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The probability density function, mean and variance of the inverse square-root transformed left-truncated $N(1, \sigma^2)$ error component $e_t^* \left(= \frac{1}{\sqrt{e_t}} \right)$ of the multiplicative time series model were established. A comparison of key-statistical properties of e_t^* and e_t confirmed normality with mean 1 but with $Var(e_t^*) \approx \frac{1}{4} Var(e_t)$ when $\sigma \leq 0.14$. Hence $\sigma \leq 0.14$ is the required condition for successful transformation.

Keywords: Multiplicative time series model, Error component, Left truncated normal distribution, Inverse square root transformation, Successful transformation, Moments

Introduction

The general multiplicative time series model for descriptive time series analysis is

$$X_t = T_t S_t C_t e_t, \quad t = 1, 2, \ldots n \tag{1}$$

where for time t, X_t denotes the observed value of the series, T_t is the trend, S_t, the seasonal component, C_t the cyclical term and e_t is the random or irregular component of the series. Model (1) is regarded as adequate when the irregular component is purely random. For a short period of time, the cyclical component is

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superimposed into the trend (Chatfield, 2004) to yield a trend-cycle component denoted by M_t and hence

$$X_t = M_t \, S_t \, e_t \tag{2}$$

where e_t are independent identically distributed normal errors with mean 1 and variance $\sigma^2 > 0$ $\left(e_t \sim N\left(1, \sigma^2\right)\right)$

According to Uche (2003), the left truncated normal distribution $\left(N\left(\mu, \sigma^2\right)\right)$ for X is

$$f^*(x) = \begin{cases} 0 & -\infty < x \le 0 \\ \dfrac{k e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\sigma\sqrt{2\pi}} & 0 < x < \infty \end{cases} \tag{3}$$

Using Equation 3, Iwueze (2007) obtained the left truncated normal distribution $\left(N\left(1, \sigma^2\right)\right)$ for $e_t \left(= X\right)$ as

$$f_{LTN}(x) = \begin{cases} 0 & -\infty < x \le 0 \\ \dfrac{e^{-\frac{1}{2}\left(\frac{x-1}{\sigma}\right)^2}}{\sigma\sqrt{2\pi}\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]} & 0 < x < \infty \end{cases} \tag{4}$$

with mean

$$E_{LTN}(X) = 1 + \frac{\sigma e^{-\frac{1}{2\sigma^2}}}{\sqrt{2\pi}\left(1 - \Phi\left(-\frac{1}{\sigma}\right)\right)} \tag{5}$$

and

$$Var_{LTN}\left(X\right)=\frac{\sigma^2}{2\left(1-\Phi\left(-\frac{1}{\sigma}\right)\right)}\left[1+\Pr\left(\chi^2_{(1)}<\frac{1}{\sigma^2}\right)\right] - \frac{\sigma e^{-\frac{1}{2\sigma^2}}}{\sqrt{2\pi}\left(1-\Phi\left(-\frac{1}{\sigma}\right)\right)} - \left[1+\frac{\sigma e^{-\frac{1}{2\sigma^2}}}{\sqrt{2\pi}\left(1-\Phi\left(-\frac{1}{\sigma}\right)\right)}\right]^2 \quad (6)$$

Iwueze (2007) also showed that $f_{LTN}\left(x\right) > 0$ provided $\sigma < 0.30$.

Data transformations are the application of mathematical modifications to values of a variable. There are a great variety of possible data transformations, including $\log_e\left(X_t\right), \sqrt{X_t}, \frac{1}{X_t}, \frac{1}{\sqrt{X_t}}, X_t^2$, and $\frac{1}{X_t^2}$. In practice many multiplicative time series data do not meet the assumptions of a parametric statistical analysis; they are not normally distributed, the variances are not homogenous or both. In analyzing such data, there are two choices:

i. Adjusting the data to fit the assumptions by making a transformation, or
ii. Developing new methods of analysis with assumptions which fit the data in its "original" form.

If a satisfactory transformation can be found, it will almost always be easier and simpler to use it rather than developing new methods of analysis (Turkey, 1957). Hence the need for this work which aims at finding conditions for satisfactory inverse square root transformation with respect to the error component of the multiplicative time series model from a study of its distribution. A transformation is considered satisfactory or successful, if the basic assumptions of the model are not violated after transformation. (Iwueze et al., 2008)The basic assumptions of a multiplicative time series model placed on the error component are: (i) unit mean (ii) constant variance (iii) Normality. According to Roberts (2008), transforming data made it much easier to work with - It was like sharpening a knife. For more information on choice of appropriate transformations see Osborne (2002), Osborne (2010) and Watthanacheewakul (2012).

Data Classification

For a time series data to be classified appropriate for inverse square root transformation,

i. the data must be amenable to the multiplicative time series model. The appropriateness of the multiplicative model is accessed by (a) displaying the data in the Buy's-Ballot Table. (b) Plotting the periodic (yearly) means (μ_i) and standard deviations σ_i against the period (year) i. If there is a dependency relationship between μ_i and σ_i, then the multiplicative model is appropriate.

ii. the variance must be unstable. The stability of the variance of the time series is ascertained by observing both the row and column means and standard deviations. If the variance is not stable the appropriate transformation is determined using Bartlett (1947) as was applied by Akpanta and Iwueze (2009);

$$Y = \begin{cases} \log_e X & , \quad \beta = 1 \\ X^{1-\beta} & , \quad \beta \neq 1 \end{cases} \tag{7}$$

The linear relationship between the natural log of periodic standard deviations ($\log_e \sigma_i$) and natural log of the periodic means ($\log_e \mu_i$) is given as

$$\log_e \sigma_i = \alpha + \beta \log_e \mu_i \tag{8}$$

The value of slope β according to Bartlett (1947) should be approximately 1.5 for the inverse square root transformation (see Table 1).

Table 1. Bartlett's transformations for some values of β

β	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	3	-1
Transformation	No transformation	$\sqrt{X}$	$\log_e X$	$\frac{1}{\sqrt{X}}$	$\frac{1}{X}$	$\frac{1}{X^2}$	X^2

Background of the Study

Since Iwueze (2007) investigated the effect of the logarithmic transformation on the error component, ($e_t \sim N(1, \sigma^2)$) of the multiplicative time series model, a number of studies investigating the effects of data transformation on the various components of the multiplicative time series model have been carried out. (See Iwueze et al., 2008; Iwu et al., 2009; Otuonye et al., 2011; Nwosu et al., 2013; and Ohakwe et al., 2013). The overall aim of such studies is to determine the conditions for successful transformation. That is, to establish the conditions where:

a. the required basic assumptions of the model are not violated after transformation, with respect to (i) the error term (ii) the seasonal component.
b. with respect to the trend component, there is no alteration in the form of the trend curve. In other words the form of the trend curve in the original series is maintained in the transformed series.

Iwueze (2007) found that the logarithmic transformation of the error component e_t $\left(e_t \sim N\left(1, \sigma^2\right)\right)$ to $e_t^* \left(= \log_e e_t\right)$ is normal with mean 0 and variance σ_1^2 provided $\sigma < 0.1$, in which case $\sigma_1 = \sigma$. It was established that the assumption for the error term e_t^*, for the additive model obtained after the logarithmic transformation, is valid if and only if $\sigma_1 < 0.10$. Observe from Table 1 that $\beta \approx 1$ for a time series data to be classified fit for logarithmic transformation.

Otuonye et al. (2011) investigated the distribution and properties of the error component of the multiplicative time series model under square root transformation, and found that the square root transformed error component $e_t^* \left(= \sqrt{e_t}\right)$ is normally distributed with mean ≈ 1 and variance $\approx \frac{1}{4}$ times that of the untransformed error component. That is $Var\left(e_t^*\right) = \frac{1}{4}\left[Var\left(e_t\right)\right]$ when $0 < \sigma \leq 0.3$. Thus $0 < \sigma \leq 0.3$ is the recommended condition for successful square root transformation. Only time series data with $\beta \approx \frac{1}{2}$ are classified fit for square root transformation. Similarly, Nwosu et al. (2013), while investigating the distribution of the inverse transformed error component of the multiplicative time series model $e_t^*\left(=\frac{1}{e_t}\right)$, obtained that the desirable statistical properties of e_t and e_t^* were found to be approximately the same and normally distributed with unit mean for $\sigma \leq 0.10$. Hence, $\sigma \leq 0.10$ is the recommended condition for successful inverse transformation of the multiplicative time series model. Time series data classified fit for inverse transformation must have $\beta \approx 2$. Also, Ohakwe et al. (2013) found that for the square transformation $e_t^*\left(=e_t^2\right)$ that $e_t^* \sim N(1,1)$ in the interval $0 < \sigma \leq 0.027$. Hence, $0 < \sigma \leq 0.027$is the condition for successful square transformation. Observe that a time series data is classified fit for square transformation when $\beta \approx$ -1.

Note that the overall aim of these works is to establish conditions for successful transformation, hence provide better choice of right transformation. According to Roberts (2008), choosing a good transformation improved his analyses in three ways: (i) increase in visual clarity as graphs were made more informative (ii). Reduction or elimination of outliers (iii). Increase in statistical clarity; his statistical test became more sensitive, F and t values increased making it more likely to detect differences when they exist.

Justification for this Study

The value of the slope β, categorized time series data into mutually exclusive groups, in the sense that any time series data belongs exclusively to one and only one group hence can only be appropriately transformed by only one of the six transformations listed in Table 1. Thus despite the fact that Iwueze (2007), Otuonye et al., (2011), Nwosu et al. (2013), and Ohakwe et al. (2013) carried out similar studies with respect to the logarithmic, square root, inverse and square transformations respectively, this work on inverse square root transformation is still very necessary since results established for the above listed four transformations cannot be applied in the analysis of time series data requiring inverse square root transformation.

Inverse Square Root Transformation

When $\beta \approx \frac{3}{2}$, adopt inverse square root transformation on the multiplicative time series model given in Equation 2 to obtain

$$Y_t = \frac{1}{\sqrt{X_t}} = \frac{1}{\sqrt{M_t}}\frac{1}{\sqrt{S_t}}\frac{1}{\sqrt{e_t}} = M_t^* S_t^* e_t^* \qquad (9)$$

where $M_t^* = \frac{1}{\sqrt{M_t}}$, $S_t^* = \frac{1}{\sqrt{S_t}}$ *and* $e_t^* = \frac{1}{\sqrt{e_t}}$, $e_t > 0$

Because e_t does not admit negative or zero values, the use of the left truncated normal distribution as the pdf of e_t shall be exploited.

Thus, it will be of interest to find what the distribution of e_t^* is. Is $e_t^* \sim$ iid $N(1, \sigma_1^2)$. What is the relationship between σ_1^2 and σ^2?

Aim and Objectives

The aim of this work is to obtain the distribution of the inverse square root transformed error component of the multiplicative time series model and the objectives are:

i. to examine the nature of the distribution.
ii. to verify the satisfaction of the assumption on the mean of the error terms; $\mu = 1$.
iii. to determine the relationship between σ_1^2 and σ^2.

Methodology

To achieve the above stated objectives the following were conducted:

Let $X = e_t$ and $Y = e_t^* = \frac{1}{\sqrt{e_t}} = \frac{1}{\sqrt{X}}$

1. Obtain the pdf of e_t^*, $g(y)$.
2. Plot the curves of the two pdfs, $g(y)$ and $f_{LTN}(x)$ for various values of σ.
3. Obtain the region where $g(i)$ satisfies the following normality conditions (Bell-shaped conditions).

   i. Mode $\approx 1 \approx$ Mean.
   ii. Median $\approx$ Mean $\approx$ 1.

iii. Approved normality test, Anderson Darling's test statistic (AD) was used to confirm the normality of the simulated error terms e_t and the inverse square root transformed error term.

$Y = e_t^* = \dfrac{1}{\sqrt{e_t}} = \dfrac{1}{\sqrt{X}}$ for some values of σ

iv. Obtain and use the functional expressions for the mean and variance of e_t^* to validate some of the results obtained using simulated data.

The probability density function of $Y = \dfrac{1}{\sqrt{x}}$, $g(y)$

Given the pdf of X in Equation 4 and the transformation

$$Y = \frac{1}{\sqrt{x}}$$

then

$$X = \frac{1}{y^2} \text{ and } \frac{dx}{dy} = -\frac{2}{y^3}$$

using the transformation of variable technique

$$g(y) = f_{LTN}(x)\left|\frac{dx}{dy}\right|$$

(see Freund & Walpole, 1986). Hence

$$g(y) = \begin{cases} \dfrac{2}{y^3\sigma\sqrt{2\pi}\left[1-\Phi\left(\dfrac{-1}{\sigma}\right)\right]} e^{\frac{-1}{2\sigma^2}\left(\frac{1}{y^2}-1\right)^2} & ,0 < y < \infty \\ 0 & -\infty < y < 0 \end{cases} \tag{10}$$

Plot of the Probability density curves $f_{LTN}(x) = f^*(x)$ and *g*(*y*)

Using the pdf of the two variables given in Equation 4 and Equation 10, the curves $f^*(x)$ and $g(y)$ were plotted for some values of $\sigma \in (0, 0.4]$. For want of space only five are shown in Figures 1 to 5.

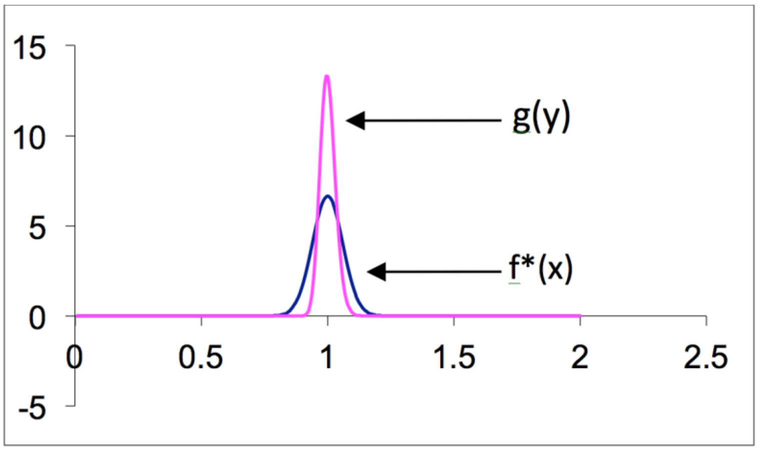

Figure 1. Curve Shapes for $\sigma = 0.06$

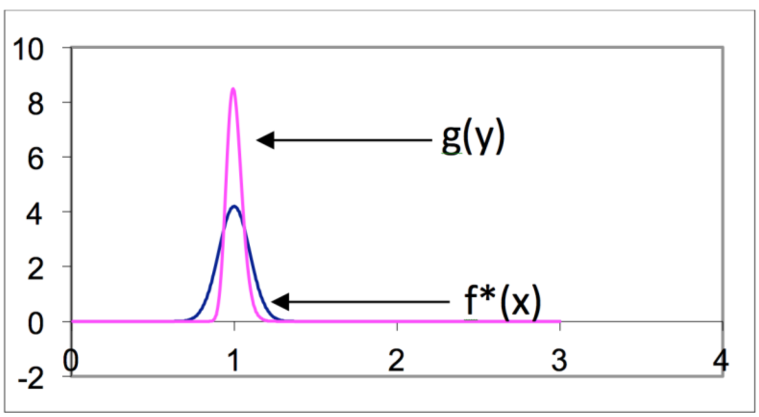

Figure 2. Curve Shapes for $\sigma = 0.095$

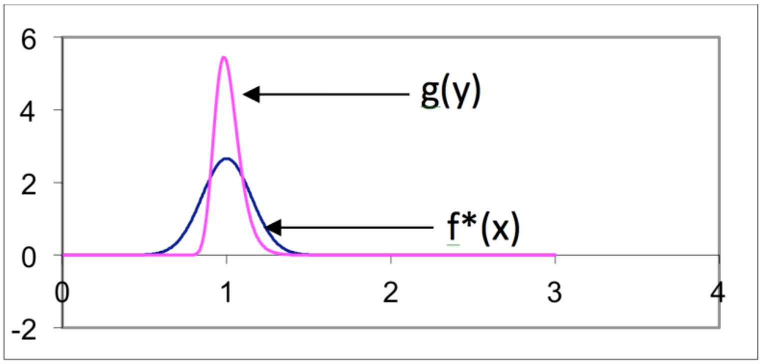

Figure 3. Curve Shapes for σ = 0.15

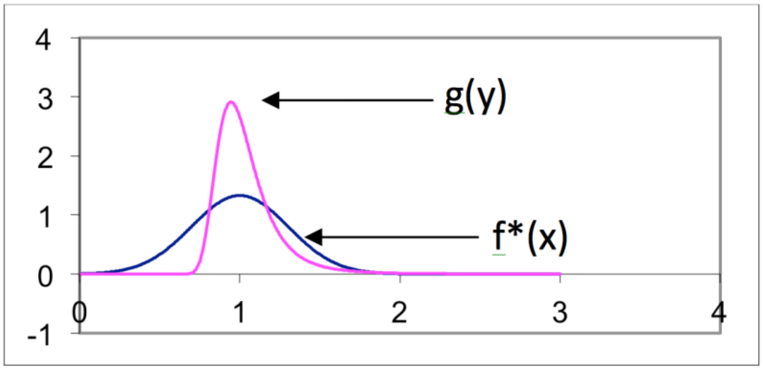

Figure 4. Curve Shapes for σ = 0.3

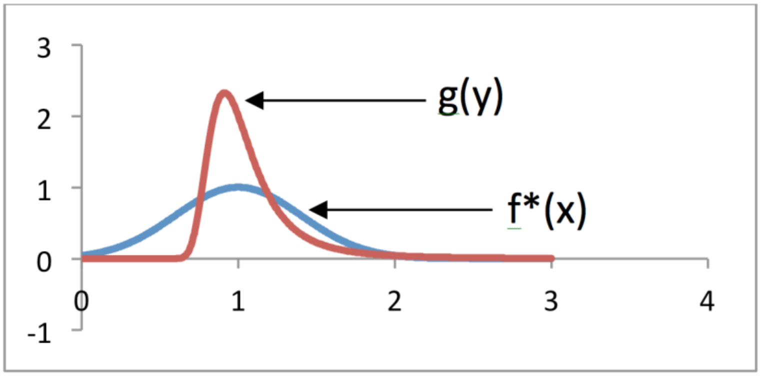

Figure 5. Curve Shapes for σ = 0.4

Observations:

i. The curve $g(y)$ is positively skewed for $\sigma > 0.15$ (see Figures 3-5).
ii. $f^*(x)$ is positively skewed for $\sigma > 0.30$ (see Figure 5) as reported in Iwueze (2007).

Normality Region for *g*(*y*)

From Figures 1 to 5, it is clear that the curve $g(y)$ has one maximum point, $y_{\max}$ (mode), and one maximum value, $g(y_{\max})$, for all values of σ. To obtain the values of σ that satisfy the symmetric and bell-shaped condition of mode = mean, we invoke Rolle's Theorem and proceed to obtain the maximum point (mode) for a given value of σ.

Differentiating $g(y)$ in Equation 10 gives

$$g'(y) = \frac{2}{\sigma\sqrt{2\pi\left[1-\Phi\left(\frac{-1}{\sigma}\right)\right]}}\left[-3y^{-4}e^{\frac{-1}{2\sigma^2}\left(\frac{1}{y^2}-1\right)^2} + y^{-3}\left(\frac{-1}{2\sigma^2}\right)\left(\frac{-4}{y^3}\right)\left(\frac{1}{y^2}-1\right)e^{\frac{-1}{2\sigma^2}\left(\frac{1}{y^2}-1\right)^2}\right] \quad (11)$$

$$= \frac{2e^{\frac{-2}{2\sigma^2}\left(\frac{1}{y^2}-1\right)^2}}{\sigma\sqrt{2\pi\left[1-\Phi\left(\frac{-1}{\sigma}\right)\right]}}\left[\frac{2(1-y^2)}{\sigma^2 y^8}-\frac{3}{y^4}\right]$$

Equating $g`(y) = 0$, gives

$$\frac{2(1-y^2)}{\sigma^2 y^8}-\frac{3}{y^4}=0$$

$$3\sigma^2 y^4 + 2y^2 - 2 = 0 \quad (12)$$

Putting $w = y^2$ in Equation 12, gives

$$3\sigma^2 w^2 + 2w - 2 = 0 \tag{13}$$

Solving Equation 13, gives

$$w = \frac{-1 \pm \sqrt{1+6\sigma^2}}{3\sigma^2}$$

Because $y_{\max}$ is positive
then

$$w = \frac{-1 + \sqrt{1+6\sigma^2}}{3\sigma^2}$$

hence

$$y = \pm\sqrt{\frac{-1 + \sqrt{1+6\sigma^2}}{3\sigma^2}}$$

and

$$y_{\max} = \sqrt{\frac{-1 + \sqrt{1+6\sigma^2}}{3\sigma^2}}$$

The bell-shaped condition would imply $y_{\max} \approx 1$, see Table 2 for the numerical computation of

$$y_{\max} = \sqrt{\frac{-1 + \sqrt{1+6\sigma^2}}{3\sigma^2}}$$

Table 2. Computation of $y_{\max} = \sqrt{\frac{-1+\sqrt{1+6\sigma^2}}{3\sigma^2}}$, for $\sigma\ \varepsilon$ [0.01, 0.3]

σ	$y_{\max} = \sqrt{\frac{-1+\sqrt{1+6\sigma^2}}{3\sigma^2}}$	$1-y_{\max}$	σ	$y_{\max} = \sqrt{\frac{-1+\sqrt{1+6\sigma^2}}{3\sigma^2}}$	$1-y_{\max}$
0.010	0.99992502	0.000075	0.155	0.94470721	0.055293
0.015	0.99970031	0.000300	0.160	0.94163225	0.058368
0.020	0.99932659	0.000673	0.165	0.93852446	0.061476
0.025	0.99880501	0.001195	0.170	0.93538739	0.064613
0.030	0.99813720	0.001863	0.175	0.93222440	0.067776
0.035	0.99732519	0.002675	0.180	0.92903869	0.070961
0.040	0.99637147	0.003629	0.185	0.92583333	0.074167
0.045	**0.99527886**	0.004721	0.190	0.92261120	0.077389
0.050	0.99405059	0.005949	0.195	0.91937505	0.080625
0.055	0.99269018	0.007310	0.200	0.91612748	0.083873
0.060	0.99120149	0.008799	0.205	0.91287093	0.087129
0.065	0.98958860	0.010411	0.210	0.90960772	0.090392
0.070	0.98785584	0.012144	0.215	0.90634001	0.093660
0.075	0.98600775	0.013992	0.220	0.90306986	0.096930
0.080	0.98404899	0.015951	0.225	0.89979918	0.100201
0.085	0.98198438	0.018016	0.230	0.89652976	0.103470
0.090	0.97981881	0.020181	0.235	0.89326328	0.106737
0.095	0.97755725	0.022443	0.240	0.89000132	0.109999
0.100	0.97520469	0.024795	0.245	0.88674534	0.113255
0.105	0.97276613	0.027234	0.250	0.88349669	0.116503
0.110	0.97024653	0.029753	0.255	0.88025665	0.119743
0.115	0.96765082	0.032349	0.260	0.87702640	0.122974
0.120	0.96498387	0.035016	0.265	0.87380702	0.126193
0.125	0.96225045	0.037750	0.270	0.87059952	0.129400
0.130	0.95945523	0.040545	0.275	0.86740484	0.132595
0.135	0.95660279	0.043397	0.280	0.86422383	0.135776
0.140	0.95369754	0.046302	0.285	0.86105729	0.138943
0.145	**0.95074378**	0.049256	0.290	0.85790594	0.142094
0.150	0.94774567	0.052254	0.295	0.85477043	0.145230
			0.300	0.85165139	0.148349

Thus $g(y)$ is symmetrical about 1 with Mode $\approx 1 \approx$ Mean correct to two decimal places when $0 < \sigma < 0.045$ and correct to one decimal place when $0 < \sigma < 0.045$.

Use of simulated error terms

To find the region where the bell-shaped conditions (ii-iii) listed in methodology are satisfied, we made use of artificial data generated from $N\left(1,\sigma^2\right)$ for e_t, subsequently transformed to obtain $e_t^* = \frac{1}{\sqrt{e_t}}$ for $0.05 \le \sigma \le 0.20$. Values of the required statistical characteristics were obtained for each variable e_t and e_t^* as shown in Tables 3 to 6. For each configuration of (n = 100, 0.05 ≤ σ ≤ 0.15), 1000 replications were performed for values of σ in steps of 0.01. For want of space the results of the first 25 replications are shown for the configurations, (n = 100, σ = 0.06), (n = 100, σ = 0.1), (n = 100, σ = 0.15), and (n = 100, σ = 0.2).

Functional expressions for the mean and variance of *g*(*y*)

By definition, the mean of Y, $E(Y)$ is given by:

$$E(Y) = \int_0^\infty y g(y) dy = \frac{2}{\sigma\sqrt{2\pi}\left[1-\Phi\left(\frac{1}{\sigma}\right)\right]}\int_0^\infty \frac{1}{y^2} e^{\frac{-1}{2\sigma^2}\left(\frac{1}{y^2}-1\right)^2} dy \qquad (14)$$

let $u = \frac{1}{y^2}$, then $y = \frac{1}{u^{\frac{1}{2}}}$ and $dy = \frac{-du}{2u^{\frac{3}{2}}}$, for $\infty < u < 0$

$$\therefore E(Y) = k\int_\infty^0 u e^{\frac{-1}{2}\left(\frac{u-1}{\sigma}\right)^2} \frac{-du}{2u^{\frac{3}{2}}} \cdot = \frac{k}{2}\int_0^\infty u^{\frac{-1}{2}} e^{\frac{-1}{2}\left(\frac{u-1}{\sigma}\right)^2} du \cdots \qquad (15)$$

$$\text{where } k = \frac{2}{\sigma\sqrt{2\pi}\left[1-\Phi\left(\frac{-1}{\sigma}\right)\right]}$$

let $z = \frac{u-1}{\sigma}$, then $z\sigma + 1 = u$ and $du = \sigma dz$ for $\frac{-1}{\sigma} < z < \infty$

$$\therefore E(Y) = \frac{k}{2}\int_{\frac{-1}{\sigma}}^\infty (1+z\sigma)^{\frac{-1}{2}} e^{\frac{-1}{2}z^2} \sigma dz = \frac{\sigma k}{2}\int_{\frac{-1}{\sigma}}^\infty (1+z\sigma)^{\frac{-1}{2}} e^{\frac{-z^2}{2}} dz \qquad (16)$$

Table 3. Simulation Results when σ = 0.06

$$X = e_t \sim N\left(1,\sigma^2\right), \sigma = 0.06 \qquad Y = e_t^* = \frac{1}{\sqrt{e_t}}, e_t \sim N\left(1,\sigma^2\right), \sigma = 0.06$$

Mean	StD	Variance	Median	AD	p-value	Mean	StD	Variance	Median	AD	p-value
1	0.06	0.0036	0.9927	.235	.788	1.0013	0.0303	0.000918	1.0037	.206	.867
1	0.06	0.0036	1.0009	.183	.908	1.0013	0.0302	0.000914	0.9995	.298	.580
1	0.06	0.0036	1.0002	.195	.889	1.0013	0.0303	0.000916	0.9999	.275	.654
1	0.06	0.0036	1.0029	.234	.790	1.0013	0.0303	0.000917	0.9985	.334	.505
1	0.06	0.0036	1.0037	.178	.918	1.0013	0.0302	0.000915	0.9982	.312	.546
1	0.06	0.0036	1.0045	.435	.294	1.0013	0.0301	0.000908	0.9978	.364	.433
1	0.06	0.0036	1.0037	.178	.918	1.0013	0.0302	0.000915	0.9982	.312	.546
1	0.06	0.0036	1.0013	.137	.976	1.0013	0.0302	0.000910	0.9993	.213	.851
1	0.06	0.0036	0.9941	.196	.888	1.0013	0.0302	0.000911	1.0030	.302	.569
1	0.06	0.0036	1.0017	.250	.739	1.0014	0.0304	0.000924	0.9991	.453	.266
1	0.06	0.0036	1.0004	.200	.880	1.0013	0.0302	0.000915	0.9998	.314	.540
1	0.06	0.0036	1.0045	.435	.294	1.0013	0.0301	0.000908	0.9978	.364	.433
1	0.06	0.0036	0.9991	.183	.908	1.0013	0.0303	0.000916	1.0005	.214	.846
1	0.06	0.0036	0.9983	.250	.739	1.0013	0.0301	0.000908	1.0009	.206	.866
1	0.06	0.0036	1.0010	.209	.859	1.0013	0.0300	0.000901	0.9995	.241	.767
1	0.06	0.0036	1.0028	.195	.889	1.0013	0.0302	0.000913	0.9986	.284	.625
1	0.06	0.0036	1.0031	.141	.972	1.0013	0.0302	0.000911	0.9985	.208	.862
1	0.06	0.0036	0.9975	.310	.550	1.0013	0.0299	0.000894	1.0012	.232	.795
1	0.06	0.0036	1.0006	.262	.699	1.0014	0.0304	0.000924	0.9997	.385	.387
1	0.06	0.0036	0.9983	.182	.911	1.0013	0.0302	0.000913	1.0009	.318	.531
1	0.06	0.0036	0.9958	.150	.962	1.0013	0.0303	0.000916	1.0021	.218	.835
1	0.06	0.0036	0.9938	.290	.606	1.0013	0.0299	0.000896	1.0031	.185	.906
1	0.06	0.0036	0.9931	.450	.270	1.0013	0.0300	0.000903	1.0035	.336	.503
1	0.06	0.0036	0.9950	.199	.882	1.0013	0.0301	0.000907	1.0025	.390	.376
1	0.06	0.0036	0.9987	.216	.841	1.0013	0.0302	0.000914	1.0006	.315	.538
1	0.06	0.0036	0.9942	.311	.546	1.0013	0.0300	0.000899	1.0029	.165	.940

*Note. For each row, $\frac{Var(e_t)}{Var(e_t^*)}$ equals 4.

Table 4. Simulation Results when $\sigma = 0.1$

$$X = e_t \sim N\left(1, \sigma^2\right), \sigma = 0.1 \qquad Y = e_t^* = \frac{1}{\sqrt{e_t}}, e_t \sim N\left(1, \sigma^2\right), \sigma = 0.1$$

Mean	StD	Variance	Median	AD	p-value	Mean	StD	Variance	Median	AD	p-value
1	0.1	0.01	0.9878	.235	0.788	1.0038	0.0514	0.00265	1.0061	.298	.582
1	0.1	0.01	1.0016	.183	0.908	1.0038	0.0511	0.00262	0.9992	.457	.260
1	0.1	0.01	1.0003	.195	0.889	1.0038	0.0513	0.00263	0.9998	.428	.306
1	0.1	0.01	1.0049	.234	0.790	1.0038	0.0513	0.00264	0.9976	.502	.201
1	0.1	0.01	1.0062	.178	0.918	1.0038	0.0512	0.00262	0.9969	.495	.211
1	0.1	0.01	1.0074	.435	0.294	1.0038	0.0509	0.00259	0.9963	.424	.313
1	0.1	0.01	1.0062	.178	0.918	1.0038	0.0512	0.00262	0.9969	.495	.211
1	0.1	0.01	1.0022	.137	0.976	1.0038	0.0509	0.00259	0.9989	.357	.450
1	0.1	0.01	0.9902	.196	0.888	1.0038	0.0510	0.00260	1.0050	.464	.251
1	0.1	0.01	1.0029	.250	0.739	1.0038	0.0516	0.00267	0.9986	.685	.071
1	0.1	0.01	1.0007	.200	0.880	1.0038	0.0512	0.00262	0.9997	.495	.210
1	0.1	0.01	1.0074	.435	0.294	1.0038	0.0509	0.00259	0.9963	.424	.313
1	0.1	0.01	0.9984	.183	0.908	1.0038	0.0513	0.00263	1.0008	.326	.516
1	0.1	0.01	0.9971	.250	0.739	1.0038	0.0509	0.00259	1.0014	.272	.664
1	0.1	0.01	1.0016	.209	0.859	1.0037	0.0505	0.00255	0.9992	.359	.445
1	0.1	0.01	1.0047	.195	0.889	1.0038	0.0511	0.00261	0.9977	.446	.277
1	0.1	0.01	1.0052	.141	0.972	1.0038	0.0510	0.00260	0.9974	.346	.477
1	0.1	0.01	0.9959	.310	0.550	1.0037	0.0502	0.00252	1.0021	.278	.642
1	0.1	0.01	1.0011	.262	0.699	1.0038	0.0516	0.00266	0.9995	.554	.150
1	0.1	0.01	0.9971	.182	0.911	1.0038	0.0511	0.00261	1.0014	.499	.205
1	0.1	0.01	0.9931	.150	0.962	1.0038	0.0513	0.00263	1.0035	.368	.424
1	0.1	0.01	0.9897	.290	0.606	1.0037	0.0503	0.00253	1.0052	.221	.827
1	0.1	0.01	0.9884	.450	0.270	1.0037	0.0506	0.00256	1.0058	.366	.428
1	0.1	0.01	0.9917	.306	0.559	1.0038	0.0508	0.00258	1.0042	.547	.156
1	0.1	0.01	0.9979	.199	0.882	1.0038	0.0511	0.00261	1.0011	.497	.207
1	0.1	0.01	0.9904	.216	0.841	1.0037	0.0504	0.00254	1.0048	.226	.815

*Note. For each row, $\frac{Var\left(e_t\right)}{Var\left(e_t^*\right)}$ equals 4.

Table 5. Simulation Results when $\sigma = 0.15$

$$X = e_t \sim N\left(1, \sigma^2\right), \sigma = 0.15$$

$$Y = e_t^* = \frac{1}{\sqrt{e_t}}, e_t \sim N\left(1, \sigma^2\right), \sigma = 0.15$$

Mean	StD	Variance	Median	AD	p-value		Mean	StD	Variance	Median	AD	p-value
1	0.15	0.0225	0.9818	.235	.788	*	1.0089	0.0803	0.00645	1.0092	.582	.126
1	0.15	0.0225	1.0024	.183	.908		1.0088	0.0791	0.00626	0.9988	.761	.046
1	0.15	0.0225	1.0005	.195	.889		1.0088	0.0798	0.00637	0.9997	.756	.047
1	0.15	0.0225	1.0073	.234	.790		1.0088	0.0798	0.00636	0.9964	.857	.027
1	0.15	0.0225	1.0093	.178	.918		1.0088	0.0792	0.00628	0.9954	.842	.029
1	0.15	0.0225	1.0111	.435	.294		1.0087	0.0788	0.00620	0.9945	.646	.089
1	0.15	0.0225	1.0093	.178	.918		1.0088	0.0792	0.00628	0.9954	.842	.029
1	0.15	0.0225	1.0034	.137	.976		1.0087	0.0786	0.00618	0.9983	.656	.085
1	0.15	0.0225	0.9853	.196	.888	*	1.0087	0.0788	0.00621	1.0075	.785	.040
1	0.15	0.0225	1.0043	.250	.739		1.0089	0.0804	0.00646	0.9979	1.109	.005
1	0.15	0.0225	1.0010	.200	.880		1.0088	0.0793	0.00628	0.9995	.860	.026
1	0.15	0.0225	1.0111	.435	.294		1.0087	0.0788	0.00620	0.9945	.646	.089
1	0.15	0.0225	0.9976	.183	.908		1.0088	0.0796	0.00633	1.0012	.596	.119
1	0.15	0.0225	0.9957	.250	.739		1.0087	0.0788	0.00621	1.0022	.486	.221
1	0.15	0.0225	1.0025	.209	.859		1.0086	0.0775	0.00601	0.9988	.620	.104
1	0.15	0.0225	1.0070	195	889		1.0088	0.0791	0.00626	0.9965	.779	.042
1	0.15	0.0225	1.0077	141	.972		1.0087	0.0787	0.00619	0.9962	.635	.095
1	0.15	0.0225	0.9938	.310	.550		1.0085	0.0770	0.00593	1.0031	.450	.271
1	0.15	0.0225	1.0016	.262	.699		1.0089	0.0799	0.00639	0.9992	.880	.023
1	0.15	0.0225	0.9957	.182	.911		1.0087	0.0789	0.00622	1.0022	.838	.030
1	0.15	0.0225	0.9896	.500	.962		1.0088	0.0798	0.00636	1.0052	.701	.065
1	0.15	0.0225	0.9846	.290	.606		1.0085	0.0770	0.00593	1.0078	.398	.361
1	0.15	0.0225	0.9826	.450	.270		1.0086	0.0781	0.00609	1.0088	.545	.157
1	0.15	0.0225	0.9876	.306	.559		1.0087	0.0782	0.00611	1.0063	.868	.025
1	0.15	0.0225	0.9968	.199	.882		1.0088	0.0790	0.00624	1.0016	.860	.026
1	0.15	0.0225	0.9856	.216	.841		1.0085	0.0772	0.00596	1.0073	.419	.322

*Note. For each row, $\frac{Var(e_t)}{Var(e_t^*)}$ equals 4 except where indicated by *. For those rows, $\frac{Var(e_t)}{Var(e_t^*)}$ equals 3.

Table 6. Simulation Results when σ = 0.2

$$X = e_t \sim N\left(1, \sigma^2\right), \sigma = 0.2$$

$$Y = e_t^* = \frac{1}{\sqrt{e_t}}, e_t \sim N\left(1, \sigma^2\right), \sigma = 0.2$$

Mean	StD	Variance	Median	AD	p-value	Mean	StD	Variance	Median	AD	p-value
1	0.2	0.04	0.9757	.235	0.788	1.0167	0.1147	0.0132	1.0124	1.176	<0.005
1	0.2	0.04	1.0032	.183	0.908	1.0162	0.1107	0.0123	0.9984	1.220	<0.005
1	0.2	0.04	1.0007	.195	0.889	1.0165	0.1127	0.0127	0.9997	1.315	<0.005
1	0.2	0.04	1.0097	.234	0.790	1.0164	0.1124	0.0126	0.9952	1.435	<0.005
1	0.2	0.04	1.0124	.178	0.918	1.0163	0.1109	0.0123	0.9939	1.353	<0.005
1	0.2	0.04	1.0148	.435	0.294	1.0161	0.1105	0.0122	0.9927	1.097	0.007
1	0.2	0.04	1.0124	.178	0.918	1.0163	0.1109	0.0123	0.9939	1.353	<0.005
1	0.2	0.04	1.0045	.137	0.976	1.0161	0.1095	0.0120	0.9978	1.117	0.006
1	0.2	0.04	0.9803	.196	0.888	1.0161	0.1100	0.0121	1.0100	1.276	<0.005
1	0.2	0.04	1.0057	.250	0.739	1.0166	0.1133	0.0128	0.9971	1.734	<0.005
1	0.2	0.04	1.0013	.200	0.880	1.0163	0.1110	0.0123	0.9994	1.418	<0.005
1	0.2	0.04	1.0149	.435	0.294	1.0161	0.1105	0.0122	0.9927	1.097	0.007
1	0.2	0.04	0.9968	.183	0.908	1.0164	0.1120	0.0125	1.0016	1.072	0.008
1	0.2	0.04	0.9943	.250	0.739	1.0162	0.1107	0.0123	1.0029	0.915	0.019
1	0.2	0.04	1.0033	.209	0.859	1.0157	0.1072	0.0115	0.9984	1.026	0.010
1	0.2	0.04	1.0094	.195	0.889	1.0162	0.1109	0.0123	0.9953	1.293	<0.005
1	0.2	0.04	1.0103	.141	0.972	1.0161	0.1097	0.0120	0.9949	1.084	0.007
1	0.2	0.04	0.9917	.310	0.550	1.0156	0.1066	0.0114	1.0042	0.768	0.045
1	0.2	0.04	1.0021	.260	0.699	1.0165	0.1119	0.0125	0.9989	1.371	<0.005
1	0.2	0.04	0.9942	.182	0.911	1.0162	0.1100	0.0121	1.0029	1.331	<0.005
1	0.2	0.04	0.9862	.150	0.962	1.0165	0.1128	0.0127	1.007	1.267	<0.005
1	0.2	0.04	0.9795	.290	0.606	1.0156	0.1064	0.0113	1.0104	0.745	0.051
1	0.2	0.04	0.9768	.450	0.270	1.0159	0.109	0.0119	1.0118	0.933	0.017
1	0.2	0.04	0.9835	.306	0.559	1.0159	0.1084	0.0118	1.0084	1.348	<0.005
1	0.2	0.04	0.9958	.199	0.882	1.0162	0.1101	0.0121	1.0021	1.402	<0.005
1	0.2	0.04	0.9808	.216	0.841	1.0156	0.1066	0.0114	1.0097	0.766	0.045

*Note. For each row, $\frac{Var(e_t)}{Var(e_t^*)}$ equals 3.

Using the binomial expansion ,

$$(1+x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \ldots \tag{17}$$

(Smith and Minton, 2008).

$$\therefore (1+z\sigma)^{\frac{-1}{2}} = \cdot 1 + \frac{\left(\frac{-1}{2}\right)z\sigma}{1!} + \frac{\frac{-1}{2}\left(\frac{-1}{2}-1\right)(z\sigma)^2}{2!} + \frac{\frac{-1}{2}\left(\frac{-1}{2}-1\right)\left(\frac{-1}{2}-2\right)(z\sigma)^3}{3!} + \ldots$$

$$= 1 - \frac{z\sigma}{2} + \frac{3(z\sigma)^2}{8} - \frac{15(z\sigma)^3}{48} + \ldots \tag{18}$$

$$\therefore E(Y) = \frac{\sigma k}{2}\int_{\frac{-1}{\sigma}}^{\infty}\left[1 - \frac{z\sigma}{2} + \frac{3(z\sigma)^2}{8} - \frac{15(z\sigma)^3}{48} + \ldots\right] e^{\frac{-z^2}{2}} d \tag{19}$$

$$= \frac{1}{\sqrt{2\pi}\left[1-\Phi\left(\frac{-1}{\sigma}\right)\right]}\left[\int_{\frac{-1}{\sigma}}^{\infty} e^{\frac{-z^2}{2}} dz - \int_{\frac{-1}{\sigma}}^{\infty} \frac{z\sigma}{2} e^{\frac{-z^2}{2}} dz + \int_{\frac{-1}{\sigma}}^{\infty} \frac{3(z\sigma)^2}{8} e^{\frac{-z^2}{2}} dz - \int_{\frac{-1}{\sigma}}^{\infty} \frac{15(z\sigma)^3}{48} e^{\frac{-z^2}{2}} dz + \ldots\right]$$

$$E(Y) = \frac{1}{\left[1-\Phi\left(\frac{-1}{\sigma}\right)\right]}\left[\begin{array}{l} \frac{1}{\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{\infty} e^{\frac{-z^2}{2}} dz - \frac{1}{\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{\infty} \frac{z\sigma}{2} e^{\frac{-z^2}{2}} dz \\ + \frac{1}{\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{\infty} \frac{3(z\sigma)^2}{8} e^{\frac{-z^2}{2}} dz - \frac{1}{\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{\infty} \frac{15(z\sigma)^3}{48} e^{\frac{-z^2}{2}} dz + \ldots \end{array}\right]$$

$$E(Y)=\frac{1}{\left[1-\Phi\left(\frac{-1}{\sigma}\right)\right]}\left[\begin{array}{l}\frac{1}{\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{\infty}e^{\frac{-z^2}{2}}dz-\frac{\sigma}{\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{\infty}\frac{z}{2}e^{\frac{-z^2}{2}}dz\\+\frac{\sigma^2}{\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{\infty}\frac{3z^2}{8}e^{\frac{-z^2}{2}}dz-\frac{\sigma^3}{\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{\infty}\frac{15z^3}{48}e^{\frac{-z^2}{2}}dz+\ldots\end{array}\right]$$

$$=\frac{1}{\left[1-\Phi\left(\frac{1}{\sigma}\right)\right]}\left[\begin{array}{l}\Pr\left(z>\frac{-1}{\sigma}\right)-\frac{\sigma}{2\sqrt{2\pi}}e^{\frac{-1}{2\sigma^2}}+\frac{3}{8}\left(\frac{\sigma^2}{2}-\frac{\sigma e^{\frac{-1}{2\sigma^2}}}{\sqrt{2\pi}}+\frac{\sigma^2}{2}\Pr\left(x^2{}_{(1)}<\frac{1}{\sigma^2}\right)\right)\\-\frac{5}{16}\left(\frac{\sigma}{\sqrt{2\pi}}e^{\frac{-1}{2\sigma^2}}-\frac{2\sigma^3}{\sqrt{2\pi}}e^{\frac{-1}{2\sigma^2}}\right)+\ldots\end{array}\right]$$

$$=\frac{1}{\left[1-\Phi\left(\frac{-1}{\sigma}\right)\right]}\left[\begin{array}{l}\left[1-\Phi\left(\frac{-1}{\sigma}\right)\right]-\frac{\sigma}{2\sqrt{2\pi}}.e^{\frac{-1}{2\sigma^2}}+\frac{3\sigma}{8\sqrt{2\pi}}e^{\frac{-1}{2\sigma^2}}-\frac{5\sigma}{16\sqrt{2\pi}}e^{\frac{-1}{2\sigma^2}}\\+\frac{3\sigma^2}{16}\left[1+\Pr\left(\chi^2{}_{(1)}<\frac{1}{\sigma^2}\right)\right]-\frac{10\sigma^3}{26\sqrt{2\pi}}e^{\frac{-1}{2\sigma^2}}+\ldots\end{array}\right]$$

$$E(Y)=\frac{1}{\left[1-\Phi\left(\frac{-1}{\sigma}\right)\right]}\left[\begin{array}{l}\left[1-\Phi\left(\frac{-1}{\sigma}\right)\right]-\frac{\sigma\left(8e^{\frac{-1}{2\sigma^2}}+6e^{\frac{-1}{2\sigma^2}}+5e^{\frac{-1}{2\sigma^2}}\right)}{16\sqrt{2\pi}}\\+\frac{3\sigma^2}{16}\left[1+\Pr\left(\chi^2{}_{(1)}<\frac{1}{\sigma^2}\right)\right]-\frac{10\sigma^3}{16\sqrt{2\pi}}e^{\frac{-1}{2\sigma^2}}+\ldots\end{array}\right]$$

$$\begin{array}{l}\therefore E(Y)=1-\frac{19\sigma e^{\frac{-1}{2\sigma^2}}}{16\sqrt{2\pi}\left[1-\Phi\left(\frac{1}{\sigma}\right)\right]}+\frac{3\sigma^2}{16\left[1-\Phi\left(\frac{1}{\sigma}\right)\right]}\left[1+\Pr\left(\chi^2{}_{(1)}<\frac{1}{\sigma^2}\right)\right]\\-\frac{10\sigma^3 e\frac{-1}{e^{2\sigma^2}}}{16\sqrt{2\pi}\left[1-\Phi\left(\frac{1}{\sigma}\right)\right]}+\ldots\end{array}\qquad(20)$$

To find the variance, first obtain the second moment;

$$E(Y^2) = \int_0^\infty y^2 g(y)dy$$

$$= \frac{2}{\sigma\sqrt{2\pi}\left[1-\Phi\left(\frac{-1}{\sigma}\right)\right]}\int_0^\infty \frac{1}{y} e^{\frac{-1}{2\sigma^2}\left(\frac{1}{y^2}-1\right)^2} dy$$

let $u = \frac{1}{y^2}$ then $du = \frac{-2}{y^3}dy,$ and $dy = \frac{-du}{2u^{\frac{3}{2}}}$ for $\infty < u < 0$

$$\therefore E(Y^2) = -k\int_\infty^0 u^{\frac{1}{2}} e^{\frac{-1}{2\sigma^2}(u-1)^2} \frac{du}{2u^{\frac{3}{2}}} = \frac{k}{2}\int_0^\infty u^{-1} e^{\frac{-1}{2}\left(\frac{u-1}{\sigma}\right)^2} du \tag{21}$$

where $k = \frac{2}{\sigma\sqrt{2\pi}\left[1-\Phi\left(\frac{-1}{\sigma}\right)\right]}$

let $z = \frac{u-1}{\sigma}$ then $u = z\sigma + 1$ and $du = \sigma dz$ for $\frac{-1}{\sigma} < z < \infty$

$$\therefore E(Y^2) = .\frac{k}{2}\int_{\frac{-1}{\sigma}}^\infty (1+z\sigma)^{-1} e^{\frac{-z^2}{2}} \sigma dz$$

Using the binomial expansion on $(1+z\sigma)^{-1}$, given in Equation 16 we have

$$(1+z\sigma)^{-1} = 1 - 1z\sigma + 1(z\sigma)^2 - 1(z\sigma)^3 + \ldots$$

$$\therefore E(Y^2) = \frac{\sigma k}{2}\int_{\frac{-1}{\sigma}}^\infty [1 - z\sigma + (z\sigma)^2 - (z\sigma)^3 + \ldots]\, e^{\frac{-z^2}{2}} dz \tag{22}$$

$$E(Y^2)=\frac{1}{\left[1-\Phi\left(\frac{-1}{\sigma}\right)\right]}\left[\begin{array}{l}\frac{1}{\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{\infty}e^{\frac{-z^2}{2}}dz-\frac{\sigma}{\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{\infty}ze^{\frac{-z^2}{2}}dz\\+\frac{\sigma^2}{\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{\infty}z^2e^{\frac{-z^2}{2}}dz-\frac{\sigma^3}{\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{\infty}z^3e^{\frac{-z^2}{2}}dz+\ldots\end{array}\right]$$

$$=\frac{1}{\left[1-\Phi\left(\frac{1}{\sigma}\right)\right]}\left[\begin{array}{l}\left[\Pr\left(z>\frac{-1}{\sigma}\right)-\frac{\sigma}{\sqrt{2\pi}}e^{\frac{-1}{2\sigma^2}}+\left(\frac{\sigma^2}{2}-\frac{\sigma e^{\frac{-1}{2\sigma^2}}}{\sqrt{2\pi}}+\frac{\sigma^2}{2}\Pr\left(x^2_{(1)}<\frac{1}{\sigma^2}\right)\right)\right]\\-\left(\frac{\sigma e^{\frac{-1}{2\sigma^2}}}{\sqrt{2\pi}}-\frac{2\sigma^3 e^{\frac{-1}{2\sigma^2}}}{\sqrt{2\pi}}\right)+\ldots\end{array}\right]$$

$$=\frac{1}{\left[1-\Phi\left(\frac{-1}{\sigma}\right)\right]}\left[\begin{array}{l}\left(1-\Phi\left(\frac{-1}{\sigma}\right)\right)-\frac{3\sigma}{\sqrt{2\pi}}e^{\frac{-1}{2\sigma^2}}\\+\frac{\sigma^2}{2}\left[1+\Pr\left(x^2_{(1)}<\frac{1}{\sigma^2}\right)\right]-\frac{2\sigma^3 e^{\frac{-1}{2\sigma^2}}}{\sqrt{2\pi}}+\ldots\end{array}\right]$$

$$\begin{aligned}E(Y^2)=1&-\frac{3\sigma}{\sqrt{2\pi}\left[1-\Phi\left(\frac{-1}{\sigma}\right)\right]}e^{\frac{-1}{2\sigma^2}}+\frac{\sigma^2}{2\left[1-\Phi\left(\frac{-1}{\sigma}\right)\right]}\left[1+\Pr\left(\chi^2_{(1)}<\frac{1}{\sigma^2}\right)\right]\\&-\frac{2\sigma^3 e^{\frac{-1}{2\sigma^2}}}{\sqrt{2\pi}\left[1-\Phi\left(\frac{-1}{\sigma}\right)\right]}\end{aligned}\qquad(23)$$

Observe the following:

1. Subsequent terms in series (20) and (23) for $E(Y)$ and $E(Y^2)$ respectively all have $e^{\frac{-1}{2\sigma^2}}$ as a factor.

2. $e^{\frac{-1}{2\sigma^2}} = 0$ for $\sigma \leq 0.22$ correct to 4 decimal places. (See Table 7, column 3)
3. Conditions (1) and (2) imply that all subsequent terms for $E(Y)$ and $E(Y^2)$ are all zeros for $\sigma \leq 0.22$.

Thus, without loss of generality

$$E(Y) = 1 + \frac{3\sigma^2}{16\left[1-\Phi\left(\frac{-1}{\sigma}\right)\right]}\left[1+\Pr\left(\chi^2_{(1)} < \frac{1}{\sigma^2}\right)\right] \text{ for } \sigma \leq 0.22 \tag{24}$$

and

$$E(Y^2) = 1 + \frac{\sigma^2}{2\left[1-\Phi\left(\frac{-1}{\sigma}\right)\right]}\left[1+\Pr\left(\chi^2_{(1)} < \frac{1}{\sigma^2}\right)\right] \text{ for } \sigma \leq 0.2 \tag{25}$$

thus

$$Var(Y) = E(Y^2) - [E(Y)]^2$$

$$Var(Y) = 1 + \frac{\sigma^2}{2\left[1-\Phi\left(\frac{-1}{\sigma}\right)\right]}\left[1+\Pr\left(\chi^2_{(1)} < \frac{1}{\sigma^2}\right)\right] - \left[1 + \frac{3\sigma^2}{16\left[1-\Phi\left(\frac{-1}{\sigma}\right)\right]}\left[1+\Pr\left(\chi^2_{(1)} < \frac{1}{\sigma^2}\right)\right]\right]^2$$

$$= \frac{\sigma^2}{8\left[1-\Phi\left(\frac{-1}{\sigma}\right)\right]}\left[1+\Pr\left(\chi^2_{(1)} < \frac{1}{\sigma^2}\right)\right] - \left[\frac{3\sigma^2}{16\left[1-\Phi\left(\frac{-1}{\sigma}\right)\right]}\left[1+\Pr\left(\chi^2_{(1)} < \frac{1}{\sigma^2}\right)\right]\right]^2 \qquad (26)$$

Numerical computations of mean and variance of $Y\left(=e_t^*\right)$

Now compute the values of $E(Y)$ and Var(Y) for $\sigma \in [0.01, 0.22]$ using the functional expressions obtained in Equations 24 and 26, respectively. Table 7 shows the computations of $E(Y)$ and Var(Y). For these computations we write

$$E(Y) = 1 + \frac{3\sigma^2 B}{8(2A)} \qquad \sigma < 0.22$$

and

$$Var(Y) = \frac{\sigma^2 B}{8A} - \left(\frac{3\sigma^2 B}{16A}\right)^2$$

where $\mathrm{A} = 1 - \Phi\left(-\frac{1}{\sigma}\right)$ and $\mathrm{B} = 1 + \Pr\left(\chi^2_{(1)} < \frac{1}{\sigma^2}\right)$

From Table 7, columns 4 and 5, A = 1 and B = 2 for $\sigma < 0.22$

$$\therefore E(Y) = 1 + \frac{3\sigma^2}{8} \qquad \sigma < 0.22 \qquad (27)$$

and

$$Var(Y) = \frac{\sigma^2}{4} - \left(\frac{3\sigma^2}{8}\right)^2 \qquad \sigma < 0.22 \qquad (28)$$

Equation 27 is the relationship observed with simulated data in Tables 3-6.

Results

The following results were obtained from the investigations carried out on the pdf of $e_t^*\left(=\frac{1}{\sqrt{e_t}}\right)$, $g(y)$ where $e_t \sim N\left(1,\sigma^2\right)$, left truncated at 0.

i. The curve shapes are bell-shaped, with mode $\approx$ mean ≈ 1 when $0 < \sigma \leq 0.145$ correct to 1 decimal place.

Using simulated data, whenever $\sigma < 0.15$

ii. Median $\approx$ Mean ≈ 1

iii. $E\left(e_t^*\right)=1+\frac{3}{8}\sigma^2$

iv. $\frac{Var\left(e_t\right)}{Var\left(e_t^*\right)}=4$, thus $\text{var}(e_t^*) \approx \frac{1}{4}Var(e_t)$

v. e_t^* is normally distributed when $\sigma \leq 0.14$. It was observed that the normality of a pdf curve at a point b implied normality at points $0<a\leq b\in\Re$.

Using the functional expressions for mean and variance of e_t^*

vi. $E\left(e_t^*\right)=1+\frac{3}{8}\sigma^2 \qquad \sigma \leq 0.22$
≈ 1 correct to 2 decimal places (dp) when $\sigma \leq 0.11$
correct to 1 dp when $\sigma \leq 0.22$

vii. $Var(e_t^*)=\frac{\sigma^2}{4}-\left(\frac{3\sigma^2}{8}\right)^2 \quad \sigma \leq 0.22$

viii. $\frac{Var\left(e_t\right)}{Var\left(e_t^*\right)} \approx 4$
correct to 2 dp when $\sigma \leq 0.04$
correct to 1 dp when $\sigma \leq 0.14$

Table 7. Computations of $E(Y)$ & $Var(Y)$ for $\sigma \in [0.01, 0.3]$

σ	σ^2	$e^{\frac{-1}{2\sigma^2}}$	A	B	$E(Y)$	$Var(Y)$	$VarX / Var\ Y$
0.01	0.0001	0.0000000	1.00000	2.00000	1.00004	0.0000250	4.00023
0.02	0.0004	0.0000000	1.00000	2.00000	1.00015	0.0001000	4.00090
0.03	0.0009	0.0000000	1.00000	2.00000	1.00034	0.0002249	4.00203
0.04	0.0016	0.0000000	1.00000	2.00000	1.00060	0.0003996	4.00360
0.05	0.0025	0.0000000	1.00000	2.00000	1.00094	0.0006241	4.00563
0.06	0.0036	0.0000000	1.00000	2.00000	1.00135	0.0008982	4.00812
0.07	0.0049	0.0000000	1.00000	2.00000	1.00184	0.0012216	4.01106
0.08	0.0064	0.0000000	1.00000	2.00000	1.00240	0.0015942	4.01445
0.09	0.0081	0.0000000	1.00000	2.00000	1.00304	0.0020158	4.01831
0.10	0.0100	0.0000000	1.00000	2.00000	1.00375	0.0024859	4.02263
0.11	0.0121	0.0000000	1.00000	2.00000	1.00454	0.0030044	4.02741
0.12	0.0144	0.0000000	1.00000	2.00000	1.00540	0.0035708	4.03266
0.13	0.0169	0.0000000	1.00000	2.00000	1.00634	0.0041848	4.03839
0.14	0.0196	0.0000000	1.00000	2.00000	1.00735	0.0048460	4.04459
0.15	0.0225	0.0000000	1.00000	2.00000	1.00844	0.0055538	4.05127
0.16	0.0256	0.0000000	1.00000	2.00000	1.00960	0.0063078	4.05844
0.17	0.0289	0.0000000	1.00000	2.00000	1.01084	0.0071075	4.06610
0.18	0.0324	0.0000002	1.00000	2.00000	1.01215	0.0079524	4.07425
0.19	0.0361	0.0000010	1.00000	2.00000	1.01354	0.0088417	4.08291
0.20	0.0400	0.0000037	1.00000	2.00000	1.01500	0.0097750	4.09207
0.21	0.0441	0.0000119	1.00000	2.00000	1.01654	0.0107515	4.10175
0.22	0.0484	0.0000326	1.00000	1.99999	1.01815	0.0117706	4.11195
0.23	0.0529	0.0000785	0.99999	1.99999	1.01984	0.0128315	4.12268
0.24	0.0576	0.0001699	0.99998	1.99997	1.02160	0.0139334	4.13394
0.25	0.0625	0.0003355	0.99997	1.99994	1.02344	0.0150757	4.14575
0.26	0.0676	0.0006134	0.99994	1.99988	1.02535	0.0162574	4.15811
0.27	0.0729	0.0010503	0.99989	1.99979	1.02734	0.0174777	4.17104
0.28	0.0784	0.0016993	0.99982	1.99964	1.02940	0.0187356	4.18454
0.29	0.0841	0.0026181	0.99972	1.99944	1.03154	0.0200304	4.19862
0.30	0.0900	0.0038659	0.99957	1.99914	1.03375	0.0213609	4.21330

From the probability density curves, the results obtained from simulated data and the functional expressions for the mean and variance, $\sigma \leq 0.14$ (intersecting region) is the recommended condition for successful inverse square root transformation.

The results of this investigation together with findings from similar investigations with respect to the error term $e_t \sim N(1, \sigma^2)$ under other types of

Table 8. Summary of this and similar findings with respect to the error term $e_t \sim N(1,\sigma^2)$ under different transformations

e_t^*	Distribution of e_t^*	Condition for successful transformation	Relationship between σ and σ_1
$\log_e e_t$	$e_t^* \sim N(0,\sigma_1^2)$	$\sigma < 0.1$	$\sigma_1 \approx \sigma$
$\frac{1}{e_t}$	$e_t^* \sim N(1,\sigma_1^2)$	$\sigma \le 0.1$	$\sigma_1 \approx \sigma$
$\sqrt{e_t}$	$e_t^* \sim N(1,\sigma_1^2)$	$\sigma \le 0.59$	$\sigma_1 \approx \frac{1}{2}\sigma$
e_t^2	$e_t^* \sim N(1,\sigma_1^2), \sigma_1^2 = 1$	$\sigma \le 0.027$	$\sigma_1 > \sigma$
$\frac{1}{\sqrt{e_t}}$	$e_t^* \sim N(1,\sigma_1^2)$	$\sigma \le 0.14$	$\sigma_1 \approx \frac{1}{2}\sigma$

Conclusion

From the results of the investigations of the distributions of the error term (e_t) of the multiplicative time series model and its inverse square root transformed error term (e_t^*), it is clear that the condition for successful inverse square root transformation is $\sigma < 0.14$. This is because the two stochastic processes e_t and e_t^* are normally distributed with mean 1, but with the variance of inverse square root transformed error term being one quarter of the variance of the untransformed error component whenever $\sigma < 0.14$, outside this region transformation is not advisable since the basic assumption on the error term are violated after the transformation. This relationship between the two variances, $Var(e_t^*) \approx \frac{1}{4} Var(e_t)$, agrees with findings of Otuonye et al. (2011) under square root transformation, however the region of successful transformation obtained is closer to the region obtained for the logarithmic and inverse transformations by Iwueze (2007) and Nwosu et al. (2013).

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