lambda u^2 + | \bigtriangledown u|^2)^{1/2}$. They also showed the sharpness of this bound if one attacks the problem from this approach. (It is a "natural" lower bound!)

4.4 Further investigation

We outline some potential connections between these two approaches and possible improvement on lower bound of $\mathcal{H}^{n-1}(\mathcal{N})$ here.

(i) Can we better utilize the BMO estimates to get improved lower bounds?

(ii) As we see in §4.2, doubling condition is crucial for estimating size of nodal set locally, so

(A) how is it related with the local scale?

(B) Sogge's L^p estimates give the best known lower bound on the number of good balls (i.e., local balls satisfying good doubling condition), can we say anything on the reverse side? That is, can doubling condition imply L^p bounds of eigenfunctions?

(iii) Theorem 4.8 indicates a rather loose existence result on the good balls: There is at least one good ball in $B(p, R)$, here R depends only on $\mathbb{M}$, i.e., independent of the distribution of $|u|^2$ on $\mathbb{M}$, which is the main obstacle for improving known results. One probably have to cooperate with proper covering lemmas to further utilize this advantage (since one only needs a good covering of $\mathbb{M}$).

(iv) A modification of d-good balls as we take consideration of $\lambda \to \infty$:

$$\int_{2B} u^2 \leq \lambda^\epsilon \int_B u^2, \tag{4.8}$$

for $\epsilon > 0$, and we call a ball satisfies (4.8) is ϵ-good.

(v) Instead of balls, we consider good points and discuss with more details in the following.

Definition 4.10 (d-good point)**.** *$p \in \mathbb{M}$ is d-good (or ϵ-good as in (4.8) above) if there exists a d-good $B \subset \mathbb{M}$ with radius*[i] *R such that $B \ni p$.*

One immediately see that if $\Omega = \{p \in \mathbb{M} | \text{p is } d\text{-good}\}$, then $\Omega \subset \mathbb{M}$ is open, furthermore, we derive from growth estimate in Theorem 2.2 as follows.

Proposition 4.11. *If p_0 is d-good, then p is cd-good for all $p \in B(p_0, \lambda^{-1/2} R)$.*

[i]One can consider different local scales, e.g. $r_\lambda = O(\lambda^{-1/2})$.

Proof of Proposition 4.11. Since p_0 is d-good, then there exists $B(p_1, R) \subset \mathbb{M}$ such that $B(p_1, R) \ni p_0$ and

$$\int_{B(p_1,2R)} u^2 \leq d \int_{B(p_1,R)} u^2,$$

and for each $p \in B(p_0, \lambda^{-1/2}R)$, there exists $p_2 \in B(p_1, \lambda^{-1/2}R)$ such that $p \in B(p_2, R)$, and we have that facts that

$$B(p_1, R) \subset B(p_2, (1 + \lambda^{-\frac{1}{2}})R), \text{ and } B(p_2, 2R) \subset B(p_1, 2(1 + \lambda^{-\frac{1}{2}})R),$$

and therefore, by growth estimate in Theorem 2.2,

$$\int_{B(p_2,2R)} \leq \int_{B(p_1,2(1+\lambda^{-\frac{1}{2}})R)} \leq c \int_{B(p_1,2R)} \leq cd \int_{B(p_1,R)} \leq cd \int_{B(p_2,(1+\lambda^{-\frac{1}{2}})R)} \leq c^2 d \int_{B(p_2,R)},$$

which means $B(p_2, R) \ni p$ is cd-good, and thus, p is cd-good. $\square$

5 Hausdorff measure estimates of nodal sets: upper bounds

In this chapter, we discuss the upper bounds of $\mathcal{H}^{n-1}$ measure estimates of nodal sets of eigenfunctions, i.e., the right-hand side of Conjecture 1.1. As mentioned in Chapter 4 that it was verified by Donnelly and Fefferman [DF1] on analytic manifolds, the known results on smooth manifolds are

- Two dimensional: $\lambda^{\frac{3}{4}}$ (Donnelly and Fefferman [DF2], Dong [D1])
- Higher dimensional: $\lambda^{\sqrt{\lambda}}$ (Hardt and Simon [HS])

We also point out that Lin [Lin] further obtained an optimal upper bound estimate of Hausdorff measures of nodal sets for solutions to second order elliptic equations with analytic coefficients and parabolic equations with time independent analytic coefficients, sequential works were done in this direction, and we refer to Han and Lin's book [HL] for details.

Recall Dong-Sogge-Zelditch's integral formula in Theorem 2.7:

$$\int_{\mathcal{N}} f|\bigtriangledown u|d\mathcal{H}^{n-1} = \frac{1}{2}\int_{\mathbb{M}}\big[(\Delta+\lambda)f\big]|u|d\mathcal{H}^{n}.$$

In [D1], in order to study $\mathcal{H}^{n-1}$ estimates of nodal sets, a special case of the above integral formula was proved:

$$\mathcal{H}^{n-1}(\mathcal{N}) = \frac{1}{2}\int_{\mathbb{M}}\frac{\Delta|u|+\lambda|u|}{\sqrt{q}}, \tag{5.1}$$

in which q is given in (2.3):

$$q = |\bigtriangledown u|^2 + \frac{\lambda u^2}{n}.$$

In fact, (5.1) is true for any $\Omega \subset \mathbb{M}$ with smooth boundary:

$$\begin{aligned} \mathcal{H}^{n-1}(\mathcal{N} \cap \Omega) &= \frac{1}{2} \int_\Omega \frac{\Delta|u| + \lambda|u|}{\sqrt{q}} \\ &\leq \frac{1}{4} \int_\Omega |\bigtriangledown \log q| + \sqrt{n\lambda}|\Omega| + |\partial\Omega|, \end{aligned}$$

therefore, one proves the upper bounds in Conjecture 1.1 if one has

Conjecture 5.1 (Dong)**.**

$$\int_{B(p,r)} |\bigtriangledown \log q| \leq c\sqrt{\lambda} r^{n-1}.$$

5.1 In dimension two

In two dimensional case, the singular set $\mathcal{S} = \{u = |\bigtriangledown u| = 0\}$ consists of isolated points on Riemannian surfaces, then the nodal sets, as nodal lines, have more tractable structure, Done [D1] further proved

$$\int_{B(p,r)} |\bigtriangledown \log q| \leq c\sqrt{\lambda} r + c\lambda r^3.$$

By choosing a covering of $\mathbb{M}$ of balls B with radius $\lambda^{-1/4}$, one deduces

$$\int_B |\bigtriangledown \log q| \leq c\lambda^{\frac{1}{4}},$$

which implies

$$\mathcal{H}^{n-1}(\mathcal{N} \cap B) \leq c\lambda^{\frac{1}{4}},$$

and

$$\mathcal{H}^{n-1}(\mathcal{N}) \leq c\lambda^{\frac{3}{4}}$$

by noting the cardinality of the covering $\sim \lambda^{1/2}$. However, we are still looking for generalization in higher dimensional cases. See [DF2] for a different approach on Riemannian surfaces and [HS] for $\lambda^{\sqrt{\lambda}}$ upper bounds in higher dimensions.

5.2 A byproduct: more on BMO estimates

An interesting consequence of Conjecture 5.1 was proposed by Dong [D2]:

Conjecture 5.2 (Dong)**.**

$$\|\log q\|_{BMO} \le c\sqrt{\lambda}.$$

He [D2] proved it is true if the right-hand side is replaced by $\lambda^n \log \lambda$ in higher dimensions. By using the argument concerning the BMO bound of $\log |u|$, (The conditions (3.8) and (3.9) in Lemma 3.9 also hold for $w = q$, see [D2].) we improve this results to

Theorem 5.3 (BMO estimates of $\log q$)**.**

$$\|\log q\|_{BMO} \le c\lambda^{\frac{3n}{4}} (\log \lambda)^2.$$

We omit the proof here, and it can be found in [HaLu1].

Remark. It is quite intriguing to think about the reverse of Conjecture 5.2: can this BMO bound of $\log q$ (or its proof) in Theorem 5.3 imply polynomially increasing upper bounds of $\mathcal{H}^{n-1}$ estimates of nodal sets?

Part II

Hardy-Littlewood-Sobolev inequalities on the Heisenberg group

6 Introduction

We begin our investigation on Hardy-Littlewood-Sobolev inequalities on the Heisenberg group and start by reviewing the history in this area.

6.1 Results on the Euclidean space

Recall the famous Hardy-Littlewood-Sobolev (shorted as HLS in the following context) inequality on $\mathbb{R}^N$: Let $1 < r, s < \infty$ and $0 < \lambda < N$ such that $\frac{1}{r} + \frac{1}{s} + \frac{\lambda}{N} = 2$, then

$$\left| \int \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\overline{f(x)} g(y)}{|x-y|^\lambda} dxdy \right| \leq C \|f\|_r \|g\|_s \tag{6.1}$$

for any $f \in L^r(\mathbb{R}^N)$ and $g \in L^s(\mathbb{R}^N)$, where $\|\cdot\|_r$ and $\|\cdot\|_s$ are the L^r and L^s norms on $\mathbb{R}^N$, respectively, and $0 < C < \infty$ is a constant depending on r, λ, and N only.

This inequality was introduced by Hardy and Littlewood [HaLi1, HaLi2, HaLi3] on $\mathbb{R}^1$ and generalized by Sobolev [S] to $\mathbb{R}^N$. We denote by $C_{r,\lambda,N}$ the sharp constant that we can put into (6.1), finding and proving sharp constant $C_{r,\lambda,N}$ and its maximizers[i] (functions which, when inserted into (6.1), the equality holds with the smallest constant $C_{r,\lambda,N}$) have driven a lot people's attention. In Lieb's paper [Li], existence of the maximizers was proved.

[i]They are also referred as optimizers or extremals in some literature.

Furthermore, when $r = s = 2N/(2N - \lambda)$, he gave explicit formulae of sharp constants $C_{\lambda,n}$ and maximizers. Precisely,

Theorem 6.1. *Let $1 < r, s < \infty$, $0 < \lambda < N$, and $\frac{1}{r} + \frac{1}{s} + \frac{\lambda}{N} = 2$, then there exists a sharp constant $C_{p,\lambda,N}$, maximizers of $f \in L^r(\mathbb{R}^N)$ and $g \in L^s(\mathbb{R}^N)$ such that*

$$\left| \int\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\overline{f(x)} g(y)}{|x-y|^\lambda} dxdy \right| = C_{r,\lambda,N} \|f\|_r \|g\|_s. \tag{6.2}$$

If $r = s = 2N/(2N - \lambda)$, then

$$C_{r,\lambda,N} = C_{\lambda,N} = \pi^{\lambda/N} \frac{\Gamma(N/2 - \lambda/2)}{\Gamma(N - \lambda/2)} \left(\frac{\Gamma(N/2)}{\Gamma(N)} \right)^{\frac{\lambda - N}{N}}.$$

In this case (6.2) holds if and only if $f \equiv (const.)g$ and

$$f(x) = \frac{c}{(d + |x-a|^2)^{\frac{2N-\lambda}{2}}} \tag{6.3}$$

for some $c \in \mathbb{C}$, $0 \neq d \in \mathbb{R}$, and $a \in \mathbb{R}^N$.

Remark.

(1) The proof of the above theorem can also be found in Lieb and Loss's monograph [LL] with more details, in which they also proved that the sharp constant $C_{r,\lambda,N}$ satisfies

$$C_{r,\lambda,N} \leq \frac{N}{rs(N-\lambda)} \left(\frac{\omega_{N-1}}{N} \right)^{\lambda/N} \left[\left(\frac{\lambda/N}{1 - 1/r} \right)^{\frac{\lambda}{N}} + \left(\frac{\lambda/N}{1 - 1/s} \right)^{\frac{\lambda}{N}} \right], \tag{6.4}$$

where ω_{N-1} is the area of unit sphere in $\mathbb{R}^N$, i.e., $\omega_{N-1} = 2\pi^{N/2}/\Gamma(N/2)$. The original proof by Lieb [Li] applies rearrangement methods, a new rearrangement-free proof was provided by Frank and Lieb [FL1, FL3].

(2) The existence of maximizers were also proved by Lions (§2.1 in [Lio4]), which is an application of the concentration compactness principle introduced by him in a series of

papers [Lio1, Lio2, Lio3, Lio4]. See also §II.4 in [Str] about the application on sharp Sobolev inequalities.

(3) The uniqueness of maximizers (6.3) was proposed by Lieb [Li] as an open problem, and was answered by Chen, Li, and Ou [CLO2], in which they used moving plane method for integral equations, (A different approach using moving sphere method for integral equations has been done by Li [L].) a related work by Chen, Li, and Ou [CLO1] studied the integral systems using the similar method. The formula of the maximizers (after dilations and translations) assume

$$\frac{1}{(1+|x|^2)^{\frac{2N-\lambda}{2}}}.$$

(4) We shall point out that (6.4) is not sharp, even when $r = s$, and neither sharp constant $C_{r,\lambda,N}$ nor maximizers are known yet when $r \neq s$.

In 1950s, Stein and Weiss [StWe] introduced the weighted HLS inequality, that is,

$$\left| \int\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{\overline{f(x)}g(y)}{|x|^\alpha|x-y|^\lambda|y|^\beta}dxdy \right| \leq C_{\alpha,\beta,r,\lambda,N}\|f\|_r\|g\|_s, \tag{6.5}$$

where $1 < r, s < \infty$, $0 < \lambda < N$, and $\alpha + \beta \geq 0$ such that $\lambda + \alpha + \beta \leq N$, $\alpha < N/r'$, $\beta < N/s'$, and $\frac{1}{r} + \frac{1}{s} + \frac{\lambda+\alpha+\beta}{N} = 2$. The sharp constant in the weighted HLS inequality (Stein-Weiss) inequality (6.5) is still unknown as far as we are aware of, even in the special case when $r = s$. When $\lambda = N - 2$, the Euler-Lagrange system of (6.5) consists of two Poisson's equations. Chen and Li [CL] studied the integral systems in this case. Caristi, D'Ambrosio and Mitidieri [CDM] also studied the integral systems (inequalities) associated with the Stein-Weiss inequalities and nonexistence of solutions to such systems. Lieb [Li] proved that the maximizers exist when $\lambda+\alpha+\beta < N$, and do not exist when $\lambda+\alpha+\beta = N$.

6.2 Structure of the Heisenberg group

We move our attention to the Heisenberg group, and the n-dimensional Heisenberg group is $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ with

- group structure:

$$uv = (z,t)(z',t') = (z+z', t+t'+2Im(z\cdot\overline{z}))$$

  for any two points $u = (z,t), v = (z',t') \in \mathbb{H}^n$, where $z, z' \in \mathbb{C}^n$, $t, t' \in \mathbb{R}$, and $z\cdot\overline{z} = \sum_{j=1}^n z_j\overline{z_j'}$.

- Haar measure: the Lebesgue measure $du = dzdt$, in which $z = x+iy$ with $x, y \in \mathbb{R}^n$.

- Lie algebra: generated by the left invariant vector fields

$$T = \frac{\partial}{\partial t},\ X_j = \frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t},\ Y_j = \frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t}.$$

- dilation: $\delta_d u = \delta_d(z,t) = (dz, d^2t)$ for each real number $d \in \mathbb{R}$.

- homogeneous norm: $|u| = |(z,t)| = (|z|^4+t^2)^{1/4}$, that is, $|\delta_d u| = |d||u|$.

- $d(u,v) := |u^{-1}v| = |uv^{-1}|$: a left-invariant metric[i].

- homogeneous dimension: $Q = 2n+2$, that is, $|B_r(u)| \sim r^Q$ with $B_r(u) = \{v \in \mathbb{H}^n | |u^{-1}v| < r\}$.

[i] One can easily verify that d is a quasi-metric, i.e.,

$$d(u_1,u_3) \leq c(d(u_1,u_2) + d(u_2,u_3))$$

for some $c \geq 1$. See, e.g. Section 4 in [N]. It was proved by Cygan [C] that it is indeed a metric, i.e., one can take $c = 1$ in the above triangular inequality.

6.3 Main questions on the Heisenberg group

In this section, we lay out the main questions concerning the analogous HLS and Stein-Weiss inequalities on the Heisenberg group. The HLS inequality was announced by Stein [St] and proved by Folland and Stein [FS] in terms of fractional integral (Proposition 8.7 and Lemma 15.3 in [FS]).

Theorem 6.2. *Let $1 < r, s < \infty$, $0 < \lambda < Q$, and $\frac{1}{r} + \frac{1}{s} + \frac{\lambda}{Q} = 2$, then there exists a constant C independent of $f \in L^r(\mathbb{H}^n)$ and $g \in L^s(\mathbb{H}^n)$, such that*

$$\left| \int\int_{\mathbb{H}^n \times \mathbb{H}^n} \frac{\overline{f(u)} g(v)}{|u^{-1}v|^{\lambda}} dudv \right| \leq C \|f\|_r \|g\|_s. \tag{6.6}$$

Here $u = (z, t)$ and $v = (z', t')$, $u^{-1} = (-z, -t)$, and without causing any confusion, we denote $\|\cdot\|_r$ and $\|\cdot\|_s$ as the L^r and L^s norms on $\mathbb{H}^n$.

Towards the sharp version of (6.6), Jerison and Lee [JL] provided sharp constant and maximizer when $\lambda = Q - 2$ and $p = q = 2Q/(2Q - \lambda) = 2Q/(Q + 2)$. Very recently, Frank and Lieb [FL2] generalized their results to all $0 < \lambda < Q$ as the following theorem.

Theorem 6.3. *Let $0 < \lambda < Q$ and $r = 2Q/(2Q - \lambda)$, then for any $f, g \in L^r(\mathbb{H}^n)$,*

$$\left| \int\int_{\mathbb{H}^n \times \mathbb{H}^n} \frac{\overline{f(u)} g(v)}{|u^{-1}v|^{\lambda}} dudv \right| \leq \left(\frac{\pi^{n+1}}{2^{n-1} n!} \right)^{\frac{\lambda}{Q}} \frac{n! \Gamma((Q - \lambda)/2)}{\Gamma^2((2Q - \lambda)/4)} \|f\|_r \|g\|_r,$$

with equality if and only if

$$f(u) = cH(d(a^{-1}u)), \;\; g(v) = c'H(d(a^{-1}v))$$

for some $c, c' \in \mathbb{C}$, $d > 0$, $a \in \mathbb{H}^n$ (unless $f \equiv 0$ or $g \equiv 0$), and

$$H = \frac{1}{[(1 + |z|^2)^2 + t^2]^{\frac{2Q-\lambda}{4}}}.$$

Their results also justified Branson, Fontana, and Morpurgo's guess [BFM] about the maximizer H. However, little about maximizers and sharp constants are known when $r \neq s$. Some other related results concerning the sharp constants of Moser-Trudinger inequalities on the Heisenberg group and $\mathbb{C}^n$ are [CoLu1, CoLu2].

Recently, by the author, Lu, and Zhu [HLZ], the weighted HLS inequalities on $\mathbb{H}^n$ were studied, and two versions of them with different weights were given therein. We state the theorems here and refer to [HLZ] for detailed analysis on these inequalities.

Theorem 6.4 ($|u|$ weighted HLS inequality)**.** *For $1 < r, s < \infty$, $0 < \lambda < Q$, and $\alpha + \beta \geq 0$ such that $\lambda + \alpha + \beta \leq Q$, $\alpha < Q/r'$, $\beta < Q/s'$, and $\frac{1}{r} + \frac{1}{s} + \frac{\lambda+\alpha+\beta}{Q} = 2$, there exists a positive constant $C_{\alpha,\beta,r,\lambda,n}$ independent of the functions f and g such that*

$$\left| \int\int_{\mathbb{H}^n \times \mathbb{H}^n} \frac{\overline{f(u)} g(v)}{|u|^\alpha |u^{-1}v|^\lambda |v|^\beta} dudv \right| \leq C \|f\|_r \|g\|_s.$$

Theorem 6.5 ($|z|$ weighted HLS inequality)**.** *For $1 < r, s < \infty$, $0 < \lambda < Q$, and $0 \leq \alpha + \beta \leq n\lambda$ such that $\lambda + \alpha + \beta \leq Q$, $\alpha < 2n/r'$, $\beta < 2n/s'$, and $\frac{1}{r} + \frac{1}{s} + \frac{\lambda+\alpha+\beta}{Q} = 2$,*

$$\left| \int\int_{\mathbb{H}^n \times \mathbb{H}^n} \frac{\overline{f(u)} g(v)}{|z|^\alpha |u^{-1}v|^\lambda |z'|^\beta} dudv \right| \leq C_{\alpha,\beta,r,\lambda,n} \|f\|_r \|g\|_s.$$

Here, $u = (z, t)$ and $v = (z', t')$.

In Chapter 7, we concentrate on the sharp version of HLS inequality (6.6), and show the existence of maximizers in general cases of r and s, also a upper bound of sharp constant is given, in which we use similar approach as in [LL].

7 Sharp Hardy-Littlewood-Sobolev inequalities

In this chapter, we study the sharp version of HLS inequality in Theorem 6.2

$$\left| \int\int_{\mathbb{H}^n \times \mathbb{H}^n} \frac{\overline{f(u)} g(v)}{|u^{-1}v|^{\lambda}} dudv \right| \leq C \|f\|_r \|g\|_s, \tag{6.6}$$

and prove the following two theorems concerning its maximizers and sharp constants. The material in this chapter has been published in [H].

A closed related inequality is about fractional integrals, by which we transfer our maximizing problem: Define for $f \in L^p(\mathbb{H}^n)$

$$I_{\lambda}(f)(u) = \int_{\mathbb{H}^n} \frac{f(v)}{|u^{-1}v|^{\lambda}} dv,$$

for $u \in \mathbb{H}^n$. Then, by using Hölder's inequality, one can easily obtain the equivalent form of the sharp version of (6.6) as

$$C_{p,\lambda,n} = \sup_{\|f\|_p = 1} \|I_{\lambda}(f)\|_q < \infty \tag{7.1}$$

under the condition that

$$\frac{1}{q} = \frac{1}{p} - \frac{Q-\lambda}{Q}. \tag{7.2}$$

Theorem 7.1 (Existence of maximizers)**.** *Let $\{f_j\}$ be a maximizing sequence of problem (7.1) and (7.2), then there exists $\{u_j\} \subset \mathbb{H}^n$ and $\{d_j\} \subset (0, \infty)$ such that the new maximizing sequence $\{h_j\}$ defined by*

$$h_j(u) = \frac{1}{d_j^{Q/p}} f_j \left(\frac{u_j u}{d_j} \right)$$

is relatively compact in $L^p(\mathbb{H}^n)$. In particular, there exists a maximum of (7.1) and (7.2).

Remark. Under a special case when $p = 2Q/(2Q-\lambda)$ and $q = 2Q/\lambda$, we derive from the above theorem a different approach to prove the existence of maximizers as shown in §4 of [FL2].

Having confirmed existence of maximizers, furthermore, we give upper bounds for sharp constants as follows.

Theorem 7.2 (Upper bounds of sharp constants)**.** *In (6.6), C can be chosen as*

$$\frac{Q|B_1(0)|^{\frac{\lambda}{Q}}}{rs(Q-\lambda)}\left[\left(\frac{\lambda/Q}{1-1/r}\right)^{\frac{\lambda}{Q}}+\left(\frac{\lambda/Q}{1-1/s}\right)^{\frac{\lambda}{Q}}\right],$$

in which $B_1(0)$ is the unit Heisenberg ball, that is, $B_1(0)\subset\mathbb{H}^n=\{u\in\mathbb{H}^n||u|<1\}$ with $|B_1(0)|$ as its volume. Precisely, from [CoLu1],

$$|B_1(0)|=\frac{2\pi^{\frac{Q-2}{2}}\Gamma(1/2)\Gamma((Q+2)/4)}{(Q-2)\Gamma((Q-2)/2)\Gamma((Q+4)/4)}.$$

7.1 Existence of maximizers

Let us first outline the difficulties to be faced: The loss of compactness is caused by a large group of actions consisting of dilations and translations. One can use symmetrization to exclude some actions and ensure the existence of maximizers on $\mathbb{R}^N$. However, symmetrization can not be expected to work on $\mathbb{H}^n$ because of its dilation structure. Therefore, different approach is needed to study the compactness here. It is worthwhile to remark now that in the process we frequently extract to subsequences of the maximizing sequence as needed.

A crucial lemma that refines Fatou's lemma is due to Brézis and Lieb [BL].

Lemma 7.3 (Brézis-Lieb lemma)**.** *Let $0<p<\infty$, $\{f_j\}\subset L^p(\mathbb{H}^n)$ satisfying $\|f\|_p\le C$ and $f_j\to f$ a.e., then*

$$\lim_{j\to\infty}\int_{\mathbb{H}^n}\left||f_j(u)|^p-|f(u)-f_j(u)|^p-|f(u)|^p\right|du=0.$$

Now suppose that we are given a maximizing sequence $\{f_j\}$ for (7.1) and (7.2), and without loss of generality we assume that $\|f_j\|_p=1$, our goal is to generate $\{h_j\}$ as stated in Theorem 7.1. To recover from the loss of compactness from dilations and translations, we first need the following concentration compactness lemma from [Lio1], we provide a proof here for completeness.

Lemma 7.4. *For simplicity, we denote by* $\rho_j = |f_j|^p$ *as a nonnegative measure on* $\mathbb{H}^n$*, thus* $\int_{\mathbb{H}^n} \rho_j = 1$*. Then, there exists a subsequence of* $\{\rho_j\}$ *(and we still denote as* $\{\rho_j\}$*) such that one of the following holds.*

1. *For all* $R > 0$*, we have*

$$\lim_{j\to\infty} \left(\sup_{u\in\mathbb{H}^n} \int_{B_R(u)} \rho_j \right) = 0.$$

2. *There exists* $\{u_j\} \subset \mathbb{H}^n$ *such that for each* $\epsilon > 0$ *small enough, we can find* $R_0 > 0$ *with*

$$\int_{B_{R_0}(u_j)} \rho_j \geq 1 - \epsilon$$

for all $j \in \mathbb{N}$*.*

3. *There exists* $0 < k < 1$ *such that for each* $\epsilon > 0$ *small enough, we can find* $R_0 > 0$ *and* $\{u_j\} \subset \mathbb{H}^n$ *such that given any* $R \geq R_0$*, there exist* ρ_j^1 *and* ρ_j^2 *as two nonnegative measures satisfying*

   (a) $\rho_j^1 + \rho_j^2 = \rho_j$*.*

   (b) supp$(\rho_j^1) \subset B_R(u_j)$ *and supp*$(\rho_j^2) \subset B_R^c(u_j)$*.*

   (c)

$$\limsup_{j\to\infty} \left(\left| k - \int_{\mathbb{H}^n} \rho_j^1 \right| + \left| (1-k) - \int_{\mathbb{H}^n} \rho_j^2 \right| \right) \leq \epsilon.$$

Proof of Lemma 7.4. We define the Levy concentration function for ρ_j on $\mathbb{H}^n$ as

$$Q_j(R) = \sup_{u \in \mathbb{H}^n} \int_{B_R(u)} \rho_j$$

for $R \in [0, \infty]$. It is obvious that $Q_j \in \mathrm{BV}[0, \infty]$[i] is nonnegative and nondecreasing with

$$Q_j(0) = 0, \text{ and } Q_j(\infty) = 1$$

for all $j \in \mathbb{N}$. Therefore, we can find a nonnegative and nondecreasing function $Q \in \mathrm{BV}[0, \infty]$ such that by passing to a subsequence of $\{Q_j\}$ if necessary (and we still denote without causing any confusion by $\{Q_j\}$)

$$\lim_{j \to \infty} Q_j(R) = Q(R)$$

for all $R \in [0, \infty)$.

Now we write

$$k = \lim_{R \to \infty} Q(R),$$

and thus $0 \leq k \leq 1$.

1. If $k = 0$, then easily we have

   $$\lim_{j \to \infty} \left(\sup_{u \in \mathbb{H}^n} \int_{B_R(u)} \rho_j \right) = 0$$

   for all $R > 0$.

2. If $k = 1$, then we first choose $R_1 > 0$ such that $Q(R_1) > \frac{3}{4}$, and for fixed $0 < \epsilon < \frac{1}{4}$, we choose R_2 such that $Q(R_2) > 1 - \frac{\epsilon}{2} > \frac{3}{4}$. Because

   $$Q_j(R) = \sup_{u \in \mathbb{H}^n} \int_{B_R(u)} \rho_j,$$

[i] See [Fo] for reference on BV (bounded variation) functions.

we let $u_j, v_j \in \mathbb{H}^n$ satisfy

$$\int_{B_{R_1}(u_j)} \rho_j \geq Q_j(R_1) - \frac{1}{j}$$

and

$$\int_{B_{R_2}(v_j)} \rho_j \geq Q_j(R_2) - \frac{1}{j}$$

for all $j \in \mathbb{N}$. We compute that

$$\begin{aligned} & \int_{B_{R_1}(u_j)} \rho_j + \int_{B_{R_2}(v_j)} \rho_j \\ \geq \ & Q_j(R_1) + Q_j(R_2) - \frac{2}{j} + o(1) \\ > \ & 1 \\ = \ & \int_{\mathbb{H}^n} \rho_j \end{aligned}$$

for j large, which means that $B_{R_1}(u_j) \cap B_{R_2}(v_j) \neq \emptyset$. Therefore,

$$B_{R_2}(v_j) \subset B_{R_1+2R_2}(u_j),$$

in which we use the fact that the quasi-metric defined as $d(u, v) = |u^{-1}v|$ on $\mathbb{H}^n$ is a metric. Now compute that

$$\int_{B_{R_1+2R_2}(u_j)} \rho_j \geq Q_j(R_2) - \frac{1}{j} \geq Q(R_2) + o(1) - \frac{1}{j} \geq 1 - \epsilon$$

for $j > j(\epsilon)$. Furthermore, we select R_3 such that

$$\int_{B_{R_3}(0)} \rho_j \geq 1 - \epsilon$$

for $j = 1, 2, ..., j(\epsilon)$. Then, one arrives at the conclusion in (ii) by taking $R_0 = R_1 + 2R_2 + R_3$.

3. If $0 < k < 1$, then $\forall \epsilon > 0$, choose R_0 such that $Q(R_0) > k - \frac{\epsilon}{8}$. For $j > j(\epsilon)$, we have

$$k - \frac{\epsilon}{4} < Q_j(R_0) < k + \frac{\epsilon}{4},$$

and therefore, there is $\{u_j\}$ such that

$$\int_{B_{R_0}(u_j)} \rho_j > k - \frac{\epsilon}{2}.$$

Similarly, we can enlarge $j(\epsilon)$ if necessary to get a sequence $\{R_j\}$ with $R_j \to \infty$ such that

$$\int_{B_{R_j}(u_j)} \rho_j < k + \frac{\epsilon}{2}$$

for all $j > j(\epsilon)$.

For any given $R \geq R_0$, we may assume $R \leq R_j$ for all $j \in \mathbb{N}$. This means that there exists $\{u_j\} \subset \mathbb{H}^n$ such that

$$k - \frac{\epsilon}{2} \leq \int_{B_{R_0}(u_j)} \rho_j \leq \int_{B_R(u_j)} \rho_j \leq \int_{B_{R_j}(u_j)} \rho_j \leq k + \frac{\epsilon}{2}.$$

Set

$$\rho_j^1 = \rho_j \chi_{B_R(u_j)} \text{ and } \rho_j^2 = \rho_j \chi_{B_R^c(u_j)},$$

thus,

$$\begin{aligned} & \left| k - \int_{\mathbb{H}^n} \rho_j^1 \right| + \left| (1-k) - \int_{\mathbb{H}^n} \rho_j^2 \right| \\ = & \left| k - \int_{B_R(u_j)} \rho_j \right| + \left| (1-k) - \int_{B_R^c(u_j)} \rho_j \right| \\ \leq & \ \epsilon. \end{aligned}$$

$\square$

Remark. Back to our maximizing problem, let $\{f_j\} \subset L^p(\mathbb{H}^n)$ be a maximizing sequence of (7.1) and (7.2) satisfying $\|f_j\| = 1$, then with the help of dilations (as $\{d_j\}$ in Theorem 7.1), we can always assume that

$$Q_j(1) = \sup_{u \in \mathbb{H}^n} \int_{B_1(u)} \rho_j = \frac{1}{2}$$

as defined in the proof of Lemma 7.4 without affecting the maximizing problem since (7.1) is dilation-invariant, therefore we are able to eliminate the case in (1). Next we prove that the case in (3) can not happen, either, from which we only need to focus on the case in (2).

Proposition 7.5. *Let $\{f_j\} \subset L^p(\mathbb{H}^n)$ be a maximizing sequence of (7.1) and (7.2) satisfying $\|f_j\| = 1$, then (3) in Lemma 7.4 can not occur.*

Proof of Proposition 7.5. We argue by contradiction. If (3) in Lemma 7.4 occurs, then there exist $0 < k < 1$ and a subsequence of $\{f_j\}$ (which we still denote by $\{f_j\}$) such that for each $\epsilon > 0$ small enough, we can find $R_0 > 0$ and $\{u_j\} \subset \mathbb{H}^n$ such that given any $R \geq R_0$,

$$\|f_j \chi_{B_R(0)}\|_p^p = k + O(\epsilon) \text{ and } \|f_j \chi_{B_R^c(0)}\|_p^p = 1 - k + O(\epsilon).$$

Without loss of generality, we may assume $u_j = 0$ for all $j \in \mathbb{Z}$ since (7.1) is translation-invariant. Thus, for any $u \in \mathbb{H}^n$, let $R = j|u|$ for $j \geq j(\epsilon, |u|)$ such that $j|u| > R_0$, we observe that $|u| \leq \frac{1}{j}|v|$ for all $v \in B_R^c(0)$, then

$$|u^{-1}v| \geq |v| - |u| \geq \frac{j-1}{j}|v|,$$

and therefore,

$$\begin{aligned}
& |I_\lambda(f_j)(u) - I_\lambda(f_j\chi_{B_R(0)})(u)| \\
= & \ |I_\lambda(f_j\chi_{B_R^c(0)})(u)| \\
= & \ \left|\int_{\mathbb{H}^n} \frac{(f_j\chi_{B_R^c(0)})(v)}{|u^{-1}v|^\lambda} dv\right| \\
\leq & \ \left[\int_{B_R^c(0)} |f_j(v)|^p dv\right]^{1/p} \left[\int_{B_R^c(0)} |u^{-1}v|^{-\lambda p'} dv\right]^{1/p'} \\
\leq & \ C\left(\frac{j}{j-1}\right)^\lambda \left(\int_{j|u|}^\infty r^{Q-\lambda p'-1} dr\right)^{1/p'} \\
\leq & \ C\left(\frac{j}{j-1}\right)^\lambda \left(\frac{1}{\lambda p' - Q}\right)^{1/p'} \left(j|u|\right)^{(Q-\lambda p')/p'} \\
\to & \ 0
\end{aligned}$$

as $j \to \infty$. Here, C depends only on $\mathbb{H}^n$, and the integral is finite because $Q < \lambda p'$ from (7.2):

$$\frac{1}{q} = \frac{1}{p} - \frac{Q-\lambda}{Q} > 0.$$

Now we apply the Brézis-Lieb lemma in Lemma 3 because $I_\lambda(f_j) \to I_\lambda(f_j\chi_{B_R(0)})$ a.e. and get

$$\|I_\lambda(f_j)\|_q^q = \|I_\lambda(f_j\chi_{B_R(0)})\|_q^q + \|I_\lambda(f_j\chi_{B_R^c(0)})\|_q^q + o(1),$$

in which the left-hand side goes to $C_{p,\lambda,n}^q$ since $\{f_j\}$ maximizes (7.1), while the right-hand side

$$
\begin{aligned}
& \|I_\lambda(f_j\chi_{B_R(0)})\|_q^q + \|I_\lambda(f_j\chi_{B_R^c(0)})(u)\|_q^q + o(1) \\
\leq \;& C_{p,\lambda,n}^q\|f_j\chi_{B_R(0)}\|_p^q + C_{p,\lambda,n}^q\|f_j\chi_{B_R^c(0)}\|_p^q + o(1) \\
\leq \;& C_{p,\lambda,n}^q(k+O(\epsilon))^{\frac{q}{p}} + C_{p,\lambda,n}^q(1-k+O(\epsilon))^{\frac{q}{p}} + o(1) \\
\leq \;& C_{p,\lambda,n}^q\left[k^{\frac{q}{p}} + (1-k)^{\frac{q}{p}}\right] + O(\epsilon) + o(1) \\
< \;& C_{p,\lambda,n}^q,
\end{aligned}
$$

if $0 < k < 1$ for large j because $\frac{q}{p} > 1$, and we conclude the contradiction. □

We can now proceed under the scope of (2) in Lemma 7.4: There exists $\{u_j\} \subset \mathbb{H}^n$ such that for R large, we have

$$\int_{B_R(u_j)} |f_j|^p \geq 1 - \epsilon(R).$$

Due to translations $f_j(v) \to f_j(u_jv)$, we use $\{f_j\}$ to denote the new maximizing sequence satisfying

$$\int_{B_R(0)} |f_j|^p \geq 1 - \epsilon(R), \tag{7.3}$$

and we derive the following corollary, in which the byproduct (7.4) serves as an important ingredient in the proof of Theorem 7.1.

Corollary 7.6. *Let $\{f_j\} \subset L^p(\mathbb{H}^n)$ be a maximizing sequence of (7.1) and (7.2) satisfying $\|f_j\| = 1$ and*

$$\int_{B_R(0)} |f_j|^p \geq 1 - \epsilon(R),$$

we may assume that $f_j \to f$ weakly in $L^p(\mathbb{H}^n)$ (by passing to a subsequence if necessary). Then, by passing to a subsequence again if necessary,

$$I_\lambda(f_j) \to I_\lambda(f) \;\; a.e..$$

Proof of Corollary 7.6. We show that $I_\lambda(f_j) \to I_\lambda(f)$ in measure to ensure the existence of a pointwisely convergent subsequence of $\{f_j\}$. Observe that for $M > R$,

$$\begin{aligned}
& \|I_\lambda(f_j)\chi_{B^c_M(0)}\|_q \\
\leq\ & \|I_\lambda(f_j\chi_{B_R(0)})\chi_{B^c_M(0)}\|_q + \|I_\lambda(f_j\chi_{B^c_R(0)})\chi_{B^c_M(0)}\|_q \\
\leq\ & \|I_\lambda(f_j\chi_{B_R(0)})\chi_{B^c_M(0)}\|_q + C_{p,\lambda,n}\|f_j\chi_{B^c_R(0)}\|_p \\
\leq\ & \|I_\lambda(f_j\chi_{B_R(0)})\chi_{B^c_M(0)}\|_q + \epsilon(R),
\end{aligned}$$

in which we apply Minkowski's integral inequality to estimate the first term, noticing that $|u^{-1}v| \geq |u| - R$ for $|v| \leq R < M \leq |u|$,

$$\begin{aligned}
& \|I_\lambda(f_j\chi_{B_R(0)})\chi_{B^c_M(0)}\|_q \\
=\ & \left(\int_{B^c_M(0)} |I_\lambda(f_j\chi_{B_R(0)})(u)|^q du\right)^{1/q} \\
=\ & \left(\int_{|u|\geq M} \left|\int_{|v|\leq R} \frac{f_j(v)}{|u^{-1}v|^\lambda} dv\right|^q du\right)^{1/q} \\
\leq\ & \|f_j\chi_{B_R(0)}\|_1 \left(\int_{|u|\geq M} \frac{1}{(|u|-R)^{\lambda q}} du\right)^{1/q} \\
\leq\ & C(R,p,n)(M-R)^{(Q-\lambda q)/q} \\
\to\ & 0
\end{aligned}$$

for every fixed R as $M \to \infty$ since $Q < \lambda q$ from (7.1). Therefore, we have

$$\|I_\lambda(f_j)\chi_{B^c_M(0)}\|_q \leq \epsilon(M), \tag{7.4}$$

and that is,

$$\|I_\lambda(f_j) - I_\lambda(f_j)\chi_{B_M(0)}\|_q \leq \epsilon(M).$$

Since $f_j \to f$ weakly in $L^p(\mathbb{H}^n)$, we have

$$\|f\chi_{B^c_R(0)}\|_p^p \leq \liminf_{j\to\infty} \|f_j\chi_{B^c_R(0)}\|_p^p \leq \epsilon(R).$$

Similarly, one can derive for f,

$$\|I_\lambda(f) - I_\lambda(f)\chi_{B_M(0)}\|_q \leq \epsilon(M).$$

Therefore, given $k > 0$,

$$\begin{aligned}
& |\{|I_\lambda(f_j)(u) - I_\lambda(f)(u)| \geq 15k|\}| \\
\leq\ & |\{|I_\lambda(f_j)(u) - I_\lambda(f_j)(u)\chi_{B_M(0)}(u)| \geq 5k\}| + \\
& |\{|I_\lambda(f_j)(u)\chi_{B_M(0)}(u) - I_\lambda(f)(u)\chi_{B_M(0)}(u)| \geq 5k\}| + \\
& |\{|I_\lambda(f)(u)\chi_{B_M(0)}(u) - I_\lambda(f)(u)| \geq 5k\}| \\
\leq\ & 2\left[\frac{\epsilon(M)}{5k}\right]^q + |\{|I_\lambda(f_j)(u) - I_\lambda(f)(u)| \geq 5k\} \cap B_M(0)|. \qquad (7.5)
\end{aligned}$$

Thus, it remains to estimate the second term above. Denote

$$I_\lambda^\eta(f)(u) = \int_{B_\eta^c(u)} \frac{f(v)}{|u^{-1}v|^\lambda} dv,$$

then

$$I_\lambda^\eta(f_j\chi_{B_R(0)})(u) \to I_\lambda^\eta(f\chi_{B_R(0)})(u)$$

for all $u \in \mathbb{H}^n$ because

$$|u^{-1}v|^{-\lambda}\chi_{B_R(0)}\chi_{B_\eta^c(u)} \in L^{p'}(\mathbb{H}^n)$$

for any fixed $u \in \mathbb{H}^n$ and $\eta > 0$. Therefore, $I_\lambda^\eta(f_j\chi_{B_R(0)}) \to I_\lambda^\eta(f\chi_{B_R(0)})$ locally in measure, which means

$$\left|\{|I_\lambda^\eta(f_j\chi_{B_R(0)})(u) - I_\lambda^\eta(f\chi_{B_R(0)})(u)| \geq k\} \cap B_M(0)\right| = o(1). \qquad (7.6)$$

On the other hand, we compute that for any fixed $m \in (1, Q/\lambda)$ by applying Minkowski's integral inequality,

$$\begin{aligned}
& \|I_\lambda(f_j\chi_{B_R(0)}) - I_\lambda^\eta(f_j\chi_{B_R(0)})\|_m \\
= & \left(\int_{\mathbb{H}^n}\left|\int_{B_\eta(u)}\frac{f_j(v)\chi_{B_R(0)}(v)}{|u^{-1}v|^\lambda}dv\right|^m du\right)^{1/m} \\
\leq & \|f_j\chi_{B_R(0)}\|_1\left(\int_{B_\eta(v)}\frac{1}{|v^{-1}u|^{\lambda m}}du\right)^{1/m} \\
\leq & C(R,p,n)\eta^{(Q-\lambda m)/m} \\
\to & 0
\end{aligned}$$

for every fixed R as $\eta \to 0$ since $Q > \lambda m$. That is,

$$\|I_\lambda(f_j\chi_{B_R(0)}) - I_\lambda^\eta(f_j\chi_{B_R(0)})\|_m \leq O(\eta). \tag{7.7}$$

Similarly, we can derive the analogous statement for f,

$$\|I_\lambda(f\chi_{B_R(0)}) - I_\lambda^\eta(f\chi_{B_R(0)})\|_m \leq O(\eta). \tag{7.8}$$

Also notice that

$$\|I_\lambda(f_j) - I_\lambda(f_j\chi_{B_R(0)})\|_q \leq C_{p,\lambda,n}\|f\chi_{B_R^c(0)}\|_p \leq \epsilon(R) \tag{7.9}$$

and

$$\|I_\lambda(f) - I_\lambda(f\chi_{B_R(0)})\|_q \leq C_{p,\lambda,n}\|f\chi_{B_R^c(0)}\|_p \leq \epsilon(R). \tag{7.10}$$

Now returning to the estimate in (7.5), combining (7.6)–(2.11), we have for any $k > 0$,

$$
\begin{aligned}
& |\{|I_\lambda(f_j)(u) - I_\lambda(f)(u)| \geq 5k\} \cap B_M(0)| \\
\leq\ & |\{|I_\lambda(f_j)(u) - I_\lambda(f_j\chi_{B_R(0)})(u)| \geq k\}| + |\{|I_\lambda(f_j\chi_{B_R(0)})(u) - I_\lambda^\eta(f_j\chi_{B_R(0)})(u)| \geq k\}| + \\
& |\{|I_\lambda^\eta(f_j\chi_{B_R(0)})(u) - I_\lambda^\eta(f\chi_{B_R(0)})(u)| \geq k\} \cap B_M(0)| + \\
& |\{I_\lambda^\eta(f\chi_{B_R(0)})(u) - I_\lambda(f\chi_{B_R(0)})(u)| \geq k\}| + |\{|I_\lambda(f_j\chi_{B_R(0)})(u) - I_\lambda(f)(u)| \geq k\}| \\
\leq\ & 2\left[\frac{\epsilon(R)}{k}\right]^q + 2\left[\frac{O(\eta)}{k}\right]^m + o(1).
\end{aligned}
$$

We can now conclude the convergence in measure of $\{f_j\}$ by properly choosing ϵ, R, M, and η. $\square$

Now we only need to verify the weak limit f in preceding corollary satisfies $\|f\|_p = 1$ to complete Theorem 7.1. We borrow Lemma 2.1 in [Lio4] as follows to proceed.

Lemma 7.7. *Let $f_j \to f$ weakly in $L^p(\mathbb{H}^n)$ and $I_\lambda(f_j) \to I_\lambda(f)$ weakly in $L^q(\mathbb{H}^n)$, assume that (7.3) and (7.4) hold, and $|f_j|^p \to \mu$ and $|I_\lambda(f_j)|^q \to \nu$ weakly for two nonnegative measures μ and ν in $L^1(\mathbb{H}^n)$. Then, there exist two at most countable families (possibly empty) $\{u_j\} \subset \mathbb{H}^n$ and $\{k_j\} \subset (0, \infty)$ such that*

$$\nu = |I_\lambda(f)|^q + \sum_j C_{p,\lambda,n} k_j^{q/p} \delta_{u_j} \tag{7.11}$$

and

$$\mu \geq |f|^p + \sum_j k_j \delta_{u_j}, \tag{7.12}$$

in which δ_{u_j} is the Dirac function at u_j.

Remark. The original version of the above lemma is on $\mathbb{R}^N$, but it can be carried out as what we did in the proof of Lemma 7.4 on $\mathbb{H}^n$ because the crucial ingredient as Lemma 1.2 in [Lio3] is valid in an arbitrary measure space. (See Remark 1.5 at the end of its proof.)

With the help of Lemma 7.7, we now prove Theorem 7.1.

Proof of Theorem 7.1. We show that $\|f\|_p = 1$ by contradiction, then $f_j \to f$ strongly in $L^p(\mathbb{H}^n)$, which implies the theorem.

Observe that $\mu(\mathbb{H}^n) = 1$ and $\nu(\mathbb{H}^n) = C^q_{p,\lambda,n}$ since $|f_j|^p \to \mu$ and $|I_\lambda(f_j)|^q \to \nu$ weakly in $L^1(\mathbb{H}^n)$. If $\|f\|_p^p = k < 1$, then from (7.12),

$$\sum_j k_j \le \mu(\mathbb{H}^n) - \|f\|_p^p = 1 - k,$$

therefore by (7.11),

$$\begin{aligned}
\nu(\mathbb{H}^n) &= \|I_\lambda(f)\|_q^q + \sum_j C^q_{p,\lambda,n} k_j^{q/p} \\
&\le C^q_{p,\lambda,n}\|f\|_p^q + C^q_{p,\lambda,n}\sum_j k_j^{q/p} \\
&\le C^q_{p,\lambda,n} k^{q/p} + C^q_{p,\lambda,n}(\sum_j k_j)^{q/p} \\
&\le C^q_{p,\lambda,n} k^{q/p} + C^q_{p,\lambda,n}(1-k)^{q/p} \\
&< C^q_{p,\lambda,n},
\end{aligned}$$

which contradicts with the fact that $\nu(\mathbb{H}^n) = C^q_{p,\lambda,n}$, and we complete Theorem 7.1. □

7.2 Upper bounds of sharp constants

Theorem 7.2 is an analogue on $\mathbb{H}^n$ of §4.3 in [LL]. Without loss of generality, we assume that both f and g are nonnegative with normalized norms $\|f\|_r = \|g\|_s = 1$. Denote χ as the characteristic function, derive that

$$\int_0^\infty \chi_{\{f>a\}}(u)da = \int_0^{f(u)} da = f(u),$$

$$\int_0^\infty \chi_{\{g>b\}}(u)db = \int_0^{g(u)} db = g(u),$$

and

$$\lambda \int_0^\infty c^{-\lambda-1}\chi_{\{|u|<c\}}(u)dc = \lambda \int_{|u|}^\infty c^{-\lambda-1}dc = |u|^{-\lambda}.$$

Inserting the above quantities into the left hand side of (6.6), we obtain

$$\begin{aligned} I &:= \int\int_{\mathbb{H}^n\times\mathbb{H}^n} f(u)|u^{-1}v|^{-\lambda}g(v)dudv \qquad (7.13)\\ &= \lambda\int_0^\infty\int_0^\infty\int_0^\infty\int_{\mathbb{H}^n}\int_{\mathbb{H}^n} c^{-\lambda-1}\chi_{\{f>a\}}(u)\chi_{\{g>b\}}(v)\\ &\quad \chi_{\{|u^{-1}v|<c\}}(u^{-1}v)dudvdadbdc.\\ &\leq \lambda\int_0^\infty\int_0^\infty\int_0^\infty c^{-\lambda-1}I(a,b,c)dadbdc, \end{aligned}$$

in which

$$\begin{aligned} \alpha(a) &= \int_{\mathbb{H}^n}\chi_{\{f>a\}}(u)du,\\ \beta(b) &= \int_{\mathbb{H}^n}\chi_{\{g>b\}}(v)dv, \end{aligned}$$

and

$$\gamma(c) = \int_{\mathbb{H}^n}\chi_{\{|u^{-1}v|<c\}}(u^{-1}v)du \text{ (or } dv) = |B_c(u)| = |B_1(0)|c^Q,$$

where $B_c(u)\subset\mathbb{H}^n = \{v\in\mathbb{H}^n||u^{-1}v|<c\}$ and

$$I(a,b,c) := \frac{\alpha(a)\beta(b)\gamma(c)}{\max\{\alpha(a),\beta(b),\gamma(c)\}}. \qquad (7.14)$$

First we estimate the integral over c, and consider $\int_0^\infty c^{-\lambda-1}I(a,b,c)dc$ in two cases.

Case A: $\alpha(a)\geq\beta(b)$. In the view of (7.13), (7.14), and definition of $\alpha(a)$, $\beta(b)$, and $\gamma(c)$, we compute

$$\begin{aligned}
&\int_0^\infty c^{-\lambda-1} I(a,b,c)dc \\
= \;& \int_{\gamma(c)\le\alpha(a)} c^{-\lambda-1}\beta(b)\gamma(c)dc + \int_{\gamma(c)>\alpha(a)} c^{-\lambda-1}\alpha(a)\beta(b)dc \\
= \;& |B_1(0)|\beta(b)\int_{\gamma(c)\le\alpha(a)} c^{Q-\lambda-1}dc + \alpha(a)\beta(b)\int_{\gamma(c)>\alpha(a)} c^{-\lambda-1}dc \\
= \;& |B_1(0)|\beta(b)\int_0^{(\alpha(a)/|B_1(0)|)^{\frac{1}{Q}}} c^{Q-\lambda-1}dc + \alpha(a)\beta(b)\int_{(\alpha(a)/|B_1(0)|)^{\frac{1}{Q}}}^\infty c^{-\lambda-1}dc \\
= \;& |B_1(0)|\beta(b)\frac{(\alpha(a)/|B_1(0)|)^{\frac{Q-\lambda}{Q}}}{Q-\lambda} + \alpha(a)\beta(b)\frac{(\alpha(a)/|B_1(0)|)^{-\frac{\lambda}{Q}}}{\lambda} \\
= \;& \frac{Q|B_1(0)|^{\frac{\lambda}{Q}}}{\lambda(Q-\lambda)}\alpha^{\frac{Q-\lambda}{Q}}(a)\beta(b).
\end{aligned}$$

Case B: $\alpha(a) < \beta(b)$. Similar to Case A, we have

$$\int_0^\infty c^{-\lambda-1} I(a,b,c)dc = \frac{Q|B_1(0)|^{\frac{\lambda}{Q}}}{\lambda(Q-\lambda)}\alpha(a)\beta^{\frac{Q-\lambda}{Q}}(b).$$

Thus, we have estimated the integral over c, plugging Cases A and B into (7.13), we obtain

$$I \le \frac{Q|B_1(0)|^{\frac{\lambda}{Q}}}{Q-\lambda}\int_0^\infty\int_0^\infty \min\{\alpha^{\frac{Q-\lambda}{Q}}(a)\beta(b), \alpha(a)\beta^{\frac{Q-\lambda}{Q}}(b)\}dadb, \tag{7.15}$$

and note that $\alpha(a) \ge \beta(b)$ if and only if $\alpha(a)\beta^{\frac{Q-\lambda}{Q}}(b) \ge \alpha^{\frac{Q-\lambda}{Q}}(a)\beta(b)$.

To estimate the integral in (7.15) over a and b, we split into two parts,

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \min\{\alpha^{\frac{Q-\lambda}{Q}}(a)\beta(b), \alpha(a)\beta^{\frac{Q-\lambda}{Q}}(b)\}dadb \\
= \;& \int_0^\infty \alpha(a)\int_0^{a^{\frac{p}{q}}} \beta^{\frac{Q-\lambda}{Q}}(b)dbda + \int_0^\infty \alpha^{\frac{Q-\lambda}{Q}}(a)\int_{a^{\frac{p}{q}}}^\infty \beta(b)dbda \\
:= \;& I_1 + I_2,
\end{aligned}$$

in which we take $t = (s-1)(1-\lambda/Q)$, and by Hölder's inequality with indices $Q/(Q-\lambda)$

and Q/λ,

$$\begin{aligned}
&\int_0^{a^{\frac{r}{s}}} \beta^{\frac{Q-\lambda}{Q}}(b)db \\
= \;& \int_0^{a^{\frac{r}{s}}} \beta^{\frac{Q-\lambda}{Q}}(b)b^t b^{-t} db \\
\leq \;& \left[\int_0^{a^{\frac{r}{s}}} \left(\beta^{\frac{Q-\lambda}{Q}}(b)b^t db\right)^{\frac{Q}{Q-\lambda}}\right]^{\frac{Q-\lambda}{Q}} \left[\int_0^{a^{\frac{r}{s}}} (b^{-t})^{\frac{Q}{\lambda}} db\right]^{\frac{\lambda}{Q}} \\
= \;& \left[\int_0^{a^{\frac{r}{s}}} \beta(b)b^{s-1} db\right]^{\frac{Q-\lambda}{Q}} \left(\int_0^{a^{\frac{r}{s}}} b^{-\frac{tQ}{\lambda}} db\right)^{\frac{\lambda}{Q}}.
\end{aligned}$$

One can check that $tQ < \lambda$ and

$$\left(\int_0^{a^{\frac{r}{s}}} b^{-\frac{tQ}{\lambda}} db\right)^{\frac{\lambda}{Q}} = \left[\frac{\lambda}{Q - s(Q-\lambda)}\right]^{\frac{\lambda}{Q}} a^{r-1}.$$

Recall for normalized f and g,

$$1 = \|f\|_r^r = r\int_0^\infty a^{r-1}\alpha(a)da$$

and

$$1 = \|g\|_s^s = s\int_0^\infty b^{s-1}\beta(b)db.$$

Then,

$$\begin{aligned}
I_1 &= \int_0^\infty \alpha(a) \int_0^{a^{\frac{r}{s}}} \beta^{\frac{Q-\lambda}{Q}}(b)dbda \\
&\leq \left(\frac{\lambda}{Q - s(Q-\lambda)}\right)^{\frac{\lambda}{Q}} \int_0^\infty \alpha(a)a^{r-1} \left(\int_0^{a^{\frac{r}{s}}} \beta(b)b^{s-1}db\right)^{\frac{Q-\lambda}{Q}} da \\
&= \frac{1}{r}\left(\frac{\lambda}{Q - s(Q-\lambda)}\right)^{\frac{\lambda}{Q}} \left(\frac{1}{s}\right)^{\frac{Q-\lambda}{Q}} \\
&= \frac{1}{rs}\left(\frac{\lambda/Q}{1-1/r}\right)^{\frac{\lambda}{Q}}.
\end{aligned}$$

To estimate I_2, by Fubini's theorem, it is easy to see

$$\begin{aligned}
I_2 &= \int_0^\infty \alpha^{\frac{Q-\lambda}{Q}}(a) \int_{a^{\frac{r}{s}}}^\infty \beta(b) db da \\
&= \int_0^\infty \beta(b) \int_0^{b^{\frac{r}{s}}} \alpha^{\frac{Q-\lambda}{Q}}(a) da db \\
&\leq \frac{1}{rs}\left(\frac{\lambda/Q}{1-1/s}\right)^{\frac{\lambda}{Q}}.
\end{aligned}$$

Thus, back to (7.15), we have

$$\begin{aligned}
I &\leq \frac{Q|B_1(0)|^{\frac{\lambda}{Q}}}{Q-\lambda} \int_0^\infty \int_0^\infty \min\{\alpha^{\frac{Q-\lambda}{Q}}(a)\beta(b), \alpha(a)\beta^{\frac{Q-\lambda}{Q}}(b)\} da db \\
&= \frac{Q|B_1(0)|^{\frac{\lambda}{Q}}}{Q-\lambda}(I_1 + I_2) \\
&\leq \frac{Q|B_1(0)|^{\frac{\lambda}{Q}}}{rs(Q-\lambda)}\left[\left(\frac{\lambda/Q}{1-1/r}\right)^{\frac{\lambda}{Q}} + \left(\frac{\lambda/Q}{1-1/s}\right)^{\frac{\lambda}{Q}}\right].
\end{aligned}$$

Therefore, we complete the proof of Theorem 7.2 by noticing

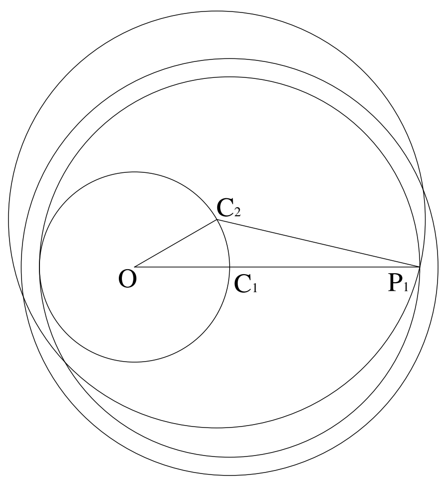
$$|B_1(0)| = \frac{2\pi^{\frac{Q-2}{2}}\Gamma(1/2)\Gamma((Q+2)/4)}{(Q-2)\Gamma((Q-2)/2)\Gamma((Q+4)/4)}$$

from [CoLu1].

APPENDIX A:
THE SHARPNESS OF CONJECTURE 3.2

By an example here we show that if it is replaced by $c\delta^{-\frac{n-1}{2}}$ in the right-hand side of (3.1) in Theorem 3.1, then it is the sharp estimate.

Let $S(x_0, r)$ denote the boundary of the ball $B(x_0, r)$ in $\mathbb{R}^n$. First, let us give two unit balls centered at C_1 and C_2 such that $C_1, C_2 \in S(O, \frac{1}{2}) \subset \mathbb{R}^n$, the sphere centered at the origin O with radius $1/2$ (as shown in the figure below).



Take P_1 such that O, C_1, and P_1 are on the same line, and

$$\overline{OC_1} = \overline{OC_2} = \frac{1}{2},$$

$$\overline{C_1P_1} = 1,\ \overline{C_2P_1} = 1 + \delta.$$

Consider the plane formed by O, C_1, C_2, and P_1, and let $\theta = \angle C_1OC_2$, we have

$$\cos\theta = \frac{\overline{OC_2}^2 + \overline{OP_1}^2 - \overline{C_2P_1}^2}{2\overline{OC_2} \cdot \overline{OP_1}} \sim 1 - \frac{\theta^2}{2},$$

$$\theta \sim \sqrt{\frac{8\delta + 4\delta^2}{3}} \sim 2\sqrt{\frac{2\delta}{3}},$$

and

$$\overline{C_1C_2} \sim \frac{1}{2}\theta \sim \sqrt{\frac{2\delta}{3}}.$$

We now consider the given family of unit balls $\{B_1, \cdots, B_N\}$ with centers of lattices $C_i \in S(O, \frac{1}{2})$ for $i = 1, \cdots, N$ of size $\sqrt{\delta}$ (roughly speaking), thus it satisfies

$$\overline{C_iC_j} \geq \sqrt{\delta}$$

for any two centers C_i and C_j.

Since $\overline{C_iC_j} \geq \sqrt{\delta} > \sqrt{2\delta/3}$, $\exists P_j \in B_j$ such that $P_j \notin \bigcup_{i=1, i\neq j}^{N}(1+\delta)B_i$ for $j = 1, \cdots, N$, and each ball must be selected such that the requirements in the covering lemma are satisfied.

Obviously,

$$N \sim \frac{c(1/2)^{n-1}}{(\sqrt{\delta})^{n-1}} \sim c\delta^{-\frac{n-1}{2}},$$

and

$$\sum_{i=1}^{N} \chi_{B_i}(O) = N \sim c\delta^{-\frac{n-1}{2}}.$$

In fact, the example in $\mathbb{R}^2$ is more illustrative:

Example 1. Consider a family of unit discs centered at

$$\frac{1}{2}(\cos\theta_i, \sin\theta_i) \in S(O, \frac{1}{2}) \subset \mathbb{R}^2,$$

where $\theta_i = 2i\sqrt{\delta}$, $i = 1, \cdots, N = \lfloor \frac{\pi}{\sqrt{\delta}} \rfloor - 1$. Then, for any two centers C_i and C_j, $\overline{C_iC_j} \gtrsim \sqrt{\delta} > \sqrt{2\delta/3}$ when δ is small enough, and each disc must be selected to satisfy the criterion in the covering theorem. Thus,

$$\sum_{i=1}^{N} \chi_{B_i}(O) = \lfloor \frac{\pi}{\sqrt{\delta}} \rfloor - 1 \sim c\delta^{-\frac{1}{2}}.$$

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ABSTRACT

NODAL GEOMETRY OF EIGENFUNCTIONS ON SMOOTH MANIFOLDS AND HARDY-LITTLEWOOD-SOBOLEV INEQUALITIES ON THE HEISENBERG GROUP

by

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AUGUST 2012

Advisor: Dr. Guozhen Lu

Major: Mathematics

Degree: Doctor of Philosophy

Part I Let $(\mathbb{M}, g)$ be a n dimensional smooth, compact, and connected Riemannian manifold without boundary, consider the partial differential equation (PDE) on $\mathbb{M}$:

$$-\Delta u = \lambda u,$$

in which Δ is the Laplace-Beltrami operator. That is, u is an eigenfunction with eigenvalue λ. We analyze the asymptotic behavior of eigenfunctions as $\lambda \to \infty$ (i.e., limit of high energy states) in terms of the following aspects.

- Local and global properties of eigenfunctions, including several crucial estimates for further investigation.
- BMO (bounded mean oscillation) estimates of eigenfunctions, and local geometric estimates of nodal domains (connected components of nonzero region).
- Write the nodal set of u as $\mathcal{N} = \{u = 0\}$, estimate the size of $\mathcal{N}$ using Hausdorff measure. Particularly, surrounding the conjecture that the $n-1$ dimensional Hausdorff measure is comparable to $\sqrt{\lambda}$, we discuss separately on lower bounds and upper bounds.
- A covering lemma which is used in the above estimates, it is of independent interest, and we also propose a conjecture concerning its sharp version.

Part II Let $\mathbb{H}^n$ be the Heisenberg group with homogeneous dimension $Q = 2n + 2$, we study the Hardy-Littlewood-Sobolev (HLS) inequality on $\mathbb{H}^n$:

$$\left| \int \int_{\mathbb{H}^n \times \mathbb{H}^n} \frac{\overline{f(u)} g(v)}{|u^{-1} v|^{\lambda}} du dv \right| \leq C \|f\|_r \|g\|_s,$$

and particularly its sharp version. Weighted Hardy-Littlewood-Sobolev inequalities with different weights shall also be investigated, and we solve the following problems.

- Establish the existence results of maximizers.
- Provide a upper bound of sharp constants.

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